A GENERALIZATION OF MAHLER'S CLASSIFICATION TO SEVERAL VARIABLES

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1. Introduction and results.

In 1932, K. Mahler [6] introduced the classification of all (real or complex) numbers into four disjoint classes A,S,T and U (see the detailed treatment of this classification and of an equivalent one by J.F. Koksma in Th. Schneider [9], Kapitel III and A. Baker [1], Chapter 8). This classification has the *Invariance Property*, i.e., two numbers which are algebraically equivalent over $\mathbb{Q}$† belong to the same class. In the present paper, a generalization of Mahler's classification to several variables, i.e. a classification of all points (in $\mathbb{R}^n$ or $\mathbb{C}^n$) into $3n + 1$ disjoint classes A^n, S^n_t, T^n_t, U^n_t, $t = 1,2,\ldots,n$, will be introduced. We will prove that this classification possesses the *Invariance Property*, i.e., any two points, which (i.e. the two sets of whose coordinates) are algebraically equivalent over $\mathbb{Q}$, belong to the same class. We will show that each of the $3n + 1$ classes are nonempty. We will classify T^n_n (referred to as T^n in the sequel) further into continuum many disjoint classes $T^n(\alpha) : T^n = \bigcup_{n\le\alpha\le\infty} T^n(\alpha)$, and prove that any two algebraically equivalent points of

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† We say that two nonempty subsets B_1 and B_2 of $\mathbb{C}$ are algebraically equivalent over $\mathbb{Q}$ if and only if every element of B_1 is algebraic over $\mathbb{Q}(B_2)$ and *vice versa*; i.e., if and only if $\overline{\mathbb{Q}(B_1)} = \overline{\mathbb{Q}(B_2)}$, where for any subfield F of $\mathbb{C}$, $\bar{F}$ denotes its algebraic closure contained in $\mathbb{C}$.

T^n belong to the same class $T^n(\alpha)$ and that there exist infinitely many α with $n \leq \alpha \leq \infty$ such that $T^n(\alpha) \cap \mathbb{R}^n \neq \phi$. We should like to refer to that K. Mahler [7] in 1971 introduced a new classification of $\mathbb{C}$, a generalization of which to $\mathbb{C}^n$ was obtained by A. Durand [2].

The following notations will be used. For every $P(x_1,\dots,x_n) \in \mathbb{C}[x_1,\dots,x_n]$, denote by $\deg P$ its total degree, by $H(P)$ the maximum of the absolute values of its coefficients, by $L(P)$ the sum of the absolute values of its coefficients. $L(P)$ has the two properties

$$(1) \qquad L(P+Q) \leq L(P) + L(Q)\,, \quad L(pQ) \leq L(P)L(Q)\ .$$

Let F be the set of nonnegative functions of integral variables $D \geq 0$ and $H \geq 1$, which are nondecreasing in D and H, respectively. For $a(D,H)$ and $b(D,H)$ in F we write

$$a(D,H) << b(D,H)$$

if there exist positive integers k_1, k_2, k_3, D_0, H_0 and a positive number γ such that the inequality

$$(2) \qquad a(D,H) \leq \gamma b(k_1 D, k_2^D H^{k_3})$$

holds for all $D \geq D_0$ and $H \geq H_0$. If $a(D,H) << b(D,H)$

and $b(D,H) \ll a(D,H)$, we write

$$a(D,H) >< b(D,H) .$$

Evidently, this defines an equivalence relation. Let G be the set of nondecreasing sequences of nonnegative numbers a_D, $D = 0,1,2,\ldots$. For a_D, b_D in G we write $a_D \ll b_D$ if there exist positive integers k, D_0 and a positive number γ such that the inequality

$$a_D \leq \gamma b_{kD}$$

holds for $D \geq D_0$. If $a_D \ll b_D$ and $b_D \ll a_D$, we write $a_D >< b_D$. This defines also an equivalence relation.

Put $P_n(D,H) = \{P \in \mathbb{Z}[x_1,\ldots,x_n] \mid P \neq 0,\ \deg P \leq D,\ H(P) \leq H\}$ for $D \geq 0$, $H \geq 1$. For any $\xi = (\xi_1,\ldots,\xi_n) \in \mathbb{C}^n$, set

$$w_D(H|\xi) = \min |P(\xi)|,$$

where the minimum is taken over all the $P \in P_n(D,H)$ with $P(\xi) \neq 0$. Clearly, $w_D(H|\xi) \leq 1$, since $1 \in P_n(D,H)$. Let

$$\Theta(D,H\ |\xi) = -\log w_D(H|\xi),$$

then $\Theta(D,H|\xi)$ belongs to F . Set

$$w_D(\xi) = \overline{\lim_{H\to\infty}} \frac{\Theta(D,H|\xi)}{\log H} .$$

Denote by $t(\boldsymbol{\xi})$ the transcendence degree of $\mathbb{Q}(\xi_1,\ldots,\xi_n)$ over $\mathbb{Q}$. When $t(\xi) = t \geq 1$, put

$$w(\xi) = \overline{\lim_{D\to\infty}} \frac{w_D(\xi)}{D^t} .$$

We put $\mu(\boldsymbol{\xi}) = \infty$ if $w_D(\boldsymbol{\xi}) < \infty$ for all D, otherwise let $\mu(\boldsymbol{\xi})$ be the least D such that $w_D(\boldsymbol{\xi}) = \infty$. Let

$$A^n = \{\boldsymbol{\xi} \in \mathbb{C}^n | t(\xi) = 0\} = \{\boldsymbol{\xi} | \xi_i \in \overline{\mathbb{Q}}, i = 1,\ldots,n\}$$

$$S_t^n = \{\xi \in \mathbb{C}^n | t(\xi) = t,\ w(\boldsymbol{\xi}) < \infty ,\ \mu(\xi) = \infty\}$$

$$T_t^n = \{\boldsymbol{\xi} \in \mathbb{C}^n | t(\boldsymbol{\xi}) = t,\ w(\xi) = \infty ,\ \mu(\xi) = \infty\}$$

$$U_t^n = \{\boldsymbol{\xi} \in \mathbb{C}^n | t(\xi) = t,\ w(\xi) = \infty ,\ \mu(\xi) < \infty\}$$

$$t = 1,2,\ldots,n .$$

Note that A^1, S_1^1, T_1^1, U_1^1 are exactly Mahler's A, S,T,U, respectively.

Theorem 1. Let

$$\sigma(\boldsymbol{\xi}) = \begin{cases} 1 , & \text{if } \boldsymbol{\xi} \in \mathbb{R}^n , \\ 2 , & \text{otherwise} \end{cases} .$$

Suppose that $\xi_1,\ldots,\xi_n$ are algebraically independent over $\mathbb{Q}$. Then there exists a constant $c_1 > 0$ depending only on $\xi_1,\ldots,\xi_n$ and n such that the inequality

$$(3) \qquad \Theta(D,H|\boldsymbol{\xi}) \geq (\sigma(\xi)^{-1}\tbinom{D+n}{n} - 1) \log (H - 1) - c_1 D$$

holds for $D \geq 1$, $H \geq 2$, whence

$$w_D(\boldsymbol{\xi}) \geq \sigma(\xi)^{-1}\binom{D+n}{n}-1 \quad , \quad D \geq 1 .$$

<u>Theorem 2.</u> Suppose that $\xi_1,\ldots,\xi_p$, $\eta_1,\ldots,\eta_q$ are not all algebraic numbers and the sets $\{\xi_1,\ldots,\xi_p\},\{\eta_1,\ldots,\eta_q\}$ are algebraically equivalent over $\mathbb{Q}$. Then

(4) $\quad \Theta(D,H|\boldsymbol{\xi}) >< \Theta(D,H|\boldsymbol{\eta})$

and

$$w_D(\boldsymbol{\xi}) >< w_D(\boldsymbol{\eta}) \; .$$

By virtue of Theorem 2, we can reduce the investigation on S_t^n, T_t^n, U_t^n $(n = 1,2,\ldots,\; t = 1,\ldots,n)$ to the investigation on S_n^n, T_n^n, U_n^n (referred to as S^n, T^n, U^n in the sequel), $n = 1,2,\ldots$. In particular, to show that S_t^n, T_t^n, U_t^n $(t = 1,\ldots,n)$ are nonempty, it suffices to show so are S^n, T^n, U^n, $n = 1,2,\ldots$. The Siegel-Shidlovsky theory for E-functions furnishes many examples of points $\boldsymbol{\xi} = (\xi_1,\ldots,\xi_n)$ in S^n with $w_D(\boldsymbol{\xi}) \leq \binom{D+n}{n} - 1$ (see, for example, N.I. Feldman and A.B. Shidlovsky [3], pp. 58-59), whence by Theorem 1 $w_D(\xi) = \binom{D+n}{n} - 1$ if $\boldsymbol{\xi} \in \mathbb{R}^n$. In particular, $(e^{\alpha_1},\ldots,e^{\alpha_n})$ is such a point for any algebraic $\alpha_1,\ldots,\alpha_n$ linearly independent over $\mathbb{Q}$. By the inequality $w_D(\xi_1,\ldots,\xi_n) \geq \max_{1\leq i\leq n} w_D(\xi_i)$ we see that if $\xi_1,\ldots,\xi_n$ are algebraically independent over $\mathbb{Q}$ and at least one from $\xi_1,\ldots,\xi_n$ is Mahler's U-number, then $(\xi_1,\ldots,\xi_n)$ belongs to U^n. Thus, for instance, the work of I. Shiokawa [10] and Y.C. Zhu [13] provides many examples of points in U^n ; we see that $(\xi_1,\ldots,\xi_n)$, where $\xi_i = \sum_{l=1}^{\infty} g_i^{-l!}$ with $g_i \geqq 2$ being distinct positive integers, belongs to U^n. We now classify T^n further. Suppose that $\xi = (\xi_1,\ldots,\xi_n)$ is in T^n . Write $\alpha = \alpha(\xi)$ for the infimum of the positive numbers α with $w_D(\xi) = O(D^\alpha)$ as $D \longrightarrow \infty$. By Theorem 1, we have $\alpha(\xi) \geq n$ for any $\xi \in T^n$. For each α with $n \leq \alpha \leq \infty$ set

$$T^n(\alpha) = \{\xi \in T^n \mid \alpha(\xi) = \alpha\}.$$

Note that Satz 3' in G. Wüstholz [12], p. 388 implies particularly that if $\wp(z)$ is a Weierstrass elliptic function with algebraic invariants g_2, g_3 and complex multiplication over the imaginary quadratic field k and if $\alpha_1,\ldots,\alpha_n$ are algebraic numbers linearly independent over k , then $(\wp(\alpha_1),\ldots,\wp(\alpha_n))$ belongs to either S^n or $T^n(n)$.

<u>Theorem 3.</u> All points in $\mathbb{C}^n$ are classified into the disjoint nonempty classes

$$A^n, S_t^n, T_t^n, U_t^n \ , \quad t = 1,2,\ldots,n.$$

Any two algebraically equivalent (over $\mathbb{Q}$) points in $\mathbb{C}^n$ fall into the same class. All points in T^n are classified into the disjoint classes: $T^n(\alpha)$, $n \leqq \alpha \leqq \infty$. Any two algebraically equivalent (over $\mathbb{Q}$) points in T^n

fall into the same class $T^n(\alpha)$.

Proof. The assertion that S^n_t, T^n_t, U^n_t $(t = 1,,\ldots,n)$ are nonempty follows from Theorem 4 below and the remark made after the formulation of Theorem 2. The remaining part of the theorem is a direct consequnce of Theorem 2.

The assertion that there exist infinitely many α such that $T^n(\alpha) \cap \mathbb{R}^n \neq \phi$ follows from the following

Theorem 4. Let $\alpha_2 \geq 3$ and $\alpha_n > n$ $(n = 3,4,\ldots)$ be any positive numbers. Then for $n = 2,3,\ldots$ there exists $\zeta = \zeta(n,\alpha_n)$ with $\alpha_n \leq \zeta \leq 2^{n-1}(\alpha_n + 1) - 1$ such that

$$T^n(\zeta) \cap \mathbb{R}^n \neq \phi \ .$$

To prove Theorem 4 our starting point is W.M. Schmidt's famous result

Theorem S. For any α with $3 \leq \alpha \leq \infty$,

$$T^1(\alpha) \cap \mathbb{R} \neq \phi \ .$$

This follows from W.M. Schmidt [8], p. 278, Corollary 3. One needs only to note that Schmidt's $\kappa_D(\xi)$ is just Koksma's $w^*_D(\xi) + 1$ (See Th. Schneider [9], p. 73) and

$$w_D(\xi) \geq w^*_D(\xi) \geq w_D(\xi) - D + 1$$

(See E. Wirsing [11], p. 68).

2. Proof of Theorem 1.

The theorem is a direct consequence of Th. Schneider [9], pp. 139-140, Hilfssatz 27 and 28. When $\sigma(\xi) = 1$, H is even, on taking in Hilfssatz 27 $M = 1$, $N = \binom{D+n}{n}$, $X = H.$, $A = (\prod_{i=1}^{n} \max(|\xi_i|,1))^D$ and noting that $\xi_1,\dots,\xi_n$ are algebraically independent over $\mathbb{Q}$, we see that there exists $p \in P_n(D,H)$ such that

$$0 < |P(\xi)| < \binom{D+n}{n} (\prod_{i=1}^{n} \max(|\xi_i|,1))^D H^{1-\binom{D+n}{n}} .$$

Thus (3) follows at once with $c_1 = n + \log \prod_{i=1}^{n} \max(|\xi_i|,1)$. The remaining cases can be similarly verified.

3. Proof of Theorem 2.

We need three lemmas

Lemma 1. Let $P_{ij} \in \mathbb{C}[x_1,\dots,x_q]$ $(1 \le i,j \le \ell)$ and $\Delta = \det(P_{ij})$. Then

(5) $$\deg \Delta \le \sum_{i=1}^{\ell} \max_{1\le j\le \ell} \deg P_{ij}$$

and

(6) $$L(\Delta) \le \prod_{i=1}^{\ell} \sum_{j=1}^{\ell} L(P_{ij}) .$$

Proof. (5) is trivially true. If $\ell = 1$, (6) is obvious. Suppose that (6) holds for $\ell - 1$ with $\ell \ge 2$. Let

$$\Delta = \sum_{j=1}^{\ell} (-1)^{j-1} P_{1j}A_j$$

be the expansion of Δ according to the first row. By the inductive hypothesis we have for $j = 1,\dots,n$

$$L(A_j) \le \prod_{i=2}^{n} \sum_{\substack{k=1 \\ k\neq j}}^{n} L(P_{ik}) \le \prod_{i=2}^{n} \sum_{j=1}^{n} L(P_{ij}) .$$

Hence, by (1), we have

$$L(\Delta) \le \sum_{j=1}^{\ell} L(P_{1j})L(A_j) \le \prod_{i=1}^{n} \sum_{j=1}^{n} L(P_{ij}).$$

Thus, the lemma is proved.

Recall the definitions of F and G introduced in Sect. 1. For any $a(D,H)$ in F, write

$$a_D = \overline{\lim_{H\to\infty}} \frac{a(D,H)}{\log H} .$$

Clearly $a_D \in G$.

Lemma 2. Suppose that $a(D,H), b(D,H)$ in F satisfy $a(D,H) >< b(D,H)$. Then $a_D >< b_D$.

Proof. It suffices to show that $a(D,H) << b(D,H)$ implies $a_D << b_D$. In fact from (2), we get

$$\frac{a(D,H)}{\log H} \le \frac{\gamma b(k_1 D, k_2^D H^{k_3})}{\log(k_2^D H^{k_3})} \cdot \frac{\log(k_2^D H^{k_3})}{\log H} ,$$

provided $D \geq D_0$ and $H \geq H_0$, whence

$$a_D \leq k_3 \gamma b_{k_1 D}$$

provided $D \geq D_0$, i.e. $a_D << b_D$.

Lemma 3 Suppose that $t = t(\xi) = t(\xi_1,\ldots,\xi_n) \geq 1$ and η is algebraic over $\mathbb{Q}(\xi_1,\ldots,\xi_n)$. Then for any $P \in P_{n+1}(D,H)$ $(D \geq 1, H \geq 2)$ with $P(\xi_1,\ldots,\xi_n,\eta) \neq 0$, there exist positive integers $c_2,\ldots,c_5$ depending only on $\xi_1,\ldots,\xi_n,\eta$ and n such that the inequality

$$(7) \qquad |P(\xi_1,\ldots,\xi_n,\eta)| \geq \exp(-\theta(c_2 D, c_3^D H^{c_4}|\xi))\, c_5^{-D} H^{-c_4}$$

holds for $D \geq 1, H \geq 2$, whence

$$(8) \qquad \theta(D,H|\xi_1,\ldots,\xi_n,\eta) << \theta(D,H|\xi_1,\ldots,\xi_n).$$

Proof. We first prove (7). Let $\ell = \deg_y P(x_1,\ldots,x_n,y)$. If $\ell = 0$, (7) is trivial. So we may assume $\ell \geq 1$. Clearly $\ell \leq D$. Let $m \geq 1$ be the degree of η over $\mathbb{Q}(\xi_1,\ldots,\xi_n)$. Then there exist $f_i(x_1,\ldots,x_n) \in \mathbb{Z}[x_1,\ldots,x_n]$ $(i = 0,1,\ldots,m)$ with g.c.d. $(f_0,f_1,\ldots,f_m) = 1$ and $f_0(\xi_1,\ldots,\xi_n) \neq 0$ such that $F(\xi_1,\ldots,\xi_n,\eta) = 0$, where

$$F(x_1,\ldots,x_n,y) = \sum_{i=0}^{m} f_i(x_1,\ldots,x_n) y^{m-i} .$$

Obviously, there exist constants $d_0 > 0$, $h_0 > 0$ depending only on $\xi_1,\ldots,\xi_n,\eta$ such that

(9) $\deg f_i \le d_0$, $H(f_i) \le h_0$ $(i = 0,1,\ldots,m)$.

Write

$$P(x_1,\ldots,x_n,y) = \sum_{i=0}^{\ell} g_i(x_1,\ldots,x_n)y^{\ell-i} .$$

We have

(10) $\deg g_i \le D$, $H(g_i) \le H$ $(i = 0,1,\ldots,\ell)$.

Let $R(x_1,\ldots,x_n)$ be the y-resultant of $F(x_1,\ldots,x_n,y)$ and $P(x_1,\ldots,x_n,y)$:

$$R(x_1,\ldots,x_n) = \underbrace{\begin{vmatrix} f_0 f_1 \cdots f_m & & \\ & f_0 f_1 \cdots f_m & \\ & \cdots\cdots\cdots & \\ & & f_0 f_1 \cdots f_m \\ g_0 g_1 \cdots g_\ell & & \\ & g_0 g_1 \cdots g_\ell & \\ & \cdots\cdots\cdots & \\ & & g_0 g_1 \cdots g_\ell \end{vmatrix}}_{\ell+m} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} \ell \\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} m \end{matrix}$$

$$= \begin{vmatrix} f_0 f_1 \cdots f_m & & & y^{\ell-1}F \\ & f_0 f_1 \cdots f_m & & y^{\ell-2}F \\ & \cdots\cdots\cdots & & \\ & & f_0 f_1 \cdots f_{m-1} & F \\ g_0 g_1 \cdots g_\ell & & & y^{m-1}P \\ & g_0 g_1 \cdots g_\ell & & y^{m-2}P \\ & \cdots\cdots\cdots & & \\ & & g_0 g_1 \cdots g_{\ell-1} & P \end{vmatrix} .$$

On expending the determinant according to the last column, we obtain

(11) $$R(x_1,\ldots,x_n) = F\cdot(y^{\ell-1}Q_1+\ldots+Q_\ell)+P(y^{m-1}Q_{\ell+1}+\ldots+Q_{\ell+m}) ,$$

where $Q_j(x_1,\ldots,x_n) \in \mathbb{Z}[x_1,\ldots,x_n]$. By Lemma 1 and (9),(10) we get

(12) $$\deg R(x_1,\ldots,x_n) \leq \ell d_0 + mD \leq c_2 D ,$$

(13) $$\begin{aligned} H(R(x_1,\ldots,x_n)) &\leq L(R(x_1,\ldots,x_n)) \\ &\leq \Big(\sum_{i=0}^{m} L(f_i)\Big)^{\ell} \Big(\sum_{j=0}^{\ell} L(g_j)\Big)^{m} \\ &\leq \Big((m+1)\binom{d_0+n}{n}h_0\Big)^{\ell} \Big((\ell+1)\binom{D+n}{n}H\Big)^{m} \\ &\leq c_3^D H^{c_4} . \end{aligned}$$

Similarly, we obtain for $j = 1,2,\dots,m$

$$\deg Q_{\ell+j} \leq c_2 D,$$
$$L(Q_{\ell+j}) \leq c_3^D H^{c_4}.$$

Hence

$$(14)\qquad \Big|\sum_{j=1}^{m} \eta^{m-j} Q_{\ell+j}(\boldsymbol{\xi})\Big| \leq m(\max(|\eta|,1))^{m-1} \max_{1\leq j\leq m} L(Q_{\ell+j}) \Big(\prod_{i=1}^{n} \max(|\xi_i|,1)\Big)^{c_2 D} \leq c_5^D H^{c_4} .$$

On substituting x_i with ξ_i, y with η in (11) and noting that $F(\xi_1,\dots,\xi_n,\eta) = 0$, we obtain by (14)

$$(15)\qquad |R(\boldsymbol{\xi})| = |R(\xi_1,\dots,\xi_n)| = |P(\xi_1,\dots,\xi_n,\eta)|\,\Big|\sum_{j=1}^{m} \eta^{m-j} Q_{\ell+j}(\boldsymbol{\xi})\Big| \leq |P(\xi_1,\dots,\xi_n,\eta)|\; c_5^D H^{c_4} .$$

We assert that $R(\boldsymbol{\xi}) \neq 0$, for otherwise $f_0(\boldsymbol{\xi}) \neq 0$ and the fact that $F(\boldsymbol{\xi},y)$ is irreducible over $\mathbb{Q}(\xi_1,\dots,\xi_n)$ would imply $F(\boldsymbol{\xi},y)$ devides $P(\boldsymbol{\xi},y)$ in $\mathbb{Q}(\xi_1,\dots,\xi_n)[y]$, a contradiction to the hypothesis that $P(\boldsymbol{\xi},\eta) \neq 0$.
Thus, by (12), (13), we see that

$$|R(\boldsymbol{\xi})| \geq \exp(-\theta(c_2 D, c_3^D H^{c_4}|\boldsymbol{\xi}|)),$$

and (7) follows from this and (15) immediately. Further, without loss of generality, we may assume that $\xi_1,\dots,\xi_t$

are algebraically independent over $\mathbb{Q}$. By Theorem 1, we have

$$(16)\qquad D+\log H \leq c_6\theta(D,H|\xi_1,\ldots,\xi_t) \leq c_6\theta(c_2D, c_3^D H^{c_4}|\xi_1,\ldots,\xi_n).$$

Now choose $P \in P_{n+1}(D,H)$ such that

$$|P(\xi_1,\ldots,\xi_n,\eta)| = w_D(H|\xi_1,\ldots,\xi_n,\eta)$$
$$= \exp(-\theta(D,H|\xi_1,\ldots,\xi_n,\eta)),$$

then (7) and (16) imply (8) at once. This completes the proof of the lemma.

Proof of Theorem 2. In virtue of Lemma 2, it suffices to prove only (4). By the hypotheses, $t(\boldsymbol{\xi}) = t(\boldsymbol{\eta}) = t \geq 1$. Since $\eta_1,\ldots,\eta_q$ are algebraic over $\mathbb{Q}(\xi_1,\ldots,\xi_p)$, we see, by Lemma 3, that

$$\theta(D,H|\xi_1,\ldots,\xi_p,\eta_1,\ldots,\eta_q) \ll \theta(D,H|\xi_1,\ldots,\xi_p,\eta_1,\ldots,\eta_{q-1}) \ll$$
$$\ll \ldots \ll \theta(D,H|\xi_1,\ldots,\xi_p,\eta_1) \ll \theta(D,H|\xi_1,\ldots,\xi_p).$$

On the other hand, by the definition,

$$\theta(D,H|\xi_1,\ldots,\xi_p) \ll \theta(D,H|\xi_1,\ldots,\xi_p,\eta_1,\ldots,\eta_q).$$

Thus

(17) $\theta(D,H|\xi_1,\ldots,\xi_p) >< \theta(D,H|\xi_1,\ldots,\xi_p,\eta_1,\ldots,\eta_q)$.

Similarly, we have

(18) $\theta(D,H|\eta_1,\ldots,\eta_q) >< \theta(D,H|\eta_1,\ldots,\eta_q,\xi_1,\ldots,\xi_p)$

$$= \theta(D,H|\xi_1,\ldots,\xi_p,\eta_1,\ldots,\eta_q) \ .$$

(17) and (18) yield (4), since $><$ is an equivalence relation. The theorem is proved.

4. Proof of Theorem 4.

We use some idea from E. Wirsing [11]. In this section we suppose $(\xi_1,\ldots,\xi_{n-1})$ is in $\mathbb{C}^{n-1}$ with $\xi_1,\ldots,\xi_{n-1}$ algebraically independent over $\mathbb{Q}$. Let $K = \mathbb{Q}(\xi_1,\ldots,\xi_{n-1})$ and $\beta \in \mathbb{C}$ be algebraic over K. Clearly, there exists an irreducible polynomial $F(x_1,\ldots,x_n) \in \mathbb{Z}[x_1,\ldots,x_n]$ with coprime coefficients and $\deg_{x_n} F \geq 1$ such that

$$F(\xi_1,\ldots,\xi_{n-1},\beta) = 0.$$

Since F is determined by β up to a factor $u = \pm 1$ (see S. Lang [5], pp. 197-199), we can define

$$d(\beta) = \deg F, \quad H(\beta) = H(F).$$

For any $\eta \in \mathbb{C}$, write $w_D^*(\eta;\xi_1,\ldots,\xi_{n-1})$ for the supremum of the numbers $w > 0$ such that there exist infinitely many β algebraic over K of $d(\beta) \leq D$ satisfying

$$0 < |\eta - \beta| < H(\beta)^{-1-\omega} .$$

In this section we use $\ll$ in the sense different from that defined in Sect. 1, i.e., we use it as the Vinogradov's symbol, the constant involved in $\ll$ may depend on $\xi_1,\ldots,\xi_{n-1}, D$ and n, but independent of H; in the proof of Lemma 7 below it may also depend on η. If E is a measurable subset of $\mathbb{R}$, we write $\mu(E)$ for its Lebesgue measure.

Lemma 4. The inequality

$$w_D^*(\eta;\xi_1,\ldots,\xi_{n-1}) \leq \binom{D+n}{n} - 1$$

holds for almost all real numbers η.

Proof. Let

$$E = \{\eta \in \mathbb{R} \mid w_D^*(\eta;\xi_1,\ldots,\xi_{n-1}) > \binom{D+n}{n} - 1\} ,$$

$$E_k = \{\eta \in \mathbb{R} \mid w_D^*(\eta;\xi_1,\ldots,\xi_{n-1}) \geq \binom{D+n}{n} - 1 + \frac{2}{k}\} .$$

Clearly $E = \bigcup_{k=1}^{\infty} E_k$. To prove the lemma, it suffices to prove $\mu(E_k) = 0$, $k = 1,2,\ldots$. If $\eta \in E_k$, by definition, there exist infinitely many β algebraic over K with

$d(\beta) \leq D$ satisfying

$$(19) \qquad 0 < |\eta - \beta| < H(\beta)^{-\binom{D+n}{n} - \frac{1}{k}} .$$

Let S_H be the set of β algebraic over K with $H(\beta) = H$ and $d(\beta) \leq D$. Let $C(\beta)$ be the disc centered at β with radius $H(\beta)^{-\binom{D+n}{n}-\frac{1}{k}}$. Set $R_H = \bigcup_{\beta \in S_H} (C(\beta) \cap \mathbb{R})$.
Obviously

$$(20) \qquad \mu(R_H) \ll H^{-(1+\frac{1}{k})} ,$$

since the cardinal of the set S_H is at most

$$\binom{D+n}{n} (2H+1)^{\binom{D+n}{n}-1} D << H^{\binom{D+n}{n}-1} .$$

We have by (19)

$$E_k \subset \bigcup_{H=N}^{\infty} R_H \quad (N = 1,2,\ldots) .$$

This and (20) imply, by Borel-Cantelli lemma, that $\mu(E_k) = 0$, whence the lemma follows at once.

Lemma 5. (see E. Wirsing [11], p. 70, Hilfssatz 2.)

Suppose that $\xi \in \mathbb{C}$ and $Q(x) = a_0(x - \alpha_1) \ldots (x - \alpha_m) \in \mathbb{C}[x]$ with $a_0 \neq 0$. Then there exists $c_7 = c_7(\xi, m) > 0$ such that

$$|a_0| \prod_{\substack{i=1 \\ |\xi-\alpha_i| \geq 1}}^{m} |\xi - \alpha_i| \leq c_7 H(Q) ,$$

where a possibly empty product means 1.

Lemma 6. (A.O. Gelfond [4], p. 135) Suppose that $P_1(x_1,\dots,x_s),\dots,P_m(x_1,\dots,x_s)$ are arbitrary polynomials in s variables with heights $H_1,\dots,H_m$. Denote the height and degrees of the polynomial $P(x_1,\dots,x_s) = \prod_{i=1}^{m} P_i(x_1,\dots,x_s)$ by H and $n_1,\dots,n_s$ in the variables $x_1,\dots,x_s$, respectively. Then we have the inequality

$$H \geq e^{-n} H_1 H_2 \dots H_m \ , \ n = \sum_{i=1}^{s} n_i \ .$$

Lemma 7. Suppose $\xi_1,\dots,\xi_{n-1},\eta$ (in $\mathbb{C}$) are algebraically independent over $\mathbb{Q}$. Then the inequality

$$(21) \qquad w_D(\xi_1,\dots,\xi_{n-1},\eta) \leq (D-\tfrac{1}{2})\, w_{[\frac{3}{2}D(D-1)]}(\xi_1,\dots,\xi_{n-1}) + w_D^*(\eta;\xi_1,\dots,\xi_{n-1}) + D - 1$$

holds for $D \geq 2$.

Proof. By the definition of $w_D(\xi_1,\dots,\xi_{n-1},\eta)$ and Lemma 6 , we see that for any $w' < w_D(\xi_1,\dots,\xi_{n-1},\eta)$ there exist infinitely many irreducible polynomials $P \in \mathbb{Z}[x_1,\dots,x_n]$ with $\deg P \leq D$ and coprime coefficients such that

$$(22) \qquad 0 < |P(\xi_1,\dots,\xi_{n-1},\eta)| < H(P)^{-w'} \ .$$

Write P as

(23) $$P(x_1,\dots,x_{n-1},y) = \sum_{i=0}^{m} p_i(x_1,\dots,x_{n-1})y^{m-i} ,$$

where $m = \deg_{x_n} P$. Evidently

(24) $$\deg p_i \le D - m + i,\ H(p_i) \le H(P),\quad i = 0,1,\dots,m.$$

If for any $w' < w_D(\xi_1,\dots,\xi_{n-1},\eta)$ among the P in (22) there are infinitely many P with $m = 0$, then $w_D(\xi_1,\dots,\xi_{n-1}) \ge w'$, whence $w_D(\xi_1,\dots,\xi_{n-1}) \ge w_D(\xi_1,\dots,\xi_{n-1},\eta)$ and (21) holds. Further, suppose that for any $w' < w_D(\xi_1,\dots,\xi_{n-1},\eta)$ among the P in (22) there are infinitely many P with $m = 1$. Let β be the zero of $P(\xi_1,\dots,\xi_{n-1},y) = p_0(\xi_1,\dots,\xi_{n-1})y + p_1(\xi_1,\dots,\xi_{n-1})$. Recalling the definition of $d(\beta)$ and $H(\beta)$, we have

(25) $$d(\beta) \le D,\ H(\beta) = H(P) .$$

Note that by (24)

$$|p_0(\xi_1,\dots,\xi_{n-1})|^{-1} \le H(P)^{\theta(D,H(P)|\xi_1,\dots,\xi_{n-1})/\log H(P)} .$$

This with (22),(25) gives

$$0 < |\eta-\beta| = \left|\frac{P(\xi_1,\dots,\xi_{n-1},\eta)}{p_0(\xi_1,\dots,\xi_{n-1})}\right| < H(\beta)^{-w'+\theta(D,H(\beta)|\xi_1,\dots,\xi_{n-1})/\log H(\beta)}$$

for infinitely many β . Hence $w_D^*(\eta;\xi_1,\dots,\xi_{n-1}) \ge w'-w_D(\xi_1,\dots,\xi_{n-1})$, therefore $w_D(\xi_1,\dots,\xi_{n-1},\eta) \le w_D(\xi_1,\dots,\xi_{n-1}) + w_D^*(\eta;\xi_1,\dots,\xi_{n-1})$, i.e. (21) holds. Thus we may assume that for any $w' < w_D(\xi_1,\dots,\xi_{n-1},\eta)$ the infinitely many P in (22) are all irreducible with $m = \deg_{x_n} P \geq 2$ and coprime coefficients, and have the expression (23). Denote by $D_P(x_1,\dots,x_{n-1})$ the discriminant of $P(x_1,\dots,x_{n-1},y)$ as a polynomial in y. Since $P(x_1,\dots,x_{n-1},y)$ is irreducible in $\mathbb{Z}[x_1,\dots,x_{n-1},y]$, we see, by Gauss lemma, that $P(x_1,\dots,x_{n-1},y)$ is an irreducible polynomial in y over the field $\mathbb{Q}(x_1,\dots,x_{n-1})$. Hence

$$(26) \qquad D_P(x_1,\dots,x_{n-1}) \neq 0 .$$

It follows from the definition of discriminant that

$$R\left(P,\frac{\partial P}{\partial y}\right) = (-1)^{\frac{m(m-1)}{2}} p_0(x_1,\dots,x_{n-1}) D_P(x_1,\dots,x_{n-1}),$$

where the left-hand side is the resultant of $P(x_1,\dots,x_{n-1},y)$ and $\frac{\partial P(x_1,\dots,x_{n-1},y)}{\partial y}$ as polynomials in y. So we have

$$(-1)^{\frac{m(m-1)}{2}} D_P(x_1,\dots,x_{n-1}) = \underbrace{\begin{vmatrix} 1 & p_1 & \cdots & p_m & & & \\ & p_0 & p_1 & \cdots & p_m & & \\ & & \cdots & \cdots & \cdots & & \\ & & & & p_0 & p_1 \cdots & p_m \\ m & (m-1)p_1 & \cdots & p_{m-1} & & & \\ & mp_0 & (m-1)p_1 & \cdots & p_{m-1} & & \\ & & \cdots & \cdots & \cdots & & \\ & & & mp_0 & (m-1)p_1 & \cdots & p_{m-1} \end{vmatrix}}_{2m-1} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} m-1 \\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} m \end{matrix} .$$

On applying Lemma 1 and (24), we obtain

$$(27)\qquad H(D_P(x_1,\ldots,x_{n-1})) \le \left((m+1)\binom{D+n-1}{n-1}H(P)\right)^{m-1}\left(\frac{m(m+1)}{2}\binom{D+n-1}{n-1}H(P)\right)^{m}$$
$$\le c_8(H(P))^{2D-1},$$

where c_8 is a positive integer depending only on D,n. On utilizing (24) and Lemma 1 to the transposed determinant of $D_P(x_1,\ldots,x_{n-1})$, we get

$$(28)\qquad \deg D_P(x_1,\ldots,x_{n-1}) \le \sum_{i=1}^{m-1}(D-m+i) + (m-1)D$$
$$= (2m-2)D - \frac{m(m-1)}{2}$$
$$\le \tfrac{3}{2}D(D-1),$$

since $1 \le m \le D$. By (26) and the hypothesis that $\xi_1,\ldots,\xi_{n-1},\eta$ are algebraically independent over $\mathbb{Q}$, we have

$$D_P(\xi_1,\ldots,\xi_{n-1}) \neq 0 .$$

This together with (27), (28) gives (writing $D_0=[\frac{3}{2}D(D-1)]$)

$$(29)\qquad |D_P(\xi_1,\ldots,\xi_{n-1})| \ge \exp(-\theta(D_0,c_8H(P)^{2D-1}|\xi_1,\ldots,\xi_{n-1})) .$$

By the definition of $w_{D_0}(\xi_1,\ldots,\xi_{n-1})$ we see that for any given $\delta > 0$ the inequality

$$(30) \qquad \theta(D_0, c_8 H(P)^{2D-1} | \xi_1, \ldots, \xi_{n-1}) / \log H(P)$$
$$\leq (2D-1) w_{D_0}(\xi_1, \ldots, \xi_{n-1}) + 2\delta$$

holds for P with $H(P)$ being sufficiently large. It follows from (29), (30) that

$$(31) \qquad |D_P(\xi_1, \ldots, \xi_{n-1})|^{-\frac{1}{2}} \leq H(P)^{(D-\frac{1}{2}) w_{D_0}(\xi_1, \ldots, \xi_{n-1}) + \delta},$$

provided $H(P)$ is sufficiently large.

On the other hand, let $\beta_1, \ldots, \beta_m$ be the zeros of $P(\xi_1, \ldots, \xi_{n-1}, y)$ so arranged that $q_i = |\eta - \beta_i|$ $(i = 1, \ldots, m)$ satisfy

$$q_1 \leq q_2 \leq \ldots \leq q_m .$$

Then for i, j with $1 \leq i < j \leq m$,

$$(32) \qquad |\beta_i - \beta_j| = |\beta_i - \eta + \eta - \beta_j| \leq 2q_j .$$

On applying Lemma 5 to

$$P(\xi_1, \ldots, \xi_{n-1}, y) = p_0(\xi_1, \ldots, \xi_{n-1})(y - \beta_1) \ldots (y - \beta_m),$$

we see, by (24), that

$$(33) \qquad |p_0(\xi_1, \ldots, \xi_{n-1})| \prod_{\substack{i=1 \\ q_i \geq 1}}^{m} q_i \leq c_7(\eta, m) \max_{0 \leq i \leq m} |p_i(\xi_1, \ldots, \xi_{n-1})|$$
$$\leq c_7 \binom{D+n-1}{n-1} H(P) \Big(\prod_{j=1}^{n-1} \max(|\xi_j|, 1)\Big)^D \leq c_9 H(P),$$

where c_9 depends only on $\xi_1,\ldots,\xi_{n-1},\eta,D,n$.

It is well-known that

$$D_P(\xi_1,\ldots,\xi_{n-1}) = (p_0(\xi_1,\ldots,\xi_{n-1}))^{2m-2} \prod_{1\leq i<j\leq m} (\beta_i-\beta_j)^2 .$$

We have, by (32),

$$|D_P(\xi_1,\ldots,\xi_{n-1})|^{\frac{1}{2}} \leq |p_0(\xi_1,\ldots,\xi_{n-1})|^{m-1} \prod_{j=2}^{m} (2q_j)^{j-1}$$

$$\ll |p_0(\xi_1,\ldots,\xi_{n-1})|^{m-1} \prod_{j=2}^{m} q_j^{j-1} .$$

By (33) we obtain

$$(34) \qquad q_1|D_P(\xi_1,\ldots,\xi_{n-1})|^{\frac{1}{2}}$$

$$\ll |p_0(\xi_1,\ldots,\xi_{n-1})|q_1q_2\ldots q_m|p_0(\xi_1,\ldots,\xi_{n-1})|^{m-2} \prod_{j=2}^{m} q_j^{j-2}$$

$$\leq |P(\xi_1,\ldots,\xi_{n-1},\eta)|\,|p_0(\xi_1,\ldots,\xi_{n-1})|^{m-2} \prod_{\substack{j=1\\ q_j\geq 1}}^{m} q_j^{m-2}$$

$$\ll |P(\xi_1,\ldots,\xi_{n-1},\eta)|H(P)^{D-2} .$$

On noting the fact that $q_1 = |\eta-\beta_1|$, $H(\beta_1) = H(P)$, it follows from (22), (31) and (34) that

$$(35) \qquad |\eta-\beta_1| \ll H(\beta_1)^{-w'+(D-\frac{1}{2})w_{D0}(\xi_1,\ldots,\xi_{n-1})+D-2+\delta} ,$$

provided $H(\beta_1) = H(P)$ is sufficiently large. Note that $d(\beta_1) = \deg P \leq D$, w' is any given number with $0 < w' < w_D(\xi_1,\ldots,\xi_{n-1},\eta)$ and δ can be arbitrarily small.

So (35) implies that

$$1 + w_D^*(\eta;\xi_1,\ldots,\xi_{n-1}) \geq w_D(\xi_1,\ldots,\xi_{n-1},\eta) - (D-\tfrac{1}{2})w_{D_0}(\xi_1,\ldots,\xi_{n-1}) - D + 2 .$$

Recalling $D_0 = [\frac{3}{2}D(D-1)]$, (21) follows at once. The proof of the lemma is complete.

Proof of Theorem 4. We prove the theorem by induction on n. When $n = 2$, we can choose, by Theorem S, $\xi_1 \in \mathbb{R} \cap T^1(\alpha_2)$. By Lemma 4, there exists $\eta \in \mathbb{R}$ such that ξ_1, η are algebraically independent over $\mathbb{Q}$ and

$$(36) \qquad w_D^*(\eta;\xi_1) \leq \binom{D+2}{2} - 1,$$

since the set of real numbers algebraic over $\mathbb{Q}(\xi_1)$ is countable, whence it is of measure zero. Now

$$w_D(\xi_1,\eta) \geq w_D(\xi_1) \quad ,$$

so $\alpha(\xi_1,\eta) \geq \alpha(\xi_1) = \alpha_2 \geq 3$. On the other hand, Lemma 7 and (36) give

$$w_D(\xi_1,\eta) \leq (D - \tfrac{1}{2})w_{[\frac{3}{2}D(D-1)]}(\xi_1) + \binom{D+2}{2} + D - 2,$$

whence $\alpha(\xi_1,\eta) \leq 2\alpha(\xi_1) + 1 = 2\alpha_2 + 1$. Obviously $(\xi_1,\eta) \in T^2 \cap \mathbb{R}^2$. Thus the theorem holds for $n = 2$. Suppose that the theorem holds for $n - 1$ with $n \geq 3$, we proceed to prove that it holds for n. On applying the inductive hypothesis with $\alpha_{n-1} = \alpha_n > n \geq 3$, we see that there exists $(\xi_1,\ldots,\xi_{n-1}) \in T^{n-1} \cap \mathbb{R}^{n-1}$ with

(37) $$\alpha_n \leq \alpha(\xi_1,\ldots,\xi_{n-1}) \leq 2^{n-2}(\alpha_n + 1) - 1 .$$

By Lemma 4, there exists $\eta' \in \mathbb{R}$ such that $\xi_1,\ldots,\xi_{n-1},\eta'$ are algebraically over $\mathbb{Q}$ and

(38) $$w_D^*(\eta';\xi_1,\ldots,\xi_{n-1}) \leq \binom{D+n}{n} - 1 ,$$

since the set of real numbers algebraic over $K = \mathbb{Q}(\xi_1,\ldots,\xi_{n-1})$ is countable, whence it has measure zero. By virtue of $w_D(\xi_1,\ldots,\xi_{n-1},\eta') \geq w_D(\xi_1,\ldots,\xi_{n-1})$ and (37), we see that

$$\alpha(\xi_1,\ldots,\xi_{n-1},\eta') \geq \alpha_n > n .$$

On the other hand, Lemma 7 and (38) give

$$w_D(\xi_1,\ldots,\xi_{n-1},\eta') \leq (D - \tfrac{1}{2})w_{[\frac{3}{2}D(D-1)]}(\xi_1,\ldots,\xi_{n-1}) + \binom{D+n}{n} + D-2 ,$$

so

$$\begin{aligned}\alpha(\xi_1,\ldots,\xi_{n-1},\eta') &\leq 2\alpha(\xi_1,\ldots,\xi_{n-1}) + 1 \\ &\leq 2^{n-1}(\alpha_n + 1) - 1\end{aligned}$$

by (37). Obviously $(\xi_1,\ldots,\xi_{n-1},\eta') \in T^n \cap \mathbb{R}^n$. Thus the theorem holds for n. The proof of the theorem is complete.

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References

[1] *A. Baker*, Transcendental number theory, Cambridge 1979.

[2] *A. Durand*, Fonction θ-ordre et classification de $\mathbb{C}^p$, C.R. Acad. Sc. Paris, t. 280 (5 mai 1975), Série A 1085-1088.

[3] *N.I. Feldman* and *A.B. Shidlovsky*, The development and present state of the theory of transcendental numbers, Russian Math. Surveys 22 (1967), 1-79.

[4] *A.O. Gelfond*, Transcendental and algebraic numbers, New York 1960.

[5] *S. Lang*, Algebra, second edition, Addison-Wesley Publishing Company, Inc. 1984.

[6] *K. Mahler*, Zur Approximation der Exponentialfunktion und des Logarithmus. I, J. reine angew. Math. 166 (1932), 118-136.

[7] *K. Mahler*, On the order function of a transcendental number, Acta Arith. 18 (1971), 63-76.

[8] *W.M. Schmidt*, Mahler's T-numbers, Proc. Symposia Pure Math., vol. 20 (Amer. Math. Soc. 1971), 275-286.

[9] *Th. Schneider*, Einführung in die transzendenten Zahlen, Berlin-Göttingen-Heidelberg 1957.

[10] *I. Shiokawa*, Algebraic independence of certain gap series, Arch. Math. 38 (1982), 438-442.

[11] *E. Wirsing*, Approximation mit algebraischen Zahlen beschränkten Grades, J. reine angew. Math. 206 (1961), 67-77.

[12] *G. Wüstholz*, Über das Abelsche Analogon des Lindemannschen Satzes I, Invent. math. 72 (1983), 363-388.

[13] *Y.C. Zhu*, On the algebraic independence of certain power series of algebraic numbers, Chin. Ann. Math. 5B:1(1984), 109-117.