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Calculation of Riemann's zeta function via interpolating determinants

by

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Abstract

Using intensive computer calculations, the author empirically discovered unusual methods for calculating high-precision approximations to the non-trivial zeroes of Riemann's zeta function, its values and values of its derivative on the whole complex plane. So far no theoretical explanation to these phenomena is known.

This paper is a slightly extended presentation of a talk given by the author on March 15, 2013 at the *Number Theory Lunch Seminar* in the Max Planck Institute for Mathematics at Bonn; more information related to this talk and the whole ongoing research can be found at `http://logic.pdmi.ras.ru/~yumat/personaljournal/artlessmethod`.

1 General settings

1.1 The Questions

Suppose that we have some function F from any set into some commutative ring R and have found $N-1$ distinct zeroes of F:

$$F(x_1) = \cdots = F(x_{N-1}) = 0. \tag{1}$$

Question 1. Knowing only $x_1, \ldots, x_{N-1}$, how could we construct some function $\tilde{F}_N(x)$ defined on the domain of F with values in the same ring R such that

$$\tilde{F}(x_1) = \cdots = \tilde{F}(x_{N-1}) = 0? \tag{2}$$

Question 2. What are other zeroes of $\tilde{F}_N(x)$? In particular, could some of them be close to some zeroes of $F(x)$?

Question 3. Could $\tilde{F}_N$ be used for calculating the values of F at other points?

1.2 Interpolating determinants

One way to answer Question 1 above is as follows. Let us select any N functions

$$f_1(x), \ldots, f_N(x) \tag{3}$$

defined on the domain of F with values in the same ring R and take for the role of the function $\tilde{F}_N(x)$ the *interpolating determinant*

$$\tilde{\Delta}_N(x) = \begin{vmatrix} f_1(x_1) & \dots & f_1(x_{N-1}) & f_1(x) \\ \vdots & \ddots & \vdots & \vdots \\ f_N(x_1) & \dots & f_N(x_{N-1}) & f_N(x) \end{vmatrix}. \tag{4}$$

Clearly,

$$\tilde{\Delta}_N(x_1) = \cdots = \tilde{\Delta}_N(x_{N-1}) = 0. \tag{5}$$

1.2.1 An example

With such a definition of $\tilde{F}_N(x)$ the possibility to give answers to Questions 2 and 3 above depends heavily on our choice of functions (3). If we define $f_n(x) = x^{n-1}$, then the function $\tilde{F}_N(x)$ will be simply the well known interpolating polynomial

$$C \prod_{k=1}^{N-1} (x - x_k), \tag{6}$$

evidently having no further zeroes.

1.2.2 Case when it works

If

$$F(x) = f_1(x) + \cdots + f_N(x) \tag{7}$$

then the determinant (4) vanishes at *every* zero of $F(x)$ because in such a case summing up all the rows results in the row containing only zeroes.

1.3 Generalization

The main interest for us are situations where instead of (3) we have an infinite sequence of functions

$$f_1(x), \dots, f_N(x), f_{N+1}(x), \dots \tag{8}$$

and the finite sum from (7) is replaced by the infinite sum:

$$F(x) \sim f_1(x) + \cdots + f_N(x) + f_{N+1}(x) + \dots. \tag{9}$$

In the case when $\sim$ is the equality, i.e., when the series converges, we might hope that the determinants (4) will still have extra zeroes close to certain zeroes of $F(x)$ because each column would sum up to a small value approaching zero with the growth of N.

However, most interesting (and surprising) are the cases when the series in (9) diverges so the ideograph $\sim$ has only some symbolic meaning. Two choices for $F(x), x_1, \dots, x_{N-1}$, and $f_1(x), f_2(x), \dots$ will now be considered.

2 First example: Riemann's zeta function

2.1 Definitions

Riemann's zeta function can be defined by Dirichlet series

$$\zeta(s) = 1^{-s} + 2^{-s} + 3^{-s} + \dots \tag{10}$$

The series converges only for $\mathrm{Re}(s) > 1$ but the function defined by it in that half-plane can be analytically extended to the whole complex plane except the point $s = 1$ which is its only pole.

Negative even integers are known as *trivial zeroes* of the zeta function; they won't be used for constructing the determinants (4).

All others, the *non-trivial* zeroes, come in conjugate pairs:

$$\dots = \zeta(\overline{\rho_3}) = \zeta(\overline{\rho_2}) = \zeta(\overline{\rho_1}) = 0 = \zeta(\rho_1) = \zeta(\rho_2) = \zeta(\rho_3) = \dots \tag{11}$$

Assuming that all zeroes satisfy *Riemann's Hypothesis* and are simple, we write

$$\rho_n = \frac{1}{2} + \mathrm{i}\gamma_n \tag{12}$$

where

$$0 < \gamma_1 < \gamma_2 < \gamma_3 \dots \tag{13}$$

We will always select an odd value for N, $N = 2M + 1$, and use $2M$ non-trivial zeroes with the smallest (in absolute value) imaginary parts:

$$\tilde{\Delta}_N(s) = \begin{vmatrix} 1 & 1 & \dots & 1 & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ n^{-\overline{\rho_1}} & n^{-\rho_1} & \dots & n^{-\overline{\rho_M}} & n^{-\rho_M} & n^{-s} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ N^{-\overline{\rho_1}} & N^{-\rho_1} & \dots & N^{-\overline{\rho_M}} & N^{-\rho_M} & N^{-s} \end{vmatrix}. \tag{14}$$

2.2 Numerical examples

It turns out that the determinants (14) indeed have zeroes very close to the zeroes of the zeta function $\rho_{M+1}, \rho_{M+2}, \dots$ not used in (14).

2.2.1 Case $N = 17$

We have:

$$\begin{aligned}
0 &= \Delta_{17}(\rho_9 - 4.396\dots\cdot 10^{-3} + 5.711\dots\cdot 10^{-3}\mathrm{i}) && (15)\\
0 &= \Delta_{17}(\rho_{10} - 1.141\dots\cdot 10^{-2} - 3.345\dots\cdot 10^{-3}\mathrm{i}) && (16)\\
0 &= \Delta_{17}(\rho_{11} - 1.498\dots\cdot 10^{-2} + 1.762\dots\cdot 10^{-3}\mathrm{i}) && (17)\\
0 &= \Delta_{17}(\rho_{12} - 1.158\dots\cdot 10^{-2} + 2.264\dots\cdot 10^{-2}\mathrm{i}) && (18)\\
0 &= \Delta_{17}(\rho_{13} - 1.317\dots\cdot 10^{-2} + 7.545\dots\cdot 10^{-2}\mathrm{i}) && (19)\\
0 &= \Delta_{17}(\rho_{14} - 7.400\dots\cdot 10^{-2} - 5.559\dots\cdot 10^{-4}\mathrm{i}) && (20)\\
0 &= \Delta_{17}(\rho_{15} + 4.486\dots\cdot 10^{-2} + 8.379\dots\cdot 10^{-2}\mathrm{i}) && (21)
\end{aligned}$$

2.2.2 Case $N = 101$

For larger N the approximation is better and we can approximate more zeroes of the zeta function:

$$
\begin{aligned}
0 &= \Delta_{101}(\rho_{51} + 3.469\ldots\cdot 10^{-15} - 1.283\ldots\cdot 10^{-15}\mathrm{i}) && (22)\\
0 &= \Delta_{101}(\rho_{52} + 1.472\ldots\cdot 10^{-14} - 4.170\ldots\cdot 10^{-15}\mathrm{i}) && (23)\\
0 &= \Delta_{101}(\rho_{53} - 3.949\ldots\cdot 10^{-13} + 1.223\ldots\cdot 10^{-14}\mathrm{i}) && (24)\\
0 &= \Delta_{101}(\rho_{54} - 4.684\ldots\cdot 10^{-13} - 9.387\ldots\cdot 10^{-13}\mathrm{i}) && (25)\\
0 &= \Delta_{101}(\rho_{55} - 5.303\ldots\cdot 10^{-12} + 2.129\ldots\cdot 10^{-12}\mathrm{i}) && (26)\\
0 &= \Delta_{101}(\rho_{56} + 2.104\ldots\cdot 10^{-11} + 4.691\ldots\cdot 10^{-11}\mathrm{i}) && (27)\\
0 &= \Delta_{101}(\rho_{57} + 1.054\ldots\cdot 10^{-10} + 1.430\ldots\cdot 10^{-10}\mathrm{i}) && (28)\\
0 &= \Delta_{101}(\rho_{58} + 1.081\ldots\cdot 10^{-10} + 2.883\ldots\cdot 10^{-10}\mathrm{i}) && (29)\\
0 &= \Delta_{101}(\rho_{59} + 6.849\ldots\cdot 10^{-10} - 9.371\ldots\cdot 10^{-10}\mathrm{i}) && (30)\\
0 &= \Delta_{101}(\rho_{60} + 5.453\ldots\cdot 10^{-9} - 8.730\ldots\cdot 10^{-9}\mathrm{i}) && (31)\\
0 &= \Delta_{101}(\rho_{61} - 5.038\ldots\cdot 10^{-9} - 9.649\ldots\cdot 10^{-9}\mathrm{i}) && (32)\\
0 &= \Delta_{101}(\rho_{62} - 2.178\ldots\cdot 10^{-8} - 1.230\ldots\cdot 10^{-8}\mathrm{i}) && (33)\\
0 &= \Delta_{101}(\rho_{63} - 8.237\ldots\cdot 10^{-8} + 6.583\ldots\cdot 10^{-8}\mathrm{i}) && (34)\\
0 &= \Delta_{101}(\rho_{64} - 1.142\ldots\cdot 10^{-7} - 8.478\ldots\cdot 10^{-8}\mathrm{i}) && (35)\\
0 &= \Delta_{101}(\rho_{65} + 1.023\ldots\cdot 10^{-7} + 5.621\ldots\cdot 10^{-7}\mathrm{i}) && (36)\\
0 &= \Delta_{101}(\rho_{66} - 6.315\ldots\cdot 10^{-8} + 7.740\ldots\cdot 10^{-7}\mathrm{i}) && (37)\\
0 &= \Delta_{101}(\rho_{67} + 5.274\ldots\cdot 10^{-7} + 8.361\ldots\cdot 10^{-7}\mathrm{i}) && (38)\\
0 &= \Delta_{101}(\rho_{68} + 2.072\ldots\cdot 10^{-6} - 4.269\ldots\cdot 10^{-7}\mathrm{i}) && (39)\\
0 &= \Delta_{101}(\rho_{69} + 4.560\ldots\cdot 10^{-6} - 1.954\ldots\cdot 10^{-6}\mathrm{i}) && (40)\\
0 &= \Delta_{101}(\rho_{70} + 9.541\ldots\cdot 10^{-6} - 2.034\ldots\cdot 10^{-6}\mathrm{i}) && (41)\\
0 &= \Delta_{101}(\rho_{71} - 2.469\ldots\cdot 10^{-5} - 9.102\ldots\cdot 10^{-6}\mathrm{i}) && (42)\\
0 &= \Delta_{101}(\rho_{72} + 1.104\ldots\cdot 10^{-5} - 3.538\ldots\cdot 10^{-5}\mathrm{i}) && (43)\\
0 &= \Delta_{101}(\rho_{73} - 5.557\ldots\cdot 10^{-6} - 2.750\ldots\cdot 10^{-5}\mathrm{i}) && (44)\\
0 &= \Delta_{101}(\rho_{74} - 6.747\ldots\cdot 10^{-5} + 8.847\ldots\cdot 10^{-6}\mathrm{i}) && (45)\\
0 &= \Delta_{101}(\rho_{75} - 1.254\ldots\cdot 10^{-4} + 7.809\ldots\cdot 10^{-5}\mathrm{i}) && (46)\\
0 &= \Delta_{101}(\rho_{76} - 1.437\ldots\cdot 10^{-4} - 4.558\ldots\cdot 10^{-5}\mathrm{i}) && (47)\\
0 &= \Delta_{101}(\rho_{77} - 2.782\ldots\cdot 10^{-5} + 1.655\ldots\cdot 10^{-4}\mathrm{i}) && (48)\\
0 &= \Delta_{101}(\rho_{78} + 9.818\ldots\cdot 10^{-5} + 3.774\ldots\cdot 10^{-4}\mathrm{i}) && (49)\\
0 &= \Delta_{101}(\rho_{79} - 2.381\ldots\cdot 10^{-4} + 4.799\ldots\cdot 10^{-4}\mathrm{i}) && (50)\\
0 &= \Delta_{101}(\rho_{80} + 6.954\ldots\cdot 10^{-4} + 2.673\ldots\cdot 10^{-4}\mathrm{i}) && (51)
\end{aligned}
$$

2.2.3 Case $N = 3001$

$$
\begin{aligned}
0 &= \Delta_{3001}(\rho_{1501} - 4.005\ldots\cdot 10^{-1113} + 1.113\ldots\cdot 10^{-1113}\mathrm{i}) && (52)\\
0 &= \Delta_{3001}(\rho_{1601} - 5.155\ldots\cdot 10^{-952} - 3.960\ldots\cdot 10^{-952}\mathrm{i}) && (53)\\
0 &= \Delta_{3001}(\rho_{1701} - 7.652\ldots\cdot 10^{-849} + 1.788\ldots\cdot 10^{-848}\mathrm{i}) && (54)\\
0 &= \Delta_{3001}(\rho_{1801} + 1.966\ldots\cdot 10^{-766} + 3.803\ldots\cdot 10^{-766}\mathrm{i}) && (55)\\
0 &= \Delta_{3001}(\rho_{1901} + 1.044\ldots\cdot 10^{-696} - 4.253\ldots\cdot 10^{-696}\mathrm{i}) && (56)\\
0 &= \Delta_{3001}(\rho_{2001} + 1.021\ldots\cdot 10^{-636} - 8.184\ldots\cdot 10^{-636}\mathrm{i}) && (57)\\
0 &= \Delta_{3001}(\rho_{2101} - 5.402\ldots\cdot 10^{-582} + 8.070\ldots\cdot 10^{-583}\mathrm{i}) && (58)\\
0 &= \Delta_{3001}(\rho_{2201} + 9.843\ldots\cdot 10^{-535} + 5.389\ldots\cdot 10^{-535}\mathrm{i}) && (59)\\
0 &= \Delta_{3001}(\rho_{2301} - 7.327\ldots\cdot 10^{-492} - 5.590\ldots\cdot 10^{-491}\mathrm{i}) && (60)\\
0 &= \Delta_{3001}(\rho_{2401} + 6.471\ldots\cdot 10^{-452} + 8.088\ldots\cdot 10^{-452}\mathrm{i}) && (61)\\
0 &= \Delta_{3001}(\rho_{2501} + 1.523\ldots\cdot 10^{-416} - 2.324\ldots\cdot 10^{-416}\mathrm{i}) && (62)\\
0 &= \Delta_{3001}(\rho_{2601} - 6.612\ldots\cdot 10^{-384} - 2.011\ldots\cdot 10^{-384}\mathrm{i}) && (63)\\
0 &= \Delta_{3001}(\rho_{2701} + 6.698\ldots\cdot 10^{-354} + 3.094\ldots\cdot 10^{-353}\mathrm{i}) && (64)\\
0 &= \Delta_{3001}(\rho_{2801} + 5.714\ldots\cdot 10^{-326} + 6.670\ldots\cdot 10^{-326}\mathrm{i}) && (65)\\
0 &= \Delta_{3001}(\rho_{2901} + 6.513\ldots\cdot 10^{-300} - 2.414\ldots\cdot 10^{-300}\mathrm{i}) && (66)\\
0 &= \Delta_{3001}(\rho_{3001} - 5.997\ldots\cdot 10^{-277} - 2.977\ldots\cdot 10^{-276}\mathrm{i}) && (67)\\
0 &= \Delta_{3001}(\rho_{3101} - 5.538\ldots\cdot 10^{-255} - 7.989\ldots\cdot 10^{-254}\mathrm{i}) && (68)\\
0 &= \Delta_{3001}(\rho_{3201} - 2.033\ldots\cdot 10^{-233} - 9.025\ldots\cdot 10^{-234}\mathrm{i}) && (69)\\
0 &= \Delta_{3001}(\rho_{3301} - 8.552\ldots\cdot 10^{-215} + 2.085\ldots\cdot 10^{-214}\mathrm{i}) && (70)\\
0 &= \Delta_{3001}(\rho_{3401} - 1.457\ldots\cdot 10^{-196} + 3.140\ldots\cdot 10^{-197}\mathrm{i}) && (71)\\
0 &= \Delta_{3001}(\rho_{3501} + 5.426\ldots\cdot 10^{-180} - 4.737\ldots\cdot 10^{-181}\mathrm{i}) && (72)\\
0 &= \Delta_{3001}(\rho_{3601} + 1.967\ldots\cdot 10^{-164} + 2.281\ldots\cdot 10^{-165}\mathrm{i}) && (73)\\
0 &= \Delta_{3001}(\rho_{3701} + 2.146\ldots\cdot 10^{-150} - 4.669\ldots\cdot 10^{-150}\mathrm{i}) && (74)\\
0 &= \Delta_{3001}(\rho_{3801} + 1.289\ldots\cdot 10^{-136} + 1.372\ldots\cdot 10^{-136}\mathrm{i}) && (75)\\
0 &= \Delta_{3001}(\rho_{3901} - 3.114\ldots\cdot 10^{-124} + 1.867\ldots\cdot 10^{-124}\mathrm{i}) && (76)\\
0 &= \Delta_{3001}(\rho_{4001} + 5.678\ldots\cdot 10^{-113} + 1.233\ldots\cdot 10^{-112}\mathrm{i}) && (77)\\
0 &= \Delta_{3001}(\rho_{4101} - 6.271\ldots\cdot 10^{-102} - 1.398\ldots\cdot 10^{-102}\mathrm{i}) && (78)\\
0 &= \Delta_{3001}(\rho_{4201} - 1.532\ldots\cdot 10^{-92} - 1.135\ldots\cdot 10^{-91}\mathrm{i}) && (79)\\
0 &= \Delta_{3001}(\rho_{4301} + 6.145\ldots\cdot 10^{-83} + 5.745\ldots\cdot 10^{-83}\mathrm{i}) && (80)\\
0 &= \Delta_{3001}(\rho_{4401} + 3.781\ldots\cdot 10^{-74} - 5.207\ldots\cdot 10^{-74}\mathrm{i}) && (81)\\
0 &= \Delta_{3001}(\rho_{4501} + 4.241\ldots\cdot 10^{-69} - 2.521\ldots\cdot 10^{-66}\mathrm{i}) && (82)
\end{aligned}
$$

The author does not have a full explanation for the high accuracy of the zeroes of determinants $\Delta_N(s)$ as approximations to the zeroes of $\zeta(s)$. Two heuristic hints will be presented below.

2.3 First partial explanation

Each determinant (14) can be expanded according to the last column:

$$\tilde{\Delta}_N(s) = \sum_{n=1}^{N} \tilde{\delta}_{N,n} n^{-s}. \tag{83}$$

Clearly, the numbers $\tilde{\delta}_{N,n}$ can be calculated from the initial zeroes of the zeta function as signed minors:

$$\tilde{\delta}_{N,n} = (-1)^{n+1} \times$$
$$\begin{vmatrix} 1 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (n-1)^{-\overline{\rho_1}} & (n-1)^{-\rho_1} & \dots & (n-1)^{-\overline{\rho_M}} & (n-1)^{-\rho_M} \\ (n+1)^{-\overline{\rho_1}} & (n+1)^{-\rho_1} & \dots & (n+1)^{-\overline{\rho_M}} & (n+1)^{-\rho_M} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ N^{-\overline{\rho_1}} & N^{-\rho_1} & \dots & N^{-\overline{\rho_M}} & N^{-\rho_M} \end{vmatrix}. \tag{84}$$

Since at the moment we are interested only in the zeroes of $\tilde{\Delta}_N(s)$, we can perform the normalization

$$\delta_{N,n} = \frac{\tilde{\delta}_{N,n}}{\tilde{\delta}_{N,1}} \tag{85}$$

and deal with the finite Dirichlet series

$$\Delta_N(s) = \sum_{n=1}^{N} \delta_{N,n} n^{-s} = \frac{\tilde{\Delta}_N(s)}{\tilde{\delta}_{N,1}} \tag{86}$$

having the same zeroes as $\tilde{\Delta}_N(s)$.

The coefficients $\delta_{N,n}$ turn out to be very interesting numbers encoding a lot of information about Riemann's zeta function and prime numbers.

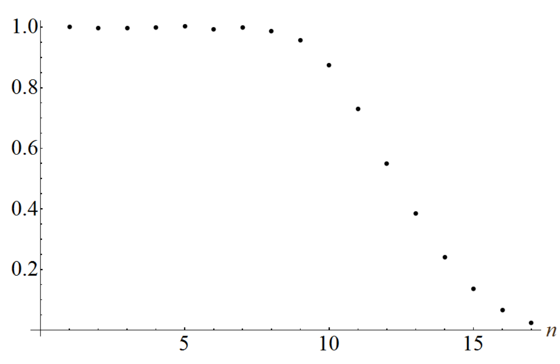

Figure 1: Coefficients $\delta_{17,n}$

Figure 1 justifies our writing

$$\Delta_{17}(s) = \sum_{n=1}^{17} \delta_{17,n} n^{-s} \rightleftharpoons \sum_{n=1}^{\infty} n^{-s} = \zeta(s) \tag{87}$$

with the ideograph $\leftrightharpoons$ having here and the sequel a very weak sense: *a few intial coefficients of the two Dirichlet series are approximately equal.*

We see from Figure 1 that $\Delta_{17}(s)$ is not a *sharp* but a *smooth* truncation of the divergent series (10). It is known that smooth truncations can accelerate convergence of a series and can even transform a divergent series into a convergent one. The smoothness of the truncation might be the first "reason" why the summands of the divergent series (10) are useful for calculation of the zeroes.

Usually a smooth truncation is to be invented and it isn't evident in advance what smooth truncation will turn out to be suitable; the numbers $\delta_{N,n}$ seem to give a *natural* smooth truncation.

2.4 Approximations at other points on the critical line

2.4.1 Case $N = 17$

Figure 2 shows that $\Delta_{17}(\frac{1}{2} + \mathrm{i}t)$ approximates $\zeta(\frac{1}{2} + \mathrm{i}t)$ very well between the eight zeroes used for constructing the former function, and a bit further.

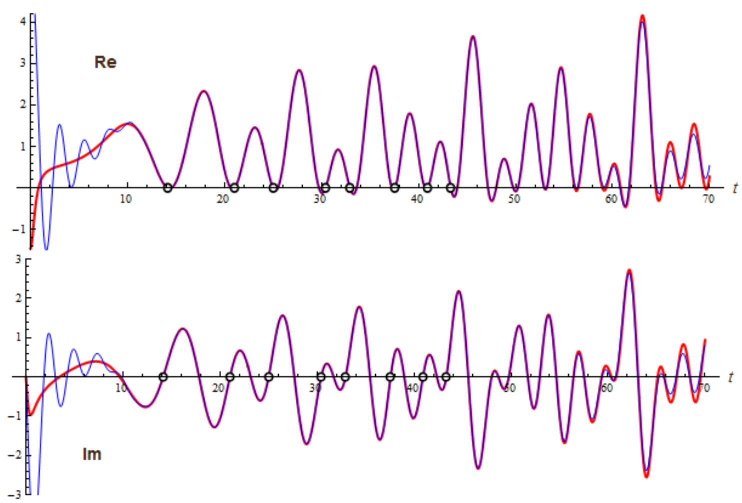

Figure 2: Re and Im of $\zeta(\frac{1}{2} + \mathrm{i}t)$ and $\Delta_{17}(\frac{1}{2} + \mathrm{i}t)$

2.4.2 Case $N = 101$

According to (15)–(21) and (22)–(51), the extra zeroes of $\Delta_{101}(\frac{1}{2}+\mathrm{i}t)$ are much closer to certain zeroes of $\zeta(\frac{1}{2}+\mathrm{i}t)$ than the extra zeroes of $\Delta_{17}(\frac{1}{2}+\mathrm{i}t)$; however, Figure 3 shows that $\Delta_{101}(\frac{1}{2}+\mathrm{i}t)$ doesn't give a good approximation to $\zeta(\frac{1}{2}+\mathrm{i}t)$ at points that aren't in the vicinity of points $t = \gamma_1, \gamma_2, \ldots$. How is it possible?

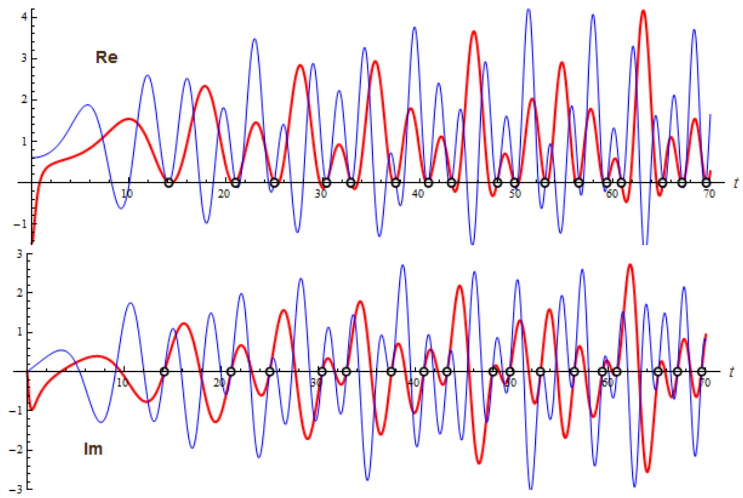

Figure 3: Re and Im of $\zeta(\frac{1}{2}+\mathrm{i}t)$ and $\Delta_{101}(\frac{1}{2}+\mathrm{i}t)$

An explanation comes from Figure 4. It shows that for $N = 101$, instead of (87), we should write

$$\Delta_{101}(s) = \sum_{n=1}^{101} \delta_{101,n} n^{-s} \leftrightharpoons \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s} = (1 - 2 \cdot 2^{-s})\zeta(s). \tag{88}$$

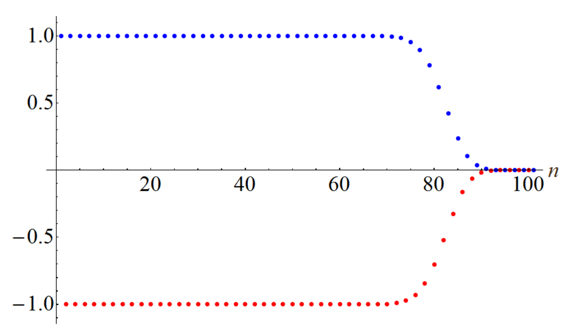

Figure 4: Coefficients $\delta_{101,n}$ for even n and odd n

Respectively, Figures 5 and 6 show that $\Delta_{101}(\frac{1}{2}+\mathrm{i}t)$ approximates the product $(1-2\cdot 2^{-\frac{1}{2}-\mathrm{i}t})\zeta(\frac{1}{2}+\mathrm{i}t)$ very well indeed.

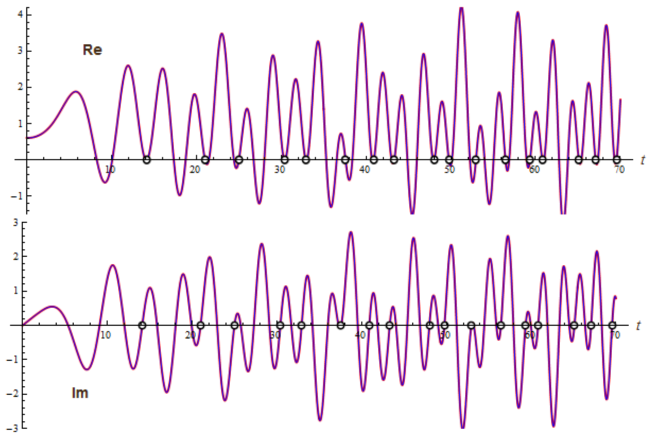

Figure 5: $(1-2\cdot 2^{-\frac{1}{2}-\mathrm{i}t})\zeta(\frac{1}{2}+\mathrm{i}t)$ and $\Delta_{101}(\frac{1}{2}+\mathrm{i}t)$

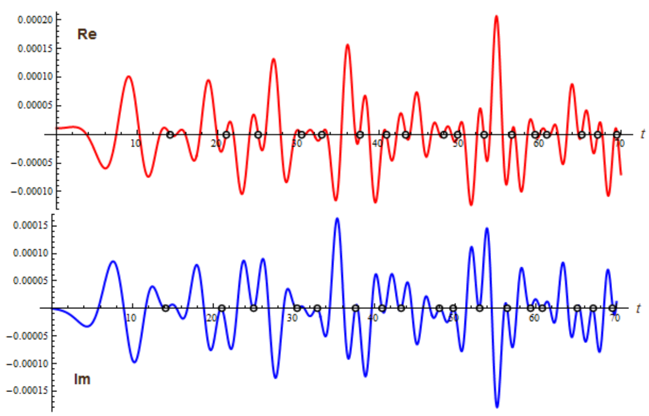

Figure 6: Re and Im of $(1-2\cdot 2^{-\frac{1}{2}-\mathrm{i}t})\zeta(\frac{1}{2}+\mathrm{i}t)-\Delta_{101}(\frac{1}{2}+\mathrm{i}t)$

2.5 Second partial explanation

The factor $1-2\cdot 2^{-s}$ from (88) was used already by Euler for assigning values to $\zeta(s)$ for $s < 1$. The alternating series in (88) converges for $\mathrm{Re}(s) > 0$, and this can be viewed as the second "reason" for the high quality of approximation demonstrated by $\Delta_{101}(s)$.

However, in the next two section we'll see that alternation does not play such an important role.

2.6 Convergence on the real axis

Figures 7 and 8 show that determinants $\Delta_N(s)$, constructed from non-trivial zeroes of the zeta functions, "know" also about the existence and positions of some trivial zeroes and give good approximations to the values of $\zeta(\sigma)$ for not too small negative values of σ where the alternating series from (88) diverges.

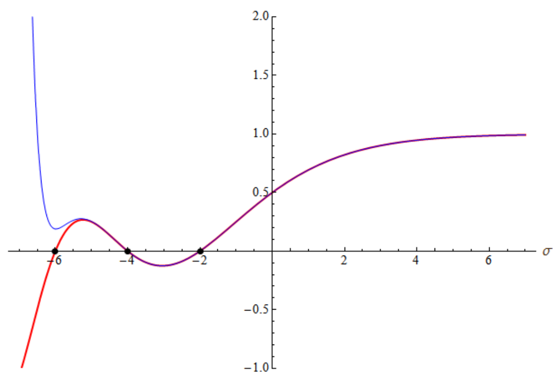

Figure 7: $(1-2\cdot 2^{-\sigma})\zeta(\sigma)$ and $\Delta_{101}(\sigma)$

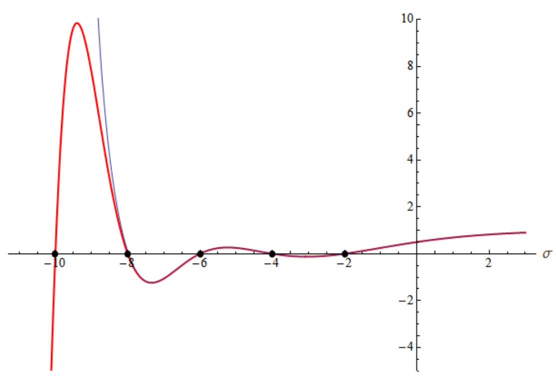

Figure 8: $(1-2\cdot 2^{-\sigma})\zeta(\sigma)$ and $\Delta_{121}(\sigma)$

2.7 Alternation

Two "reasons" for the efficiency of $\Delta_N(s)$ as an approximant were indicated above:

- the smoothness of truncation,
- convergence of the alternating series in (88).

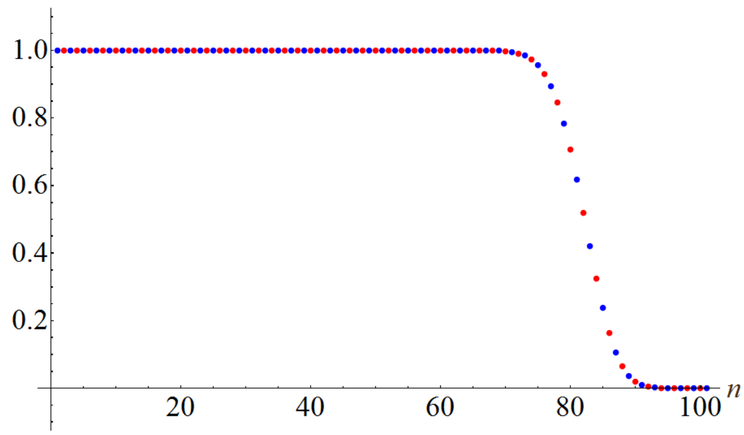

Figure 9: Alternating coefficients $(-1)^{n+1}\delta_{101,n}$for even n and odd n

Now we are to get rid of the second "reason". Figure 9 shows that

$$\sum_{n=1}^{101}(-1)^{n+1}\delta_{101,n}n^{-s} \risingdotseq \sum_{n=1}^{\infty}n^{-s} = \zeta(s), \tag{89}$$

so we define

$$\nabla_N(s) = \sum_{n=1}^{N}(-1)^{n+1}\delta_{N,n}n^{-s} \tag{90}$$

in the hope that values of $\zeta(s)$ will be well approximated by $\nabla_N(s)$. Figure 10 shows that this is indeed so for $N = 101$ but the approximation isn't as good as it was in the case of $\Delta_{101}(s)$ (see Figure 3). However, if we compare Figures 6 with the plot of the analogous difference for $\nabla_N(s)$ on Figure 11, we shall see much more regular curves. Moreover, Figure 12 shows that the absolute value of the difference doesn't oscillate at all (for not too big values of t).

The plots on Figure 11 look similar to plots of several classic functions but so far the author wasn't able to identify the difference $\zeta(\frac{1}{2} + \mathrm{i}t) - \nabla_{101}(\frac{1}{2} + \mathrm{i}t)$ as an approximation to any such function.

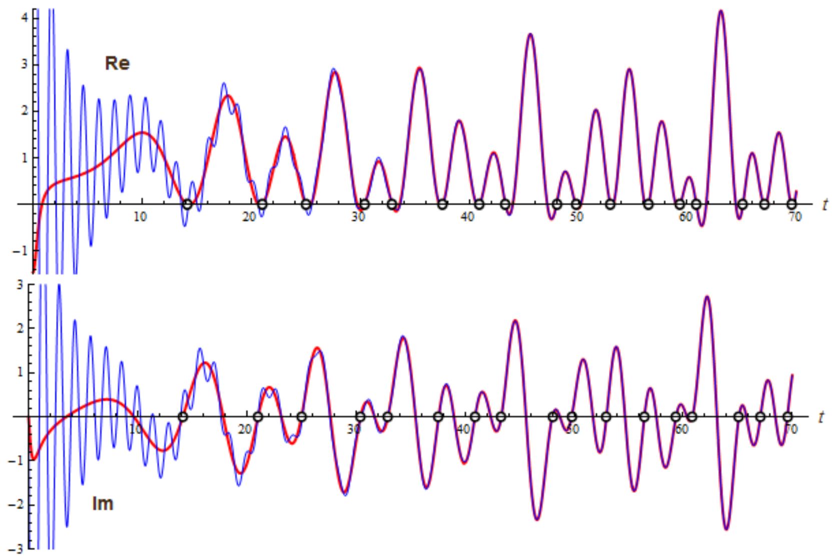

Figure 10: Re and Im of $\zeta(\frac{1}{2}+\mathrm{i}t)$ and $\nabla_{101}(\frac{1}{2}+\mathrm{i}t)$

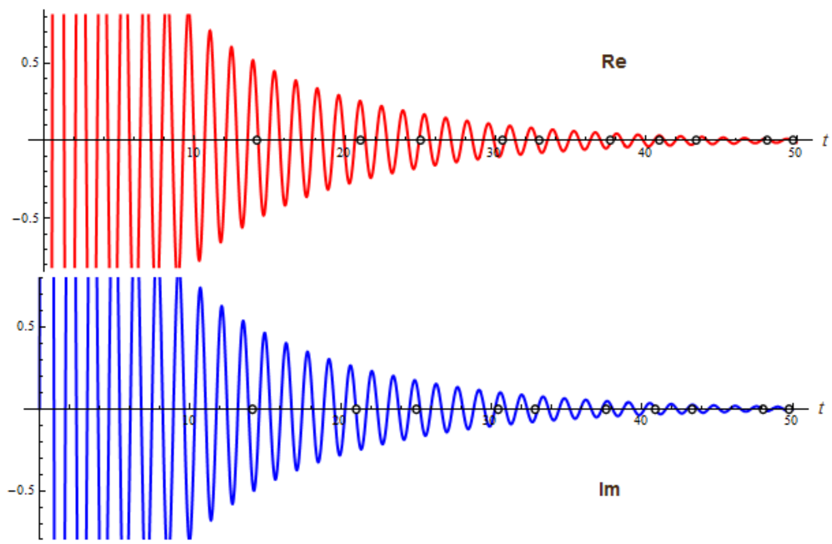

Figure 11: Re and Im of $\zeta(\frac{1}{2}+\mathrm{i}t)-\nabla_{101}(\frac{1}{2}+\mathrm{i}t)$

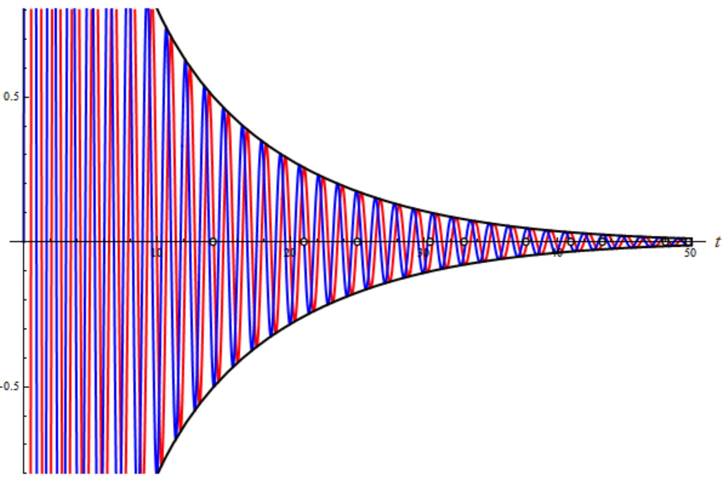

Figure 12: Re and Im of $\zeta(\frac{1}{2}+\mathrm{i}t)-\nabla_{101}(\frac{1}{2}+\mathrm{i}t)$, $\pm|\zeta(\frac{1}{2}+\mathrm{i}t)-\nabla_{101}(\frac{1}{2}+\mathrm{i}t)|$

2.8 Finer structure of the coefficients $\delta_{N,n}$

We now look at the finer structure of $\delta_{N,n}$ in the case $N = 3001$. Similar to (88), for small n these numbers are very close to $(-1)^{n+1}$. Figure 13 exhibits differences $\delta_{3001,1}-1,\ldots,\delta_{3001,300}-1$ with logarithmic scale.

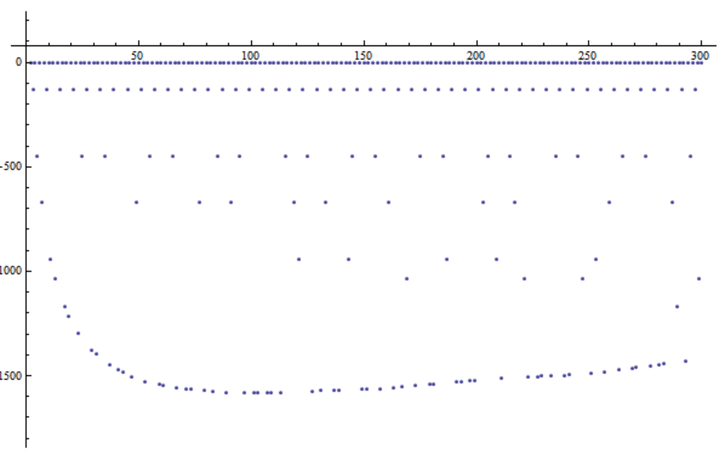

Figure 13: $\log_{10}|\delta_{3001,n}-1|$

The top row corresponds to even values of n for which $\delta_{3001,n}$ is close to -1.

The second row corresponds to odd values of n divisible by 3.

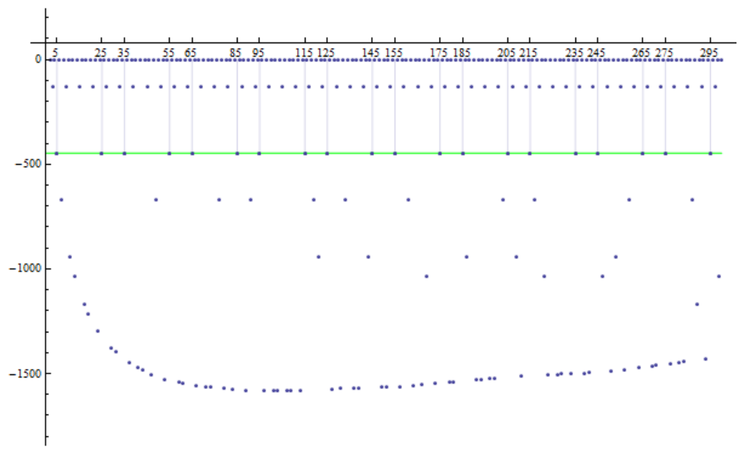

Figure 14: $\log_{10} |\delta_{3001,n} - 1|$

The third row corresponds to those values of n that are divisible by 5 but are relatively prime to $2 \cdot 3$ (see Figure 14).

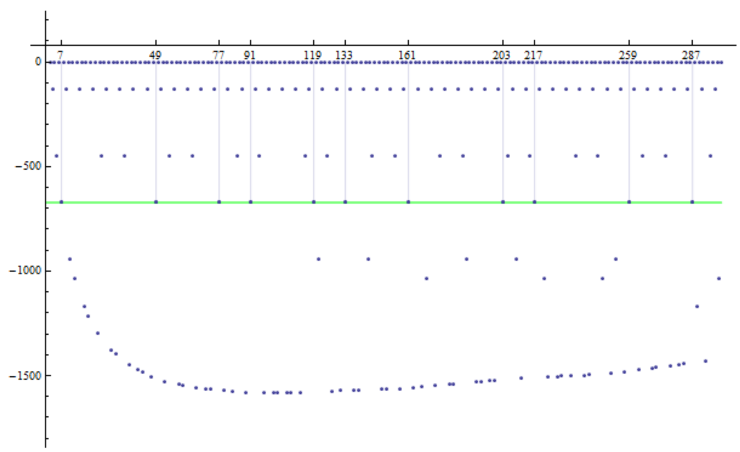

Figure 15: $\log_{10} |\delta_{3001,n} - 1|$

The fourth row corresponds to those values of n that are divisible by 7 but are relatively prime to $2 \cdot 3 \cdot 5$ (see Figure 15).

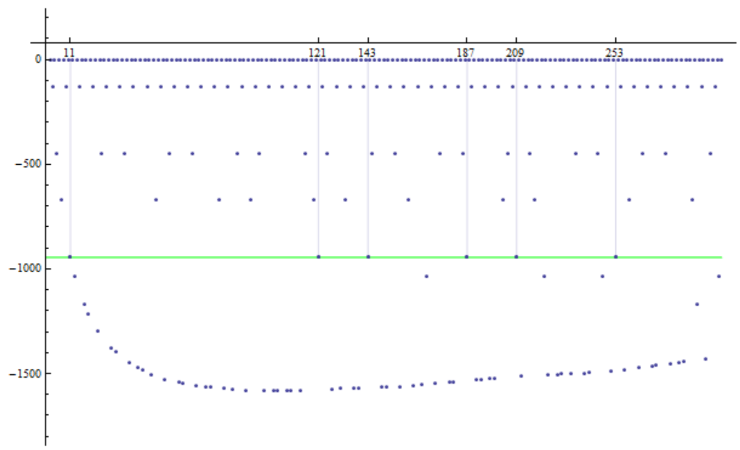

Figure 16: $\log_{10} |\delta_{3001,n} - 1|$

The fifth row corresponds to those values of n that are divisible by 11 but are relatively prime to $2 \cdot 3 \cdot 5 \cdot 7$ (see Figure 16).

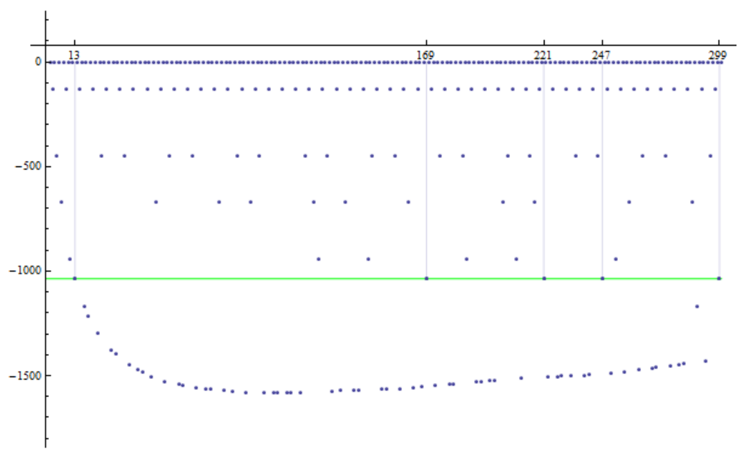

Figure 17: $\log_{10} |\delta_{3001,n} - 1|$

The sixth row corresponds to those values of n that are divisible by 13 but are relatively prime to $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$ (see Figure 17).

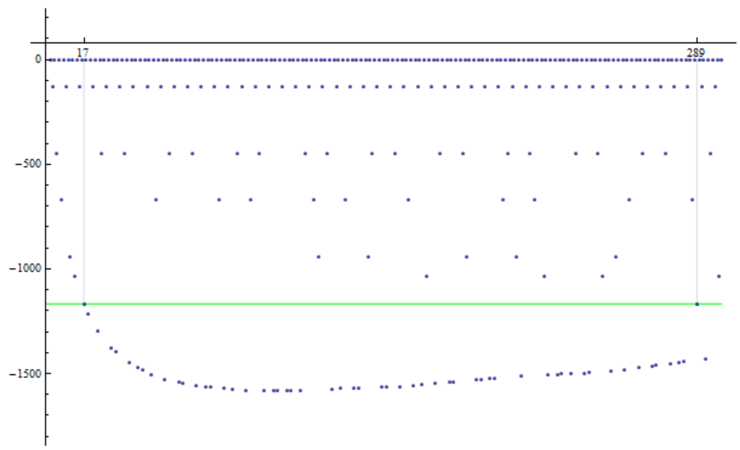

Figure 18: $\log_{10}|\delta_{3001,n} - 1|$

The seventh row corresponds to those values of n that are divisible by 17 but are relatively prime to $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ (see Figure 18).

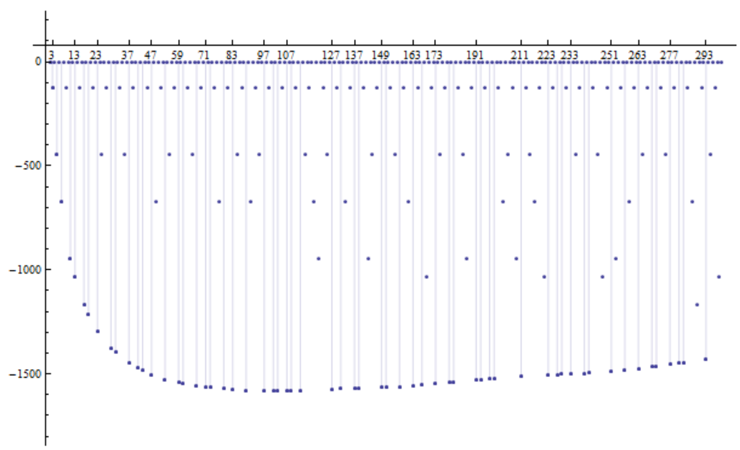

Figure 19: $\log_{10}|\delta_{3001,n} - 1|$

The remaining dots correspond to prime values of n (see Figure 19). So we can say that the initial part of the plot of $\log_{10}|\delta_{3001,n} - 1|$ represents the *Sieve of Eratosthenes*.

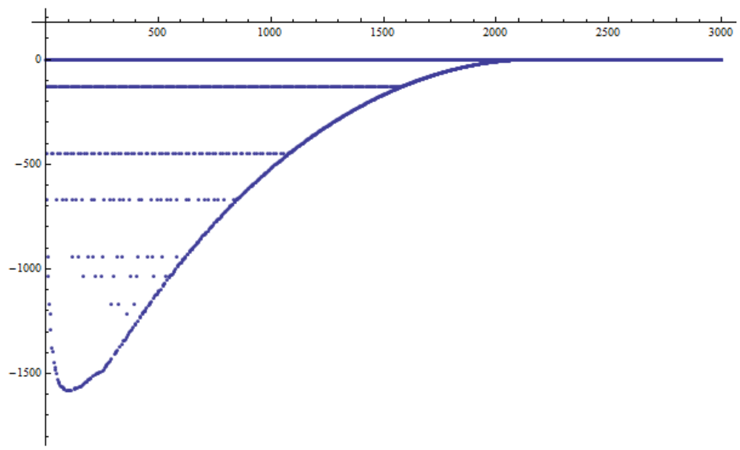

Figure 20: $\log_{10}|\delta_{3001,n} - 1|$

Figure 20 extends Figure 13 up to $n = 3001$. We see that horizontal rows corresponding to values of n divisible by 2, by 3, ... break off when they touch a "smooth curve" of increasing values of $\log_{10}|\delta_{3001,n} - 1|$. These horizontal rows will be called *Eratosthenes levels*.

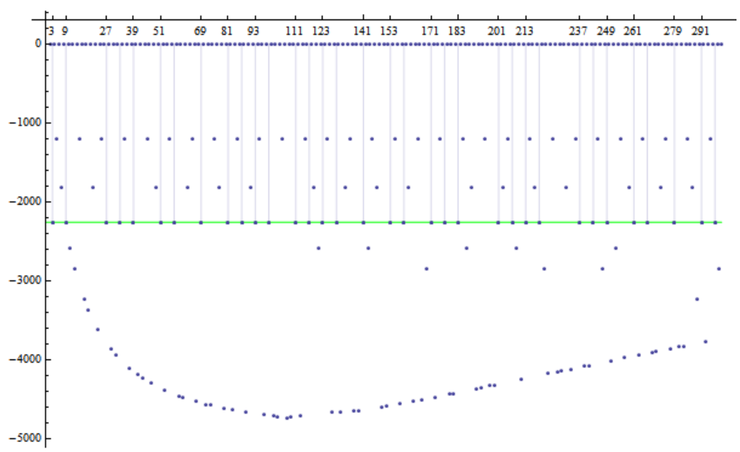

Figure 21: $\log_{10}|\delta_{7999,3m} - \delta_{7999,3}|$

Closer examination reveals that each Eratosthenes level in its turn contains sublevels corresponding to a slightly modified Sieve of Eratosthenes. Figure 21 shows such sublevels for the main Eratosthenes level corresponding to prime $p = 3$ in the case $N = 7999$. These sublevels correspond to deleting composite numbers according to their divisibility at first by 2, then by 5, 7, 3, 11, 13, ...

In the general case deleting composite numbers divisible by p happens between deleting composite numbers divisible by consecutive primes q_1

and q_2 such that $q_1 < p^2 < q_2$. It seems that the sublevels contain subsublevels and so on.

2.9 Calculation of values of $\zeta(s)$

The factor $1 - 2 \cdot 2^{-s}$ appeared in (88) from the visual observation (made from Figure 4) that initial coefficients are close to $(-1)^{n+1}$. Now we know that they have a finer structure and wish to replace $1 - 2 \cdot 2^{-s}$ by a "correct" factor. Namely, we define numbers $\mu_{N,n}$ via formal division of Dirichlet series:

$$\frac{\Delta_N(s)}{\zeta(s)} = \frac{\sum_{n=1}^{N} \delta_{N,n} n^{-s}}{\sum_{n=1}^{\infty} n^{-s}} = \sum_{n=1}^{\infty} \mu_{N,n} n^{-s}. \tag{91}$$

By Möbius inversion, we can give explicit expressions for these numbers

$$\mu_{N,n} = \sum_{m|n} \mu\left(\tfrac{n}{m}\right) \delta_{N,n} \tag{92}$$

where $\mu(k)$ is the Möbius function and we assume that $\delta_{N,n} = 0$ for $n > N$. Defining

$$\nu_N(s) = \sum_{n=1}^{\infty} \mu_{N,n} n^{-s} \tag{93}$$

we have the equality

$$\Delta_N(s) = \nu_N(s)\zeta(s) \tag{94}$$

where the right-hand side is understood as the formal product of the two Dirichlet series. Shouldn't we expect that numerically

$$\Delta_N(s) \approx \nu_{N,M}\zeta(s) \tag{95}$$

where

$$\nu_{N,M}(s) = \sum_{n=1}^{M} \mu_{N,n} n^{-s} \tag{96}$$

is a truncation of (93)? In other words, can $\zeta(s)$ be well approximated by the ratio of two finite Dirichlet series:

$$\zeta(s) \approx \frac{\Delta_N(s)}{\nu_{N,M}(s)} = \frac{\sum_{n=1}^{N} \delta_{N,n} n^{-s}}{\sum_{n=1}^{M} \mu_{N,n} n^{-s}}. \tag{97}$$

To begin with, let us look at Figures 22 and 23 exhibiting the values of $\log_{10}|\mu_{3001,n}|$. These plots look similar to the plots on Figures 13 and 20. Indeed, according to (92), for a prime n we have $\mu_{N,n} = \delta_{N,n} - 1$ so points corresponding to the prime n occupy the same positions. For composite n corresponding to points not lying on the Eratosthenes levels, the differences $\mu_{N,n} - (\delta_{N,n} - 1)$ are very small so the right parts on Figures 20 and 23 are visually the same. But points lying on the Eratosthenes levels and corresponding to composite n on Figures 13 and 20 drop down to fit, on Figures 22 and 23, onto a "smooth curve" with the other points.

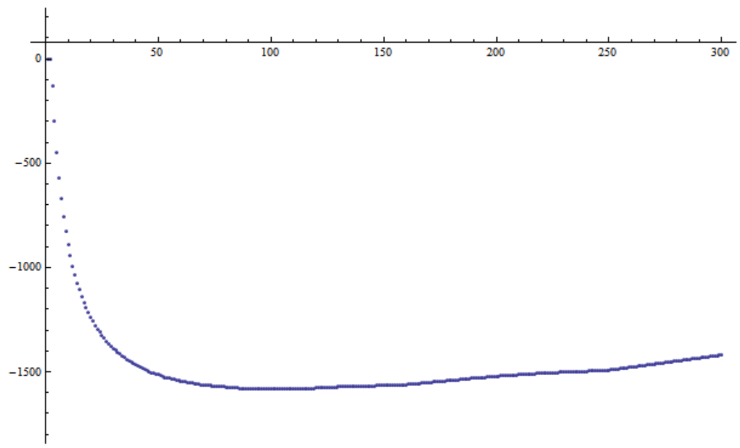

Figure 22: $\log_{10}|\mu_{3001,n}|$

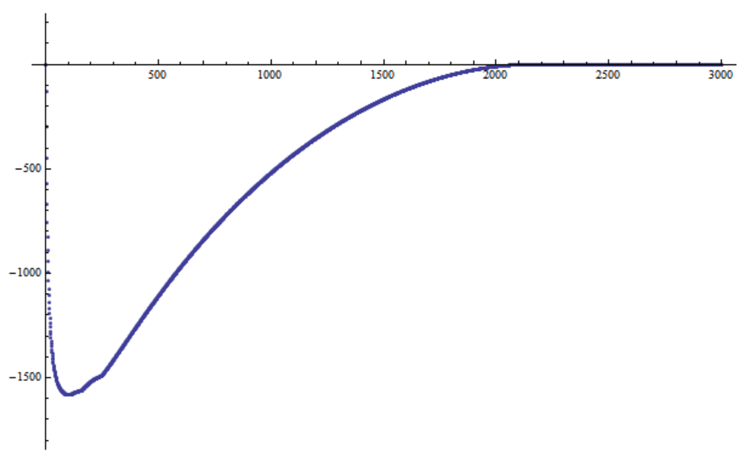

Figure 23: $\log_{10}|\mu_{3001,n}|$

Table 1: Calculation of $\zeta(s)$ at $s = \frac{1}{4} + 1000i$ for N=3001

M	$\mu_{N,M}$	$\left\lvert \zeta(s) - \frac{\Delta(s)}{\nu_{N,M}(s)} \right\rvert$
2	$-2 + 1.43 \ldots \cdot 10^{-127}$	$2.24128 \ldots \cdot 10^{-127}$
3	$-2.14787 \ldots \cdot 10^{-127}$	$1.57968 \ldots \cdot 10^{-299}$
4	$-1.62673 \ldots \cdot 10^{-299}$	$4.85859 \ldots \cdot 10^{-448}$
5	$+5.29034 \ldots \cdot 10^{-448}$	$1.00748 \ldots \cdot 10^{-569}$
6	$-1.14817 \ldots \cdot 10^{-569}$	$1.83153 \ldots \cdot 10^{-672}$
7	$+2.16930 \ldots \cdot 10^{-672}$	$3.15150 \ldots \cdot 10^{-756}$
8	$-3.85941 \ldots \cdot 10^{-756}$	$2.34266 \ldots \cdot 10^{-829}$
9	$-2.95462 \ldots \cdot 10^{-829}$	$3.17791 \ldots \cdot 10^{-891}$
10	$+4.11503 \ldots \cdot 10^{-891}$	$6.45307 \ldots \cdot 10^{-946}$
11	$-8.55748 \ldots \cdot 10^{-946}$	$6.55682 \ldots \cdot 10^{-994}$
12	$+8.88627 \ldots \cdot 10^{-994}$	$1.00011 \ldots \cdot 10^{-1036}$
13	$+1.38282 \ldots \cdot 10^{-1036}$	$2.32048 \ldots \cdot 10^{-1074}$
14	$-3.26844 \ldots \cdot 10^{-1074}$	$1.18994 \ldots \cdot 10^{-1108}$
15	$+1.70521 \ldots \cdot 10^{-1108}$	$7.70890 \ldots \cdot 10^{-1142}$
16	$-1.12267 \ldots \cdot 10^{-1141}$	$7.13768 \ldots \cdot 10^{-1168}$
17	$-1.05536 \ldots \cdot 10^{-1167}$	$1.30877 \ldots \cdot 10^{-1193}$
18	$-1.96297 \ldots \cdot 10^{-1193}$	$2.02873 \ldots \cdot 10^{-1217}$
19	$-3.08422 \ldots \cdot 10^{-1217}$	$4.26737 \ldots \cdot 10^{-1239}$
20	$-6.57127 \ldots \cdot 10^{-1239}$	$2.13752 \ldots \cdot 10^{-1259}$
21	$-3.33194 \ldots \cdot 10^{-1259}$	$4.48286 \ldots \cdot 10^{-1278}$
22	$-7.06955 \ldots \cdot 10^{-1278}$	$5.73053 \ldots \cdot 10^{-1295}$
23	$+9.13814 \ldots \cdot 10^{-1295}$	$8.80688 \ldots \cdot 10^{-1311}$
24	$+1.41940 \ldots \cdot 10^{-1310}$	$1.24696 \ldots \cdot 10^{-1325}$
25	$-2.03034 \ldots \cdot 10^{-1325}$	$1.15887 \ldots \cdot 10^{-1339}$
26	$-1.90550 \ldots \cdot 10^{-1339}$	$5.47316 \ldots \cdot 10^{-1353}$
27	$+9.08470 \ldots \cdot 10^{-1353}$	$1.16240 \ldots \cdot 10^{-1365}$
28	$+1.94705 \ldots \cdot 10^{-1365}$	$4.81799 \ldots \cdot 10^{-1377}$
29	$-8.14135 \ldots \cdot 10^{-1377}$	$5.73845 \ldots \cdot 10^{-1388}$
30	$+9.77926 \ldots \cdot 10^{-1388}$	$5.76237 \ldots \cdot 10^{-1398}$
31	$-9.90086 \ldots \cdot 10^{-1398}$	$1.22940 \ldots \cdot 10^{-1407}$
32	$-2.12919 \ldots \cdot 10^{-1407}$	$1.65327 \ldots \cdot 10^{-1416}$
33	$-2.88538 \ldots \cdot 10^{-1416}$	$4.62738 \ldots \cdot 10^{-1425}$
34	$+8.13647 \ldots \cdot 10^{-1425}$	$5.22887 \ldots \cdot 10^{-1433}$
35	$-9.26096 \ldots \cdot 10^{-1433}$	$1.53755 \ldots \cdot 10^{-1440}$
36	$-2.74243 \ldots \cdot 10^{-1440}$	$3.53025 \ldots \cdot 10^{-1448}$
37	$+6.33998 \ldots \cdot 10^{-1448}$	$1.28345 \ldots \cdot 10^{-1454}$
38	$-2.32037 \ldots \cdot 10^{-1454}$	$2.55684 \ldots \cdot 10^{-1461}$
39	$+4.65266 \ldots \cdot 10^{-1461}$	$4.67633 \ldots \cdot 10^{-1468}$
40	$-8.56354 \ldots \cdot 10^{-1468}$	$2.42671 \ldots \cdot 10^{-1473}$

Table 1 shows that taking larger and larger values of M we get better and better approximations of $\zeta(s)$ via (97). Probably, the situation here is similar to what we have with asymptotic expansions: one has to stop at a certain optimal number of summands.

2.10 Special values of $\nu_{N,M}(s)$

Let

$$M = \mathrm{LCM}(1,2,3,4,5,6,7,8,9,10) = 2520, \qquad N = M+1 = 2521. \tag{98}$$

We have the following interesting expansion into a simple continued fraction:

$$2M \cdot \nu_{N,M}(1) = 0.9998015873172093... \tag{99}$$

$$= \cfrac{1}{1+\cfrac{1}{5039+\cfrac{1}{2520+\cfrac{1}{1680+\cfrac{1}{1260+\cfrac{1}{1008+\cfrac{1}{840+\cfrac{1}{720+\cfrac{1}{630+\cfrac{1}{560+\cfrac{1}{504+\cfrac{1}{\ddots}}}}}}}}}}} \tag{100}$$

It is rather unusual that the partial quotients have such big values (according to *Gauss-Kuzmin distribution* partial quotients of a "random" real number are mainly rather small). Moreover, these large partial quotients have the following structure motivating our choice of values in (98):

$$5039 = 2M-1,\ 2520 = \tfrac{2M}{2},\ 1680 = \tfrac{2M}{3},\ 1260 = \tfrac{2M}{4}, 1008 = \tfrac{2M}{5},$$
$$840 = \tfrac{2M}{6},\ 720 = \tfrac{2M}{7},\ 630 = \tfrac{2M}{8},\ 560 = \tfrac{2M}{9},\ 504 = \tfrac{2M}{10}. \tag{101}$$

The continued fraction (100) suggests consideration of the function

$$\phi(M) \quad = \quad \frac{1}{2M} \cdot \cfrac{1}{1+\cfrac{1}{2M-1+\cfrac{1}{\frac{2M}{2}+\cfrac{1}{\frac{2M}{3}+\cfrac{1}{\frac{2M}{4}+\cfrac{1}{\frac{2M}{5}+\ddots}}}}}} \tag{102}$$

It seems that

$$\phi(M) = \frac{1}{2}\psi\left(\frac{M}{2}+1\right) - \frac{1}{2}\psi\left(\frac{M+1}{2}\right) \tag{103}$$

where, as usual,

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}. \tag{104}$$

Function (103) turns out to be a good approximation to $\nu_{N,M}(1)$ with M and N from (98) and for other values of these parameters as well:

$$\frac{\nu_{2521,2520}(1)}{\phi(2520)} = 1 - 1.063066513532... \cdot 10^{-108} \tag{105}$$

$$\frac{\nu_{3001,3000}(1)}{\phi(3000)} = 1 + 7.158776770618... \cdot 10^{-128} \tag{106}$$

$$\frac{\nu_{6001,6000}(1)}{\phi(6000)} = 1 + 5.411860996641659... \cdot 10^{-259} \tag{107}$$

It seems that among the two arguments of ν, the second is more impotant:

$$\frac{\nu_{7001,6000}(1)}{\phi(6000)} = 1 + 5.258535208832606... \cdot 10^{-209} \tag{108}$$

The author came to the expected equality (103) by calculating a few initial coefficients of expansions of the both sides into series over $\frac{1}{M}$; D. Zagier [2] verified that all coefficients coincide by giving yet another representation for ϕ via a continued fraction studied in [1]:

$$\phi(M) = x - x \cdot \cfrac{x}{1 + \cfrac{1 \cdot 2 \cdot x^2}{1 + \cfrac{2 \cdot 3 \cdot x^2}{1 + \cfrac{3 \cdot 4 \cdot x^2}{1 + \cfrac{4 \cdot 5 \cdot x^2}{1 + \cfrac{5 \cdot 6 \cdot x^2}{1 + \ddots}}}}}} \tag{109}$$

where $x = \frac{1}{2M}$.

The question about the convergence of the continued fractions remains open.

3 Second example: Riemann's xi function

It is well-known that Riemann's zeta function and gamma function have a close relationship; however, the appearance (via (103)) of the gamma function in (105)–(108) seems to be new. Traditionally, the gamma function is used for defining the function

$$\xi(s) = g(s)\zeta(s) \tag{110}$$

where

$$g(s) = \pi^{-\frac{s}{2}}(s-1)\Gamma\left(\tfrac{s}{2}+1\right). \tag{111}$$

In terms of the function $\xi(s)$ the *functional equation* takes the simple form

$$\xi(s) = \xi(1-s). \tag{112}$$

3.1 Coefficients of new interpolating determinants

In order to use $\xi(s)$ in the role of $F(x)$ in (9), we need to select new functions $f_1(s), f_2(s), \dots$. According to (10) and (110)

$$\xi(s) = \quad g(s)\zeta(s) \quad = \sum_{n=1}^{\infty} g(s)n^{-s}. \tag{113}$$

Due to (112) we also have:

$$\xi(s) = \quad g(1-s)\zeta(1-s) \quad = \sum_{n=1}^{\infty} g(1-s)n^{s-1}. \tag{114}$$

The series in (113) converges only for $\mathrm{Re}(s) > 1$, the series in (114) converges only for $\mathrm{Re}(s) < 0$, nevertheless we define

$$f_n(s) = \frac{g(s)n^{-s} + g(1-s)n^{s-1}}{2} \tag{115}$$

and formally write

$$\xi(s) \sim \sum_{n=1}^{\infty} f_n(s). \tag{116}$$

The zeroes of $\xi(s)$ are exactly the non-trivial zeroes (12) of the zeta function, and (assuming the Riemann hypothesis, as in Section 2)

$$\overline{\rho_n} = 1 - \rho_n. \tag{117}$$

Functions (115) trivially satisfy a counterpart of the functional equation:

$$f_n(s) = f_n(1-s). \tag{118}$$

Due to (117) and (118), we need not (and cannot!) use zeroes with negative imaginary parts and we define

$$\tilde{\Delta}^{\Gamma}_N(s) = \begin{vmatrix} f_1(\rho_1) & \dots & f_1(\rho_{N-1}) & f_1(s) \\ \vdots & \ddots & \vdots & \vdots \\ f_N(\rho_1) & \dots & f_N(\rho_{N-1}) & f_N(s) \end{vmatrix} = \sum_{n=1}^{N} \tilde{\delta}^{\Gamma}_{N,n} f_n(s). \tag{119}$$

Clearly,

$$\tilde{\Delta}^{\Gamma}_N(\overline{\rho_{N-1}}) = \dots = \tilde{\Delta}^{\Gamma}_N(\overline{\rho_1}) = 0 = \tilde{\Delta}^{\Gamma}_N(\rho_1) = \dots = \tilde{\Delta}^{\Gamma}_N(\rho_{N-1}). \tag{120}$$

Similar to (85) and (86) we define

$$\delta^{\Gamma}_{N,n} = \frac{\tilde{\delta}^{\Gamma}_{N,n}}{\tilde{\delta}^{\Gamma}_{N,1}} \tag{121}$$

and

$$\Delta^{\Gamma}_N(s) = \sum_{n=1}^{N} \delta^{\Gamma}_{N,n} n^{-s}. \tag{122}$$

Notice that $\Delta_N^\Gamma(s)$, as so defined, is *not* a normalization of $\tilde{\Delta}_N(s)$, they are related in the following way:

$$\tilde{\Delta}_{N,n}^\Gamma(s) = \tilde{\delta}_{N,1}^\Gamma \left(g(s)\Delta_{N,n}^\Gamma(s) + g(1-s)\Delta_{N,n}^\Gamma(1-s) \right). \tag{123}$$

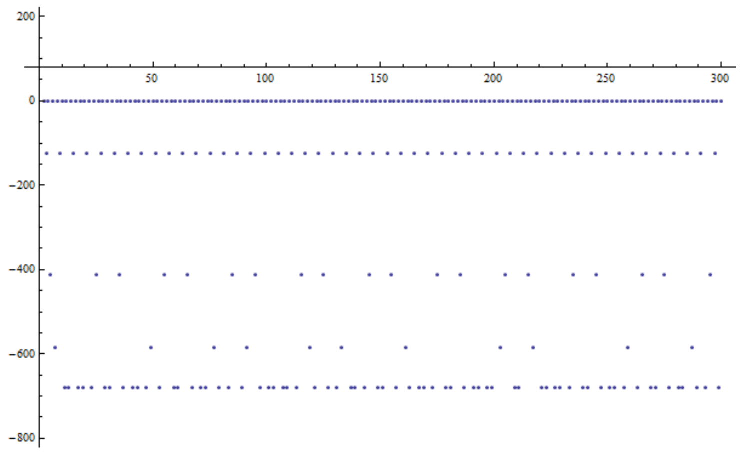

Figure 24: $\log_{10}(|\delta_{3000,n}^\Gamma - 1|)$

3.2 Sieve of Eratosthenes

Coefficients $\delta_{N,n}^\Gamma$ behave similar to coefficients $\delta_{N,n}$ but there are certain distinctions. The plot of $\log_{10}(|\delta_{3000,n}^\Gamma - 1|)$ for initial values of n looks as if construction of the sieve of Eratosthenes has been broken at some stage–on Figure 24 we see only four Eratosthenes levels. In the case $N = 3000$, the lowest row contains both prime and composite numbers.

3.3 Calculation of values of $\zeta(s)$

Similar to (91) we define

$$\frac{\Delta_N^\Gamma(s)}{\zeta(s)} = \frac{\sum_{n=1}^N \delta_{N,n}^\Gamma n^{-s}}{\sum_{n=1}^\infty n^{-s}} = \sum_{n=1}^\infty \mu_{N,n} n^{-s} \tag{124}$$

or directly

$$\mu_{N,n}^\Gamma = \sum_{m|n} \mu\left(\tfrac{n}{m}\right) \delta_{N,n}^\Gamma \tag{125}$$

by assuming that $\delta_{N,n}^\Gamma = 0$ for $n > N$. Further, similar to (96) we define

$$\nu_{N,M}^\Gamma(s) = \sum_{n=1}^M \mu_{N,n}^\Gamma n^{-s} \tag{126}$$

and consider approximations of $\zeta(s)$ by the ratio

$$\frac{\sum_{n=1}^{N} \delta_{N,n}^{\Gamma} n^{-s}}{\sum_{n=1}^{M} \mu_{N,n}^{\Gamma} n^{-s}}. \tag{127}$$

Table 2: Calculation of $\zeta(s)$ at $s = \frac{1}{4} + 1000\mathrm{i}$ for N=3000

M	$\mu_{N,M}^{\Gamma}$	$\left\lvert \zeta(s) - \frac{\Delta^{\Gamma}(s)}{\nu_{N,M}^{\Gamma}(s)} \right\rvert$
2	$-2 - 4.93\ldots \cdot 10^{-126}$	$7.72773\ldots \cdot 10^{-126}$
3	$+7.40565\ldots \cdot 10^{-126}$	$2.77621\ldots \cdot 10^{-284}$
4	$+2.85890\ldots \cdot 10^{-284}$	$1.70620\ldots \cdot 10^{-411}$
5	$-1.85782\ldots \cdot 10^{-411}$	$2.24778\ldots \cdot 10^{-509}$
6	$-2.56167\ldots \cdot 10^{-509}$	$4.24006\ldots \cdot 10^{-585}$
7	$-5.02202\ldots \cdot 10^{-585}$	$3.59049\ldots \cdot 10^{-641}$
8	$+4.39701\ldots \cdot 10^{-641}$	$6.70503\ldots \cdot 10^{-681}$
9	$+1.08444\ldots \cdot 10^{-681}$	$6.85250\ldots \cdot 10^{-681}$
10	$+1.90599\ldots \cdot 10^{-716}$	$6.85250\ldots \cdot 10^{-681}$
11	$+2.37291\ldots \cdot 10^{-681}$	$8.03260\ldots \cdot 10^{-681}$
12	$-8.19041\ldots \cdot 10^{-753}$	$8.03260\ldots \cdot 10^{-681}$
13	$+2.53822\ldots \cdot 10^{-681}$	$7.77032\ldots \cdot 10^{-681}$
14	$-8.82811\ldots \cdot 10^{-755}$	$7.77032\ldots \cdot 10^{-681}$
15	$+7.90462\ldots \cdot 10^{-756}$	$7.77032\ldots \cdot 10^{-681}$
16	$+6.85925\ldots \cdot 10^{-682}$	$7.90768\ldots \cdot 10^{-681}$
17	$+2.80369\ldots \cdot 10^{-681}$	$6.13061\ldots \cdot 10^{-681}$
18	$+9.79035\ldots \cdot 10^{-755}$	$6.13061\ldots \cdot 10^{-681}$
19	$+2.91376\ldots \cdot 10^{-681}$	$7.51911\ldots \cdot 10^{-681}$
20	$+8.85123\ldots \cdot 10^{-755}$	$7.51911\ldots \cdot 10^{-681}$
21	$-1.28032\ldots \cdot 10^{-754}$	$7.51911\ldots \cdot 10^{-681}$
22	$-1.44488\ldots \cdot 10^{-754}$	$7.51911\ldots \cdot 10^{-681}$
23	$+3.10282\ldots \cdot 10^{-681}$	$5.58306\ldots \cdot 10^{-681}$
24	$-5.83115\ldots \cdot 10^{-755}$	$5.58306\ldots \cdot 10^{-681}$
25	$+1.59266\ldots \cdot 10^{-681}$	$5.89767\ldots \cdot 10^{-681}$
26	$-7.27060\ldots \cdot 10^{-755}$	$5.89767\ldots \cdot 10^{-681}$
27	$+1.08716\ldots \cdot 10^{-681}$	$6.50614\ldots \cdot 10^{-681}$
28	$-1.61246\ldots \cdot 10^{-755}$	$6.50614\ldots \cdot 10^{-681}$
29	$+3.33221\ldots \cdot 10^{-681}$	$4.79343\ldots \cdot 10^{-681}$
30	$+1.00523\ldots \cdot 10^{-755}$	$4.79343\ldots \cdot 10^{-681}$

Table 2 shows that for $N = 3000$ increasing the value of M stops to improve the accuracy of the approximation rather soon in contrast to the case of Table 1. Indeed, numbers $\mu_{N,n}^{\Gamma}$ behave differently from numbers $\mu_{N,n}$, namely, Table 2 and Figure 25 show that after $n = 11$ the values of $\mu_{N,n}^{\Gamma}$ begin to oscillate between values of orders 10^{-681} and 10^{-754}.

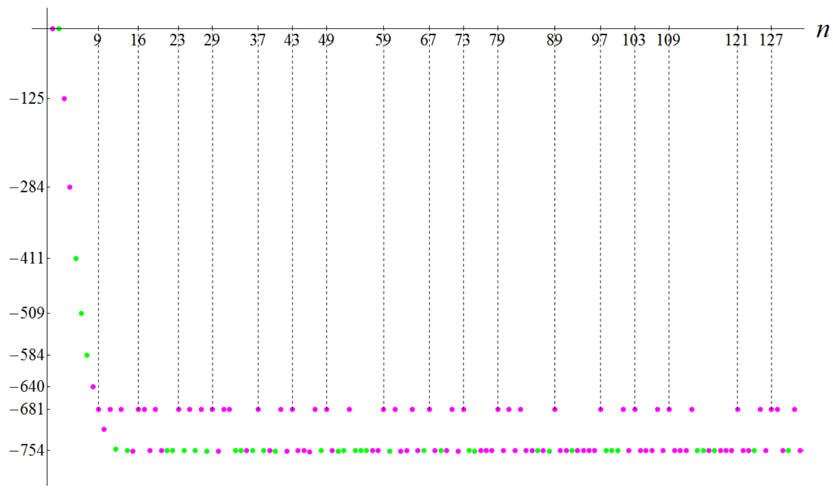

Figure 25: $\log|\mu^{\Gamma}_{3000,n}|$, `magenta`, if $\mu^{\Gamma}_{3000,n} > 0$, `green` otherwise

What is remarkable is the way in which this splitting happens: values close to 10^{-681} correspond to those values of n that are either primes or powers of primes. Figure 26 demonstrate futher splitting–the upper "curve" correspond to genuine prime values of n.

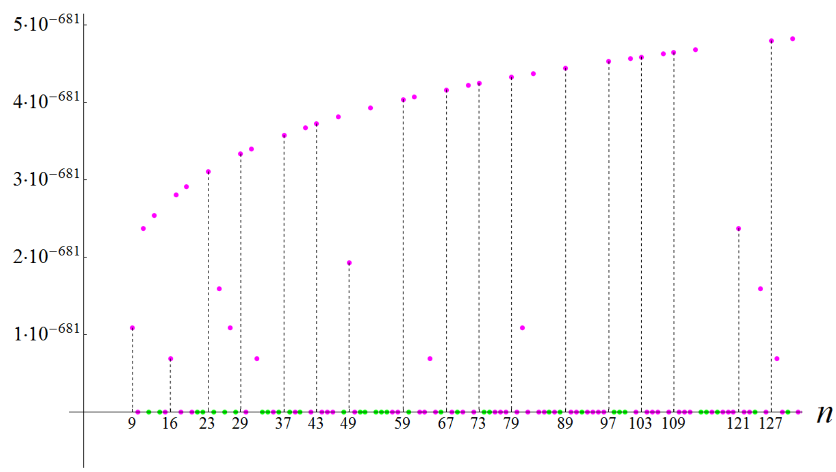

Figure 26: $\mu^{\Gamma}_{3000,n}$, `magenta`, if $\mu^{\Gamma}_{3000,n} > 0$, `green` otherwise

The "curve" on Figure 26 looks like a plot of the logarithmic function, and indeed after divison by $\log(n)$ points on Figure 27 lie on several horizontal lines.

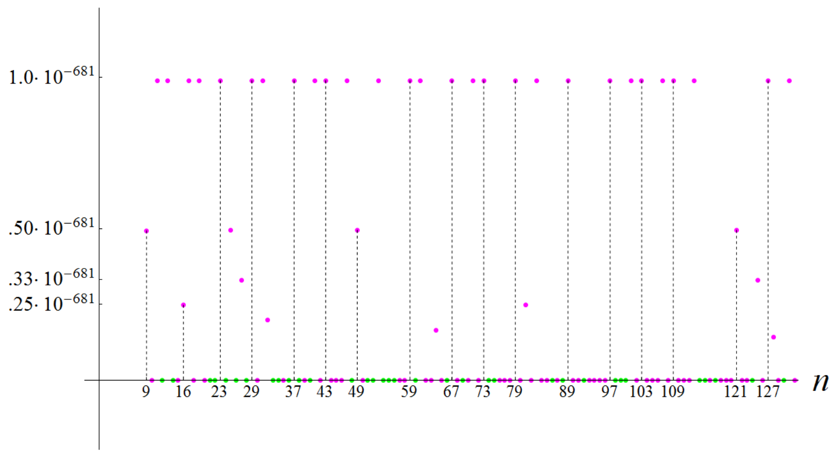

Figure 27: $\frac{\mu^{\Gamma}_{3000,n}}{\log(n)}$, magenta, if $\mu^{\Gamma}_{3000,n} > 0$, green otherwise

This further suggests that we should divide $\mu^{\Gamma}_{N,n}$ not by $\log(n)$ but by the von Mangoldt function $\Lambda(n)$, and this results in points corresponding to prime and prime power values of n lying on the same line on Figure 28. Indeed, let

$$\omega_{3000} = \frac{\mu^{\Gamma}_{3000,13}}{\ln(13)} = 9.895811... \cdot 10^{-682}, \tag{128}$$

then for a prime p such that $13 \leq p^k \leq 419$ we have

$$\left| \frac{\mu^{\Gamma}_{3000,p^k} / \ln(p)}{\omega_{3000}} - 1 \right| < 3.85... \cdot 10^{-73} \tag{129}$$

and

$$\left| \frac{\mu^{\Gamma}_{3000,p^k}}{\ln(p)} - \omega_{3000} \right| < 3.81... \cdot 10^{-754}. \tag{130}$$

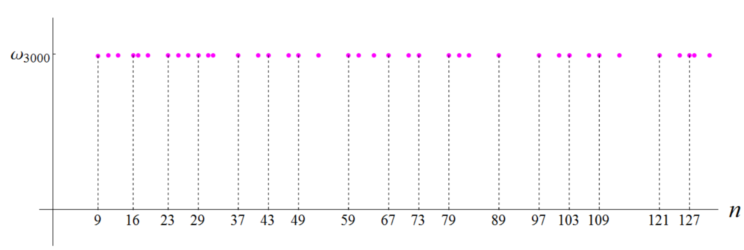

Figure 28: $\frac{\mu^{\Gamma}_{3000,n}}{\Lambda(n)}$

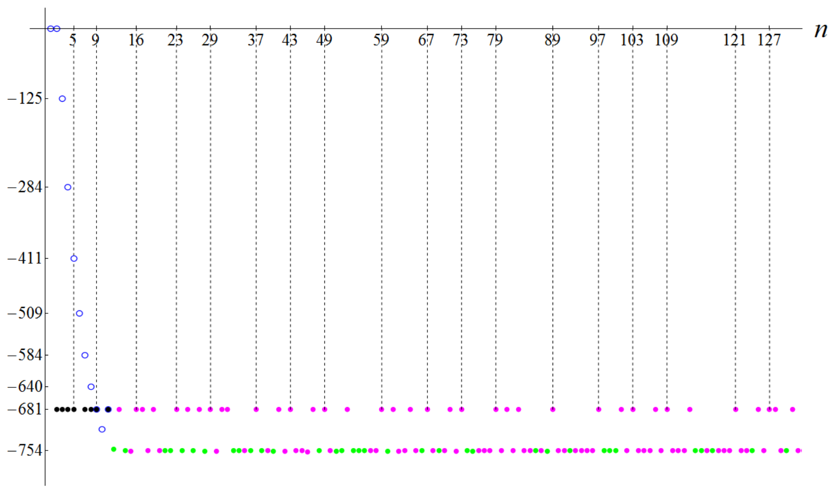

Figure 29: Modified series (91)

3.4 Calculating zeta derivative

By modifying the first 11 summands in (91) (see Figure 29) we can write

$$\sum_{n=1}^{11}\left(\omega_{3000}\Lambda(n)-\mu^{\Gamma}_{3000,n}\right)n^{-s}+\frac{\Delta^{\Gamma}_{3000}(s)}{\zeta(s)}\leftrightharpoons\sum_{n=1}^{\infty}\omega_{3000}\Lambda(n)n^{-s}. \quad (131)$$

For $\mathrm{Re}(s)>1$ the values of the right hand side in (131) are well-known:

$$\sum_{n=1}^{\infty}\omega_{3000}\Lambda(n)n^{-s}=-\omega_{3000}\frac{\zeta'(s)}{\zeta(s)}. \quad (132)$$

So shouldn't we expect that

$$\zeta(s)\sum_{n=1}^{11}\left(\omega_{3000}\Lambda(n)-\mu^{\Gamma}_{3000,n}\right)n^{-s}+\Delta^{\Gamma}_{3000}(s)\approx-\omega_{3000}\zeta'(s), \quad (133)$$

and even if s is inside the critical strip? At first, this seem to be rather implausible because the ideograph $\leftrightharpoons$ in (131) has a very weak meaning: Figure 30 shows that from some n the values of the coefficients in the left hand side (equal to $\mu^{\Gamma}_{3000,n}$) become in absolute value many orders larger than the coefficients in the right hand side (having the absolute value at most $|\omega_{3000}\Lambda(n)|$). Nevertheless, the left hand side of (133) produces very good approximations to the right hand side.

3.4.1 Calculating zeta derivative at zero

We start by choosing for s a zero of $\zeta(s)$—in this case (133) simplifies to

$$\Delta^{\Gamma}_{3000}(\rho_k) \quad \approx \quad -\omega_{3000}\zeta'\rho_k) \quad (134)$$

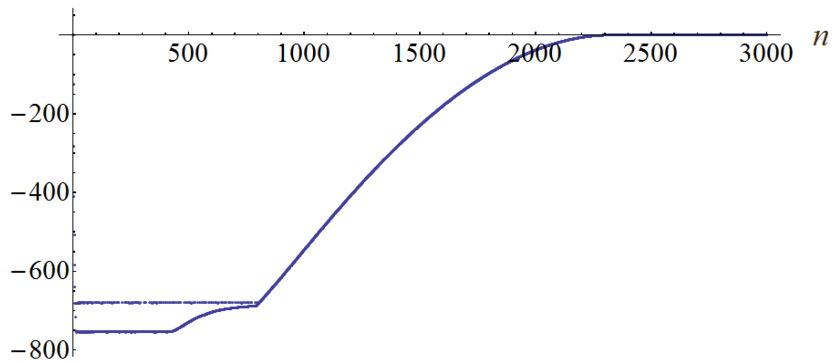

Figure 30: $\log_{10}|\mu^{\Gamma}_{3000,n}|$

(but the right hand side in (132) turns into ∞ for such s). Indeed, we have:

$$\left|\frac{\Delta^{\Gamma}_{3000}\left(\rho_{100}\right)}{-\omega_{3000}} - \zeta'(\rho_{100})\right| = 4.092... \cdot 10^{-36}, \tag{135}$$

$$\left|\frac{\Delta^{\Gamma}_{3000}\left(\rho_{500}\right)}{-\omega_{3000}} - \zeta'\left(\rho_{500}\right)\right| = 1.063... \cdot 10^{-74}. \tag{136}$$

3.4.2 Calculating zeta derivative inside the critical strip

For $s = \frac{1}{4} + 1000\mathrm{i}$ we have:

$$\left|\frac{\zeta(s)\sum_{n=1}^{11}\left(\omega_{3000}\Lambda(n) - \mu^{\Gamma}_{3000,n}\right)n^{-s} + \Delta^{\Gamma}_{3000}(s)}{-\omega_{3000}} - \zeta'\left(s\right)\right| = 1.61... \cdot 10^{-71}. \tag{137}$$

3.4.3 Calculating both zeta and its derivative. I

Using (133) for calculating $\zeta'(s)$ requries knowledge of $\zeta(s)$ with great precision. Instead of this, we can use two copies of (133) with sufficiently different values of N. For example, solving the system

$$\zeta(s)\sum_{n=1}^{11}\left(\omega_{3000}\Lambda(n) - \mu^{\Gamma}_{3000,n}\right)n^{-s} + \Delta^{\Gamma}_{3000}(s) \approx -\omega_{3000}\zeta'(s) \tag{138}$$

$$\zeta(s)\sum_{n=1}^{11}\left(\omega_{3500}\Lambda(n) - \mu^{\Gamma}_{3500,n}\right)n^{-s} + \Delta^{\Gamma}_{3500}(s) \approx -\omega_{3500}\zeta'(s) \tag{139}$$

for $s = \frac{1}{4} + 1000\mathrm{i}$ produces 908 correct decimal digits for $\zeta(s)$ and 72 correct decimal digits for $\zeta'(s)$.

3.4.4 Calculating both zeta and its derivative. II

We can avoid the necessity to calculate $\delta_{N,n}$ for two different values of N by using a copy of (133) with s replaced by $1-s$:

$$\zeta(1-s)\sum_{n=1}^{11}\left(\omega_{3000}\Lambda(n)-\mu^{\Gamma}_{3000,n}\right)n^{s-1}+\Delta^{\Gamma}_{3000}(1-s)\approx \\ -\omega_{3000}\zeta'(1-s). \quad (140)$$

Then we can differentiate both sides of the functional equation

$$g(s)\zeta(s)=g(1-s)\zeta(1-s) \quad (141)$$

and get a fourth relation

$$g'(s)\zeta(s)+g(s)\zeta'(s)=-g'(1-s)\zeta(1-s)-g(1-s)\zeta'(1-s) \quad (142)$$

between $\zeta(s)$, $\zeta(1-s)$, $\zeta'(s)$, and $\zeta'(1-s)$. Solving the system of four equations (133), (140), (141), and (142) for $s=\frac{1}{4}+1000\mathrm{i}$ produces 752 correct decimal digits for $\zeta(s)$ and 72 correct decimal digits for $\zeta'(s)$.

3.5 Approximation of $\nu^{\Gamma}_{N,M}(1)$

It turns out that the same function (103) gives a good approximation not only to $\nu_{N,M}(1)$ but to $\nu^{\Gamma}_{N,M}(1)$ as well:

$$\frac{\nu^{\Gamma}_{2520,2520}(1)}{\phi(2520)} = 1+3.822274405191727\ldots\cdot 10^{-105} \quad (143)$$

$$\frac{\nu^{\Gamma}_{3000,3000}(1)}{\phi(3000)} = 1-2.468278393149214\ldots\cdot 10^{-126} \quad (144)$$

$$\frac{\nu^{\Gamma}_{6000,6000}(1)}{\phi(6000)} = 1+2.663814892833696\ldots\cdot 10^{-253} \quad (145)$$

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