Normal subgroups of gauge groups

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MPI/88-4

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1.Introduction

Lie groups G have been used in mathematical physics for a long time. Recently, infinite–dimensional groups started to appear there too (as gauge and current groups). Some of those groups have the form $G(A)$, where G is a simple connected Lie group and A is a ring of continuous functions on a topological space X (in this notation, we use the algebraic group structure on adG ,see below). When $A = \mathbb{R}^X$ is the ring of all continuous functions $X \to \mathbb{R}$ (where $\mathbb{R}$ is the reals), then $G(A)$ is just the group G^X of all continuous maps $X \to G$. In the case when X is a circle, those groups are known as loop groups [9].

To give a precise definition of $G(A)$, we consider the adjont representation ad of G on its Lie algebra $\mathfrak{g}$. Picking a basis in $\mathfrak{g}$, we obtain a group morphism $ad : G \to \mathrm{GL}_m\mathbb{R}$, where $m = \dim(\mathfrak{g})$. We assume that G is a simple Lie group, i.e. its Lie algebra $\mathfrak{g}$ is simple. Then ad is a local isomorphism from G to $\mathrm{Aut}(\mathfrak{g})$, and the kernel of ad is the center of G. We do not assume that G is connected, so G could be the whole $\mathrm{Aut}(\mathfrak{g})$.

By definition, the group $G(A)$ consists of all $g \in G^X = G(\mathbb{R}^X)$ such that $ad(g) \in \mathrm{GL}_mA$. Assuming that A contains all constants, this does not depend on choice of basis in $\mathfrak{g}$. For any ideal B of A, let $G(B)$ denote the group of all g in $G(A)$ such that $ad(g)$ is congruent to the identity matrix 1_m modulo B.

We assume that A contains all constants and that all functions in A are bounded. We endow A with the topology of uniform convergence, and consider the induced topology on $G(A)$. For every ideal B of A, let $G(B)^0$ denote its connected component of identity in $G(B)$.

We will impose on A the following condition with a natural number N depending on G:

(1) $a^{1/N} \in \mathrm{GL}_1 A$ for any a in A sufficiently closed to 1

In particular, this condition with $N = 1$ says that $\mathrm{GL}_1 A$ is open in A. By the way, $a^{1/N}$ means the positive root, which is defined for positive functions a. Note that the condition (1) with some N implies this condition with N replaced by any divisor of any power of N.

Sometimes (depending on G), we will impose a stronger than (1) condition on A:

(2) $a^{1/N} \in \mathrm{GL}_1 A[i]$ for any $a \in A[i]$ sufficiently closed to 1.

Here $A[i]$ is the subring of all complex functions a on X with both real and imaginary part in A. For any a in $\mathbb{C}^X$ with $\sup |a - 1| < 1$, $a^{1/N}$ is defined by the series $a^{1/N} = 1 + (a - - 1))^{1/N} = 1 + (a - 1)/N + (a - 1)^2 (1/N)(1/N - 1)/2 + \ldots$.

Note that the conditions (1) and (2) hold for all natural numbers N when A is closed in $\mathbb{R}^X$ (then A is a Banach algebra). They also hold for all natural numbers N when A is the ring of all smooth bounded functions on a manifold X or the ring of real analytic bounded functions on an analytic set X.

Here is the main result of this paper.

Theorem 3. Let X be a topological space, A a ring of bounded real continuous functions on X containing all constants and satisfying the condition (2) for all natural numbers N. Then for any simple Lie group G of classical type a subgroup H of $G(A)$ is normalized by $G(A)^0$ if and only if $G(B)^0 \subset H \subset G(B)$ for an ideal B of A.

When X is a single point, so $A = \mathbb{R}$, the theorem says that the group $G(\mathbb{R})^0$ modulo its center is simple (as an abstract group). This fact goes back to Dickson [3], van der Waerden [12], E.Cartan [1].

We will prove Theorem 3 together with the following theorem.

Theorem 4. Under the conditions of Theorem 3:

(a) the group $G(B)^0$ is arcwise connected for any ideal B of A;

(b)there is a natural number N' such that for any ideal B of A and any open neighbourhood U of 1 in G, the set of all products of N' commutators of the form $[g, h]$, where $g \in G(A) \cap U^X$ and $h \in G(B) \cap U^X$, is open in $G(B)$; so

$[G(A), G(B)^0] = [G(A)^0, G(B)] = [G(A)^0, G(B)^0] = G(B)^0$ for any ideal B of A.

It seems that the restriction that G is of classical type in the theorems is redundant. When G is a Chevalley group (for example, G is a complex Lie group, i.e. G is not absolutely simple as algebraic group over $\mathbb{R}$), the conclusions of Theorems 3, 4 follows easily from [15] (rank ≥ 2) and [16] (rank = 1, i.e. $G(\mathbb{R})$ is locally isomorphic to $SL_2\mathbb{R}$ or $SL_2\mathbb{C}$). In this case $G(A)^0$ is generated by all its "root subgroups" with respect to a fixed maximal torus as well as by all its elements g with unipotent $ad(g)$. This seems to be true whenever G is isotropic, i.e. $G(\mathbb{R})$ is not compact, i.e. its $\mathbb{R}$–rank is not 0, i.e. there are non–trivial unipotent elements in G (for G of classical type, this can be shown by our methods).

We will use results of [16] to prove Theorems 3, 4 in some cases. For orthogonal groups of $\mathbb{R}$–rank ≥ 2 and for some other classical groups of $\mathbb{R}$–rank ≥ 3, we could use results of [19], [20]. However we do not do this here, because we have to deal also with small $\mathbb{R}$–rank cases, and when $\mathbb{R}$–rank is 0, the methods of [19], [20] using unipotent elements do not work.

Note that the structure of anisotropic groups over rings is much more intricate than that of isotropic groups. This is true even when the ring is a field. Some work on anisotropic case was done in [2], [4]– [11] .

In particular, de la Harpe [4] proved that every maximal normal subgroup of $G(A)^0$ has the form $G(B) \cap G(A)^0$, where B is a maximal ideal of A, provided that G is compact and A is the ring of all bounded smooth functions on a smooth manifold X. He noted that the analogous result look plausible for the real analytic functions A and non–compact G. We cofirm this here for classical G, and our Theorem 3 describes all normal subgroups of $G(A)^0$ (not only the maximal ones).We can handle a wider than in [4] class of rings A, because our methods do not use partitions of 1. Our methods allow in fact to handle also power serries rings [2], [8], but in this paper we restrain ourselfs to subrings A of $\mathbb{R}^X$.

The result of [4] on the maximal normal subgroups of $G(A)^0$ was used there to describe all automorphisms of this group, improving a previous result of [9]. Similarly, our results allow to describe all automorphisms of $G(A)^0$ (under the conditions of Theorem 3), but we do not pursude this in the present paper.

Often the structure of Lie group on G in Theorem 3 can be refined to the structure of an algebraic group over $\mathbb{R}$ (when the center of G is finite). In this case, we can consider the group G_A of the points over A for any commutative $\mathbb{R}$–algebra A. When G is adjoint, i.e. has trivial center, $G_A = G(A)$. It is not clear whether G_A and $G(A)$ could be different. In any case, we will see that ${G_A}^0$ and $G(A)^0$. We use the groups $G(A)$ rather than G_A for the following two reasons: they make sense even if G is not algebraic, and they behave nicely under local isomorphisms.

Namely, in the next section, we show that the statements of Theorems 3, 4 for locally isomorphic groups are equivalent. Then we give a complete list of simple Lie groups of classical types up to local isomorphisms. We also indicate the number N in the condition (1) or (2) used. In Sections 3–13 we prove Theorems 3, 4 for each G on the list.

Note that the "if" part in the conclusion of Theorem 3 is obvious (it follows from the inclusion $G(B)^0 \supset [G(A)^0, G(B)]$). So it suffices to show the "only if" part, i.e. the following statement:

(5) for any subgroup H of $G(A)$ which is normalized by $G(A)^0$ there is an ideal B of A such that $G(B)^0 \subset H \subset G(B)$

However to show the equivalence of (4) for locally isomorphic groups we had to couple it with the conclusion (b) of Theorem 4. It is conceivable that one can take $N' = 2$ independently of G in Theorem 4(b), but minimization of N' is not a goal of this paper. Also it seems that in the case when the fundamental $\pi_1 G(\mathbb{R})$ is finite and the dimension of X is finite, every element of $G(A)^0$ is the product of a bounded number commutators, where the bound depends only on G and the dimension of X. See [23] about the case $G = \mathrm{SL}_n\mathbb{R}$ or $\mathrm{SL}_n\mathbb{C}$.

2. Locally isomorphic groups

It is clear that if the conclusion (a)of Theorem 4 and the first half of 4(b) (which obviously implies the second half of (b)) hold for a simple Lie group G, then the corresponding conclusions also hold for any Lie group G' which is locally isomorphic to G.

Here we show that if the statement (5) and 4(b) is true for a simple Lie group G, then the corresponding statements are also true for any Lie group G' locally isomorphic to G. It suffices to show that in the case when the other group is $G' = \mathrm{Aut}(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G. We will use the symbol *ad* (adjoint representation) for the canonical homomorphism $G(A) \to G'(A)$ for any ring A.

By our definition, $G(B)$ is the inverse image of $G'(B)$ for any ideal B of A. Since G and G' are locally isomorphic, $ad(G(B)^0) = G'(B)^0$ for any ideal B of A.

Assume first that (5) holds for G, and let us prove it for G'. Let H' be a subgroup of $G'(A)$ which is normalized by $G'(A)^0$. Then its inverse image H in $G(A)$ is normalized by $G(A)^0$. By (5) for G, we have $G(B)^0 \subset H \subset G(B)$ for some ideal B of A. Applying *ad*, we obtain that $ad(G(B)^0) = G'(B)^0 \subset H' \subset ad(G(B) \subset G'(B)$. (We did not use 4(b) .)

Assume now the statements (5) and 4(b) with G' instead of G, and let us prove (5) for G. Let H be a subgroup of $G(A)$ which is normalized by $G(A)^0$. Then $H' = ad(H)$ is a subgroup of $G'(A)$ which is normalized by $G'(A)^0$. By (5) for G', $G'(B)^0 \subset H' \subset G'(B)$ for an ideal B of A. Taking the inverse image, we conclude that $\ker(ad)G(B)^0 \subset H \subset G(B)$. Now we have to rid off $\ker(ad)$ (which is the center of $G(A)$) here. That is, we want to show that $f \in H$ for any f in $G(B)$ sufficiently close to 1.

Using the condition 4(b) (for G or, equivalently, for G'), we pick a neighborhood U of 1 in $G(\mathbb{R})$ such that no product of $4N'$ elements of U is a non-trivial element of the center of $G(\mathbb{R})$ (which is a discrete subgroup of $G(\mathbb{R})$). Then we use 4(b) with this U. Any $f \in G(B)$ sufficiently close to 1 can be written as $f = [g_1, h_1] \dots [g_{N'}, h_{N'}]$ with g_m in $G(A) \cap U^X$ and h_m in $G(B) \cap U^X$. Replace h_m here by $h_m c_m \in H$ with c_m in the center of $G(A)$, we obtain that $f = [g_1, h_1 c_1] \dots [g_{N'}, h_N c_{N'}] \in H$. Thus, we have obtained the conclusion (5) for G.

Under the conditions of Theorem 3, the group $G(\mathbb{R})$ is locally isomorphic to one of the following Lie groups.

Complex Lie groups: $G(\mathbb{R}) = SU_n\mathbb{C}$, $n \geq 2$; $G(\mathbb{R}) = Sp_{2n}\mathbb{C}$, $n \geq 2$; $G(\mathbb{R}) = SO_n\mathbb{C}$, $n \geq 7$.
Real of type A: $G(\mathbb{R}) = SL_n\mathbb{R}$, $n \geq 2$; $G(\mathbb{R}) = SL_n(\mathbb{H})$, $n \geq 1$; $G(\mathbb{R}) = SU(p,q; \mathbb{C})$, $p \geq q \geq 0$, $p + q \geq 3$.
Real of types B *and* D : $G(\mathbb{R}) = SO(p, q; \mathbb{R})$, $p \geq q \geq 0, p + q \geq 7$; $G(\mathbb{R}) = SO(n, \mathbb{H})$, $n \geq 3$.
Real of type C: $G(\mathbb{R}) = Sp_{2n}\mathbb{R}$, $n \geq 2$; $G(\mathbb{R})$) $= Sp(p, q; \mathbb{H})$, $p \geq q \geq 0, p + q \geq 2$.

In Sections 3-13 we remind definitions of these groups, and we prove (5) and Theorem 4 in each of these cases.

Every group on the list has its own algebraic group structure over $\mathbb{R}$. It is not clear whether the corresponding group G_A may differ from $G(A)$. In any case, we will see that $G(B)^0 = G_B{}^0$ for all ideals B of A (including $B = A$) under the condition (1) or (2) with an appropriate N.

Now we specify N in the condition (1) or (2) used in our proof. We will use the following condition:

the condition (1) with $N = n$ in the case $G(\mathbb{R}) = SL_n\mathbb{R}$; the condition (1) with $N = 2n$ in the case $G(\mathbb{R}) = SL_n(\mathbb{H})$; the condition (2) with $N = n$ in the cases $G(\mathbb{R}) = SU_n\mathbb{C}$, and $SU(p, q; \mathbb{C})$, $n = p + q$; the condition (1) with $N = 2$ in all other cases.

Note that this N is the order of the center of $G(\mathbb{C})$ in the case $G(\mathbb{R}) = SL_n\mathbb{R}$, and N is the order of the center of $G(\mathbb{R})$ in all other cases, except the cases $G(\mathbb{R}) = SO(p, q; \mathbb{R})$with odd $p + q$ and $G(\mathbb{R}) = SO_n\mathbb{C}$ with odd n (in which cases the center is trivial). It can be shown that under the condition (2) with $N = 1$, for each listed G, the above condition is necessary for (5) or 4(b) to be true.

In all cases above, $G(\mathbb{R})$ is realized as a subgroup of GL_nF for some n, where $F = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ is a finite-dimensional division algebra over $\mathbb{R}$. Its Lie algebra $\mathfrak{g}$ will be realized as a subalgebra of the Lie algebra of all n by n matrices over F. The adjoint representation is given by $ad(g)h = ghg^{-1}$, where $g \in G$ and $h \in \mathfrak{g}$.

Sometimes we will use improved versions of the conditions (1), (2) which in fact follow from them:

Lemma 6. Under the condition (1) (resp. (2)),let B be an ideal of A. Then $a^{1/N} \in \mathrm{GL}_1 B$ (resp. $a^{1/N} \in \mathrm{GL}_1 B[i]$) for any $a \in A$ such that $a - 1 \in B$ and $\sup |a - 1| < 1$. Moreover, under the condition (1), $a^{1/N} \in \mathrm{GL}_1 A$ for any $a \in A$ with $\inf(a) > 0$.

Proof. Consider first the case when a is very close to 1. Then $a^{1/N} = 1 + x \in \mathrm{GL}_1 A$ (resp. $a^{1/N} \in \mathrm{GL}_1 A[i]$). Moreover, from $(1 + x)^N = a$, we see that $x(N + N(N - 1)x/2 + \dots) = a - 1 \in B$. For a close to 1, x is close to 0, hence $N + N(N - 1)x/2 + \dots \in \mathrm{GL}_1 A$ (resp. $\mathrm{GL}_1 A[i]$) and $x \in B$ (resp. $B[i]$). So $a^{1/N} \in \mathrm{GL}_1 B$ (resp. $a^{1/N} \in \mathrm{GL}_1 B[i]$).
Now, given any a as in the lemma, and any natural number M, we can approximate a arbitrary closely by the M-th power of a trunncated Taylor serries b for $(1 + (a - 1))^{1/M}$. So $(a/b^M)^{1/N} \in \mathrm{GL}_1 B$ (resp. $\mathrm{GL}_1 B[i]$). Taking a large M we can make b arbitrary close to 1, so that $b^{1/N} \in \mathrm{GL}_1 B$ (resp. $\mathrm{GL}_1 B[i]$). Then $a^{1/N} \in \mathrm{GL}_1 B$ (resp. $\mathrm{GL}_1 B[i]$).The first part of the lemma is proved. To prove the second part, we set $c = \sup(a) \in \mathbb{R}$. Then $\sup |a/c - 1| = 1 - \inf(a)/c < 1$. So $(a/c)^{1/N} \in \mathrm{GL}_1 A$, hence $a^{1/N} \in \mathrm{GL}_1 A$. The lemma is proved.

It is clear that the condition (2) implies (1) (with the same N). In Section 6 we will show that the conditions (1) and (2) with $N = 2$ are equivalent.

3. The case $G(\mathbb{R}) = \mathrm{SL}_n\mathbb{R}$, $n \geq 2$

The Lie algebra $\mathfrak{g}$ of $G(\mathbb{R})$ consists of all n by n real matrices of trace 0. Its dimension is $n(n - 1/2$. The group G acts on $\mathfrak{g}$ by $ad(g)(h) = ghg^{-1}$, where $g \in G$ and $h \in \mathfrak{g}$.

We pick the basis of $\mathfrak{g}$ consisting of all matrix units $e_{p,q}$ with $1 \leq p \neq q \leq n$ and all $e_{p,p} - e_{q,q}$ with $1 \leq p < q \leq n$. Then it is clear that $G(A)$ consists of all g in $\mathrm{SL}_n\mathbb{R}^X$ such that $xy \in A$ for any entry x of g and any entry y of g^{-1}.

In particular, multiplying the equality $\det(g^{-1}) = 1$ by x^n, we obtain that $x^n \in A$ for every entry x of g..

Now we assume the condition (1) with $N = n$ (note that this N is the order of the center of the complexification $\mathrm{SL}_n\mathbb{C}$ of $G(\mathbb{R})$). Then, given $g = (g_{p,q})$ in $G(A)$ sufficiently close to 1_n, the above inclusion $x^n \in A$ implies that all diagonal entries x of g belong to A. Now the inclusions $xy \in A$ imply that all entries of g^{-1} belong to A. Thus, SL_nA contains a neighbourhood of 1_n in $G(A)$, hence $G(A)^0 = (\mathrm{SL}_nA)^0 = \mathrm{E}_nA$ (see [16].[18], [21] , [22] about the last equality).

Similarly, $G(B)^0 = (\mathrm{SL}_nB)^0 = \mathrm{E}_n(A , B)$ for any ideal B of A. (We have used that, by Lemma 6, $(1 + b)^{1/n} \in 1 + B$ for any b in B sufficiently close to 0.)

Here E_nA is the subgroup of SL_nA generated by all elementary matrices $a^{p,q}$ with $a \in A$, $1 \leq p \neq q \leq n$. The group $\mathrm{E}_n(A , B)$ is the normal subgroup of E_nA generated by elementary matrices in SL_nB. Now it is clear, that $G(B)^0 = (\mathrm{SL}_nB)^0$ is arcwise connected.

By [16], $\mathrm{E}_n(A , B) = [\mathrm{E}_n(A , B), \mathrm{E}_nA] = [\mathrm{E}_nB, \mathrm{E}_nA]$. So the second part of 4(b) for our $G(\mathbb{R}) = \mathrm{SL}_n\mathbb{R}$ is proved. To prove the first part of 4(b), we have to make some estimates on the number and size of commutators.

For any real $\partial > 0$, let U_∂ denote the set of matrices g in $G(\mathbb{R})$ such that $|(1_n - g)_{p,q} < \partial$ for all p, q. These U_∂ and their left shifts genetrate the topology on G. For $\partial < 1/n$, every matrix h in $U_\partial^X \cap G(B)$ is the product of an upper triangular matrix in $U_{\partial/(1-\partial n)}{}^X \cap G(B)$ with ones along the diagonal and a lower diagonal matrix in $U_{\partial/(1-\partial n)}{}^X \cap G(B)$. So h is a product of $n(n - 1)$ elementary matrices in $U_{\partial/(1-\partial n)}{}^X \cap G(B)$ and a diagonal matrix in $U_{\partial/(1-\partial n)}{}^X \cap G(B)$.

Every elementary $b^{p,q}$ in $U_{\partial/(1-\partial n)}{}^X \cap G(B)$ is a commutator of an elementary matrix in $U_{\partial'}{}^X \cap G(B)$ and a diagonal matrix in $U_{\partial'}$, where $\partial' = (1 + (\partial/(1-\partial n))^{1/2})^{1/2} - 1$. For example,

$$b^{1,2} = [\mathrm{diag}(1+\partial', (1+\partial')^{-1}), (b/((1+\partial')^{-2})^{1,2}].$$

Every diagonal matrix in $U_{\partial/(1-\partial n)}{}^X \cap G(B)$ is a product of $n-1$ diagonal matrices of the form $\mathrm{diag}(d,1\ldots1, d^{-1},1\ldots)$ in $U_{\partial''}{}^X \cap G(B)$, where ∂'' is a number depending on ∂ and n which tends to 0 as $\partial \to 0$. Every matrix $\mathrm{diag}(d, d^{-1})$ in $\mathrm{SL}_2B \cap U_{\partial''}{}^X$ can be written as $\mathrm{diag}(d, d^{-1})$

$$=(-\varepsilon d^{-1})^{2,1}((d-1)/\varepsilon)^{1,2}(\varepsilon)^{2,1}((1-d^{-1})/\varepsilon)^{1,2}$$
$$=[(-\varepsilon d^{-1})^{2,1},((d-1)/\varepsilon)^{1,2}]((d-1)/\varepsilon)^{1,2}(\varepsilon-\varepsilon d^{-1})^{2,1}((1-d^{-1})/\varepsilon)^{1,2},$$

where $\varepsilon = \partial''^{1/2} \to 0$ as $\partial \to 0$. So $\mathrm{diag}(d, d^{-1})$ is a product of 4 commutators of the form $[g, h]$ where $g \in G(A) \cap U_{\varepsilon'}{}^X$ and $h \in G(B) \cap U_{\varepsilon'}{}^X$ with $\varepsilon' \to 0$ as $\partial \to 0$. Adding up the number of commutators involved, we see that every matrix h in $U_\partial{}^X \cap G(B)$ is the product of $N' = n(n-1) + 4(n-1)$ commutators of the form $[g, h]$ where $g \in G(A) \cap U_\lambda{}^X$ and $h \in G(B) \cap U_\lambda{}^X$ with some number $\lambda \to 0$ as $\partial \to 0$. Theorem 4 is proved.

Now we prove (5). Let H be a subgroup of $G(A)$ which is normalized by $G(A)^0 = (\mathrm{SL}_nA)^0 = \mathrm{E}_nA$. Set $H' = H \cap \mathrm{SL}_nA$. By[16], there is an ideal B of A such that $\mathrm{E}_n(A, B) \subset H' \subset G(B) \cap \mathrm{SL}_nA$. We claim that $H \subset G(B)$, i.e. $h \in G(B)$ for an arbitrary element $h \in H$.

Indeed, since $[g, h] \subset H' \subset G(B) \cap \mathrm{SL}_nA$ for every g in E_nA, the assignment $g \mapsto [g, h]$ modulo B gives a homomorphism from the group E_nA to the group of scalar matrices in $\mathrm{SL}_n(A/B)$. Since the first group is perfect and the second one is abelian, this homomorphism must be trivial. That is, $hgh^{-1} \equiv g \pmod B$ for any g in E_nA. Now we use that $\mathrm{E}_n\mathbb{R} \cap \mathfrak{g}$ spans $\mathfrak{g}$ over $\mathbb{R}$ (in fact, the following matrioces in $\mathrm{E}_n\mathbb{R}$ span $\mathfrak{g}$ over $\mathbb{R}$: $(-1)^{q,p}2^{p,q}$, $(-2)^{q,p}1^{p,q}$, $2^{q,p}(-1)^{p,q}$ with $1 \le p < q \le n$). So $hgh^{-1} \equiv g \pmod B$ for any g in $\mathfrak{g}$, hence $h \in G(B)$.

4. The case $G(\mathbb{R}) = \mathrm{SL}_n\mathbb{C}$, $n \geq 2$

The proof of (4), (5) in this case repeats the proof given in the previous section. The only difference is that now we use the condition (2) with $N = n$ instead of (1) with $N = n$. The condition (2) with $N = 1$ (which follows from (2) with $N = n$) implies that $\mathrm{SL}_n A[i]^0 = \mathrm{E}_n A[i]$ and $(\mathrm{SL}_n B')^0 = \mathrm{E}_n(A[i], B')$ for any ideal B' of $A[i]$. The condition (2) with $N = n$ gives that $G(A)^0 = \mathrm{SL}_n A[i]^0$. Since every ideal B' of $A[i]$ has the form $B + Bi$, where B is an ideal of A, we obtain also that $(\mathrm{SL}_n B')^0 = G(B)^0$.

5. The case $G(\mathbb{R}) = \mathrm{Sp}_{2n}\mathbb{R}$, $n \geq 2$

We proceed as in Section 3 with the following changes. The group $G(\mathbb{R}) = \mathrm{Sp}_{2n}\mathbb{R}$ consists of all matrices g in $\mathrm{SL}_n\mathbb{R}$ such that $g^{\mathrm{T}}Jg = J$, where J is a non-singular alternating $2n$ by $2n$ matrix in $\mathrm{SL}_{2n}\mathbf{Z}$ which can be chosen to be in a standard form. The Lie algebra $\mathfrak{g}$ of G consists of all $2n$ by $2n$ matrices h over $\mathbb{R}$ with $h^{\mathrm{T}}J + Jh = 0$. The adjont representation is given by $ad(g)(h) = ghg^{-1}$.

The group $G(A)$ consists of all matrices g in $\mathrm{Sp}_{2n}\mathbb{R}^X$ such that ghg^{-1} is a matrix over A for all h in $\mathfrak{g}$. Then the product of any entry of g with the diagonal entry in the same row belongs to A.

Assuming the condition (1) with $N = 2$, we conclude that $G(A)^0 = (\mathrm{Sp}_{2n}A)^0 = \mathrm{Ep}_{2n}A$ (the last equality follows from the condition (1) with $N = 1$). For any ideal B of A we have $G(B)^0 = (\mathrm{Sp}_{2n}B)^0 = \mathrm{Ep}_{2n}(A, B)$. The standard description of all subgroups H of $\mathrm{Sp}_{2n}A$ normalized by $\mathrm{Ep}_{2n}A$ is obtained in [15] for any commutative ring $A = 2A$, $n \geq 2$ (when $n = 1$, $\mathrm{Sp}_2 A = \mathrm{SL}_2 A$ and see Section 3 above). This gives (5), (Proving (5) we have to use that the least ideal B of A such that $H \subset G(B)$ coincides with the least ideal B of A such that $H \cap \mathrm{Sp}_{2n}A \subset G(B)$. This follows from the fact that $\mathfrak{g}$ is spanned over $\mathbb{R}$ by its elements of the form $g - 1_{2n}$ with g in $G(\mathbb{R}) = \mathrm{Sp}_{2n}\mathbb{R} = \mathrm{Ep}_{2n}\mathbb{R}$.)

6. The case $G(\mathbb{R}) = \mathrm{Sp}_{2n}\mathbb{C}$, $n \geq 2$

The proof of Theorem 4 and (5) in this case repeats the proof given in the previous section. The only difference is that now we use the condition (2) with $N = 2$ instead of (1) with $N = 2$. This condition implies that $G(B)^0 = (\mathrm{Sp}_{2n}B')^0 = \mathrm{Ep}_{2n}(A[i], B')$ for any ideal $B' = B[i] = B + Bi$, of $A[i]$. Finally, we observe the following fact.

Lemma 7. When $N = 2$, the conditions (1) and (2) are equivalent.

Proof. Clearly, (2) implies (1) for any N. Let us show now that (1) implies (2) when $N = 2$. We will show that $h^{1/2} \in A[i]$ for any $h \in A[i]$ with $\sup|h - 1| < 1$. We write $h = a + bi$ and $h^{1/2} = x + yi$. Then $a = x^2 - y^2 \in A$, $b = 2xy \in A$, and $a^2 + b^2 = (x^2 + y^2)^2 \in A$, hence $x^2 + y^2 = (a^2 + b^2)^{1/2} \in A$ by Lemma 6. So $x = ((a + (a^2 + b^2)^{1/2})/2)^{1/2} \in \mathrm{GL}_1 A$, hence $y = b/2x \in A$.

7. The case $G(\mathbb{R}) = \mathrm{SO}_n\mathbb{C}$, $n \geq 5$

Now $G(\mathbb{R}) = \mathrm{SO}_n\mathbb{C}$, consists of all matrices g in $\mathrm{SL}_n\mathbb{C}$ such that $g^{\mathrm{T}}g = 1_n$. When $n = 2$ (resp. $n = 4$) this group is not simple, and it is locally isomorphic to $\mathrm{GL}_1\mathbb{C}$ (resp. $\mathrm{SL}_2\mathbb{C} \times \mathrm{SL}_2\mathbb{C}$). So we assume that $n \geq 5$. When $n = 5$ (resp. $n = 6$), this group is locally isomorphic with $\mathrm{Sp}_4\mathbb{C}$ (resp. $\mathrm{SL}_4\mathbb{C}$),

The Lie algebra $\mathfrak{g}$ of $G(\mathbb{R})$ consists of all alternating (skew-symmetric) n by n matrices over $\mathbb{C}$. Its dimension over $\mathbb{R}$ is $n(n-1)$.

The group $G(A)$ consists of all matrices g in $\mathrm{SO}(\mathbb{C}^X)$ such that all 2 by 2 minors of g belong to $A[i]$. When n is odd (i.e. the center of G is trivial), every entry of g^{-1} is a polynomial in these minors with coefficients ± 1, so $g^{-1} \in \mathrm{SO}_n A$. So $G(A) = \mathrm{SO}_n A[i]$. for odd n, and, simalarly, $G(B) = \mathrm{SO}_n B[i]$ for any ideal B of A

When n is even, to prove that $G(B)^0 = (\mathrm{SO}_n B[i])^0$, we assume the condition (1) with $N = 2$ (note that 2 is the order of the center of $G(\mathbb{R})$ for even n). By Lemma 7, the condition (2) with $N = 2$ holds.

Every matrix g in $G(A)$ sufficiently close to 1_n has all its diagonal minors in $\mathrm{GL}_1\mathbb{C}^X$. Consider the submatrix g' of g which is formed by the first 4 rows and columns. Its determinant $\det(g')$ is a polynomial with coefficients ± 1 in 2 by 2 minors of g, so $\det(g') \in A[i]$. Now we use the entry $x = (g')_{1,1} = g_{1,1} \in \mathrm{GL}_1\mathbb{C}^X$ of g' to eliminate the off diagonal entries in the first row and column of g' by addition operations. We obtain a matrix of the form $x \oplus h$ with a 3 by 3 matrix h such that xy is a 2 by 2 minor of g' for every entry y of h. So $xy \in A[i]$, and $x^2\det(g') = x^3\det(h) \in A[i]$.

Using (2) with $N = 1$ (which follows from (2) with $N = 2$), we see that $x^2 \in \mathrm{GL}_1 A[i]$. Using (2) with $N = 2$, we conclude that $x \in \mathrm{GL}_1 A[i]$. Now we can take any n -1 by n -1 submatrix g' of g containing $x = g_{1,1}$ and bring it by addition operations to the form $x \oplus h$ with a matrix h of size n -2 by n -2 such that xy is a 2 by 2 minor of g' for every entry y of h. So all entries of h are in $A[i]$, hence $\det(g') \in A[i]$. Similarly, $\det(g') \in A[i]$ for an arbitrary n -1 by n -1 submatrix g' of g, i.e. $g^{-1} \in \mathrm{SO}_n A[i]$, i.e. $g \in \mathrm{SO}_n A[i]$.

Thus, $G(A)^0 = (\mathrm{SO}_n A[i])^0$. similarly, $G(B) = \mathrm{SO}_n B[i]$ for any ideal B of A.

Note now that the quadratic form $x_1{}^2 + ... + x_n{}^2$ is isomorphic over $\mathbb{C}$ to the quadratic form $y_1y_2 + ... + y_{n-1}y_n$ when n is even, and it isomorphic to $y_1y_2 + ... + y_{n-2}y_{n-1} + y_n{}^2$ when n is odd. (Here $y_1 = x_1 + ix_2$, $y_2 = x_1 - ix_2$, ...) So G is a Chevalley group. of rank $[n/2]$, and we can use results of [15] to describe all subgroups of $\mathrm{SO}_n(A[i])$ which are normalized by $\mathrm{EO}_n(A[i])$.

It is easy to see that $\mathrm{EO}_n(A) = G(A)^0 = (\mathrm{SO}_n A[i])^0$. Indeed, every matrix g in $(\mathrm{SO}_n A[i])$ sufficiently close to 1_n is the product of $2n - 4$ elementary orthogonal matrices and a matrix of the form $\mathrm{diag}(z, z^{-1}) \oplus h$, where h is a matrix of size $n - 2$ by n -2 acting on y_m with $m \geq 2$, and $z = g_{1,1}$. So if n is odd, we are reduced to the case $n = 5$ when $G(\mathbb{R})$ is locally isomorphic to $\mathrm{Sp}_4\mathbb{C}$, and if n is even, we are reduced to the case $n = 6$ when $G(\mathbb{R})$ is locally isomorphic to $\mathrm{SL}_4\mathbb{C}$,

Details are similar to those in Section 3.

7. The case of $G(\mathbb{R}) = \mathrm{SL}_1\mathbb{H}$

Now $G(\mathbb{R})$ consists of all Hamilton quaternions $g = a + bi + cj + dk$ with the reduced norm $|g|^2 = a^2 + b^2 + c^2 + d^2 = 1$. As topological space, $G(\mathbb{R})$ is the sphere S^3. The Lie algebra $\mathfrak{g}$ of $G(\mathbb{R})$ consists of all quaternions $bi + cj + dk$ of trace 0. Its dimension is 3, and we take i, j, k as a basis.

The group $G(\mathbb{R}^X)$ can be identified with all $g = a + bi + cj + dk$, where a, b, c, d are in $\mathbb{R}^X$,$|g| = a^2 + b^2 + c^2 + d^2 = 1$, and $k = ij = -ji$, $i^2 = j^2 = -1$. The group $G(A)$ consists of all g as above with $xy \in A$ for all $x, y \in \{a, b, c, d\}$. For an ideal B of A, $G(B)$ consists of g as above with all xy above in B, except for a^2, for which we have $a^2 - 1 \in B$.

Now we assume the condition (1) with $N = 2$ (this N is the order of the center of $G(\mathbb{R})$ or $G(\mathbb{C})$).

Then for any g sufficiently close to 1 we have a,b,c,d in A. If B is an ideal of A, then for any g in $G(B)$ sufficiently close to 1 we have $a-1, b, c, d$ in B. So $a - 1, b,c,d \in B$. for any $g = a + bi + cj + dk$ in $G(B)^0$.

Let G_1 (resp., G_2,resp. G_3) denote the (algebraic) subgroup of G consisting of $g = a + bi + cj + dk$ with a, b, c, d in A , $a^2 + b^2 + c^2 + d^2 = 1$ and with $d = c = 0$ (resp, $d = b = 0$, resp. $b = c = 0$). So the Lie groups $G_1(\mathbb{R})$, $G_2(\mathbb{R})$, and $G_3(\mathbb{R})$ are all isomorphic to the circle S^1.

Lemma 8. If B is an ideal of A, $g = a + bi \in G_1(B)$ and $|g - 1| < 1$, then $g = h^2$ with h in the path-component of 1 in $G_1(B)$.

Proof. We have $2 - 2a = |g - 1|^2 < 1$, hence $1/2 < a \le 1$. By (1) with $N = 1$ (see also Lemma 6), $1 + a \in \mathrm{GL}_1 A$. Set $t = b/(1 + a) \in B$. Then $g = a + bi = (1 - t^2)/(1 + t^2) + 2t/(1 + t^2)i$. We set now $s = t/((1 + t^2)^{1/2} + 1) \in B$ and $h = (1 - s^2)/(1 + s^2) + 2s/(1 + s^2)i$. Then $t = 2s/(1 - s^2)$ and $g = h^2$. Replacing here s by $s\mu$ with real μ ranging from 0 to 1, we obtain a continuous path in $G_1(B)$ connecting 1 with h.

Corollary 9. If B', B'' are ideals of A, then the subgroup $G_1(B'' + B'')^0$ of $G(A)$ is generated by its subgroups $G_1(B')^0$ and $G_1(B'')^0$.

Proof. It suffices to show that every $g = a + bi$ in $G_1(B' + B'')$ with $|g - 1| < 1$ is in $G_1(B')^0 G_1(B'')^0$. We write $b = b' + b''$ with b' in B' and b'' in B'', and set

$$g' = (a + b'i)/(a^2 + b'^2)h^{1/2} \in G_1(B')^0.$$

Then $g'^{-1}g \in G_1(B'')^0$. So $g \in G_1(B')^0 G_1(B'')^0$.

Lemma 10. For any ideal B of A every element $g = a + bi + cj + dk$, of $G(B)$ with $|g - 1| < (0.4)^{1/2}$ has the form $g = g_1 g_2 g_3$ with g_m in $G_m(B)$ and $|g_m - 1| \leq 2\,|g - 1|$.

Proof. We want to find an element g_3 in $G_3(B)$ such that
$(a + bi + cj + dk)g_3^{-1} \in G_1(B)G_2(B)$. Writing $g_3^{-1} =$
$(1 + xk)(1 + x^2)^{-1/2} =$
$((a - dx) + (b + cx)i + (c - bx)j + (d + ax)k)(1 + x^2)^{-1/2}$, we obtain the following equation for x:
$(a - dx)(d + ax) = (b + cx)(c - bx)$,
or $x^2(bc - ad) + x(a^2 + b^2 - c^2 - d^2) + ad - bc = 0$,
or $2x/(1 - x^2) = 2(bc - ad)/(a^2 + b^2 - c^2 - d^2)$.
So we set $g_3 = (a^2 + b^2 - c^2 - d^2 + (bc - ad)k)/w$, where
$w = ((a^2 + b^2 - c^2 - d^2)^2 + (bc - ad)^2)^{1/2}$.

The condition $|g - 1| < (0.4)^{1/2}$ means that $a \geq 0.8$. So $a^2 + b^2 - c^2 - d^2 \geq 2a^2 - 1 \geq 0.28$, hence $g_3 \in G_3(B)^0$. Note also that $w^2 = (a^2 + b^2 - c^2 - d^2)^2 + (bc - ad)^2 \leq a^2 + b^2 \leq 1$. So $g_3^{-2} = a' + d'k$ with $a' \geq 2a^2 - 1$. This implies that $|g_3 - 1| = |g_3^{-1} - 1| \leq |g - 1|$.

Therefore $|gg_3^{-1} - 1| \leq 2\,|g - 1|$, so when we write gg_3^{-1} as $g_1 g_2$ with $\mathrm{tr}(g_1)$, $\mathrm{tr}(g_2) > 0$ (this can be done uniquely), we have $|g_m - 1| \leq |gg_3^{-1} - 1| \leq 2\,|g - 1|$ for $m = 1, 2$.

Corollary 11. For any ideal B of A the group $G(B^0$ is generated by its subgroups $G_1(B)^0$, $G_2(B)^0$ and $G_3(B)^0$. Also $G(B^0$ is the normal subgroup of $G(A)^0$ generated by $G_1(B)^0$.

To prove the second conclusion, we use that G_2 and G_3 are conjugated to G_1 by elements from $G(A)^0$. Namely $a + bi = g(a + bj)g^{-1} = h(a + bk)h^{-1}$, where $g = (1 + k)/2^{1/2} \in G_3(\mathbb{R}) = G_3(\mathbb{R})^0$ and $h = (1 + j)/2^{1/2} \in G_2(\mathbb{R}) = G_2(\mathbb{R})^0$.

Corollary 12. If $b, c, d \in A$ and $b > 1$, then there is $g \in G(A)^0$ such that $g(bi + cj + dk)g^{-1} = (b^2 + c^2 + d^2)^{1/2}i$.

Proof. Set $b' = (b^2 + c^2)^{1/2} \in \mathrm{GL}_1 A$ and write $(b + dj)/b' = h^2$ with h in $G(A)^0$. Then $h(bi + cj + dk)h^{-1} = cj + hi(b + dj)h^{-1} = cj + i(b + dj)h^{-2} = cj + b'i$. Now we set $b'' = (b'^2 + c^2)^{1/2} \in \mathrm{GL}_1 A$ and write $(ck - b')/b'' = h'^2$ with h' in $G(A)^0$. Then for $g = h'h$ we have $g(bi + cj + dk)g^{-1} = h'(cj + b'i.)h'^{-1} = h'^2(ck + b')i = b''i = (b^2 + c^2 + d^2)^{1/2}i$.

Lemma 13. Let $h = a + bi \in G_1A$. Then every element $f + bzi$ in G_1A with $f, z \in A$, $|z| \le 2$, $\inf(f) > 0$ can be written as $f + bzi = g[u, h]g^{-1}$ with
$u = (1 - z^2/(2 + 2f))^{1/2} + z(2 + 2f)^{-1/2}j$ and $g \in G(A)^0$.

Proof. To take advantage of trigonometry in the following computations, we write
$u = (1 - z^2/(2 + 2f))^{1/2} + z(2 + 2f)^{-1/2}j = \cos\theta + j\sin\theta$
with $\theta \in \mathbb{R}^X$, $|\theta| < \pi/2$. Then $[u, h] = (a + bu^2j)(a - bi)$
$= 1 + b(u^2 - 1)j(a - bi)$
$= a^2 + b^2\cos 2\theta + bai(\cos 2\theta - 1) - bak \sin 2\theta - b^2 j \sin 2\theta$
$= a^2 + b^2\cos 2\theta - 2b \sin\theta(ai \sin\theta + ak \cos\theta + bj \cos\theta)$.
By Corollary 12, there is $g \in G(A)^0$ such that
$g(ai \sin\theta + ak \cos\theta + bj \cos\theta)g^{-1} = i(\cos^2\theta + a^2\sin^2\theta)^{1/2}$
(we used that i, j, k are similar in $G(\mathbb{R}) \subset G(A)^0$). So $h' = g[u, h] g^{-1}$
$= a^2 + b^2\cos 2\theta - 2bi(\cos^2\theta + a^2\sin^2\theta)^{1/2}\sin\theta = 1 - 2b^2\sin^2\theta + 2bi(1 - b^2\sin^2\theta)^{1/2}\sin\theta$.

Note that $1 - 2b^2\sin^2\theta = 1 - z^2b^2/(1 + f)$
$= (1 + f - z^2b^2)/(1 + f) = (f + f^2)/(1 + f) = f$
and that both $2b(1 - b^2\sin^2\theta)^{1/2}\sin\theta$ and bz have the same sign as $\sin\theta$ (or θ). So $h' = g[u, h] g^{-1} = f + bzi$.

Corollary 14. Let H be a subgroup of $G(A)$ normalized by $G(A^0$ and containing $h = a + bi$ with a, b in A. Then $H \supset G_1(Ab)^0$.

Proof. We have to prove that any $h' = f + zbi$ in $G_1(Ab)$, where $f, z \in A$, belongs to H provided that it is sufficiently close to 1. We assume that $|h' - 1| < 1$, i.e. $|f - 1| < 1/2$. As in Lemma 8, we can write $h' = h''^2$ with $h'' = f_1 + z_1bi$, where $f_1, z_1 \in A$ and $z_1 = z(1 + f)^{-1}((1 + ((bz/(1 + t)^2)^{1/2} + 1)$. Note that $|z_1| \le |z|/2$. Iterating this, we can write h' as a power of an element $f_n + z_nbi$ with an arbitrary small $|z|$. So without loss of generality we can assume that $|z| \le 2$ and $\inf(f) > 0$. Then $h' = f + zbi \in G_1(Ab)$ by Lemma 13.

Corollary 15. Let H be a subgroup of $G(A)$ normalized by $G(A^0$, and let $h = a + bi + cj + dk \in H$ with a, b, c, d in A and $|h - 1| < 1/2$.. Then $H \supset G(Ab + Ac + Ad)^0$.

Proof. By Lemma 10, it suffices to show that $H \supset G_m(Ab + Ac + Ad)^0$ for $m = 1, 2, 3$. Since G_2 and G_3 are conjugated to G_1 by elements from $G(A)^0$ (see the proof of Corollary 11 above), it suffices to do this for $m = 1$. By Corollary 9, it suffices to show that $H \supset G_1(Ab)$, $G_1(Ac)$, $G_1(Ad)$. By symmetry it suffices to prove the first inclusion, i.e. that $H \supset G_1(Ab)$.

Since $i \in G(\mathbb{R}) = G(\mathbb{R})^0$, we have $h' = ihi^{-1} = a + bi - cj - dk \in H$. So $h'' = hh' = 1 + 2hbi = 1 - 2b^2 + 2abi + 2bdj - 2bck \in H$. By Corollary 12, there is $g \in G(A)^0$ such that $g(ai + dj + ck)g^{-1} = a'i$ with $a' = (a^2 + d^2 + c^2)^{1/2} \in \mathrm{GL}_1 A$.

Then $gh''g^{-1} = 1 - 2b^2 + 2ba'i \in H$. The condition $|h - 1| < 1/2$ means that $a > 3/4$, hence $b^2 \leq 1 - a^2 < 7/16$ and $1 - 2b^2 > 0$. By Corollary 13, $H \supset G_1(A(2ba'))^0 = G_1(Ab)^0$, because $2a' \in \mathrm{GL}_1 A$.

Proof of Theorem 4 for $G(\mathbb{R}) = \mathrm{SL}_1(\mathbb{H})$. The part (a) follows from Lemma 8, Part (b) with $N' = 3$ follows from Lemma 10 and Lemma 13 (where we can take z to be a small real number (note that the neighborhoods $|h - 1| < \varepsilon$ of 1 in G are invariant under inner automorphisms).

Proof of (5) for $G(\mathbb{R}) = \mathrm{SL}_1(\mathbb{H})$. Let H be a subgroup of $G(A)$ normalized by $G(A)^0$. We denote by B the largest ideal of A such that $H \supset G(B)^0$ (see Corollary 9 and Lemma 10. We have to prove that $G(B) \supset H$, i.e. $h'gh'^{-1} \equiv g \pmod B$ for every quternion g in $\mathfrak{g}$. By Corollary 14, applied to $h = [h', g]$, we have $h'gh'^{-1} \equiv g \pmod B$ for all g in $G(\mathbb{R})$ sufficiently close to 1. Therefore we have this for all g in $G(\mathbb{R})^0 = G(\mathbb{R})$. Since $\mathbb{H}$ is spanned over $\mathbb{R}$ by $G(\mathbb{R})$ (in fact, the standard basis $1, i, j, k$ is contained in $G(\mathbb{R}) = \mathrm{SL}_1\mathbb{H}$), we conclude that $h'gh'^{-1} \equiv g \pmod B$ for all g in $\mathbb{H}$.

9. The case of $G(\mathbb{R}) = \mathrm{SL}_n\mathbb{H}$, $n > 2$

The group $\mathrm{SL}_n\mathbb{H}$ consists of all n by n matrices g over the quaternions $\mathbb{H}$ with the reduced norm 1. If we write every entry of an n by n matrix g over $\mathbb{H}$ as $a + bi + cj + dk$ with real numbers a, b, c, d ("real coordinates" of this entry) and replace it by the matrix $\begin{pmatrix} a+bi & -c+di \\ c+di & a-bi \end{pmatrix}$, then the reduced norm of g is the determinant of the obtained $2n$ by $2n$ matrix over $\mathbb{C}$. In particular, the reduced norm of g is a homogeneous polynomial of degee $2n$ with integral coefficients in real coordinats of entries of g. It is well-known that $\mathrm{SL}_n\mathbb{H} = \mathrm{E}_n\mathbb{H}$

The Lie algebra $\mathfrak{g}$ consists of all n by n matrices over $\mathbb{H}$ with the trace of the sum of diagonal entries = 0. The adjoint representation is given by $ad(g)h = ghg^{-1}$.

An element g of G^X belongs to $G(A)$ if an only if $xy \in A$ for any real coordinate x of any entry in g and any real coordinate y of any entry in g^{-1}. Multiplzing the equation that the reduced norm of g^{-1} is 1 by x^{2n}, where x is as above, we obtain that $x^{2n} \in A$.

Now we assume the condition (1) with $N = 2n$. Then $G(B)^0 = (\mathrm{SL}_nB)^0 = \mathrm{E}_n(A, B)$ for any ideal B of A. Now we can use results of [16] for the ring $A' = A + Ai + Aj + Ak$ to obtain (5). We use that every ideal B' of A' has the form $B' = B + Bi + Bj + Bj$, where $B = B' \cap A$ is an ideal of A. In the case $n = 2$ we have to use that $A' = A + D$, where A is the center of A' and D is the ideal of A' generated by all additive commutators $xy - yx$ in A'.

10.The case of $G(\mathbb{R}) = \mathrm{SU}(p, q: \mathbb{C}), p \geq q \geq 0, p+q \geq 2$

Now $G(\mathbb{R})= \mathrm{SU}(p, q: \mathbb{C})$ consists of g in $\mathrm{SL}_n\mathbb{C}$ such that $g^*Fg = F$, where $n = p + q$, and $F = 1_p \oplus (\angle 1)_q$ is the diagonal matrix with p ones and q minus ones. If we extend scalars from $\mathbb{R}$ to $\mathbb{C}$, then the algebraic group G becomes isomorphic to $\mathrm{SL}_n\mathbb{C}$. Assuming the condition (2) with $N = n$, we conclude that $G(A)^0 = \mathrm{SU}(p, q; A[i])^0 = G_A{}^0$.

The group $G_A = \mathrm{SU}(p, q; A[i]) \subset G(A)$ acts on the $A[i]$-module $A[i]^n$ of all columns v over $A[i]$ by left multiplications preserving the hermition squares $v^*Fv \in A$. Let e_m denote the standard basis of $\mathbb{R}^n \subset A[i]^n$.

When $n = 2$, the Lie group $G(\mathbb{R})$ has been dealt with above. Namaly, $\mathrm{SU}(1, 1: \mathbb{C})$ is isomorphic to $\mathrm{SL}_2\mathbb{R}$, and $\mathrm{SU}(2, 0: \mathbb{C})$ is isomorphis to $\mathrm{SL}_1\mathbb{H}$. So we can assume that $n \geq 3$, and we will proceed by induction on n.

Let $G(m,m';A)$ denote the subgroup of $G_A = \mathrm{SU}(p, q; A[i])$ consisting of matrices acting only on the coordinates m and m', where $1 \leq m < m' \leq n$. So $G(m,m'; \mathbb{R})$ is isomorphic to both $\mathrm{SU}(2,0; \mathbb{C})$ and $\mathrm{SL}_1\mathbb{H}$ when $m' \leq p$ or $m > p$, and it is isomorphic to both $\mathrm{SU}(1, 1; \mathbb{C})$ and $\mathrm{SL}_2\mathbb{R}$ when $m \leq p < m'$. It is conjugated in $\mathrm{SU}(p, q; \mathbb{C})$ with $G(1,n; \mathbb{R})$ or with $G(1,1;\mathbb{R})$.

Lemma 16. Let B be an ideal of A, $\varepsilon \in \mathbb{R}$ such that $0 < \varepsilon < 1$, $v = (v_m) \in A[i]^n$. Suppose that $v_1 -1, v_m \in B[i]$, $|v_m| \leq \varepsilon$ for $m \geq 2$, $|v_1 - 1| \leq \varepsilon$, $v^*Fv \geq (1 - \varepsilon)^2$, Then there are g_m in $G(1,m; B)$ such that $g_n \dots g_2 v = e_1 v^*Fv$ and $|x| \leq \varepsilon/(1 - \varepsilon)$ for every entry x of every matrix $g_m - 1_2$, $m = 2,\dots, n$.

Proof. Set $w_1 = v_1$ and $w_m =(v_1{}^*\partial_1 v_1 + \dots + v_m{}^*\partial_m v_m)^{1/2} \in A$ for $m = 2,\dots, n$, where $\partial_m = 1$ for $m \leq p$ and $\partial_m = -1$ for $m \geq p+ 1$ (so $\partial_m = e_m{}^*Fe_m$ and $F = \mathrm{diag}(\partial_1, \dots, \partial_n)$). Then $w_n{}^2 = v^*Fv$ and $w_m \geq 1 - \varepsilon$ for all $m \geq 2$. Now we set

$$g_m = \begin{pmatrix} v_{m-1}{}^*/w_m & \partial_m v_m{}^*/w_m \\ - v_m/w_m & v_{m-1}/w_m \end{pmatrix} \in G_m(B)$$

for $m = 2,\dots, n$. Then the first entry of the column $(g_m \dots g_2)v$ is w_m, the next $m - 1$ entries are 0, and the rest is the same as in v. In particular $g_n \dots g_2 v = e_1 w_n = e_1 v^*Fv$. It is clear that $|v_m{}^*/w_m| = |-v_m/w_m| \leq \varepsilon/(1 - \varepsilon)$.

Corollary 17. For any open neighbourhood U of 1 in $G(\mathbb{R})$ and any ideal B of A, the set of all products $\prod g_\alpha$ is open in $G(B)$, where α runs over all pairs (m, m') of integers such that $1 \le m < m' \le n$, $g_\alpha \in G(\alpha;B) \cap U^X$, and the the factors in the product are ordered such that m' grows from left to right and, when m' is constant, m also grows from left to right.

This corollary follows from the lemma by induction on n.

Proof of Theorem 4 for $G(\mathbb{R}) = \mathrm{SU}(p, q: \mathbb{C}), p \ge q \ge 0, p + q \ge 2$. Corollary 17 reduces it to the case $p + q = 2$, which has been dealt with.

Proof of (5). Let $B = L(H)$ be the largest ideal of A such that $H \supset G(B)^0$. We have to prove that $G(B) \supset H$. We proceed by induction by $n = p + q$. The case $n = 2$ has been dealt with. So we assume that $n \ge 4$.

Take an arbitrary h in H. We want to prove that $h \in G(B)$.

Case 1: $he_n = e_n$. Then $h \in G'(A)$ for $G'(\mathbb{R}) = \mathrm{SU}(p, q\text{-}1; \mathbb{C})$. By the induction hypothesis, either $h \in G'(B) \subset G(B)$ or $G'(B') \subset H \cap G'(A) \subset H$ for an ideal B' of A such that $B \ne B' \supset B$.

Let us show that $H \supset G(B')$ in the second case, so it is impossible. Since $H \supset G(m, m'; B')$ for $m' < n$, it remeins to show that $H \supset G(m, n; B')$ for $m = 1,..., n-1$. Since $H \supset G(1, m; B')$, we conclude that $H \supset G(m; B')$, where $G(m) = G(1, m) \cap G(m, n)$ is the subgroup of G acting only on the coordinate m (so $G(m, \mathbb{R})$ is isomorphic to the circle S^1). Working now inside $G(m, n)$, we conclude that $H \supset G(m, n; B')$.

Case 2: $he_m = e_m$ for some m. This case is similar to Case 1.

Case 3: $h \in G_A$, $e_2{}^*he_n = 0$, and $|x| \le 0.1/n$ for every entry x of the column $ge_2 - e_2$. We set

$g = \mathrm{diag}(\lambda, \lambda^{1-n}, \lambda,..., \lambda) \in G(\mathbb{R}) \subset G(A)^0$,

where $\lambda = 0.8 + 0.6i \in \mathbb{C}$. Then the matrix $ghg^{-1} \in H$ has the same last column as h, so $[g, h]e_n = e_n$. By Case 1, $[g, h] \in G(B)$. This implies that $he_2 \equiv e_2 \pmod B$. By Lemma 16, there is $h' \in G(B)^0 \subset H$ such that $h'ge_2 = e_2$. By Case 2 with $m = 2$, $h'g \in G(B$, hence $g \in G(B)$.

Case 4: $h \in G_A$ and $|x| \leq 0.1/n$ for every entry x of the column $he_1 - e_1$. For any μ in $\mathbb{R}$, we set

$\lambda = (1 - \mu^2 + 2\mu i)/(1 + \mu^2) \in \mathbb{C}$

and

$g = \mathrm{diag}(\lambda^{1-n}, \lambda, \ldots, \lambda) \in G(\mathbb{R}) \subset G(A)^0$.

The matrix $gh^{-1}g^{-1}$ differs from h^{-1} only in the first row and column. The last column $[h, g]e_n$ of the matrix $h' = [h, g] = hgh^{-1}g^{-1} \in H$ has the form $e_n + (h_1)x$, where $x = e_1{}^*h^{-1}e_n(\lambda^{-n} - 1)$. Clearly, $h' \to 1_n$ as $\mu \to 0$.

Set

$$g_2 = \begin{pmatrix} h_{1,1}{}^*/w & h_{2,1}{}^*/w \\ -h_{2,1}/w & h_{1,1}/w \end{pmatrix} \in G_2(A)^0 \subset G(A)^0,$$

where $h_{m,1} = e_m{}^*he_1$ and $w = (|h_{1,1}|^2 + |h_{2,1}|^2)^{1/2}$.

By Case 3, $(g_2 \oplus 1_{n-2})h'(g_2 \oplus 1_{n-2})^{-1} \in G(B)$, hence $h' \in G(B)$ (for all μ sufficiently close to 0). It follows that $he_1 \equiv e_1 \pmod B$. By Lemma 16, there is $h'' \in G(B)^0 \subset H$ such that $h''he_1 = e_1$. By Case 2 with $m = 1$, $h''h \in G(B)$, hence $h \in G(B)$.

General case. By Case 4, $[g, h] \in G(B)$, i.e. $ad(h)g = hgh^{-1} \equiv g \pmod B$ for every g in $G(\mathbb{R})$ sufficiently close to 1_n. Since the vector space in n by n complex matrices spanned by any open neighbourhood U of 1_n in $G(\mathbb{R})$ contains $\mathfrak{g}$, we obtain that $hgh^{-1} \equiv g \pmod B$ for all g in $\mathfrak{g}$. So $h \in G(B)$.

11.The case of $G(\mathbb{R}) = SO(p, q: \mathbb{R}), p \geq q \geq 0, p+q \geq 5$

Now $G(\mathbb{R}) = SU(p, q: \mathbb{C}) \cap SL_n\mathbb{R}$, where $n = p + q$. Since $G(\mathbb{C}) = SO_n\mathbb{C}$, we can use results of Section 7 above to conclude that $G(A) = G_A = SO(p, q; A)$ when n is odd, and that $G(B)^0 = G_B{}^0$ for any n and any ideal B of A under the condition (1) with $N = 2$.

The group $G_A = SO(p, q; A) \subset G(A)$ acts on the A-module A^n of all columns v over A by left multiplications preserving the scalar squares $v^T F v = v^* F v \in A$. Let e_m denote the standard basis of $\mathbb{R}^n \subset A^n$.

When $n = 2$, the Lie group $G(\mathbb{R})$ is not simple. Namely,
$SO(1, 1: \mathbb{R})$ is locally isomorphic to $GL_1\mathbb{R}$, and $SO(2, 0: \mathbb{R})$ is isomorphic to the circle S^1.

When $n = 3$, the Lie group $G(\mathbb{R})$ has been dealt with above. Namely, $SO(2, 1: \mathbb{R})$ is locally isomorphic to $SL_2\mathbb{R}$, and $SO(2, 0: \mathbb{R})$ is locally isomorphic to $SL_1\mathbb{H}$.

When $n = 4$, the Lie group $G(\mathbb{R})$ is not simple or has been dealt with above. Namely, $SO(4, 0: \mathbb{R})$ is locally isomorphic to $SL_1\mathbb{H} \times SL_1\mathbb{H}$, $SO(2, 2: \mathbb{R})$ is locally isomorphic to $SL_2\mathbb{R} \times SL_2\mathbb{R}$, and $SO(3, 1: \mathbb{R})$ is locally isomorphic to $SL_2\mathbb{C}$.

So we can assume that $n \geq 5$, and we will proceed by induction on n.

Let $G(m,m';A)$ denote the subgroup of $G_A = SO(p, q; A)$ consisting of matrices acting only on the coordinates m and m', where $1 \leq m < m' \leq n$. So $G(m,m'; \mathbb{R})$ is isomorphic to both $SO(2,0; \mathbb{R})$ and S^1 when $m' \leq p$ or $m > p$, and it is locally isomorphic to $GL_1\mathbb{R}$ when $m \leq p < m'$. It is conjugated in $SO(p, q; \mathbb{R})$ with $G(1, n; \mathbb{R})$ or with $G(1, 1; \mathbb{R})$.

Lemma 18. Let B be an ideal of A, $\varepsilon \in \mathbb{R}$ such that $0 < \varepsilon < 1$ $v = (v_m) \in A^n$. Suppose that $v_1 - 1, v_m \in B$, $|v_m| \leq \varepsilon$ for $m \geq 2$, $|v_1 - 1| \leq \varepsilon$, $v^*Fv \geq (1-\varepsilon)^2$, Then there are g_m in $G(1,m; B)$ such that $g_n \ldots g_2 v = e_1 v^*Fv$ and $|x| \leq \varepsilon/(1-\varepsilon)$ for every entry x of every matrix $g_m - 1_2$, $m = 2,\ldots, n$.

Proof. See the proof of Lemma 16 above.

Corollary 19. Corollary 17 holds for our $G(\mathbb{R}) = SO(p, q: \mathbb{R}), p + q \geq 3$.

Proof of Theorem 4 for $G(\mathbb{R}) = SO(p, q: \mathbb{R}), p \geq q \geq 0, p + q \geq 5$. Corollary 19 reduces it to the case $p + q = 3$, which has been dealt with.

Proof of (5) for $G(\mathbb{R}) = SO(p, q: \mathbb{R}), p \geq q \geq 0, p + q \geq 5$.. Let $B = L(H)$ be the largest ideal of A such that $H \supset G(B)^0$. We have to prove that $G(B) \supset H$. We proceed by induction by $n = p + q$. The case $n = 3$ has been dealt with. So we assume that $n \geq 5$.

Take an arbitrary h in H. We want to prove that $h \in G(B)$.

Case 1: $he_n = e_n$. Then $h \in G'(A)$ for $G'(\mathbb{R}) = SO(p, q-1; \mathbb{C})$. By the induction hypothesis, either $h \in G'(B) \subset G(B)$ or $G'(B') \subset H \cap G'(A) \subset H$ for an ideal B' of A such that $B \neq B' \supset B$.

Let us show that $H \supset G(B')$ in the second case, so it is impossible. When $q \neq 1$ we conclude that $H \supset G_n(B')^0$, because G_n is similar to G_{n-1}, hence $H \supset G(B')$ by Corollary 17 (with B' instead of B). When $q = 1$, we consider the subgroup G'' of G which fixes e_m with $m \leq n/3$. Then $G''(\mathbb{R})$ is isomorphic to both $SU(1,1; \mathbb{C})$ and $SL_2\mathbb{R}$. Since $G''(A) \cap H \supset G''(A) \cap G'(B')^0$, we conclude that $G''(A) \cap H \supset G''(B')^0$. So we obtain again that $H \supset G(B')$ (using again Corollary 17).

Case 1:$he_n = e_n$. Then $h \in G'(A)$ for $G'(\mathbb{R}) = SO(p, q-1; \mathbb{R})$. By the induction hypothesis, either $h \in G'(B) \subset G(B)$ or $G'(B') \subset H \cap G'(A) \subset H$ for an ideal B' of A such that $B \neq B' \supset B$.

Let us show that $H \supset G(B')$ in the second case, so it is impossible. Since $H \supset G(m, m'; B')$ for $m' < n$, it remains to show that $H \supset G(m, n; B')$ for $m = 1,\ldots, n-1$. Since $H \supset G(1, m; B')$, we conclude that $H \supset G(1, m, n; B')$, where $G(1,m, n)$ is the subgroup of G acting only on the coordinate m (so $G(1,m,n; \mathbb{R})$ is isomorphic to $SO(3,0; \mathbb{R})$ or $SO(2,1; \mathbb{R})$). Therefore $H \supset G(m, n; B')$.

Case 2:$he_m = e_m$ for some m. This case is similar to Case 1.

Case 3: $h \in G_A$, $e_3{}^*he_{n-1} = e_3{}^*he_n = 0$, and $|x| \leq 0.1/n$ for every entry x of the column $ge_n - e_n$. Let $g \in G(n-1, n; \mathbb{R}) \subset G(\mathbb{R}) \subset G(A)^0$. Then the matrix $ghg^{-1} \in H$ has the same third row as h, so $e_3{}^*[g, h] = e_3{}^*$, i.e. $[g, h]e_3 = e_3$. By Case 2 with $m = 3$, $[g, h] \in G(B)$. This implies that $he_n \equiv e_n \pmod B$. By Lemma 16, there is $h' \in G(B)^0 \subset H$ such that $h'ge_n = e_n$. By Case 1, $h'g \in G(B)$, hence $g \in G(B)$.

Case 4: $h \in G_A$ and $|x| \leq 0.1/n$ for every entry x of the columns $he_1 - e_1$ and $he_2 - e_2$. For any μ in $\mathbb{R}$, we set

$$g = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \in G(1, 2;\mathbb{R})^0 \subset G(A)^0 .$$

The matrix $gh^{-1}g^{-1}$ differs from h^{-1} only in the first and second rows and columns. The last two columns $[h, g](e_{n-1}, e_n)$ of the matrix $h' = [h, g] = hgh^{-1}g^{-1} \in H$ has the form $(e_{n-1}, e_n) + (he_1, he_2)x$, where x is a 2 by 2 matrix over A. Clearly, $h' \to 1_n$ as $\mu \to 0$.

Now we find matrices g_2 in $G(1, 2;A)^0$ and g_3 in $G(1, 3;A)^0$ such that $e_m{}^*g_2g_3he_1 = 0$ for $m = 2, 3$ (we use Lemma 17 with $p + q = 3$, $B = A$). Then we find g_1 in $G(2, 3;\mathrm{A})^0$ such that $e_3{}^*g_1g_2g_3he_2 = 0$. Then for $g' = g_1g_2g_3$ we have $e_3{}^*g'he_1 = e_3{}^*g'he_2 = 0$. So $e_3{}^*g'h'g'^{-1}e_{n-1} = e_3{}^*g'h'g'^{-1}e_n = 0$.

By Case 3, $g'h'g'^{-1} \in G(B)$, hence $h' \in G(B)$ (for all μ sufficiently close to 0). It follows that $he_1 \equiv e_1 \pmod B$. By Lemma 16, there is $h'' \in G(B)^0 \subset H$ such that $h''he_1 = e_1$. By Case 2 with $m = 1$, $h''h \in G(B)$, hence $h \in G(B)$.

General case. By Case 4, $[g, h] \in G(B)$, i.e. $ad(h)g = hgh^{-1} \equiv g \pmod B$ for every g in $G(\mathbb{R})$ sufficiently close to 1_n Since the vector space in n by n complex matrices spanned by any open neighbourhood U of 1_n in $G(\mathbb{R})$ contains $\mathfrak{g}$, we obtain that $hgh^{-1} \equiv g \pmod B$ for all g in $\mathfrak{g}$. So $h \in G(B)$.

12. The case of $G(\mathbb{R}) = \mathrm{SO}(n, \mathbb{H})$, $n \geq 3$

Now $G(\mathbb{R}) = \mathrm{SO}(n, \mathbb{H})$ consists of all matrices g in $\mathrm{SL}_p\mathbb{H}$ such that $g^*g = 1_n$, where the involution on matrices is induced by the following involution on the quaternions $\mathbb{H}$:

$$(a + bi + cj + dk)^* = a + bi - cj + dk.$$

When $n = 1$ (resp. $n = 2$), the Lie group $G(\mathbb{R})$ is not simple; it is isomorphic to the circle S^1 (resp. locally isomorphic to $\mathrm{SL}_1\mathbb{H} \times \mathrm{SL}_2\mathbb{R}$).

When $n = 3$ (resp. $n = 4$) the group $G(\mathbb{R})$ is locally isomorphic to $\mathrm{SU}(3, 1; \mathbb{C})$ (resp. isomorphic to $\mathrm{SO}(6, 2; \mathbb{R})$), so it has been dealt with.

Thus, we can assume that $n \geq 5$. We proceed by induction on n as in Section 10. For any ideal B of A, we set $B' = B + Bi + Bj + Bk \subset \mathbb{H}^X$. We assume the condition (1) with $N = 2$.

Lemma 20. If $x = x^* \in B'$ and $|x| < 1/2$, then there is $y \in B'$ such that $y = y^*$, $1 + x = (1 + y)^*(1 + y)$, and $|y| \leq |x|$.

Proof. We write $x = a - 1 + bi + dk$ with $a - 1, b, d \in B$, and set $y = z - 1 + bi/(2z) + dk/(2z)$. Then the equality $1 + x = (1 + y)^*(1 + y)$ takes the form
$1 + x = a + bi + dk = (1 + y)^2 = (z + bi/(2z) + dk/(2z))^2$, or $z^4 - az^2 - (b^2 + d^2)/4 = 0$.
Using (1) with $N = 2$, we find the solution $z = u^{1/2} \in 1 + B$, where
$2u = a + (a^2 + b^2 + d^2)^{1/2} \in 1 + B$.

Let $G(m,m';A)$ denote the subgroup of $G_A = \mathrm{SO}(n, A')$ consisting of matrices acting only on the coordinates m and m', where $1 \leq m < m' \leq n$. So $G(m,m'; \mathbb{R})$ is isomorphic to $\mathrm{SO}(2, \mathbb{H})$. It is conjugated in $\mathrm{SO}(2, \mathbb{H})$ with $G(1,1;\mathbb{R})$.

Lemma 21. Let B be an ideal of A, $\varepsilon \in \mathbb{R}$ such that $0 < \varepsilon < 1$ $v = (v_m) \in A'^n$. Suppose that $v_1 - 1, v_m \in B$, $|v_m| \leq \varepsilon/n$ for $m \geq 2$, $|v_1 - 1| \leq \varepsilon/n$, Then there are g_m in $G(1,m; B)$ such that $g_n \cdots g_2 v = e_1 u_m$ with $u_m \in \mathrm{GL}_1 B$, $u_m^* u_m = v^* v$, and $|x| \leq \varepsilon/(1 - \varepsilon)$ for every entry x of every matrix $g_m - 1_2$, $m = 2, \ldots, n$.

Proof. Set $u_1 = v_1$. We define $g_m \in G_m(B)$ and u_m, $w_m \in \mathrm{GL}_1 B$, $m = 2,\dots, n$, inductively as follows: $w_m = |v_m u_{m-1}^{-1}(v_m u_{m-1}^{-1})^* + 1)^{1/2}$ (see Lemma 20),

$$g_m = \begin{pmatrix} w_m^{-1} & w_m^{-1}(v_m u_{m-1}^{-1})^* \\ - w_m^{-1} v_m u_{m-1}^{-1} & w_m^{-1} \end{pmatrix} \in G_m(B),$$

and for $u_m = w_m^{-1} u_{m-1} + w_m^{-1}(v_m u_{m-1}^{-1})^* v_m$ for $m = 2,\dots, n$. Then the first entry of the column $(g_m \dots g_2)v$ is u_m, th next $m - 1$ entries are 0, and the rest is the same as in v. In particular $g_n \dots g_2 v = e_1 u_n$. The rest of the proof is also similar to that in Section 10.

13.The case of $G(\mathbb{R}) = \mathrm{Sp}(p, q; \mathbb{H})$, $p \geq q \geq 0$, $p+q \geq 2$

Now the group $G(\mathbb{R}) = \mathrm{Sp}(p, q; \mathbb{H})$, consists of matrices $g \in \mathrm{GL}_n\mathbb{H}$ such that $g^*Fg = F$, where the involution on matrices is induced by the usual involution on the quternions $\mathbb{H}$ and where $F = 1_p \oplus (-1_q)$ as before. Note that $g \in \mathrm{SL}_n\mathbb{H}$ (i.e. the reduced norm of g is 1) automatically.

When $n = p + q = 1$, i.e. $p = 1 = q + 1$, the Lie group $G(\mathbb{R}) = \mathrm{SL}_1\mathbb{H}$ has been dealt with. When $n = p + q = 2$ the Lie group $G(\mathbb{R})$ has also been dealt with. Namely, $\mathrm{Sp}(2, 0; \mathbb{H})$ is locally isomorphic with $\mathrm{SO}(5, 0; \mathbb{R})$ and $\mathrm{Sp}(1, 1; \mathbb{H})$ is locally isomorphic with $\mathrm{SO}(4, 1; \mathbb{R})$. So we can assume that $n = p + q \geq 3$.

The rest of the proof in the case at hand is so similar to those in Sections 10–12, so we leave it to the reader.

14. Subnormal subgroups

Combining methods of this paper with those of [17] and [18], one obtain the following result.

Theorem 22. Under the conditions of Theorem 3, a subgroup H of $G(A)^0$ is subnormal if and only if $G(B^m) \subset H \subset G(B)$ for an ideal B of A and a natural number m.

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