HOW TO REGULARIZE SINGULAR VECTORS AND KILL THE DYNAMICAL WEYL GROUP

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Abstract. Let $\mathfrak{g}$ be a simple Lie algebra, and let M_λ be the Verma module over $\mathfrak{g}$ with highest weight λ. For a finite-dimensional $\mathfrak{g}$-module U we introduce a notion of a regularizing operator, acting in U, which makes the meromorphic family of intertwining operators $\Phi : M_{\lambda+\mu} \to M_\lambda \otimes U$ holomorphic, and conjugates the dynamical Weyl group operators $A_w(\lambda) \in \mathrm{End}(U)$ to constant operators. We establish fundamental properties of regularizing operators, including uniqueness, and prove the existence of a regularizing operator in the case $\mathfrak{g} = \mathfrak{sl}_3$.

1. Introduction

Let $\mathfrak{g}$ be a simple Lie algebra. Let M_λ be the Verma module over $\mathfrak{g}$ with highest weight λ, and $\mathbf{1}_\lambda \in M_\lambda$ the highest weight vector. A fundamental object of the representation theory of $\mathfrak{g}$ is a family of intertwining operators

$$\Phi^u_\lambda \ : \ M_{\lambda+\mu} \ \to \ M_\lambda \otimes U, \qquad \mathbf{1}_{\lambda+\mu} \ \mapsto \ \mathbf{1}_\lambda \otimes u \ + \ \text{ lower order terms },$$

where U is a finite-dimensional $\mathfrak{g}$-module, and $u \in U[\mu]$. In this paper we study singularities of Φ^u_λ as a function of λ and u.

The intertwining operator Φ^u_λ is completely determined by the image of the highest weight vector, which we denote by $\mathrm{Sing}(\mathbf{1}_\lambda \otimes u)$. For any $u \in U[\mu]$, the singular vector $\mathrm{Sing}(\mathbf{1}_\lambda \otimes u)$ can be written in terms of the Shapovalov form on M_λ, and is unique for generic λ. It is a rational function of λ with possible poles if λ belongs to at least one of the Kac-Kazhdan hyperplanes $\langle \alpha, \lambda+\rho\rangle = \frac{k}{2}\langle\alpha,\alpha\rangle$. For those special values of λ the operator Φ^u_λ may fail to exist.

One of the objectives of this paper is to regularize the family of operators Φ^u_λ, that is, to construct a family of intertwining operators $\tilde\Phi^u_\lambda : M_{\lambda+\mu} \ \to \ M_\lambda \otimes U$, holomorphically depending on λ and u, such that the correspondence $u \mapsto \tilde\Phi^u_\lambda$ is still injective.

We also study the relation between the operators Φ^u_λ for different values of λ and u. Recall that for a dominant integral weight ν and any element w of the Weyl group W, the Verma module M_ν contains a submodule isomorphic to $M_{w\cdot\nu}$. We fix this identification and regard $M_{w\cdot\nu}$ as a submodule of M_ν.

If $\lambda+\mu$ is a dominant integral weight, then for any $w \in W$ the restriction of the operator Φ^u_λ to the submodule $M_{w\cdot(\lambda+\mu)}$ takes values in the submodule $M_{w\cdot\lambda}\otimes U$. The relation between

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this restriction and the operator $\Phi^u_{w\cdot\lambda}$ is given by

$$\Phi^u_\lambda|_{M_{w\cdot\lambda}} = \Phi^{A_w(\lambda)u}_{w\cdot\lambda}\,,$$

where $A_w(\lambda)$ is a rational $\mathrm{End}(U)$-valued function of λ. The operators $A_w(\lambda)$, $w \in W$, are called the dynamical Weyl group operators.

The specialization of the dynamical Weyl group operators at $\lambda = -\rho$ produces constant operators $\bar w = A_w(-\rho)$, and for a simple root reflection $s_i \in W$ we have $\bar s_i = f_i^{h_i}$. According to the classical results of Verma and Lusztig, the operators $\bar s_i$ form a representation of the Weyl group. The dynamical Weyl group operators may be regarded as meromorphic deformations of the operators $\bar w$ in the class of cocycles on the Weyl group, satisfying

$$A_{w_1w_2}(\lambda) = A_{w_1}(w_2\cdot\lambda)A_{w_2}(\lambda)\,.$$

We introduce a notion of a regularizing operator $N(\lambda)$, acting in U and polynomially depending on λ. By definition, we require that $N(\lambda)$ has the following properties.

First, the modified intertwining operators $\Phi^{N(\lambda)u}_\lambda : M_{\lambda+\mu} \to M_\lambda \otimes U$ are holomorphic in λ for any $u \in U[\mu]$.

Second, the conjugated dynamical Weyl group operators $N(w\cdot\lambda)^{-1}A_w(\lambda)N(\lambda)$ are constant operators. This implies, in particular, that the cocycle $A_w(\lambda)$ is cohomologically equivalent to a constant function of λ.

Third, we impose a minimality condition on $N(\lambda)$ by requiring that $\det N(\lambda)$ is equal to a certain explicitly specified polynomial. (The second property of $N(\lambda)$ implies the divisibility of $\det N(\lambda)$ by this polynomial.)

We prove that the regularizing operator $N(\lambda)$, if it exists, is uniquely determined up to the right multiplication by an operator, polynomially depending on λ, invertible for all values of λ, and symmetric with respect to the Weyl group action. If $\mathfrak{g} = \mathfrak{sl}_2$ it is easy to check that the regularizing operator exists for any irreducible finite-dimensional $\mathfrak{g}$-module U; in each weight subspace of U the operator $N(\lambda)$ acts as a scalar, determined by the determinant condition. In this paper we prove the existence property in the case $\mathfrak{g} = \mathfrak{sl}_3$. We conjecture that a regularizing operator exists for any simple Lie algebra $\mathfrak{g}$ and any finite-dimensional $\mathfrak{g}$-module U.

Our construction of the regularizing operator $N(\lambda)$ relies on a certain realization of finite-dimensional $\mathfrak{sl}_3$-modules, based on a special case of the $(\mathfrak{gl}_m, \mathfrak{gl}_n)$ duality. We identify weight subspaces $U[\mu]$ of an $\mathfrak{sl}_3$-module U with subspaces of singular vectors in tensor products $V_{(l_1,0)} \otimes V_{(l_2,0)} \otimes V_{(l_3,0)}$ of finite-dimensional $\mathfrak{gl}_2$-modules. In this realization the action of operators $A_w(\lambda)$ is identified with the action of suitably renormalized standard rational R-matrices, the spectral parameters in the R-matrices being determined by λ.

We use the functional realization $\mathcal{H}[\mathbf{l}]$ of the tensor product $V_{(l_1,0)} \otimes V_{(l_2,0)} \otimes V_{(l_3,0)}$, provided by the representation theory of the Yangian $Y(\mathfrak{gl}_2)$, to define operators $\mathcal{N}[\mathbf{l};\mathbf{z}] : \mathcal{H}[\mathbf{l}] \to V_{(l_1,0)} \otimes V_{(l_2,0)} \otimes V_{(l_3,0)}$, which conjugate the renormalized R-matrices to the identity. We then use results on the reducibility of the tensor product $V_{(l_1,0)}(z_1) \otimes V_{(l_2,0)}(z_2) \otimes V_{(l_3,0)}(z_3)$ of the evaluation $Y(\mathfrak{gl}_2)$-modules, to prove that for any $u \in \mathrm{Im}\,\mathcal{N}[\mathbf{l};\mathbf{z}]$ the operator Φ^u_λ is holomorphic in λ.

The final step of the construction is an alternative 'constant' identification of the functional space $\mathcal{H}[\mathbf{l}]^{\mathrm{sing}}$ with the subspace of singular vectors $\bigl(V_{(l_1,0)} \otimes V_{(l_2,0)} \otimes V_{(l_3,0)}\bigr)^{\mathrm{sing}}$, which is provided by a certain combinatorial lemma. The composition of the 'constant' identification with $\mathcal{N}[\mathbf{l};\mathbf{z}]$ gives the desired regularizing operator $N(\lambda)$ in our particular realization.

As an application of our results, we use the regularized family of intertwining operators to introduce an $\mathrm{End}(U[0])$-valued function $\Psi(\lambda, x)$ of $\lambda \in \mathfrak{h}^*, x \in \mathfrak{h}$, defined as a certain matrix trace of Φ^u_λ. This Baker-Akhiezer type function is holomorphic in λ, and satisfies certain algebraic identites (resonance conditions), relating values of $\Psi(\lambda, x)$ for different values of λ. The fact that $N(\lambda)$ conjugates the dynamical Weyl group operators $A_w(\lambda)$ to the constant operators $\bar{w}$ implies that these identities have a simple form

$$\Psi(\lambda, x)u = \Psi(s_\alpha \cdot \lambda, x)u \,,$$

for suitable λ, u. Special cases of such resonance conditions were used in [ES] to establish the algebraic integrability of the generalized quantum Calogero-Sutherland systems.

This paper is organized as follows. In Section 2 we review facts on $\mathfrak{g}$-modules and set the notation. In Section 3 we recall the construction of intertwining operators $\Phi : M_{\lambda+\mu} \to M_\lambda \otimes U$ and singular vectors in the tensor product.

In Section 4 we review facts about the Weyl group and the dynamical Weyl group. We introduce a distinguished action of W in U, and show that the dynamical Weyl group operators can be viewed as its deformation. We establish Theorems 4 and 5, which improve results of [EV, TV].

In Section 5 we introduce the notion of a regularizing operator $N(\lambda)$, and use the axiomatic description to establish properties of regularizing operators. Theorem 9 gives the uniquiness of a regularizing operator. We formulate Theorem 10 on the existence of regularizing operators for $\mathfrak{g} = \mathfrak{sl}_3$. This is the main technical result of the paper.

Section 6 provides the background necessary to construct regularizing operators for $\mathfrak{sl}_3$. We invoke the $(\mathfrak{gl}_m, \mathfrak{gl}_n)$-duality and functional spaces to give a realization of an $\mathfrak{sl}_3$-module U, and use the realization to construct the fundamental operator $\mathcal{N}[\mathbf{l}; \mathbf{z}]$, which is the main ingredient in the construction of $N(\lambda)$.

Section 7 contains the construction of regularizing operators for $\mathfrak{sl}_3$ and proofs of all their properties. The short Section 8 is devoted to resonance conditions for trace functions, renormalized by $N(\lambda)$.

In Appendix A we review the calculus of formal monomials, which is used to prove the properties of the dynamical Weyl group operators. Appendix B is devoted to the representation theory of the Yangian $Y(\mathfrak{gl}_2)$. We use Yangian modules to establish properties of the operator $\mathcal{N}[\mathbf{l}; \mathbf{z}]$. Appendix C contains a proof of a technical combinatorial Lemma 19, which is used in the main construction.

2. Notation

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ with a Cartan subalgebra $\mathfrak{h}$ and a root system Δ with a polarization $\Delta = \Delta^+ \cup \Delta^-$. We identify $\mathfrak{h}$ with $\mathfrak{h}^*$ using the Killing form on $\mathfrak{g}$, and denote the induced invariant bilinear form on $\mathfrak{h}^*$ by $\langle \cdot, \cdot \rangle$.

An element $\lambda \in \mathfrak{h}^*$ is called dominant, if $\langle \alpha, \lambda \rangle \geq 0$ for any $\alpha \in \Delta^+$. We write $\lambda \geq \mu$ for $\lambda, \mu \in \mathfrak{h}^*$, if $\lambda - \mu$ is dominant. This is a partial order on $\mathfrak{h}^*$.

An element $\lambda \in \mathfrak{h}^*$ is called integral, if $\frac{2\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$ for any $\alpha \in \Delta^+$.

Denote W the Weyl group of $\mathfrak{g}$. We introduce two actions of W in $\mathfrak{h}^*$. The standard action is defined by

$$s_\alpha v = v - \frac{2\langle \alpha, v \rangle}{\langle \alpha, \alpha \rangle}\alpha$$

for any root reflection $s_\alpha \in W$, and the "dot" action is defined by

$$s_\alpha \cdot v = s_\alpha(v+\rho) - \rho,$$

where $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$.

Let $\{e_i, h_i, f_i\}$ denote the standard generators of the Lie algebra $\mathfrak{g}$, corresponding to a simple root α_i. We denote $\mathfrak{sl}_2(\alpha_i)$ the Lie subalgebra of $\mathfrak{g}$, spanned by e_i, h_i, f_i.

We denote $\mathfrak{n}^+, \mathfrak{n}^-$ the Lie subalgebras, generated respectively by $\{e_i\}$ and $\{f_i\}$. We have a vector space decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$.

The root lattice Q is defined by $Q = \sum_{i=1}^{\dim\mathfrak{h}} \mathbb{Z}\alpha_i$. We also set $Q^+ = \sum_{i=1}^{\dim\mathfrak{h}} \mathbb{Z}_{\geq 0}\alpha_i$.

The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is a Q-graded associative algebra, with the grading defined by

$$\mathrm{wt}(e_i) = \alpha_i, \quad \mathrm{wt}(h_i) = 0, \quad \mathrm{wt}(f_i) = -\alpha_i, \qquad i = 1, \dots, \dim\mathfrak{h}.$$

The Poincare-Birkhoff-Witt theorem gives us a decomposition

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}^-) \otimes \mathcal{U}(\mathfrak{h}) \otimes \mathcal{U}(\mathfrak{n}^+),$$

and we denote $\pi_0 : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{h})$ the induced projection along the subspace $(\mathfrak{n}^-\mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g})\mathfrak{n}^+) \subset \mathcal{U}(\mathfrak{g})$.

Introduce an anti-involution ϖ of $\mathfrak{g}$ by

$$\varpi(e_i) = f_i, \qquad \varpi(f_i) = e_i, \qquad \varpi(h_i) = h_i, \qquad i = 1, \dots, \dim\mathfrak{h}.$$

Define a bilinear $\mathcal{U}(\mathfrak{h})$-valued form S on $\mathcal{U}(\mathfrak{g})$, by

$$S(x,y) = \pi_0(\varpi(x)y) \in \mathcal{U}(\mathfrak{h}), \qquad x, y, \in \mathcal{U}(\mathfrak{g}).$$

The form S is contravariant, i.e. satisfies

$$S(xy, z) = S(y, \varpi(x)z), \qquad x, y, z \in \mathcal{U}(\mathfrak{g}).$$

We will identify the commutative algebra $\mathcal{U}(\mathfrak{h})$ with the algebra $\mathbb{C}[\mathfrak{h}^*]$ of polynomial functions on $\mathfrak{h}^*$, and for any $\lambda \in \mathfrak{h}^*$ we will denote S_λ the $\mathbb{C}$-valued contravariant form on $\mathcal{U}(\mathfrak{g})$, obtained by evaluating S at λ.

We consider $\mathfrak{g}$-modules V with a weight space decomposition

$$V = \bigoplus_{\mu\in\mathfrak{h}^*} V[\mu], \qquad V[\mu] = \{v \in V \quad | \quad hv = \mu(h)v \text{ for all } h \in \mathfrak{h}\},$$

with finite-dimensional weight subspaces $V[\mu]$. We say that $v \in V$ is homogeneous of weight μ and write $\mathrm{wt}(v) = \mu$, if $v \in V[\mu]$. Then for any homogeneous $x \in \mathcal{U}(\mathfrak{g})$, we have $\mathrm{wt}(xv) = \mathrm{wt}(x) + \mathrm{wt}(v)$.

In this paper U always denotes a finite-dimensional $\mathfrak{g}$-module.

For any $\lambda \in \mathfrak{h}^*$, we denote M_λ the Verma module for $\mathfrak{g}$, generated by the highest weight vector $\mathbf{1}_\lambda$, with relations

$$\mathfrak{n}^+\mathbf{1}_\lambda = 0, \qquad h\mathbf{1}_\lambda = \lambda(h)\mathbf{1}_\lambda, \quad \text{for } h \in \mathfrak{h}.$$

We identify any Verma module M_λ with $\mathcal{U}(\mathfrak{n}^-)$ by sending $x \in \mathcal{U}(\mathfrak{n}^-)$ to $x\mathbf{1}_\lambda \in M_\lambda$.

The Verma module M_λ carries a bilinear $\mathfrak{g}$-contravariant form $(\cdot|\cdot)$, defined by

$$(x\mathbf{1}_\lambda | y\mathbf{1}_\lambda) = S_\lambda(x, y), \qquad x, y \in \mathcal{U}(\mathfrak{g}).$$

This form is called the Shapovalov form. Different weight subspaces of M_λ are orthogonal with respect to this form, and its restriction to a weight subspace $M_\lambda[\mu]$ is nondegenerate unless λ belongs to a hyperplane of the form

$$\chi_{\alpha,k}(\lambda) \stackrel{def}{=} \frac{2\langle\alpha,\lambda+\rho\rangle}{\langle\alpha,\alpha\rangle} - k = 0$$

for some $\alpha \in \Delta^+, k \in \mathbb{Z}_{\geq 0}$.

The Verma module M_λ is reducible if and only if λ belongs to at least one of those hyperplanes. The kernel of the Shapovalov form is the maximal proper submodule of M_λ. Let V_λ denotes the irreducible quotient of the Verma module M_λ; it inherits a contravariant form and a weight decomposition from M_λ; moreover, the contraviriant form on V_λ is nondegenerate. It is known that the module V_λ is finite-dimensional if and only if λ is a dominant integral weight.

3. Singular vectors and intertwining operators

A vector v in a $\mathfrak{g}$-module V is called singular, if $\mathfrak{n}^+ v = 0$. The subspace of all singular vectors in V is denoted V^{sing}. For any $\lambda \in \mathfrak{h}^*$, we have an isomorphism

$$\operatorname{Hom}_{\mathfrak{g}}(M_\lambda, V) \cong V^{\text{sing}}[\lambda],$$

constructed by associating with any intertwining operator $\Phi \in \operatorname{Hom}_{\mathfrak{g}}(M_\lambda, V)$ the image of the vector $\mathbf{1}_\lambda$.

It is known [BGG] that a Verma module M_λ contains a unique up to proportionality singular vector of weight ν if and only if there exists a finite sequence of weights

$$\lambda^{(0)} = \lambda, \lambda^{(1)}, \ldots, \lambda^{(k)} = \nu,$$

such that

$$\lambda^{(i)} = s_{\beta_i} \cdot \lambda^{(i-1)} = \lambda^{(i-1)} - n_i\beta_i, \qquad i = 1, \ldots, k,$$

for some $\beta_i \in \Delta^+, n_i \in \mathbb{Z}_{\geq 0}$. We write $\nu \prec \lambda$ if this condition is satisfied.

Theorem 1. [FFM] *Let $\lambda \in \mathfrak{h}^*$, and let $w \in W$ be such that $w \cdot \lambda \prec \lambda$. Let $w = s_{i_1} \ldots s_{i_l}$ be the reduced decomposition of w in terms of simple root reflections. Then the formal monomial*

$$F_w(\lambda) = f_{i_1}^{\gamma_1(\lambda)} \ldots f_{i_l}^{\gamma_l(\lambda)}, \qquad \gamma_k(\lambda) = \langle \alpha_{i_k}, (s_{k+1} \ldots s_l) \cdot \lambda \rangle,$$

makes sense, and $F_w(\lambda)\mathbf{1}_\lambda$ is a singular vector of weight $w \cdot \lambda$ in the Verma module M_λ.

If λ is a dominant integral weight, then $w \cdot \lambda \prec \lambda$ for any $w \in W$, and $\gamma_k(\lambda) \in \mathbb{Z}_{\geq 0}$ for all $k = 1, \ldots, l$. Therefore, $F_w(\lambda)$ is a well-defined element in $\mathcal{U}(\mathfrak{n}^-)$. The meaning of $F_w(\lambda)$ for other $\lambda \in \mathfrak{h}^*$, when we may have $\gamma_k(\lambda) \notin \mathbb{Z}_{\geq 0}$, is explained in Appendix A.

Introduce notation $v_{w\cdot\lambda} = F_w(\lambda)\mathbf{1}_\lambda$ for the singular vector, constructed in Theorem 1. We identify the submodule of M_λ, generated by $v_{w\cdot\lambda}$, with $M_{w\cdot\lambda}$ by mapping $\mathbf{1}_{w\cdot\lambda}$ to $v_{w\cdot\lambda}$.

We now study singular vectors in tensor products $M_\lambda \otimes U$, $\lambda \in \mathfrak{h}^*$. All tensor products in this paper are considered over the field $\mathbb{C}$.

Let $\{g_i\}$ be a homogeneous basis in $\mathcal{U}(\mathfrak{n}^-)$, and let $\left(S_\lambda^{-1}\right)_{ij}$ be the inverse to the matrix of the form S_λ, restricted to $\mathcal{U}(\mathfrak{n}^-)$, with respect to this basis. The matrix elements $\left(S_\lambda^{-1}\right)_{ij}$ are rational functions of $\lambda \in \mathfrak{h}^*$, with possible simple poles at the hyperplanes $\chi_{\alpha,k}(\lambda) = 0$, for some $\alpha \in \Delta^+$ and $k \in \mathbb{Z}_{\geq 0}$.

Introduce the Cartan anti-automorphism ω of the Lie algebra $\mathfrak{g}$ by

$$\omega(e_i) = -f_i, \qquad \omega(f_i) = -e_i, \qquad \omega(h_i) = h_i, \qquad i = 1, \ldots, \dim \mathfrak{h}\,.$$

Consider a rational $\mathcal{U}(\mathfrak{n}^-) \otimes \operatorname{End}_{\mathbb{C}}(U)$-valued function of $\lambda \in \mathfrak{h}^*$,

$$\Xi(\lambda) = \sum_{i,j} \left(S_\lambda^{-1}\right)_{ij} g_i \otimes \omega(g_j)\,, \tag{3.1}$$

where we regard $\omega(g_j)$ as operators acting in U. Note that the summation over i, j is finite, because $\left(S_\lambda^{-1}\right)_{ij} = 0$ for sufficiently large i and $\omega(g_j)$ acts as zero for sufficiently large j. One can check that $\Xi(\lambda)$ does not depend on the choice of the basis $\{g_j\}$.

For any $\lambda \in \mathfrak{h}^*$ and $u \in U$ such that $\Xi(\lambda)(1 \otimes u) \in \mathcal{U}(\mathfrak{n}^-) \otimes U$ is well-defined, we denote

$$\operatorname{Sing}(\mathbf{1}_\lambda \otimes u) = \Xi(\lambda)(\mathbf{1}_\lambda \otimes u) \in M_\lambda \otimes U.$$

For generic λ, the vector $\operatorname{Sing}(\mathbf{1}_\lambda \otimes u)$ is the unique singular vector of the form

$$\operatorname{Sing}(\mathbf{1}_\lambda \otimes u) = \mathbf{1}_\lambda \otimes u + \text{ lower order terms }\,,$$

where the "lower order terms" is a linear combination of vectors $v' \otimes u'$ with $\operatorname{wt}(v') < \lambda$, see [ES]. The vector $\operatorname{Sing}(\mathbf{1}_\lambda \otimes u)$ is a singular vector for any λ, when it is well-defined.

For any $u \in U[\mu]$ such that $\operatorname{Sing}(\mathbf{1}_\lambda \otimes u)$ is well defined we introduce an intertwining operator $\Phi_\lambda^u \in \operatorname{Hom}_{\mathfrak{g}}(M_{\lambda+\mu}, M_\lambda \otimes U)$, uniquely determined by the condition

$$\Phi_\lambda^u \mathbf{1}_{\lambda+\mu} = \operatorname{Sing}(\mathbf{1}_\lambda \otimes u).$$

4. Weyl group and dynamical Weyl group

The Weyl group W, associated with $\mathfrak{g}$, is generated by simple reflections $\{s_i\}$, $i = 1, \ldots, \dim \mathfrak{h}$, subject to the relations $s_i^2 = id$, and the braid relations

$$\underbrace{s_i s_j s_i \ldots}_{n_{ij} \text{ factors}} = \underbrace{s_j s_i s_j \ldots}_{n_{ij} \text{ factors}}, \tag{4.1}$$

for $i \neq j$, where $n_{ij} = 2, 3, 4, 6$ if $\frac{4\langle \alpha_i, \alpha_j \rangle^2}{\langle \alpha_i, \alpha_i \rangle \langle \alpha_j, \alpha_j \rangle} = 0, 1, 2, 3$ respectively.

The length $l(w)$ of any $w \in W$ is defined to be the smallest positive integer l such that w can be written as $w = s_{i_1} \ldots s_{i_l}$ for some $i_1, \ldots, i_l$. For the identity element $id \in W$ we set $l(id) = 0$.

Let U be a finite-dimensional $\mathfrak{g}$-module. For any $i = 1, \ldots, \dim \mathfrak{h}$, introduce an operator $\bar{s}_i \in \operatorname{End}_{\mathbb{C}}(U)$ by setting

$$\bar{s}_i f_i^k v = f_i^{n-k} v, \qquad k = 0, 1, \ldots, n, \tag{4.2}$$

for any $v \in U$ such that $e_i v = 0$ and $h_i v = nv$ for some $n \in \mathbb{Z}_{\geq 0}$. One can think of operators $\bar{s}_i$ as operators $f_i^{h_i}$.

The following lemma is well-known; see [EV] and references therein.

Lemma 2. *There exists a linear representation $w \mapsto \bar{w}$ of the Weyl group W in U, such that the generators s_i are mapped to the operators $\bar{s}_i$.*

Obviously, for any $w \in W$ we have

$$\bar{w} = \bigoplus \bar{w}_\mu, \qquad \bar{w}_\mu \in \operatorname{Hom}_{\mathbb{C}}(U[\mu], U[w\mu]).$$

Note that if $w \in W$ and $\mu \in \mathfrak{h}^*$ are such that $w\mu = \mu$, then $\bar{w}_\mu = \mathrm{Id}_{U[\mu]}$. One can show that the equivalence class of the representation $w \mapsto \bar{w}$ of W is completely determined by these conditions.

Remark. The operators $\bar{w}$ do not preserve the Shapovalov form on U; one might try to modify the formula (4.2), and introduce operators $\tilde{s}_i$ by

$$\tilde{s}_i f_i^k v = \frac{k!}{(n-k)!} f_i^{n-k} v, \qquad k = 0, 1, \dots, n, \tag{4.3}$$

for any $v \in U$ such that $e_i v = 0$ and $h_i v = nv$ for some $n \in \mathbb{Z}_{\geq 0}$. The new operators $\tilde{s}_i$ clearly preserve the Shapovalov form.

Conjecture 3. *There exists a representation of the Weyl group W in U, such that the generators s_i are mapped to the operators $\tilde{s}_i$.*

This conjecture can be verified in certain cases, including adjoint representations for arbitrary $\mathfrak{g}$; however, no general proof is available.

We now study the dynamical Weyl group, introduced in [TV], [EV]. The following Theorem 4 and Theorem 5 are refinements of previous results.

Let $\mathbb{C}[\mathfrak{h}^*], \mathbb{C}(\mathfrak{h}^*)$ denote respectively the algebras of polynomial and rational functions on $\mathfrak{h}^*$. If V is a vector space and $F \in V \otimes \mathbb{C}[\mathfrak{h}^*]$, we denote $F(\lambda) \in V$ the value of the V-valued polynomial F at point $\lambda \in \mathfrak{h}^*$.

Theorem 4. *For any $w \in W$ there exists an operator $A_w \in \mathrm{End}_{\mathbb{C}}(U) \otimes \mathbb{C}(\mathfrak{h}^*)$, satisfying the following property.*

Suppose $\lambda, \mu \in \mathfrak{h}^*$ are such that $w \cdot (\lambda + \mu) \prec \lambda + \mu$, and $u \in U[\mu]$ is such that $\mathrm{Sing}(\mathbf{1}_\lambda \otimes u)$ *is well-defined. Then $A_w(\lambda)u$ is well-defined, and*

1. *If $w \cdot \lambda \prec \lambda$, then*

$$\Phi_\lambda^u v_{w\cdot(\lambda+\mu)} = v_{w\cdot\lambda} \otimes A_w(\lambda)u + \textit{ lower order terms }. \tag{4.4}$$

2. *If the condition $w \cdot \lambda \prec \lambda$ fails, then $A_w(\lambda)u = 0$.*

This theorem is proved in Appendix A.

The collection of operators $A_w \in \mathrm{End}_{\mathbb{C}}(U) \otimes \mathbb{C}(\mathfrak{h}^*), w \in W$, is called the dynamical Weyl group. Equation (4.4) implies that

$$A_w = \bigoplus_\mu A_w^\mu, \qquad A_w^\mu \in \mathrm{Hom}_{\mathbb{C}}(U[\mu], U[w\mu]) \otimes \mathbb{C}(\mathfrak{h}^*).$$

Example. Let U be a finite-dimensional $\mathfrak{sl}_2$-module, generated by a highest weight vector v of weight $n \in \mathbb{Z}_{\geq 0}$. Then the dynamical Weyl group consists of two operators: $A_{id} \equiv \mathrm{Id}$ and A_{s_1}, determined by

$$A_{s_1}(\lambda) f^k v = (-1)^n \frac{(-t-2)(-t-3)\dots(-t-n+k-1)}{t(t-1)\dots(t-k+1)} \frac{k!}{(n-k)!} f^{n-k} v, \qquad k = 0, 1, \dots, n, \tag{4.5}$$

where $t = (\lambda, \alpha_1)$.

For arbitrary $\mathfrak{g}$, the operators $A_{s_i}(\lambda)$ act in any irreducible $\mathfrak{sl}_2(\alpha_i)$ submodule of U by formula (4.5), with $t = \frac{2\langle\lambda,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle}$.

Theorem 5. *The dynamical Weyl group operators A_w, $w \in W$, acting in a finite-dimensional $\mathfrak{g}$-module U, have the following properties.*

1. *The dynamical Weyl group operators satisfy the cocycle condition,*
$$A_{w_1w_2}(\lambda)=A_{w_1}(w_2\cdot\lambda)A_{w_2}(\lambda),\qquad w_1,w_2\in W. \tag{4.6}$$
2. *The operator A_w, regarded as an $\mathrm{End}_{\mathbb{C}}(U)$-valued rational function on $\mathfrak{h}^*$, has at most simple poles, which may occur only at the hyperplanes*
$$\chi_{\alpha,k}(\lambda)=0,\qquad \alpha\in\Delta^+\cap w(\Delta^-),\ k\in\mathbb{Z}_{\geq 0}.$$
3. *If we identify $U[\mu]$ and $U[w\mu]$ in any way not depending on λ, then for generic $\lambda\in\mathfrak{h}^*$ we have*
$$\det A^\mu_w(\lambda)=const\prod_{\alpha\in\Delta^+\cap w(\Delta^-)}\frac{\prod_{k=0}^\infty\chi_{\alpha,k}(w\cdot\lambda)^{\dim U[w\mu+k\alpha]}}{\prod_{k=0}^\infty\chi_{\alpha,k}(\lambda)^{\dim U[\mu+k\alpha]}}.$$
where the constant depends on the choice of identification.

Remark. The dynamical Weyl group operators $A_w(\lambda)$ were introduced in [TV], [EV], using a different normalization of the singular vectors:
$$v_{w\cdot\lambda}=\frac{f_{i_1}^{n_1}}{n_1!}\dots\frac{f_{i_l}^{n_l}}{n_l!}\mathbf{1}_\lambda,\qquad n_k=\langle\alpha_{i_k},(s_{k+1}\dots s_l)\cdot\lambda\rangle.$$
Equation (4.4) was observed in [TV], [EV] only for $\lambda\gg\mu$. Part 2 of Theorem 4 is new.

In [TV], [EV], the cocycle condition was satisfied only for $w_1,w_2\in W$ such that $l(w_1w_2)=l(w_1)+l(w_2)$; equivalently, operators $A_w(\lambda)$ formed a cocycle on the braid group. Our operators satisfy the additional conditions $A_{s_\alpha}(s_\alpha\cdot\lambda)A_{s_\alpha}(\lambda)=\mathrm{Id}$, and thus form a cocycle on the Weyl group.

Proof of Theorem 5. It follows directly from the construction (see also [TV, EV]) that the cocycle condition (4.6) is satisfied when $w_1,w_2\in W$ are such that $l(w_1w_2)=l(w_1)+l(w_2)$. Therefore, it suffices to show that
$$A_{s_i}(s_i\cdot\lambda)A_{s_i}(\lambda)=\mathrm{Id},\qquad i=1,\dots,\dim\mathfrak{h}.$$
This follows from the explicit formula (4.5) for the action of $A_{s_i}(\lambda)$ in any irreducible $\mathfrak{sl}_2(\alpha_i)$-submodule of U.

Now, let $w\in W$ have a reduced decomposition $w=s_{i_1}\dots s_{i_l}$. Then the cocycle condition implies
$$A_w(\lambda)=A_{s_{i_1}}(s_{i_2}\dots s_{i_l}\cdot\lambda)\dots A_{s_{i_l}}(\lambda),$$
and singularities of $A_w(\lambda)$ are determined by the singularities of $A_{s_{i_k}}(s_{i_{k+1}}\dots s_{i_l}\cdot\lambda)$ for $k=1,\dots,\dim\mathfrak{h}$. From the explicit formula (4.5) it follows that $A_{s_{i_k}}(s_{i_{k+1}}\dots s_{i_l}\cdot\lambda)$ may only have simple poles at the hyperplanes $\chi_{\alpha_{i_k},n}(s_{i_{k+1}}\dots s_{i_l}\cdot\lambda)=0$, or equivalently
$$\chi_{\beta_k,n}(\lambda)=0,\qquad \beta_k=s_{i_l}\dots s_{i_{k+1}}\alpha_{i_k}. \tag{4.7}$$
It is known that if $w=s_{i_1}\dots s_{i_l}$ is a reduced expression, then $\{\beta_k\}$ form (without repetitions) the set $\Delta^+\cap w(\Delta^-)$. Therefore, the hyperplanes (4.7) are all distinct, and the second assertion follows.

Finally, from the cocycle property we obtain
$$\det A^\mu_w(\lambda)=\det A_{s_{i_1}}(s_{i_2}\dots s_{i_l}\cdot\lambda)\dots\det A_{s_{i_l}}(\lambda)$$

and the proof of the general formula reduces to verification of the formula

$$\det A^{\mu}_{s_i}(\lambda) = const \;\; \frac{\prod_{k=0}^{\infty} \chi_{\alpha_i,k}(s_i \cdot \lambda)^{\dim U[s_i\mu+k\alpha_i]}}{\prod_{k=0}^{\infty} \chi_{\alpha_i,k}(\lambda)^{\dim U[\mu+k\alpha_i]}}.$$

which follows from (4.5). □

The cocycle condition (4.6) immediately implies that the operators $A_w(-\rho), w \in W$, form a representation of the Weyl group. One can easily check that in fact $A_w(-\rho) = \bar{w}$, where operators $\bar{w}$ are defined by (4.2).

5. Regularizing operators and their properties

In this section we define the main object of our study — regularizing operators for finite-dimensional representations of a semisimple Lie algebra $\mathfrak{g}$, and establish some of their properties.

Let $\mathfrak{g}$ be a simple Lie algebra, and let U be a finite-dimensional $\mathfrak{g}$-module. An operator $N \in \mathrm{End}_{\mathbb{C}}(U) \otimes \mathbb{C}[\mathfrak{h}^*]$ is called a regularizing operator, if it satisfies the following conditions.

1. The operator N preserves weight subspaces, i.e. it can be decomposed as

$$N = \bigoplus_{\mu} N_{\mu}, \qquad N_{\mu} \in \mathrm{End}_{\mathbb{C}}(U[\mu]) \otimes \mathbb{C}[\mathfrak{h}^*].$$

2. For any $\mu \in \mathfrak{h}^*$, there exists a nonzero constant c_{μ} such that

$$\det N_{\mu}(\lambda) = c_{\mu} \prod_{\alpha \in \Delta^+} \prod_{k=0}^{\infty} \chi_{\alpha,k}(\lambda)^{\dim U[\mu+k\alpha]}. \tag{5.1}$$

3. There exist operators $a_w \in \mathrm{End}_{\mathbb{C}}(U)$, such that for any $w \in W$ and generic $\lambda \in \mathfrak{h}^*$ we have

$$a_w = N(w \cdot \lambda)^{-1} A_w(\lambda) N(\lambda). \tag{5.2}$$

4. The operator $\Xi_N(\lambda) : U \to \mathcal{U}(\mathfrak{n}^-) \otimes U$, defined by

$$\Xi_N(\lambda) u = \Xi(\lambda)(1 \otimes N(\lambda) u), \qquad u \in U,$$

depends polynomially on λ. Here $\Xi(\lambda)$ is given by (3.1).

Remark. The last condition is equivalent to the requirement that for any $u \in U$, the singular vector $\mathrm{Sing}(\mathbf{1}_{\lambda} \otimes N(\lambda)u) \in M_{\lambda} \otimes U$ is polynomial in λ.

The following Lemma is convenient for establishing the formula (5.1).

Lemma 6. *Suppose $N \in \mathrm{End}_{\mathbb{C}}(U) \otimes \mathbb{C}[\mathfrak{h}^*]$ satisfies conditions (1) and (3) above, and is such that N^{-1} is regular outside the hyperplanes $\chi_{\alpha,k}(\lambda) = 0$ for all $\alpha \in \Delta^+$ and $k \in \mathbb{Z}_{\geq 0}$. Then N also satisfies condition (2).*

Proof. The formula (5.2) implies that for any $w \in W$ and generic $\lambda \in \mathfrak{h}^*$, we have

$$\det A^{\mu}_{w}(\lambda) \det N_{\mu}(\lambda) = const \;\; \det N_{w\mu}(w \cdot \lambda). \tag{5.3}$$

The right-hand side of this equation polynomially depends on λ, and therefore in the left-hand side $\det N_{\mu}(\lambda)$ must be divisible by the denominator of $\det A^{\mu}_{w}(\lambda)$. In the special case when $w = w_0$ is the longest element of the Weyl group, this denominator is precisely $\prod_{\alpha\in\Delta^+}\prod_{k=0}^{\infty}\chi_{\alpha,k}(\lambda)^{\dim U[\mu+k\alpha]}$. Therefore, we conclude that

$$c_\mu(\lambda) = \frac{\det N_\mu(\lambda)}{\prod_{\alpha\in\Delta^+}\prod_{k=0}^{\infty}\chi_{\alpha,k}(\lambda)^{\dim U[\mu+k\alpha]}}$$

is polynomial in λ, and (5.3) implies that $c_\mu(\lambda) = const\ c_{w_0\mu}(w_0\cdot\lambda)$.

By the assumption of the Lemma, $c_\mu(\lambda)$ may only vanish at hyperplanes $\chi_{\alpha,k}(\lambda) = 0$, and similarly $c_{w_\mu}(w_0\cdot\lambda)$ may only vanish at hyperplanes $\chi_{\alpha,k}(w_0\cdot\lambda) = 0$. Since these two families of hyperplanes are disjoint, we conclude that $c_\mu(\lambda)$ is a polynomial in λ which never vanishes, i.e. a constant polynomial. The condition (2) follows. □

The argument above also shows that the condition (2) can be thought of as a minimality condition for N.

Here are some other immediate corollaries of the definition.

Proposition 7. *Let U be a finite-dimensional $\mathfrak{g}$-module, and let N be a regularizing operator.*

1. *The operators $a_w \in \operatorname{End}_{\mathbb{C}}(U)$ form a representation of the Weyl group.*
2. *The inverse operator N^{-1}, regarded as an $\operatorname{End}_{\mathbb{C}}(U)$-valued rational function on $\mathfrak{h}^*$, may have only simple poles. The poles may only occur at hyperplanes $\chi_{\alpha,k}(\lambda) = 0$ for $\alpha\in\Delta^+$ and $k\in\mathbb{Z}_{\geq 0}$ such that $U[\mu+k\alpha]\neq 0$.*
3. *Let $C\in\operatorname{End}_{\mathbb{C}}(U)$ be a weight-preserving operator, i.e.*
$$C = \bigoplus_\mu C_\mu, \qquad C_\mu \in \operatorname{End}(U[\mu]).$$
Then the operator $\bar N = NC$ is also a regularizing operator.

Proof. For any $w_1, w_2 \in W$, we compute

$$a_{w_1w_2} = N(w_1w_2\cdot\lambda)^{-1}A_{w_1w_2}(\lambda)N(\lambda) = N(w_1w_2\cdot\lambda)^{-1}A_{w_1}(w_2\cdot\lambda)A_{w_2}(\lambda)N(\lambda) =$$
$$\big(N(w_1w_2\cdot\lambda)^{-1}A_{w_1}(w_2\cdot\lambda)N(w_2\cdot\lambda)\big)\big(N(w_2\cdot\lambda)^{-1}A_{w_2}(\lambda)N(\lambda)\big) = a_{w_1}a_{w_2},$$

which shows that the correspondence $w\mapsto a_w$ gives a representation of the Weyl group.

From the formula for $\det N(\lambda)$, we see that N^{-1} may only have poles at hyperplanes $\chi_{\alpha,k}(\lambda) = 0$, for some $\alpha\in\Delta^+$ and $k\in\mathbb{Z}_{\geq 0}$ such that $U[\mu+k\alpha]\neq 0$. Fix $\alpha\in\Delta_+$ and $k\in\mathbb{Z}_{\geq 0}$. We have

$$N(\lambda)^{-1} = a_{s_\alpha}^{-1}N(s_\alpha\cdot\lambda)^{-1}A_{s_\alpha}(\lambda)\,.$$

since $N(s_\alpha\cdot\lambda)^{-1}$ does not have a pole at the hyperplane $\chi_{\alpha,k}(\lambda) = 0$, the singularity of $N(\lambda)^{-1}$ may come only from the operator $A_{s_\alpha}(\lambda)$, which have at most simple pole there.

The operators $\bar N(\lambda) = N(\lambda)C$ are obviously polynomial in λ, and the conditions on C_μ ensure weight preserving and determinant properties of $\bar N(\lambda)$. Finally,

$$\bar a_w = \bar N(w\cdot\lambda)^{-1}A_w(\lambda)\bar N(\lambda) = C^{-1}N(w\cdot\lambda)^{-1}A_w(\lambda)N(\lambda)C = C^{-1}a_wC$$

do not depend on λ, and for any $u\in U$ the vector

$$\Xi_{\bar N}(\lambda)u = \Xi(\lambda)(1\otimes\bar N(\lambda)u) = \Xi(1\otimes N(\lambda)Cu) = \Xi_N(\lambda)(Cu)$$

is polynomial in λ. □

Theorem 8. *Let U be a finite-dimensional $\mathfrak{g}$-module, and let $N\in\operatorname{End}_{\mathbb{C}}(U)\otimes\mathbb{C}[\mathfrak{h}^*]$ be a regularizing operator. Fix any $\lambda_0\in\mathfrak{h}^*$. Then*

1. *The linear map*

$$\Xi_N(\lambda_0) : U \to (M_{\lambda_0} \otimes U)^{\mathrm{sing}}, \qquad u \mapsto \mathrm{Sing}(\mathbf{1}_{\lambda_0} \otimes N(\lambda_0)u)$$

is injective.

2. *We have the following description of the image of $N_\mu(\lambda_0)$:*

$$\mathrm{Im}\, N(\lambda_0) = \{\, u \in U \ \mid \ \mathrm{Sing}(\mathbf{1}_{\lambda_0} \otimes u) \text{ is well-defined} \}\,.$$

Proof. Let Δ_0 be the root subsystem of Δ, defined by

$$\Delta_0 = \{\, \alpha \in \Delta \mid \langle \alpha, \lambda_0 \rangle \in \mathbb{Z} \}\,.$$

We set $\Delta_0^+ = \Delta_0 \cap \Delta^+$, and denote by W_0 the subgroup of W, generated by root reflections $s_\alpha, \alpha \in \Delta_0$.

Let $w \in W_0$ be such that $w \cdot \lambda_0$ is antidominant with respect to Δ_0^+. Then, $\det N(w \cdot \lambda_0) \neq 0$, and $N(w \cdot \lambda_0)$ is invertible.

According to Theorem 4,

$$\Phi_{\lambda_0}^{N(\lambda_0)u} v_{w\cdot(\lambda_0+\mu)} = v_{w\cdot\lambda_0} \otimes A_w(\lambda_0) N(\lambda_0) u + \text{ lower order terms }.$$

Suppose for some $u \in U[\mu]$ we have $\mathrm{Sing}(\mathbf{1}_{\lambda_0} \otimes N(\lambda_0))u = 0$. Then the intertwining operator $\Phi_{\lambda_0}^{N(\lambda_0)u}$ is identically zero, and in particular

$$N(w \cdot \lambda_0) a_w u = A_w(\lambda_0) N(\lambda_0) u = 0.$$

Since $N(w \cdot \lambda_0)$ and a_w are both invertible, we conclude that $u = 0$, and thus $\Xi_N(\lambda_0)$ is injective.

Now, let $u \in U[\mu]$ be such that $\mathrm{Sing}(\mathbf{1}_{\lambda_0} \otimes u)$ is well-defined. Then the vector $u' = A_w(\lambda_0)u$ is also well-defined, and we have

$$u = A_w(\lambda_0)^{-1} u' = N(\lambda_0) a_w^{-1} N(w \cdot \lambda_0)^{-1} u' \in \mathrm{Im}\, N(\lambda_0).$$

Conversely, property (4) of $N(\lambda_0)$ guarantees that $\mathrm{Sing}(\mathbf{1}_{\lambda_0} \otimes u)$ is well-defined for any $u \in \mathrm{Im}\, N_\mu(\lambda_0)$. □

Finally, we prove that the regularizing operator is determined uniquely up to right multiplication by a polynomially invertible matrix $C(\lambda)$, satisfying certain W-invariance conditions.

Theorem 9. *Let $N, \bar{N}$ be regularizing operators, and let $a_w, \bar{a}_w$ be the associated constant representations of the Weyl group. Then the operator*

$$C(\lambda) = N(\lambda)^{-1} \bar{N}(\lambda) \in \mathrm{End}_{\mathbb{C}}(U)$$

is polynomial in λ, has a decomposition

$$C(\lambda) = \bigoplus_{\mu} C_\mu(\lambda), \qquad C_\mu(\lambda) \in \mathrm{End}_{\mathbb{C}}(U[\mu]),$$

and satisfies

$$C(w \cdot \lambda) \bar{a}_w = a_w C(\lambda).$$

Moreover, the inverse operator $C(\lambda)^{-1}$ is also polynomial in λ, and the Weyl group representations $a_w, \bar{a}_w$ are equivalent.

Proof. Since regularizing operators preserve weight subspaces, we have

$$C_\mu(\lambda) = N_\mu(\lambda)^{-1}\bar{N}_\mu(\lambda) \in \mathrm{End}_{\mathbb{C}}(U[\mu]).$$

The only possible singularities of $C(\lambda) = N(\lambda)^{-1}\bar{N}(\lambda)$ may be simple poles at one of the hyperplanes $\chi_{\alpha,k}(\lambda) = 0$. Fix $\alpha \in \Delta^+, k \in \mathbb{Z}_{\geq 0}$, and write

$$N(\lambda)^{-1} = \frac{X(\lambda)}{\chi_{\alpha,k}(\lambda)} + Y(\lambda)$$

for some $X(\lambda), Y(\lambda) \in \mathrm{End}_{\mathbb{C}}(U)$, regular at the hyperplane $\chi_{\alpha,k}(\lambda) = 0$.

It follows from the identity $N(\lambda)^{-1}N(\lambda) = \mathrm{Id}$ that for generic λ_0 from this hyperplane, we have $X(\lambda_0)N(\lambda_0) = 0$. Hence,

$$\ker \mathrm{X}(\lambda_0) \supset \mathrm{Im}\, N(\lambda_0) = \mathrm{Im}\, \bar{N}(\lambda_0)\,,$$

which implies that $C(\lambda)$ is well-defined at λ_0. Therefore, $C(\lambda)$ does not have a pole at the hyperplane $\chi_{\alpha,k}(\lambda) = 0$, which was arbitrary, and thus $C(\lambda)$ is polynomial in λ.

In view of (5.1), we conclude that $\det C_\mu(\lambda) = (\det N_\mu(\lambda))^{-1} \det \bar{N}_\mu(\lambda)$ is a nonzero constant. Hence, each $C_\mu(\lambda)$ is polynomially invertible.

Finally, we compute

$$C(w\cdot\lambda)^{-1}a_w C(\lambda) = \bar{N}(w\cdot\lambda)^{-1}N(w\cdot\lambda)a_w N(\lambda)^{-1}\bar{N}(\lambda) = \bar{N}(w\cdot\lambda)^{-1}A_w(\lambda)\bar{N}(\lambda) = \bar{a}_w.$$

The invariance condition follows. Setting $\lambda = -\rho$, we get

$$\bar{a}_w = C(-\rho)^{-1}a_w C(-\rho)$$

and the representations $a_w, \bar{a}_w$ of the Weyl group are equivalent. □

Example: Let $\mathfrak{g} = \mathfrak{sl}_2$. We identify $\mathfrak{h}^*$ with $\mathbb{C}$ by associating $\mathfrak{h}^* \ni \lambda \leftrightarrow \langle\alpha_1, \lambda\rangle \in \mathbb{C}$. Under this identification, $\alpha_1 \equiv 2$ and $\rho \equiv 1$.

Let U be an irreducible finite-dimensional $\mathfrak{sl}_2$-module with highest weight $\Lambda \in \mathbb{Z}_{\geq 0}$. For $u \in U[\Lambda - 2k]$, $k \in \mathbb{Z}_{\geq 0}$, we have

$$\mathrm{Sing}(\mathbf{1}_\lambda \otimes u) = \sum_{j=0}^{k} \frac{(-1)^j}{j!\prod_{j=0}^{k}(\lambda - j)} f^j\mathbf{1}_\lambda \otimes e^j u\,.$$

The weight subspaces of U are one-dimensional, the operators $N_{\Lambda-2k}(\lambda) \in \mathrm{End}(U[\Lambda - 2k]) \cong \mathbb{C}$ being scalars, and we take

$$N_{\Lambda-2k}(\lambda) = \frac{1}{k!}\prod_{j=0}^{k-1}\chi_{\alpha,j}(\lambda) = \frac{1}{k!}\prod_{j=0}^{k-1}(\lambda - j)\,.$$

One can check that the operator a_s acts in U by

$$a_s : f^k\mathbf{1}_\Lambda \mapsto f^{\Lambda-k}\mathbf{1}_\Lambda, \qquad k = 0, 1, \ldots, \Lambda\,.$$

In particular, if Λ is an even integer and $u = f^{\Lambda/2}\mathbf{1}_\Lambda \in U[0]$, we have $a_s u = u$.

We now state our main result on the existence of regularizing operators.

Theorem 10. *Let $\mathfrak{g} = \mathfrak{sl}_3$, and let U be an irreducible finite-dimensional $\mathfrak{g}$-module. Then there exists a regularizing operator $N \in \mathrm{End}_{\mathbb{C}}(U)\otimes\mathbb{C}(\mathfrak{h}^*)$. Moreover, the associated operators a_{s_i}, corresponding to simple root reflections in W, coincide with the operators $\bar{s}_i$ given by the formulas* (4.2).

This theorem will be proved in Section 7.

Example. Let U be the adjoint representation of $\mathfrak{sl}_3$. The zero weight subspace $U[0]$ is the Cartan subalgebra $\mathfrak{h}$; the generators $h_1, h_2 \in \mathfrak{sl}_3$ form a basis in $U[0]$.

The action of operators $A_w^{[0]}(\lambda)$ on $U[0]$ is completely determined by the matrices

$$A_{s_1}^{[0]}(\lambda) \begin{pmatrix} \frac{-\lambda_1-2}{\lambda_1} & \frac{-\lambda_1-1}{\lambda_1} \\ 0 & 1 \end{pmatrix}, \qquad A_{s_2}^{[0]}(\lambda) \begin{pmatrix} 1 & 0 \\ \frac{-\lambda_2-1}{\lambda_2} & \frac{-\lambda_2-2}{\lambda_2} \end{pmatrix},$$

where $\lambda_1 = \langle \alpha_1, \lambda \rangle, \lambda_2 = \langle \alpha_2, \lambda \rangle$. The operator $N_{[0]}(\lambda) : U[0] \to U[0]$ is given by the matrix

$$N_{[0]}(\lambda) = \frac{1}{3} \begin{pmatrix} 2\lambda_1^2 + 2\lambda_1\lambda_2 - \lambda_2^2 + 4\lambda_1 - \lambda_2 & 2\lambda_1^2 + 2\lambda_1\lambda_2 - \lambda_2^2 + 2\lambda_1 - 2\lambda_2 \\ 2\lambda_2^2 + 2\lambda_1\lambda_2 - \lambda_1^2 - 2\lambda_1 + 2\lambda_2 & 2\lambda_2^2 + 2\lambda_1\lambda_2 - \lambda_1^2 - \lambda_1 + 4\lambda_2 \end{pmatrix}.$$

It is a straightforward computation to show that

$$N_{[0]}(s_1 \cdot \lambda)^{-1} A_{s_1}^{[0]}(\lambda) N_{[0]}(\lambda) \equiv \mathrm{Id}_{U[0]}, \qquad N_{[0]}(s_2 \cdot \lambda)^{-1} A_{s_2}^{[0]}(\lambda) N_{[0]}(\lambda) \equiv \mathrm{Id}_{U[0]},$$

and that

$$\det N_{[0]}(\lambda) = \lambda_1\lambda_2(\lambda_1 + \lambda_2 + 1) = \chi_{\alpha_1,1}(\lambda)\chi_{\alpha_2,1}(\lambda)\chi_{\alpha_1+\alpha_2,1}(\lambda).$$

6. The $(\mathfrak{gl}_m, \mathfrak{gl}_n)$ duality, and functional realizations of $\mathfrak{gl}_n$-modules

In Section 7 we will construct the regularizing operators for $\mathfrak{g} = \mathfrak{sl}_3$ using a concrete realization of irreducible finite-dimensional $\mathfrak{sl}_3$-modules. This section is devoted to a review of some underlying results on representations of the reductive Lie algebras $\mathfrak{gl}_n$.

The space $\mathbb{C}^{mn} = (\mathbb{C}^m)^{\otimes n} = (\mathbb{C}^n)^{\otimes m}$ has natural commuting actions of $\mathfrak{gl}_m$ and $\mathfrak{gl}_n$. The algebra $\mathbb{P}_{m,n} = S(\mathbb{C}^{mn})$ of polynomial functions on $\mathbb{C}^{mn}$ becomes a module over $\mathfrak{gl}_m \otimes \mathfrak{gl}_n$. It has decompositions

$$\mathbb{P}_{m,n} = \bigoplus_{\mu_1,\dots,\mu_m} \mathbb{P}_{m,n}^{\mathfrak{gl}_m}[(\mu_1, \dots, \mu_m)] = \bigoplus_{\nu_1,\dots,\nu_n} \mathbb{P}_{m,n}^{\mathfrak{gl}_n}[(\nu_1, \dots, \nu_n)]$$

where the subspaces

$$\mathbb{P}_{m,n}^{\mathfrak{gl}_m}[(\mu_1, \dots, \mu_m)] \cong S^{\mu_1}(\mathbb{C}^n) \otimes \dots \otimes S^{\mu_m}(\mathbb{C}^n),$$
$$\mathbb{P}_{m,n}^{\mathfrak{gl}_n}[(\nu_1, \dots, \nu_n)] \cong S^{\nu_1}(\mathbb{C}^m) \otimes \dots \otimes S^{\nu_n}(\mathbb{C}^m),$$

are the weight subspaces with respect to the algebras $\mathfrak{gl}_m$ and $\mathfrak{gl}_n$ respectively.

In the explicit realization of $\mathbb{P}_{m,n}$ as the polynomial algebra in mn variables $\mathbb{C}[\{x_{ij}\}], 1 \le i \le m, 1 \le j \le n$, the actions of $\mathfrak{gl}_m, \mathfrak{gl}_n$ are given by

$$E_{ab}^{\mathfrak{gl}_m} = \sum_{i=1}^{n} x_{ai} \frac{\partial}{\partial x_{bi}}, \qquad a, b = 1, \dots, m,$$

$$E_{ij}^{\mathfrak{gl}_n} = \sum_{a=1}^{m} x_{ai} \frac{\partial}{\partial x_{aj}}, \qquad i, j = 1, \dots, n,$$

where $E_{rs}^{\mathfrak{gl}_k}$ are the standard generators of the Lie algebra $\mathfrak{gl}_k$, corresponding to the elementary matrices with 1 at the intersection of the r-th row and s-th column, and 0 elsewhere.

We have the following theorem, see [Zh, H].

Theorem 11. *The $\mathfrak{gl}_m \otimes \mathfrak{gl}_n$ module $\mathbb{P}_{m,n}$ has the decomposition*

$$\mathbb{P}_{m,n} = \bigoplus_{\Lambda \in \mathcal{P}_{\min(m,n)}} V_\Lambda^{\mathfrak{gl}_m} \otimes V_\Lambda^{\mathfrak{gl}_n}$$

where $\mathcal{P}_j$ denotes the collection of finite sequences Λ of nonnegative integers $\Lambda_1 \geq \cdots \geq \Lambda_j$, and $V_\Lambda^{\mathfrak{gl}_k}$ for $k \geq j$ denotes the irreducible $\mathfrak{gl}_k$-module with highest weight $(\Lambda_1, \ldots, \Lambda_j, 0, \ldots, 0)$.

In particular, this theorem implies that a $\mathfrak{gl}_n$-module $V_\Lambda^{\mathfrak{gl}_n}$ may be realized as the subspace of $\mathfrak{gl}_m$-singular vectors of weight Λ in $\mathbb{P}_{m,n}$.

We will use the special case of the $(\mathfrak{gl}_2, \mathfrak{gl}_3)$ duality, which gives the following decomposition of the polynomial algebra $\mathbb{P}_{2,3}$:

$$\mathbb{P}_{2,3} = \bigoplus_{m_1 \geq m_2 \geq 0} V_{(m_1,m_2)}^{\mathfrak{gl}_2} \otimes V_{(m_1,m_2,0)}^{\mathfrak{gl}_3}.$$

We identify a module $V_{(m_1,m_2,0)}^{\mathfrak{gl}_3}$ with the the subspace of $\mathfrak{gl}_2$-singular vectors in the algebra $\mathbb{P}_{2,3}$ of weight (m_1, m_2) with respect to $\mathfrak{gl}_2$:

$$V_{(m_1,m_2,0)}^{\mathfrak{gl}_3} \cong \bigoplus_{l_1+l_2+l_3=m_1+m_2} \left(V_{(l_1,0)}^{\mathfrak{gl}_2} \otimes V_{(l_2,0)}^{\mathfrak{gl}_2} \otimes V_{(l_3,0)}^{\mathfrak{gl}_2} \right)^{\mathrm{sing}} [(m_1, m_2)].$$

Such identification gives the correspondence of the weight subspaces,

$$V_{(m_1,m_2,0)}^{\mathfrak{gl}_3}[(l_1, l_2, l_3)] \cong \left(V_{(l_1,0)}^{\mathfrak{gl}_2} \otimes V_{(l_2,0)}^{\mathfrak{gl}_2} \otimes V_{(l_3,0)}^{\mathfrak{gl}_2} \right)^{\mathrm{sing}} [(m_1, m_2)].$$

Our next goal is to describe a realization of $V_{(l_1,0)}^{\mathfrak{gl}_2} \otimes V_{(l_2,0)}^{\mathfrak{gl}_2} \otimes V_{(l_3,0)}^{\mathfrak{gl}_2}$ as a certain functional space. In the remaining part of this section $\mathbf{l} = (l_1, l_2, l_3)$ will denote a triple of non-negative integers.

To shorten formulas, we will omit superscripts for $\mathfrak{gl}_2$-modules and operators, and write $V_{(l,0)}, E_{ab}$ instead of $V_{(l,0)}^{\mathfrak{gl}_2}, E_{ab}^{\mathfrak{gl}_2}$, respectively. By a slight abuse of notation, we also write $\mathbf{1}_l$ for the highest weight vector of the $\mathfrak{gl}_2$-module $V_{(l,0)}$.

Let $\mathbb{C}[\mathbf{z}], \mathbb{C}(\mathbf{z})$ denote respectively the algebras of polynomial and rational functions in complex variables $\mathbf{z} = (z_1, z_2, z_3)$.

Let $\mathcal{H}_k$ denote the space of symmetric polynomials in variables $t_1, \ldots, t_k$ of degree at most 2 in each $t_1, \ldots, t_k$, with coefficients in $\mathbb{C}[\mathbf{z}]$. We also set

$$\mathcal{H} = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \mathcal{H}_k \,.$$

For any function φ of $t_1, \ldots, t_k$, we denote

$$Sym(\varphi(t_1, \ldots, t_k)) = \sum_{\sigma \in S_k} \varphi(t_{\sigma(1)}, \ldots, t_{\sigma(k)})$$

Proposition 12. [TV3] *The space $\mathcal{H}$ has a structure of a $\mathfrak{gl}_2$-module, depending on $\mathbf{l}$, such that the action of generators $\{E_{ab}\}$ on a function $\varphi \in \mathcal{H}_k$ is given by*

$$(E_{11}\varphi)(t_1,\dots,t_k) = (l_1+l_2+l_3-k)\varphi(t_1,\dots,t_k),$$
$$(E_{22}\varphi)(t_1,\dots,t_k) = k\varphi(t_1,\dots,t_k),$$
$$(E_{12}\varphi)(t_1,\dots,t_{k-1}) = \lim_{t_k\to\infty}\frac{\varphi(t_1,\dots,t_k)}{t_k^2},$$
$$(E_{21}\varphi)(t_1,\dots,t_{k+1}) = \frac{1}{k!}\,Sym\left(\varphi(t_1,\dots,t_k)\left(\prod_{a=1}^{3}(t_{k+1}-z_a+l_a)\prod_{i=1}^{k}\frac{t_{k+1}-t_i-1}{t_{k+1}-t_i} - \prod_{a=1}^{3}(t_{k+1}-z_a)\prod_{i=1}^{k}\frac{t_{k+1}-t_i+1}{t_{k+1}-t_i}\right)\right).$$

One can easily check that multiplication by any function of $\mathbf{z}$ commutes with the $\mathfrak{gl}_2$ action above; thus, $\mathcal{H}$ can be regarded as a $\mathfrak{gl}_2\otimes\mathbb{C}[\mathbf{z}]$ - module.

Proposition 13. [TV3] *The linear map*

$$\phi[\mathbf{l}] : V_{(l_1,0)}\otimes V_{(l_2,0)}\otimes V_{(l_3,0)} \to \mathcal{H}\,,$$

$$\phi[\mathbf{l}]\left(E_{21}^{k_1}\mathbf{1}_{l_1}\otimes E_{21}^{k_2}\mathbf{1}_{l_2}\otimes E_{21}^{k_3}\mathbf{1}_{l_3}\right)\frac{l_1!l_2!l_3!}{(l_1-k_1)!(l_2-k_2)!(l_3-k_3)!} =$$
$$Sym\left(\prod_{a=1}^{3}\left(\prod_{i=1}^{k_1+\dots+k_{a-1}}(t_i-z_a)\prod_{j=k_1+\dots+k_a+1}^{k_1+k_2+k_3}(t_j-z_a+l_a)\right)\prod_{i<j}\frac{t_i-t_j+1}{t_i-t_j}\right), \tag{6.1}$$

is a homomorphism of $\mathfrak{gl}_2$-modules.

Let $\mathcal{H}_k[\mathbf{l}]$ denote the subspace of $\mathcal{H}_k$, consisting of functions $\varphi\in\mathcal{H}_k$, satisfying additional "admissibility" conditions

$$\varphi(z_i, z_i-1,\dots,z_i-l_i, t_{l_i+2},\dots,t_k)\equiv 0, \tag{6.2}$$

imposed if $k>l_i$, $\quad i=1,2,3$. We also denote

$$\mathcal{H}[\mathbf{l}] = \bigoplus_{k\in\mathbb{Z}_{\ge 0}}\mathcal{H}_k[\mathbf{l}]\,.$$

One can check that $\mathcal{H}[\mathbf{l}]$ is a $\mathfrak{gl}_2\otimes\mathbb{C}[\mathbf{z}]$-submodule of $\mathcal{H}$.

Proposition 14. 1. *The map $\phi[\mathbf{l}]$ is injective, and*

$$\mathrm{Im}\,\phi[\mathbf{l}]\otimes_{\mathbb{C}[\mathbf{z}]}\mathbb{C}(\mathbf{z}) = \mathcal{H}[\mathbf{l}]\otimes_{\mathbb{C}[\mathbf{z}]}\mathbb{C}(\mathbf{z}). \tag{6.3}$$

2. *There exists a homomorphism of $\mathfrak{gl}_2\otimes\mathbb{C}[\mathbf{z}]$-modules*

$$\mathcal{I}[\mathbf{l}] : \mathcal{H}[\mathbf{l}] \to \left(V_{(l_1,0)}\otimes V_{(l_2,0)}\otimes V_{(l_3,0)}\right)\otimes\mathbb{C}(\mathbf{z}),$$

such that

$$(\mathcal{I}[\mathbf{l}]\circ\phi[\mathbf{l}])\,v = v, \qquad \text{for any } v\in V_{(l_1,0)}\otimes V_{(l_2,0)}\otimes V_{(l_3,0)}. \tag{6.4}$$

Moreover, for any $\varphi\in\mathcal{H}[\mathbf{l}]$, the vector $\mathcal{I}[\mathbf{l}]\varphi$, rationally depending on $\mathbf{z}$, may have at most simple poles, located at the hyperplanes

$$z_j - r = z_i - l_i, \qquad r = 0,1,\dots,\min(l_i,l_j)-1, \quad 1\le i\le j\le 3.$$

The Proposition is proved in Appendix B.

Introduce $\mathcal{X}[\mathbf{l}] \in \mathbb{C}[\mathbf{z}]$ by

$$\mathcal{X}[\mathbf{l};\mathbf{z}] = \prod_{r=1}^{l_2}(z_1 - l_1 - z_2 + r)\prod_{r=1}^{l_3}(z_1 - l_1 - z_3 + r)\prod_{r=1}^{l_3}(z_2 - l_2 - z_3 + r),$$

and set

$$\mathcal{N}[\mathbf{l}] = \mathcal{X}[\mathbf{l}]\ \mathcal{I}[\mathbf{l}]. \tag{6.5}$$

Corollary 15. *The formula* (6.5) *defines a homomorphism of* $\mathfrak{gl}_2 \otimes \mathbb{C}[\mathbf{z}]$*-modules*

$$\mathcal{N}[\mathbf{l}] : \mathcal{H}[\mathbf{l}] \to \big(V_{(l_1,0)} \otimes V_{(l_2,0)} \otimes V_{(l_3,0)}\big) \otimes \mathbb{C}[\mathbf{z}].$$

Proof. For any $\varphi \in \mathcal{H}[\mathbf{l}]$, the possible simple poles of $\mathcal{I}[\mathbf{l}]\varphi$ are offset by $\mathcal{X}[\mathbf{l}]$. Therefore, $\mathcal{N}[\mathbf{l}]\varphi$ is polynomial in $\mathbf{z}$. The $\mathfrak{gl}_2$-intertwining property for $\mathcal{N}[\mathbf{l}]$ follows from the fact that $\mathcal{I}[\mathbf{l}]$ is a $\mathfrak{gl}_2$-homomorphism, and the commutativity of multiplication by any element from $\mathbb{C}[\mathbf{z}]$ with the action of $\mathfrak{gl}_2$. □

Now we let formal variables $\mathbf{z}$ take particular complex values; specifying $\mathbf{z} = \mathbf{z}_0$ for some $\mathbf{z}_0 \in \mathbb{C}^3$ will be reflected by adding $\mathbf{z}_0$ to the notation. For example, $\phi[\mathbf{l};\mathbf{z}_0]$ will denote the composition of $\phi[\mathbf{l}]$ with the homomorphism $\mathbb{C}[\mathbf{z}] \to \mathbb{C}$ of evaluation at $\mathbf{z} = \mathbf{z}_0$, and $\mathcal{H}_k[\mathbf{l};\mathbf{z}_0]$ will consist of symmetric polynomials in $t_1, \dots, t_k$ with complex coefficients, obtained from elements of $\mathcal{H}_k[\mathbf{l}]$ by specializing $\mathbf{z} = \mathbf{z}_0$.

We now establish certain properties of operators $\phi[\mathbf{l};\mathbf{z}_0], \mathcal{N}[\mathbf{l};\mathbf{z}_0]$ for special values of $\mathbf{z}_0$. Recall the following standard decompositions of the tensor product of $\mathfrak{gl}_2$-modules:

$$V_{(l_1,0)} \otimes V_{(l_2,0)} \otimes V_{(l_3,0)} \cong \left(\bigoplus_{i=0}^{\min(l_1,l_2)} V_{(l_1+l_2-i,i)}\right) \otimes V_{(l_3,0)} \cong V_{(l_1,0)} \otimes \left(\bigoplus_{i=0}^{\min(l_2,l_3)} V_{(l_2+l_3-i,i)}\right).$$

Proposition 16. *Let* $\mathbf{z}_0 \in \mathbb{C}^3$.

1. *If* $\mathbf{z}_0$ *is generic from the hyperplane* $z_1 - z_2 - l_1 + s = 0, \quad 0 \le s < \min(l_1, l_2)$*, then*

$$\ker \phi[\mathbf{l};\mathbf{z}_0] = \left(\bigoplus_{i=s+1}^{\min(l_1,l_2)} V_{(l_1+l_2-i,i)}\right) \otimes V_{(l_3,0)}\,.$$

2. *If* $\mathbf{z}_0$ *is generic from the hyperplane* $z_2 - z_3 - l_2 + s = 0, \quad 0 \le s < \min(l_2, l_3)$*, then*

$$\ker \phi[\mathbf{l};\mathbf{z}_0] = V_{(l_1,0)} \otimes \left(\bigoplus_{i=s+1}^{\min(l_2,l_3)} V_{(l_2+l_3-i,i)}\right),$$

This Proposition will be proved in Appendix B. As a consequence, we obtain the following properties of operators $\mathcal{N}[\mathbf{l};\mathbf{z}_0]$.

Corollary 17. *Let* $\mathbf{z}_0 \in \mathbb{C}^3$.

1. *If* $\mathbf{z}_0$ *is generic from the hyperplane* $z_1 - l_1 - z_2 + s = 0, \quad 0 \le s < \min(l_1, l_2)$*, then*

$$\operatorname{im} \mathcal{N}[\mathbf{l};\mathbf{z}_0] \subset \left(\bigoplus_{i=s+1}^{\min(l_1,l_2)} V_{(l_1+l_2-i,i)}\right) \otimes V_{(l_3,0)}$$

2. *If* $\mathbf{z}_0$ *is generic from the hyperplane* $z_2 - l_2 - z_3 + s = 0, \quad 0 \le s < \min(l_2, l_3)$, *then*

$$\operatorname{im} \mathcal{N}[\mathbf{l}; \mathbf{z}_0] \subset V_{(l_1,0)} \otimes \left(\bigoplus_{i=s+1}^{\min(l_2,l_3)} V_{(l_2+l_3-i,i)} \right)$$

Proof. Let $\varphi \in \mathcal{H}[\mathbf{l}; \mathbf{z}_0]$. Then it follows from the definitions and (6.3) that

$$\phi[\mathbf{l}; \mathbf{z}_0] \circ \mathcal{N}[\mathbf{l}; \mathbf{z}_0]\varphi = \mathcal{X}[\mathbf{l}; \mathbf{z}_0]\varphi \,. \tag{6.6}$$

For any point $\mathbf{z}_0$ from the hyperplane $z_1 - l_1 - z_2 + s = 0$ with $0 \le s < \min(l_1, l_2)$, we have that $\mathcal{X}[\mathbf{l}; \mathbf{z}_0] = 0$, and therefore $\mathcal{N}[\mathbf{l}; \mathbf{z}_0]\varphi \subset \ker \phi[\mathbf{l}; \mathbf{z}_0]$. Hence, the first assertion follows from Proposition 16.

The proof of the second assertion is similar. □

Note that admissibility conditions (6.2) are invariant under simultaneous permutations of triples $\mathbf{l}$ and $\mathbf{z}$, i.e. we have

$$\mathcal{H}[\mathbf{l}; \mathbf{z}] = \mathcal{H}[\sigma(\mathbf{l}); \sigma(\mathbf{z})], \qquad \sigma \in S_3.$$

Proposition 18. *There exists a family of linear operators* $\check{\mathcal{R}}_{l,l'}(u) : V_{(l,0)} \otimes V_{(l',0)} \to V_{(l',0)} \otimes V_{(l,0)}$, *rationally depending on a complex parameter* u, *such that*

$$\check{\mathcal{R}}_{l,l'}(u)\mathbf{1}_l \otimes \mathbf{1}_{l'} = \frac{\prod_{j=1}^{l'}(u - l + j)}{\prod_{j=1}^{l}(-u - l' + j)} \mathbf{1}_{l'} \otimes \mathbf{1}_l, \tag{6.7}$$

and the following diagram is commutative for generic $\mathbf{z}$*:*

$$\begin{array}{ccc}
\mathcal{H}[s_1(\mathbf{l}); s_1(\mathbf{z})] & \xrightarrow{\mathcal{N}[s_1(\mathbf{l}); s_1(\mathbf{z})]} & V_{(l_2,0)} \otimes V_{(l_1,0)} \otimes V_{(l_3,0)} \\
\Big\| & & \Big\uparrow \scriptstyle{\check{\mathcal{R}}_{l_1 l_2}(z_1 - z_2) \otimes 1} \\
\mathcal{H}[\mathbf{l}; \mathbf{z}] & \xrightarrow{\mathcal{N}[\mathbf{l}; \mathbf{z}]} & V_{(l_1,0)} \otimes V_{(l_2,0)} \otimes V_{(l_3,0)} . \\
\Big\| & & \Big\downarrow \scriptstyle{1 \otimes \check{\mathcal{R}}_{l_2 l_3}(z_2 - z_3)} \\
\mathcal{H}[s_2(\mathbf{l}); s_2(\mathbf{z})] & \xrightarrow{\mathcal{N}[s_2(\mathbf{l}); s_2(\mathbf{z})]} & V_{(l_1,0)} \otimes V_{(l_3,0)} \otimes V_{(l_2,0)}
\end{array} \tag{6.8}$$

Proposition 18 is proved in Appendix B, where it is shown that the operators $\check{\mathcal{R}}_{l,l'}(u)$ are the standard rational R-matrices, renormalized by (6.7).

7. The main construction

Let U be an irreducible finite-dimensional $\mathfrak{sl}_3$-module with highest weight Λ. Set

$$m = \langle \alpha_1 + 2\alpha_2, \Lambda \rangle, \qquad k = \langle \alpha_2, \Lambda \rangle.$$

The $\mathfrak{gl}_3$-module $V^{\mathfrak{gl}_3}_{(m-k,k,0)}$, regarded by restriction as an $\mathfrak{sl}_3$-module, is isomorphic to U. We fix an identification (which is unique up to a scalar)

$$\Theta : U \to \bigoplus_{\mathbf{l} \in \mathbb{Z}^3_{\ge 0}} \left(V_{(l_1,0)} \otimes V_{(l_2,0)} \otimes V_{(l_3,0)} \right)^{\rm sing} [(m-k, k)],$$

so that Θ maps a weight subspace $U[\mu]$ to $\big(V_{(l_1,0)}\otimes V_{(l_2,0)}\otimes V_{(l_3,0)}\big)^{\rm sing}[(m-k,k)]$, where the corresponding $\mathbf{l}=(l_1,l_2,l_3)\in\mathbb{Z}^3_{\geq 0}$ is determined by

$$l_1-l_2=\langle\alpha_1,\mu\rangle,\qquad l_2-l_3=\langle\alpha_2,\mu\rangle,\qquad l_1+l_2+l_3=m. \tag{7.1}$$

Under this identification the Weyl group for $\mathfrak{sl}_3$ coincides with the symmetric group S_3 naturally acting on $\mathbb{Z}^3_{\geq 0}$.

For any $\mathbf{l}\in\mathbb{Z}^3_{\geq 0}$ we fix a basis $\{v_i[\mathbf{l}]\}$ of the space $\big(V_{(l_1,0)}\otimes V_{(l_2,0)}\otimes V_{(l_3,0)}\big)^{\rm sing}[(m-k,k)]$. We assume that $v_i[w(\mathbf{l})]=\bar w v_i[\mathbf{l}]$, where the operators $\bar w$ correspond to the action of the Weyl group given by Lemma 2. Note that if $w(\mathbf{l})=\mathbf{l}$, then the restriction of $\bar w$ on $\big(V_{(l_1,0)}\otimes V_{(l_2,0)}\otimes V_{(l_3,0)}\big)^{\rm sing}[(m-k,k)]$ is the identity operator.

We have the following Lemma, proved in Appendix C.

Lemma 19. *For any* $\mathbf{l}=(l_1,l_2,l_3)\in\mathbb{Z}^3_{\geq 0}$, *and* $k\in\mathbb{Z}_{\geq 0}$, *we have*

$$\dim\mathcal{H}_k[\mathbf{l};\mathbf{z}]^{\rm sing}=\dim\big(V_{(l_1,0)}\otimes V_{(l_2,0)}\otimes V_{(l_3,0)}\big)^{\rm sing}[(m-k,k)]\,,$$

and there exists a $\mathbb{C}[\mathbf{z}]$*-basis* $\{\varsigma_i[\mathbf{l}]\}$ *of* $\mathcal{H}_k[\mathbf{l}]^{\rm sing}$, *such that*

$$\varsigma_i[\mathbf{l};\mathbf{z}]=\varsigma_i[\sigma(\mathbf{l});\sigma(\mathbf{z})],\qquad \sigma\in S_3. \tag{7.2}$$

Define the operators $\Upsilon[\mathbf{l}]:\big(V_{(l_1,0)}\otimes V_{(l_2,0)}\otimes V_{(l_3,0)}\big)^{\rm sing}[(m-k,k)]\to\mathcal{H}[\mathbf{l}]^{\rm sing}$ by

$$\Upsilon[\mathbf{l}]v_i[\mathbf{l}]=\varsigma_i[\mathbf{l}].$$

By construction, the operators $\Upsilon[\mathbf{l}]$ satisfy

$$\begin{tikzcd}
\big(V_{(l_2,0)}\otimes V_{(l_1,0)}\otimes V_{(l_3,0)}\big)^{\rm sing}[(m-k,k)] \arrow[r, "{\Upsilon[s_1(\mathbf{l});s_1(\mathbf{z})]}"] & \mathcal{H}_k[s_1(\mathbf{l});s_1(\mathbf{z})]^{\rm sing} \\
\big(V_{(l_1,0)}\otimes V_{(l_2,0)}\otimes V_{(l_3,0)}\big)^{\rm sing}[(m-k,k)] \arrow[u, "\bar s_1"] \arrow[d, "\bar s_2"'] \arrow[r, "{\Upsilon[\mathbf{l};\mathbf{z}]}"] & \mathcal{H}_k[\mathbf{l};\mathbf{z}]^{\rm sing} \arrow[u, equal] \arrow[d, equal] \\
\big(V_{(l_1,0)}\otimes V_{(l_3,0)}\otimes V_{(l_2,0)}\big)^{\rm sing}[(m-k,k)] \arrow[r, "{\Upsilon[s_2(\mathbf{l});s_2(\mathbf{z})]}"] & \mathcal{H}_k[s_2(\mathbf{l});s_2(\mathbf{z})]^{\rm sing}
\end{tikzcd}\,. \tag{7.3}$$

We now introduce our main object — the operators $N_\mu(\lambda)\in{\rm End}_{\mathbb{C}}(U[\mu])$ — by setting

$$N_\mu(\lambda)=\ \Theta^{-1}\circ\mathcal{N}[\mathbf{l};\mathbf{z}]\circ\Upsilon[\mathbf{l};\mathbf{z}]\circ\Theta\,. \tag{7.4}$$

Here μ and l_1,l_2,l_3 are as in (7.1), and z_1,z_2,z_3 satisfy

$$(z_1-l_1)-(z_2-l_2)=\langle\alpha_1,\lambda+\rho\rangle,\qquad (z_2-l_2)-(z_3-l_3)=\langle\alpha_2,\lambda+\rho\rangle\,. \tag{7.5}$$

Theorem 20. *The operator* $N(\lambda)=\bigoplus_\mu N_\mu(\lambda)$ *is a regularizing operator for the* $\mathfrak{g}$*-module* U. *The corresponding Weyl group representation* $w\mapsto a_w$ *is such that* $a_{s_i}=\bar s_i$.

Proof. Corollary 15 implies that the operator $N(\lambda)$ depends on $\lambda\in\mathfrak{h}^*$ polynomially.

To check (5.2), we use the result of [TV2], where it was shown that the action of the dynamic Weyl group operators A_w in $\mathbb{P}_{2,3}$ coincides with the action of operators $\check{\mathcal{R}}$. More precisely, we have the following commutative diagram:

$$\begin{array}{ccc} U[s_1\mu] & \longrightarrow & \left(V_{(l_2,0)}\otimes V_{(l_1,0)}\otimes V_{(l_3,0)}\right)^{\mathrm{sing}}[(m-k,k)] \\ {\scriptstyle A^{\mu}_{s_1}(\lambda)}\Big\uparrow & & \Big\uparrow{\scriptstyle \check{\mathcal{R}}_{l_1l_2}(z_1-z_2)\otimes 1} \\ U[\mu] & \longrightarrow & \left(V_{(l_1,0)}\otimes V_{(l_2,0)}\otimes V_{(l_3,0)}\right)^{\mathrm{sing}}[(m-k,k)] \\ {\scriptstyle A^{\mu}_{s_2}(\lambda)}\Big\downarrow & & \Big\downarrow{\scriptstyle 1\otimes\check{\mathcal{R}}_{l_2l_3}(z_2-z_3)} \\ U[s_2\mu] & \longrightarrow & \left(V_{(l_1,0)}\otimes V_{(l_3,0)}\otimes V_{(l_2,0)}\right)^{\mathrm{sing}}[(m-k,k)] \end{array} \qquad (7.6)$$

Combining (7.6) with (6.8) and (7.3), we get the commutative diagram

$$\begin{array}{ccc} U[s_1\mu] & \xrightarrow{N_{s_1\mu}(s_1\cdot\lambda)} & U[s_1\mu] \\ {\scriptstyle \bar{s}_1}\Big\uparrow & & \Big\uparrow{\scriptstyle A^{\mu}_{s_1}(\lambda)} \\ U[\mu] & \xrightarrow{N_{\mu}(\lambda)} & U[\mu] \\ {\scriptstyle \bar{s}_2}\Big\downarrow & & \Big\downarrow{\scriptstyle A^{\mu}_{s_2}(\lambda)} \\ U[s_2\mu] & \xrightarrow{N_{s_2\mu}(s_2\cdot\lambda)} & U[s_2\mu] \end{array},$$

which implies (5.2).

Next, the operator $\mathcal{N}[\mathbf{l};\mathbf{z}]^{-1} = \mathcal{X}[\mathbf{l};\mathbf{z}]^{-1}\phi[\mathbf{l};\mathbf{z}]$ is singular only at the hyperplanes

$$\begin{aligned} z_1 - l_1 - z_2 + s &= 0, \qquad 0 \le s < l_2\,, \\ z_2 - l_2 - z_3 + s &= 0, \qquad 0 \le s < l_3\,, \\ z_1 - l_1 - z_3 + s &= 0, \qquad 0 \le s < l_3\,, \end{aligned}$$

which correspond respectively to the hyperplanes

$$\begin{aligned} \chi_{\alpha_1,l_2-s}(\lambda) &= 0\,, \qquad 0 \le s < l_2\,, \\ \chi_{\alpha_2,l_3-s}(\lambda) &= 0\,, \qquad 0 \le s < l_3\,, \\ \chi_{\alpha_1+\alpha_2,l_3-s}(\lambda) &= 0 \qquad 0 \le s < l_3\,. \end{aligned}$$

Now it follows from Lemma 6 that $N(\lambda)$ satisfies (5.1).

Finally, let us show that for any $u \in U[\mu]$, the vector $\Xi_N(\lambda)u \in \mathcal{U}(\mathfrak{n}^-)\otimes U$ polynomially depends on λ. According to (3.1), this is equivalent to regularity of the sum $\sum_j \left(S_\lambda^{-1}\right)_{ij}\omega(g_j)N(\lambda)u$ for any i at any hyperplane $\chi_{\alpha,r}(\lambda)=0, \quad \alpha\in\Delta^+, r=1,2,\dots$.

Consider the case $\alpha=\alpha_1$.

Let $\{g_j\}$ be the Poincare-Birkhoff-Witt basis of $\mathcal{U}(\mathfrak{n}^-)$, corresponding to an ordering of Δ^+ with α_1 being the last root. It is known (see [ES]) that $\left(S_\lambda^{-1}\right)_{ij}$ is regular at the hyperplane $\chi_{\alpha_1,r}(\lambda)=0$ unless $g_j=\tilde{g}_jf_1^r$ for some $\tilde{g}_j\in\mathcal{U}(\mathfrak{n}^-)$. In the last case $\left(S_\lambda^{-1}\right)_{ij}$ may have a simple pole there, and the desired regularity will follow if $\omega(\tilde{g}_j)e_1^rN(\lambda)u$ vanishes at the hyperplane $\chi_{\alpha_1,r}(\lambda)=0$.

We prove that $e_1^rN(\lambda)u=0$ at the hyperplane $\chi_{\alpha_1,r}(\lambda)=0$ using the explicit realization Θ of the module U in $\mathbb{P}_{2,3}$. The operator $e_1^r : U[\mu]\to U[\mu+r\alpha_1]$ corresponds to the operator

$$\left(E_{12}^{\mathfrak{gl}_3}\right)^r : \left(V_{(l_1,0)}\otimes V_{(l_2,0)}\otimes V_{(l_3,0)}\right)^{\mathrm{sing}}[(m-k,k)] \to \left(V_{(l_1+r,0)}\otimes V_{(l_2-r,0)}\otimes V_{(l_3,0)}\right)^{\mathrm{sing}}[(m-k,k)]\,,$$

if $r \le l_2$, and is equal to zero, if $r > l_2$. There is nothing to prove in the latter case. The claim in the former case is implied by the following lemma.

Lemma 21. *Assume that $r \le l_2$. Let $\mathbf{z}_0$ be a generic point from the hyperplane $(z_1 - l_1) - z_2 + (l_2 - r) = 0$. Then the following composition of operators vanishes:*

$$\mathcal{H}[\mathbf{l};\mathbf{z}_0] \xrightarrow{\mathcal{N}[\mathbf{l};\mathbf{z}_0]} V_{(l_1,0)} \otimes V_{(l_2,0)} \otimes V_{(l_3,0)} \xrightarrow{\left(E_{12}^{\mathfrak{gl}_3}\right)^r} V_{(l_1+r,0)} \otimes V_{(l_2-r,0)} \otimes V_{(l_3,0)}. \tag{7.7}$$

Proof. Since $r \le l_2$, the product $\mathcal{X}[\mathbf{l};\mathbf{z}]$ contains the factor $z_1 - l_1 - z_2 + (l_2 - r)$, corresponding to $\chi_{\alpha_1,r}(\lambda)$, and $\mathcal{X}[\mathbf{l};\mathbf{z}_0] = 0$ If $l_2 - r > \min(l_1, l_2)$, then $\mathcal{I}[\mathbf{l}]\varphi$ is regular at $\mathbf{z} = \mathbf{z}_0$ for any $\varphi \in \mathcal{H}[\mathbf{l}]$, and the operator $\mathcal{N}[\mathbf{l};\mathbf{z}_0] = \mathcal{X}[\mathbf{l};\mathbf{z}_0]\,\mathcal{I}[\mathbf{l};\mathbf{z}_0]$ vanishes.

If $l_2 - r \le \min(l_1, l_2)$, then by Proposition 17, the image of $\mathcal{N}[\mathbf{l};\mathbf{z}_0]$ is contained in $\left(\bigoplus_{i=l_2-r+1}^{\min(l_1,l_2)} V_{(l_1+l_2-i,i)}\right) \otimes V_{(l_3,0)}$. Since $\left(E_{12}^{\mathfrak{gl}_3}\right)^r$ commutes with the $\mathfrak{gl}_2$ action and

$$V_{(l_1+r,0)} \otimes V_{(l_2-r,0)} \otimes V_{(l_3,0)} \cong \left(\bigoplus_{i=0}^{l_2-r} V_{(l_1+l_2-i,i)}\right) \otimes V_{(l_3,0)}\,,$$

the composition (7.7) vanishes. □

The case $\alpha = \alpha_2$ can be considered similarly to the case $\alpha = \alpha_1$.

Let now $\alpha = \alpha_1 + \alpha_2$. In this case we have $\alpha = s_2\alpha_1$, and therefore

$$\chi_{\alpha,r}(\lambda) = \langle \alpha, \lambda + \rho\rangle - r = \langle s_2\alpha_1, \lambda + \rho\rangle - r = \langle \alpha_1, s_2 \cdot \lambda + \rho\rangle - r = \chi_{\alpha_1,r}(s_2 \cdot \lambda).$$

Recall that $\Xi_N(\lambda)u$ is regular at the hyperplane $\chi_{\alpha,r}(\lambda) = 0$ if and only if the operator $\Phi_\lambda^{N(\lambda)u}$ is regular at the same hyperplane.

Consider λ from the hyperplane $\langle \alpha_2, \lambda + \rho\rangle = -n$, for a large enough nonnegative integer n. Then M_λ is contained as a proper submodule in $M_{s_2\cdot\lambda}$, and we have

$$\Phi_\lambda^{N(\lambda)u} = \Phi_{s_2\cdot\lambda}^{A(s_2\cdot\lambda)N(s_2\cdot\lambda)u}|_{M_\lambda}\,.$$

Since the operator $\Phi_{s_2\cdot\lambda}^{N(s_i\cdot\lambda)u}$ is regular at the hyperplanes $\chi_{\alpha_1,r}(s_2 \cdot \lambda) = 0$, so is the operator $\Phi_\lambda^{N(\lambda)u}$. Hence $\Xi_N(\lambda)u$ has no pole at the hyperplane $\chi_{\alpha,r}(\lambda) = \chi_{\alpha_1,r}(s_2 \cdot \lambda) = 0$, provided $\langle \alpha_2, \lambda + \rho\rangle = -n$.

The rational function $\chi_{\alpha,r}(\lambda)\Xi_N(\lambda)u$ is regular at the hyperplane $\chi_{\alpha,r}(\lambda) = 0$, and vanishes at infinitely many points of the hyperplane. Therefore, the function must vanish at the hyperplane identically, that is $\Xi_N(\lambda)u$ has no pole at the hyperplane $\chi_{\alpha,r}(\lambda) = 0$. □

8. Resonance conditions

Let $\mathfrak{g}$ be a semisimple Lie algebra, and let U be an irreducible finite-dimensional $\mathfrak{g}$-module with nontrivial zero weight subspace $U[0]$.

Let $N(\lambda) \in \mathrm{End}_{\mathbb{C}}(U)$ be a regularizing operator. Introduce an $\mathrm{End}_{\mathbb{C}}(U[0])$-valued function $\Psi(\lambda, x)$, where $\lambda \in \mathfrak{h}^*, x \in \mathfrak{h}$, by

$$\Psi(\lambda, x)u = \mathrm{Tr}\,|_{M_\lambda}\left(\Phi_\lambda^{N(\lambda)u} e^x\right).$$

Special cases of the function $\Psi(\lambda, x)$ were used in [ES] to give a representation-theoretic proof of the algebraic integrability of Calogero-Sutherland systems. In [EV] it was shown that the trace functions satisfy remarkable differential and difference equations.

An important property of the function $\Psi(\lambda, x)$ is that it satisfies certain resonance conditions.

Proposition 22. *Let $N = N(\lambda)$ be a regularizing operator acting in a finite-dimensional $\mathfrak{g}$-module U with $U[0] \neq 0$, and let $\Psi(\lambda, x)$ be the corresponding trace function.*

Let $\alpha \in \Delta^+$ and $k \in \mathbb{Z}_{\geq 0}$ be such that $U[k\alpha] \neq 0$. Then for any $\lambda \in \mathfrak{h}^*$ such that $\chi_{\alpha,k}(\lambda) = 0$, and any $u \in \ker N(\lambda) \subset U[0]$, we have

$$\Psi(\lambda, x)u = \Psi(s_\alpha \cdot \lambda, x)u\,.$$

Proof. Let λ be generic from the hyperplane $\chi_{\alpha,k}(\lambda) = 0$. Then M_λ contains a unique proper submodule $M_{s_\alpha \cdot \lambda}$, and $N(\lambda)u = 0$ implies that the image of $\Phi_\lambda^{N(\lambda)u}$ is contained in $M_{s_\alpha\cdot\lambda} \otimes U$. Therefore, the trace of the operator $\Phi_\lambda^{N(\lambda)u}$ in M_λ is equal to the trace of its resrtiction to $M_{s_\alpha\cdot\lambda}$. Moreover, the restriction of $\Phi_\lambda^{N(\lambda)u}$ to the submodule $M_{s_\alpha\cdot\lambda}$ coincides with $\Phi_{s_\alpha\cdot\lambda}^{N(s_\alpha\cdot\lambda)u}$, and we have

$$\Psi(\lambda, x)u = \mathrm{Tr}\,|_{M_\lambda}\left(\Phi_\lambda^{N(\lambda)u} e^x\right) = \mathrm{Tr}\,|_{M_{s_\alpha\cdot\lambda}}\left(\Phi_\lambda^{N(\lambda)u} e^x\right) =$$

$$\mathrm{Tr}\,|_{M_{s_\alpha\cdot\lambda}}\left(\Phi_{s_\alpha\cdot\lambda}^{N(s_\alpha\cdot\lambda)u} e^x\right) = \Psi(s_\alpha \cdot \lambda, x)u\,.$$

□

Example: Let $\mathfrak{g} = \mathfrak{sl}_2$. We identify $\mathfrak{h}^*$ with $\mathbb{C}$ by associating $\mathfrak{h}^* \ni \lambda \leftrightarrow \langle \alpha, \lambda \rangle \in \mathbb{C}$.

Let U be an irreducible finite-dimensional $\mathfrak{sl}_2$-module with even highest weight Λ. In this case the weight subspace $U[0]$ is one-dimensional, $\mathrm{End}(U[0]) \cong \mathbb{C}$, and the function $\Psi(\lambda, x)$ is scalar-valued. It satisfies the resonance conditions

$$\Psi(\lambda, x) = \Psi(s \cdot \lambda, x), \qquad \lambda = 0, 1, 2, \ldots, \Lambda/2 - 1\,.$$

Appendix A. Formal monomials

To prove Theorem 4 we use the calculus of formal monomials in $\mathcal{U}(\mathfrak{n}^-)$, developed in [FFM].

Lemma 23. *Let $f \in \mathfrak{n}^-, X \in \mathfrak{g}$. For any $n \in \mathbb{Z}_{\geq 0}$ we have the following identity in $U(\mathfrak{n}^-)$,*

$$X f^n = \sum_{k=0}^{\infty} \binom{n}{k} f^{n-k} \underbrace{[[[X, f], \ldots,], f]}_{k \text{ times}}. \tag{A.1}$$

This lemma is proved by induction. Note that the summation over k is finite.

Introduce an associative algebra $\mathbb{A}$, with unit $\mathbf{1}$ and generators f_i^γ, $i = 1, \ldots, \dim \mathfrak{h}$, $\gamma \in \mathbb{C}$, subject to the relations

$$\underbrace{[f_i, [, \ldots, [f_i, f_j] \ldots]]}_{a_{ij} \text{ times}} = 0, \qquad a_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}, \tag{A.2}$$

$$f_i^\beta f_i^\gamma = f_i^{\beta+\gamma}, \qquad \beta, \gamma \in \mathbb{C}, \tag{A.3}$$

$$f_i f_j^\gamma = \sum_{k=0}^{\infty} \binom{\gamma}{k} f_j^{\gamma-k} \underbrace{[[\ldots[f_i, f_j], \ldots,], f_j]}_{k \text{ times}}, \qquad \gamma \in \mathbb{C}. \tag{A.4}$$

where the commutator $[\cdot, \cdot]$ is defined as $[X, Y] = XY - YX$.

Lemma 24. *There exists an algebra homomorphism $\iota : \mathcal{U}(\mathfrak{n}^-) \to \mathbb{A}$ such that $\iota(f_i) = f_i$ for any $i = 1, \dots, \dim \mathfrak{h}$. The homomorphism ι is injective.*

Proof. It is known that $U(\mathfrak{n}^-)$ is the quotient of the free Lie algebra with generators $\{f_i\}$, by the ideal, generated by the Serre relations (A.2). Therefore, ι extends from the generators to the entire algebra $\mathcal{U}(\mathfrak{n}^-)$.

Next, we note that the algebra $\mathbb{A}$ is nonzero, i.e. the ideal, generated by the relations (A.4) is proper. Indeed, a linear functional φ on $\mathbb{A}$, defined on monomials by

$$\varphi(f_{i_1}^{\gamma_1} \dots f_{i_l}^{\gamma_l}) = 1,$$

is well-defined, and is nonzero, so $\mathbb{A} \neq 0$.

Let $\lambda \in \mathfrak{h}^*$ be generic. Introduce a $\mathfrak{g}$-module $\tilde{M}_\lambda$, which is equal to $\mathbb{A}$ as a vector space, with f_i acting naturally on the left, $h \in \mathfrak{h}$ acting by

$$h f_{i_1}^{\gamma_1} \dots f_{i_l}^{\gamma_l} = \langle h, \lambda - \sum_{k=1}^{l} \gamma_k \alpha_{i_k} \rangle f_{i_1}^{\gamma_1} \dots f_{i_l}^{\gamma_l}.$$

The action of operators e_i is determined by the condition $e_i \mathbf{1} = 0$, and the commutation formula

$$e_i f_j^\gamma = \sum_{k=0}^{\infty} \binom{\gamma}{k} f_j^{\gamma-k} \underbrace{[[\dots[e_i, f_j], \dots,], f_j]}_{k \text{ times}}, \qquad \gamma \in \mathbb{C}.$$

The module $\tilde{M}_\lambda$ is an obvious analog of the Verma module M_λ.

For any $X, Y \in \mathcal{U}(\mathfrak{n}^-)$, consider the action of $\omega(Y) \in \mathcal{U}(\mathfrak{g})$ on the element $\iota(X)\mathbf{1} \in \tilde{M}_\lambda$. We have $\omega(Y)\iota(X)\mathbf{1} = S_\lambda(X, Y)\mathbf{1}$, and therefore $\iota(X) = 0$ implies $S_\lambda(X, Y) = 0$ for any $Y \in \mathcal{U}(\mathfrak{n}^-)$. Since the Shapovalov form S_λ is nondegenerate for generic λ, this means that $X = 0$. Thus, ι is an injection. □

We identify $\mathcal{U}(\mathfrak{n}^-)$ with its image $\iota(\mathcal{U}(\mathfrak{n}^-)) \subset \mathbb{A}$. We say that an element $X \in \mathbb{A}$ makes sense, if $X \in \mathcal{U}(\mathfrak{n}^-)$.

Example 1. Let $\mathfrak{g} = \mathfrak{sl}_3$, and let $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{C}$. If $\gamma_1 + \gamma_3, \gamma_2 \in \mathbb{Z}_{\geq 0}$, and $\gamma_1 + \gamma_3 \geq \gamma_2$, then the monomial $f_1^{\gamma_1} f_2^{\gamma_2} f_1^{\gamma_3}$ makes sense, and we can write

$$f_1^{\gamma_1} f_2^{\gamma_2} f_1^{\gamma_3} = \sum_{j=0}^{\infty} \binom{\gamma_2}{j} \binom{\gamma_3}{j} f_1^{\gamma_1+\gamma_3-j} f_2^{\gamma_2-j} [f_2, f_1]^j.$$

Consider the completion $\mathbb{A} \hat{\otimes} \mathcal{U}(\mathfrak{n}^-)$, consisting of possibly infinite sums $\sum_i X_i \otimes Y_i$ with $X_i \in \mathbb{A}$ and homogeneous $Y_i \in \mathcal{U}(\mathfrak{n}^-)$, such that $\mathrm{wt}(Y_i) \to \infty$ when $i \to \infty$.

We following two Lemmas can be easily verified; we give sketches of proof, and leave more technical details to a reader.

Lemma 25. *There exists an algebra homomorphism*

$$\Delta : \mathbb{A} \to \mathbb{A} \hat{\otimes} \mathcal{U}(\mathfrak{n}^-),$$

such that

$$\Delta(f_i^\gamma) = \sum_{j=0}^{\infty} \binom{\gamma}{j} f_i^{\gamma-j} \otimes f_i^j.$$

□

Sketch of proof. We have to check that the comultiplication above can be extended to an algebra homomorphism, i.e. that it respects the defining relations (A.4). For each such relation of homogeneous weight γ, it amounts to verifying a family of identities in weight subspaces $\mathbb{A}[\nu] \otimes \mathcal{U}(\mathfrak{n}^-)[\gamma-\nu]$. These identities are are valid for any nonnegative integer γ, because in this case they are equivalent to the consistency of the comultiplication in $\mathcal{U}(\mathfrak{n}^-)$. Since these identities are polynomial in γ, they are satisfied for arbitrary γ. □

It is easy to see that if $X \in \mathcal{U}(\mathfrak{n}^-) \subset \mathbb{A}$, then $\Delta(X) \in \mathcal{U}(\mathfrak{n}^-) \otimes \mathcal{U}(\mathfrak{n}^-) \subset \mathbb{A} \hat{\otimes} \mathcal{U}(\mathfrak{n}^-)$.

Lemma 26. *Let $J \in \mathbb{Z}_{\geq 0}, \gamma_1, \dots, \gamma_l \in \mathbb{C}$. Let $\{g_i\}$ be a homogeneous basis of $\mathcal{U}(\mathfrak{n}^-)$. Then there exist $J_1, \dots, J_l \in \mathbb{Z}_{\geq 0}$, such that any monomial $f_{i_1}^{\gamma_1 - j_1} \dots f_{i_l}^{\gamma_l - j_l}$ with $j_1, \dots, j_l \leq J$, can be written as*

$$f_{i_1}^{\gamma_1 - j_1} \dots f_{i_l}^{\gamma_l - j_l} = f_{i_1}^{\gamma_1 - J_1} \dots f_{i_l}^{\gamma_l - J_l} \sum_i q_i(\gamma_1, \dots, \gamma_l) g_i,$$

for some polynomials $q_i(\gamma_1, \dots, \gamma_l)$. □

Sketch of proof. The statement is obvious for $l = 1$. Using induction on l, we can write

$$f_{i_1}^{\gamma_1 - j_1} \dots f_{i_{l-1}}^{\gamma_{l-1} - j_{l-1}} f_{i_l}^{\gamma_l - j_l} = f_{i_1}^{\gamma_1 - J_1} \dots f_{i_{l-1}}^{\gamma_{l-1} - J_{l-1}} \left(\sum_i q_i(\gamma_1, \dots, \gamma_l) g_i \right) f_{i_l}^{\gamma_l - j_l}.$$

Then we can move all f_{i_l} to the left by commuting them with $\{g_i\}$; nilpotency of the adjoint action of f_{i_l} will guarantee that all only finite number of f_{i_l}'s can be absorbed. □

We now use these resulst to give a proof of Theorem 4.

Proof of Theorem 4. We represent the left side of (4.4) as

$$\begin{aligned}
&\Phi_\lambda^u v_{w\cdot(\lambda+\mu)} = \Phi_\lambda^u F_w(\lambda+\mu) \mathbf{1}_{\lambda+\mu} = \Delta(F_w(\lambda+\mu)) \Xi(\lambda)(\mathbf{1}_\lambda \otimes u) = \\
&\sum_{j_1, \dots, j_l \in \mathbb{Z}_{\geq 0}} \sum_{i,j} q_{j_1, \dots, j_l}(\gamma_1(\lambda+\mu), \dots, \gamma_l(\lambda+\mu)) \times \\
&\qquad f_{i_1}^{\gamma_1(\lambda+\mu) - j_1} \dots f_{i_l}^{\gamma_l(\lambda+\mu) - j_l} g_i \mathbf{1}_\lambda \otimes \left(S_\lambda^{-1}\right)_{ij} f_{i_1}^{j_1} \dots f_{i_l}^{j_l} \omega(g_j) u. \qquad (A.5)
\end{aligned}$$

Here the coefficients $q_{j_1, \dots, j_l}(\gamma_1(\lambda+\mu), \dots, \gamma_l(\lambda+\mu))$ are certain polynomials in $\gamma_1(\lambda+\mu), \dots, \gamma_l(\lambda+\mu)$. The summation in (A.5) is actually finite, because the element $f_{i_1}^{j_1} \dots f_{i_l}^{j_l}$ acts as zero in U if any of $j_1, \dots, j_l$ is big enough.

Next, we have that $\gamma_k(\lambda+\mu) = \gamma_k(\lambda) + \langle \beta_k, \mu \rangle$ for some $\beta_k \in \Delta^+$. Hence, for all nonzero monomials in the sum, we have

$$\gamma_k(\lambda+\mu) - j_k = \gamma_k(\lambda) + \langle \beta_k, \mu \rangle - j_k \geq \gamma_k(\lambda) - J, \qquad k = 1, \dots, l,$$

for some $J \in \mathbb{Z}_{\geq 0}$. Lemma 26 implies that there exist $J_1, \dots, J_l \in \mathbb{Z}_{\geq 0}$, such that we can write

$$\Delta(F_w(\lambda+\mu)) \Xi(\lambda) = \left(f_{i_1}^{\gamma_1(\lambda) - J_1} \dots f_{i_l}^{\gamma_l(\lambda) - J_l} \otimes 1 \right) \left(\sum_i g_i \otimes P_i(\lambda) \right), \qquad (A.6)$$

for a certain finite collection of $\mathrm{End}_{\mathbb{C}}(U)$-valued rational functions $P_i(\lambda)$.

We now make two claims, which imply Theorem 4.

First, in the summation formula (A.6) we have

$$P_i(\lambda) \equiv 0, \qquad \text{if } \operatorname{wt}(g_i) \notin \operatorname{wt}(f_{i_1}^{J_1} \dots f_{i_l}^{J_l}) - Q^+. \qquad (A.7)$$

Second, for some rational $\mathrm{End}_{\mathbb{C}}(U)$-valued functions $A_w(\lambda)$, we have

$$\left(f_{i_1}^{\gamma_1(\lambda)-J_1}\dots f_{i_l}^{\gamma_l(\lambda)-J_l}\otimes 1\right)\left(\sum_i{}' g_i\otimes P_i(\lambda)\right) = F_w(\lambda)\otimes A_w(\lambda), \tag{A.8}$$

where $\sum_i{}'$ denotes the sum of terms with $\mathrm{wt}(g_i)=\mathrm{wt}(f_{i_1}^{J_1}\dots f_{i_l}^{J_l})$.

To verify these claims, we consider a dominant integral weight λ such that $\lambda\gg 0$. One can check that for any $u\in U$ we have

$$\Delta(F_w(\lambda+\mu))\Xi(\lambda)(\mathbf{1}_\lambda\otimes u) = v_{w\cdot\lambda}\otimes\tilde u + \text{ lower order terms }, \tag{A.9}$$

for some $\tilde u\in U$. In particular, the term $g_i\otimes P_i(\lambda)$ is nonzero only if

$$\mathrm{wt}(f_{i_1}^{\gamma_1(\lambda)-J_1}\dots f_{i_l}^{\gamma_l(\lambda)-J_l})+\mathrm{wt}(g_i)+\lambda\le w\cdot\lambda. \tag{A.10}$$

From the definition of $\gamma_k(\lambda)$ it follows that

$$\lambda+\mathrm{wt}(f_{i_1}^{\gamma_1(\lambda)}\dots f_{i_l}^{\gamma_l(\lambda)}) = w\cdot\lambda,$$

and thus (A.9) implies (A.7) for the given λ. Since the rational functions $P_i(\lambda)$ vanish for all sufficiently large dominant integral weights, they must vanish for all λ.

Using equation (A.8), we define the dynamical Weyl group operator $A_w(\lambda)$ for integral dominant $\lambda\gg 0$ by the rule $A_w(\lambda)u=\tilde u$. We now establish that $A_w(\lambda)$ rationally depends on λ, and verify our second claim.

By Lemma 26, we can write

$$F_w(\lambda)f_{i_1}^{\gamma_1(\lambda)-J_1}\dots f_{i_l}^{\gamma_l(\lambda)-J_l}\sum_i Q_i(\lambda)g_i,$$

for some polynomials $Q_i(\lambda)$ in $\lambda\in\mathfrak{h}^*$. Equation (A.9) then becomes

$$\left(f_{i_1}^{\gamma_1(\lambda)-J_1}\dots f_{i_l}^{\gamma_l(\lambda)-J_l}\otimes 1\right)\left(\sum_i{}' g_i\mathbf{1}\otimes P_i(\lambda)u\right) = \\ \left(f_{i_1}^{\gamma_1(\lambda)-J_1}\dots f_{i_l}^{\gamma_l(\lambda)-J_l}\otimes 1\right)\left(\sum_i{}' Q_i(\lambda)g_i\mathbf{1}\otimes A_w(\lambda)u\right). \tag{A.11}$$

or equivalently

$$\sum_i{}' g_i\mathbf{1}\otimes P_i(\lambda)u = \sum_i{}' Q_i(\lambda)g_i\mathbf{1}\otimes A_w(\lambda)u.$$

It is valid for dominant integral $\lambda\gg 0$, and implies that for all i we have

$$A_w(\lambda)=\frac{P_i(\lambda)}{Q_i(\lambda)},\qquad \text{if } \mathrm{wt}(g_i)=\mathrm{wt}(f_{i_1}^{J_1}\dots f_{i_l}^{J_l}). \tag{A.12}$$

Therefore, $A_w(\lambda)$ can be extended to a rational $\mathrm{End}_{\mathbb{C}}(U)$-valued function, such that (A.12) is satisfied for all λ. Equations (A.11), (A.8), and (4.4) follow for all λ. □

Appendix B. Yangians, weight functions and R-matrices

We recall some facts about the Yangian $Y(\mathfrak{gl}_2)$, and explain how to get weight functions as matrix elements of evaluation Yangian modules. For more details on Yangians, see for example [CP].

The Yangian $Y(\mathfrak{gl}_2)$ is a Hopf algebra with generators $T_{ij}^{(k)}$, $i,j=1,2$; $k=1,2,3,\dots$. The relations are most conveniently written in terms of the generating series

$$T_{ij}(u)=\delta_{ij}+\sum_{k=1}^{\infty}T_{ij}^{(k)}u^{-k}\,,$$

and have the form

$$(u-v)[T_{ij}(u),T_{kl}(v)]=T_{kj}(v)T_{il}(u)-T_{kj}(u)T_{il}(v)\,.$$

The coproduct for $Y(\mathfrak{gl}_2)$ is given by

$$T_{ij}(u)\mapsto\sum_{k=1}^{2}T_{kj}(u)\otimes T_{ik}(u)\,.$$

The Yangian $Y(\mathfrak{gl}_2)$ contains $\mathcal{U}(\mathfrak{gl}_2)$ as a Hopf subalgebra, the embedding $\mathcal{U}(\mathfrak{gl}_2)\to Y(\mathfrak{gl}_2)$ being given by $E_{ij}\mapsto T_{ji}^{(1)}$. We identify $\mathcal{U}(\mathfrak{gl}_2)$ wih its image in $Y(\mathfrak{gl}_2)$ under this embedding.

There is a family of homomorphisms $\epsilon_z:Y(\mathfrak{gl}_2)\to\mathcal{U}(\mathfrak{gl}_2)$, depending on a complex parameter z:

$$\epsilon_z:T_{ij}(u)\mapsto\delta_{ij}+\frac{E_{ji}}{u-z}\,.$$

These homomorphism are identical on the subalgebra $\mathcal{U}(\mathfrak{gl}_2)\subset Y(\mathfrak{gl}_2)$.

For any $\mathfrak{gl}_2$-module V, we denote $V(z)$ the pullback of V through the homomorphism ϵ_z; the $Y(\mathfrak{gl}_2)$-module $V(z)$ is called an evaluation module.

The Yangian action in a tensor product of evaluation modules is the underlying structure for the functional realization $\phi[\mathbf{l}]$ of the tensor product of the corresponding $\mathfrak{gl}_2$-modules.

As before, $\mathbf{l}=(l_1,l_2,l_3)$ will always denote a triple of nonnegative integers. For any $k\in\mathbb{Z}_{\geq0}$ set

$$\mathcal{Z}_k=\{(k_1,k_2,k_3)\in\mathbb{Z}^3_{\geq0}\ \mid\ k_1+k_2+k_3=k\}\,,$$
$$\mathcal{Z}_k[\mathbf{l}]=\{(k_1,k_2,k_3)\in\mathcal{Z}_k\ \mid\ k_1\leq l_1,k_2\leq l_2,k_3\leq l_3\}.$$

Proposition 27. [KBI, TV3] *Let $\mathbf{z}=(z_1,z_2,z_3)$ be a triple of complex numbers. Consider the $Y(\mathfrak{gl}_2)$-module $V_{(l_1,0)}(z_1)\otimes V_{(l_2,0)}(z_2)\otimes V_{(l_3,0)}(z_3)$. For any $k\in\mathbb{Z}_{\geq0}$, $\mathbf{q}\in\mathcal{Z}_k[\mathbf{l}]$, we have*

$$\left(\prod_{i=1}^{3}\prod_{a=1}^{k}(t_a-z_i)\right)T_{12}(t_1)\dots T_{12}(t_k)(\mathbf{1}_{l_1}\otimes\mathbf{1}_{l_2}\otimes\mathbf{1}_{l_3})=$$
$$\sum_{\mathbf{p}\in\mathcal{Z}_k[\mathbf{l}]}w_{\mathbf{p}}[\mathbf{l};\mathbf{z}](t_1,\dots,t_k)E_{12}^{p_1}\mathbf{1}_{l_1}\otimes E_{12}^{p_2}\mathbf{1}_{l_2}\otimes E_{12}^{p_3}\mathbf{1}_{l_3},$$

$$\left(\prod_{i=1}^{3}\prod_{a=1}^{k}(t_a-z_i)\right)T_{21}(t_1)\dots T_{21}(t_k)(E_{12}^{q_1}\mathbf{1}_{l_1}\otimes E_{12}^{q_2}\mathbf{1}_{l_2}\otimes E_{12}^{q_3}\mathbf{1}_{l_3})=$$
$$w'_{\mathbf{q}}[\mathbf{l};\mathbf{z}](t_1,\dots,t_k)(\mathbf{1}_{l_1}\otimes\mathbf{1}_{l_2}\otimes\mathbf{1}_{l_3})\,,$$

where

$$w_{\mathbf{k}}[\mathbf{l};\mathbf{z}](t_1,\dots,t_k) = \frac{1}{k_1!k_2!k_3!}\, Sym\left(\prod_{i=1}^{3}\left(\prod_{a=1}^{k_1+\dots+k_{i-1}}(t_a-z_i+l_i)\prod_{b=k_1+\dots+k_i+1}^{k_1+k_2+k_3}(t_b-z_i)\right)\prod_{a<b}\frac{t_a-t_b-1}{t_a-t_b}\right)$$

$$w'_{\mathbf{k}}[\mathbf{l};\mathbf{z}](t_1,\dots,t_k) = \frac{1}{k_1!k_2!k_3!}\, Sym\left(\prod_{i=1}^{3}\left(\prod_{a=1}^{k_1+\dots+k_{i-1}}(t_a-z_i)\prod_{b=k_1+\dots+k_i+1}^{k_1+k_2+k_3}(t_b-z_i+l_i)\right)\prod_{a<b}\frac{t_a-t_b+1}{t_a-t_b}\right).$$

The functions $w_{\mathbf{k}}[\mathbf{l};\mathbf{z}]$ and $w'_{\mathbf{k}}[\mathbf{l};\mathbf{z}]$ are called the weight functions and the dual weight functions, respectively. They are polynomials in $t_1,\dots,t_k$, and z_1,z_2,z_3.

The definition (6.1) of the the map $\phi[\mathbf{l}]$ can be written as

$$\phi[\mathbf{l};\mathbf{z}]\left(E_{21}^{k_1}\mathbf{1}_{l_1}\otimes E_{21}^{k_2}\mathbf{1}_{l_2}\otimes E_{21}^{k_3}\mathbf{1}_{l_3}\right) = \frac{k_1!l_1!}{(l_1-k_1)!}\,\frac{k_2!l_2!}{(l_2-k_2)!}\,\frac{k_3!l_3!}{(l_3-k_3)!}\,w'_{\mathbf{k}}[\mathbf{l};\mathbf{z}]\,.$$

Let $\mathcal{H}'_k$ denote the space of symmetric polynomials in variables $t_1,\dots,t_k$ of degree at most 2 in each $t_1,\dots,t_k$. Set

$$\mathcal{H}' = \bigoplus_{k\in\mathbb{Z}_{\geq 0}}\mathcal{H}'_k\,.$$

It is clear that $\mathcal{H}_k = \mathcal{H}'_k\otimes\mathbb{C}[\mathbf{z}]$ and $\mathcal{H} = \mathcal{H}'\otimes\mathbb{C}[\mathbf{z}]$.

Theorem 28. [TV3] *The space $\mathcal{H}'$ has a structure of a $Y(\mathfrak{gl}_2)$-module, depending on $\mathbf{l},\mathbf{z}$, with the following properties:*

1. *the $\mathfrak{gl}_2$-module structure on $\mathcal{H}'$ coincides with that, induced by the $\mathfrak{gl}_2$-module structure on $\mathcal{H}$, described in Proposition 12;*
2. *the map $\phi[\mathbf{l};\mathbf{z}] : V_{(l_1,0)}(z_1)\otimes V_{(l_2,0)}(z_2)\otimes V_{(l_3,0)}(z_3)\to\mathcal{H}'$ is a homomorphism of $Y(\mathfrak{gl}_2)$-modules.*

We now use the representation theory of the Yangian $Y(\mathfrak{gl}_2)$ to prove the results, announced in Chapter 6. The following theorems go back to [T].

Theorem 29. [T]

1. *Let $l\in\mathbb{Z}_{\geq 0}$, and let U be an irreducible finite-dimensional $Y(\mathfrak{gl}_2)$-module. Then for generic $z\in\mathbb{C}$, the $Y(\mathfrak{gl}_2)$-modules $U\otimes V_{(l,0)}(z)$ and $V_{(l,0)}(z)\otimes U$ are irreducible and isomorphic to each other.*
2. *Let $l,l'\in\mathbb{Z}_{\geq 0}$, and let $z,z'\in\mathbb{C}$ be such that $z-l=z'-r$ for some $r=0,1,\dots,\min(l,l')-1$. Then the $Y(\mathfrak{gl}_2)$-module $V_{(l,0)}(z)\otimes V_{(l',0)}(z')$ contains a unique proper submodule $\widetilde{V}_{l,l';r}(z)$. As a $\mathfrak{gl}_2$-module, $\widetilde{V}_{l,l';r}(z)$ is isomorphic to*

$$\widetilde{V}_{l,l';s}(z)\cong\bigoplus_{s=r+1}^{\min(l,l')}V_{(l+l'-s,s)}. \tag{B.1}$$

Theorem 30. [T] *Let $l,l'\in\mathbb{Z}_{\geq 0}$. There exists a unique linear operator $R_{l,l'}(z)\in\mathrm{End}_{\mathbb{C}}(V_{(l,0)}\otimes V_{(l',0)})$, rationally depending on a complex parameter z, such that for generic z,z' the operator*

$$\check{R}_{l,l'}(z-z') = P\circ R_{l,l'}(z-z') : V_l(z)\otimes V_{l'}(z')\to V_{l'}(z')\otimes V_l(z)$$

is an isomorphism of $Y(\mathfrak{gl}_2)$*-modules,* P *being the permutation operator:* $P(u \otimes v) = v \otimes u$*, and*

$$R_{l,l'}(z)\mathbf{1}_l \otimes \mathbf{1}_{l'} = \mathbf{1}_l \otimes \mathbf{1}_{l'} \,.$$

We call the operator $R_{l,l'}(z)$ the standard rational R-matrix.

Proof of Proposition 16. Let $\mathbf{z}_0 = (z_1, z_2, z_3)$ be a generic point from the hyperplane $z_1 - z_2 + l_2 - r = 0$, with $r = 0, 1, \ldots, \min(l_1, l_2) - 1$. Then it follows from Theorem 29 that the $Y(\mathfrak{gl}_2)$-module $V_{(l_1,0)}(z_1) \otimes V_{(l_2,0)}(z_2) \otimes V_{(l_3,0)}(z_3)$ has a unique proper submodule $\widetilde{V}_{l_1,l_2;r}(z_1) \otimes V_{(l_3,0)}(z_3)$, while Proposition 27 and formula (B.1) imply that $\ker \phi[\mathbf{l}; \mathbf{z}_0]$ contains $\widetilde{V}_{l_1,l_2;r}(z_1) \otimes V_{(l_3,0)}(z_3)$. Since $\phi[\mathbf{l}; \mathbf{z}_0]$ is a non-trivial homomorphism of $Y(\mathfrak{gl}_2)$-modules, the kernel $\ker \phi[\mathbf{l}; \mathbf{z}_0]$ must be a proper $Y(\mathfrak{gl}_2)$-submodule, which proves the equality

$$\ker \phi[\mathbf{l}; \mathbf{z}_0] = \widetilde{V}_{l_1,l_2;r}(z_1) \otimes V_{(l_3,0)}(z_3) \,.$$

The proof of the second part of Proposition 16 is similar. □

Proof of Proposition 18. It follows from Proposition 27 that to determine the image of a vector $v \in V_{l_1} \otimes V_{l_2} \otimes V_{l_3}$, we must act on it by the element $\left(\prod_{i=1}^{3} \prod_{a=1}^{k} (t_a - z_i)\right) T_{21}(t_1) \ldots T_{21}(t_k)$, and then use the linear functional $(\mathbf{1}_{l_1} \otimes \mathbf{1}_{l_2} \otimes \mathbf{1}_{l_3})^*$. The $Y(\mathfrak{gl}_2)$-intertwining property of the R-matrix implies that we have the commutative diagram

$$\begin{array}{ccc}
V_{l_2} \otimes V_{l_1} \otimes V_{l_3} & \xrightarrow{\phi[s_1(\mathbf{l}); s_1(\mathbf{z})]} & \mathcal{H}[s_1(\mathbf{l}); s_1(\mathbf{z})] \\
\Big\uparrow {\scriptstyle R_{l_1 l_2}(z_1 - z_2) \otimes 1} & & \Big\| \\
V_{l_1} \otimes V_{l_2} \otimes V_{l_3} & \xrightarrow{\phi[\mathbf{l}; \mathbf{z}]} & \mathcal{H}[\mathbf{l}; \mathbf{z}] \\
\Big\downarrow {\scriptstyle 1 \otimes R_{l_2 l_3}(z_2 - z_3)} & & \Big\| \\
V_{l_1} \otimes V_{l_3} \otimes V_{l_2} & \xrightarrow{\phi[s_2(\mathbf{l}); s_2(\mathbf{z})]} & \mathcal{H}[s_2(\mathbf{l}); s_2(\mathbf{z})]
\end{array} \,. \tag{B.2}$$

Now denote

$$\check{\mathcal{R}}_{l,l'}(u) = \frac{\prod_{j=1}^{l'} (u - l + j)}{\prod_{j=1}^{l} (-u - l' + j)} \; \check{R}_{l,l'}(u) \,.$$

The commutative diagram (B.2) and the formulas

$$\frac{\prod_{j=1}^{l_2} (z_1 - z_2 - l_1 + j)}{\prod_{j=1}^{l_1} (z_2 - z_1 - l_2 + j)} = \frac{\mathcal{X}[\mathbf{l}; \mathbf{z}]}{\mathcal{X}[s_1(\mathbf{l}); s_1(\mathbf{z})]} \,,$$

$$\frac{\prod_{j=1}^{l_3} (z_2 - z_3 - l_2 + j)}{\prod_{j=1}^{l_2} (z_3 - z_2 - l_3 + j)} = \frac{\mathcal{X}[\mathbf{l}; \mathbf{z}]}{\mathcal{X}[s_2(\mathbf{l}); s_2(\mathbf{z})]} \,,$$

imply that the operators $\check{\mathcal{R}}_{l,l'}(u)$ satisfy the conditions of Proposition 18. □

In the remaining part of the Appendix we construct the map $\mathcal{I}[\mathbf{l}]$ explicitly, and describe its singularities.

Let $\mathbf{u} = (u_1, u_2, u_3)$ be complex variables, and for any $k \in \mathbb{Z}_{\geq 0}$ set $\widetilde{\mathcal{H}}_k = \mathcal{H}_k \otimes \mathbb{C}[\mathbf{u}]$. Denote also

$$\Omega_k[\mathbf{u};\mathbf{z}](t_1,\dots,t_k) = \prod_{i=1}^{3}\prod_{a=1}^{k} \frac{1}{(t_a - z_i + u_i)(t_a - z_i)} \prod_{\substack{a,b=1\\ a\neq b}}^{k} \frac{t_a - t_b}{t_a - t_b - 1}\,.$$

For any function $F = F(t_1,\dots,t_k)$, set

$$\operatorname{Res}_{(\zeta_1,\dots,\zeta_k)} F = \operatorname{Res}_{t_k=\zeta_k}\left(\operatorname{Res}_{t_{k-1}=\zeta_{k-1}}\left(\dots \operatorname{Res}_{t_1=\zeta_1} F(t_1,\dots,t_k)\right)\dots\right)\,,$$

Introduce a symmetric $\mathbb{C}(\mathbf{u};\mathbf{z})$-valued bilinear form $\langle\cdot,\cdot\rangle$ on each $\widetilde{\mathcal{H}}_k$, so that for any $\varphi,\psi \in \widetilde{\mathcal{H}}_k$ the value $\langle\varphi,\psi\rangle_{\mathbf{u},\mathbf{z}}$ of the corresponding rational function at generic $\mathbf{u},\mathbf{z}$ is given by

$$\langle\varphi,\psi\rangle_{\mathbf{u},\mathbf{z}} = \sum_{\mathbf{k}\in\mathcal{Z}_k} \operatorname{Res}_{\tau^-_{\mathbf{k}}[\mathbf{z}]}\left(\varphi[\mathbf{u};\mathbf{z}]\psi[\mathbf{u};\mathbf{z}]\Omega_k[\mathbf{u};\mathbf{z}]\right)\,, \tag{B.3}$$

where the "string" $\tau^-_{\mathbf{k}}[\mathbf{z}]$ is defined by

$$\tau^-_{\mathbf{k}}[\mathbf{z}] = (z_1,\dots,z_1-k_1+1, z_2,\dots,z_2-k_2+1, z_3,\dots,z_3-k_3+1)\,.$$

Remark. The definition above is motivated by the fact (see [TV4]) that the integral over the imaginary hyperplane $\operatorname{Re} t_1 = \dots = \operatorname{Re} t_k = 0$,

$$\frac{1}{(2\pi i)^k}\int\cdots\int \varphi[\mathbf{u};\mathbf{z}](t_1,\dots,t_k)\psi[\mathbf{u};\mathbf{z}](t_1,\dots,t_k)\Omega_k[\mathbf{u};\mathbf{z}](t_1,\dots,t_k)dt_1\dots dt_k\,,$$

converges when $\operatorname{Re}(u_i) \ll \operatorname{Re}(z_i) < 0$, $\quad i = 1,2,3$, and represents the rational function $\langle\varphi,\psi\rangle$.

Evaluating the integral by residues, and using the analytic continuation, we get the more convenient algebraic formula (B.3); one can show that it is equivalent to another iterated residue formula:

$$\langle\varphi,\psi\rangle_{\mathbf{u},\mathbf{z}} = (-1)^k \sum_{\mathbf{k}\in\mathcal{Z}_k} \operatorname{Res}_{\tau^+_{\mathbf{k}}[\mathbf{z}-\mathbf{u}]}\left(\varphi[\mathbf{u};\mathbf{z}]\psi[\mathbf{u};\mathbf{z}]\Omega_k[\mathbf{u};\mathbf{z}]\right)\,, \tag{B.4}$$

where $\tau^+_{\mathbf{k}}[\mathbf{z}-\mathbf{u}]$ is defined by

$$\tau^+_{\mathbf{k}}[\mathbf{z}-\mathbf{u}] = (z_1-u_1,\dots,z_1-u_1+k_1-1, z_2-u_2,\dots,z_2-u_2+k_2-1, z_3-u_3,\dots,z_3-u_3+k_3-1).$$

Lemma 31. *For any $\varphi,\psi \in \widetilde{\mathcal{H}}_k$, the only possible singularities of $\langle\varphi,\psi\rangle_{\mathbf{u},\mathbf{z}}$ are simple poles at the following hyperplanes:*

$$z_i - u_i = z_j - r, \qquad i,j = 1,2,3, \qquad r = 0,1,\dots,k-1. \tag{B.5}$$

Proof. Since φ,ψ are polynomials, possible singularities of $\langle\varphi,\psi\rangle_{\mathbf{u},\mathbf{z}}$ are determined by $\Omega_k[\mathbf{u};\mathbf{z}]$. Using explicit formulas, one can check that $\operatorname{Res}_{\tau^-_{\mathbf{k}}[\mathbf{z}]}(\Omega_k[\mathbf{u};\mathbf{z}])$ has only simple poles located at the hyperplanes

$$z_i - u_i = z_j - r, \qquad z_i = z_j - r, \qquad i,j = 1,2,3, \qquad r = 0,1,\dots,k-1\,.$$

Similarly, $\operatorname{Res}_{\tau^+_{\mathbf{k}}[\mathbf{z}-\mathbf{u}]}(\Omega_k[\mathbf{u};\mathbf{z}])$ has only simple poles located at the hyperplanes

$$z_i - u_i = z_j - r, \qquad z_i = z_j - u_j - r, \qquad i,j = 1,2,3, \qquad r = 0,1,\dots,k-1\,.$$

Since $\langle\varphi,\psi\rangle_{\mathbf{u},\mathbf{z}}$ is given either by formula (B.3) or (B.4), the function $\langle\varphi,\psi\rangle_{\mathbf{u},\mathbf{z}}$ can have poles only at the hyperplanes which belong to both of these lists, which gives precisely list (B.5). □

Let $\widetilde{\mathcal{H}}_k[\mathbf{l}]$, $k\in\mathbb{Z}_{\geq 0}$, denote the subspace of functions $\varphi\in\widetilde{\mathcal{H}}_k$, satisfying the following conditions for each $i=1,2,3$ such that $l_i<k$:

$$\varphi(z_i,\dots,z_i-l_i,t_{l_i+2},\dots,t_k)=0 \qquad \text{at the hyperplane } u_i=l_i\,.$$

We have two important imbeddings of $\mathcal{H}_k[\mathbf{l}]$ into $\widetilde{\mathcal{H}}_k[\mathbf{l}]$, given by $\varphi\mapsto\varphi^-$,

$$\varphi^-[\mathbf{u};\mathbf{z}](t_1,\dots,t_k)=\varphi[\mathbf{z}](t_1,\dots,t_k),$$

and $\varphi\mapsto\varphi^+$,

$$\varphi^-[\mathbf{u};\mathbf{z}](t_1,\dots,t_k)=\varphi[\mathbf{z}-\mathbf{u}+\mathbf{l}](t_1,\dots,t_k).$$

For $\mathbf{p}\in\mathcal{Z}_k$, let $w_{\mathbf{p}}=w_{\mathbf{p}}[\mathbf{u};\mathbf{z}]$ be the weight function defined in Proposition 27, with integers $\mathbf{l}=(l_1,l_2,l_3)$ replaced by variables $\mathbf{u}=(u_1,u_2,u_3)$. It is obvious that $w_{\mathbf{p}}\in\widetilde{\mathcal{H}}_k$; moreover, if $\mathbf{p}\in\mathcal{Z}_k[\mathbf{l}]$, then $w_{\mathbf{p}}\in\widetilde{\mathcal{H}}_k[\mathbf{l}]$.

Lemma 32. *Let $\mathbf{p}\in\mathcal{Z}_k[\mathbf{l}]$, and $\varphi\in\mathcal{H}_k[\mathbf{l}]$.*

1. *The function $\langle w_{\mathbf{p}},\varphi^-\rangle_{\mathbf{u},\mathbf{z}}$ may have only simple poles, located at the hyperplanes*

$$z_i-u_i=z_j-r,\qquad 1\leq i\leq j\leq 3,\qquad r=0,1,\dots,l_j-1.$$

2. *The function $\langle w_{\mathbf{p}},\varphi^+\rangle_{\mathbf{u},\mathbf{z}}$ may have only simple poles, located at the hyperplanes*

$$z_i-u_i=z_j-r,\qquad 1\leq i\leq j\leq 3,\qquad r=0,1,\dots,l_i-1.$$

Proof. Since $\varphi^-[\mathbf{u};\mathbf{z}](\tau^-_{\mathbf{q}}[\mathbf{z}])=\varphi[\mathbf{z}](\tau^-_{\mathbf{q}}[\mathbf{z}])=0$ unless $\mathbf{q}\in\mathcal{Z}_k[\mathbf{l}]$, we have

$$\langle w_{\mathbf{p}},\varphi^-\rangle_{\mathbf{u},\mathbf{z}}=\sum_{\mathbf{q}\in\mathcal{Z}_k}\varphi^-(\tau^-_{\mathbf{q}}[\mathbf{z}])\mathrm{Res}_{\tau^-_{\mathbf{q}}[\mathbf{z}]}(w_{\mathbf{p}}[\mathbf{u};\mathbf{z}]\Omega_k[\mathbf{u};\mathbf{z}])=$$
$$\sum_{\mathbf{q}\in\mathcal{Z}_k[\mathbf{l}]}\varphi^-(\tau^-_{\mathbf{q}}[\mathbf{z}])\mathrm{Res}_{\tau^-_{\mathbf{q}}[\mathbf{z}]}(w_{\mathbf{p}}[\mathbf{u};\mathbf{z}]\Omega_k[\mathbf{u};\mathbf{z}])\,.$$

Let $\mathbf{q}\in\mathcal{Z}_k[\mathbf{l}]$. From the explicit formulas for $w_{\mathbf{p}}[\mathbf{u};\mathbf{z}]$ and $\Omega_k[\mathbf{u};\mathbf{z}]$ it follows that the function $\mathrm{Res}_{\tau^-_{\mathbf{q}}[\mathbf{z}]}(w_{\mathbf{p}}[\mathbf{u};\mathbf{z}]\Omega_k[\mathbf{u};\mathbf{z}])$ has no pole at the hyperplane $z_i-u_i=z_j-r$ provided $i>j$ or $r\geq q_j$. Using the description (B.5) of all possible poles, we see that $\langle w_{\mathbf{p}},\varphi^-\rangle_{\mathbf{u},\mathbf{z}}$ may have only poles corresponding to $i\leq j$ and $r<q_j\leq l_j$, which proves the first assertion.

Similarly, using the fact that $\varphi^+[\mathbf{u};\mathbf{z}](\tau^+_{\mathbf{q}}[\mathbf{z}])\varphi[\mathbf{z}-\mathbf{u}](\tau^+_{\mathbf{q}}[\mathbf{z}-\mathbf{u}])=0$ unless $\mathbf{q}\in\mathcal{Z}_k[\mathbf{l}]$, we arrive to the formula

$$\langle w_{\mathbf{p}},\varphi^+\rangle_{\mathbf{u},\mathbf{z}}=\sum_{\mathbf{q}\in\mathcal{Z}_k[l]}\varphi^+(\tau^+_{\mathbf{q}}[\mathbf{z}-\mathbf{u}])\mathrm{Res}_{\tau^+_{\mathbf{q}}[\mathbf{z}-\mathbf{u}]}(w_{\mathbf{p}}[\mathbf{u};\mathbf{z}]\Omega_k[\mathbf{u};\mathbf{z}])\,.$$

For any $\mathbf{q}\in\mathcal{Z}_k[\mathbf{l}]$ the function $\mathrm{Res}_{\tau^+_{\mathbf{q}}[\mathbf{z}-\mathbf{u}]}(w_{\mathbf{p}}[\mathbf{u};\mathbf{z}]\Omega_k[\mathbf{u};\mathbf{z}])$ has no pole at the hyperplane $z_i-u_i=z_j-r$ provided $i>j$ or $r\geq q_i$. Therefore, it follows that $\langle w_{\mathbf{p}},\varphi^+\rangle_{\mathbf{u};\mathbf{z}}$ may only have poles corresponding to $i\leq j$ and $r<q_i\leq l_i$. □

Lemma 33. *Let $\varphi, \psi \in \widetilde{\mathcal{H}}_k[\mathbf{l}]$. The function $\langle \varphi, \psi \rangle_{\mathbf{u},\mathbf{z}}$ analytically continues to $\mathbf{u} = \mathbf{l}$, and we have*

$$\begin{aligned}\langle \varphi, \psi \rangle_{\mathbf{l},\mathbf{z}} &= \sum_{\mathbf{k} \in \mathcal{Z}_k[\mathbf{l}]} \mathrm{Res}_{\tau^-_{\mathbf{k}}[\mathbf{z}]} \left(\varphi[\mathbf{l};\mathbf{z}] \psi[\mathbf{l};\mathbf{z}] \Omega_k[\mathbf{l};\mathbf{z}] \right) \\ &= (-1)^k \sum_{\mathbf{k} \in \mathcal{Z}_k[\mathbf{l}]} \mathrm{Res}_{\tau^+_{\mathbf{k}}[\mathbf{z}-\mathbf{l}]} \left(\varphi[\mathbf{l};\mathbf{z}] \psi[\mathbf{l};\mathbf{z}] \Omega_k[\mathbf{l};\mathbf{z}] \right). \end{aligned} \tag{B.6}$$

In other words, the analytic continuation to $\mathbf{u} = \mathbf{l}$ can be computed by applying the residue formula to the specialized functions $\psi[\mathbf{l};\mathbf{z}], \varphi[\mathbf{l};\mathbf{z}]$ and $\Omega_k[\mathbf{l};\mathbf{z}]$, rather than treating $\mathbf{u}$ as variables, and the summation is reduced to $\mathbf{k} \in \mathcal{Z}_k[\mathbf{l}] \subset \mathcal{Z}_k$. Note that in general this is not true for arbitrary $\varphi, \psi \in \widetilde{\mathcal{H}}_k$.

Proof. One can check that for any $\mathbf{q} \in \mathcal{Z}_k$ the rational functions $\mathrm{Res}_{\tau^-_{\mathbf{q}}[\mathbf{z}]}(\psi[\mathbf{u};\mathbf{z}]\Omega_k[\mathbf{u};\mathbf{z}])$ and $\mathrm{Res}_{\tau^+_{\mathbf{q}}[\mathbf{z}-\mathbf{u}]}(\psi[\mathbf{u};\mathbf{z}]\Omega_k[\mathbf{u};\mathbf{z}])$ have no poles at the hyperplanes $u_i = l_i$, $i = 1,2,3$. Together with Lemma 31, this shows that the analytic continuation to $\mathbf{u} = \mathbf{l}$ is well-defined.

Since for $\mathbf{q} \notin \mathcal{Z}_k[\mathbf{l}]$, the expression $\varphi[\mathbf{u};\mathbf{z}](\tau^-_{\mathbf{k}}[\mathbf{z}])$ vanishes at $\mathbf{u} = \mathbf{l}$, it follows that the summation in the residue formulas (B.3) reduces to $\mathbf{q} \in \mathcal{Z}_k[\mathbf{l}]$. Similar conclusion holds for the redisue formula (B.4).

The statement now follows from the fact that for $\mathbf{q} \in \mathcal{Z}_k[\mathbf{l}]$ we have

$$\begin{aligned}\mathrm{Res}_{\tau^-_{\mathbf{q}}[\mathbf{z}]}(\psi[\mathbf{u};\mathbf{z}]\Omega_k[\mathbf{u};\mathbf{z}])|_{\mathbf{u}=\mathbf{l}} &= \mathrm{Res}_{\tau^-_{\mathbf{q}}[\mathbf{z}]}(\psi[\mathbf{l};\mathbf{z}]\Omega_k[\mathbf{l};\mathbf{z}])\,, \\ \mathrm{Res}_{\tau^+_{\mathbf{q}}[\mathbf{z}-\mathbf{u}]}(\psi[\mathbf{u};\mathbf{z}]\Omega_k[\mathbf{u};\mathbf{z}])|_{\mathbf{u}=\mathbf{l}} &= \mathrm{Res}_{\tau^+_{\mathbf{q}}[\mathbf{z}-\mathbf{l}]}(\psi[\mathbf{l};\mathbf{z}]\Omega_k[\mathbf{l};\mathbf{z}])\,.\end{aligned}$$

□

Introduce a partial order $\leq$ on $\mathcal{Z}_k$ by saying that $\mathbf{x} \leq \mathbf{y}$ if and only if $x_1 \leq y_1$ and $x_3 \geq y_3$.

Lemma 34. *For any $\mathbf{k}, \mathbf{p}, \mathbf{q} \in \mathcal{Z}_k$ we have*

$$\begin{aligned} w_{\mathbf{p}}[\mathbf{l},\mathbf{z}](\tau^-_{\mathbf{k}}[\mathbf{z}]) &= 0, \qquad \textit{unless}\ \ \mathbf{k} \leq \mathbf{p}\,, \\ w'_{\mathbf{q}}[\mathbf{l},\mathbf{z}](\tau^-_{\mathbf{k}}[\mathbf{z}]) &= 0, \qquad \textit{unless}\ \ \mathbf{k} \geq \mathbf{q}\,. \end{aligned}$$

Moreover, if $\mathbf{z}$ is generic, then $w'_{\mathbf{q}}[\mathbf{l},\mathbf{z}](\tau^-_{\mathbf{q}}[\mathbf{z}]) \neq 0$.

The proof is straightforward.

Lemma 35. [TV4] *Let $k \in \mathbb{Z}_{\geq}0$, and let $\mathbf{p}, \mathbf{q} \in \mathcal{Z}_k[\mathbf{l}]$. Then, we have*

$$\langle w_{\mathbf{p}}, w'_{\mathbf{q}} \rangle_{\mathbf{l},\mathbf{z}} = \delta_{\mathbf{p},\mathbf{q}} \frac{p_1! l_1!}{(l_1 - p_1)!} \frac{p_2! l_2!}{(l_2 - p_2)!} \frac{p_3! l_3!}{(l_3 - p_3)!}\,. \tag{B.7}$$

Proof. By Lemma 34 the summation in the residue formula (B.6) reduces to $\mathbf{k}$ such that $\mathbf{q} \leq \mathbf{k} \leq \mathbf{p}$, that is

$$\langle w_{\mathbf{p}}, w'_{\mathbf{q}} \rangle_{\mathbf{l},\mathbf{z}} = \sum_{\mathbf{q} \leq \mathbf{k} \leq \mathbf{p}} w'_{\mathbf{q}}[\mathbf{l};\mathbf{z}](\tau^-_{\mathbf{k}}[\mathbf{z}]) w_{\mathbf{p}}[\mathbf{l};\mathbf{z}](\tau^-_{\mathbf{k}}[\mathbf{z}]) \mathrm{Res}_{\tau^-_{\mathbf{k}}[\mathbf{z}]}(\Omega_k[\mathbf{l};\mathbf{z}])\,.$$

It follows immediately that $\langle w_{\mathbf{p}}, w'_{\mathbf{q}} \rangle_{\mathbf{l},\mathbf{z}} = 0$ unless $\mathbf{q} \leq \mathbf{p}$.

Similarly, one can check that

$$\begin{aligned} w_{\mathbf{p}}(\tau^+_{\mathbf{k}}[\mathbf{z}-\mathbf{l}]) &= 0, \qquad \text{unless}\ \ \mathbf{k} \geq \mathbf{p}\,, \\ w'_{\mathbf{q}}(\tau^+_{\mathbf{k}}[\mathbf{z}-\mathbf{l}]) &= 0, \qquad \text{unless}\ \ \mathbf{k} \leq \mathbf{q}\,, \end{aligned}$$

Thus, it follows that

$$\langle w_{\mathbf{p}}, w'_{\mathbf{q}}\rangle_{\mathbf{l},\mathbf{z}} = \sum_{\mathbf{p}\le\mathbf{k}\le\mathbf{q}} w_{\mathbf{p}}[\mathbf{l};\mathbf{z}](\tau^+_{\mathbf{k}}[\mathbf{z}-\mathbf{l}])w'_{\mathbf{q}}[\mathbf{l};\mathbf{z}](\tau^+_{\mathbf{k}}[\mathbf{z}-\mathbf{l}])\left(\operatorname{Res}_{\tau^+_{\mathbf{k}}[\mathbf{z}-\mathbf{l}]}\Omega_k[\mathbf{l},\mathbf{z}]\right).$$

and therefore, $\langle w_{\mathbf{p}}, w'_{\mathbf{q}}\rangle_{\mathbf{l},\mathbf{z}} = 0$ unless $\mathbf{p}\le\mathbf{q}$.

Hence, $\langle w_{\mathbf{p}}[\mathbf{l},\mathbf{z}], w'_{\mathbf{q}}[\mathbf{l},\mathbf{z}]\rangle_{\mathbf{l},\mathbf{z}}$ is equal to zero unless $\mathbf{p}=\mathbf{q}$. In the last case there is only one term in the residue summation formula, corresponding to $\mathbf{k}=\mathbf{p}=\mathbf{q}$, and the statement follows from the direct computation of the only remaining residue at $\tau_{\mathbf{k}}[\mathbf{z}]$. □

Finally, we have all ingredients necessary to describe explicitly the map $\mathcal{I}[\mathbf{l}]$ and to prove its properties.

Proof of Proposition 14. Lemma 34 implies that the dual weight functions $\{w'_{\mathbf{k}}[\mathbf{l};\mathbf{z}]\}$ are linearly independent as elements of $\mathcal{H}_k$, which yields the injectivity of the map $\phi[\mathbf{l}]$. Using the explicit combinatorial formulas for $w'_{\mathbf{k}}[\mathbf{l};\mathbf{z}]$, one can check that the admissibility conditions (6.2) are satisfied provided that $\mathbf{k}\in\mathcal{Z}_k[\mathbf{l}]$. Hence, $\operatorname{Im}\phi[\mathbf{l}]\otimes_{\mathbb{C}[\mathbf{z}]}\mathbb{C}(\mathbf{z})\subset\mathcal{H}_k[\mathbf{l}]\otimes_{\mathbb{C}[\mathbf{z}]}\mathbb{C}(\mathbf{z})$. To show that these spaces coincide, it is enough to check that for generic $\mathbf{z}$ one has $\dim\mathcal{H}_k[\mathbf{l};\mathbf{z}]\le\dim\operatorname{Im}\phi[\mathbf{l};\mathbf{z}]=\#\mathcal{Z}_k[\mathbf{l}]$.

Consider the maps $\varepsilon_{\mathbf{p}}[\mathbf{z}]:\mathcal{H}'_k\to\mathbb{C}$,

$$\varepsilon_{\mathbf{p}}[\mathbf{z}]:\psi\mapsto\psi(\tau_{\mathbf{p}}[\mathbf{z}]).$$

Lemma 34 implies that for generic $\mathbf{z}$ these maps are linearly independent. It is clear that for any $\psi\in\mathcal{H}[\mathbf{l};\mathbf{z}]$, $\mathbf{p}\notin\mathcal{Z}_k[\mathbf{l}]$ one has $\varepsilon_{\mathbf{p}}[\mathbf{z}](\psi)=0$, which gives $\dim\mathcal{H}_k[\mathbf{l};\mathbf{z}]\le\#\mathcal{Z}_k[\mathbf{l}]$, since $\dim\mathcal{H}'_k=\#\mathcal{Z}_k$.

Now, consider the map

$$\mathcal{I}[\mathbf{l};\mathbf{z}]:\mathcal{H}[\mathbf{l};\mathbf{z}]\to V_{(l_1,0)}\otimes V_{(l_2,0)}\otimes V_{(l_3,0)},$$

$$\mathcal{I}[\mathbf{l};\mathbf{z}]\varphi=\sum_{k\in\mathbb{Z}_{\ge0}}\sum_{\mathbf{p}\in\mathcal{Z}_k[\mathbf{l}]}\langle w_{\mathbf{p}},\varphi\rangle_{\mathbf{l},\mathbf{z}}E_{21}^{p_1}\mathbf{1}_{l_1}\otimes E_{21}^{p_2}\mathbf{1}_{l_2}\otimes E_{21}^{p_3}\mathbf{1}_{l_3}.$$

Formulas (6.1) and (B.7) imply (6.4).

According to Lemma 33, we have

$$\langle w_{\mathbf{p}},\varphi\rangle_{\mathbf{l},\mathbf{z}}=\left(\langle w_{\mathbf{p}},\varphi^-\rangle_{\mathbf{u},\mathbf{z}}\right)|_{\mathbf{u}=\mathbf{l}}\left(\langle w_{\mathbf{p}},\varphi^+\rangle_{\mathbf{u},\mathbf{z}}\right)|_{\mathbf{u}=\mathbf{l}},$$

and Lemma 32 implies that $\langle w_{\mathbf{p}},\varphi\rangle_{\mathbf{l},\mathbf{z}}$ may have poles only at the hyperplanes

$$z_i-u_i=z_j-r,\qquad 1\le i<j\le 3,\quad r=0,1,\dots,\min(l_i,l_j)-1.$$

This proves the last part of Proposition 14. □

Appendix C.

In this section we prove the existence of the distinguished polynomial basis $\{\varsigma_i[\mathbf{l}]\}$ of $\mathcal{H}_k[\mathbf{l}]^{\mathrm{sing}}$, see Lemma 19.

Proof of Lemma 19. The subspace $\mathcal{H}_k[\mathbf{l}]^{\mathrm{sing}}$ consists of polynomials of degree at most 1 in each t_j, and therefore any function $\varsigma(t_1,\dots,t_k)\in\mathcal{H}_k[\mathbf{l}]^{\mathrm{sing}}$ has the form

$$\varsigma(t_1,\dots,t_k)=\sum_{r=0}^{k}X_r(z_1,z_2,z_3)\,\sigma_r(t_1,\dots,t_k)$$

where $X_0, \dots X_k$ are some coefficients from $\mathbb{C}[\mathbf{z}]$, and

$$\sigma_r(t_1, \dots, t_k) = \sum_{1 \le i_1 < \dots < i_r \le k} t_{i_1} \dots t_{i_r}, \qquad r = 0, 1, 2, \dots;$$

in particular, $\sigma_0(t_1, \dots, t_k) \equiv 1$, and $\sigma_r(t_1, \dots, t_k) \equiv 0$ if $r > k$.

Using the combinatorial identity

$$\sigma_r(z, \dots, z-l, t_{l+2}, \dots, t_k) = \sum_{s=0}^{r} \sigma_{r-s}(z, \dots, z-l)\sigma_s(t_{l+2}, \dots, t_k)\,,$$

we see that admissibility conditions (6.2) for the function $\varsigma(t_1, \dots, t_k)$ are equivalent to the following system of equations for the coefficients $X_0, \dots X_k$:

$$\sum_{r=s}^{k} X_r\, \sigma_{r-s}(z_i, \dots, z_i - l_i) = 0, \qquad s = 0, \dots, k - l_i - 1, \quad i = 1, 2, 3. \tag{C.1}$$

Let $d = \dim \big(V_{(l_1,0)} \otimes V_{(l_2,0)} \otimes V_{(l_3,0)}\big)^{\mathrm{sing}} [(l_1 + l_2 + l_3 - k, k)]$. We will show that system (C.1) determines $X_0, \dots, X_{k-d}$ as linear combinations of free variables $X_{k-d+1}, \dots, X_k$ with coefficients, polynomial in z_1, z_2, z_3. Then we construct $\varsigma_s[\mathbf{l}]$, $\quad s = 1, \dots, d$, by taking

$$X_{k-s+1} = 1, \qquad X_{k-r+1} = 0 \quad \text{for} \;\; r \ne s, \quad 1 \le r \le d\,,$$

and computing $X_0, \dots, X_{k-d}$ from equations (C.1). The polynomials $\varsigma_s[\mathbf{l}]$ obeys invariance (7.2) by construction.

Consider the $(k - d + 1) \times (k + 1)$ matrix of system (C.1). It suffices to prove that its $(k - d + 1) \times (k - d + 1)$ minor, corresponding to the variables $X_0, \dots, X_{k-d}$, is nonzero, and divides all other $(k - d + 1) \times (k - d + 1)$ minors. Before giving a general proof of this statement, we consider an example.

Example: $l_1 = 1$, $l_2 = 2$, $k = 3$, $l_3 > 3$. Then $d = \dim \big(V_{(1,0)} \otimes V_{(2,0)} \otimes V_{(l_3,0)}\big)^{\mathrm{sing}} [(l_3, 3)]$ $= 1$, and the matrix of system of (C.1) is equal to

$$\begin{pmatrix} 1 & z_1 + (z_1 - 1) & z_1(z_1 - 1) & 0 \\ 0 & 1 & z_1 + (z_1 - 1) & z_1(z_1 - 1) \\ 1 & z_2 + (z_2 - 1) + (z_2 - 2) & z_2(z_2 - 1) + z_2(z_2 - 2) + (z_2 - 1)(z_2 - 2) & z_2(z_2 - 1)(z_2 - 2) \end{pmatrix}.$$

The 3×3 minors of this matrix are given by

$$M_{012} = 3(z_1 - z_2)(z_1 - z_2 + 1)\,,$$

$$M_{013} = (z_1 - z_2)(z_1 - z_2 + 1)(2z_1 + z_2 - 2)\,,$$

$$M_{023} = (z_1 - z_2)(z_1 - z_2 + 1)(z_1^2 + 2z_1 z_2 - 3z_1 - z_2 + 2)\,,$$

$$M_{123} = (z_1 - z_2)(z_1 - z_2 + 1)(3z_1^2 z_2 - 3z_1^2 - 3z_1 z_2 + 5z_1 + z_2 - 2)\,.$$

The leftmost minor M_{012} is nonzero and divides all other minors. The function $\varsigma(t_1, t_2, t_3)$, spanning the space $\mathcal{H}[(1, 2, l_3), (z_1, z_2, z_3)]^{\mathrm{sing}}[(l_3, 3)]$, is given by

$$\varsigma(t_1, t_2, t_3) = t_1 t_2 t_3 - \frac{1}{M_{012}} \big(M_{013}(t_1 t_2 + t_2 t_3 + t_1 t_3) - M_{023}(t_1 + t_2 + t_3) + M_{123}\big)\,.$$

We now return to the case of arbitrary $\mathbf{l}$. Without loss of generality, we may assume that $l_1 \le l_2 \le l_3$. To shorten notation, we denote

$$\sigma_r^{(i)} = \sigma_r(z_i, z_i - 1, \dots, z_i - l_i), \qquad r = 0, \dots, k, \quad i = 1, 2, 3.$$

Case 1: $k \le l_1$**.** Then, there are no admissibility conditions, $d = k + 1$, and all the variables $X_0, \dots X_k$ are free.

Case 2: $l_1 < k \le l_2$**.** Then $d = l_1 + 1$, and the matrix of system (C.1) is equal to

$$\begin{pmatrix} \sigma_0^{(1)} & \sigma_1^{(1)} & \sigma_2^{(1)} & \dots & \sigma_{k-l_1-1}^{(1)} & \sigma_{k-l_1}^{(1)} & \sigma_{k-l_1+1}^{(1)} & \dots & \sigma_k^{(1)} \\ 0 & \sigma_0^{(1)} & \sigma_1^{(1)} & \dots & \sigma_{k-l_1-2}^{(1)} & \sigma_{k-l_1-1}^{(1)} & \sigma_{k-l_1}^{(1)} & \dots & \sigma_{k-1}^{(1)} \\ 0 & 0 & \sigma_0^{(1)} & \dots & \sigma_{k-l_1-3}^{(1)} & \sigma_{k-l_1-2}^{(1)} & \sigma_{k-l_1-1}^{(1)} & \dots & \sigma_{k-2}^{(1)} \\ \vdots & \vdots & \vdots & \ddots & \dots & \dots & \dots & \ddots & \dots \\ 0 & 0 & 0 & \dots & \sigma_0^{(1)} & \sigma_1^{(1)} & \sigma_2^{(1)} & \dots & \sigma_{l_1+1}^{(1)} \end{pmatrix}$$

Note that some of the terms in the upper right corner can also vanish, because $\sigma_{l_1+2}^{(1)} = \dots = \sigma_k^{(1)} = 0$.

Since $\sigma_0^{(1)} \equiv 1$, the leftmost $(k - d + 1) \times (k - d + 1)$ minor is identically equal to 1. Therefore $X_{k-d+1}, \dots, X_k$ are free variables, and $X_0, \dots, X_{k-d}$ are their linear combinations with coefficients, polynomial in z_1, z_2, z_3.

Case 3: $l_2 < k \le l_3$**.** If $k \ge l_1 + l_2$, then $d = 0$, and there is nothing to check. Assume that $k < l_1 + l_2$, which gives $d = l_1 + l_2 - k$. The matrix of system (C.1) is equal to

$$\begin{pmatrix} \sigma_0^{(1)} & \sigma_1^{(1)} & \dots & \dots & \dots & \sigma_{k-l_1-1}^{(1)} & \dots & \sigma_k^{(1)} \\ 0 & \sigma_0^{(1)} & \dots & \dots & \dots & \sigma_{k-l_1-2}^{(1)} & \dots & \sigma_{k-1}^{(1)} \\ \vdots & \vdots & \vdots & \ddots & \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & \dots & \dots & \sigma_0^{(1)} & \dots & \sigma_{l_1+1}^{(1)} \\ \sigma_0^{(2)} & \sigma_1^{(2)} & \dots & \sigma_{k-l_2-1}^{(2)} & \dots & \dots & \dots & \sigma_k^{(2)} \\ 0 & \sigma_0^{(2)} & \dots & \sigma_{k-l_2-2}^{(2)} & \dots & \dots & \dots & \sigma_{k-1}^{(2)} \\ \vdots & \vdots & \ddots & \dots & \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & \sigma_0^{(2)} & \dots & \dots & \dots & \sigma_{l_2+1}^{(2)} \end{pmatrix} \tag{C.2}$$

We make two claims now. First, every $(k - d + 1) \times (k - d + 1)$ minor of this matrix is a polynomial in z_1, z_2, divisible by

$$D(z_1, z_2) = \prod_{j_1=1}^{k-l_1} \prod_{j_2=1}^{k-l_2} (z_1 - z_2 + j_1 - j_2)\,.$$

Second, the leftmost $(k - d + 1) \times (k - d + 1)$ minor of matrix (C.2) is a nonzero polynomial in z_1, z_2 of homogeneous degree $(k - l_1)(k - l_2)$.

These two claims together imply the desired statement. Indeed, since the homogeneous degree of the polynomial $D(z_1, z_2)$ is equal to $(k-l_1)(k-l_2)$, the leftmost minor is proportional to $D(z_1, z_2)$ with a nonzero coefficient. Therefore, every $(k - d + 1) \times (k - d + 1)$ minor is divisible by the leftmost $(k - d + 1) \times (k - d + 1)$ minor, which means that $X_0, \dots, X_{k-d}$ can be expressed as linear combinations of the free variables $X_{k-d+1}, \dots, X_k$ with coefficients, polynomially depending on z_1, z_2, z_3.

We first prove the second claim. Note that $\sigma_r^{(i)}$ is a polynomial of z_i with the highest degree term $\binom{l_i+1}{r} z_i^r$. Therefore, the homogeneous degree of the leftmost $(k-d+1)\times(k-d+1)$ minor does not exceed $(k-l_1)(k-l_2)$, while the sum of terms of degree $(k-l_1)(k-l_2)$ is given by the determinant

$$\det\begin{pmatrix} 1 & \binom{l_1+1}{1} z_1 & \dots & \dots & \dots & \binom{l_1+1}{k-l_1-1} z_1^{k-l_1-1} & \dots & \binom{l_1+1}{k-d-1} z_1^{k-d-1} \\ 0 & 1 & \dots & \dots & \dots & \binom{l_1+1}{k-l_1-2} z_1^{k-l_1-2} & \dots & \binom{l_1+1}{k-d-2} z_1^{k-d-2} \\ \vdots & \vdots & \vdots & \ddots & \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & \dots & \dots & 1 & \dots & \binom{l_1+1}{k-l_2} z_1^{k-l_2} \\ 1 & \binom{l_2+1}{1} z_2 & \dots & \binom{l_2+1}{k-l_2-1} z_2^{k-l_2-1} & \dots & \dots & \dots & \binom{l_2+1}{k-d-1} z_2^{k-d-1} \\ 0 & 1 & \dots & \binom{l_2+1}{k-l_2-2} z_2^{k-l_2-2} & \dots & \dots & \dots & \binom{l_2+1}{k-d-2} z_2^{k-d-2} \\ \vdots & \vdots & \ddots & \dots & \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & 1 & \dots & \dots & \dots & \binom{l_2+1}{k-l_1} z_2^{k-l_1} \end{pmatrix}$$

The monomial $z_1^{(k-l_1)(k-l_2)}$ enters this determinant with a nonzero coefficient

$$\det\begin{pmatrix} \binom{l_1+1}{k-l_2} & \binom{l_1+1}{k-l_2+1} & \dots & \binom{l_1+1}{2k-l_1-l_2-1} \\ \binom{l_1+1}{k-l_2-1} & \binom{l_1+1}{k-l_2} & \dots & \binom{l_1+1}{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{l_1+1}{l_1-l_2+1} & \binom{l_1+1}{l_1-l_2+2} & \dots & \binom{l_1+1}{k-l_2} \end{pmatrix} = \prod_{i=1}^{k-l_1}\prod_{j=1}^{k-l_2}\prod_{s=1}^{l_1+l_2-k+1} \frac{i+j+s-1}{i+j+s-2}\,.$$

The last equality follows from the general formula, established in [K]:

$$\det\begin{pmatrix} \binom{a+b}{a} & \binom{a+b}{a+1} & \dots & \binom{a+b}{a+c} \\ \binom{a+b}{a-1} & \binom{a+b}{a} & \dots & \binom{a+b}{a+c-1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{a+b}{a-c} & \binom{a+b}{a-c+1} & \dots & \binom{a+b}{a} \end{pmatrix} = \prod_{i=1}^{c+1}\prod_{j=1}^{a}\prod_{s=1}^{b} \frac{i+j+s-1}{i+j+s-2}\,,$$

by taking $a=k-l_2$, $b=l_1+l_2-k+1$, $c=k-l_1-1$.

Finally, we prove the first claim. Recall that we assume $l_1 \le l_2 < k < l_1+l_2$.

Take an integer j such that $-l_2 \le j \le l_1$ and let $z_1-z_2-j=0$. Then the sets $\{z_1, z_1-1, \dots, z_1-l_1\}$ and $\{z_2, z_2-1, \dots, z_2-l_2\}$ have $p(j)+1$ common points $\{\zeta, \zeta-1, \dots, \zeta-p(j)\}$, where $\zeta=z_1$ for $j<0$, $\zeta=z_2$ for $j\ge 0$, and

$$p(j)=\begin{cases} l_2+j, & -l_2 \le j < l_1-l_2\,, \\ l_1, & l_1-l_2 \le j < 0\,, \\ l_1-j, & 0 \le j \le l_1\,, \end{cases}$$

Consider the system of linear equations on variables $X_0, \dots, X_k$, corresponding to the condition

$$\varsigma(\zeta, \zeta-1, \dots, \zeta-p(j), t_{p(j)+2}, \dots, t_k) \equiv 0\,. \tag{C.3}$$

Let $L \subset \mathbb{C}^k$ denote the linear span of the rows of the corresponding matrix. Since the rows are linearly independent, we have $\dim L = k - p(j)$.

Obviously, if the function $\varsigma(t_1, \ldots, t_k)$ obeys condition (C.3), then

$$\varsigma(z_1, z_1 - 1, \ldots, z_1 - l_1, t_{l_1+1}, \ldots, t_k) \equiv 0\,,$$
$$\varsigma(z_2, z_2 - 1, \ldots, z_2 - l_2, t_{l_2+1}, \ldots, t_k) \equiv 0\,.$$

This means that the linear span of the rows of matrix (C.2) is contained in L. In particular, the rank of matrix (C.2) does not exceed $\dim L$. Therefore, if $l_1 - k < j < k - l_2$, then the rank of matrix (C.2) is less than its number of rows $2k - l_1 - l_2$ by at least $k + p(j) - l_1 - l_2$, which is positive.

The above consideration shows that every $(k-d+1) \times (k-d+1)$ minor of matrix (C.2) is divisible by the product

$$\prod_{j=l_1-k+1}^{k-l_2-1} (z_1 - z_2 - j)^{k+p(j)-l_1-l_2}\,.$$

It is easy to show that this product coincides with $D(z_1, z_2)$.

Case 4: $l_1 + l_2 < k \le l_3$**.** In this case $\dim \mathcal{H}_k(\mathbf{l}; \mathbf{z})^{\mathrm{sing}} = 0$, and there is nothing to check.

Case 5: $k > l_3$**.** There are three types of admissibility conditions in this case, related to (z_1, l_1), (z_2, l_2) and (z_3, l_3). The consideration, similar to Case 3, shows that the leftmost $(k-d+1) \times (k-d+1)$ minor of the corresponding matrix is nonzero and divides every $(k-d+1) \times (k-d+1)$ minor. This implies the required statement. □

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