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THREE HOPF ALGEBRAS FROM NUMBER THEORY, PHYSICS & TOPOLOGY, AND THEIR COMMON OPERADIC, SIMPLICIAL & CATEGORICAL BACKGROUND

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Abstract. We consider three *a priori* totally different setups for Hopf algebras from number theory, mathematical physics and algebraic topology. These are the Hopf algebras of Goncharov for multiple zeta values, that of Connes–Kreimer for renormalization, and a Hopf algebra constructed by Baues to study double loop spaces. We show that these examples can be successively unified by considering simplicial objects, cooperads with multiplication and Feynman categories at the ultimate level. These considerations open the door to new constructions and reinterpretation of known constructions in a large common framework.

Introduction

Hopf algebras have long been known to be a highly effective tool in classifying and methodologically understanding complicated structures. In this vein, we start by recalling three Hopf algebra constructions, two of which are rather famous and lie at the center of their respective fields. These are Goncharov's Hopf algebra of multiple zeta values [Gon05] whose variants lie at the heart of the recent work [Bro17], for example, and the ubiquitous Connes–Kreimer Hopf algebra of rooted forests [CK98]. The third Hopf algebra predates them but is not as well publicized: it is a Hopf algebra discovered and exploited by Baues [Bau81] to model double loop spaces. We will trace the existence of the first and third of these algebras back to a fact known to experts[1], namely that simplices form an operad. It is via this simplicial bridge that we can push the understanding of the Hopf algebra of Goncharov to a deeper level and relate it to Baues' construction which comes from an *a priori* totally different setup. Here, we prove a general theorem, that any simplicial object gives rise to a bi–algebra.

[1]As one expert put it: "Yes this is well–known, but not to many people".

The tree Hopf algebra of Connes and Kreimer fits into this picture through a map given by contracting all the internal edges of the trees. This map also furnishes an example *par excellence* of the complications that arise in this story. A simpler example is given by restricting to the sub-Hopf algebra of three-regular trees. In this case the contraction map exhibits the corresponding Hopf algebra as a pull-back of a simplicial object. This relationship is implicit in [Gon05] and is now put into a more general framework.

Another Hopf algebra that is closely related, but more complicated is the Connes–Kreimer Hopf algebra for renomalization defined on graphs.

We show that the essential key to obtain a Hopf structure in all these examples is the realization that they are quotients of bi–algebras and that these bi–algebras have a natural origin coming from Feynman categories. This explains the "raison d'être" of the co–product formulas as simply given by the dual to a partial product given by the composition in Feynman categories, which are special monoidal categories. The quotient is furthermore identified as the natural quotient making the bi–algebras connected.

In the first three examples, there is an intermediate explanation in terms of operad theory. These correspond to particularly simple Feynman categories. More precisely, we first regard co–operads and co–operad with multiplication, a new notion that we introduce.[2] We prove a general theorem which states that a co–operad with multiplication always yields a bi–algebra. In the most natural construction, one starts with a unital operad, then dualizes it to obtain a co–operad. For this co–operad, one regards the free algebra it generates, and this is a co–operad with multiplication. This is a non–connected construction first discussed in [KWZ12] and is natural from the point of view of Feynman categories. There is also an intermediate quotient, which can be seen as a q–deformation. As $q \to 1$, we obtain the Hopf algebra.

In the general setting of co–operads with multiplication, these bi–algebras are neither unital nor co–unital. While there is no problem adjoining a unit, the co–unit is a subtle issue in general and we discuss the conditions for its existence in detail. We show that the conditions are met in the special cases at hand as they stem from the dual of unital operads. A feature of the more general case is that there is a natural "depth" filtration. We furthermore elucidate the relation of the general case to the free case by proving that there is always a surjection from

[2] For the experts, we wish to point out that due to different gradings (in the operad degree) this is neither what is known as a Hopf operad nor its dual.

a free construction to the associated graded. Going further, we prove the following structural theorem: if the bi–algebra has a left co–algebra co–unit, then it is a deformation of its associated graded and moreover this associated graded is a quotient of the free construction of its first graded piece. These deformations are of interest in themselves.

Another nice result comes about by noticing that just as there are operads and pseudo-operads, there are co–operads and pseudo–co–operads. We show that these dual structures lead to bi–algebras and a version of infinitesimal bi–algebras. The operations corresponding to the dual of the partial compositions of pseudo-operads are then dual to the infinitesimal action of Brown [Bro12a]. In other words, they give the Lie-co–algebra structure dual to the pre-Lie structure.

Moving from the constructed bi–algebras to Hopf algebras is possible under the extra condition of almost connectedness. If the co–operad satisfies this condition, which technically encompasses the existence of a split bi–algebraic co–unit, then there is a natural quotient of the bi–algebra which is connected and hence Hopf. Indeed, in the three examples taking this quotient is implemented in the original constructions by assigning values to degenerate expressions.

A further level of complexity is reflected in the fact that there are several variations of the construction of the Connes–Kreimer Hopf algebra based for example on planar labelled trees, labelled trees, unlabelled trees and trees whose external legs have been "amputated" — a term common in physics. We show, in general, these correspond to non-Sigma co–operads, coinvariants of symmetric co–operads and certain colimits, which are possible in semisimplicial co–operads.

An additional degree of understanding is provided by the insight that the underlying co–operads for the Hopf algebras of Goncharov and of Baues are given by a co–simplicial structure. This also allows us to understand the origin of the shuffle product and other relations commonly imposed in the theory of multiple zeta values and motives from this angle. For the shuffle product, in the end it is as Broadhurst remarked; the product comes from the fact that we want to multiply the integrals, which are the amplitudes of connected components of disconnected graphs. In simplicial terms this translates to the compatibility of different naturally occurring free monoid constructions, in the form of the Alexander–Whitney map and a multiplication based on the relative cup product. There are more surprising direct correspondences between the extra relations, like the contractibility of a 2-skeleton used by Baues and a relation on multiple zeta values essential for the motivic co–action.

These digressions into mathematical physics bring us to the ultimate level of abstraction and source of Hopf algebras of this type: the Feynman categories of [KW17]. We show that under reasonable assumptions a Feynman category gives rise to a Hopf algebra formed by the free Abelian group of its morphisms. Here the co–product, motivated by a discussion with D. Kreimer, is deconcatenation. With hindsight, this type of co–product goes back at least as far as [JR79] or [Ler75], who considered a deconcatenation co–product from a combinatorial point of view. Feynman categories are monoidal, and this monoidal structure yields a product. Although it is not true in general for any monoidal category that the multiplication and comultiplication are compatible and form a bi–algebra, it is for Feynman categories, and hence also for their opposites. This also gives a new understanding for the axioms of a Feynman category. The case relevant for co–operads with multiplication is the Feynman category of finite sets and surjections and its enrichments by operads. The constructions of the bi–algebra then correspond to the pointed free case considered above if the co–operad is the dual of an operad. Invoking opposite categories, one can treat co–operads directly. For this one notices that the opposite Feynman category, that for co–algebras, can be enriched by co–operads. It is here that we can also say that the two constructions of Baues and Goncharov are related by Joyal duality to the operad of surjections.

There are quotients that are obtained by "dividing out isomorphisms", which amounts to dividing out by certain coideals. This again allows us to distinguish the levels between planar, symmetric, labelled and unlabelled versions. To actually get the Hopf algebras, rather than just bi–algebras, one again has to take quotients and require certain connectedness assumptions. Here the conditions become very transparent. Namely, the unit, hidden in the three examples by normalizations, will be given by the unit endomorphism of the monoidal unit $\mathbb{1}$ of the Feynman category, viz. $id_{\mathbb{1}}$. Isomorphisms keep the co–algebra from being co–nilpotent. Even if there are no isomorphisms, still all identities are group–like and hence the co–algebra is not connected. This explains the necessity of taking quotients of the bi–algebra to obtain a Hopf algebra. We give the technical details of the two quotients, first removing isomorphisms and then identifying all identity maps.

There is also a distinction here between the non–symmetric and the symmetric case. While in the non–symmetric case, there is a Hopf structure before taking the quotient, the passing to the quotient, viz. coinvariants is necessary in the symmetric case.

These construction are more general in the sense that there are other Feynman categories besides those which yield co-operads with multiplication. One of the most interesting examples going deeper into mathematical physics is the Feynman category whose morphisms are graphs. This allows us to obtain the graph Hopf algebras of Connes and Kreimer. Going further, there are also the Hopf algebras corresponding to cyclic operads, modular operads, and new examples based on 1-PI graphs and motic graphs, which yield the new Hopf algebras of Brown [Bro17]. Here several general constructions on Feynman categories, such as enrichment, decoration, universal operations, and free construction come into play and give interrelations between the examples.

The paper is organized as follows: We proceed in steps. To be self-contained, we write out the relevant definitions at work in the background at each step. We also start each step with a short overview of the following constructions and their goals.

We begin by recalling the three Hopf algebras and their variations in §1. We give all the necessary details and add a discussion after each example indicating its position within the whole theory.

In §2, we consider the non–connected or free case, in which the co–operads have free multiplication. In order to make the technical details and the build–up of complexity more transparent, we start with a road–map §2.1 that runs through the different Connes–Kreimer constructions on trees. The main results for the non–symmetric case are Theorem 2.42 and Theorem 2.52. These explain the examples of Goncharov, Baues and the planar version of Connes–Kreimer trees. The infinitesimal structure and the q deformation are summarized in Theorem 2.63. As an example, we reconstruct Brown's derivations. The results for the technically more demanding symmetric version are Theorems 2.74 and 2.75 and 2.79. We then proceed to examine the "amputated case" in §2.10 resulting in Theorem 2.83. We end the paragraph with a discussion of co–actions §2.11.

In §3, we give the definition of a co–operad with multiplication and the constructions of bi–algebras and Hopf algebras. This paragraph also contains a discussion of the filtered and graded cases. This setup is strictly more general than the three examples, which all have a free multiplication. The result for the bi–algebra structure is Theorem 3.2. The discussion of units and co–units is intricate and summarized in §3.3.6. The results about bi–algebra deformations are to be found in Theorem 3.18. The results on Hopf, infinitesimal structure and deformations all transfer to this more general setting under the assumption of having a bi–algebra unit.

Given that the origin of the co–operad structure for Goncharov's and Baues' Hopf algebras is simplicial, we develop the general theory for the simplicial setting in §4. We give a particularly clean construction for the bi–algbras starting from the observation that simplices form an operad, yielding Proposition 4.8. We then discuss the examples from Baues and Goncharov in this setting. Further results pertain to the cubical structure §4.5 and to a co–lax monoidal structure given by simplical strings §4.3. Both these observations have further ramifications which will be explored further in the future.

It is §5 that contains the generalization to Feynman categories. Here we realize the examples in the more general setting and give several pertinent constructions. We start by giving an overview of the results of the rest of the paper in §5.1. We then treat the non–symmetric case, where the bi–algebra equation follows directly from the conditions on a Feynman category, viz. Theorem 5.20, with more work, there is a version for symmetric Feynman categories; see Theorem 5.21. Under certain conditions, there is again a Hopf quotient, see 5.6. In order to get a practical handle, we consider graded Feynman categories. The result is then Theorem 5.37. We conclude the sectopm with a discussion of functoriality; §5.7. This analysis explains why there is no Hopf algebra map from the Hopf algebra of Connes–Kreimer to that of Goncharov.

The shorter §6 gives further constructions and twists. It contains the original construction on indecomposables as well as a different quotient construction.

Having the whole theory at hand, we give a detailed discussion of examples in §7. Here we first treat the examples introduced in §1 as well as the Connes–Kreimer category for graphs. This discussion also identifies the construction of §2 and §3 as the special case of Feynman categories with trivial vertex set. We then review constructions from [KW17] to put these special cases into a larger context. These include decorations (§7.5), enrichments (§7.8) and universal operations (7.9). These explain the underlying mechanisms and allow for alterations for future applications. Among the special cases of these general construction is the motic Hopf algebra of Brown. The enrichment adds another layer of sophistication and is kept short referring to [KW17] for additional details.

This section also contains a detailed discussion of simplicial structures and the relationship with Joyal duality; §7.6. The latter is of independent interest, since this duality explains the ubiquitous occurrence of two types of formulas, those with repetition and those without repetition, in the contexts of number theory, mathematical physics and algebraic topology. This also explains the two graphical versions used in this type of calculations, polygons vs. trees, which are now just Joyal duals of each other, see especially §§7.6.4–7.6.4.

In §8, we give a short summary of the given constructions, their interrelations and specializations to the original examples and end with an outlook to further results.

To be self-contained the paper also has three appendices: one on graphs, one on co–algebras and Hopf algebras and one on the definition of Joyal duality.

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Notation. As usual for a set X with an action of a group G, we will denote the the invariants by $X^G = \{x|g(x) = x\}$ and the co–invariants by $X_G = X/\sim$ where $x \sim y$ iff there exists a $g \in G : g(x) = y$.

For an object V in a moniodal category, we denote by TV the free unital algebra on V, that is $TV = \bigoplus_n V^{\otimes n}$, in the case of an Abelian monoidal category, and by $\bar{T}V$ the free algebra on V, that is reduced the tensor algebra on $\bar{T}V = \bigoplus_{n\geq 1} V^{\otimes n}$ in the case of an Abelian monoidal category. Similarly $SV = \bigoplus_{n\geq 0} V^{\odot n}$ denotes the free symmetric algebra and $\bar{S}V$ the free non–unital symmetric algebra. We use the notation $\odot$ for the symmetric aka. symmetrized, aka. commutative tensor product: $V^{\odot^n} = (V^{\otimes n})_{\mathbb{S}_n}$ where $\mathbb{S}_n$ permutes the tensor factors.

Furthermore, we use $\underline{n} = \{1, \ldots, n\}$ and denote by $[n]$ to be the category with $n+1$ objects $\{0, \ldots, n\}$ and morphisms generated by the chain $0 \to 1 \to \cdots \to n$.

Contents

1. Preface: Three Hopf algebras

In this section, we will review the construction of the main Hopf algebras which we wish to put under one roof and generalize. After each example we will give a discussion paying special attention to their unique features.

1.1. **Multiple zeta values.** We briefly recall the setup of Goncharov's Hopf algebra of multiple zeta values. Given r natural numbers $n_1, \ldots, n_{r-1} \geq 1$ and $n_r \geq 2$, one considers the real numbers

$$\zeta(n_1, \ldots, n_r) := \sum_{1 \leq k_1 \leq \cdots \leq k_r} \frac{1}{k_1^{n_1} \ldots k_r^{n_r}} \tag{1.1}$$

The value $\zeta(2) = \pi^2/6$, for example, was calculated by Euler.

Kontsevich remarked that there is an integral representation for these, given as follows. If $\omega_0 := \frac{dz}{z}$ and $\omega_1 := \frac{dz}{1-z}$ then

$$\zeta(n_1, \ldots, n_r) = \int_0^1 \omega_1 \underbrace{\omega_0 \ldots \omega_0}_{n_1-1} \omega_1 \underbrace{\omega_0 \ldots \omega_0}_{n_2-1} \ldots \omega_1 \underbrace{\omega_0 \ldots \omega_0}_{n_r-1} \tag{1.2}$$

Here the integral is an iterated integral and the result is a real number. The *weight* of (1.2) is $N = \sum_1^r n_i$ and its *depth* is r.

Example 1.1. As it was already known by Leibniz,

$$\zeta(2) = \int_0^1 \omega_1\omega_0 = \int_{0\leq t_1 \leq t_2 \leq 1} \frac{dt_1}{1-t_1}\frac{dt_2}{t_2}$$

One of the main interests is the independence over $\mathbb{Q}$ of these numbers: some relations between the values come directly from their representation as iterated integrals, see e.g. [Bro12b] for a nice summary. As we will show in Chapter 4 many of these formulas can be understood from the fact that simplices form an operad and hence simplicial objects form a co–operad.

1.1.1. **Formal symbols.** Following Goncharov, one turns the iterated integrals into formal symbols $\hat{I}(a_0; a_1, \ldots, a_{n-1}; a_n)$ where the $a_i \in \{0, 1\}$. That is, if w is an arbitrary word in $\{0, 1\}$ then $\hat{I}(0; w; 1)$ represents the iterated integral from 0 to 1 over the product of forms according to w, so that

$$\hat{I}(0; 1, \underbrace{0, \ldots, 0}_{n_1-1}, 1, \underbrace{0, \ldots, 0}_{n_2-1}, \ldots, 1, \underbrace{0 \ldots 0}_{n_r-1}; 1)$$

is the formal counterpart of (1.2). The weight is now the length of the word w and the depth is the number of 1s. Note that the integrals (1.2) converge only for $n_r \geq 2$, but may be extended to arbitrary words using a regularization described e.g. in [Bro12b, Lemma 2.2].

1.1.2. **Goncharov's first Hopf algebra.** Taking a more abstract viewpoint, let $\mathscr{H}_G$ be the polynomial algebra on the formal symbols $\hat{I}(a; w; b)$ for elements a, b and any nonempty word w in the set $\{0, 1\}$, and let

$$\hat{I}(a; \varnothing; b) = \hat{I}(a; b) = 1 \tag{1.3}$$

On $\mathscr{H}_G$ define a comultiplication Δ whose value on a polynomial generator is

$$\Delta(\hat{I}(a_0; a_1, \dots, a_{n-1}; a_n)) = \sum_{\substack{k\geq 0 \\ 0=i_0<i_1<\cdots<i_k=n}} \hat{I}(a_{i_0}; a_{i_1}, \dots; a_{i_k})$$
$$\otimes \hat{I}(a_{i_0}; a_{i_0+1}, \dots; a_{i_1})\hat{I}(a_{i_1}; a_{i_1+1}, \dots; a_{i_2}) \cdots \hat{I}(a_{i_{k-1}}; a_{i_{k-1}+1}, \dots; a_{i_k}) \tag{1.4}$$

Theorem 1.2. *[Gon05] If we assign $\hat{I}(a_0; a_1, \dots, a_m; a_{m+1})$ degree m then $\mathscr{H}_G$ with the co–product (1.4) (and the unique antipode) is a connected graded Hopf algebra.*

Remark 1.3. The fact that it is unital and connected follows from (1.3).

Remark 1.4. The letters $\{0, 1\}$ are actually only pertinent insofar as to get multiple zeta values at the end; the algebraic constructions work with any finite set of letters S. For instance, if S are complex numbers, one obtains polylogarithms.

1.1.3. **Goncharov's second Hopf algebra and the version of Brown.** There are several other conditions one can impose, which are natural from the point of view of iterated integrals or multiple zeta values, by taking quotients. They are

(1) The shuffle formula

$$\hat{I}(a; a_1, \dots, a_m; b)\hat{I}(a; a_{m+1}, \dots, a_{m+n}; b) = \sum_{\sigma \in Ш_{m,n}} \hat{I}(a; a_{\sigma(1)}, \dots, a_{\sigma(m+n)}; b) \tag{1.5}$$

where $Ш_{m,n}$ is the set of (m, n)-shuffles.

(2) The path composition formula

$$\forall x \in \{0, 1\} : \hat{I}(a_0; a_1, \dots, a_m; a_{m+1}) = \sum_{k=1}^{m} \hat{I}(a_0; a_1, \dots, a_k; x)\hat{I}(x; a_{k+1}, \dots, a_m; a_{m+1}) \tag{1.6}$$

(3) The triviality of loops

$$\hat{I}(a; w; a) = 0 \tag{1.7}$$

(4) The inversion formula

$$\hat{I}(a_0; a_1, \dots, a_n; a_{n+1}) = (-1)^n \hat{I}(a_{n+1}, a_n, \dots, a_1; a_0) \tag{1.8}$$

(5) The exchange formula

$$\hat{I}(a_0; a_1, \dots, a_n; a_{n+1}) = \hat{I}(1 - a_{n+1}; 1 - a_n, \dots, 1 - a_1; 1 - a_0) \tag{1.9}$$

Here the map $a_i \mapsto 1 - a_i$ interchanges 0 and 1.

(6) 2–skeleton equation

$$\hat{I}(a_0; a_1; a_2) = 0 \tag{1.10}$$

Definition 1.5. $\tilde{\mathscr{H}}_G$ be the quotient of $\mathscr{H}_G$ with respect to the following homogeneous relations stemming from conditions (1),(2),(3) and (4), let $\mathscr{H}_B$ be the quotient of $\mathscr{H}_G$ with respect to relations of the conditions (1), (3), (4) and let $\tilde{\mathscr{H}}_B$ be the quotient by the relations given in the conditions (1),(2),(4),(5) and (6).

Again one can generalize to a finite set S.

Theorem 1.6. *[Gon05, Bro12a, Bro12b] Δ and the grading descend to $\tilde{\mathscr{H}}_G$ and using the unique antipode is a graded connected Hopf algebra. Furthermore (1), (2), (3) imply (4). $\mathscr{H}_B$ and $\tilde{\mathscr{H}}_B$ are graded connected Hopf algebras as well.*

1.1.4. **Discussion.** In the theory of multiple zeta values it is essential that there are two parts to the story. The first is the motivic level. This is represented by the Hopf algebras and co–modules over them. The second are the actual real numbers that are obtained through the iterated integrals. The theory is then an interplay between these two worlds, where one tries to get as much information as possible from the motivic level. This also explains the appearance of the different Hopf algebras since the evaluation in terms of iterated integrals factors through these quotients. In our setting, we will be able to explain many of the conditions naturally. The first condition (1.3) turns a naturally occurring non-connected bi–algebra into a connected bi–algebra and hence a Hopf algebra. The existence of the bi–algebra itself follows from a more general construction stemming from co–operad structure with multiplication. One example of this is given by simplicial objects and the particular co–product (1.4) is of this simplicial type. This way, we obtain the generalization of $\mathscr{H}_G$. Condition (1.3) is understood in the simplicial setup in Chapter 4 as the contraction of a 1-skeleton of a simplicial object. The relation (2) is actually related to a second algebra structure, the so-called path algebra structure [Gon05], which we will discuss in the future. The relation (3) is a normalization, which is natural from iterated integrals. The condition (1) is natural within the simplicial setup, coming from the Eilenberg–Zilber and Alexander–Whitney maps and interplay between two naturally occurring monoids. That is we obtain a generalization of $\mathscr{H}_B$ used in the work of Brown [Bro15, Bro12a].

The Hopf algebra $\tilde{\mathscr{H}}_B$ is used in [Bro12b]. The relation (5), in the simplicial case, can be understood in terms of orientations. Finally, the equation (6) corresponds to contracting the 2-skeleton of a simplicial object. It is intriguing that on one hand (6) is essential for the coaction [Bro16] while it is essential in a totally different context to get a model for chains on a double loopspace [Bau98], see below.

Moreover, in his proofs, Brown essentially uses operators D_r which we show to be equal to the dual of the $\circ_i$ map used in the definition of a pseudo-co–operad, see §2.7.1. There is a particular normalization issue with respect to $\zeta(2)$ which is handled in [Bro15] by regarding the Hopf comodule $\mathscr{H}_B \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta^{\mathfrak{m}}(2))$ of $\mathscr{H}_B$. The quotient by the second factor then yields the Hopf algebra above, in which the element representing $\zeta(2)$ vanishes. Natural coactions are discussed in §2.11.

1.2. **Connes–Kreimer.**

1.2.1. **Rooted forests without tails.** We will consider graphs to be given by vertices, flags or half-edges and their incidence conditions; see Appendix A for details. There are two ways to treat graphs: either with or without tails, that is, free half-edges. In this section, we will recapitulate the original construction of Connes and Kreimer and hence use graphs without tails.

A tree is a contractible graph and a forest is a graph all of whose components are trees. A rooted tree is a tree with a marked vertex. A rooted forest is a forest with one root per tree. A rooted subtree of a rooted tree is a subtree which shares the same root.

1.2.2. **Connes–Kreimer's Hopf algebra of rooted forests.** We now fix that we are talking about isomorphism classes of trees and forests. In particular, the trees in a forest will have no particular order.

Let $\mathscr{H}_{CK}$ be the free *commutative* algebra, that is, the polynomial algebra, on rooted trees, over a fixed ground commutative ground ring k. A forest is thus a monomial in trees and the empty forest $\varnothing$, which is equal to "the empty rooted tree", is the unit 1_k in k. We denote the commutative multiplication by juxtaposition and the algebra is graded by the number of vertices.

Given a rooted subtree τ_0 of a rooted tree τ, we define $\tau \setminus \tau_0$ to be the forest obtained by deleting all of the vertices of τ_0 and all of the edges incident to vertices of τ_0 from τ: it is a rooted forest given by a collection of trees whose root is declared to be the unique vertex that has an edge in τ connecting it to τ_0.

One also says that $\tau \setminus \tau_0$ is given by an admissible cut [CK98].

Define the co–product on rooted trees as:

$$\Delta(\tau) := \tau \otimes 1_k + 1_k \otimes \tau + \sum_{\substack{\tau_0 \text{ rooted subtree of } \tau \\ \tau_0 \neq \tau}} \tau_0 \otimes \tau \setminus \tau_0 \qquad (1.11)$$

and extend it multiplicatively to forests, $\Delta(\tau_1\tau_2) = \tau_1^{(1)}\tau_2^{(1)} \otimes \tau_1^{(2)}\tau_2^{(2)}$ in Sweedler notation. One may include the first two terms in the sum by considering also $\tau_0 = \tau$ and $\tau_0 = \varnothing = 1_k$ (the empty subforest of τ), respectively, by declaring the empty forest to be a valid rooted sub–tree. In case τ_0 is empty $\tau \setminus \tau_0 = \tau$ and in case $\tau_0 = \tau$: $\tau \setminus \tau_0 = \varnothing = 1_k$.

Theorem 1.7. *[CK98] The comultiplication above together with the grading define a structure of connected graded Hopf algebra.*

Note that, since the Hopf algebra is graded and connected, an antipode exists.

1.2.3. **Other variants.** There is a planar variant, using planar planted trees. Another variant which is important for us is the one using trees with tails. This is discussed in §2.3 and §8 and Appendix A. There is also a variant where one uses leaf labelled trees. For this it is easier not to pass to isomorphism classes of trees and just keep the names of all the half edges during the cutting. These will be introduced in the text, see also [Foi02b, Foi02a].

Finally there are algebras based on graphs rather than trees, which are possibly super-graded commutative by the number of edges. In this generality, we will need Feynman categories to explain the naturality of the constructions. Different variants of interest to physics and number theory are discussed in §7.4.

1.2.4. **Discussion.** This Hopf algebra, although similar, is more complicated than the example of Goncharov. This is basically due to three features which we would like to discuss. First, we are dealing with isomorphism classes, secondly, in the original version, there are no tails and lastly there is a sub-Hopf algebra of linear trees. Indeed the most natural bi–algebra that will occur will be on planar forests with tails. To make this bi–algebra into a connected Hopf algebra, one again has to take a quotient analogous to the normalization (1.3), implemented by the identification of the forests with no vertices (just tails) with the unit in k. To obtain the commutative, unlabelled case, one has to pass to coinvariants. Finally, if one wants to get rid of tails, one has to be able to 'amputate' them. This is an extra structure, which in the case of labelled trees is simply given by forgetting a tail together with its label. Taking a second colimit with respect to this forgetting construction yields the original Hopf algebra of Connes and Kreimer. The final complication is given by the Hopf subalgebra of forests of linear, i.e. trees with only binary vertices. This Hopf subalgebra is again graded and connected. In the more general setting, the connectedness will be an extra check that has to be performed. It is related to the fact that for an operad $\mathcal{O}$, $\mathcal{O}(1)$ is an algebra and dually for a co–operad $\check{\mathcal{O}}$, $\check{\mathcal{O}}(1)$ is a co–algebra, as we will explain. If $\mathcal{O}$ or $\check{\mathcal{O}}$ is not reduced (i.e. one dimensional generated by a unit, if we are over k), then this extra complication may arise and in general leads to an extra connectedness condition.

1.3. **Baues' Hopf algebra for double loop spaces.** The basic starting point for Baues [Bau81] is a simplicial set X, from which one passes to the chain complex $C_*(X)$. It is well known that $C_*(X)$ is a co–algebra under the diagonal approximation chain map $\Delta : C_*(X) \to C_*(X) \otimes C_*(X)$, and to this co–algebra one can apply the cobar construction: $\Omega C_*(X)$ is the free algebra on $\Sigma^{-1}C_*(X)$, with a natural differential which is immaterial to the discussion at this moment.

The theorem by Adams and Eilenberg–Moore is that if ΩX is connected then $\Omega C_*(X)$ is a model for chains on the based loop space ΩX of X. This raises the question of iterating the construction, but, unlike ΩX, which can be looped again, $\Omega C_*(X)$ is now an algebra and thus does not have an obvious cobar construction. To remedy this situation Baues introduced the following comultiplication map:

$$\Delta(x) = \sum_{\substack{k\geq 0\\ 0=i_0<i_1<\cdots<i_k=n}} x_{(i_0,i_1,\dots,i_k)} \otimes x_{(i_0,i_0+1,\dots,i_1)} x_{(i_1,i_1+1,\dots,i_2)} \cdots x_{(i_{k-1},i_{k-1}+1,\dots,i_k)} \quad (1.12)$$

where $x \in X_n$ is an $(n-1)$-dimensional generator of $\Omega C_*(X)$, and $x_{(\alpha)}$ denotes its image under the simplicial operator specified by a monotonic sequence α.

Theorem 1.8. *[Bau81] If X has a reduced one skeleton $|X|^1 = *$, then the comultiplication, together with the free multiplication and the given grading, make $\Omega C_*(X)$ into a Hopf algebra. Furthermore if $\Omega\Omega|X|$ is connected, i.e. $|X|$ has trivial 2-skeleton, then $\Omega\Omega C_*(X)$ is a chain model for $\Omega\Omega|X|$.*

1.3.1. **Discussion.** Historically, this is actually the first of the types of Hopf algebras we are considering. With hindsight, this is in a sense the graded and noncommutative version of Goncharov and gives the

Hopf algebra of Goncharov a simplicial backdrop. There are several features, which we will point out. In our approach, the existence of the diagonal (co–product), written by hand in [Bau81], is derived from the fact that simplices form an operad. This can then be transferred to a co–operad structure on any simplicial set. Adding in the multiplication as a free product (as is done in the cobar construction), we obtain a bi–algebra with our methods. The structure can actually be pushed back into the simplicial setting, rather than just living on the chains, which then explains the appearance of the shuffle products.

To obtain a Hopf algebra, we again need to identify 1 with the generators of the one skeleton. This quotient passes through the contraction of the one skeleton, where one now only has one generator. This is the equivalent to the normalization (1.3). We speculate that the choice of the *chemin droit* of Deligne can be seen as a remnant of this in further analysis. We expect that this gives an interpretation of (1.9). The condition (1.8) can be viewed as an orientation condition, which suggests to work with dihedral instead of non-Σ operads, see e.g. [KL16]. Again this will be left for the future.

Lastly, the condition (1.10) corresponds to the triviality of the 2-skeleton needed by Baues for the application to double loop spaces. At the moment, this is just an observation, but we are sure this bears deeper meaning.

2. Special case I: Bi– and Hopf Algebras from (Co)-Operads

In this section, we give a general construction, which encompasses all the examples discussed in §1. We start by collecting together the results needed about operads, which we will later dualize to co–operads in §2.3 in order to generalize the constructions. There are many sources for further information about operads. A standard reference is [MSS02] and [Kau04a] contains the essentials with figures for the relevant examples. Before going into the technical details, we will consider various Connes–Kreimer type examples of tress and forests for concreteness. This will also lay out a blueprint for the constructions. This includes a discussions of the symmetric and non–symmetric case, an infinitesimal version together with co-derivations, Connes–Kreimer type amputations and grading.

There is an even more general theory using co-operads and co–operads with multiplication which is treated in §3.

2.1. **Connes–Kreimer trees as a road map.** There are several versions of the Connes-Kreimer Hopf algebra which we will discuss and generalize. Here we give a first look.

2.1.1. **Planar planted trees/forests.** The first are planar planted trees. In a planar planted tree, the leaves are naturally ordered by the planar embedding. A planar planted tree has a marked half–edge or flag at the root vertex that is unpaired, viz. is not part of an edge. All other unpaired half–edges are called leaves. Leaf vertices are not allowed. These planar planted trees form an operad, by gluing the operations $\tau_1 \circ_i \circ \tau_2$ which glues the tree τ_2 to τ_1 by joining the root half–edge to the i–th leaf forming a new edge, see Figure 1.

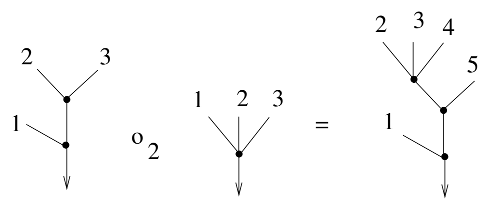

FIGURE 1. Grafting trees with labelled leaves. The tree is grafted onto the leaf number 2.

If τ has k labelled leaves, there is a full gluing operation $\gamma(\tau, \tau_1, \dots, \tau_k)$ which simultaneously glues all the trees τ_i, where τ_i is glued onto the i–th leaf. We allow the tree $|$ that consists only of one leaf. This is a sort of identity. Consider the linear version —that is the free Abelian group or vector space based on the set of leaf labelled trees. This is a graded space $\bigoplus_{n\geq 1} \mathcal{O}(n)$ where $\mathcal{O}(n)$ is generated by n labelled trees, and we insist that there is at least one leaf.

In the dual $\check{\mathcal{O}}(n) = Hom(\mathcal{O}(n), k)$ and consider the trees as their dual characteristic functions: $\tau \leftrightarrow \delta_\tau$ where $\delta_\tau(\tau) = 1$ and $\delta_\tau(\tau') = 0$ for all $\tau' \neq \tau$. One gets an operation dual to gluing that cuts off trees using an admissible cut c: $\check{\gamma}_c(\tau) = \tau_0 \otimes (\tau_1 \otimes \cdots \otimes \tau_k)$, where τ_0 is the left over stump which has k leaves and c is a collection of edges or leaves, such that if they are cut, only trees appear. The leaves and root half–edge cut into two leaves, see the Figure 2, where we have replaced the free multiplication $\otimes$ by juxtaposition. If n_i are the number of leaves of the τ_i and n is the number of leaves of τ, then $\sum_{i=1}^{k} n_i = n$, see Figure 2.

Let $\check{\mathcal{O}} = \bigoplus_n \check{\mathcal{O}}(n)$. Summing over all possible cuts, one gets the map $\check{\gamma} : \check{\mathcal{O}} \to T\check{\mathcal{O}}$ dual to γ, where $T\check{\mathcal{O}}$ is the tensor algebra, which is the free algebra on $\check{\mathcal{O}}$. One can then obtain a bi–algebra $\mathscr{B}' = T\check{\mathcal{O}}$ by extending $\check{\gamma}$ to $T\check{\mathcal{O}}$ via the bi–algebra equation: $\Delta(ab) = \sum a^{(1)} b^{(1)} \otimes$

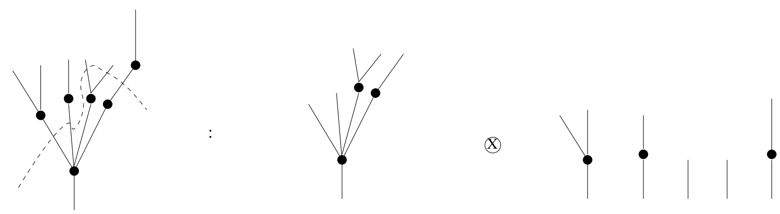

FIGURE 2. A cut in the non–symmetric planar case. Then the right side is an ordered tensor product. Alternatively, the same cut can be seen as a term on the full coinvariants, viz. the unlabeled version where on the right side, the forest is a symmetric product.

$a^{(2)}b^{(2)}$ in Sweedler notation, again replacing the free multiplication $\otimes$ by juxstaposition. The bi-algebra $\mathscr{B}'$ is the bi–algebra of planar planted forests. It is unital and co-unital, with the co-unit evaluating to 1 on a generator of the form $||\cdots|$ and 0 on all other generators. Here the empty forest is the unit $1 \in k \subset T\check{\mathcal{O}}$. The bi–algebra is graded. The degree of a forest is the total number of leaves n minus the number of trees in the forest l. The generalization of this construction is is Theorem 2.42

To obtain a Hopf algebra, one needs to take the quotient by the two sided ideal $\mathcal{I}$ generated by $1 - |$, i.e. $\mathscr{H} = \mathscr{B}'/\mathcal{I}$. The result is a connected bi–algebra and hence Hopf. This is generalized by Theorem 2.52 where we obtain a non–commutative Hopf algebra from a non-Σ operad under certain conditions. The conditions guarantee that the quotient is connected bi–algebra. The grading descends to the quotient, and is related to the co–radical degree, see Example 2.92 and §2.12.

2.1.2. **Leaf–labelled rooted trees/forests.** The construction above works with modification in the case of leaf labelled rooted trees. In this case there is no natural order on the leaves, which is why one has to add labels to them in order to define the gluing. Labelling the leaves of a tree with n leaves by $1, \dots, n$, we can again define the $\tau_1 \circ_i \tau_2$, as well as γ. Here we have to take care about the new labeling; this is done in the standard operadic way, keep the labelling of the unglued tails of τ_1 up to label $i-1$ as before, this is followed by the enumerated tails of τ_2 in their increasing order, and then continues with the rest of the unglued tails of τ_1 in their order. When we want to dualize, we, however see, that cut would yield no labels on the leaves of the stump τ_0, see Figure 3.

Furthermore, since there is no labeling, there is also no order on the forest that remains when cutting or deleting τ_0. This is why one should

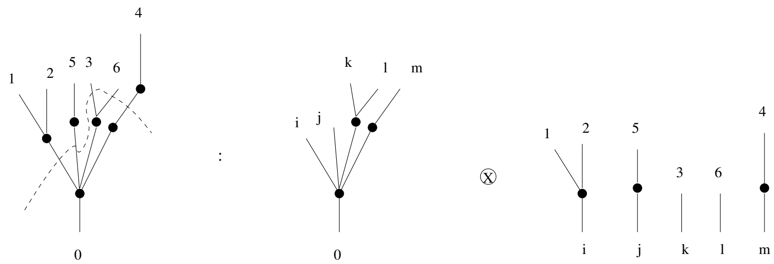

FIGURE 3. One term of the dual to γ as given by a cut in the labelled case. The labels i, j, k, l, m indicate the parring of the half edges after severing the edges.

consider the unlabelled duals, that is the coinvariants $\check{\mathcal{O}}(n)_{\mathbb{S}_n}$, where the symmetric group $\mathbb{S}_n$ permutes the labels. Set $\check{\mathcal{O}}_{\mathbb{S}} = \bigoplus_n \check{\mathcal{O}}(n)_{\mathbb{S}_n}$ then a cut on the unlabelled tree gives a morphism morphism $\bar{\gamma}^\vee : \check{\mathcal{O}}_{\mathbb{S}} \to \check{\mathcal{O}}_{\mathbb{S}} \otimes S\check{\mathcal{O}}_{\mathbb{S}}$ where S is the free symmetric algebra on $\check{\mathcal{O}}_{\mathbb{S}}$, see Figure 2 for an example.

To obtain a Hopf algebra, we again take the quotient by the two sided ideal $\mathcal{I}$ generated by $1 - |$. This is generalized by Theorem 2.75, where we obtain a commutative Hopf algebra from an operad, again under certain conditions that guarantee that the quotient $\mathscr{H} = \mathscr{B}'/\mathcal{I}$ is connected. The grading is as in the planar case.

There are several intermediate cases, one of which uses the equivariant tensor product, see Figure 4 and Remark 2.69. Another version is given by incorporating certain symmetry factors and also [CK98, CK00, CK01, CL01] and §6.

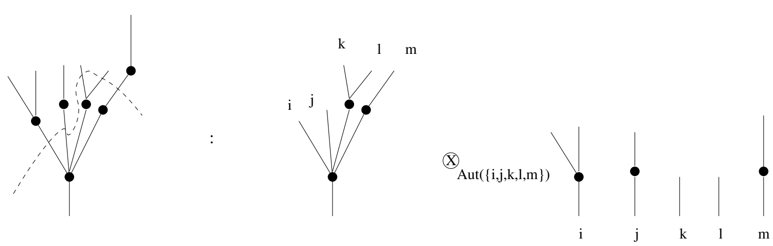

FIGURE 4. An example of a cut for the coinvarants yielding the relative tensor product according to Remark 2.69 part (2).

2.1.3. **Original version.** In the original version of Connes and Kreimer [CK98], the trees are rooted, but have no half–edges or tails. There is a planar and a non–planar version, see e.g. [Foi02b, Foi02a]. These trees are not glued, but only cut using admissible cuts. During the cutting both half edges of a cut edge are removed, see Figure 5. In order to obtain this structure from the ones above, one has to amputate the leaves. In the specific example this can easily be done, but in the general setup this can be achieved by adding certain structure maps, see §2.10. An alternative view of this procedure is given by adding trees without tails, see §2.10.1.

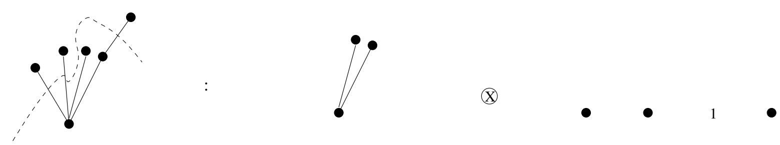

FIGURE 5. co–product for the amputated version. The same example for the amputated version: First all tails are removed. After cutting all newly formed tails are amputated and empty trees/forests are represented by $1 = 1_K$. Notice that indeed $||$ from Figure 2 is set to 1_K as is done in the Hopf quotient.

Notice that the amputation of tails identifies $|$ with 1 and also does not preserve the degree. The grading on $\mathscr{H}_{CK}$ is instead given by the co-radical degree, which in the case at hand is the number of vertices. The relationship is discussed in §2.12.

2.2. **Operads.** We now formalize and generalize the construction above. We start by reviewing operads in §2.2.1. In general, for a non–Σ operad satisfying some additional conditions, dualizing the simultaneous gluing γ, yields a co–product on the free algebra over the graded dual to the operad. The single gluing operations $\circ_i$ assemble to a pre–Lie product, which dually gives the structure of an infinitesimal co–pre–Lie algebra, see §2.7.1. For symmetric operads, we obtain similar structure on the free symmetric algebra over the graded dual, see §2.9. To obtain Hopf algebras, one takes a quotient by an ideal, see §2.5. The result is connected –and hence Hopf— if certain conditions are met. Finally, the amputated version is discussed in §2.10.

2.2.1. **Non-Σ pseudo-operads.** There are several sources for a full definition, see e.g. [MSS02, Kau04a], which we refer to. Loosely an operad is a collection of "somethings" with n inputs and one output, like functions of several variables. And just like for functions there are permutations of variables and substitution operations.

To make things concrete: consider the category $\mathcal{V}ect_k$ of k–vector spaces with the monoidal structure $\otimes$ given by the tensor product $\otimes_k$.

A non-Σ *pseudo-operad* in this category is given by a collection $\mathcal{O}(n)$ of Abelian groups, together with structure maps

$$\circ_i : \mathcal{O}(k) \otimes \mathcal{O}(m) \to \mathcal{O}(k+m-1) \text{ for } 1 \leq i \leq k \tag{2.1}$$

which are associative in the appropriate sense, that is:

$$(- \circ_i -) \circ_j - = \begin{cases} - \circ_i (- \circ_{j-i+1} -) & \text{if } i \leq j < m+i \\ ((- \circ_j -) \circ_{i+n-1} -)\pi & \text{if } 1 \leq j < i. \end{cases} \tag{2.2}$$

Here $\pi = (23) : \mathcal{O}(k) \otimes \mathcal{O}(m) \otimes \mathcal{O}(n) \cong \mathcal{O}(k) \otimes \mathcal{O}(n) \otimes \mathcal{O}(m)$.

Remark 2.1. The data we need to write down the equations above are a monoidal, aka. tensor, product $\otimes_k$—to write down the morphisms $\circ_i$, associativity constraints $A_{UVW} : (U \otimes V) \otimes W \xrightarrow{\cong} U \otimes (V \otimes W) : (u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$, —these are needed to re–bracket— and commutativity constraints $C_{UV} : U \otimes V \xrightarrow{\cong} \otimes U : u \otimes v \mapsto v \otimes u$ —these are needed to permute the factors—. Additional data for a monoidal category are a unit $\mathbb{1}$, $\mathbb{1} = k$ in $\mathcal{V}ect_k$, and the unit constraints $U_L : V \otimes \mathbb{1} \xrightarrow{\cong} V, U_R : k \otimes V \to V$.

Thus, in general the $\mathcal{O}(n)$ are objects in a symmetric monoidal category, which is the following data: a category $\mathcal{C}$ together with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, the $A_{XYZ}, C_{XY}, \mathbb{1}, U_L, U_R$ which have to satisfy natural compatibility axioms, see e.g. [Kas95].

The categories we will consider are the category of sets $\mathcal{S}et$ with disjoint union $\amalg$ and unit $\varnothing$, k–vector spaces k-$\mathcal{V}ect$ with $\otimes_k$, differential graded k vector spaces $dg\mathcal{V}ect$ with $\otimes_k$, unit k in degree 0, the usual additive grading $\deg(u \otimes v) = \deg(u) + \deg(v)$ and the graded commutativity constraint $C_{UV}(u \otimes v) = (-1)^{\deg(u)+\deg(v)}(v \otimes u)$, Abelian groups $\mathcal{A}b$ with $\otimes_{\mathbb{Z}}$ with unit $\mathbb{Z}$, or $g\mathcal{A}b$ graded Abelian groups with $\otimes_{\mathbb{Z}}$, unit $\mathbb{Z}$ in degree 0 and the additive grading and graded commutativity. The associativity and unit constraints are the obvious ones.

We call $\mathcal{O}$ reduced if $\mathcal{O}(1) = \mathbb{1}$, the unit of the monoidal category.

2.2.2. **Pseudo-operads.** If we add the condition that each $\mathcal{O}(n)$ has an action of the symmetric group $\mathbb{S}_n$ and that the $\circ_i$ are equivariant with respect to the symmetric group actions in the appropriate sense, we arrive at the definition of a pseudo-operad. Given a non–Σ pseudo–operad, we can always produce a pseudo operad by tensoring $\mathcal{O}(n)$ with the regular representation of $\mathbb{S}_n$.

Example 2.2. A very instructive example is that of multivariate functions, given by the collection $\{End(X)(n) = Hom(X^{\otimes n}, X)\}$. The $\circ_i$ act as substitutions, that is, $f_1 \circ_i f_2$ substitutes the function f_2 into the ith variable of f_1. The symmetric group action permutes the variables.

The equivariance then states that it does not matter if one permutes first and then substitutes or the other way around, provided that one uses the correct permutation.

As it is defined above $\{End(X)(n)\}$ is just an peudo–operad in sets. If X is a vector space V over k, and $\otimes$ is the tensor product over k and the functions are multilinear and $Hom(V^{\otimes n}, V)$ are again a vectors spaces. That is Hom is actually what is called an internal hom, denoted by $\underline{Hom}$, i.e. $\underline{Hom}(V^{\otimes n}, V)$ is the vector space of multilinear maps. If one takes X to be a set or a compactly generated Hausdorff space $\otimes$ stands for the Cartesian product $\times$ and one uses the compact-open topology on the set of maps to obtain a space — again an internal hom. More generally, if the monoidal category $\mathcal{C}$ is closed, that means that internal homs exist and $\otimes$ and $\underline{Hom}$ are adjoint, viz. $Hom(U\otimes V, W) \leftrightarrow Hom(U, \underline{Hom}(V, W))$ are in natural bijection, then the $\underline{End}(n) = \underline{Hom}(V^{\otimes n}, V)$ form an operad in $\mathcal{C}$.

2.2.3. **The main examples.** Here we give the main examples which underlie the three Hopf algebras above. Notice that not all of them directly live in $\mathcal{A}b$ or $\mathcal{V}ect_k$, but for instance live in $\mathcal{S}et$. There are then free functors, which allow one to carry these over to $\mathcal{A}b$ or $\mathcal{V}ect_k$ as needed.

Example 2.3. The operad of leaf-labelled rooted trees. We consider the set of rooted trees with n-labelled leaves, which means a bijection is specified between the set of leaves and $\{1, \dots, n\}$. Given a n-labelled tree τ and an m-labelled tree τ', we define an $(m+n-1)$-labelled tree $\tau \circ_i \tau'$ by grafting the root of τ' onto the ith leaf of τ to form a new edge. The root of the tree is taken to be the root of τ and the labelling first enumerates the first $i-1$ leaves of τ, then the leaves of τ' and finally the remaining leaves of τ, see Figure 1.

The action of $\mathbb{S}_n$ is given by permuting the labels.

There are several interesting suboperads, such as that of trees whose vertices all have valence k. Especially interesting are the cases $k = 2$ and 3: also known as the linear and the binary trees respectively. Also of interest are the trees whose vertices have valence at least 3.

Example 2.4. The non-Σ operad of (unlabelled) planar planted trees. A planar planted tree is a planar rooted tree with a linear order at the root. Planar means that there is a cyclic order for the flags at each vertex. Adding a root promotes the cyclic order at all of the non-root vertices to a linear order, the flag in the direction of the root being the first element. For the root vertex itself, there is no canonical choice for a first vertex, and planting makes a choice for first flag, which sometimes called the root flag. With these choices, there is a linear order on all the flags and in particular there is a linear order to all the leaves. Thus, we do not have to give them extra labels for the gluing: there is an unambiguous i-th leaf for each planar planted tree with $\geq i$ leaves, and $\tau \circ_i \tau'$ is the tree obtained by grafting the root flag of τ' onto that i-th leaf.

Restricting the valency of the vertices to be either constant, e.g. 3–valent vertices only, or less or equal to a given bound yields suboperads.

Example 2.5. The operad of order preserving surjections, also known as planar labelled corollas, or just the associative operad. Consider n-labelled planar corollas, that is, rooted trees with one vertex, tails labelled by $\underline{n} = \{1, \dots, n\}$ and an order on them. For an n-labelled planar corolla c_n and an m-labelled planar corolla c_m define $c_n \circ_i c_m = c_{n+m-1}$. This is the $(n+m-1)$-labelled planar corolla with the same relabelling scheme as in example 2.3 above. This corresponds to splicing together the orders on the sets. Alternatively, the gluing can be thought of as the gluing on planar labelled trees followed by the edge contraction of the new edge, see Figure 6.

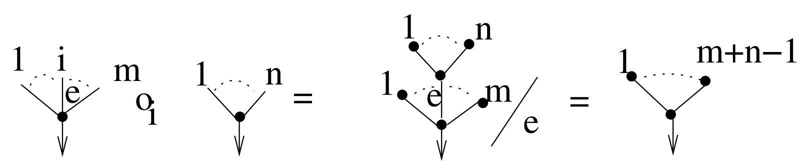

FIGURE 6. Grafting of rooted corollas as first grafting trees and then contracting the new edge.

Alternatively we can think of such a corolla as the unique order preserving map from the ordered set $(\underline{n} = \{1, \dots, n\}, <)$, to the one element set $\underline{1} = \{1\}$ with its unique order. The composition $\circ_i$, of the maps is given by substitution, that splicing in $(\underline{n}, <)$ into the position of i. This corresponds to gluing the planar corollas. The $\mathbb{S}_n$ action permutes the labels and acts effectively on the possible orders. Forgetting the $\mathbb{S}_n$ operations this is a non-Σ operad.

There is another non-Σ version, that of unlabelled planar corollas. If c_n is unique unlabeled planar corolla, then the operations are $c_n \otimes c_m = c_{n+m-1}$. We obtain back the symmetric operad above as $c_n \times \mathbb{S}_n$, where the operad structure on $\mathbb{S}_n$ is given by block permutations, see e.g. [MSS02, Kau04a] and $\mathbb{S}_n$ acts on itself. The identification uses that an element σ of $\mathbb{S}_n$ gives a unique order to the set $\underline{n} : \sigma(1) < \cdots < \sigma(n)$ and the block permutation corresponds to splicing in the orders, which is alternatively just the re–labelling scheme, see [Kau04a]. Forgetting the $\mathbb{S}_n$ action this also identifies the unlabelled planar corollas, with

$$0' \longrightarrow 1' \longrightarrow \dots \longrightarrow n' \qquad \circ_i \qquad 0'' \longrightarrow 1'' \longrightarrow \dots \longrightarrow m''$$

$$\begin{tikzcd}
 & & & \circ_i(0'') \arrow[r] \arrow[d, equal] & \circ_i(1'') \arrow[r] \arrow[d, equal] & \dots \arrow[r] & \circ_i(m'') \arrow[d, equal] & & \\
0 \arrow[r] \arrow[d, equal] & 1 \arrow[r] \arrow[d, equal] & \dots \arrow[r] & i-1 \arrow[r] \arrow[d, equal] & i \arrow[r] & \dots \arrow[r] & i+m-1 \arrow[r] \arrow[d, equal] & \dots \arrow[r] & m+n-1 \arrow[d, equal] \\
\circ_i(0') \arrow[r] & \circ_i(1') \arrow[r] & \dots \arrow[r] & \circ_i((i-1)') \arrow[rrr] & & & \circ_i(i') \arrow[r] & \dots \arrow[r] & \circ_i(n')
\end{tikzcd}$$

FIGURE 7. Splicing together simplices. Primes and double primes are mnemonics only

the the non–Σ sub–operad of order preserving surjections of the sets $\underline{n}$ with their natural order. Vice–versa, the unlabelled version is given by the $\mathbb{S}_n$ co–invariants.

Example 2.6. Simplices form a non-Σ operad (see also Proposition 4.3 for another dual operad structure). We consider $[n]$ to be the category with $n+1$ objects $\{0, \dots, n\}$ and morphisms generated by the chain $0 \to 1 \to \dots \to n$. The i–th composition of $[m]$ and $[n]$ is given by the following functor $\circ_i : [m] \sqcup [n] \to [m+n-1]$. On objects of $[m] : \circ_i(l) = l$ for $l < i$ and $\circ_i(l) = l+n-1$ for $l \geq i$. On objects of $[n] : \circ_i(l) = i-1+l$. On morphisms: the morphism $l-1 \to l$ of $[m]$ is sent to the morphism $l-1 \to l$ of $[m+n-1]$ for all $l < i$, the morphism $i-1 \to i$ of $[m]$ is sent to the composition of $i-1 \to i \dots \to i+n-1$ in $[m+n-1]$, the morphism $l-1 \to l$ of $[m]$ to $l+n-1 \to l+n$ of $[m+n-1]$ for $l > i$ and finally sends the morphism $k \to k+1$ of $[n]$ to $k+i \to k+1+i$.

In words, one splices the chain $[n]$ into $[m]$ by replacing the i-th link, see Figure 7. This is of course intimately related to the previous discussion of order preserving surjections. In fact the two are related by Joyal duality, cf. C.1, as we will explain in §7.3.2, see in particular Figure 11.

2.2.4. **The $\circ$-product aka. pre-Lie structure.** One important structure going back to Gerstenhaber [Ger64] is the following bilinear map:

$$a \circ b := \sum_{i=1}^{n} a \circ_i b \text{ if } a \text{ has operad degree } n \tag{2.3}$$

This product is neither commutative nor associative but preLie, which means that it satisfies the equation $(a \circ b) \circ c - a \circ (b \circ c) = (a \circ c) \circ b - a \circ (c \circ b)$.

An important consequence is that $[a, b] = a \circ b - b \circ a$ is a Lie bracket.

Remark 2.7. One considers a graded version with "shifted" degrees in which $\mathcal{O}(n)$ has degree $n-1$. The operation $\circ_i$ are then of degree 1 and one sets:

$$a \circ b := \sum_{i=1}^{n} (-1)^{(i-1)(n-1)} a \circ_i b \text{ if } a \in \mathcal{O}(n) \tag{2.4}$$

The algebra is graded pre–Lie [Ger64] and the commutator is odd Lie. This is done e.g. in the co–bar construction and is highly relevant for several constructions, see [KWZ12] for a full discussion

2.2.5. **(Non-Σ) Operads.** Another almost equivalent way to encode the above data is as follows. A non–Σ operad is a collection $\mathcal{O}(n)$ together with structure maps

$$\gamma = \gamma_{k;n_1,\dots,n_k} : \mathcal{O}(k) \otimes \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k) \to \mathcal{O}(\sum_{i=1}^{k} n_i) \tag{2.5}$$

Such that map γ is associative in the sense that if $(n_1, \dots, n_k)$ is k a partition of n, and $(n_1^i, \dots, n_{l_i}^i)$ are l_i partitions of the n_i, $l = \sum_{i=1}^k l_i$ then

$$\gamma_{k;n_1,\dots,n_k} \circ id \otimes \gamma_{l_1;n_1^1,\dots,n_{l_1}^1} \otimes \gamma_{l_2;n_1^2,\dots,n_{l_2}^2} \otimes \cdots \otimes \gamma_{l_k n_1^k,\dots,n_{l_k}^k} = \\ \gamma_{l;n_1^1,\dots,n_{l_1}^1,n_1^2,\dots,n_{l_2}^2,\dots,n_1^k,\dots,n_{l_k}^k} \circ \gamma_{k;l_1,\dots,l_k} \otimes id^{\otimes l} \circ \pi \tag{2.6}$$

as maps $\mathcal{O}(k) \otimes \bigotimes_{i=1}^k (\mathcal{O}(l_i) \otimes \bigotimes_{j=1}^{l_i} \mathcal{O}(n_i^j)) \to \mathcal{O}(n)$, where π permutes the factors of the $\mathcal{O}(l_i)$ to the right of $\mathcal{O}(k)$. Notice that we chose to index the operad maps, since this will make the operations easier to dualize. The source and target of the map are then determined by the length k of the index, the indices n_i and their sum.

For an operad, aka. symmetric operad, one adds the data of an $\mathbb{S}_n$ action on each $\mathcal{O}(n)$ and demands that the map γ is equivariant, again in the appropriate sense, see Example 2.2 or [MSS02, Kau04a].

Definition 2.8. A (non–Σ) operad is called *locally finite*, if any element $a_n \in \mathcal{O}(n)$ is in the image of only finitely many $\gamma_{k;n_1,\dots,n_k}$, where $n = \sum_i n_i$.

Lemma 2.9. *If $\mathcal{O}(0)$ is empty in $\mathcal{S}et$ or zero in an Abelian category, then $\mathcal{O}$ is locally finite.*

Proof. There are only finitely many partitions of n into k non–zero elements. $\square$

Example 2.10. If we consider planar planar leaf labelled trees and allow leaf vertices, that is vertices with no inputs, then there is $\check{\mathcal{O}}(0)$, namely trees without any leaves, but the operad is still locally finite. Indeed the number of vertices is conserved under gluing, so there are only finitely many possible pre–images.

2.2.6. **Morphisms.** Morphisms of (pseudo)–operads $\mathcal{O}$ and $\mathcal{P}$ are given by a family of morphisms $f_n : \mathcal{O}(n) \to \mathcal{P}(n)$ that commute with the structure maps. E.g. $f_n(a) \circ_i^{\mathcal{P}} f_m(b) = f_{n+m-1}(a \circ_i b)$. If there are symmetric group actions, then the maps f_n should be $\mathbb{S}_n$ equivariant.

Example 2.11. If we consider the operad of rooted leaf labelled trees $\mathcal{O}$ there is a natural map to the operad of corollas $\mathcal{P}$ given by $\tau \mapsto \tau/E(\tau)$, where $\tau/E(\tau)$ is the corolla that results from contracting all edges of τ. This works in the planar and non–planar version as well as in the pseudo-operad setting, the operad setting and the symmetric setting. This map contracts all linear trees and identifies them with the unit corolla. Furthermore, it restricts to operad maps for the sub-operads of k-regular or at least k-valent trees.

An example of interest considered in [Gon05] is the map restricted to planar planted 3-regular tress (sometimes called binary). The kernel of this map is the operadic ideal generated by the associativity equation between the two possible planar planted binary trees with three leaves.

2.2.7. **Units.** The two notions of pseudo-operads and operads become equivalent if one adds a unit.

Definition 2.12. A *unit* for a pseudo-operad is an element $u \in \mathcal{O}(1)$ such that $u \circ_1 b = b$ and $b \circ_i u = b$ for all m, for all $1 \leq i \leq m$ and all $b \in \mathcal{O}(m)$.

A unit for an operad is an element $u \in \mathcal{O}(1)$ such that

$$\gamma_{1;n}(u; a) = a \text{ and } \gamma_{n;1,\dots,1}(a; u, \dots, u) = a \tag{2.7}$$

There is an equivalence of categories between unital pseudo–operads and unital operads. It is given by the following formulas: for $\alpha \in \mathcal{O}(n), b \in \mathcal{O}(m)$ with $n \geq i$

$$a \circ_i b = \gamma_{n,1,\dots,1,m,1,\dots,1}(a; u, \dots, u, b, u, \dots, u) \text{ } b \text{ in the } i\text{–th place} \tag{2.8}$$

and vice–versa for $a \in \mathcal{O}(k)$:

$$\gamma_{k;n_1,\dots,n_k}(a; b_1, \dots, b_k) = ((\dots((a \circ_k b_k) \circ_{k-1} b_{k-1}) \dots) \circ_1 b_1) \tag{2.9}$$

Morphisms for (pseudo)–operads with units should preserve the unit.

Remark 2.13. The component $\mathcal{O}(1)$ always forms an algebra via $\gamma : \mathcal{O}(1) \otimes \mathcal{O}(1) \to \mathcal{O}(1)$. If there is an operadic unit, then this algebra is unital. More precisely, the algebra is over $R = Hom(\mathbb{1}, \mathbb{1})$. In the case of operads in K-vector spaces the algebra is a $K = Hom(K, K)$ algebra, in the case of operads in Abelian groups, this is a $\mathbb{Z}$ algebra and in in the case of operads in sets, being an algebra reduces to being monoid.

Remark 2.14. In order to transport *Set* operads to Abelian groups and vector spaces or R-modules, we can consider the free Abelian groups generated by the sets $\mathcal{O}(n)$ and furthermore extend coefficients. In particular, we can induce co–operads in different categories, by extending coefficients, say from $\mathbb{Z}$ to $\mathbb{Q}$ or a field K in general. More generally, we can consider, the adjoint to the forgetful functor [Kel82] for any enriched category.

2.2.8. **Non–connected version of an operad.**

Assumption 2.15. In this section for concreteness, we will assume that we are in an Abelian monoidal categories whose bi–product distributes over tensors. and use $\bigoplus$ for the biproduct. The usual categories to keep in mind are those of Abelian groups $\mathcal{A}b$, graded Abelian groups $g\mathcal{A}b$, $\mathcal{V}ect_k$ vector spaces over a field k. If $\mathcal{O}$ is a $\mathcal{S}et$ operad, we tacitly consider its Abelianization, that is $\{\mathbb{Z}\mathcal{O}(n)\}$ which we still denote by $\mathcal{O}$. We will also assume that $\mathcal{O}$ is locally finite.

In [KWZ12, §A.1], we introduced the non–connected operad corresponding to an operad.

$$\mathcal{O}^{nc}(n) := \bigoplus_{k,(n_1,\dots,n_k):\sum_i n_i = n} \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k) \tag{2.10}$$

There is a natural multiplication $\mu := \otimes : \mathcal{O}^{nc}(n) \otimes \mathcal{O}^{nc}(m) \to \mathcal{O}^{nc}(n+m)$ which identifies $\bigoplus_n \mathcal{O}^{nc}(n) = \bar{T}(\bigoplus_n \mathcal{O}(n))$ that is the reduced tensor algebra. The pseudo–operad structure naturally extends, see [KWZ12] and the operad structure extends via

$$\gamma^{nc}(a_k \otimes b_l; a_{n_1}, \dots, a_{n_k}, b_{m_1}, \dots, b_{m_l}) = \gamma(a_k; a_{n_1}, \dots, a_{n_k}) \otimes \gamma(b_l; b_{m_1}, \dots, b_{m_l}) \tag{2.11}$$

for $a_i, b_i \in \mathcal{O}(i)$. More precisely, $\gamma^{nc} \circ (\mu \otimes id) \circ \pi = \mu \circ (\gamma \otimes \gamma)$ where π is the permutation permutes correct factor into the second place. This association $\mathcal{O} \to \mathcal{O}^{nc}$ is functorial.

Remark 2.16. In fact, the operad structure on $\mathcal{O}^{nc}$ is up to permutation exactly the operation that appears in l.h.s. of the associativity equation (2.6). The nc–version also appears naturally in the formulation in terms of Feynman categories, cf. 7.8.

Example 2.17. In the example of planar planted trees, $\mathcal{O}^{nc}(n)$ are planar planted forests, these are ordered collections of planar planted trees, with n leaves and in particular $\mathcal{O}(n_1)\otimes\cdots\otimes\mathcal{O}(n_k)$ contains the forests with k trees. The ith tree has n_i leaves and the total number of leaves is n. The operad gluing grafting a forest of n trees onto a forest with n leaves, but grafting the i–th tree to the i–th label. The pseudo–operad structure does one of these graftings and leaves the other trees alone, but shifting them into the right position, see [KWZ12] or use (2.8).

2.2.9. **Bi–grading and algebra over monoid structure.** There is another way to view the operad $\{\mathcal{O}^{nc}(n)\}$. First notice that there is an internal grading by tensor length. Set $\mathcal{O}=\bigoplus_k\mathcal{O}(k)$ and let $\mathcal{O}^{nc}(n,k)\subset\mathcal{O}^{nc}(k)$ be the tensors of length k. Then $\mathcal{O}^{nc}(n)=\bigoplus_k\mathcal{O}^{nc}(n,k)$ an set $\mathcal{O}^{nc}=\bigoplus_n\mathcal{O}^{nc}(n)$. Summing the $\gamma_{k;n_1,\dots,n_k}$ over the partitions $(n_1,\dots,n_k)$ with fixed k and n, one obtains maps $\gamma_{k,n}:\mathcal{O}(k)\otimes\mathcal{O}^{nc}(n,k)\to\mathcal{O}(n)$. Further summing over the $\gamma_{k,n}$ over k, we obtain a map $\gamma_n:\mathcal{O}\otimes\mathcal{O}^{nc}(n)\to\mathcal{O}(n)$, lastly summing over n, we obtain a map $\gamma:\mathcal{O}\otimes\mathcal{O}^{nc}\to\mathcal{O}$. Note that by un–bracketing tensors there is an identification: $(\mathcal{O}^{nc})^{nc}=\bar{T}(\bar{T}\mathcal{O})\simeq\bar{T}\mathcal{O}=\mathcal{O}^{nc}$.

Proposition 2.18. *Under the assumption 2.15, the associativity of γ implies that $\mathcal{O}^{nc}$ is an associative monoid induced by γ^{nc} and $\mathcal{O}$ is a left module over $\mathcal{O}^{nc}$ via γ.*

Proof. Indeed $\tilde{\gamma}^{nc}:\mathcal{O}^{nc}\otimes\mathcal{O}^{nc}\overset{\simeq}{\to}\mathcal{O}^{nc}\otimes(\mathcal{O}^{nc})^{nc}\overset{\gamma^{nc}}{\to}\mathcal{O}^{nc}$ is a multiplication. The multiplication $\tilde{\gamma}^{nc}$ is associative by the associativity of γ^{nc}, which follows from that of γ via the definition. The associativity diagram corresponding to (2.6) is

$$\begin{tikzcd}
\mathcal{O}\otimes\mathcal{O}^{nc}\otimes\mathcal{O}^{nc} \arrow[r, "id\otimes\gamma^{nc}"] \arrow[d, "\tilde{\gamma}\otimes id"] & \mathcal{O}\otimes\mathcal{O}^{nc} \arrow[d, "\gamma"'] \\
\mathcal{O}\otimes\mathcal{O}^{nc} \arrow[r, "\gamma"'] & \mathcal{O}
\end{tikzcd} \tag{2.12}$$

which is, at the same time, the statement that $\mathcal{O}$ is a left module over $\mathcal{O}^{nc}$. □

2.3. Co–operads.

The relevant constructions will all involve the dual notion to operads, that is co–operads. In terms of trees this provides the transition from grafting to cutting.

2.3.1. **Non-Σ co–operads.** Dualizing the notion of an operad, we obtain the notion of a co–operad. That is, there are structure maps dual to the ones (2.5) of for all n, k and partitions $(n_1, \dots, n_k)$ of m,

$$\check{\gamma}_{k;n_1,\dots,n_k} : \check{\mathcal{O}}(n) \to \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(n_1) \otimes \cdots \otimes \check{\mathcal{O}}(n_k) \tag{2.13}$$

which satisfy the dual relations to (2.6). That is,

$$id \otimes \check{\gamma}_{l_1;n_1^1,\dots,n_{l_1}^1} \otimes \check{\gamma}_{l_2;n_1^2,\dots,n_{l_2}^2} \otimes \cdots \otimes \check{\gamma}_{l_k;n_1^k,\dots,n_{l_k}^k} \circ \check{\gamma}_{k;n_1,\dots,n_k} = \tag{2.14}$$

$$\pi \circ (\check{\gamma}_{k;l_1,\dots,l_k} \otimes id^{\otimes l}) \circ \check{\gamma}_{l;n_1^1,\dots,n_{l_1}^1,n_1^2,\dots,n_{l_2}^2,\dots,n_1^k,\dots,n_{l_k}^k} \tag{2.15}$$

as maps $\check{\mathcal{O}}(n) \to \check{\mathcal{O}}(k) \otimes \bigotimes_{i=1}^k (\check{\mathcal{O}}(l_i) \otimes \bigotimes_{j=1}^{l_i} \check{\mathcal{O}}(n_i^j))$, for any k-partition $(n_1, \dots, n_k)$ of n and l_i-partitions $(n_1^i, \dots, n_{l_i}^i)$ of n_i. Either side of the relation determines these partitions and hence determines the other side. Here $l = \sum l_i$ and π is the permutation permuting the factors $\check{\mathcal{O}}(l_i)$ to the left of the factors $\check{\mathcal{O}}(n_i^j)$.

The example of a co–operad that is pertinent to the three constructions is given by dualizing an operad $\mathcal{O}$. In particular, if $\mathcal{O}$ is an operad in (graded) Abelian groups $\check{\mathcal{O}}(n) = \underline{Hom}(\mathcal{O}(n), \mathbb{Z})$, that is the group homomorphisms considered as a (graded) Abelian group is the dual co–operad. For (dg)–k–Vector spaces the dual co–operad is $\check{\mathcal{O}}(n) = \underline{Hom}(\mathcal{O}(n), k)$. This construction works in any closed monoidal category by setting $\check{\mathcal{O}}(n) = \underline{Hom}(\mathcal{O}, \mathbb{1})$, where $\underline{Hom}$ is the internal hom.

Lemma 2.19. *The dual of an operad, $\check{\mathcal{O}}(n) = \underline{Hom}(\mathcal{O}, \mathbb{1})$, in a closed monoidal category is a co–operad and this association is functorial. Likewise, if the objects in the monoidal category are graded, the graded dual of $\mathcal{O}$ is also functorially a co–operad.*

Proof. The association $\mathcal{O}(n) \to \check{\mathcal{O}}(n)$ is contravariant and all the diagrams to check are the dual diagrams. The functoriality is straight–forward. □

Remark 2.20. In a linear category the maps $\check{\gamma}_{k,n_1,\dots,n_k}$ can be 0. If this is not the case, e.g. in $\mathcal{S}et$, one can weaken the conditions to state that the $\check{\gamma}_{k,n_1,\dots,n_k}$ are partially defined functions, (2.14) holds whenever it is defined, and the r.h.s. exists, if and only if the l.h.s. does.

2.3.2. **Examples based on free constructions on $\mathcal{S}et$ operads.** If the operad $\{\mathcal{O}(n)\}$ is a $\mathcal{S}et$ operad then we can consider the free Abelian groups $\mathbb{Z}[\mathcal{O}(n)] \subset Hom(\mathcal{O}(n), \mathbb{Z})$ or (free) vector spaces $k[\mathcal{O}(n)] \subset Hom(\mathcal{O}(n), k)$ in $\mathcal{A}b$ or in k-$\mathcal{V}ect$. In this case, there is standard notation. For an element $\tau \in \mathcal{O}(n)$, we have the characteristic function $\delta_\tau : \mathcal{O}(n) \to \mathbb{1}$ given by $\delta_\tau(\tau') = 1$ if $\tau = \tau'$ and 0 else. Then an element in the free Abelian group or the vectors space generated by

$\mathcal{O}(n)$ is just a finite formal sum $\sum_{i\in I} n_i\delta_{\tau_i}$. By abuse of notation this is often written as $\sum_{i\in I} n_i\tau_i$. The dual on these spaces are then again formal sums of characteristic functions $\sum_{\alpha\in J} n_\alpha\tau^*_\alpha$, where the $\tau^* = ev_\tau$ are the evaluation maps at τ. This is course the known embedding $V \to (\check{V})^\vee$ for vector spaces and the fact that the dual of a direct sum is a product.

Remark 2.21. We collect the following straight–forward facts:

(i) If the $\mathcal{O}(n)$ are finite sets or finite free then the formal sums reduce to finite formal sums. Dropping the superscript $*$, we again can identify elements of $\check{\mathcal{O}}(n)$ as finite formal linear combinations.
(ii) In the general case: Finite formal sums are a sub–co–operad if and only if $\mathcal{O}(n)$ is locally finite.
(iii) The analogous statements hold for graded duals of free graded $\mathcal{O}(n)$. If the internal grading is preserved by γ and the bi–graded pieces of $\mathcal{O}^{nc}$ are finite dimensional, then the graded dual has only finite sums. This is usually the starting point for the constructions mentioned in the introduction.

Example 2.22. In the case of leaf labelled rooted trees, the sets $\mathcal{O}(n)$ are finite precisely if one excludes vertices of valence 2. Otherwise, the $\mathcal{O}(n)$ are infinite. There is an internal grading by the number of vertices. This grading is respected by the operad and hence the co–operad structure. Adding the internal grading, the graded pieces $\mathcal{O}_r(n)$ with r vertices are finite. The bi–graded pieces $\mathcal{O}^{nc}_r(n)$ are all finite dimensional. This is a consequence of the fact that $\mathcal{O}(0)$ has only positive internal degree —no non–vertex trees without leaves. Using the vertex grading, one can also directly see that the operad is locally finite and the graded dual are only finite sums.

Remark 2.23. There is also the notion of a partial or colored operad. This means that there is a restriction on the gluing morphisms that the inputs can only be glued to like outputs, cf. §7.3. The dual of such a partial structure is actually a co–operad. The point is that in the dual there is no restrictions as one only decomposes what has previously been composed, see [Kau04a] for this point of view.

Example 2.24 (The overlapping sequences (co)–operad)**.** Consider a set S. Let $\check{\mathcal{O}}(n)$ = the free Abelian group (or vector space) on the set of finite sequences of length $n+1$ in S. We define the co–operad structure as the decomposition of the set into overlapping ordered partitions:

$$\check{\gamma}_{k,n_1,\dots,n_k}(a_0;a_1,\dots,a_n-1;a_n)= \sum_{0=i_0<i_1<\dots<i_k=n}(a_0=a_{i_0};a_{i_1},a_{i_2},\dots;a_{i_k}=a_n) \otimes(a_0=a_{i_0};a_1,\dots,a_{i_1-1};a_{i_1})\otimes(a_{i_1};a_{i_1+1},\dots,a_{i_2-1};a_{i_2})\otimes\dots\otimes (a_{i_{k-1}};a_{i_{k-1}+1},\dots,a_{n-1};a_{i_k}=a_n) \quad (2.16)$$

The co–operad structure is dual to the free extension of the partial (a.k.a. colored) $\mathcal{S}et$ operad structure, where if $a_i=b_0$ and $a_{i+1}=b_{n+1}$: $(a_0;a_1,\dots,a_n;a_{n+1})\circ_i(b_0;b_1,\dots,,b_m;b_{m+1}) = (a_0;a_1,\dots,a_{i-1},a_i = b_0,b_1,\dots,b_m,b_{m+1} = a_{i+1},a_{i+2},\dots,a_n;a_{n+1})$. This, the the connection Goncharov's Hopf algebra, to Joyal duality C.1, §7.3.2 is why we chose the notation using semi–colons as it gives the colors, and the fact that there are double base–points, the first and the last element.

Note that this co–operad has sub–co–operads given by fixing a particular sequence and considering all subsequences.

This is the example relevant for Goncharov's Hopf algebra and that of Baues when suitably shifted, see Remark 2.36. It will be further discussed in §2.8.1 below. A more in depth consideration explaining the existence of this co–operad is given in §4.

Remark 2.25. Note that the indices of the sequence are given by $\check{\gamma}(0,\dots,n)$ where $(0,\dots,n)$ is thought of as a sequence in $\mathbb{N}_0$. The iterations of the indices, then correspond to splitting or splicing intervals in $\mathbb{N}_0$. This makes contact with the operad structure on simplices and is the basis for the simplicial considerations section §4. The partial operad structure becomes natural when considering Feynman categories where the partial operad structure corresponds to the partial structure of composition of morphisms in a category. The sequences can also be thought of as a , as decorations of angles of the corollas, See Figure 11 and more precisely as decoration in the technical sense for Feynman categories, see §7.3.2.

2.3.3. **Co–operadic co–units.** A morphism $\epsilon:\check{\mathcal{O}}(1)\to\mathbb{1}$ is a left co–operadic co–unit if its extension ϵ_1 by 0 on the $\check{\mathcal{O}}(n), n\neq 1$ satisfies[3]:

$$\sum_k(\epsilon_1\otimes id^{\otimes k})\circ\check{\gamma}=id \quad (2.17)$$

[3]Here and in the following, we suppress the unit constraints in the monoidal category and tacitly identify $V\otimes\mathbb{1}\simeq V\simeq\mathbb{1}\otimes V$.

and a right co–operadic unit if

$$\sum_k (id \otimes \epsilon_1^{\otimes k}) \circ \check{\gamma} = id \tag{2.18}$$

A co–operadic co–unit is a right and left co–operadic co–unit. We will use $\epsilon_1 : \check{\mathcal{O}} \to \mathbb{1}$ for its extension by 0 on all $\check{\mathcal{O}}(n) : n \neq 1$.

Remark 2.26. Note, if there is only one tensor factor on the right, then the left factor has to be $\check{\mathcal{O}}(1)$ by definition. If ϵ_1 would have support outside $\check{\mathcal{O}}(1)$, the $\check{\gamma}$ would have to vanish on the right side for all elements having that left hand side, which is rather non–generic. This is why we assume ϵ vanishes outside $\check{\mathcal{O}}(1)$. It is also the notion naturally dual to an operadic unit.

Lemma 2.27. *The dual of a unital operad is a co–unital co–operad and this association is functorial.*

Proof. A unit $u \in \mathcal{O}(1)$, can be thought of as a map of $u : \mathbb{1} \to \mathcal{O}(1)$, where $\mathbb{1}$ is $\mathbb{Z}$ for Abelian groups or in general the unit object, e.g. k for $\mathcal{V}ect_k$. Its dual is then a morphism $\check{u} := \check{\mathcal{O}}(1) \to \mathbb{1}$. Now, $\check{u} := \epsilon$ is a left/right co–operadic co–unit if it satisfies he equations (2.17) and (2.18), but these are the diagrams dual to the equations (2.7). Functoriality is straight–forward. □

2.3.4. **Morphisms.** Morphisms of co–operads $\check{\mathcal{O}}$ and $\check{\mathcal{P}}$ are given by a family of morphisms $f_n : \check{\mathcal{O}}(n) \to \check{\mathcal{P}}(n)$ that commute with the structure maps

$$\check{\gamma}^{\check{\mathcal{P}}}_{k;n_1,\dots,n_k} \circ f_n = (f_k \otimes f_{n_1} \otimes \cdots \otimes f_{n_k}) \circ \check{\gamma}^{\check{\mathcal{O}}}_{k;n_1,\dots,n_k}$$

2.3.5. **Completeness and Finiteness Assumptions.**

Assumption 2.28 (Completeness Assumption)**.** If the monoidal category in which the co–operad lives is complete and certain limits (in particular, products) commute with taking tensors, then we can define

$$\check{\gamma}_n : \check{\mathcal{O}}(m) \to \prod_k \prod_{(n_1,\dots,n_k):\sum_{i=1}^k n_i = n} \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(n_1) \otimes \cdots \otimes \check{\mathcal{O}}(n_k). \tag{2.19}$$

For the applications, we will use free algebras, which are based on finite products of the $\check{\mathcal{O}}(n)$. In the Abelian monoidal categories of (graded) vector spaces k-$\mathcal{V}ect$, differential graded vector spaces $dg\mathcal{V}ect$, Abelian groups $\mathcal{A}b$, or $g\mathcal{A}b$ graded Abelian groups, these finite product are direct sums. In order to write down the multiplication and the co–multiplication, we will need the maps $\check{\gamma}(m)$ to be locally finite.

Definition 2.29. We call $\{\check{\mathcal{O}}(n)\}$ *locally finite* if for any $a_n \in \check{\mathcal{O}}(n)$: $\check{\gamma}_{k;n_1,\dots,n_k}(a_n) \neq 0$ only for finitely many k partitions of n.

Lemma 2.30. *If there is no $\check{\mathcal{O}}(0)$, then $\{\check{\mathcal{O}}(n)\}$ is locally finite.*

Proof. There are only finitely many partitions $(n_1, \dots, n_k)$ of n with $n_i \geq 1$. □

This implies that in the limits and the limits reduce to finite limits as there are only finitely many maps.

Assumption 2.31 (Basic assumption)**.** We will assume that the co–operads are locally finite and that the co–operads are in an Abelian category with bi–product $\oplus$ which are distributive with the tensor product.

Set $\check{\mathcal{O}} = \bigoplus_m \check{\mathcal{O}}(m)$, summing over the m we obtain morphism $\check{\gamma}$: $\check{\mathcal{O}} \to \check{\mathcal{O}} \otimes \bigoplus_k \check{\mathcal{O}}^{\otimes k} = \check{\mathcal{O}} \otimes \bar{T}\check{\mathcal{O}}$. The right hand side is actually multi–graded. Set

$$\check{\mathcal{O}}^{nc}(n) := \bigoplus_{k,(n_1,\dots,n_k):\sum_i n_i = n} \check{\mathcal{O}}(n_1) \otimes \dots \otimes \check{\mathcal{O}}(n_k) \tag{2.20}$$

2.4. Bi–algebra structure on the non–connected dual of a non–Σ operad.

2.4.1. **Non–connected co–operad.** Just like for an operad, we can define a non–connected version for a co–operad. For trees this is again the transitions to forests.

Composition of tensors, or un–bracketing, gives a multiplication μ : $\check{\mathcal{O}}^{nc}(n) \otimes \check{\mathcal{O}}^{nc}(m) \to \check{\mathcal{O}}(n+m)$. Set $\check{\mathcal{O}} = \bigoplus_n \check{\mathcal{O}}(n)$ and let $\mathscr{B} = \bar{T}\check{\mathcal{O}} = \bigoplus_{n \geq 1} \check{\mathcal{O}}^{nc}(n)$ be the free algebra on $\check{\mathcal{O}}$, then μ is just the free multiplication. This is indeed again a co–operad, which we will use to generalize in §3.

Proposition 2.32. *Under the basic assumption, $\check{\mathcal{O}}^{nc}$ is a cooperad with respect to $\check{\gamma}^{nc}$ defined by*

$$\check{\gamma}^{nc}(a_n \otimes b_m) = \sum_{\substack{k,(n_1,\dots,n_k):\sum_i n_i = n \\ l,(m_1,\dots,m_k):\sum_i m_i = m}} \sum (a_k^{(0)} \otimes b_l^{(0)}) \otimes a_{n_1}^{(1)} \otimes \dots \otimes a_{n_k}^{(k)} \otimes b_{m_1}^{(1)} \otimes \dots \otimes b_{m_l}^{(l)} \tag{2.21}$$

using a multi–Sweedler notation for the $\check{\gamma}$ *and indication the co–operadic degree by subscripts. More precisely,*

$$\check{\gamma}^{nc} \circ \mu = (\mu \otimes \mu) \circ \pi \circ (\check{\gamma} \otimes \check{\gamma}) f \tag{2.22}$$

where π *permutes the first factor of the second* $\check{\gamma}^{nc}$ *into the second place.*

If $\check{\mathcal{O}}$ *is the dual of a locally finite operad* $\mathcal{O}$*, then* $\check{\mathcal{O}}^{nc}$ *is the dual co–operad of the operad* $\mathcal{O}^{nc}$*.*

Proof. Using the equation iteratively, we see that the components of $\check{\gamma}^{nc}$ on $\check{\mathcal{O}}^{nc}(n)$ are $\check{\mathcal{O}}(n_1) \otimes \cdots \otimes \check{\mathcal{O}}(n_k) \to (\bigoplus_{(l_1,\dots,l_k)} \check{\mathcal{O}}(l_1) \otimes \cdots \otimes \check{\mathcal{O}}(l_k)) \otimes \bigoplus \check{\mathcal{O}}(n_1^1) \otimes \cdots \otimes \check{\mathcal{O}}(n_{l_1}^1) \otimes \bigoplus \check{\mathcal{O}}(n_1^2) \otimes \cdots \otimes \check{\mathcal{O}}(n_{l_2}^2) \otimes \cdots \otimes \bigoplus \check{\mathcal{O}}(n_1^k) \otimes \cdots \otimes \check{\mathcal{O}}(n_{l_k}^k) \subset \check{\mathcal{O}}^{nc}(l) \otimes \check{\mathcal{O}}^{nc}(n_1) \otimes \cdots \otimes \check{\mathcal{O}}^{nc}(n_l)$ where the sums are over partitions and $\sum_i l_i = l, \sum_k n_k^i = n_k$ and $\sum_k n_k = n$

The co–associativity for the co–operad structure $\check{\gamma}^{nc}$ follows readily from that of $\check{\gamma}$. The last statement follows from the fact that (2.21) is the dual of (2.11). □

Example 2.33 (Bar of an operad/algebra)**.** A natural way to obtain a co–operad from an operad it given by the operadic bar transform, see e.g. [MSS02]. One can then consider the free algebra on this co–operad. This is much bigger than just doing the tensor algebra on the dual of an operad. A reasonably small version is provided by the bar construction of an algebra, which is a co–algebra, which can also be thought of as a co–operad as an algebra is a operad with only $\mathcal{O}(1)$, see also §7.23 and §7.6.8.

2.4.2. **From non-Σ co–operads to bi–algebras.** There is another way to interpret the co–operad structure on $\check{\mathcal{O}}^{nc}$ in which $\check{\gamma}^{nc}$ becomes a co–multiplication. Let $\check{\mathcal{O}} = \bigoplus_{n\geq 1} \check{\mathcal{O}}(n)$, $\mathscr{B} = \bar{T}\check{\mathcal{O}}$ and $\mathscr{B}' = T\check{\mathcal{O}} = \mathbb{1} \oplus \mathscr{B}$ be the free and free unital associative algebras on $\check{\mathcal{O}}$, then we can decompose $\mathscr{B} = \bigoplus_n \mathscr{B}(n)$:

$$\mathscr{B}(n) = \bigoplus_k \bigoplus_{(n_1,\dots,n_k):\sum_i n_i = n} \check{\mathcal{O}}(n_1) \otimes \cdots \otimes \check{\mathcal{O}}(n_k) = \check{\mathcal{O}}^{nc}(n) \tag{2.23}$$

then the free multiplication is composition of tensors and on components is given by $\mu_{n,m} : \mathscr{B}(n) \otimes \mathscr{B}(m) \to \mathscr{B}(n+m)$.

For $\mathscr{B}'$ we let $\mathscr{B}(0) = \mathbb{1}$ and tacitly use the unit constraints $u_R : X \otimes \mathbb{1} \xrightarrow{\sim} X$ and $u_L : \mathbb{1} \otimes X \xrightarrow{\sim} \mathbb{1}$ to shorten any tensor containing a unit factor, and hence make $\mathscr{B}'$ unital. This defines the unit components of μ: $\mu_{0,n} : \mathscr{B}(0) \otimes \mathscr{B}(n) \to \mathscr{B}(n)$ and $\mu_{n,0} : \mathscr{B}(n) \otimes \mathscr{B}(0) \to \mathscr{B}(n)$.

In this notation, the maps $\check{\gamma}_{k,n_1,\dots,n_k}$ can be seen as maps: $\check{\mathcal{O}}(n) \to \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}^{nc}(n) = \check{\mathcal{O}}(k) \otimes \mathscr{B}(n)$. Summing the $\check{\gamma}_{k;n_1,\dots,n_k}$ over all non–empty k partitions of n, we obtain a map

$$\check{\gamma}_{k,n} : \check{\mathcal{O}}(n) \to \check{\mathcal{O}}(k) \otimes \mathscr{B}(k) \tag{2.24}$$

The extension $\check{\gamma}$ to $\check{\mathcal{O}}^{nc}$ gives maps

$$\Delta_{k,n} : \mathscr{B}(n) \to \mathscr{B}(k) \otimes \mathscr{B}(n) \tag{2.25}$$

If we sum the $\Delta_{k,n}$ over k and n we obtain maps $\Delta_n = \sum_k \Delta_{k,n}$ which define $\Delta = \sum_n \Delta_n$

$$\Delta : \mathscr{B} \to \mathscr{B} \otimes \mathscr{B} \tag{2.26}$$

In $\mathscr{B}'$ we let $\mathscr{B}'(0) = \mathbb{1}$ and Δ extends to $\mathscr{B}'$ via $\Delta(1) = 1 \otimes 1$, for $1 = id_{\mathbb{1}} \in Hom(\mathbb{1}, \mathbb{1}) = \mathbb{1}$. Thus, on $\mathscr{B}'(0,0)$, we have the additional component $\Delta_{0,0} : \mathscr{B}'(0,0) \to \mathscr{B}'(0) \otimes \mathscr{B}'(0)$. This also extends the co–operadic structure of $\check{\mathcal{O}}^{nc}$ to $\mathscr{B}' = \mathbb{1} \oplus \check{\mathcal{O}}^{nc}$.

2.4.3. **Grading.** We have the grading by co-operadic degree deg with degree of $\check{\mathcal{O}}(n)$ being n. In this grading, the graded dual of $\mathcal{O}$ is $\check{\mathcal{O}}$: $(\bigoplus_n \mathcal{O}(n))^\vee = \bigoplus_n \check{\mathcal{O}}(n)$. On $T\check{\mathcal{O}}$, the natural degrees are additive degrees, i.e. for $a \in \check{\mathcal{O}}^{nc}(n)$, $\deg(a) = \sum n_i = n$. We set the degree of elements in $\mathbb{1}$ to be zero. This coincides with the co–operadic grading for $\check{\mathcal{O}}^{nc}$. We also have the grading in $\mathscr{B}$ by tensor length l. This gives a double grading $B = \bigoplus_{n,p} \mathscr{B}(n,p)$ where

$$\mathscr{B}(n,p) = \check{\mathcal{O}}^{nc}(n,p) = \bigoplus_{(n_1,\dots,n_p), \sum n_i = n} \check{\mathcal{O}}(n_1) \otimes \cdots \otimes \check{\mathcal{O}}(n_p) \tag{2.27}$$

In $\mathscr{B}'$ the unit is defined to have length 0: $\mathbb{1} = \mathscr{B}'(0,0)$. Using the bi–grading the components of μ maps are:

$$\mu : \mathscr{B}'(n_1,p_1) \otimes \mathscr{B}'(n_1,p_2) \to \mathscr{B}'(n_1+n_2, p_1+p_2) \tag{2.28}$$

and the non–vanishing components of Δ are:

$$\Delta_{k,n} : \mathscr{B}(n,p) \to \mathscr{B}(k,p) \otimes \mathscr{B}(n,k) \qquad 1 \le p \le k \le n \tag{2.29}$$

the restriction comes from the fact that Δ is an algebra morphism, and hence it does not change the length of the first factor. By definition the degree of the first factor is the length of the second factor and hence the $\Delta_{k,n}$ are the only non–zero components, so that $\Delta_k = \sum_{n \ge k} \Delta_{k,n}$ and $\Delta = \sum_{k,n:k \le n} \Delta_{k,n}$. On $\mathscr{B}'(0,0)$, we have the additional component $\Delta_{0,0} : \mathscr{B}'(0,0) \to \mathscr{B}'(0,0) \otimes \mathscr{B}'(0,0)$.

We define the weight grading on $\mathscr{B}'$ to be given by $wt = \deg - l$. Incorporating an additional internal grading by considering operads in *dg*-$\mathcal{V}ect$ or $g\mathcal{A}b$ is done by adding the external and internal gradings; as usual.

Proposition 2.34. *$\mathscr{B}$ is a bi–algebra and $\mathscr{B}'$ is a unital bi–algebra both graded with respect to* wt. *This association is functorial.*

Proof. We have to check that Δ and μ satisfy the bi–algebra equation that is, if c is decomposable $c = \mu(a \otimes b) = a \otimes b$, then it must satisfy

$$\Delta \circ \mu(a \otimes b) = (\mu \otimes \mu) \circ \pi_{2,3} \circ (\Delta(a) \otimes \Delta(b)) \tag{2.30}$$

where $\pi_{2,3}$ permutes the second and third factor in $\mathscr{B}' \otimes \mathscr{B}' \otimes \mathscr{B}' \otimes \mathscr{B}'$. But, this equation is (2.22) when interpreted in terms of $\mathscr{B}$. Co–associativity follows from (2.15). The extension to $\mathscr{B}'$ follows readily.

For the grading, looking at (2.28), we obtain $n_1 + n_2 - p_1 - p_2$ as the degree for both sides of the multiplication. For the co–product, we see that on both sides of (2.29) the degree is $n - p = k - p + n - k$. The functoriality is clear. ☐

Remark 2.35. To obtain the bi–algebra, we could have alternatively just defined $\mathscr{B} = \bar{T}\check{\mathcal{O}}^{nc}$, as the free algebra, defined $\rho = \check{\gamma} : \check{\mathcal{O}} \to \check{\mathcal{O}} \otimes \mathscr{B}$ and then extended ρ to all of $\mathscr{B}$ as $\Delta : \mathscr{B} \to \mathscr{B} \otimes \mathscr{B}$ via (2.30), without defining the co–operad structure on $\check{\mathcal{O}}^{nc}$. Note that ρ makes $\check{\mathcal{O}}$ into a co–module over $\mathscr{B}$ and $\mathscr{B}'$. The way, it is set up now —co–operad with multiplication— will allow us to generalize the structure in §3.

Remark 2.36 (Shifted version)**.** One obtains the weight naturally using the suspensions $\check{\mathcal{O}}(n)[1]$ of the $\check{\mathcal{O}}(n)$ in (2.20). The suspended operadic degree is the weight. This is analogous to the conventions of signs in the graded pre–Lie structure and in general to using odd operads [KWZ12].

This is also very similar to the co–bar transform ΩC for a co–algebra C, but without the differential. The differential is instead replaced by the co–operad structure, or the co–product. A similar situation is what happens in Baues' construction. Here one can think of a co-bar transform of an algebra of simplicial objects, where the simplicial structure gives the (co)operad structure, see §4.

Remark 2.37. This shifted version only a small part of the operadic co–bar construction which would have components for any tree and the ones in the shifted construction correspond in a precise sense only to level trees that are of height 2. The two constructions are related by enrichment of Feynman categories and B_+ operators. We will not go in to full details here and refer to [KW17, §3,§4] and future analysis.

2.4.4. **Co–module structure.** Dual to (2.12) we can write the co–associativity of $\check{\gamma}$ as

$$\begin{tikzcd} \check{\mathcal{O}} \otimes \check{\mathcal{O}}^{nc} \otimes \check{\mathcal{O}}^{nc} & \check{\mathcal{O}} \otimes \check{\mathcal{O}}^{nc} \arrow[l, "id\otimes\check{\gamma}^{nc}"'] \\ \check{\mathcal{O}} \otimes \check{\mathcal{O}}^{nc} \arrow[u, "\check{\gamma}\otimes id"] & \check{\mathcal{O}} \arrow[l, "\check{\gamma}"'] \arrow[u, "\check{\gamma}"'] \end{tikzcd} \tag{2.31}$$

From this, we obtain the dual to §2.18.

Proposition 2.38. *Under the basic assumption, the co–associativity of $\check{\gamma}$ implies that $\check{\mathcal{O}}^{nc}$ is a co–associative co–monoid induced by $\check{\gamma}^{nc}$ and $\check{\mathcal{O}}$ is a left co–module over $\check{\mathcal{O}}^{nc}$ via $\check{\gamma}$.* □

2.4.5. **Finiteness, $\check{\mathcal{O}}(0)$ and co–modules.** If $\check{\mathcal{O}}(0)$ is empty or zero, then $\{\check{\mathcal{O}}(n)\}$ is locally finite. If there is an internal grading for the $\check{\mathcal{O}}(n)$ that is preserved under $\check{\gamma}$ and positive on $\check{\mathcal{O}}(0)$ then again, $\{\check{\mathcal{O}}(n)\}$ is locally finite. In these cases, there is no problem in considering $\{\check{\mathcal{O}}(n)\}, n \geq 0$. Generally, $\{\check{\mathcal{O}}(n)\}, \geq 1$ is a sub–cooperad of $\{\check{\mathcal{O}}(n)\}, n \geq 0$.

Remark 2.39. One may in this case also consider an $\check{\mathcal{O}}(0)$ of rooted trees without leaves, but not without vertices. The leaf labelled trees are then replaced by rooted trees in general. Algebras over this are algebras over the operad together with a module. Dually, this yields the co–algebra over the co–operad structure.

2.4.6. **Operadic units, co–operadic co–units and bi–algebraic co–units.** If ϵ is a co–unit for the co–operad $\{\check{\mathcal{O}}(n)\}$, $\epsilon : \check{\mathcal{O}}(1) \to \mathbb{1}$, we extend it by 0 to $\epsilon_1 : \check{\mathcal{O}} \to \mathbb{1}$. We further extend ϵ_1 to the co–operad $\check{\mathcal{O}}^{nc}$ by $\epsilon_{tot} = \bigoplus_{n\geq 1} \epsilon_1^{\otimes n}$ and to $\mathscr{B}'$ by $\epsilon_1^{\otimes 0} = id : \mathscr{B}(0) \to \mathbb{1}$ that is $\epsilon_{tot} = \bigoplus_{n\geq 0} \epsilon_1^{\otimes n} : \mathscr{B}' \to \mathbb{1}$.

Proposition 2.40. *The map $\epsilon_{tot} : \mathcal{O}^{nc} \to \mathbb{1}$ is a co–operadic co–unit for $\check{\gamma}$. As a map $\epsilon_{tot} : \mathscr{B}' \to \mathbb{1}$ is a bi–aglebraic co-unit for $\mathscr{B}'$ and its restriction to $\mathscr{B}$ is a bi–algebraic unit for $\mathscr{B}$. Vice–versa, for $\mathscr{B}'$ to have a co–unit, $\check{\mathcal{O}}$ has to be co–unital with co–operadic co–unit ϵ with $\epsilon_1 = \epsilon_{tot}|_{\check{\mathcal{O}}}$ where $\check{\mathcal{O}} \subset \mathscr{B}'$ as $\check{\mathcal{O}} = \bigoplus_{n\geq 1} \mathscr{B}(n, 1)$.*

Moreover, the associations $\{\check{\mathcal{O}}(n)\} \to \mathscr{B}$ and $\{\check{\mathcal{O}}(n)\} \to \mathscr{B}'$ of a co–unital respectively unital and co–unital graded bi–algebra to a co-unital co–operad are functorial.

Proof. On the indecomposables $\mathcal{O}^{nc}$ of $\mathscr{B}$ the fact that ϵ_{tot} is a co–unit is just the fact that ϵ_1 is an operadic co–unit, i.e. satisfies (2.17) and (2.18). For decomposables, we can use induction by using the bi–algebra equation (2.30) and the fact that $\epsilon_{tot} \circ \mu = \mu_{\mathbb{1}} \circ (\epsilon_{tot} \otimes \epsilon_{tot})$ where $\mu_{\mathbb{1}} : \mathbb{1} \otimes \mathbb{1} \to \mathbb{1}$ is given by the unit constraints. This is also the compatibility of the multiplication and the co–unit. The compatibility of the unit and the co–multiplication says that 1 is group–like. Finally, by definition $\epsilon_{tot}(1) = \epsilon_1^{\otimes 0}(1) = 1$.

The fact that this is a necessary condition is Proposition 3.19. For the functoriality: Any map $\mathcal{P} \to \mathcal{O}$ induces a dual map $\check{\mathcal{O}} \to \check{\mathcal{P}}$, which in turn induces a morphism on the free algebras. It is straightforward to check that this map is also a morphism of bi–algebras preserving grading and the unit. In the case of a co–unital co– operad case the co-unit is preserved by the morphisms $\epsilon_{\check{\mathcal{O}}} = \epsilon_{\check{\mathcal{P}}} \circ f$, by definition, and hence also the bi–algebraic co–unit. □

Example 2.41. In the example of leaf labelled rooted forests. A unit for the operad is given by the "degenerate" tree or leaf $|$, where gluing on a leaf leaves the tree invariant. This means that $\mathcal{O}(1) = \mathbb{1} \oplus \bar{\mathcal{O}}(1)$, where $\mathbb{1}$ is spanned by $|$ and $\bar{\mathcal{O}}(1)$ has generators τ which are "ladders", viz. all vertices are bi–valent. The dual co–operadic co–unit is the characteristic function $\epsilon = \partial_{|}$. Indeed $\check{\gamma}(\tau)$ is the sum over all cuts. Applying $\sum_{k\geq 1} \epsilon_1 \otimes id^{\otimes k}$, we see that only the cut which cuts the root half–edge, that is the term $| \otimes \tau$ evaluates to a non–zero value and $\sum_{k\geq 1}(\epsilon_1 \otimes id^k) \circ \check{\gamma}(\tau) = \tau$. As for the right co–unit property, we see that if $\tau \in \mathcal{O}(n)$, then the only terms that survive $id \otimes \epsilon_{tot}$ are the ones with only factors of $|$ on the right, that is $\tau' \otimes | \otimes \cdots \otimes |$. There is precisely one such term with n occurrences of $|$ on the right, which comes from the cut through all the leaves. Indeed, $(\mathrm{id} \otimes \epsilon_{tot}) \circ \check{\gamma}(\tau) = \tau$.

When looking at decomposables in $\mathscr{B}$ or $\mathscr{B}'$, these are forests with more than on tree. The only terms surviving $\epsilon_{tot} \otimes id$ are the ones with only $|$ on the right, which is just one term corresponding to the cut cutting all root half–edges. Similarly to be non–zero under $id \otimes \epsilon_{tot}$ the right terms must all be $|$, and again there is only one cut, namely the cut that cuts all leaves of all the trees in the forest. So, indeed we get $(\epsilon_{tot} \otimes id) \circ \Delta = id = (id \otimes \epsilon_{tot}) \circ \Delta$.

Summing up the results:

Theorem 2.42. *Under the basic assumption, given a co–operad $\{\check{\mathcal{O}}(n)\}$ $\mathscr{B}$ is a graded bi–algebra and $\mathscr{B}'$ is a graded unital bi–algebra. If on only if the underlying co–operad $\{\check{\mathcal{O}}(n)\}$ is a co–unital co–operad, the bi–algebras $\mathscr{B}$ and $\mathscr{B}'$ also have a co–unit. The association of (co–unital)* *operads to graded (unital), (co–unital) bi-algebras is a covariant functor. The association of (unital) operads to graded (unital), (co–unital) bi-algebras is a contravariant functor.*

Remark 2.43. This is an example which comes from the enriched Feynman categories $\mathfrak{F}\mathfrak{S}_{\mathcal{O}}$, see [KW17] and §5, especially §7.8.

Example 2.44 (Morphisms of operads and co–operads for types of trees)**.** Let $\mathcal{O}$ be the operad of planar planted leaf–labelled tress, $\mathcal{O}_3$ the sub–operad of planar planted trivalent trees and $\mathcal{P}$ the the operad of planar corollas.

(1) The inclusion $\mathcal{O}_3 \to \mathcal{O}$ gives a morphism $\mathscr{B}_{\mathcal{O}} \to \mathscr{B}_{\mathcal{O}_3}$. This is the map that maps all δ_τ for non–trivalent τ to 0.
(2) There is also an inclusion of $\check{\mathcal{O}}_3 \to \check{\mathcal{O}}$ which is defined by the inclusion of the generating set. This yields a morphism $\mathscr{B}_{\mathcal{O}_3} \to \mathscr{B}_{\mathcal{O}}$ which is again inclusion.
(3) There is a morphism of operads $con : \mathcal{O} \to \mathcal{P}$ given by contracting all internal edges of a tree τ. This restricts to $\mathcal{O}_3 \to \mathcal{P}$. These morphisms give rise to maps $\mathscr{B}_{\mathcal{P}} \to \mathscr{B}_{\mathcal{O}}$ and $\mathscr{B}_{\mathcal{P}} \to \mathscr{B}_{\mathcal{O}_3}$. These morphisms send δ_c to $\sum_{\tau \in con^{-1}(c)} \delta_\tau$, where the sum is over all the trivalent pre–images for $\mathscr{B}_{\mathcal{O}_3}$. This morphism and its angle decoration is considered in [Gon05], see also §7.3.2. Combinatorially this corresponds to associating to each multiplication $(a_1 \cdot \dots \cdot a_n)$ all possible bracketings as such it can be seen as the boundary for the associahedra and the co–operad map is the boundary map. The fact that one gets a bi–algebra morphism then states that the operad map on associahedra is a dg–map.

2.5. **Hopf algebra structure for co–connected co–operads.** Under certain conditions a quotient of the bi–algebra $\mathscr{B}'$ is a Hopf algebra. These conditions guarantee connectedness and co–nilpotence of the quotient. When considering Hopf algebras, we will *always* make the following assumption:

Assumption 2.45. The tensor structure and kernels commute. Under this assumption the notions of conilpotent and connected are equivalent.

For example, this is the case if we are working in k–Vect.

2.5.1. **(Co)–connected (co)–operads.** For a co–unital co–operad, we will say that the co–unit ϵ is split if $\check{\mathcal{O}}(1) = \mathbb{1} \oplus ker(\epsilon) = \mathbb{1} \oplus \check{\mathcal{O}}(1)^{red}$. This is automatic if we are in the category of vector spaces or the $\mathcal{O}(n)$ are free Abelian groups, e.g. if they come from an underlying $\mathcal{S}et$

operad. In the case that ϵ is split, let $|$ be the generator of $\mathbb{1}$ with $\epsilon(|) = 1$.

For an operad, we will say that a unit u is *split* if $\mathcal{O}(1) = \mathbb{1} \oplus \bar{\mathcal{O}}(1)$, where $u = 1 \in \mathbb{1}$. If u is split, dualizing $\mathcal{O}(1) = \mathbb{1} \oplus \bar{\mathcal{O}}(1)$ to $\check{\mathcal{O}}(1) = (\mathcal{O}(1))^\vee = \mathbb{1} \oplus \check{\mathcal{O}}(1)^{red}$ where $| = \delta_u = 1 \in \mathbb{1} \subset \check{\mathcal{O}}(1)$ and $\check{\mathcal{O}}(1)^{red} = (\bar{\mathcal{O}}(1))^\vee = ker(\epsilon_1)$. Whence the dual of a split unital operad is a split co–unital co–operad.

Remark 2.46. In an operad $\mathcal{O}(1)$ forms an algebra via the restriction of γ: $\gamma_{1;1} : \mathcal{O}(1) \otimes \mathcal{O}(1) \to \mathcal{O}(1)$. Dually $\check{\mathcal{O}}(1)$ forms a co–algebra via $\Delta := \check{\gamma}_{1;1} : \check{\mathcal{O}}(1) \to \check{\mathcal{O}}(1) \otimes \check{\mathcal{O}}(1)$. Δ is the restriction of the co–product on $\mathscr{B}$ to $\check{\mathcal{O}}(1)$. If the co–operad has a split co–unit, this co-algebra is pointed by the element $|$.

Definition 2.47. A co–unital operad is co–connected, if

(1) The co–unit is split.
(2) The element $|$ is group–like: $\Delta(|) = \check{\gamma}(|) = | \otimes |$
(3) $(\check{\mathcal{O}}(1), |, \epsilon)$ is connected as a pointed co-algebra in the sense of Quillen [Qui67] (see Appendix B).

An unital operad is called reduced if $\mathcal{O}(1) = \mathbb{1}$ it is then automatically split and $|$ is group–like. Likewise a co–unital co–operad is reduced if $\check{\mathcal{O}}(1) = \mathbb{1}$. It then automatically satisfies (1) and (2).

As the dual of a split unital operad is a split co–unital co–operad, we can illustrate the conditions (2) and (3) in a practical fashion. Consider $\mathcal{O}(1)$ as a an algebra.:

Lemma 2.48. *The dual split co–unital co–operad of a split unital operad is satisfies (2) if and only if $O(1)$ does not contain any left or right invertible elements except for multiples of the identity. It satisfies (3) if and only if*

(3') *any element $a \in \mathcal{O}(1)$ the decompositions $a = \prod_{i\in I} a_i$ with all $\epsilon(a_i) = 0$, have bounded length, i.e. $|I|$ is bounded.*

Proof. Recall that the co–product in $\check{\mathcal{O}}(1)$ is dual to multiplication in $\mathcal{O}(1)$, that is, the co–proiduct is decomposition. Let u be the unit, then $| = \delta_u$. Now, $\Delta(|)(a \otimes b) \neq 0$ means that $\gamma(a, b) = u$ and hence, a is a left inverse to b and b is a right inverse to a. If $|$ is group like, then we need that $a, b \in \mathbb{1} \subset \mathbb{1} \oplus \bar{\mathcal{O}}(1)$, which is the first statement. Likewise, since, the co–product is decomposition, being co–nilpotent is equivalent to the given finiteness condition. □

An obstruction to being co–connected are group like elements in $\check{\mathcal{O}}(1)^{red}$. Such a group like element will be dual to an idempotent.

Corollary 2.49. *If $\mathcal{O}(1)$ contains any isomorphisms or idempotents except for multiples of the unit, then $\check{\mathcal{O}}$ is not co–connected. More precisely, if $\mathcal{O}(1)$ splits as $\mathbb{1} \oplus \bar{\mathcal{O}}(1)$, then $\bar{\mathcal{O}}(1)$ may not contain any invertible elements or any idempotent elements.*

Proof. Indeed, if $a \in \bar{\mathcal{O}}(1)$ is an isomorphism it has factorizations of any unbounded length: $a = a(a^{-1}a)^n$ for any n and $\epsilon(a) = \epsilon(a^{-1}) = 0$. Likewise, if $p \in \bar{\mathcal{O}}(1)$ is an idempotent then $p = p^n$, again for any n yielding infinitely many factorizations. $\square$

Example 2.50.

(1) If the unital operad $\mathcal{O}$ is reduced, that is $\mathcal{O}(1) \simeq \mathbb{1}$ its dual is also reduced. This is the case for the surjection and the simplex operads.

(2) More generally, if for a split co–unital co–operad, $\check{\mathcal{O}}(1)^{red}$ is free of finite rank as a co–monoid, then $\check{\mathcal{O}}$ is co–connected. This is the case for the dual co–operad of an operad whose $\mathcal{O}(1)$ is a free unital algebra of finite rank. An example are planar planted trees, where $\check{\mathcal{O}}(1)^{red}$ is free of rank 1 with the generator being the rooted corolla with one tail. As previously, the generator corresponding to the dual of the identity can be depicted as the degenerate "no vertex" corolla with one input and output $|$ and the other generator as $\bullet\!\!\!\mid$.

    This is linked to the considerations of [Moe01] in the rank 1 case and those of higher rank to [vdLM06a], see §7.2.2, where the generators can the thought of as $\bullet\!\!\!\mid c$, where c is a color index.

(3) Assume $\mathcal{O}(1)$ is split unital and (2) holds. If $\mathcal{O}(1)$ as an algebra is an algebra presented by homogenous relations, then $\mathcal{O}$ is co–connected. This follows, since homogenous relations do not change the length of a decomposition.

For a split co—unital co–operad, let $\mathcal{I}$ be the two-sided ideal of $\mathscr{B}'$ spanned by $1 - |$. Set

$$\mathscr{H} := \mathscr{B}'/\mathcal{I} \tag{2.32}$$

Notice that in $\mathscr{H}$ we have that $|^k \equiv 1 \mod \mathcal{I}$ for all k.

Proposition 2.51. *If $\{\check{\mathcal{O}}(n)\}$ has a split co–unit and $|$ is grouplike, then $\mathcal{I}$ is a coideal of $\mathscr{B}'$ and hence $\mathscr{H}$ is a co–algebra. The unit η descends to a unit $\bar{\eta} : \mathbb{1} \to \mathscr{H}$ and the co–unit ϵ_{tot} factors as $\bar{\epsilon}$ to make $\mathscr{H}$ into a bi–algebra.*

Proof. $\Delta(1-|) = 1\otimes 1 - |\otimes| = (1-|)\otimes| + 1\otimes(1-|) \in \mathcal{I}\otimes\mathscr{B} + \mathscr{B}\otimes\mathcal{I}$ and $\epsilon_{tot}(1-|) = 1 - 1 = 0$. $\square$

Theorem 2.52. *If $\{\mathcal{O}(n)\}$ is co–connected then $\mathscr{H}$ is co–nilpotent and hence admits a unique structure of Hopf algebra.*

Proof. Let $\pi = id - \bar{\epsilon} \circ \bar{\eta}$ be the projection $\mathscr{H} = \mathbb{1} \oplus \bar{\mathscr{H}} \to \bar{\mathscr{H}}$ to the augmentation ideal. We have to show that each element lies in the kernel of some $\pi^{\otimes m} \circ \Delta^m$. For $\mathbb{1}$ this is clear, for the image of $\check{\mathcal{O}}(1) = \check{\mathcal{O}}^{nc}(1)$ this follows from the assumptions, from the Lemma above and the identification of 1 and $|$ in the quotient. Now we proceed by induction on n, namely, for $a \in \mathscr{B}(n)$, we have that $\Delta(a) \in \bigoplus_{k,n} \mathscr{B}(k) \otimes \mathscr{B}(n)$. Since the co–product is co–associative, we see that all summands with $k < n$ are taken care of by the induction assumption. This leaves the summands with $k = n$. Then the right hand side of the tensor product is the product of elements which are all in $\mathscr{B}(1) = \check{\mathcal{O}}(1)$. Since Δ is compatible with the multiplication, we are done by the assumption on $\check{\mathcal{O}}(1)$ using co–associativity. □

Example 2.53 (Hopf algebra of leaf labelled planar planted trees)**.** In the example of leaf labelled planar planted trees the Hopf algebra is one of the versions found in [Foi02b, Foi02a]. It simply means that all occurrences of $|$ are replaced by 1 that is just eliminated unless all the factors are $|$ in which case it is replace by 1. In this case, we get $\Delta_{\mathscr{H}}(\tau) = 1 \otimes \tau + \tau \otimes 1 + \sum_{\tau_0 \subset \tau} \tau_0 \otimes \tau \setminus \tau_0$ as in (1.11), where now all the cut off leaves and cut off root half edges are ignored, or better set to 1.

Example 2.54 (The operad of order preserving surjections, aka. planar planted corollas)**.** In the example of ordered surjections or planar planted corollas, which is isomorphic to order preserving surjections. Let c_n be the planar planted corolla with n leaves and let $n : n \twoheadrightarrow 1$ the unique surjection. The isomorphism is given by $n \leftrightarrow c_n$. The non–Σ pseudo–operad structure is $c_n \circ_i c_m = c_{n+n-1}$. The operad structure is given by $\gamma(c_k; c_{n_1}, \dots, c_{n_k}) = c_n$ with $n = \sum_i n_i$. As surjections this is the composition of maps $\gamma(k, n_1, \dots, n_k) = (n_1 \twoheadrightarrow 1) \amalg \cdots \amalg (n_k \twoheadrightarrow 1) \circ k \twoheadrightarrow 1 : n = n_1 \amalg \cdots \amalg n_k \twoheadrightarrow k \twoheadrightarrow 1$. Keeping the notation c_n we get $\Delta(n) = \sum_{k,n_1,\dots,n_k : \sum n_i = n, n_i \geq 1} c_k \otimes c_{n_1} \otimes \cdots \otimes c_{n_k}$. In $\mathscr{H}$: $\Delta_{\mathscr{H}}(c_n) = 1 \otimes c_n + c_n \otimes 1 + \sum_{1<k<n} \sum_{(n_1,\dots,n_j)\, \sum n_i + k - j = n, n_i \geq 2} \binom{k}{j} c_k \otimes c_{n_1} \otimes \cdots \otimes c_{n_j}$. Here the c_1 have made into units in $1 \in \mathbb{1}$ and hence the sum is over partitions not containing ones, but there are now multiplicities according to the number of spots where they were originally inserted. In particular, $\Delta_{\mathscr{B}}(c_3) = 1 \otimes c_3 + c_3 \otimes c_1 \otimes c_1 \otimes c_1 + c_2 \otimes c_1 \otimes c_2 + c_2 \otimes c_2 \otimes c_1$ while $\Delta_{\mathscr{H}}(c_3) = 1 \otimes c_3 + c_3 \otimes 1 + 2c_2 \otimes c_2$.

2.6. **The Hopf algebra as a deformation.** Rather then taking the approach above, we can produce the Hopf algebra in two seprate steps.

Without adding a unit, we will first mod out by the two-sided ideal $\mathcal{C}$ of $\mathscr{B}$ generated by $|a-a|$. This forces $|$ to lie in the centre. We denote the result by $\mathscr{H}_q := \mathscr{B}/\mathcal{C}$, where the image of $|$ under this quotient is denoted by q. This allows us to view q as a deformation parameter and view $\mathscr{H}$ as the classical limit $q \to 1$ of $\mathscr{H}_q$.

Proposition 2.55. *If $\{\check{\mathcal{O}}(n)\}$ is split co–unital and $|$ is grouplike, then $\mathcal{C}$ is a co–ideal and hence $\mathscr{H}_q$ is a co–unital bi–algebra.*

Proof. Using Sweedler notation:

$$\begin{aligned}\Delta(|a-a|) &= |a^{(1)} \otimes |a^{(2)} - a^{(1)}| \otimes a^{(2)}| \\ &= (|a^{(1)} - a^{(1)}|) \otimes |a^{(2)} + a^{(1)}| \otimes (|a^{(2)} - a^{(2)}|) \in \mathcal{C} \otimes \mathscr{B} + \mathscr{B} \otimes \mathcal{C}\end{aligned}$$

Furthermore $\epsilon(|a-a|) = \epsilon(a) - \epsilon(a) = 0$. □

In the case of a split co–unital co–operad, we split $\check{\mathcal{O}} = \mathbb{1} \oplus \check{\mathcal{O}}^{red}$ where $\check{\mathcal{O}}^{red} = ker(\epsilon_1)$ that is $\check{\mathcal{O}}(1)^{red} = ker(\epsilon) \subset \check{\mathcal{O}}(1)$ and $\check{\mathcal{O}}(n)^{red} = \check{\mathcal{O}}(n)$. We set

$$\check{\mathcal{O}}^{nc,red}(n) = \bar{T}\check{\mathcal{O}}^{red} = \bigoplus_{k\geq 1} \bigoplus_{(n_1,\dots n_k):\sum n_i = n} \check{\mathcal{O}}^{red}(n_1) \otimes \cdots \otimes \check{\mathcal{O}}^{red}(n_k) \tag{2.33}$$

Notice that the image of $|^n$ is q^n and if we give q the degree and length 1, then the grading by operadic degree passes to the quotient as well as the length grading and any combination of them. In particular, $\mathscr{H}_q$ decomposes as $\mathscr{H}_q = \bigoplus_d \mathscr{H}_q(d)$ according to the length grading. Furthermore, q is group-like and $\epsilon(q) = 1$.

Proposition 2.56. *For a split unital $\mathcal{O}$ with a grouplike $|$,*

$$\mathscr{H}_q(d) \simeq \bigoplus_{n \leq d} q^{n-d}\check{\mathcal{O}}^{red,nc}(n) \text{ and } \mathscr{H}_q \simeq \bar{T}\check{\mathcal{O}}^{red}[q] \tag{2.34}$$

Proof. In $\mathscr{H}_q$ one can move all the images of $|$, that is factors q to the left. This leaves terms of the form $q^k a$ with a in the image of $\check{\mathcal{O}}^{red,nc}$. This yields a unique standard form for any element in $\mathscr{H}_q$ and establishes the first isomorphism. The second isomorphism is a reformulation using (2.33). □

Corollary 2.57. *$\mathscr{H}_q$ is a deformation of $\mathscr{H}$ given by $q \to 1$.*

Proof. Setting $q = 1$ corresponds to taking the quotient $\mathscr{H}'_q = T\check{\mathcal{O}}^{red}[q] = \mathbb{1} \oplus \mathscr{H}_q$ by the bi–algebra ideal $\mathcal{I}'$ generated by $1-q$ and as $(\mathscr{B}'/1 \oplus \mathscr{J}) = \mathscr{H}'_q$ we see that indeed the double quotient satisfies $\mathscr{H}'_q/I' = ((\mathscr{B}'/1 \oplus \mathscr{J})/\mathcal{I}') = \mathscr{B}'/\mathcal{I} = \mathscr{H}$. □

2.7. **The infinitesimal structure.** The infinitesimal version corresponds to dualizing the pseudo operad structure, notably $\circ$. In order to obtain the infinitesimal structure, we consider the filtration in terms of factors of $|$. Using the construction of the double quotient, that is first identifying $|$ with q, also gives context to the name infinitesimal.

Assumption 2.58. In this subsection, we will assume that $\check{\mathcal{O}}(0)$ is empty or 0. The arguments also work if $\check{\mathcal{O}}$ is locally finite. For the general case, see §2.11.

2.7.1. **Pseudo co–operads and the co–pre–Lie Poisson.** As before, if we denote by $\epsilon_1 : \check{\mathcal{O}} \to \mathbb{1}$ the dual of $u : \mathbb{1} \to \mathcal{O}(1)$ extended to all of $\check{\mathcal{O}}$ by first projecting to $\check{\mathcal{O}}(1)$ — in the linear case this is just the extension by 0. The dual of the $\circ_i$ expressed as in (2.8) becomes a morphism $\check{\circ}_i : \check{\mathcal{O}}(n) \to \bigoplus_{k=1}^{n-1} \check{\mathcal{O}}(k) \otimes \mathbb{1}^{\otimes i-1} \otimes \check{\mathcal{O}}(n-k) \otimes \mathbb{1}^{\otimes k-i}$ which for $a \in \check{\mathcal{O}}(n)$, $1 \le i \le n$ is defined by

$$\check{\circ}_i(a) = \sum_{k=1}^{n-1} (id \otimes \epsilon_1^{\otimes i-1} \otimes id \otimes \epsilon_1^{\otimes k-i})(\check{\gamma}_{k;1,\dots 1,n-k,1\cdot\dots\cdot 1}(a)) \tag{2.35}$$

with id in the 1st and $i+1$-st place. Using the unit constraints for the monoidal category to eliminate tensor factors of $\mathbb{1}$ resulting from the maps ϵ_1, we get maps

$$\delta_i : \check{\mathcal{O}}(n) \to \bigoplus_{k=1}^{n-1} \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(n-k+1) \text{ for } 1 \le i \le n \tag{2.36}$$

These maps and their compatibilities constitute a (non–Σ pseudo–co–operad structure. One can add symmetric group actions as well and postulate equivariance as before to obtain the notion of a pseudo co–operad.

One can reconstruct the $\check{\gamma}$ from the $\check{\circ}_i$ and the δ_i. This fact is used with great skill in [Bro17, Bro15, Bro12b]. Summing over all the δ_i we get a map

$$\delta_c : \check{\mathcal{O}}(n) \to \bigoplus_{k=1}^{n} \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(n-k+1) \tag{2.37}$$

The subscript c stands for connected. As $\check{\circ}$ is dual to the pre–Lie product δ_c it is co–pre–Lie.

We will extend δ_c to $\mathscr{B}$ by

$$\delta \circ \mu \quad = \quad (\mu \otimes id) \circ \pi_{23} \circ (\delta \otimes \mathrm{id}) + (\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta) \tag{2.38}$$

where $\pi_{2,3}$ switches the second and third factor, i.e. $\delta(ab) = \sum a^{(1)} b \otimes a^{(2)} + \sum ab^{(1)} \otimes b^{(2)}$, which can be thought of as a Poisson condition for the co–pre–Lie–product δ that is obtained by dualizing the Poisson bracket to δ, but keeping the multiplication in the usual Poisson equation.

Example 2.59. In the example of leaf labelled planar planted forests, δ corresponds to cutting a single edge of the forest. In simplicial terms, δ defines the $\cup_1$ product; see §7.6.9 for details.

For the operad of planted planar corollas, we have $\delta(c_1) = c_1 \otimes c_1$ and for $n \geq 2$: $\delta(c_n) = \sum_{k=2}^{n} k\, c_k \otimes c_{n-k+1}$.

2.7.2. **Filtration and structure of infinitesimal co–algebra.** Let $\mathscr{J}$ be the two-sided ideal of $\mathscr{B}$ spanned by $|$. Then there is an exhaustive filtration of $\mathscr{B}$ by the powers of $\mathscr{J}$. That is $\bar{T}\check{\mathcal{O}}^{red} = \mathscr{J}^{\leq 0} \subset \mathscr{J}^{\leq 1} \subset \cdots \subset \mathscr{J}^{\leq k} \subset \cdots \subset \mathscr{B} = \bar{T}\check{\mathcal{O}}$. An element $a \in \mathscr{J}^{\leq k}$ if its summands contains less or equal to k occurrences of $|$. This filtration survives the quotient by $\mathcal{C}$ and gives a filtration in powers of q that can be viewed as a deformation over a formal disc, with the central fiber $z = 0$, corresponding to $q = \exp 2\pi i z$ and $\exp(2\pi i 0) = 1$, so that $q \to 1$ corresponds to $z \to 0$.

Lemma 2.60. *If $\mathcal{O}$ is split unital and $|$ is grouplike then $\Delta(|^n) = |^n \otimes |^n$, and for $a \in \check{\mathcal{O}}^{nc}(n,p)$ with $\epsilon_{tot}(a) = 0$:*

$$\begin{aligned}
\Delta(a) &= |^p \otimes a + a \otimes |^n + \bar{\Delta}(a) \ \textit{with}\ \bar{\Delta} = \sum_{k=p}^{n} \bar{\Delta}_k \ \textit{and} \\
\bar{\Delta}_k(a) &= \sum_{i=0}^{k-1} a_k^{(i,1)} \otimes |^i a_{n-k+1}^{(i,2)} |^{k-i-1} + R \ \textit{with}\ R \in \mathscr{J}^{\leq p-1} \otimes \mathscr{J}^{\leq k-2} \\
&\quad \textit{where}\ \ a_k^{(i,1)} \in \mathscr{J}^{\leq p-1}, a_{n-k+1}^{(i,2)} \in \check{\mathcal{O}}^{red}(n-k+1)
\end{aligned} \tag{2.39}$$

Proof. The first statement follows from the bi–algebra equation (2.30) and the fact that $|$ is group like. The second statement follows from the fact that ϵ_{tot} is a left and right co–unit. If we compute $(\epsilon_{tot} \otimes id) \circ \Delta(a)$ only terms of $\Delta(a)$ of the form $|^k \otimes f_k$ with $l(f_k) = k$ survive. If a has length p, then $(\epsilon_{tot} \otimes id) \circ \Delta(a) = a$, i.e. $(\epsilon_{tot} \circ id)(\sum_k |^k \otimes f_k) = \sum_k f_k = a$, forces all terms of length not equal to p to vanish, $f_k = 0, k \neq p$ and $f_p = a$. Similarly computing $(id \otimes \epsilon_{tot}) \circ \Delta(a)$ only terms of the form $f_k \otimes |^k$ survive where now $\deg(f_k) = k$, and we find that $f_k = 0, k \neq n$ and $f_n = a$.

In general one can count the factors of $|$ that may occur on the right in Δ. Such factors can only come from factors of $\check{\mathcal{O}}(1)$. Fixing the length on the right side of Δ to be k i.e. considering $\Delta_{k,n}$, the maximal number of factors of $|$ appearing in $\Delta_{k,n}(a)$ is k, that is $\Delta_{n,k}(a) \subset \mathscr{B} \otimes \mathscr{J}^{\leq k}$. By the above, the maximal number k is only obtained if $k = \deg(a) = n$ the next leading order in $|$ is then $k-1$ which means that only one factor on the right is not $|$. By the degree grading, this has to lie in $\check{\mathcal{O}}(n-k-1)$ and if $k = n$, so that $n-k+1 = 1$ then the factor cannot be $|$, since the maximal number is $|^{k-1}$. This is the first term in $\bar{\Delta}$. By the above, we have already seen that the only term with the maximal number of $|$ on the left, i.e. $|^p$, is $|^p \otimes a$, so that all remaining terms are indeed in $\mathscr{J}^{\leq p-1} \otimes \mathscr{J}^{\leq k-2}$. □

Corollary 2.61. *For elements $a \in \check{\mathcal{O}}(n,p)$:*

$$\delta(a) = \sum_{k=p}^{n} \sum_{i=1}^{k-1} a_k^{(i,1)} \otimes a_{n-k+1}^{(i,2)} \tag{2.40}$$

where the $\sum a_k^{(i,1)} \otimes a_{n-k+1}^{(i,2)}$ are given in (2.39) and can be extracted via

$$\sum a_k^{(i,1)} \otimes a_{n-k+1}^{(i,2)} = \mathbf{u} \circ (id \otimes \epsilon_1^{\otimes i-1} \otimes id \otimes \epsilon_1^{\otimes k-i})(\bar{\Delta}_k(a)) \tag{2.41}$$

where $\mathbf{u} = id \otimes u_L^{\otimes i-1} \otimes id \otimes u_R^{\otimes k-i-1}$ are the unit constraints to get rid of the factors $\mathbb{1}$ stemming from the images of ϵ_1.

Proof. We proceed by induction. For elements of length 1 the claim follows from (2.37) via (2.35). For length p we use (2.38) and induction to plug in. The formula (2.41) follows directly from (2.37), since it is non–vanishing only on $im(\bar{\Delta}_k) \cap \mathscr{J}^{k-1}$. □

When we descend to $\mathscr{H}_q$ the filtration in $|$ corresponds to the powers of q: $im(\mathscr{J}^{\leq k}) = \{p : p \in \mathscr{H}_q = \bar{T}\check{\mathcal{O}}^{red}[q], \deg_q(p) \leq k\}$

Corollary 2.62. *The co–pre–Lie Poisson δ descends to $\mathscr{H}_q$, and its terms are given by terms of $\bar{\Delta}_k$ whose degree on the right is q^{k-1}. Alternatively, setting $\bar{\Delta}_{q,k} = (id \otimes q^{-k}) \circ \bar{\Delta}_k$, to compensate the grading on the right side, $\delta = res_{-1}(\sum_k \bar{\Delta}_{q,k})$ where res_{-1} is the residue in q.*

Proof. This follows directly from (2.39) and Corollary 2.61. □

2.8. **Derivations and irreducibles.** In the case of multiple zeta values, applying the Hopf quotient to the infinitesimal structure and restricting to indecomposables, yields Brown's operators D_n [Bro12a] that determine his co–action, see also §2.24 and §2.11. This holds more generally:

Theorem 2.63. *Taking the limit $q \to 1$ or equivalently regarding the quotient $\mathscr{H}$, the co-pre–Lie structure induces a co-Lie algebra structure on the indecomposables $\bar{\mathscr{H}}/\bar{\mathscr{H}}\bar{\mathscr{H}}$, where $\bar{\mathscr{H}}$ is reduced version of $\mathscr{H}$.*

Proof. By The indecomposables in reduced Hopf algebra are precisely given by $\check{\mathcal{O}}^{red}$ and the co–pre–Lie structure is δ_c of (2.37). Its co–commutator yields to co–Lie structure corresponding dually the usual Lie structure of Gerstenhaber [Ger64]. □

2.8.1. **The example of overlapping sequences and Goncharov's Hopf algebra.** Considering Example 2.24, the bi–algebra structure $\mathscr{B}$ is simply given by re–interpreting the r.h.s. of (2.16) as a free product. There is a co–unit for the co–operad which is given by $\epsilon(a_0; a_1) = 1$ and 0 else. Indeed under $\epsilon \otimes id$ the only non–zero term of the r.h.s. of (2.16) is $(a_0; a_n) \otimes (a_0; a_1, \dots, a_{n-1}; a_n)$ and under $id \otimes \epsilon^{\otimes k}$ the only non–zero term occurs when $k = n$ and occurs for $(a_0; a_1, \dots, a_{n-1}; a_n) \otimes (a_0; a_1) \otimes (a_1; a_2) \otimes \dots \otimes (a_n; a_{n+1})$. In the Hopf quotient $(a_0; a_n)$ and the the $(a_i; a_{i+1})$ will be set to 1.

Proposition 2.64. *Consider the projection $\pi : \mathscr{B}' = T\check{\mathcal{O}} \to S\check{\mathcal{O}}$ Let $\pi(a_0; a_1 \dots a_{n-1}; a_n) = \hat{I}(a_0; a_1, \dots, a_{n-1}; a_n)$. In the case of $S = \{0, 1\}$ Then after taking the Hopf quotient, that is considering the induced map $\pi_{\mathscr{H}} : \mathscr{H} \to \mathscr{H}_{com}$, where $\mathscr{H}_{com} = \mathscr{H}_{com}/[\mathscr{H}_{com}, \mathscr{H}_{com}]$ is the Abeliniazation, we obtain Goncharov's first Hopf algebra $\mathscr{H}_{com} = \mathscr{H}_G$. The operadic degree of $(a_0; \dots; a_n)$ is n and hence the weight is $n - 1$ $\mathrm{wt}(a_0; a_1, \dots, a_{n-1}; a_n) = n - 1$ is the weight of $\hat{I}(a_0; a_1, \dots, a_{n-1}; a_n)$ as defined previously.*

Proof. The relations on the symbols $\hat{I}$ are precisely that they commute and (1.3). The latter condition already holds in $\mathscr{H}$, namely $(a_0, a_1) = 1$ and hence $\hat{I}(a_0, a_1) = 1$. □

Remark 2.65.
1. In the case $S \subset \mathbb{C}$, we get the Hopf algebra for for the polylogs, [Gon05].
2. The role of the depth as the number of 1s is not as clear in this formulation, but see Remark 7.13 for a possible explanation using Joyal duality. That remark also links it to the depth filtration used §3.

Proposition 2.66. *The action of δ is given by*

$$\delta(\hat{I}(a_0; a_1, \dots, a_{n-1}; a_{N+1}) = \sum_{n=3}^{N-3} \sum_{p=0}^{N-n}$$
$$\hat{I}(a_0; a_1, \dots, a_p, a_{p+n-1}, \dots, a_N; a_{N+1}) \otimes \hat{I}(a_p; a_{p+1}, \dots, a_{p+n}; a_{p+n-1}) \tag{2.42}$$

Proof. This is a straightforward calculations. Note that the sub–sequences of length 2 are in $\check{\mathcal{O}}(1)$ and are set to q respectively 1 in the different quotients, so that the reduced structure starts with words of length 3 on the left and stops with words of length 3 on the right. $\square$

Corollary 2.67. *Incorporating the projection and the the co–action this is recovers the co–derivations D_n of Brown [Bro12a, (3.4)] as the degree n part. Moreover, this is exactly in q degree $N-n$.*

2.9. **Bi– and Hopf algebras from symmetric (co)–operads.** To dualize symmetric operads, it will be important to go the generality of indexing by in arbitrary sets, see e.g. [MSS02]. This means that for any finite set S we have an $\mathcal{O}(S)$ and any isomorphism $\sigma : S \to S'$ an isomorphism $\mathcal{O}(S) \to \mathcal{O}(S')$. The composition is then defined for any map $f : S \to T$ as a morphism $\mathcal{O}(T) \otimes \bigotimes_{t\in T} \mathcal{O}(f^{-1}(t)) \to \mathcal{O}(S)$ which is equivariant for any diagram of the form

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\sigma'"'] & T \arrow[d, "\sigma"] \\ S' \arrow[r, "f'"] & T' \end{tikzcd} \tag{2.43}$$

Thus, $f' = \sigma \circ f \circ \sigma'^{-1}$ and the partition $S = \amalg_{t\in T} f^{-1}(t)$ maps to the partition $S' = \amalg_{t'\in T'} f'^{-1}(t') = \amalg_{\sigma(t):t\in T} \sigma'(f^{-1}(t))$. That is σ' maps the fibers to the fibers and σ permutes the fibers.

Recall that if we are only given the $\mathcal{O}(n)$ then the extension to finite sets is given by $\mathcal{O}(S) := \mathrm{colim}_{f:S\leftrightarrow \underline{n}} \mathcal{O}(n)$. Where $\underline{n} = \{1, \dots, n\}$ and the co–limit is over bijections. Concretely in an Abelian category this is $\mathcal{O}(S) = (\bigoplus_{\phi:S \xrightarrow{1-1} \underline{n}} \mathcal{O}(n))_{\mathbb{S}_n}$ where $\mathbb{S}_n$ acts by post–composing. This actually yields an equivalence of categories between operads over finite sets and operads. Notice that $\underline{0} = \varnothing$ and we can restrict the considerations to non–empty sets and surjections with the skeleton consisting of $\underline{n}, n \geq 1$ and surjections.

Lemma 2.68. *Using the equivariance of the maps γ_f one obtains an induced map on invariants $\bar{\gamma}_{k;n_1,\dots,n_k} : \mathcal{O}(k)^{\mathbb{S}_k} \otimes Symm(\bigotimes_{i=1}^k \mathcal{O}(n_i)^{\mathbb{S}(n_i)}) \to \mathcal{O}(n)^{\mathbb{S}_n}$, where $Symm(\bigotimes_{i=1}^k \mathcal{O}(n_i)^{\mathbb{S}(n_i)})$ are the symmetric tensors.*

Proof. The following proof is technical and involves limits. In the concrete example of vector spaces, the morphisms i and $\imath$ are natural inclusions.

Let $|S| = |S'| = k$, $n_t = |f^{-1}(t)| = |f'^{-1}(\sigma(t))|$ and $\sigma'_t : f^{-1}(t) \to (f')^{-1}(\sigma(t))$ be the restriction. For all pairs of isomorphisms (σ', σ) given above, we have a natural diagram

$$
\begin{tikzcd}
\mathcal{O}(T) \otimes \bigotimes_{t \in T} \mathcal{O}(f^{-1}(t)) \arrow[rr, "\gamma_f"] \arrow[dd, bend right, "\sigma \otimes \bigotimes_T \sigma'_t"'] & & \mathcal{O}(S) \arrow[dd, bend left, "\sigma'"] \\
(\mathcal{O}(k) \otimes \bigotimes_{i=1}^k \mathcal{O}(n_i)^{\mathbb{S}(n_i)})^{\mathbb{S}_k} \arrow[u, "i"] \arrow[d, "i"] & \mathcal{O}(k)^{\mathbb{S}_k} \otimes Symm(\bigotimes_{i=1}^k \mathcal{O}(n_i)^{\mathbb{S}(n_i)}) \arrow[l, "\imath"'] \arrow[r, "\bar{\gamma}_{k;n_1,\dots,n_k}"] \arrow[ur, "\imath \circ i \circ \gamma_f"] \arrow[dr, "\imath \circ i \circ \gamma_{f'}"'] & \mathcal{O}(n)^{\mathbb{S}_n} \arrow[u] \arrow[d] \\
\mathcal{O}(T') \otimes \bigotimes_{t' \in T'} \mathcal{O}(f'^{-1}(t')) \arrow[rr, "\gamma_{f'}"'] & & \mathcal{O}(S')
\end{tikzcd}
\tag{2.44}
$$

in which the outer square of the diagram below commutes. Moreover, this diagram exists for all σ' and any fixed $f : T \to S$ whose k fibers have the right cardinalities n_i One choice is given by $\sigma' = id$ and $f' = \sigma \circ f$. The morphisms are defined as follows: let $Iso(n, k)$ be the category with objects the surjections $S \to T$ with $|S| = n$ and $|T| = k$ and morphisms the commutative diagrams of the type (2.43) with σ, σ' bijections and f, f' surjections, and $Iso(n)$ the category with objects S, with $|S| = n$ and bijections. Then

(1) $\lim_{Iso(n)} \mathcal{O} = \mathcal{O}(n)^{\mathbb{S}_n}$ are the invariants.
(2) $\lim_{Iso(n.k)} \mathcal{O} = \bigoplus_{(n_1,\dots,n_k):\sum n_i = n} \left(\mathcal{O}(k) \otimes \bigotimes_{i=1}^k \mathcal{O}(n_i)^{\mathbb{S}(n_i)}\right)^{\mathbb{S}_k}$ where on $\mathcal{O}(f : S \to T) = \mathcal{O}(T) \otimes_T \bigotimes_T \mathcal{O}(f^{-1}(t))$ $Aut(T) \simeq \mathbb{S}_k$ acts anti–diagonally as $\sigma \otimes \sigma^{-1}$ these include into the $\mathcal{O}(T) \otimes \bigotimes_T \mathcal{O}(f^{-1}(t))$ by virtue of being a limit.
(3) The invariants under the full $\mathbb{S}_k \times \mathbb{S}_k$ action on $\mathcal{O}(k) \otimes \bigotimes_{i=1}^k \mathcal{O}(n_i)^{\mathbb{S}(n_i)}$ are are $\mathcal{O}(k)^{\mathbb{S}_k} \otimes Symm(\bigotimes_{i=1}^k \mathcal{O}(n_i)^{\mathbb{S}(n_i)})$, where $Symm$ is the subspace of symmetric tensors, These include into the invariants of only the $\mathbb{S}_k$ action using the anti–diagonal embedding $\mathbb{S}_k \subset \mathbb{S}_k \times \mathbb{S}_k$. Technically, since this is again a limit.
(4) The map $\bar{\gamma}$ exists by the universal property of limits applied to $\mathcal{O}(n)^{\mathbb{S}_n}$ and the cone given by the $\gamma_f \circ i \circ \imath$.

□

Remark 2.69. These are exactly universal operations in the sense of [KW17, §6] for the Feynman category $\mathfrak{F}_{May}$, see [KW17, §6] and §7.9 below.

2.9.1. **Symmetric co–operads.** The dual notion is a symmetric co–operads indexed by finite sets. This is a collection of objects $\{\check{\mathcal{O}}(S)\}$ for all finite sets S, isomorphisms $\sigma^* : \check{\mathcal{O}}(S') \to \check{\mathcal{O}}(S)$ for bijections

$\sigma : S \to S'$ for any surjection $f : S \twoheadrightarrow T$

$$\check{\gamma}_f : \check{\mathcal{O}}(S) \to \otimes \bigotimes_{t\in T} \check{\mathcal{O}}(f^{-1}(t)) \tag{2.45}$$

that are equivariant with respect to the isomorphisms.

Lemma 2.70. *The degree–wise dual of a symmetric operad is a symmetric co–operad and the association is functorial.* □

Proposition 2.71. *The maps $\check{\gamma}_f$ descend to the co–invariats as a map $\bar{\gamma}^{\vee}_{k;n_1,\dots,n_k} : \check{\mathcal{O}}(k)_{\mathbb{S}_k} \otimes \bigodot_{i=1}^k \mathcal{O}(n_k)_{\mathbb{S}(k)} \to \check{\mathcal{O}}(n)_{\mathbb{S}_n}$, where $\bigodot$ denotes the symmetric (aka. commutative) tensor product.*

Proof. The dual diagrams, to (2.44) are

$$\begin{tikzcd}
\check{\mathcal{O}}(S) \arrow[rr, "\check{\gamma}_f"] \arrow[d] \arrow[dr, "\gamma_f\circ p\circ\pi"] \arrow[dd, bend right, "\sigma'"'] & & \check{\mathcal{O}}(T) \otimes \bigotimes_{t\in T} \check{\mathcal{O}}(f^{-1}(t)) \arrow[d, "p"] \arrow[dd, bend left, "\sigma\otimes\bigotimes_T \sigma'_t"] \\
\check{\mathcal{O}}(n)_{\mathbb{S}_n} \arrow[r, "\bar{\gamma}^{\vee}_{k;n_1,\dots,n_k}"] & \check{\mathcal{O}}(k)_{\mathbb{S}_k} \otimes \bigodot_{i=1}^k \mathcal{O}(n_k)_{\mathbb{S}(k)} & \check{\mathcal{O}}(k) \otimes_{\mathbb{S}_k} \bigotimes_{i=1}^k \check{\mathcal{O}}(n_k) \arrow[l, "\pi"'] \\
\check{\mathcal{O}}(S') \arrow[rr, "\check{\gamma}_{f'}"] \arrow[u] \arrow[ur, "\gamma_f\circ p\circ\pi"'] & & \check{\mathcal{O}}(S') \otimes \bigotimes_{t'\in T'} \check{\mathcal{O}}(f')^{-1}(t') \arrow[u, "p"']
\end{tikzcd} \tag{2.46}$$

where

(1) $\mathrm{colim}_{Iso(n)}\check{\mathcal{O}} = \mathcal{O}(n)_{\mathbb{S}_n}$ are the coinvariants.
(2) $\mathrm{colim}_{Iso(n.k)}\check{\mathcal{O}} = (\check{\mathcal{O}}(k) \otimes_{\mathbb{S}_k} \bigotimes_{i=1}^k \check{\mathcal{O}}(n_k)$, where $\mathbb{S}(k)$ acting anti–diagonally yields the relative tensor product.
(3) These project to the full coinvariants under the $\mathbb{S}(k) \times \mathbb{S}(k)$ action: $\check{\mathcal{O}}(k)_{\mathbb{S}_k} \otimes \bigodot_{i=1}^k \check{\mathcal{O}}(n_k)_{\mathbb{S}(k)}$
(4) The map $\bar{\gamma}^{\vee}_{k;n_1,\dots,n_k}$: exists by the universal property of co–limits applied to $\check{\mathcal{O}}(n)_{\mathbb{S}_k}$ and the co–cone given by the $\pi\circ p\circ\gamma_f$. Again fixing an f with the correct cardinality of the fibers, we have morphisms f' defined by any given σ', and some choice of σ, e.g. $\sigma = id$.

□

Remark 2.72. In order to obtain a system of representatives one has to "enumerate everything" in order to rigidify. This means that for S and T one fixes an isomorphism to $\{1,\dots,|S| = n\}$ and $\{1,\dots,|T| = k\}$ *and* considers them as ordered sets. Then the unique map f with fiber cardinalities $n_1,\dots,n_k$ yields the representative for $\bar{\gamma}^{\vee}_{k;n_1,\dots,n_k}$

Remark 2.73. Using the co–cone $\gamma_f \circ p$ there is the intermediate possibility to have the co–multiplication $\check{\gamma}_{k,n_1,\dots,n_k}$ as a morphism $\check{\gamma}: \check{\mathcal{O}}(n)_{\mathbb{S}(n)} \to \check{\mathcal{O}}(k) \otimes_{\mathbb{S}_k} \bigotimes_{i=1}^k \check{\mathcal{O}}(n_k)_{\mathbb{S}(k)}$. This is an interesting structure that has appeared for instance in [DEEFG16]. This corresponds to Figure 4.

2.9.2. **Bi–algebras and Hopf algebras in the symmetric case.** We now proceed similarly to the non–Σ case, but using symmetric algebras on the co–invariants instead.

For a locally finite symmetric co–operad $\{\check{\mathcal{O}}\}$, set $\check{\mathcal{O}}_{\mathbb{S}} = \bigoplus_k \check{\mathcal{O}}(k)_{\mathbb{S}_k}$ and define $\mathscr{B} = \bar{S}\check{\mathcal{O}}_{\mathbb{S}}$, the reduced symmetric algebra on $\check{\mathcal{O}}_{\mathbb{S}}$, aka. the free commutative algebra, and $\mathscr{B}' = S\check{\mathcal{O}}_{\mathbb{S}}$, the free unital symmetric algebra, aka. the free unital commutative algebra. By summing over the k and n_i, we obtain the morphism $\bar{\gamma}^\vee : \check{\mathcal{O}}_{\mathbb{S}} \to \check{\mathcal{O}}_{\mathbb{S}} \otimes \mathscr{B}$. We extend to a co–product Δ using the bi–algebra equation (2.30) to obtain a co–multiplication on $\mathscr{B}$ and define $\Delta(1) = 1 \otimes 1$ to extend to $\mathscr{B}'$.

$$\Delta : \mathscr{B} \to \mathscr{B} \otimes \mathscr{B} \text{ and } \Delta : \mathscr{B}' \to \mathscr{B}' \otimes \mathscr{B}' \tag{2.47}$$

$\mathscr{B}$ and $\mathscr{B}$" still have the double grading by degree and length $\mathscr{B} = \bigoplus_{n \geq k \geq 1} \mathscr{B}(n,k)$ and $\mathscr{B}' = \bigoplus_{n \geq k \geq 0} \mathscr{B}'(n,k) = \mathbb{1} \oplus \mathscr{B}$. Given a co–unit for the co–operad[4], we see that $\epsilon^{\otimes k}$ is symmetric and hence gives a morphisms $\epsilon_k : \check{\mathcal{O}}(k)_{\mathbb{S}_k} \to \mathbb{1}$.Set $\epsilon_{tot} = \sum_k \epsilon_k : \check{\mathcal{O}}_{\mathbb{S}} \to \mathbb{1}$. We set $\check{\mathcal{O}}^{red}(n) = \check{\mathcal{O}}(n)$ for $n \geq 2$ and $\check{\mathcal{O}}^{red}_{\mathbb{S}} = \bigoplus_{k \geq 1} \check{\mathcal{O}}^{red}_{\mathbb{S}_k}(k)$.

Theorem 2.74. *Δ is co–associative, $\mathscr{B}$ is a commutative bi–algebra and $\mathscr{B}'$ is a commutative unital bi–algebra. The bi–algebra $\mathscr{B}$ and the unital bi–algebra $\mathscr{B}'$ have a co–unit if and only if the co–operad $\{\check{\mathcal{O}}(n)\}$ has a split co–unit. In this case, $\mathscr{B}' \simeq \bar{S}\check{\mathcal{O}}^{red}_{\mathbb{S}}[q]$. These associations are functorial.*

Proof. The fact that the co–product Δ is co–associative follows from the equivariance and co–associativity of the co–operad $\{\check{\mathcal{O}}(n)\}$ in a straightforward fashion. The statements about $\mathscr{B}$ and $\mathscr{B}'$ then follow from their definition. The fact that ϵ_{tot} is a co–unit is verified as in the non–Σ case and the other direction follows from Proposition 3.19. If the co–operad has a split co–unit, we denote by q the image of $|$ in $\check{\mathcal{O}}_{\mathbb{S}}$. Since $\mathscr{B}'$ is already commutative, q commutes with everything and we can collect the powers of q in each monomial leading to the identification with the polynomial ring over $\bar{S}\check{\mathcal{O}}^{red}$.

Finally, it is clear that a morphism of symmetric operads $\mathcal{O} \to \mathcal{P}$ induces a morphism $\check{\mathcal{P}} \to \check{\mathcal{O}}$ and due to the compatibility of the $\mathbb{S}_n$

[4]There is no additional assumption as the symmetry group that acts on $\check{\mathcal{O}}(1)$ is trivial

actions a morphism from $\check{\mathcal{P}}_{\mathbb{S}} \to \check{\mathcal{O}}_{\mathbb{S}}$ which is compatible with the $\bar{\gamma}^\vee$. The free functors are also functorial showing the functorialtiy of $\mathscr{B}$ and $\mathscr{B}'$. $\square$

We say that a symmetric operad $\mathcal{O}$ is co–connected, if it is split unital, $|$ is group–like for Δ and $(\check{\mathcal{O}}(1), |, \Delta|_{\check{\mathcal{O}}(1)})$ is connected. Let $\mathcal{I}$ be the two sided ideal of $\mathscr{B}'$ generated by $1 - |$. It is straightforward to check that this is also a co–ideal using the proof of Proposition 2.51.

Theorem 2.75. *If $\mathcal{O}$ is co–connected, $\mathscr{H} = \mathscr{B}'/\mathcal{I}$ is a Hopf algebra and under the identification $\mathscr{B} = \bar{S}\check{\mathcal{O}}_{\mathbb{S}}^{red}[q]$ is a deformation and as $q \to 1$ yields $\mathscr{H}$.*

Proof. Analogous to Theorem 2.52 and Corollary 2.57. $\square$

Example 2.76. In the examples of leaf–labelled rooted trees and surjections, extra multiplicities appear from considering the symmetric product, which identifies certain contributions. In particular, let $[c_n]$ be the class of an un-labelled non–planar corolla then

$$\Delta([c_n]) = \sum_{\substack{k, n_1 < \cdots < n_l, m_1, \ldots, \ldots, m_l : \\ \sum m_l n_i = n, \sum m_j = k, n_i, m_j \geq 1}} \binom{k}{m_1 \cdots m_k} [c_k] \otimes [c_{n_1}][c_{n_2}] \cdots [c_{n_k}] \tag{2.48}$$

Remark 2.77. Since $\mathscr{H}$ is commutative in the symmetric case, its dual is co-commutative and $\mathscr{H}^* = U(Prim(\mathscr{H}^*))$ by the Cartier–Milnor–Moore theorem. This relates to the considerations of [Kau07, CL01]. We leave the complete analysis for further study.

Example 2.78. Reconsidering the examples in this new fashion, we see that:

(1) For the ordered surjections, in the symmetric version, we get all the surjections, since the permutation action induces any order. These are pictorially represented by forests of non–planar corollas. Taking coinvariants makes these forests unlabelled and the forests commutative.
(2) This carries on to the graded case like in Baues.
(3) For the trees, we go from planar planted trees to leaf–labelled rooted trees in the symmetric version. The trees/forrests are again un–labelled on the co–invariants.

2.9.3. **Infinitesimal stucture.** For the infinitesimal structure, we notice that although the individual $\circ_i$ are not well defined on the invariants $\check{\mathcal{O}}(n)_{\mathbb{S}_n}$ their sum $\circ$ is. This was first remarked in [KM01]. Again, these are universal operations for the Feynman category for pseudo–operads $\mathfrak{O}$, see [KW17, §6]. Thus, the dual $\check{\circ} = \delta_c$ is well defined and we can again use (2.38) to extend to maps

$$\delta : \mathscr{B} \to \mathscr{B} \otimes \mathscr{B} \text{ and } \delta : \mathscr{B}' \to \mathscr{B}' \otimes \mathscr{B}' \tag{2.49}$$

this map is again pre–co–Lie and Poison. The analogue of Lemma 2.60 and Corollary 2.61 hold, as does:

Theorem 2.79. *The co-pre–Lie structure induces a co-Lie algebra structure on the indecomposables $\bar{\mathscr{H}}/\bar{\mathscr{H}}\bar{\mathscr{H}}$, where $\bar{\mathscr{H}}$ is reduced version of $\mathscr{H}$.*

If $\mathcal{O}$ is split unital, the terms of the co–pre–Lie Poisson δ on $\mathscr{B} = \bar{S}\check{\mathcal{O}}^{red}_{\mathbb{S}}[q]$ are given by terms of $\bar{\Delta}_k$ whose q–degree on the right is q^{k-1}. Alternatively, setting $\bar{\Delta}_{q,k} = (id \otimes q^{-k}) \circ \bar{\Delta}_k$, to compensate the grading on the right side, $\delta = res_{-1}(\sum_k \bar{\Delta}_{q,k})$ where res_{-1} is the residue in q.

Proof. Analogous to Theorem 2.63 and Corollary 2.62. □

Remark 2.80. There are actually no new symmetry factors in the examples appearing for δ since it only involves products with one term that is not $|$.

2.10. **Connes–Kreimer co–limit quotient.** To obtain the original Hopf algebra of Connes and Kreimer and its planar analogues, we have to take one more step, which in the general case is only possible if an additional structure is present.

Definition 2.81. A clipping or amputation structure, for a non-Σ co-ooperad $\check{\mathcal{O}}$ is a co–semisimplical structure that is compatible with the operad compositions, i. e.

(1) There are maps $\sigma_i : \check{\mathcal{O}}(n) \to \check{\mathcal{O}}(n-1)$, and for $i \leq j : \sigma_j \circ \sigma_i = \sigma_i \circ \sigma_{j+1}$. In case that there is no $\check{\mathcal{O}}(0)$, these maps are defined for $n \geq 2$.
(2) For all n, and partition $(n_1, \dots, n_k)$ of n and each $1 \leq i \leq n$, with i in the n_j component of the partition and i_j its the position within this block, i.e. $i = \sum_{k=1}^{j-1} n_j + i_j$:

$$\check{\gamma}_{n_1,\dots,n_j-1,\dots,n_k} \circ \sigma_i = (id \otimes id \otimes \dots \otimes \sigma_{i_j} \otimes id \otimes \dots \otimes id) \circ \check{\gamma}_{n_1,\dots,n_j,\dots,n_k} \tag{2.50}$$

with the factor σ_{i_j} in the $j+1$–st position acting on $\check{\mathcal{O}}(n_j)$.

In case that there is no $\check{\mathcal{O}}(0)$, we demand that $n \geq 2$ and also that $n_j \geq 2$.

For symmetric co–operads, in the set indexed version, the data and compatibilities for and amputation structure are given by a functor $\check{\mathcal{O}}$ from the category of finite set and surjections together with a compatible co–operad structure on the $\check{\mathcal{O}}(S)$. This boils down to:

(1') Amputation morphisms $\sigma_s : \check{\mathcal{O}}(S) \to \check{\mathcal{O}}(S \setminus s)$ for all $s \in S$. Which commute $\sigma_s \circ \sigma_{s'} = \sigma_{s'} \circ \sigma_s$, and are compatible with isomorphisms induced by bijections. Namely, for $\phi : S \to T$ a bijection, let $\phi|_{S\setminus\{s\}} : S \setminus \{s\} \to T \setminus \{\phi(s)\}$ be the bijection induced by restriction, then:

$$\sigma_{\phi(s)} \circ \check{\mathcal{O}}(\phi) = \check{\mathcal{O}}(\phi|_{S\setminus\{s\}}) \circ \sigma_s \tag{2.51}$$

Where $\check{\mathcal{O}}(\phi) : \check{\mathcal{O}}(S) \to \check{\mathcal{O}}(T)$ are the structural isomorphisms, see §2.9. Note that if $S = \{s\}$ then $\sigma_s : \mathcal{O}(\{s\}) \to \mathcal{O}(\varnothing)$ and this is excluded, if we make the assumption that there is no $\check{\mathcal{O}}(0) = \check{\mathcal{O}}(\varnothing)$.

(2') Given a surjection $f : S \twoheadrightarrow T$ for any $s \in S$, let $f|_{S\setminus\{s\}}$ be the restriction. Let $t = f(s)$ then the map $f|_{S\setminus\{s\}:S\setminus\{s\}} \twoheadrightarrow T$ is surjective as long as $f^{-1}(t) \neq \{s\}$. In this case, we set $f_{S\setminus s} = f|_{S\setminus s}$. This is the only case, if we exclude $\check{\mathcal{O}}(\varnothing)$. In case $\check{\mathcal{O}}(\varnothing)$ is allowed, if $f^{-1}(t) = \{s\}$ then we define $f_{S\setminus s} = f|_{S\setminus s} : S \setminus \{s\} \to T \setminus \{t\}$. With this notation the compatibility equation is

$$\check{\gamma}_{f_{S\setminus\{s\}}} \circ \sigma_s = \sigma_s \circ \check{\gamma}_f \tag{2.52}$$

Example 2.82. If $\mathcal{O}(n)$ is free, e.g. if it comes from a co–simplicial $\mathcal{S}et$ operad, then one can use the maps maps $\sigma_i : \mathcal{O}(n) \to \mathcal{O}(n-1)$, to induce map $\check{\mathcal{O}}(n) \to \check{\mathcal{O}}(n-1)$ by using a basis δ_τ of $\check{\mathcal{O}}(n)$ and sending $\delta_\tau \to \delta_{\sigma_i(\tau)}$. If these are compatible with the operad composition and in the symmetric case with the permutation group action, then one obtains an amputation structure. Note, we do not change the variance which would be the case if we were to use functoriality. Instead, we are using a basis to retain the variance.

The paradigmatic example being deletion of tails planar planted trees or labelled rooted trees. Here a σ_i removes the i-th tail. This is either the i-th tail in the planar order or the tail labelled by i. Likewise σ_s removes a tail labelled by s. The equations (2.50) and (2.52) then just state that it does not matter whether one first removes a tail and then cuts an internal edge, or vice–versa.

For a non–Σ cooperad without and $\check{\mathcal{O}}(0)$, with an amputation structure, we define $\mathscr{B}^{amp} = \mathrm{colim}_{\mathbb{A}}\check{\mathcal{O}}$, where the co–limit is taken along the directed system of amputation morphisms σ_i encoded by $\mathbb{A}$. In the symmetric case, again without $\check{\mathcal{O}}(\varnothing)$ we take the co–limit over the directed system $\mathbb{A}S$ induced by the σ_s $\mathscr{B}^{amp} = \mathrm{colim}_{\mathbb{A}S}\check{\mathcal{O}}$. Due to the equivariance the co–limit factors through to the coinvariants.

In particular, under our standard assumption, the co–limit is the quotient $\mathscr{B}^{amp} = \bigoplus_n \check{\mathcal{O}}(n)/\sim$ where $\sim$ is the equivalence relation generated by $\mathcal{O}(n) \ni a_n \sim b_{n-1} \in \mathcal{O}(n-1)$ if there exists an i such that $\sigma_i(a_n) = b_{n-1}$. In the symmetric case the co–limit can be computed as $\mathscr{B}^{amp} = \bigoplus_n \check{\mathcal{O}}(n)_{\mathbb{S}_n}/\sim$ where now $\mathcal{O}(n)_{\mathbb{S}_n} \ni [a_n] \sim [b_{n-1}] \in \mathcal{O}(n-1)_{\mathbb{S}_{n-1}}$ if there exists an i such that $\sigma_i(a_n) = b_{n-1}$.

Theorem 2.83. *For an $\mathcal{O}$ with an amputation structure, $\mathscr{B}^{amp}$ is a bi–algebra. In both cases, these co–limits descend to the quotient $\mathscr{H}$, set $\mathscr{H}^{amp} = colim_\sigma \check{\mathcal{O}}/\mathcal{I}$. It has a co–unit iff $\mathcal{O}$ has a split unit. If $\mathcal{O}$ is co-connected, $\mathscr{H}^{amp}$ is a Hopf algebra, and if $\mathcal{O}$ is symmetric co–connected $\mathscr{H}^{amp}$ is a commutative Hopf algebra.* □

Proof. The co–product is defined via the $\check{\gamma}$ and these are compatible with the amputation, so that the co–product descends. The product and compatibilities also descend as the σ_i and σ_s commute with disjoint unions. Since there is no $\check{\mathcal{O}}(0)$ the generator of the ideal $1 - |$ remains untouched. □

Example 2.84. Taking this co–limit on the planar planted trees or the labelled rooted trees yields the Hopf algebras of Connes and Kreimer. What is left in the co–limit are representatives which are trees without tails, sometimes called amputated trees [Kre99, BBM13]. The root half–edge, which is not amputated by the co–limit, simply indicates the planar order at the root in the planar case and can be forgotten in the rooted case, by marking the root vertex instead. The co–limit also removes any tails after an edge is cut, thus it also removes any tails of appearing from the cut edges effectively deleting the edges; the leaf half edge is clipped, and the non–deleted half–edges of the cut edges are all root half–edges, and can also be forgotten by the procedure above. Also notice that if the tail itself is cut, the condition that $| = 1$ takes care of the cuts "above" a leaf vertex or "below" the root vertex in the conventions of Connes and Kreimer and hence the co–product is exactly that of Connes and Kreimer, both in the commutative/symmetric and noncommutative/non–Σ case. A pictorial representation is given in Figure 5.

2.10.1. **Adding a formal $\check{\mathcal{O}}(0)$ to compute $\mathscr{H}^{amp}$.** It is convenient to think of the elements representing the co–limits as being elements of a $\check{\mathcal{O}}(0)$. In this case, $\check{\mathcal{O}}(0)$ becomes a final object and the co–limits are more easily computed. Notice that $\check{\mathcal{O}}(1)$ is not a final object for the clipping structure, as there are n morphisms from $\check{\mathcal{O}}(n)$ to $\check{\mathcal{O}}(1)$ "forgetting" all but one i. Considering the co–limits to lie in an additional $\check{\mathcal{O}}(0)$ yields an equivalent formulation whose construction is more involved, but whose pictorial representations are more obvious.

We assume that $\mathcal{O}$ is split unital and, we now allow $\check{\mathcal{O}}(0)$ and consider $\sigma_1 : \check{\mathcal{O}}(1) \to \check{\mathcal{O}}(0)$ with the conditions that

(1) $\check{\mathcal{O}}(0)$ only contains elements that are images of σ_i from higher $\check{\mathcal{O}}(n)$ and $\sigma_1(a) = [a]$ that is the class that a represents in the co–limit if $a \notin \mathbb{1}$ and $\sigma_1(|) = 1$.
(2) Define a co–product and product structure on $\check{\mathcal{O}}(0)$ by using representatives in $\check{\mathcal{O}}(n), n \geq 1$ and the co–operad and multiplication structure for the $\check{\mathcal{O}}(n), n \geq 1$, where $\sigma_1(|) = 1$ is identified with as the unit of the product on $\check{\mathcal{O}}(0)$.

It is clear then, that $\check{\mathcal{O}}(0)$ is just the co–limit $(\mathbb{1} \oplus \mathscr{B}^{amp})/(1-|) = \mathscr{H}^{amp}$ with the induced structures.

Proposition 2.85. *Enlarging $\check{\mathcal{O}}$ in this way and taking the co–limit directly yields $\check{\mathcal{O}}(0) = \mathscr{H}^{amp}$.* □

The computation in the co–limit version can be made using the formalism of Appendix A, §A.4.

2.11. **Coaction.**

2.11.1. $\check{\mathcal{O}}(0)$ **and coaction.** As we have seen in the last section, although it sometimes does make sense to include $\check{\mathcal{O}}(0)$, in general $\check{\mathcal{O}}(0)$ may prevent local finiteness and is hence be a potential hindrance to being co-nilpotent. It may also cause issues for summing over the Δ_k.

There is a remedy in which $\check{\mathcal{O}}(0)$ can be viewed as a co–algebra over a co–operad and then as a co–module over the Hopf algebra. For this consider a co–operad with $\check{\mathcal{O}}(0)$. Then set $\mathscr{B} = \bar{T}\check{\mathcal{O}}$ or $\mathscr{B} = \bar{S}\check{\mathcal{O}}$ in the symmetric case, as before, omitting the zero summand $\check{\mathcal{O}} = \bigoplus_{n \geq 1} \check{\mathcal{O}}$. Using the co–operad structure which is defined by using only the terms of the co-product of the form $\check{\gamma}_{n_1,\dots,n_k}$ with k and $n_i > 0$. Then $C = \bar{T}\check{\mathcal{O}}(0) = \check{\mathcal{O}}^{nc}(0)$ becomes a co-algebra over the co–operad via $\check{\gamma}_{0^k} := \check{\gamma}_{0,\dots,0} : \check{\mathcal{O}}(0) \to \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(0)^{\otimes k}$, $k > 0$. We will assume that this co-action is locally finite, that is for any $a \in \check{\mathcal{O}}(0)$ there sum over all $\check{\gamma}_{0^k}(a)$ is finite. Denote by μ the free (symmetric) multiplication. Notice that $\mu : \check{\mathcal{O}}^{nc}(0) \otimes \check{\mathcal{O}}^{nc}(0) \to \check{\mathcal{O}}^{nc}(0)$ which makes $\check{\mathcal{O}}^{nc}(0)$ into an algebra whose multiplication is compatible with the $\check{\gamma}_{0^k}$ by definition. Summing over all the $\check{\gamma}_{0^k}$ and post–composing with the multiplication on the factors of $\check{\mathcal{O}}(0)^{\otimes k}$ we get a co–algebra map

$$\check{\rho} : C \to \mathscr{B} \otimes C \tag{2.53}$$

where now $C = \check{\mathcal{O}}(0)$.

2.11.2. **Motivating examples.** Consider $\mathcal{O}$ with $\mathcal{O}(0)$ then $\mathcal{O}(0)$ is an algebra over $\mathcal{O}^{\oplus} = \bigoplus_{n\geq 1}\mathcal{O}(n)$ via: $\gamma : \mathcal{O}(k)\otimes\mathcal{O}(0)^{\otimes k}\to\mathcal{O}(0)$. Dually, we see that $\check{\mathcal{O}}(0)$ is a co–algebra over the co–operad $\check{\mathcal{O}}(n)$. This construction extends to $\check{\mathcal{O}}^{nc}$, where $\check{\mathcal{O}}^{nc}(0) = T\check{\mathcal{O}}(0) = \bigoplus_n \check{\mathcal{O}}(0)^{\otimes n}$. Now, we do have a well defined co–operad with multiplication structure on $\mathscr{B} = \bigoplus_{n\geq 1}\check{\mathcal{O}}^{nc}(n)$ and by restriction, we have a co–module given by extending

$$\check{\mathcal{O}}(0) \xrightarrow{\check{\rho}} \bigoplus_{k\geq 1}\check{\mathcal{O}}(k)\otimes\check{\mathcal{O}}(0)^{\otimes k} \tag{2.54}$$

Example 2.86. In the case of trees, we can also consider trees without tails. These will have leaf vertices, i.e. unary non-root vertices. Making admissible cuts those that leave a trunk that has only tails and no leaves, and branches that only have leaf vertices and no tails, we get a natural co–algebra structure. This is precisely the co–action (2.54).

The construction is related to §2.10.1 in that the latter differs from the former only in applying the co–limit deleting the tails on the left hand side as well.

2.11.3. **Co-algebras.** The construction above, immediately generalizes to any co–algebra C over a co–operad $\check{\mathcal{O}}$ with a compatible multiplication. Such a co–algebra by definition has $\check{\rho}^n : C \to \check{\mathcal{O}}(n)\otimes C^{\otimes n}, n > 0$. The extra datum is an associative algebra structure for C: $\mu_C : C\otimes C\to C$, which is compatible with the co–algebra over the co–operad $\check{\mathcal{O}}$ in the usual way. We furthermore assume that the $\check{\rho}^n$ are locally finite. Then we can define co–algebra maps $\check{\rho} : C\to\mathscr{B}\otimes C$ by setting $\mathscr{B} = \bigoplus_{n\geq 1}\check{\mathcal{O}}(n)$ as usual and defining the co–action by $\check{\rho} = \sum_n \mu^{\otimes n-1}\circ\check{\rho}^n$.

2.11.4. **Half–infinite chains, co–algebra.** One interesting algebra comes from adding $\vert\!\!\bullet\infty$, representing a half–infinite rooted chain, with $\Delta(\vert\!\!\bullet\infty) = \sum_{n\geq 0}\vert\!\!\bullet n\otimes\vert\!\!\bullet\infty$. This is an example where there is a bi–grading in which the co–product is finite in each bi–degree, the degrees lying in $\mathbf{N_0}\cup\{\infty\}$. With $s = \sum_{n\geq 0}\vert\!\!\bullet n$, we see from the associativity that $\Delta^n(\vert\!\!\bullet\infty) = s^{\otimes n-1}\otimes\vert\!\!\bullet\infty$ and s is a group like element. This fact leads to interesting physics, [Kre08].

We can also treat the half–infinite chain as a co–algebra $C = span(\vert\!\!\bullet\infty)$, and $\mathscr{H}$ being the graded Hopf algebra of trees, graded by the number of vertices or its sub–algebra of finite linear trees, where $\check{\rho}(\vert\!\!\bullet\infty) = \sum_{n\geq 0}\vert\!\!\bullet n\otimes\vert\!\!\bullet\infty$. Everything is finite in each degree.

Lastly, we can condenser the larger co–algebra is spanned by Dirac–trees that is rooted trees with semi–infinite chains as leaves. The co–action is to cut the semi–infinite tree with a cut that leaves a finite base tree and infinite Dirac–sea–type tree branches.

Having infinite chains is not that easy, but this will be considered elsewhere.

2.12. **Grading for the quotients and Hopf algebras.** Before taking the Hopf quotient there was the grading by $n-p$ for the graded case and filtered accordingly in the general case. Now, $|-1$ has degree 0, so the grading descends to the quotient $\mathscr{H}=\mathscr{B}/\mathcal{I}$. This works in the symmetric and the non–symmetric case. In the amputation construction, this grading will not prevail as the co–limit kills the operadic grading. However, in the case that $\mathscr{H}$ is indeed a connected Hopf algebra, there is the additional filtration by the co–radical degree r. We can lift this co–radical filtration to $\mathscr{B}$ and $\mathscr{H}_q$.

Lemma 2.87. *For an almost connected (symmetric or non–Σ) co–operad with multiplication, the co–radical filtration lifts to $\mathscr{B}$ and $\mathscr{H}_q$. The lifted co–radical filtration $\mathcal{R}$ is compatible with multiplication and co–multiplication and in particular satisfies. $\bar{\Delta}(\mathcal{R}^d)\subset\mathcal{R}^{d-1}\otimes\mathcal{R}^{d-1}$. This filtration descends to $\mathscr{H}$ and $\mathscr{H}^{amp}$ respectively.*

Proof. The co–radical degree of an element a is given by its quotient image, in which any occurrence of $|$ is replaced by 1. Since the lift or $|$ will lie in $\mathcal{R}^0$ and both 1 and $|$ are group like due to the bi–algebra equation the filtration is compatible with the multiplication and the co–multiplication. Due to the form of Δ in Proposition 3.24, see (3.11), we see that the first term of Δ descends as the only term of the type to $1\otimes a+a\otimes 1$ and hence $\bar{\Delta}$ descends to $\bar{\Delta}$ on $\mathscr{H}$. This shows the claimed property of $\bar{\Delta}$ on $\mathscr{B}$. The fact that the filtration descends through amputation is clear. ☐

Definition 2.88. We call the co–radical filtration of $\mathscr{B}$ and consequentially of $\mathscr{H}_q,\mathscr{H}$ well behaved, if

$$\bar{\Delta}(\mathcal{R}^i)\subset\bigoplus_{p+q=i}\mathcal{R}^p\otimes\mathcal{R}^q \tag{2.55}$$

We will use the same terminology for all the cases that is $\mathscr{H}$ and $\mathscr{H}^{amp}$ in both the symmetric and the non–Σ case.

Since μ is always additive in the co–radical filtration due to the bi–algebra equation, we have that the co–radical filtration respects both multiplication and co–multiplication.

Proposition 2.89. *If the coradical filtration is well behaved, then $\mathscr{H}$ and $\mathscr{H}^{amp}$ are graded by the co-radical degree.* □

Lemma 2.90. *For $\check{\mathcal{O}}^{nc}$, if $\check{\mathcal{O}}(1)$ is reduced, then the maximal coradical degree for an element in $\check{\mathcal{O}}(n,p)$ is indeed $n-p$.*

Proof. In $\check{\mathcal{O}}(n)$, applying $\bar{\Delta}$ we generically get a term $\check{\mathcal{O}}(n-1)\otimes\check{\mathcal{O}}(2)\otimes\check{\mathcal{O}}(1)^{\otimes n-2}$, repeating this procedure and "peeling off" an $\check{\mathcal{O}}(2)$ then the maximal co–radical degree will be $n-1$. Both n and p are additive under μ which finishes the argument. □

Example 2.91. Goncharov's and Baues' Hopf algebras are examples where this maximum is attained. Indeed any surjection $\underline{n} \twoheadrightarrow \underline{1}$ factors as $(\pi \amalg id \amalg \cdots \amalg id) \circ \cdots \circ (\pi \amalg id) \circ \pi : \underline{n} \twoheadrightarrow \underline{n-1} \twoheadrightarrow \cdots \twoheadrightarrow \underline{1}$ where $\pi : \underline{2} \twoheadrightarrow \underline{1}$ is the unique surjection. Using Joyal duality, see Appendix C.1, the same holds true for base–point preserving injections.

Example 2.92. Another instructive example is the Connes–Kreimer Hopf–algebra of rooted forests with tails. Here the co–radical degree of a tree is simply $E+1=V$, where E is the number of edges and V is the number of vertices. This is so, since each application of $\bar{\Delta}$ will cut at least one edge and cutting just one edge is possible. Since we are dealing with a tree $E+1=V$. For a forest with p trees, this it is $E+p=V$

This is the grading that descends to $\mathscr{H}^{amp}$. The same reasoning holds for the symmetric and the non–symmetric case.

Now there are two different gradings. The co–radical degree and the original grading by $n-p$. There is a nice relationship here. Notice that n is the number of tails, p is the number of roots thus for a forest the number of flags $F=n+p+2E=n-p+2(E-p)=n-p+2V$ and this is a third grading that is preserved. It is important to note that in the Hopf algebra $|=1$ and does not count as a flag.

Vice–versa since the flag grading and the $n-p$ grading are preserved it follows that the co–radical grading is preserved, giving an alternative explanation of it.

Proposition 2.93. *For a unital operad $\mathcal{O}$,with an almost connected $\check{\mathcal{O}}(1)$, $\check{\mathcal{O}}^{nc}$ has a well behaved co–radical grading.*

Proof. Fix an element $\check{a}$ dual to $a \in \mathcal{O}(n)$. Due to the conditions there are finitely many iterated γ compositions that result in a. Each of these can be presented by a level tree whose vertices v are decorated by elements of $\mathcal{O}(|v|-1)$. If we delete the nodes decorated by the identities, we remain with trees with vertices decorated by non–identity operad elements, see [KW17][2.2.1] for details about this construction. The number of edges of the tree then represent the number of operadic concatenations, and dually the number of co–operad and Δ operations that are necessary to reach the decomposition. It follows that the coradical degree is equal to the maximal value of $E + 1 = V$. This is easily seen to be additive under μ and preserved under Δ. The computations is parallel to the one above for the Connes–Kreimer algebra of trees, only that now the vertices are decorated by operad elements. In this picture, V is also the word length of the expression of an element as an itereated application of $\circ_i$ operations. □

Remark 2.94. This proposition also reconciles the two examples, Goncharov and Connes–Kreimer. Futhermore it explains the "lift" of Goncharov to the Hopf algebra of trivalent forests. Indeed the expression in Example 2.91 is word of length $n-1$ represented by a binary rooted tree. See also Example 5.44.

The fact that there is no general Hopf algebra morphism between the two Hopf algebras is explained in §5.7

3. A Generalization of Case I: Hopf algebras from co–operads with multiplication

In this section, we generalize §2 by replacing the free algebra with a more general multiplicative structure. This is captured in the concept of a co-operad with multiplication which is enough to obtain a bi–algebra. The theory already at this level is more involved as units and co–units for the bi–algebra become more difficult and lead to restrictions. Another aspect is that the natural gradings in the free case only correspond filtrations. However, regarding the associated graded objects to the filtered objects, we can prove a theorem stating that if units and co–units exist, the bi–algebra is a deformation of the associated graded, which in turn is the quotient of a free algebra on a co–operad.

3.1. From non-Σ co–operads with multiplication to bi–algebras.

We will now generalize the co–operad $\{\mathscr{B}(n)\} = \{\check{\mathcal{O}}^{nc}(n)\}$ of §2.3 with the free multiplication $\mu = \otimes : \mathscr{B}(m) \otimes \mathscr{B}(n) \to \mathscr{B}(m+n)$ to a co–operad with a compatible multiplication, where compatibility guarantees the bi–algebra structure. $(\{\mathscr{B}(m) = \check{\mathcal{O}}^{nc}(m)\}, \otimes)$ is a special case and will be called the free construction and all theorems apply to this special case.

Definition 3.1. A *non–Σ co–operad with multiplication* μ is a non–Σ co–operad $(\check{\mathcal{O}}, \check{\gamma})$ together with a family of maps, $n, m \geq 0$,

$$\mu_{n,m} : \check{\mathcal{O}}(n) \otimes \check{\mathcal{O}}(m) \to \check{\mathcal{O}}(n+m),$$

which satisfy the following compatibility equations:

(0) The maps $\mu_{n,m}$ are associative: $\mu_{l,n+m} \circ (id \otimes \mu_{n,m}) = \mu_{l+n,m} \circ (\mu_{l,n} \otimes id)$.

(1) For any $n, n' \geq 1$ and partitions $m_1 + \cdots + m_k = n$ and $m'_1 + \cdots + m'_{k'} = n'$, write $\check{\gamma}$ and $\check{\gamma}'$ for $\check{\gamma}_{k;m_1,\dots,m_k}$ and $\check{\gamma}_{k';m'_1,\dots,m'_{k'}}$ respectively, and write $\check{\gamma}''$ for $\check{\gamma}_{k+k';m_1,\dots,m_k,m'_1,\dots,m'_{k'}}$. Then the following diagram commutes

$$\begin{tikzcd}
\check{\mathcal{O}}(n) \otimes \check{\mathcal{O}}(n') \arrow[r, "\pi\,(\check{\gamma}\otimes\check{\gamma}')"] \arrow[d, "\mu_{n,n'}"'] & \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(k') \otimes \bigotimes_{r=1}^{k} \check{\mathcal{O}}(m_r) \otimes \bigotimes_{r'=1}^{k'} \check{\mathcal{O}}(m'_{r'}) \arrow[d, "\mu_{k,k'}\otimes \mathrm{id}"] \\
\check{\mathcal{O}}(n+n') \arrow[r, "\check{\gamma}''"] & \check{\mathcal{O}}(k+k') \otimes \bigotimes_{r=1}^{k} \check{\mathcal{O}}(m_r) \otimes \bigotimes_{r'=1}^{k'} \check{\mathcal{O}}(m'_{r'})
\end{tikzcd} \tag{3.1}$$

Here π is the isomorphism which permutes the $k+k'+2$ tensor factors according to the $(k+1)$-cycle $(2\,3\,\dots\,k+2)$.

(2) If $m''_1 + \cdots + m''_{k''} = n + n'$ is a partition of $n+n'$ which does not arise as the concatenation of a partition of n and a partition of n' (that is, there is no k such that $m''_1 + \cdots + m''_k = n$ and $m''_{k+1} + \cdots + m''_{k''} = n'$) then the composite

$$\check{\mathcal{O}}(n) \otimes \check{\mathcal{O}}(n') \xrightarrow{\mu_{n,n'}} \check{\mathcal{O}}(n+n') \xrightarrow{\check{\gamma}_{k'';m''_1,\dots,m''_{k''}}} \check{\mathcal{O}}(k'') \otimes \bigotimes_{r''=1}^{k''} \check{\mathcal{O}}(m''_{r''})$$

is zero.

Under the completeness assumption, the $\mu_{n,m}$ assemble into a map μ and the $\check{\gamma}_{k,n_1\cdot\dots\cdot n_k}$ assemble to a map $\check{\gamma}$. satisfying the compatibility relation

$$\check{\gamma}(\mu(a \otimes b)) = \mu(\pi(\check{\gamma}(a) \otimes \check{\gamma}(b))) \tag{3.2}$$

where π is the permutation that permutes the first factor of $\check{\gamma}(b)$ next to the first factor of $\check{\gamma}(a)$.

A *morphism of co–operads with multiplication* $f : \check{\mathcal{O}} \to \check{\mathcal{P}}$ is a morphism of co–operads which commutes with the multiplication, $f_{m+n}\mu_{n,m} = \mu_{n,m}(f_n \otimes f_m)$.

Denote by μ^k the k-th iteration of the associative product μ, e.g.: $\mu^1 = \mu, \mu^k = \mu \circ (\mu^{k-1} \otimes id)$.

Theorem 3.2. *Using the basic assumption 2.31, let $\check{\mathcal{O}}$ be a co–operad with compatible multiplication μ in an Abelian symmetric monoidal category with unit $\mathbb{1}$. Then*

$$\mathscr{B} := \bigoplus_n \check{\mathcal{O}}(n)$$

is a (non-unital, non-co–unital) bi–algebra, with multiplication μ, and comultiplication Δ given in short form notation by $(\mathrm{id} \otimes \mu)\check{\gamma}$: In more details: $\Delta = \sum_k \Delta_k$, $\Delta_n = \sum_k \Delta_{k,n}$ with $\Delta_{k,n} = (\mathrm{id} \otimes \mu^{k-1})\check{\gamma}_{k,n}$, where $\check{\gamma}_{k,n}$ is the sum over all $\check{\gamma}_{k;n_1,\dots,n_k}$ with $\sum_i n_i = n$.

$$\begin{tikzcd}
\check{\mathcal{O}}(n) \arrow[r, "\check{\gamma}"] \arrow[dr, bend right, "\Delta_n := \sum_k (\mathrm{id} \otimes \mu^{\otimes k-1})\check{\gamma}_{k,n}"'] & \bigoplus_{\substack{k \geq 1, \\ n = m_1 + \dots + m_k}} \left(\check{\mathcal{O}}(k) \otimes \bigotimes_{r=1}^{k} \check{\mathcal{O}}(m_r) \right) \arrow[d, "\mathrm{id} \otimes \mu^{k-1}"] \\
 & \bigoplus_{k \geq 1} \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(n).
\end{tikzcd} \tag{3.3}$$

Morphisms of co–operads with co–multiplication induce homomorphisms of bi–algebras.

Proof. The multiplication μ is associative by definition. The compatibility of μ with $\check{\gamma}$, together with the associativity of μ, shows that μ is a morphism of co–algebras, $\Delta\mu = (\mu \otimes \mu)\pi(\Delta \otimes \Delta)$:

$$\begin{tikzcd}
\check{\mathcal{O}}(n) \otimes \check{\mathcal{O}}(n') \arrow[r, "\pi\, \check{\gamma}^{\otimes 2}"] \arrow[d, "\mu_{n,n'}"'] \arrow[dr, phantom, "\text{compatibility}"] & \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(k') \otimes \bigotimes_{r=1}^{k} \check{\mathcal{O}}(m_r) \otimes \bigotimes_{r'=1}^{k'} \check{\mathcal{O}}(m'_{r'}) \arrow[r, "\mathrm{id}\otimes\mu\otimes\mu"] \arrow[d, "\mu_{k,k'}\otimes \mathrm{id}"] \arrow[dr, phantom, "\text{associativity}"] & \begin{matrix} \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(k') \\ \otimes \check{\mathcal{O}}(n) \otimes \check{\mathcal{O}}(n') \end{matrix} \arrow[d, "\mu_{k,k'}\otimes\mu_{n,n'}"] \\
\check{\mathcal{O}}(n+n') \arrow[r, "\check{\gamma}"] & \check{\mathcal{O}}(k+k') \otimes \bigotimes_{r=1}^{k} \check{\mathcal{O}}(m_r) \otimes \bigotimes_{r'=1}^{k'} \check{\mathcal{O}}(m'_{r'}) \arrow[r, "\mathrm{id}\otimes\mu"] & \check{\mathcal{O}}(k+k') \otimes \check{\mathcal{O}}(n+n').
\end{tikzcd}$$

For the co–associativity, one has to prove that $(id \otimes \Delta_{l,n})\Delta_{k,n} = (\Delta_{k,l} \otimes id)\Delta_{l,n} : \check{\mathcal{O}}(n) \to \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(l) \otimes \check{\mathcal{O}}(n)$, which can be done term by term using (2.15) and (3.1).

Explicitly fix a k–partition $n_1, \dots n_k$ of n an l partition $(m_1, \dots, m_l)$ of n. by compatibility the left hand side vanishes unless $(m_1, \dots, m_l)$ naturally decomposes into the list $(n^1_1, \dots, n^1_{l_1}, n^2_1, \dots, n^2_{l_2}, \dots, n^k_1, \dots, n^k_{l_k})$ where n^i_j is a partition of n_i. This yields the k partition $(l_1, \dots l_k)$ of l. Starting on the rhs that is with $(m_1, \dots, m_l)$ and $(l_1, \dots l_k)$, we decompose the list $(m_1, \dots, m_l)$ as above, which determines the $n_i = \sum_j n^i_j$.

Set $\Delta_{n_1,\dots,n_k} = (id \otimes \mu^{\otimes k-1}) \circ \check{\gamma}_{k;n_1,\dots,n_k}$, then:

$$
\begin{aligned}
&(id\otimes\Delta_{l;m_1,\dots,m_l})\Delta_{k;n_1,\dots n_k} = (id\otimes id\otimes\mu^{l-1})(id\otimes[\check{\gamma}_{l;m_1,\dots,m_l}\circ\mu^{k-1}])\circ\check{\gamma}_{k;n_1,\dots,n_k}\\
&= (id\otimes id\otimes\mu^{l-1})(id\otimes\mu^{k-1}\otimes id^{\otimes l})\circ\pi\circ(id\otimes\check{\gamma}_{l_1;n^1_1,\dots,n^1_{l_1}}\otimes\check{\gamma}_{l_2;n^2_1,\dots,n^2_{l_2}}\otimes\cdots\otimes\check{\gamma}_{l_k;n^k_1,\dots,n^k_{l_k}})\circ\check{\gamma}_{k;n_1,\dots,n_k}\\
&= (id\otimes\mu^{k-1}\otimes id)(id\otimes id^{\otimes k}\otimes\mu^{l-1})(\check{\gamma}_{k;l_1,\dots,l_k}\otimes id^{\otimes l})\check{\gamma}_{l;n^1_1,\dots,n^1_{l_1},n^2_1,\dots,n^2_{l_2},\dots,n^k_1,\dots,n^k_{l_k}}\\
&= [([id\otimes\mu^{k-1}]\check{\gamma}_{k;l_1,\dots,l_k})\otimes id](id\otimes\mu^{l-1})\check{\gamma}_{l;m_1,\dots,m_l} = (\Delta_{k;l_1,\dots,l_k}\otimes id)\Delta_{l;m_1,\dots,m_l}
\end{aligned}
\tag{3.4}
$$

where π is the permutation that shuffles all the right factors next to each other as before.

□

3.2. **A natural depth filtration and the associated graded.** In the free construction of §2 there is a natural grading by tensor length. In the general case, there is only a filtration, the depth filtration. The grading appears, as expected, on the associated graded object.

Definition 3.3. We define the *decreasing depth filtration* on a co–operad $\check{\mathcal{O}}$ as follows: $a \in F^{\geq p}$ if $\check{\gamma}(a) \in \bigoplus_{k\geq p}\bigoplus_{(n_1,\dots,n_k):\sum_i n_i=m} \check{\mathcal{O}}(k)\otimes \check{\mathcal{O}}(n_1)\otimes\cdots\otimes\check{\mathcal{O}}(n_k)$. So $\mathscr{B} = F^{\geq 1} \supset F^{\geq 2} \supset \dots$ and $\bigcap_p F^{\geq p} = 0$, since we assumed that there is no $\check{\mathcal{O}}(0)$ or at least that $\check{\mathcal{O}}$ is locally finite.

We define the depth of an element a to be the maximal p such that $a \in F^{\geq p}$.

This filtration induces a depth filtration $F^{\geq p}T\mathscr{B}$ on the tensor algebra $T\mathscr{B}$ by giving $F^{\geq p_1}\otimes\cdots\otimes F^{\geq p_k}$ depth $p_1+\cdots+p_k$. Note that any element in $T^p\mathscr{B}$ will have depth at least p.

Proposition 3.4. *The following statements hold for a co–operad with multiplication with empty $\check{\mathcal{O}}(0)$:*

a) *The algebra structure is filtered: $F^{\geq p}\cdot F^{\geq q} \subset F^{\geq p+q}$.*
b) *The co–operad structure satisfies $\check{\gamma}(F^{\geq p}) \subset F^{\geq p}\otimes T^{\geq p}\mathscr{B}$ where $T^{\geq p}\mathscr{B} = \bigoplus_{i=p}^{\infty}(\mathscr{B})^{\otimes i} \subset F^{\geq p}T\mathscr{B}$ and more precisely $\check{\gamma}_{k;n_1,\dots,n_k}$: $\check{\mathcal{O}}(n)\cap F^{\geq p} \to [\check{\mathcal{O}}(k)\otimes\check{\mathcal{O}}(n_1)\otimes\cdots\otimes\check{\mathcal{O}}(n_k)]\cap F^{\geq p}\otimes F^{\geq k}T\mathscr{B}$.*
c) *The co–algebra structure satisfies: $\Delta(F^{\geq p}) \subset F^{\geq p}\otimes F^{\geq p}$ and more precisely $\Delta_k(F^{\geq p}) \subset F^{\geq p}\otimes F^{\geq k}$.*
d) *$\check{\mathcal{O}}(n)\cap F^{\geq n+1} = \varnothing$.*

Proof. The first statement follows from the compatibility (3.1). The second statement follows from the Lemma 3.5 below. The more precise statement on the right part of the filtration stems from the fact that $T^k\mathscr{B} \subset F^{\geq k}T\mathscr{B}$. The third statement then follows from a) and b), since there are at least p factors on the right before applying the multiplication and the filtration starts at 1. This shows that the right factor is in $F^{\geq p}$. Finally, for $\check{\mathcal{O}}(n)$ the greatest depth that can be achieved happens when all the $n_i = 1 : i = 1, \dots, k$ and since they sum up to n this is precisely at $k = n$. $\square$

Lemma 3.5. *If $a^p \in \mathscr{B}$ of depth p let $\check{\gamma}_{k;n_1,\dots,n_k}(a^p) = \sum a^{(0)}_{(p_0)} \otimes a^{(1)}_{(p_1)} \otimes \cdots \otimes a^{(k)}_{(p_k)}$, where we used a generalized Sweedler notation for both the co–operad structure and the depth, then the terms of lowest depth will satisfy $p_0 = \sum_{i=1}^k p_i \geq p$.*

Proof. To show the equation, we use co–associativity of the co–operad structure. If we apply $\mathrm{id} \otimes \check{\gamma}^{\otimes k}$ we get least $1 + k + \sum_{i=1}^k p_i$ tensor factors from the lowest depth term, since we assumed that $\check{\mathcal{O}}(0)$ is empty. On the other hand applying $\check{\gamma} \otimes id^{\otimes k}$ to the terms of lowest depth, we obtain elements with at least $1 + p_0 + k$ tensor factors. Since elements of higher depth due to equation (2.15) produce more tensor factors these numbers have to agree. Since all the $p_i \geq 1$ their sum is $\geq p$.

$\square$

3.2.1. **The associated graded bi–algebra.** We now consider the associated graded objects $Gr^p := F^{\geq p}/F^{\geq p+1}$ and denote the image of $\check{\mathcal{O}}(n) \cap F^p$ in Gr^p by $\check{\mathcal{O}}(n,p)$. An element of depth p will have non–trivial image in Gr^p under this map. We denote the image of an element a^p of depth p under this map by $[a^p]$ and call it the principal part.

We set $Gr = \bigoplus Gr^p$, by part d) of 3.4: $Gr = \bigoplus_p \bigoplus_{n=1}^p \check{\mathcal{O}}(n,p)$ and define a grading by giving the component $\check{\mathcal{O}}(n,p)$ the total degree $n-p$.

Corollary 3.6. *By the Proposition 3.4 above we obtain maps*

- *$\mu : Gr^p \otimes Gr^q \to Gr^{p+q}$ by taking the quotient by $F^{\geq p+1} \otimes F^{q+1}$ on the left and $F^{\geq p+q+1}$ on the right*
- *$\check{\gamma}^{p,k} : Gr^p \to Gr^p \otimes (Gr^1)^{\otimes k}$ by taking the quotient by $F^{\geq p+1}$ on the left and $F^{\geq k+1}T\mathscr{B} \cap T^{\otimes k}\mathscr{B}$ on the right. In particular $\check{\gamma}(Gr^1) \subset Gr^1 \otimes TGr^1$*
- *$\Delta^{p,k} : Gr^p \to Gr^p \otimes Gr^k$ by taking the quotient by $F^{\geq p+1}$ on the left and $F^{\geq k+1}$ on the right.*
- *$\Delta^p : Gr^p \to Gr^p \otimes Gr$ via $\Delta^p = \sum_k \Delta^{p,k}$*
- *$\Delta : Gr \to Gr \otimes Gr$ via $\Delta = \sum_p \Delta^p$*

Proposition 3.7. *Gr inherits the structure of a non-unital, non-co–unital graded bi–algebra with the degree of $\check{\mathcal{O}}(n,k)$ being $n-k$. Each Gr^p is a non-co–unital comodule over Gr, and Gr^1 is a co–operad.*

Proof. Most claims are straightforward from the definitions in the corollary. For the grading we notice the multiplication preserves grading: $\check{\mathcal{O}}(n,p) \otimes \check{\mathcal{O}}(m,q) \dot{\to} \check{\mathcal{O}}(n+m, p+q)$ due to the bi-algebra equation. The degree on both sides is $m+n-p-q$. For the comultiplication we have that $\Delta_{k,n}(\check{\mathcal{O}}(n,p)) \subset \check{\mathcal{O}}(k,p) \otimes \check{\mathcal{O}}(n,k)$. The degree on the left is $n-p$ and on the right is $k-p+n-k = n-p$ and hence the comultiplication also preserves degree. $\square$

Example 3.8. For the free construction $\mathscr{B} = \check{\mathcal{O}}^{nc}$, we obtain

$$F^{\geq p} = \bigoplus_{k \geq p} \bigoplus_{(n_1,\dots,n_k)} \check{\mathcal{O}}(n_1) \otimes \cdots \otimes \check{\mathcal{O}}(n_k) \tag{3.5}$$

$$Gr^k = \bigoplus_{(n_1,\dots,n_k)} \check{\mathcal{O}}(n_1) \otimes \cdots \otimes \check{\mathcal{O}}(n_k) \tag{3.6}$$

$$\check{\mathcal{O}}^{nc}(n,k) = \bigoplus_{(n_1,\dots,n_k):\sum_i n_i = n} \check{\mathcal{O}}(n_1) \otimes \cdots \otimes \check{\mathcal{O}}(n_k) \tag{3.7}$$

This means that the depth of an element of $\mathscr{B}$ given by an elementary tensor is its length. The associated graded is isomorphic to the $\mathscr{B}$ which has a double grading by depth and operadic degree. Furthermore $Gr^1 = \check{\mathcal{O}}$ and $\mathscr{B} = (Gr^1)^{nc} = \check{\mathcal{O}}^{nc}$.

Corollary 3.9. *Since Gr^1 is a co–operad $(Gr^1)^{nc} = \bar{T}Gr^1$ yields a co–operad with (free) multiplication. The multiplication μ and its iterates define a morphism $(Gr^1)^{nc} \to Gr$ of co–operads with multiplication preserving the filtrations and hence give a morphism of (non-unital, non–co–unital) bi–algebras.*

Proof. Indeed the multiplication map gives such a map of algebras, since Gr^{nc} is the free algebra. The compatibility map (3.2) ensures that this is also a map of co–operads with multiplication. The compatibility with the filtration follows from the definitions. $\square$

3.3. **Unital and co–unital bi–algebra structure.** There is no problem in adding a unit and the existence of a bi–algebraic co–unit in the free construction $\check{\mathcal{O}}^{nc}$ the existence of a co–operadic co–unit for $\check{\mathcal{O}}$ is sufficient. For the general case, things are more complicated and worked out in detail in this section. An upshot is that in the free construction the existence of a co–operadic co–unit is also necessary.

In general, the existence of a right bi–algebra co–unit, is equivalent to the co–operad having a right co–unit, which extends to a multiplicative family. So that the existence of a right co–operadic co–unit is a necessary condition. As proven below, a co–operadic co–unit determines a unique candidate for a bi–algebra co–unit, which, however, does not automatically work in general; it does in the free construction. We give several conditions that are necessary for this, treating the cases of left and right co–units separately with care.

Having a left co–algebra co–unit for $\mathscr{B}$ fixes the structure of the associated graded as a quotient of the free construction on Gr^1 via the map of Corollary 3.9 and $\mathscr{B}$ is a deformation of this quotient, see Theorem 3.18.

3.3.1. **Unit.** If there is no element of operad degree 0 then, as the multiplication preserves operad degree, $(\mathscr{B}, \mu)$ cannot have a unit. In this case we may formally adjoin a unit 1 to $\mathscr{B}$: $\mathscr{B}' = \mathbb{1} \oplus \mathscr{B}$, with η be the inclusion of $\mathbb{1}$ and pr the projection to $\mathscr{B}$. We extend μ in the obvious way, and set $\Delta(1) = 1 \otimes 1$, making $\mathscr{B}'$ into a unital bi–algebra. In the full detail: $1 = id_{\mathbb{1}} \in Hom(\mathbb{1}, \mathbb{1})$ which is the ground ring/field. In the free construction, we think of $\mathbb{1}$ as the tensors of length 0 and in the Feynman category interpretation indeed $1 = id_{\mathbb{1}}$ where $\mathbb{1}$ is the emtpy word.

3.3.2. **Co–unit and multiplicativity.** We will denote putative co–units on $\mathscr{B}$ by $\epsilon_{tot} : \mathscr{B} \to \mathbb{1}$ and decompose $\epsilon_{tot} = \sum_{k\geq 1} \epsilon_k$ according to the direct sum decomposition on $\mathscr{B}$: $\epsilon_k : \check{\mathcal{O}}(k) \to \mathbb{1}$ extended to zero on all other components. We will also use the truncated sum $\epsilon_{\geq p} = \sum_{k \geq p} \epsilon_k$ which is set to 0 on all $\check{\mathcal{O}}(k)$ for $k < p$.

Remark 3.10. There is a 1–1 correspondence between (left/right) co–units on $\mathscr{B}$ and on $\mathscr{B}'$. This is given by adding ϵ_0 on the identity component via the definition $\epsilon_0 \circ \eta = id$ and vice–versa truncating the extended sum $\epsilon_{tot} = \sum_{k\geq 0} \epsilon_k$ at $k = 1$.

A family of morphisms $\epsilon_k : \check{\mathcal{O}}(k) \to \mathbb{1}$ is called multiplicative if $\kappa \circ (\epsilon_k \otimes \epsilon_l) = \epsilon_{k+l} \circ \mu$, where $\kappa : \mathbb{1} \otimes \mathbb{1} \to \mathbb{1}$ is the unit constraint — e.g. multiplication in the ground field in case we are in k-$Vect$ — which we will omit from now on.

Lemma 3.11. *If ϵ_{tot} is a co–unit (left or right) then the ϵ_k are a multiplicative family. More generally $\epsilon_{n_1} \otimes \cdots \otimes \epsilon_{n_k} = \epsilon_{\sum n_i} \circ \mu^{k-1}$ and in particular $\epsilon_1^{\otimes k} = \epsilon_k \otimes \mu^{k-1}$. If ϵ_k is a any multiplicative family and η_1 is a section of ϵ_1 then $\mu^{k-1} \circ \eta_1^{\otimes k}$ is a section of ϵ_k.*

Furthermore ϵ_{tot} descends to the associated graded.

Proof. The first statement is equivalent to ϵ being an algebra morphism. The other equations follow readily. Now $\epsilon_p(F^{\geq p+1}) = 0$, since $\check{\mathcal{O}}(p, p+1) = 0$ and hence each ϵ_p descends to Gr^p. The sum ϵ_{tot} then descends as the sum of the ϵ_p with each ϵ_p defined on the summand Gr^p. $\square$

3.3.3. **Right co–algebra co–units.**

Lemma 3.12. *If $\mathscr{B}$ has a right bi–algebra co–unit ϵ_{tot}, then ϵ_1 is a right co–operadic co–unit. If there are elements of depth greater than one, there can be no left co–operadic co–unit.*

Proof. For the first statement, we verify (2.18) using Lemma 3.11:

$$\sum_k (id \otimes \epsilon_1^{\otimes k}) \circ \check{\gamma} = \sum_k (id \otimes \epsilon_k) \circ \mu^{k-1} \circ \check{\gamma} = (id \otimes \epsilon_{tot}) \circ \Delta = id \quad (3.8)$$

The second statement just says that using ϵ on the left, we would need exactly one tensor factor on the right after applying $\check{\gamma}$ in order to get an identity. Indeed, if we apply $\check{\gamma}$ to $a \in F^{\geq p}$ then there are at least $p+1$ tensor factors, and ϵ will only take the leftmost tensor factor to the ground field. Thus there can be no left co–unit for elements in $F^{\geq 2}$. □

A necessary condition for the existence of a right co–unit for $\mathscr{B}$ is hence

Proposition 3.13. *ϵ_{tot} is a right bi–algebraic co–unit if and only if ϵ_1 is a right co–operadic co–unit which extends to a multiplicative family ϵ_k.*

Proof. This follows by reading equation (3.8) right to left. □

3.3.4. **Left co–algebra co–units.**

Proposition 3.14. *If $\mathscr{B}$ as a co–algebra has a left co–unit ϵ_{tot}, then $F^{\geq p} = (F^{\geq 1})^{\geq p}$, where the latter denotes the sum of the k–th powers of $F^{\geq 1}$ with $k \geq p$. Moreover, the morphism of co–operads with multiplication and of bi–algebras $(Gr^1)^{nc} \to Gr$ given by Corollary 3.9 is surjective.*

Proof. The inclusion $F^{\geq p} \supset (F^{\geq 1})^{\geq p}$ is in Proposition 3.4. For the reverse inclusion, let $a \in F^{\geq p}$, then after applying $(\epsilon_{tot} \otimes id) \circ \Delta$ we are left with a sum of products of at least p factors and hence the reverse inclusion follows.

In the same way, we see that $Gr^p = (Gr^1)^p$ and that the map in question is surjective.

□

We recall from [Ger64] that a filtered algebra/ring $(\mathscr{B}, F^{\geq p})$ is predevelopable if there exists for each p an additive mapping $q_p : Gr^p \to F^{\geq p}$ which is a section of $p_p : F^{\geq p} \to Gr^p = F^{\geq p}/F^{\geq p+1}$ i.e. $p_p \circ q_p(a) = a$ for all $a \in Gr^p$. It is developable if also $\bigcap_p F^{\geq p} = 0$ and the ring is complete in the topology induced by the filtration. In our case, due to the assumption the there is no $\check{\mathcal{O}}(0)$, the first condition is true and also since we only took finite sums, the algebra is complete.

Proposition 3.15. *If $\mathscr{B}$ has a left co–algebra co–unit then $P_p = (\epsilon_{\geq p} \otimes id) \circ \Delta$ is a projector to $F^{\geq p}$. Hence the short exact sequence $0 \to F^{\geq p+1} \to F^{\geq p} \to Gr^p \to 0$ splits and $\mathscr{B}$ is predevelopable.*

Proof. If ϵ_{tot} is a left co–algebra co–unit then using multi–Sweedler notation for $a \in \check{\mathcal{O}}(n) : a = (\epsilon_{tot} \otimes id) \circ \Delta(a) = \sum_k \epsilon_k(a_k^{(0)}) \otimes a_{n_1}^{(1)} \cdots a_{n_k}^{(1)} =: \sum_k a_k$ with a_k a product of k factors and hence in $F^{\geq k}$. Since $\epsilon_{\geq p} = 0$ on $\check{\mathcal{O}}(k) : k < p$, we see that $P_p(a) = \sum_{k \geq p}^n a_k$ and hence the image of P_p lies in $F^{\geq p}$. If on the other hand $a \in F^{\geq p}$ then $a = \sum_k a_k = \sum_{k \geq p} a_k = P_p(a)$, since all lower terms do not exist as the summation for Δ stands at p.

□

Note that $T_i(a) = [P_{i-1} \cdots P_1(a)]$ gives the development of a in Gr in the notation of [Ger64].

Corollary 3.16. *If ϵ_{tot} is a left bi–algebra unit, then for $a \in \check{\mathcal{O}}(n) \cap F^{\geq p}$ there is a decomposition $a = \sum_{k \geq p}^n a_k$ with each $a_k \in F^{\geq k}$ and (after possibly collecting terms) this gives the development of a.* □

Corollary 3.17. *If ϵ_{tot} is a left co–algebra co–unit for $\mathscr{B}$, then ϵ_p descends to a well defined map $Gr^p \to \mathbb{1}$. and on $Gr^p : (\epsilon_p \otimes id) \circ \Delta_p = id$. Thus ϵ_{tot} understood as acting on Gr^p with ϵ_p is a left co–unit for Gr. Furthermore $(\epsilon_k \otimes id) \circ \Delta|_{Gr^p} = \delta_{k,p} id$.*

Proof. First $\epsilon_p(F^{\geq p+1}) = 0$, since $\check{\mathcal{O}}(p, p+1) = 0$. The statements then follows from the development. □

It is known [Ger64] that if $\mathscr{B}$ is developable then Gr is a deformation of $\mathscr{B}$. Coupled with the results above one has:

Theorem 3.18. *if $\mathscr{B}$ has a left co–algebra co–unit, then $\mathscr{B}$ is a deformation Gr, which is a quotient of the free construction on Gr^1.*

□

3.3.5. **Units and co–units for the free case $\check{\mathcal{O}}^{nc}$.** In this section, we let $\check{\mathcal{O}}$ be a co–operad and consider $\check{\mathcal{O}}^{nc}(n) = \bigoplus_k \bigoplus_{(n_1,\dots,n_k):\sum_i n_i = n} \check{\mathcal{O}}(n_1) \otimes \cdots \otimes \check{\mathcal{O}}(n_k)$ and its bi–algebra $\mathscr{B} = \bigoplus \check{\mathcal{O}}^{nc}(n)$.

Proposition 3.19. *The bi–algebra $\mathscr{B} = \bigoplus_n \check{\mathcal{O}}^{nc}(n)$ has a bi–algebraic co–unit if, and only if, $\check{\mathcal{O}}$ has a co–operadic co–unit.*

Proof. We already know that a right co–operadic co–unit for $\check{\mathcal{O}}^{nc}$ is necessary. This yields a right co–operadic coounit for $\check{\mathcal{O}}$ by restriction to $Gr^1 = \check{\mathcal{O}}$. Then for $a \in \check{\mathcal{O}} = Gr^1$ $a = \epsilon_1 \otimes id \circ \Delta(a) = \sum_k \epsilon_1 \otimes id^{\otimes k} \circ \check{\gamma}$, since all terms with $k \neq 1$ vanish and for the term with $k = 1$ $\Delta = \check{\gamma}$. Thus ϵ_1 is also a left co–operadic co–unit for $\check{\mathcal{O}}$. We stress for $\check{\mathcal{O}}$ not for $\check{\mathcal{O}}^{nc}$.

Now assume that ϵ_1 is a co–operadic co–unit for $\check{\mathcal{O}}$. It follows that ϵ_1 is a right co–operadic co–unit for $\check{\mathcal{O}}^{nc}$ by compatibility. Now since $\mu = \otimes$: the extension $\epsilon_k = \epsilon_1^{\otimes k}$ is multiplicative and hence a right bi–algebra co–unit. It remains to check whether it is bi–algebraic, which reduces to checking that it is a left co–algebraic unit. The multiplicativity is clear, so, we only need to check on Gr^1, that is for all $a \in \check{\mathcal{O}}^{nc}(n, 1) = \check{\mathcal{O}}(n)$. On the $\check{\mathcal{O}}(n)$ the equation says exactly that ϵ_1 is a left co–operadic unit for $\check{\mathcal{O}}$. $\square$

3.3.6. **Co–units summary.** If $\mathscr{B}$ comes from $\check{\mathcal{O}}^{nc}$ then having a bi–algebra unit ϵ_{tot} is equivalent to ϵ_1 being a co–operad co–unit on $\check{\mathcal{O}}$.

In general, for $\mathscr{B}$ to have a bi–algebra co–unit, it is necessary, that

(1) ϵ_1 is a right co–operadic co–unit.
(2) $F^{\geq p} = (F^{\geq 1})^{\geq p}$.
(3) $P_k = (\epsilon_{\geq k} \otimes id) \circ \Delta$ are projectors onto $F^{\geq k}$.
(4) $\mathscr{B}$ is developable and a deformation of the associated graded Gr

On the associated graded Gr. If ϵ_{tot} is a putative bi–algebra co–unit

(1) ϵ_p is uniquely determined from ϵ_1.
(2) Lifted to $(Gr^1)^{nc}$, ϵ_1 is a co–operadic unit, which ensures that the lift of ϵ_{tot} is a bi–algebra unit.
(3) For ϵ_{tot} to descend to Gr, it needs to vanish on the kernel of the, by (2) surjective, map $\mu^{\otimes p-1} : (Gr^1)^{\otimes p} \to Gr^p$.

The first statement holds by Proposition 3.14 and Corollary 3.17 which says that $Gr^p = (Gr^1)^p$ and hence Lemma 3.11 determines ϵ_p. Since co–units are multiplicative, they lift via Proposition 3.19.

Definition 3.20. In general, we say that a co–operadic right co–unit ϵ_1 is *bi–algebraic*, if it extends to a bi–algebraic co–unit ϵ_{tot} for $\mathscr{B}$. If such an ϵ_{tot} exists, we will call $\check{\mathcal{O}}$ bi–algebraic.

3.4. **The pointed case.**

Definition 3.21. A co–operad $\check{\mathcal{O}}$ with a right co–operadic co–unit ϵ_1 is called *pointed* if the co–unit ϵ_1 is split, i.e. there is a section $\eta_1 : \mathbb{1} \to \check{\mathcal{O}}(1)$ of ϵ_1.

We call $\check{\mathcal{O}}$ *reduced* if it is pointed and η_1 is an isomorphism $\mathbb{1} \simeq \check{\mathcal{O}}(1)$; it is then automatically pointed.

A bi–algebra unit will be called split and $\mathscr{B}$ pointed if the associated right co–operadic unit ϵ_1 is split.

We will denote $| := \eta_1(1)$. For pointed co–operads Lemma 3.11 applies and we split each $\check{\mathcal{O}}(n) = \mathbb{1} \oplus \bar{\check{\mathcal{O}}}(n)$ where $\bar{\check{\mathcal{O}}}(n) = ker(\epsilon_n) = ker(\epsilon_{tot}|_{\check{\mathcal{O}}(n)})$ and $\mathbb{1}$ is the component of $|^n$. We set $\bar{B} = \bigoplus \bar{\check{\mathcal{O}}}(n)$. Notice that this is smaller than the augmentation ideal $\mathscr{B}^{red} = ker(\epsilon_{tot})$.

Example 3.22. The connection to the free construction of §2 is as follows: $\check{\mathcal{O}}^{nc}$ for a unital operad $\mathcal{O}$ is pointed if the unit morphism $u : \mathbb{1} \to \mathcal{O}(1)$ split via a morphism c. The element $|$ is then the dual element to the unit $u(1) \in \mathcal{O}(1)$. Here $| = \check{c}(1) = \eta_1(1)$ and being the dual element means that $\check{u}(|) = \epsilon_1 \circ \eta_1(1) = (c \circ u)^\vee(1) = 1$. All of the examples of §2.2.3 have this property.

Lemma 3.23. *If $\mathscr{B}$ has a split bi–algebraic co–unit, then have $\Delta(|) = | \otimes | + \bar{\Delta}(|)$ with $\bar{\Delta}(|) \in \bar{\check{\mathcal{O}}}(1) \otimes \bar{\check{\mathcal{O}}}(1)$ and hence $\Delta(|^p) = |^p \otimes |^p +$ terms of lower order in $|$. Thus the image of $|^p$ is not 0 in Gr^p and we can split $Gr^p = \mathbb{1} \oplus \overline{Gr}^p$ where $\mathbb{1}$ is the component if the image of $|^p$.*

Proof. The first statement follows since ϵ_{tot} is a bi–algebraic unit. The second statement follows, from the bi–algebra compatibility condition. □

More generally,

Proposition 3.24. *Let $\check{\mathcal{O}}$ be a co–operad with multiplication and a pointed bi–algebraic co–unit on $\mathscr{B}$, then*

$$\begin{aligned} \Delta(|) &= | \otimes | + \bar{\Delta}(|) \text{ with} \\ \bar{\Delta}(|) &\in \bar{\check{\mathcal{O}}}(1) \otimes \bar{\check{\mathcal{O}}}(1) \qquad (3.9) \\ \Delta(|^p) &= |^p \otimes |^p + \bar{\Delta}(|^p) \text{ with} \\ \bar{\Delta}(|^p) &\in \bar{\check{\mathcal{O}}}(p) \otimes \bar{\check{\mathcal{O}}}(p) \qquad (3.10) \end{aligned}$$

And for $a \in \bar{\check{\mathcal{O}}}(n) \cap F^{\geq p}$

$$\begin{aligned} \Delta(a) &= \sum_{k \geq p}^{n} |^k \otimes a_k + a \otimes |^n + \bar{\Delta}(a) \text{ with} \\ & a_k \in \bar{\check{\mathcal{O}}}(n), \bar{\Delta}(a) \in \bar{\mathscr{B}} \otimes \bar{\check{\mathcal{O}}}(n) \qquad (3.11) \end{aligned}$$

with $a = \sum_{k \geq p}^{n} a_k$ and the a_k are as in Corollary 3.16.

Likewise, in the associated graded case, for $a \in \bar{\check{\mathcal{O}}}(n,p)$

$$\Delta(a) = |^p \otimes a + a \otimes |^n + \bar{\Delta}(a) \text{ with} \\ \bar{\Delta}(a) \in \bar{Gr} \otimes \bar{Gr} \tag{3.12}$$

Again, if these equations hold having a bi–algebraic co–unit ϵ_{tot} is equivalent to ϵ_1 being a right co–operadic co–unit.

Proof. Using Corollary 3.16 and applying ϵ_{tot} on the left, we obtain the first term and applying ϵ_{tot} on the right, the second term. These are different if $a \neq |^k$ for some k. In the case $a = |^k$ the equation follows from the Lemma above. In general, the remaining terms lie in the reduced space. Replacing $\mathscr{B}$ with Gr proves the rest. □

We also get a practical criterion for a bi–algebra co–unit.

Corollary 3.25. *Conversely, assume the equations in Propositions 3.24 hold, then having a bi–algebraic counit ϵ_{tot} is equivalent to ϵ_1 being a right co–operadic co–unit.*

Proof. By Lemma 3.11, we see that ϵ_k is the projection to the factor $|^k$ of $\check{\mathcal{O}}(k) = \mathbb{1} \oplus \bar{\check{\mathcal{O}}}(k)$ and on that factor it is $\epsilon_1^k \circ \mu^{k-1}$ and hence determined by ϵ_1. Now the second term of (3.11) is equivalent to ϵ_{tot} being a right bi–algebra co–unit. Furthermore, since this is the term relevant for the right co–operad co–unit, we obtain the equivalence for the right bi–algebra co–unit. Similarly, applying the given ϵ_{tot} as a potential left bi–algebra co–unit, we see that having a left bi–algebra co–unit is equivalent to $a = \sum_k a_k$, i.e. the first term in (3.11). □

3.5. **Hopf Structure.** In this section, unless otherwise stated, we will assume that $\check{\mathcal{O}}$ is a co–operad with multiplication and a split bi–algebraic co–unit and assume Assumption 2.45.

Definition 3.26. We call a pointed co–operad $\check{\mathcal{O}}$ with multiplication and bi–algebraic co–unit ϵ_{tot} *connected* if

1. The element $|$ is group–like: $\Delta(|) = | \otimes |$.
2. $(\check{\mathcal{O}}(1), \eta_1, \epsilon_1)$ is connected as a co–algebra.

Notice that a reduced $\check{\mathcal{O}}$ is automatically connected, but this is not a necessary condition. In the free case, the co–operad $\check{\mathcal{O}}^{nc}$ is connected if $\check{\mathcal{O}}$ is. If we start with an operad $\mathcal{O}$ as in §2, the co–operad $\check{\mathcal{O}}^{nc}$ is connected if $\mathcal{O}$ is co-connected.

Remark 3.27. Notice that for a connected co–operad $\{\check{\mathcal{O}}(n)\}$ the bi–algebras $\mathscr{B} = \bigoplus \check{\mathcal{O}}(n)$ and $\mathscr{B}' = \mathbb{1} \oplus \mathscr{B}$ are usually not connected, since all powers $|^k$ are group–like: $\Delta(|^k) = |^k \otimes |^k$, $\epsilon_{tot}(|^k) = 1$.

For a pointed co–operad with multiplication, let $\mathcal{I}$ be the two-sided ideal spanned by $1 - |$. Set

$$\mathscr{H} := \mathscr{B}'/\mathcal{I} \tag{3.13}$$

Notice that in $\mathscr{H}$ we have that $|^k \equiv 1 \mod \mathcal{I}$ for all k.

Proposition 3.28. *If $\check{\mathcal{O}}$ is connected, then $\mathcal{I}$ is a coideal and hence $\mathscr{H}$ is a co–algebra. The unit η descends to a unit $\bar{\eta} : \mathbb{1} \to \mathscr{H}$ and the co–unit ϵ_{tot} factors as $\bar{\epsilon}$ to make $\mathscr{H}$ into a bi–algebra.*

Proof. Analogous to Proposition 2.51. □

Theorem 3.29. *If $\check{\mathcal{O}}$ is connected as a co–operad then $\mathscr{H}$ is co–nilpotent and hence admits a unique structure of Hopf algebra.*

Proof. Analogous to Theorem 2.52 □

3.6. **Deformation, symmetric version, amputation and grading.** Assuming that we have a co–operad with multiplication and a split bi–algebraic co–unit, we get the analogous results to §§2.6, 2.7, 2.10 and 2.12, see below.

3.6.1. **Deformations.** Let $\mathcal{C}$ be the ideal spanned by $|a - a|$ of $\mathscr{B}$. This is again a co–ideal by the calculation of Proposition 2.55. Denote by q the image of $|$ under $\pi : \mathscr{B} \to \mathscr{B}/\mathcal{C} =: \mathscr{H}_q$.

Theorem 3.30. *Let $\check{\mathcal{O}}$ be a co–operad with multiplication and a split bi–algebraic co–unit, then*

$$\mathscr{H}_q(d) \simeq \bigoplus_{n \le d} q^{n-d} \bar{\mathscr{B}}(n) \tag{3.14}$$

and $\mathscr{H}_q$ is a deformation of $\mathscr{H}$ given by $q \to 1$.

Proof. check formula □

3.6.2. **A type of bi–algebra from pseudo co–operads with multiplication and indecomposables.**

Definition 3.31. A *pseudo-cooperad with multiplication* μ is a pseudo-cooperad (cf. 2.7.1) $\check{\mathcal{O}}$ with a family of maps, $n, m \ge 0$,

$$\mu_{n,m} : \check{\mathcal{O}}(n) \otimes \check{\mathcal{O}}(m) \to \check{\mathcal{O}}(n+m)$$

which together with the co–multiplication $\delta := \check{\circ}$ of (2.37) satisfies the equation

$$\delta \circ \mu \quad = \quad (\mu \otimes id) \circ \pi_{23} \circ (\delta \otimes \mathrm{id}) + (\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta) \tag{3.15}$$

Proposition 3.32. *If $\check{\mathcal{O}}$ is a non-Σ cooperad with multiplication and multiplicative right cooperadic counit. Then the multiplication is also compatible with the non-Σ pseudo-cooperad structure.*

Proof. Straightforward using equation (3.15). □

Proposition 3.33. *If the co–operad $\check{\mathcal{O}}$ is connected, then in the Hopf quotient, the co-preLie structure induces a co-Lie algebra structure on the indecomposables $\mathscr{H}_>/\mathscr{H}_>\mathscr{H}_>$, where $\mathscr{H}_>$ is reduced version of $\mathscr{H}$.*

Proof. Analogous to the proof of 2.63. □

Example 3.34. In the free case $\check{\mathcal{O}}^{nc}$, the indecomposables are precisely given by $\check{\mathcal{O}}$ and the co-preLie structure is $\check{\circ}$. If $\check{\mathcal{O}}$ is the dual of $\mathcal{O}$ then the co-Lie structure corresponds dually to the usual Lie structure of Gerstenhaber.

3.6.3. **Symmetric version.** We now assume that the co–operad $\check{\mathcal{O}}$ is symmetric and in finite sets. We then have the same diagram as (2.46).

Definition 3.35. A co-operad with multiplication in finite sets, is a cooperad in finite sets with multiplications $\mu_{S,T} : \check{\mathcal{O}}(S) \otimes \check{\mathcal{O}}(T) \to \check{\mathcal{O}}(S \sqcup T)$, such that the following diagram commutes.

$$\begin{tikzcd}
\check{\mathcal{O}}(S) \otimes \check{\mathcal{O}}(T) \arrow[r, "\mu_{S,T}"] \arrow[d, "\sigma\otimes\sigma'"'] & \check{\mathcal{O}}(S \sqcup T) \arrow[d, "\sigma\sqcup\sigma'"] \\
\check{\mathcal{O}}(S') \otimes \check{\mathcal{O}}(T') \arrow[r, "\mu_{S',T'}"] & \check{\mathcal{O}}(S' \sqcup T')
\end{tikzcd} \tag{3.16}$$

and the analogue of (3.2) holds equivariantly.

Lemma 3.36. *For a co–operad with multiplication in finite sets the co–operad structure and the multiplication descend to the coinvariants.*

Proof. The well–definedness of $\check{\gamma}$ on the co–invariants is guaranteed by (2.46). Due to the diagram (3.16) the multiplication descends to the co–invariants as well. This means that Δ is well defined and it follows that it is co–associative. Finally, since (3.2) holds equivariantly, Δ is comptible □

Set $\mathscr{B}_{\mathbb{S}} = \bigoplus_n \check{\mathcal{O}}(n)_{\mathbb{S}_n}$. A bialgebraic counit ϵ is called invariant if for all $a_S \in \check{\mathcal{O}}(S)$ and any isomorphism $\sigma : S \to S'$, $\epsilon \circ \sigma = \epsilon$.

Proposition 3.37. *With the assumption above, $\mathscr{B}_{\mathbb{S}}$ is a non-unital, non-counital, bialgebra. If we furthermore assume that an invariant bialgebraic counit for $\mathscr{B}$ exists then $\mathscr{B}'_{\mathbb{S}} = k \oplus \mathscr{B}_{\mathbb{S}}$ is a unital and* *counital bialgebra and $\bar{\mathscr{H}} := \mathscr{B}/\bar{\mathcal{I}}$, where $\bar{\mathcal{I}}$ is the image of $\mathcal{I}$ in $\mathscr{B}'_{\mathbb{S}}$, is a bi–algebra, that is if it is connected a commutative Hopf algebra.* □

3.6.4. **The free example.** In the free example, starting with a symmetric operad, we do not only have to take the sum, but also induce the representation to $\mathbb{S}_n$ in order to obtain a symmetric co–operad with multiplication. Let

$$\check{\mathcal{O}}^{symmnc}(n) = \bigoplus_k \bigoplus_{(n_1,\dots,n_k):\sum_i n_i = n} Ind^{\mathbb{S}_n}_{(\mathbb{S}(n_1)\times\cdots\times\mathbb{S}(n_k))\wr\mathbb{S}(k)} \check{\mathcal{O}}(n_1)\otimes\cdots\otimes\check{\mathcal{O}}(n_k) \tag{3.17}$$

Remark 3.38. When taking coinvariants, this induction step is cancelled and we only have to take co–invariants with respect to $\mathbb{S}(n_1)\times\cdots\times\mathbb{S}(n_k)\times\mathbb{S}(k)$. That is

$$\mathscr{B}_{\mathbb{S}} = \bigoplus \check{\mathcal{O}}^{symmnc}(n)_{\mathbb{S}_n} = \bigoplus_k \bigoplus_{(n_1,\dots,n_k):\sum_i n_i = n} \left(\check{\mathcal{O}}(n_1)_{\mathbb{S}_{n_1}} \otimes\cdots\otimes \check{\mathcal{O}}(n_k)_{\mathbb{S}_{n_k}}\right)_{\mathbb{S}_k} = \bigoplus_{(n_1,\dots,n_k):\sum_i n_i = n} \check{\mathcal{O}}(n_1)_{\mathbb{S}_{n_1}} \odot\cdots\odot \check{\mathcal{O}}(n_k)_{\mathbb{S}_{n_k}} \tag{3.18}$$

where $\odot$ is the symmetric product.

Proposition 3.39. *The $\check{\mathcal{O}}^{symmnc}(n)$ form a symmetric cooperad with mutiplication and $\mathscr{B} = \bigoplus \check{\mathcal{O}}^{symmnc}(n)_{\mathbb{S}_n}$ forms a bialgebra, and if $\check{\mathcal{O}}$ has an operadic counit, then $\mathscr{B}'$ is a unital an non–unital bialgebra. Furthermore if $\check{\mathcal{O}}(1)$ is almost connected, then the quotient $\mathscr{B}'/\mathcal{I}$ is a Hopf algebra.*

Proof. It is clear that the free multiplication then also satisfies (3.16) and the equivariant version of (3.2) holds. A counit for a symmetric cooperad is by definition a morphism $\check{\mathcal{O}}(\{s\}) \to k$ that is invariant under isomorphism, hence so is its extension. The rest of the statements are proved analogously to the non–symmetric case.

□

3.7. Grading.

Proposition 3.40. *For a almost connected cooperad with multiplication the depth filtration descends to $\mathscr{H}_q$, $\mathscr{H}$ and $\mathscr{H}^{amp}$ in both the symmetric and non–Σ case. It satisfies $\bar{\Delta}_n(F^{\geq p}) \subset \bigoplus F^{\geq p} \otimes F^{\geq 1}$. The depth of $1 \in \mathscr{H}$ is 0.*

Proof. Since $|$ is grouplike $\Delta(a|) = \Delta(a)(| \otimes |)$, it is clear that the depth filtration descends to $\mathscr{H}_q$. Now any lift of $a \in \mathscr{H}$ to $\mathscr{H}_q$ is of the form aq^k. We define the depth of $a \in \mathscr{B}$ to be the minimal depth of a lift or equivalently for any lift the difference between the depth and the q degree. This give 1 depth 0. The relation then follows from Proposition 3.4 and Lemma 2.60. The fact that the filtration descends through amputation is clear. □

4. Case II specializing Case I: co–operads from simplicial objects

In this section, we present an important (but accessible) construction of some co–operads with multiplication. This construction is best expressed in the language of simplical objects, and so we will first recall some of the basic notions. Some of the examples, however, can be understood with no simplicial background. For an arbitrary set S, we will see that the set X of all sequences or words in S has the structure of a co–operad, and Goncharov's Hopf algebra may be obtained from the case $S = \{0, 1\}$. Elements of X can be understood as strings of consecutive edges in the complete graph (with vertex loops) K_S, and further geometric intuition can be obtained by considering also strings of triangles or more generally n-simplices. The way to encode this construction is to think of the graph K_S as defining a groupoid $G(S)$, i.e. a category whose morphisms are invertible. The set of objects is S and for any pair of objects there is a unique invertible morphism between them. The transition to the simplicial setting is then made by considering the nerve of this category.

In fact, our construction defines a co–operad with multiplication, and hence a bialgebra (or Hopf algebra) for any (reduced) simplicial set X, see Proposition 4.8. In this guise, we also recover the Hopf algebra of Baues.

4.1. **Recollections: the simplicial category and simplicial objects.** Let Δ be the small category whose objects are the finite nonempty ordinals $[n] = \{0 < 1 < \cdots < n\}$ and whose morphisms are the order-preserving functions between them. Of course, each $[n]$ can itself be regarded as a small category, with objects $0, 1, \ldots, n$ and a (unique) arrow $i \to j$ iff $i \leq j$, and order preserving functions are just functors. Thus Δ is a full subcategory of the category of small categories.

Among the order-preserving functions $[m] \to [n]$ one considers the following generators: the injections $\partial^i : [n-1] \to [n]$ which omit the value i, termed coface maps, and the surjections $\sigma^i : [n+1] \to [n]$ which repeat the value i, termed codegeneracy maps. These maps satisfy certain obvious cosimplicial relations.

For D a small category, and $\mathcal{C}$ any category, we can consider the contravariant functors or the covariant functors X from D to $\mathcal{C}$. For $D = \Delta$ these are termed the simplicial and the cosimplicial objects in $\mathcal{C}$. A functor $D^{op} \to \mathcal{S}et$ is *representable* if it is $\hom_D(-, d)$ for some object d. In general, such functors are also called pre–sheaves on $\mathcal{D}$. If $\mathcal{D}$ is monoidal then so is the category of pre–sheaves, with the product given by Day convolution. The Yoneda Lemma gives a bijection between the set of natural transformations $\hom_D(-, d) \to X$ and the set $X(d)$, and in particular $d \mapsto \hom_D(-, d)$ defines a full embedding y of D into the functor category $\mathcal{S}et^{D^{op}}$. This category together with the embedding y is also called the co–completion and has the universal property that any functor from D to a cocomplete category (one that contains all colimits) factors through it.

The following result is central to the classical theory and in particular for us it will yield the construction of a nerve of a small category.

Lemma 4.1. *Let D be a small category and $\mathcal{C}$ a cocomplete category. Any functor $r : D \to \mathcal{C}$ has a unique extension along the Yoneda embedding to a functor $R : \mathcal{S}et^{D^{op}} \to C$ with a right adjoint N,*

$$\begin{tikzcd} D \arrow[r, hook, "y"] \arrow[dr, "r"'] & \mathcal{S}et^{D^{op}} \arrow[d, bend right, "R"'] \arrow[d, phantom, "\dashv"] \\ & \mathcal{C}. \arrow[u, bend right, "N"'] \end{tikzcd}$$

If $r : D \to \mathcal{C}$ is a monoidal functor between monoidal categories, then R sends monoidal functors $D^{op} \to \mathcal{S}et$ to monoids in $\mathcal{C}$.

The functor R is sometimes denoted $(-) \otimes_D r$, where the tensor over D is thought of as giving an object of D for every pair of D^{op}- and D-objects in $\mathcal{C}$, analogously to the language of tensoring left and right modules or algebras over a ring. The right adjoint N is termed the *nerve*, and is given on objects by

$$N(C) = \hom_{\mathcal{C}}(r(\text{-}), C).$$

Now a simplicial object is determined by the sequence of objects X_n, and the face and degeneracy maps $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$, given by the images of $[n]$, and ∂^i and σ^i, and dually for cosimplicial objects. Maps $X \to Y$ of (co)simplicial objects, that is, natural transformations, are just families of maps $X_n \to Y_n$ that commute with the (co)face and (co)degeneracy maps.

We write $\Delta[n]$ for the representable simplicial set $\hom_\Delta(-, [n])$ so, by Yoneda, simplicial maps $\Delta[n] \to X$ are just elements of X_n and maps $\Delta[m] \to \Delta[n]$ are just order preserving maps $[m] \to [n]$. For such a map α we use the notation $\alpha^* = X(\alpha) : X_n \to X_m$ and

$$x_{(\alpha_0,\ldots,\alpha_m)} \in X_m$$

to denote the image under α^* of an n-simplex x in a simplicial set X.

If $D = \Delta$ and X is a simplicial set then $R(X)$ is usually called the *realization* of a simplicial set with respect to the models r. Considering for example the embedding $r : \Delta \to \mathcal{C}at$ we obtain the notion of the simplicial nerve of a category: for C a small category, there is a natural bijection between the functors from $[n]$ to C and the n-simplices of the nerve NC,

$$N(C)_n = \hom_{\mathcal{C}at}([n], C).$$

Example 4.2. Let S be a set, and let $X(S)$ be the simplicial set given by the nerve of the *contractible* $G(S)$ *with object set* S,

$$X(S) = NG(S).$$

If $S = [n]$, for example, we may identify $G(S)$ with the fundamental groupoid of $\Delta[n]$, and

$$X([n]) \cong N\pi_1\Delta[n].$$

Giving a functor from $[n]$ to the contractible groupoid $G(S)$ is the same as giving the function on the objects, so an n-simplex of $X(S)$ is just a sequence of $n+1$ elements of S,

$$X(S)_n \;=\; S^{n+1} \;=\; \{\,(a_0; a_1, a_2, \ldots, a_{n-1}; a_n) \,:\, a_i \in S\,\}.$$

In the case $S = \{0, 1\}$, the groupoid $G(S)$ is

$$\begin{tikzcd} 0 \arrow[loop left] \arrow[r, bend left] & 1 \arrow[loop right] \arrow[l, bend left] \end{tikzcd}$$

and the n-simplices of X are words of length $n+1$ in the alphabet $\{0, 1\}$.

4.2. **The operad of little ordinals.** The category of small categories, and the category of simplicial sets, can be regarded as monoidal categories with the disjoint union playing the role of the tensor product, and the initial object $\varnothing$ the neutral object. In this context, we have the following result, compare for example [DK12, Example 3.6.4].

Proposition 4.3. *The sequence of finite nonempty ordinals* $([n])_{n\geq 0}$ *forms an operad in the category of small categories. For any partition* $n = m_1 + m_2 + \cdots + m_k$*, consider the subset* $\{0 = n_0 < n_1 < n_2 <$

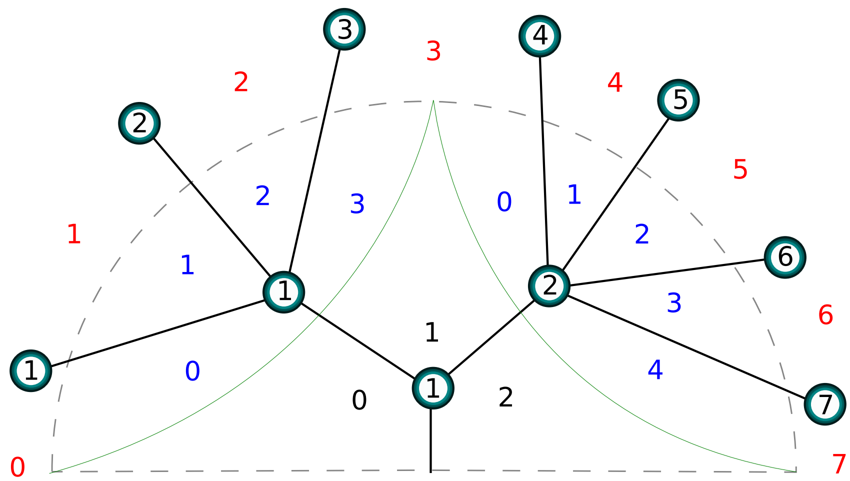

FIGURE 8. An example of a factorization $\underline{7} \to \underline{2} \to \underline{1}$ of order preserving surjections and, reading outwards from the root to the leaves, the corresponding operad structure map $\gamma_{3,4} : [2] \cup [3] \cup [4] \to [7]$.

$\cdots < n_k = n\}$ *of* $[n]$ *given by* $n_r = m_1 + \cdots + m_r$. *Then the structure map*

$$\gamma_{m_1,\ldots,m_k} = (\gamma^0, \gamma^1, \ldots, \gamma^k) : [k] \cup [m_1] \cup \cdots \cup [m_k] \to [n]$$

is defined by

$$\gamma^0(i) = n_i \quad (0 \leq i \leq k) \text{ and } \gamma^r(j) = n_r + j \quad (0 \leq j \leq m_r, 1 \leq r \leq k).$$

This operad clearly has a unit $u : \varnothing \to [1]$.

This construction is related, via Joyal duality (see Appendix C), to the factorisations of maps $\underline{n} \to \underline{1}$ into order preserving surjections $\underline{n} \to \underline{k} \to \underline{1}$, where $\underline{n} = \{1, \ldots, n\}$. Under the Joyal duality between end-point preserving ordered maps —see Appendix C— $[k] \to [n]$ and ordered maps $\underline{n} \to \underline{k}$, the injection $\gamma^0 : [k] \to [n]$ defined in the Proposition corresponds to the order preserving surjection $\underline{n} \to \underline{k}$ whose fibres over each i have cardinality m_i (see Figure 8).

The image of the operad structure in Proposition 4.3 under the Yoneda embedding gives:

Corollary 4.4. *The collection of representable simplicial sets* $(\Delta[n])_{n\geq 0}$ *forms a unital operad in the category of simplicial sets.*

If X is a simplicial set, then the unital operad structure on the sequence $\Delta[n]$, $n \geq 0$, induces a counital co–operad structure on the sequence $X_n = \hom(\Delta[n], X)$. That is, the sequence $(X_n)_{n\geq 0}$ forms a counital co–operad with

$$\begin{array}{cccccccc} X_n & \xrightarrow{\check{\gamma}_{m_1,\dots,m_k}} & X_k & \times & X_{m_1} & \times & \dots & \times \; X_{m_k} \\ x & \longmapsto & \big(x_{(n_0,n_1,\dots,n_k)}, & & x_{(n_0,n_0+1,\dots,n_1)}, & \dots, & & x_{(n_{k-1},n_{k-1}+1,\dots,n_k)})\big) \end{array} \tag{4.1}$$

where $0 = n_0 < n_1 < n_2 < \cdots < n_k = n$ are given by $n_r = m_1 + \cdots + m_r$ as usual. The counit is given by the unique map

$$X_1 \to \{*\}.$$

More generally:

Corollary 4.5. *Let X be a simplicial object in a category $\mathcal{C}$ with finite products. Then the sequence $(X_n)_{n\geq 0}$ forms a counital co–operad in $\mathcal{C}$.*

Example 4.6. The set of all words in a given alphabet S is naturally a simplicial set (see Example 4.2 above) and so by Corollary 4.5 it forms a counital co–operad X in the category of sets. The elements of arity n in this co–operad are the words of length $n+1$ in S,

$$X_n \;=\; S^{n+1} \;=\; \{\, (a_0; a_1, a_2, \dots, a_{n-1}; a_n) \;:\; a_i \in S\,\}$$

and the operation $\check{\gamma}_{m_1,\dots,m_k}$ sends such an element $(a_0; a_1, a_2, \dots, a_{n-1}; a_n)$ to

$$\big((a_{n_0}; a_{n_1}, \dots; a_{n_k}),\; (a_{n_0}; a_{n_0+1}, \dots; a_{n_1}), \dots, (a_{n_{k-1}}; a_{n_{k-1}+1}, \dots; a_{n_k})\big)$$

where $n_0 = 0$, $n_k = n$ and $n_r - n_{r-1} = m_r$.

This construction can also be carried out in an algebraic setting.

Proposition 4.7. *Let X be a simplicial set, and let $\check{\mathcal{O}}(n)$ be the free abelian group on the set X_n, for each $n \geq 0$. Then $\check{\mathcal{O}}$ forms a counital co–operad in the category of abelian groups, with the co–operadic structure given by*

$$\begin{array}{cccccccc} \check{\mathcal{O}}(n) \xrightarrow{\check{\gamma}} & \check{\mathcal{O}}(k) & \otimes & \check{\mathcal{O}}(m_1) & \otimes & \dots & \otimes & \check{\mathcal{O}}(m_k) \\ x \longmapsto & x_{(n_0,n_1,\dots,n_k)} & \otimes & x_{(n_0,n_0+1,\dots,n_1)} & \otimes & \dots & \otimes & x_{(n_{k-1},n_{k-1}+1,\dots,n_k)}) \end{array}$$

and the counit given by the augmentation

$$\check{\mathcal{O}}(1) \longrightarrow \mathbb{Z}.$$

Proof. This follows by applying free abelian group functor (which carries finite cartesian products of sets to tensor products) to the co–operad structure considered in (4.1). ∎

From section 2 we therefore have

Proposition 4.8. *Let X be a simplicial set. The co–operad structure $\check{\mathcal{O}}$ on $(\mathbb{Z}X_n)_{n\geq 1}$ of the previous proposition extends to a structure of a co–operad with (free) multiplication, and hence to a graded bialgebra structure, on the free tensor algebra*

$$\mathscr{B}(X) = \bigoplus_n \check{\mathcal{O}}^{nc}(X)(n) = \bigoplus_{n_1,n_2,\dots\geq 1} \bigotimes_i \mathbb{Z}X_{n_i}$$

generated by X, where elements of $\mathbb{Z}X_n$ have degree $n-1$.

4.2.1. **Goncharov's first Hopf algebra.** Let S be the set $\{0,1\}$. We considered in Example 4.2 the contractible groupoid $G(S)$ with object set S, and the simplicial set $X = X(S)$ given by its simplicial nerve. If we denote the simplices of X_n by tuples $(a_0; a_1, \dots, a_{n-1}; a_n)$ as in Example 4.6 and apply Proposition 4.8 we obtain a graded bialgebra

$$\mathscr{B}(X) = \mathbb{Z}[(a_0; a_1, \dots, a_{n-1}; a_n); a_i \in \{0,1\}]$$

with the coproduct that sends a polynomial generator $(a_0; a_1, \dots, a_{n-1}; a_n)$ in degree $n-1$ to

$$\sum_{0=n_0<n_1<\cdots<n_k=n} (a_{n_0}; a_{n_1}, \dots; a_{n_k}) \otimes \prod_{i=0}^{k-1} (a_{n_i}; a_{n_i+1}, \dots; a_{n_{i+1}})$$

When we identify all generators in degree 0 we obtain Goncharov's connected graded Hopf algebra $\mathscr{H}_G$, as in Theorem 1.2.

For any simplicial set X, let $C_n(X)$ be the free abelian group on the n-simplices X_n. This defines a chain complex $(C(X), d_X)$ where

$$d_X(x) = \sum_{i=0}^{n} (-1)^i d_i x.$$

Diagonal approximation makes CX a differential graded coalgebra,

$$C(X) \longrightarrow C(X \times X) \longrightarrow CX \otimes CX$$

whose classical cobar construction is the tensor algebra on the desuspension of the reduced coalgebra

$$\Omega CX = (T\Sigma^{-1}\overline{C}X, d_\Omega)$$

where the differential d_Ω is formed from d_X and the coproduct. For the moment, however, we merely observe that if one takes the symmetric rather than the tensor algebra then the underlying graded abelian group is isomorphic to Goncharov's $\mathscr{H}_G$.

4.3. **Simplicial strings.** For $(D, \otimes)$ a strict monoidal category, consider $(\Omega' D, \boxtimes)$ the strict monoidal category generated by D together with morphisms $a \boxtimes b \to a \otimes b$ for objects a, b of D, subject to the obvious naturality and associativity relations. In this way a strict monoidal functor on $\Omega' D$ is exactly a (strictly unital) *lax monoidal* functor on D: a functor F on D together with maps $Fa \otimes Fb \to F(a \otimes b)$ satisfying appropriate naturality and associativity conditions.

Definition 4.9. Let $\Delta_{*,*}$ be the strict monoidal category given as the subcategory of Δ containing just the generic (that is, end-point preserving) maps $[m] \to [n]$, with the monoidal structure $[p] \otimes [q] = [p+q]$ given by identifying $p \in [p]$ and $0 \in [q]$.

We define the category of *simplicial strings* $\Omega\Delta$ to be the strict monoidal category $\Omega' \Delta_{*,*}$.

This agrees with Baues' construction in [Bau80, Definition 2.7]. Now a contravariant monoidal functor on the category of simplicial strings is just an oplax monoidal functor on $\Delta_{*,*}^{op}$. Explicitly, if $\mathcal{C}$ is a category with the cartesian monoidal structure, then to give a monoidal functor $(\Omega\Delta)^{op} \to \mathcal{C}$ is to give a functor $X : \Delta_{*,*}^{op} \to \mathcal{C}$ together with associative natural transformations $\mu_{p,q} = (\lambda_{p,q}, \rho_{p,q}) : X_{p+q} \to X_p \times X_q$. Note that X becomes a simplicial object, if we define outer face maps $X_n \to X_{n-1}$ by $d_0 = \rho_{1,n-1}$ and $d_n = \lambda_{n-1,1}$. Moreover these determine all maps $\rho_{p,q}$ and $\lambda_{p,q}$ via the naturality conditions $(d_1^{p-1} \times \mathrm{id})\mu_{p,q} = \mu_{1,q} d_1^{p-1}$ and $(\mathrm{id} \times d_1^{q-1})\mu_{p,q} = \mu_{p,1} d_{p+1}^{q-1}$. Thus we have:

Proposition 4.10. *Let $\mathcal{C}$ be a cartesian monoidal category. Then the following categories are equivalent:*

- *The category of simplicial objects in $\mathcal{C}$,*
- *The category of oplax monoidal functors $\Delta_{*,*}^{op} \to \mathcal{C}$,*
- *The category of monoidal functors $(\Omega\Delta)^{op} \to \mathcal{C}$.*

Given a simplicial object X, the corresponding oplax monoidal functor is given by the restriction of X to the endpoint preserving maps, with the structure map

$$(d_{p+1}^q, d_0^p) : X_{p+q} \to X_p \times X_q.$$

Definition 4.11. An *interval object* [BT97] (or a *segment* [BM06]) in a monoidal category $(\mathcal{D}, \otimes, \mathbb{1})$ is an augmented monoid $(L, L^{\otimes 2} \xrightarrow{\mu} L, \mathbb{1} \xrightarrow{\eta} L, L \xrightarrow{\varepsilon} \mathbb{1})$ together with an *absorbing object*, that is, $\overline{\eta} : \mathbb{1} \to L$ satisfying $\mu(\mathrm{id}_L \otimes \overline{\eta}) = \overline{\eta}\varepsilon = \mu(\overline{\eta} \otimes \mathrm{id}_L)$, $\varepsilon\overline{\eta} = \mathrm{id}_I$.

To any augmented monoid L one associates a simplicial object or, under Joyal duality, a covariant functor $L^\bullet$ on $\Delta_{*,*}$ with $L^0 = L^1 = \mathbb{1}$,

$L^n = L^{\otimes(n-1)}$,

$$s^0 = \varepsilon \otimes \mathrm{id},\ s^n = \mathrm{id} \otimes \varepsilon,\ s^i = \mathrm{id} \otimes \mu \otimes \mathrm{id} : L^{\otimes n} \to L^{\otimes(n-1)},$$
$$d^i = \mathrm{id} \otimes \eta \otimes \mathrm{id} : L^{\otimes(n-2)} \to L^{\otimes(n-1)},$$

If in addition L has an absorbing object then $L^\bullet$ has a lax monoidal structure

$$\mathrm{id} \otimes \overline{\eta} \otimes \mathrm{id} : L^{\otimes(p-1)} \otimes L^{\otimes(q-1)} \to L^{\otimes(p+q-1)}$$

so we obtain a monoidal functor $L^\bullet : \Omega\Delta \to \mathcal{D}$.

Definition 4.12. Let X be a simplicial set, or the corresponding contravariant monoidal functor on the category of simplicial strings (Proposition 4.10). Baues' *geometric cobar construction* $\Omega_L X$ with respect to an interval object L in a cocomplete monoidal category $\mathcal{D}$ is defined as the monoid object in $\mathcal{D}$ given by the realisation functor (see Lemma 4.1),

$$\Omega_L(X) = X \otimes_{\Omega\Delta} L^\bullet$$

We have four fundamental examples:

(1) Let $L = [0,1]$ be the unit interval in the category of CW complexes, with unit and absorbing objects $0, 1 : \{*\} \to [0,1]$, and multiplication given by $\max : [0,1]^2 \to [0,1]$. Then the geometric cobar construction on a 1-reduced simplicial set is homotopy equivalent to the loop space of the realisation of X.
(2) Taking the cellular chains on the previous interval object we gives an interval object L in the category of chain complexes. In this case $\Omega_L(X)$ coincides with Adams' cobar construction, which has the same homology as the loop space on X, if X is 1-reduced.
(3) If we forget the boundary maps in example (2) we obtain an interval object L in the category of graded abelian groups, and $\Omega_L(X)$ coincides as an algebra with the object $\mathscr{B}(X)$ of Proposition 4.8: it is just the free tensor algebra whose generators in dimension n are the $n+1$-simplices of X.
(4) Let $L = \Delta[1]$ in the category of simplicial sets, with unit and absorbing object d^1 and $d^0 : \Delta[0] \to \Delta[1]$, and multiplication $\mu : \Delta[1]^2 \to \Delta[1]$ defined by

$$\mu_n([n] \xrightarrow{x} [1], [n] \xrightarrow{x'} [1]) = (i \mapsto \max(x_i, x'_i)).$$

Berger has observed that, up to group completion, $\Omega_L X$ has the same homotopy type as the simplicial loop group GX of Kan.

Note that the CW complex given by the simplicial realisation of $\Delta[1]^2$ does not have the same cellular structure as $[0,1]^2$: to relate examples

(1–3) with (4) requires appropriate diagonal approximation and shuffle maps.

In example (3) the multiplication is free, and we have seen that the co–operad structure $\check{\gamma}$ on the simplicial set X gives a comultiplication and hence a bialgebra structure on $\Omega_L(X) = \mathscr{B}(X)$. Baues showed that essentially the same coproduct gives a differential graded bialgebra structure on $\Omega_L(X)$ in example (2), and used this to iterate the classical cobar construction to obtain an algebraic model of the double loop space. In example (4) we remain in the category of simplicial sets, and we have the following result:

Proposition 4.13. *Let X be a simplicial set, and $\Omega_L(X)$ the simplicial monoid given by the geometric co–bar construction on X with respect to the interval object $L = \Delta[1]$. Then the co–operad structure $\check{\gamma}$ on X induces a map*

$$\Omega_L(X)_n \longrightarrow \prod_{m_1+\cdots+m_k=n} \Omega_L(X)_{k-1} \times \Omega_L(X)_{n-k}$$

for each $n, k \geq 1$.

Proof. Let $Y = \Omega_L(X)$. For each partition $m_1 + \cdots + m_k = n$ the co–operad structure map $\check{\gamma}_{m_1,\ldots,m_k}$ of (4.1) induces a map $Y_{n-1} \longrightarrow Y_{k-1} \times Y_{n-k}$ as follows. The map $\gamma_{m_1,\ldots,m_k}$ of Proposition 4.3 restricts to give a bijection $\underline{k-1} \cup \underline{m_1 - 1} \cup \cdots \cup \underline{m_k - 1} \to \underline{n-1}$ and hence an isomorphism

$$\Delta[1]^{n-1} \longrightarrow \Delta[1]^{k-1} \times \Delta[1]^{m_1-1} \times \cdots \times \Delta[1]^{m_k-1}.$$

Together with the map $\check{\gamma}_{m_1,\ldots,m_k}$ of (4.1) this defines a map

$$X_n \times \Delta[1]^{n-1} \longrightarrow X_k \times \Delta[1]^{k-1} \times (X_{m_1} \times \Delta[1]^{m_1-1} \times \cdots \times X_{m_k} \times \Delta[1]^{m_k-1})$$

which induces the map on Y as required. ☐

4.4. **Comparison with Goncharov's second Hopf algebra.** We have seen above that Goncharov's first Hopf algebra $\mathscr{H}_G$ and Baues Hopf algebra $\Omega_L(X)$ are closely related. The differences between Baues' and Goncharov's algebras are as follows

- Baues' Hopf algebra has a differential, and the underlying graded abelian group $\mathscr{B}(X)$ is the free tensor algebra, that is, a free associative algebra. No differential is given on Goncharov's algebra, which is a free polynomial algebra, that is, a free commutative and associative algebra.
- To obtain a model for the double loop space Baues requires X to have trivial 2-skeleton (only one vertex, one degenerate edge, and one degenerate 2-simplex), but to construct Goncharov's

bialgebra we take X to be 0-coskeletal (a unique n-simplex for any $(n+1)$-tuple of vertices). In the latter construction, however, one may still impose the relations $x \sim 1$ and $x \sim 0$ for 1- and 2-simplices x *after* taking the polynomial algebra (compare (1.3) and (1.10) respectively).

For Goncharov's second Hopf algebra $\tilde{\mathscr{H}}_G$, and the variants due to Brown, one imposes extra relations such as the shuffle formula (1.5). This has the following natural expression in the language of the cobar construction. Let $X = X(S)$, the 0-coskeletal simplicial set with vertex set $X_0 = S$. The cobar construction $\Omega_L X$ is a colimit of copies of $C(x_{n+1}) = L^{\otimes n}$ for each $(n+1)$-simplex $x_{n+1} = (s; w_n; s')$, where w_n is a word of length n in the alphabet S. In a symmetric monoidal category each $(p, n-p)$-shuffle corresponds to a natural isomorphism $L^{\otimes p} \otimes L^{\otimes(n-p)} \to L^{\otimes n}$ and the content of the shuffle relation is that this isomorphism is also obtained from the shuffle of the letters of a word w_p with a word w_{n-p} to obtain a word w_n.

4.5. **Cubical structure.** Baues' and Goncharov's comultiplications come from path or loop spaces and may be seen having natural cubical structure. The space of paths P from 0 to n in the n-simplex $|\Delta[n]|$ is a cell complex homeomorphic to the $(n-1)$-dimensional cube.

Cubical complexes have a natural diagonal approximation,

$$\delta : P = [0,1]^{n-1} \xrightarrow{\cong} \bigcup_{K \cup L = \{1,\ldots,n-1\}} \partial_K^-[0,1]^{n-1} \times \partial_L^+[0,1]^{n-1} \xrightarrow{\subset} P \times P$$

One can identify faces ∂_i^- of the cube P as the spaces of paths through the faces $x_{(0,\ldots,\widehat{i},\ldots,n)}$ of the n-simplex x. Faces ∂_i^+ are products of a $(i-1)$-cube and $(n-i-1)$-cube: the spaces of paths through $x_{(0,\ldots,i)}$ and through $x_{(i,\ldots,n)}$.

The term for $L = \{i_1, \ldots, i_{k-1}\}$ under this identification is

$$x_{(0,i_1,\ldots,i_{k-1},n)} \;\times\; x_{(0,1,\ldots,i_1)}\, x_{(i_1,i_1+1,\ldots,i_2)} \;\cdots\; x_{(i_{k-1},i_{k-1}+1,\ldots,n)}.$$

which reproduces the summands of the coproduct.

The cubical stucture is illustrated for the case of Δ^3 in Figure 9

To get into this analysis, we can choose two other alternative presentations. The first is given by a self–explanatory bar notation and the second is a parametrized notation. For the latter, we use $0 \xrightarrow{a} 1 \xrightarrow{b} 2 \xrightarrow{c} 3$. Then s, t are formal parameters. At $t = 1$ an edge disappears, while for $t = 0$ the morphisms are composed. The latter also explains the shuffles very nicely. Indeed in the usual diagonal approximation there is a shuffle of inner degeneracies. The degeneracies are the composition

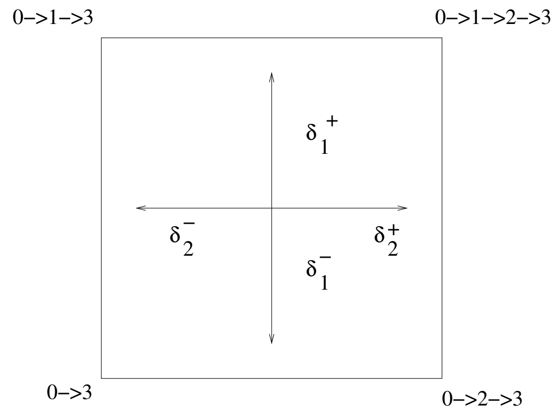

FIGURE 9. The cubical structure in the case of $n = 3$

and the square modulo the symmetric group action yields the simplex. Lifting this yields the terms in the shuffle product.

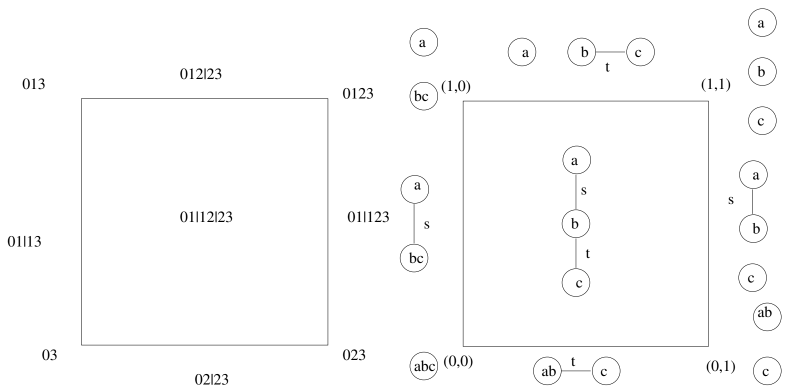

FIGURE 10. Two other renderings of the same square

The cubical stucture is also related to Cutkosky rules [Blo15, BK15, Kre16] Outer Space [CV03]. This natural appearance of cubical chains can be understood using decorated Feynman categories [KL16] and the W–construction [KW17], as explained in [BK19].

5. The general case: Bi– and Hopf algebras from Feynman categories

5.1. **Preview.** Consider the Connes–Kreimer Hopf algebra of graphs, aka. core Hopf algebra. We will show that this Hopf algebra is again a quotient of a bi–algebra, where the bi–algebra structure is particularly transparent. The main point is that the graphs (along with extra data) form the morphisms of a special type of monoidal category, a Feynman category as introduced in [KW17], see §5.3.1 below. This setup recovers all previous constructions and explains the co–product structure as de–concatenation.

In a monoidal category there are two products on morphisms, the tensor product $\otimes$ and the partially defined product of composition $\circ$. The product for morphisms will just be their tensor product. The co–product with be dual to partial composition product $\circ$. Unlike the composition, de–composition is not a partial operation, but rather unconditionally defined. The compatibility, viz. bi–algebra equation, is guaranteed by the axioms of a Feynman category.

There are three main types of examples for Feynman categories, the first are of combinatorial type and are based on sets. The second are those of graph type, where the graphs are a structure of the morphisms. These also appear in physics in the form of Feynman diagrams, whence the name. The last type are the enriched Feynman categories. These will be discussed in §7.8.

The Hopf algebras of Goncharov and Baues are combinatorial as are the tree Hopf algebras of Connes and Kreimer. The graph Hopf algebra of Connes and Kreimer is of graph type. The Hopf algebras from co–operads more generally are of enriched type, however, they still have a description of combinatorial type if the co–operad is in $\mathcal{S}et$.

There are also two flavors, depending on whether one is working in symmetric or simply monoidal (non-Σ) categories. We preview the results of this section:

Theorem 5.1. *Let $\mathfrak{F}$ be a non–Σ decomposition finite strict monoidal Feynman category. Set $\mathscr{B} = \mathbb{Z}Mor(\mathcal{F})$. Let $\mu = \otimes$ and $\Delta(\phi) = \sum_{(\phi_0,\phi_1):\phi=\phi_0\circ\phi_1} \phi_0 \otimes \phi_1$, $\epsilon(\phi) = 1$ if $\phi = id_X$ for some X, else $\epsilon(\phi) = 0$ and let $\eta(1) = id_{\mathbb{1}}$ then $(\mathscr{B}, \mu, \eta, \Delta, \epsilon)$ is a bi–algebra.*

Let $\mathfrak{F}$ be a factorization finite Feynman category. Let $\mathscr{B}^{iso}$ be the free Abelian group on the isomorphisms classes of morphisms. Then there is a bialgebra structure on $\mathscr{B}^{iso}$ given by $(\mu, \eta^{iso}, \Delta^{iso}, \epsilon^{iso})$ where μ is the tensor product on classes $\eta^{iso} = [id_{\mathbb{1}}]$, Δ^{iso} is the coproduct induced on co–invariants, and ϵ^{iso} evaluates to 1 precisely on the isomorphism classes of identities.

If $\mathfrak{F}$ is almost connected then there is a bi–ideal $\mathcal{I}$ spanned by $[id_X] - [id_{\mathbb{1}}]$ and the quotient $\mathscr{H} = \mathscr{B}^{iso}/\mathcal{I}$ is connected and Hopf.

For the notion of “almost connected” in this context, see Definition §5.35.

In the next section, we give alternative descriptions in terms of indecomposables §6.1 and in the non–Σ case we have a different construction for taking isomorphism classes using a quotient rather than co–invariants, cf. §6.2.

Proposition 5.2. *The relation of being in the same isomorphism class gives rise to a co–ideal $\mathcal{C}$ spanned by $f - g$ for any two morphisms that are isomorphic in the arrow category. In the non-Σ case $\mathscr{B}^{quot}_{\mathbb{Q}} := \mathscr{B}/\mathcal{C}\otimes\mathbb{Q}$ is a bi–algebra for a normalized ϵ^{quot}. If $\mathfrak{F}$ is almost–connected then there is an ideal $\mathscr{J}$ and the quotient $\mathscr{B}^{quot}/\mathscr{J}$ is connected and a Hopf algebra over $\mathbb{Q}$ in general.*

Here the ideal $\mathscr{J}$ is spanned by $|Aut(X)||Iso(X)|id_X - |Aut(Y)||Iso(Y)|id_Y$, where $|Aut(X)|$ is the cardinality of the automorphism group and $|Iso(X)|$ is the number of objects isomorphic to X. Both are finite if $\mathcal{F}$ is decomposition finite. If $\mathcal{F}$ is skeletal the $|Iso(X)| = 1$, and if $\mathcal{V}$ is furthermore discrete, the ideal is simply $(id_X - id_Y)$. This is the case for non–sigma co–operads, in which case the two constructions coincide. For the symmetric case, it is possible to twist the co–multiplication in certain cases, so that the bi–algebra equation holds, see Theorem 6.15 for a summary.

In order to recover the previous cases, one has to use several constructions defined in [KW17, §3, §4]. This is done in §7. In particular, case I corresponds to the Feynman level category $\mathfrak{F}^+$ and its relation to enriched Feynman categories, [KW17, §3.6,§4] applied to the Feynman category of surjections $\mathfrak{F}\mathfrak{S}$, that is the Feynman categories $\mathfrak{F}\mathfrak{S}_{\mathcal{O}}$, where $\mathcal{O}$ is an operad. The generalization comes from the nc–construction [KW17, §3.2] applied to the Feynman category for operads $\mathfrak{O}$ and a B_+ operator as given in [KW17, Example 3.5.2]. The construction of simplicial strings is captured by the nc–construction applied to the Feynman category $\Delta_{*,*}$ together with a decoration, that is the construction of $\mathfrak{F}_{dec\mathcal{O}}$, see [KL16] and [KW17, §3.3], see §7 in particular §7.8 and 7.3.2. Finally, universal operations 7.9 explain the amputation mechanism.

We will begin by considering algebra and co–algebra structures for morphisms and isomorphisms classes of morphisms. We then introduce the notion of Feynman category in the symmetric and non-Σ version. Thus allows us to prove the bi–algebra structures under standard assumptions. Afterwards, we turn to the Hopf algebras and functoriality.

5.2. **Algebra and co-algebra structures for mophisms.** Given a category $\mathcal{F}$ let $\mathscr{B} = \mathbb{Z}[Mor(\mathcal{F})] \subset Hom(Mor(\mathcal{F}), \mathbb{Z})$ be the free Abelian group on the morphisms of $\mathcal{F}$.

5.2.1. **Isomorphism classes.** Set $\mathscr{B}^{iso} = \mathscr{B}/\sim$ where $\sim$ is the equivalence relation on morphisms given by isomorphisms in $(\mathcal{F}\downarrow\mathcal{F})$. In particular, the equivalence relation $\sim$, which exists on any category, means that for given f and g: $f\sim g$ if there is a commutative diagram with isomorphisms as vertical morphisms.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\simeq\sigma"'] & Y \arrow[d, "\simeq\sigma'"] \\ X' \arrow[r, "g"] & Y' \end{tikzcd}$$

i.e.: $f = \sigma'^{-1}\circ g\circ\sigma$.

$\mathscr{B}^{iso}$ is the free Abelian group on isomorphism classes. Fixing a skeleton $\mathcal{F}^{sk}$ of $\mathcal{F}$, $\mathscr{B}^{iso} = \mathbb{Z}[\amalg_{X,Y\in Obj(\mathcal{F}^{sk})\,Aut(Y)}Hom(X,Y)_{Aut(X)}]$, that is the free Abelian group of the co–invariants of the left $Aut(Y)$ and right $Aut(X)$ action of the Hom sets of $\mathcal{F}^{sk}$. In general $\mathscr{B}^{iso}(\mathcal{F})\simeq\mathscr{B}^{iso}(\mathcal{F}^{sk})$.

Remark 5.3. The morphisms of $\mathcal{F}$ together with these isomophisms are also precisely the groupoid of vertices $\mathcal{V}'$ of the iterated Feynman cagegroy $\mathfrak{F}'$, cf. [KW17, §3.4].

Lemma 5.4. *$id_X\sim g$ if and only if $g: X'\to Y'$ is an isomorphism and $X\simeq X'\simeq Y'$.*

□

5.2.2. **Algebra of morphisms of a (strict) monoidal category.**

Proposition 5.5. *Let $(\mathcal{F},\otimes)$ be a strict monoidal category. Then $\mathscr{B}$ is a unital algebra with multiplication $\mu(\phi,\psi)=\phi\otimes\psi$ and unit $1=id_{\mathbb{1}}$.*

If $(\mathcal{F},\otimes)$ is a monoidal category then $\mathscr{B}^{iso}$ is a unital algebra with multiplication $\mu([\phi],[\psi])=[\phi\otimes\psi]$ and unit $1=[id_{\mathbb{1}}]$. If $(\mathcal{F},\otimes)$ is symmetric monoidal then $\mathscr{B}^{iso}$ is a commutative unital algebra.

Proof. Recall that strict monodial means that in the unit constraints and associativity constraints are identities. Thus $X\otimes(Y\otimes Z)=(X\otimes Y)\otimes Z$ which guarantees the associativity $(\phi_1\otimes\phi_2)\otimes\phi_3=\phi_1\otimes(\phi_2\otimes\phi_3)$. Likewise $X\otimes\mathbb{1}=X=\mathbb{1}\otimes X$ shows that indeed $\phi\otimes id_{\mathbb{1}}=\phi=\phi\otimes id_{\mathbb{1}}$.

The product is well defined on isomorphism classes, since if $\phi'\simeq\phi,\psi'\simeq\phi$ then $\phi'=\sigma^{-1}\phi\sigma'$ and $\psi'=\tau^{-1}\psi\tau'$ for isomorphisms, $\sigma,\sigma',\tau,\tau'$ and $\phi'\otimes\psi'=(\sigma^{-1}\otimes\tau^{-1})(\phi\otimes\psi)(\sigma'\otimes\tau'$, so that $[\phi\otimes\psi]=[\phi'\otimes\psi']$. Without the assumption of strictness, if $\phi_i: X_i\to Y_i, i=1,2,3$ we have $(\phi_1\otimes\phi_2)\otimes\phi_3)=A(\phi_1\otimes(\phi_2\otimes\phi_3))$ in $\mathscr{B}$ where A is given by pre- and post-composing with associativity isomorphisms a_{X_1,X_2,X_3}

and $a^{-1}_{Y_1,Y_2,Y_3}$. Thus when one passes to isomorphism classes, the algebra structure is strict. In the same way, the unit constraints provide the isomorphism, which make the unit strict on $\mathscr{B}^{iso}$. If $\mathcal{F}$ is symmetric, then the commutativity constraints $C_{X,Y}$ give the isomorphisms, proving that $[\phi] \otimes [\psi] = [\psi] \otimes [\phi]$.

□

Remark 5.6. The condition of being stict is not severe as by using MacLane's coherence theorem [ML98] one can pass from any monoidal category to an equivalent strict one. We make this assumption, so that the algebra structure will be unital and associative rather than only weakly unital and weakly associative. After taking isomorphism classes the algebra structure is strict even if the monoidal category is not. Note that if we are working in the enriched version $Hom(\mathbb{1}, \mathbb{1}) = K$ will play the role of a ground ring.

5.2.3. **The decomposition co–product.** Suppose that $\mathcal{F}$ is a decomposition finite category. This means that for each morphism ϕ of $\mathcal{F}$ the set $\{(\phi_0, \phi_1) : \phi = \phi_0 \circ \phi_1\}$ is finite. Then, $\mathscr{B}$ carries a co–associative co–product given by the dual of the composition. On generators it is given by the sum over factorizations:

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, "\phi_1"'] & & Z \\ & Y \arrow[ur, "\phi_0"'] & \end{tikzcd} \tag{5.1}$$

that is

$$\Delta(\phi) = \sum_{\{(\phi_0,\phi_1):\phi=\phi_0\circ\phi_1\}} \phi_0 \otimes \phi_1. \tag{5.2}$$

where we have abused notation to denote by ϕ the morphism $\delta_\phi(\psi)$ that evaluates to 1 on ϕ and zero on all other generators.

A co–unit is defined on the generators by:

$$\epsilon(\phi) = \begin{cases} 1 & \text{if for some object } X : \phi = id_X \\ 0 & \text{else} \end{cases} \tag{5.3}$$

The co–unit axioms are readily verified and the co–associativity follows from the associativity of composition.

Remark 5.7. One can enlarge the setting to the situation in which the sets of morphisms are graded and composition preserves the grading. In this case, one only needs degreewise composition finite. This will be the case for any graded Feynman category [KW17]. See also Example 2.11.4.

Remark 5.8. We realized with hindsight that the co–product we constructed on indecomposables, guided by remarks from D. Kreimer given below §6.1, is equivalent to the co–product above. A little bibliographical sleuthing revealed that the the co–product for any finite decomposition category appeared already in [Ler75] and was picked up later in [JR79].

5.2.4. **Coproduct of the identity morphisms.**

Remark 5.9.

$$\Delta(id_X) = \sum_{(\phi_L,\phi_R):\phi_L\circ\phi_R=id_X} \phi_L \otimes \phi_R \tag{5.4}$$

where $id_X : X \stackrel{\phi_R}{\to} X' \stackrel{\phi_L}{\to} X$. This mean that each ϕ_L has a right inverse ϕ_R, and each ϕ_R has a left inverse ϕ_L. They do not have to be invertible in general.

Corollary 5.10. *In a decomposition finite category the automorphism groups $Aut(X)$ are finite for all objects X, as are the classes $Iso(X)$ of objects isomorphic to X.*

Proof. For each automorphism ϕ of X and for each isomorphism $\phi : X \to X'$ there is a factorisation $id_X = \phi^{-1} \circ \phi$, and there are only finitely many such factorisations. $\square$

Lemma 5.11. *If $\mathcal{F}$ is decomposition finite, if the identity of an object has a factorization $id_X : X \stackrel{\phi_R}{\to} X \stackrel{\phi_L}{\to} X$ then both ϕ_R and ϕ_L are invertible.*

Proof. Using the powers of ϕ_L and ϕ_R, there are decompositions of $\phi_L = \phi_L^l \circ (\phi_R^l \circ \phi_L)$. Since $\mathcal{F}$ is decomposition finite, we have to have that $\phi_L^l = \phi_L^k$ for some $k > l$. Applying ϕ_R^l from the right, we see that $\phi_L^{k-l} = id_X$. That is ϕ_L is unipotent and hence an isomorphism. $\square$

5.2.5. **Co–algebra on isomorphisms classes.** The set $Hom(X, Y)$ has a natural action of $Aut(Y) \times Aut(X)$: $\phi \stackrel{\lambda_{\sigma_Y}\rho_{\sigma_X^{-1}}}{\longrightarrow} \sigma_Y \circ \phi \circ \sigma_X^{-1}$. We let $Aut(\phi) \subset Aut(Y) \times Aut(X)$ be the stabilizer group of ϕ. There is also an action on decompositions. There is an action of $Aut(Y)$ on $Hom(Y, Z) \times Hom(X, Y)$ given by $\bar{d} : (\sigma)(\phi_0, \phi_1) = (\phi_0 \circ \sigma^{-1}, \sigma \circ \phi_1)$, which leaves the composition map invariant: $\phi_0 \circ \phi_1 = \phi_0 \circ \sigma^{-1} \circ \sigma \circ \phi_1$. These are specializations of the actions of $I_{X,X'} = Iso(X, X')$, $I_{Y,Y'} = Iso(Y, Y')$ and $I_{Z,Z'} = Iso(Z, Z')$ which maps $Hom(Y, Z) \times Hom(X, Y) \to Hom(Y', Z') \times Hom(X', Y')$ by $(\phi_0, \phi_1) \mapsto (\phi'_0, \phi'_1) = (\sigma_Z \phi_0 \sigma_Y^{-1}, \sigma_Y \phi_1 \sigma_X^{-1}$.

There is an equivalence relation on factorizations (ϕ_0, ϕ_1) of the type (5.1) given by the action of $I_{Y,Y'}$, namely we set: $(\phi_0, \phi_1) \sim (\phi'_0, \phi'_1)$ if $\phi'_0 = \phi_0\sigma_Y^{-1}$ and $\phi'_1 = \sigma_Y\phi_1$. For a given class c under this equivalence choose a representative $c = [f] = [(\phi_0, \phi_1)]$ and consider the corresponding summand Δ_f of Δ together with the $I_{X,X'}, I_{Y,Y'}$ and $I_{Z,Z'}$ actions and co–invarians on this decomposition.

$$
\begin{tikzcd}
\phi \arrow[rr, mapsto, "\Delta_f"] \arrow[d, mapsto] \arrow[dr, mapsto, "\pi\circ p\circ\Delta_f"] \arrow[dd, bend right=60, mapsto, "\lambda_{\sigma_Z}\rho_{\sigma_X^{-1}}"'] & & (\phi_0, \phi_1) \arrow[d, mapsto, "p"] \arrow[dd, bend left=60, mapsto, "\lambda_{\sigma_Z}\rho_{\sigma_Y^{-1}}\otimes\lambda_{\sigma_Y}\rho_{\sigma_X^{-1}}"] \\
{[\phi]} \arrow[r, mapsto] & ([\phi_0], [\phi_1]) & {[(\phi_0, \phi_1)]} \arrow[l, mapsto, "\pi"'] \\
\sigma_Z\phi\sigma_X^{-1} \arrow[rr, mapsto, "\Delta_{f'}"'] \arrow[u, mapsto] \arrow[ur, mapsto, "\pi\circ p\circ\Delta_{f'}"'] & & (\sigma_Z\phi_0\sigma_Y^{-1}, \sigma_Y\phi_1\sigma_X^{-1}) \arrow[u, mapsto, "p"']
\end{tikzcd}
\tag{5.5}
$$

here $f = (\phi_0, \phi_1)$ is a factorization $f' = (\phi'_0, \phi'_1)$ is a different representative of the same class $[(\phi_0, \phi_1)]$ under the action of $\lambda \times \bar{d} \times \rho$ of $I_{Z,Z'} \times I_{Y,Y'} \times I_{X,X'}$.

For simplicity assume that $\mathcal{F}$ is skeletal. To shorten notation, we let $\mathcal{F}(X, Y) = Hom_{\mathcal{F}}(X, Y)$, and $A_X = Aut(X)$.

Fix a representative of the intermediate space Y in its isomorphism class and a choice of representative decompositions F, one for each class of of the $\phi \in \mathcal{F}(X, Z)$ this fixes a system of representatives obtained by conjugation F' .

Assume that $\mathcal{F}$ is finite if for any morphism ϕ and fixed space Y. Under this condition, the map $\Delta^{iso}_{X,Y,Z}$ in the diagram below is well defined due to the properties of co–limits and the finiteness assumption., we obtain a diagram of the type (2.46).

$$
\begin{tikzcd}
\mathcal{F}(X, Z) \arrow[rr, "\Delta_F"] \arrow[d] \arrow[dr, "\pi\circ p\circ\Delta_F"] \arrow[dd, bend right=60, "\lambda_{\sigma_Z}\rho'_{\sigma_X^{-1}}"'] & & \mathcal{F}(Y, Z) \times \mathcal{F}(X, Y) \arrow[d, "p"] \arrow[dd, bend left=60, "\lambda_{\sigma_Z}\rho_{\sigma_Y^{-1}}\otimes\lambda_{\sigma_Y}"] \\
{}_{A_Z}\mathcal{F}(X, Z)_{A_X} \arrow[r, "\Delta^{iso}_{X,Y,Z}"] & {}_{A_Z}\mathcal{F}(Y, Z)_{A_Y} \times {}_{A_Y}\mathcal{F}(X, Y)_{A_Z} & {}_{A_Z}\mathcal{F}(Y, Z) \times_{A_Y} \mathcal{F}(X, Y)_{A_X} \arrow[l, "\pi"'] \\
\mathcal{F}(X', Z') \arrow[rr, "\Delta_{F'}"] \arrow[u] \arrow[ur, "\pi\circ p\circ\Delta_{F'}"'] & & \mathcal{F}(Y', Z') \times \mathcal{F}(X', Y') \arrow[u, "p"']
\end{tikzcd}
\tag{5.6}
$$

Assume further that for any ϕ only finitely many isomorphism classes Y appear in the decompositions of ϕ. If both finiteness assumptions are satisfied, we call $\mathcal{F}$ *factorization finite*. Fixing a representative ϕ

and summing over isomorphism classes of Y, we then obtain $\Delta^{iso}([\phi])$ if $\mathcal{F}$ is factorization finite.

Lemma 5.12. *If $\mathcal{F}^{sk}$ is decomposition finite, then $\mathcal{F}$ is factorization finite.* □

Proposition 5.13. *If $\mathcal{F}$ is factorization finite, then the decomposition co–product Δ and the co–unit ϵ descend to a co–product Δ^{iso} and co–unit ϵ^{iso} on $\mathscr{B}^{iso}$ as co–invariants and $(\mathscr{B}^{iso}, \Delta^{iso}, \epsilon^{iso})$ is a co–unital co–algebra.*

Proof. Co–associativity follows readily. The co–unit ϵ is invariant under the left and right actions by automorphisms and descends to ϵ^{iso}

$$\epsilon^{iso}([\phi]) = \begin{cases} 1 & \text{if } [\phi] = [id_X] \\ 0 & \text{else} \end{cases} \tag{5.7}$$

□

5.2.6. **Direct formula for Δ^{iso}.** There is a direct way to describe the co–product, by analyzing the image of $\Delta^{iso}_{X,Y,Z}$.

We call a pair (ϕ_0, ϕ_1) of morphisms *weakly composable*, if there is an isomorphism σ, such that $\phi_1 \circ \sigma \circ \phi_0$ is composable. A weak decomposition of a morphism ϕ is a pair of morphisms (ϕ_0, ϕ_1) for which there exist isomorphisms $\sigma, \sigma', \sigma''$ such that $\phi = \sigma \circ \phi_1 \circ \sigma' \circ \phi_0 \circ \sigma''$. In particular a decomposition (ϕ_0, ϕ_1) is weakly composable. We introduce an equivalence relation on weakly composable morphisms, which says that $(\phi_0, \phi_1) \sim (\psi_0, \psi_1)$ if they are weak decompositions of the same morphism. An equivalence class of weak decompositions will be called a decomposition channel. The notation will be $([\phi_0], [\phi_1])$. In this notation, we have that $\pi([(\phi_0, \phi_1)]) = ([\phi_0], [\phi_1])$ and the image of $\Delta^{iso}_{X,Y,Z}$ are precisely the decomposition channels. These may, however, appear with multiplicities.

Proposition 5.14. *For an element/equivalence class $[\phi] \in B_{iso}$.*

$$\Delta^{iso}([\phi]) = \sum_{[(\phi_0, \phi_1)]} [\phi_0] \otimes [\phi_1] \tag{5.8}$$

where the sum is over a complete system of decompositions for a fixed representative ϕ. □

Remark 5.15. Notice that there are many ways in which two weakly composable morphisms are composable and hence may yield different compositions. Thus the right hand side may have terms that can be collected together. To obtain a representative one has to again rigidify by "enumerating everything", as in Remark 2.72.

This is also the reason that the composition dual to decomposition in $\mathscr{B}^{iso}$ in (2.44) is on invariants. A similar phenomenon is known in physics, when composing graphs [Kre06]. For Feynman categories of graph type such aspect have been previously discussed in [KW17, §2.1].

Example 5.16. In particular in §7.4 the construction is made concrete for the Connes-Kreimer Hopf algebra of graphs. An isomorphism class of a morphism is fixed by the ghost graph. The ghost graph of ϕ_1 is naturally a subgraph of the ghost graph of ϕ. The action on the intermediate space allows to "forget" the target of ϕ up to isomorphism and identify the ghost graph of ϕ_0 with the quotient graph. In the co–product one forgets the structure of being a subgraph, which is also what leads to multiplicities, cf. Example 7.9.

5.2.7. **Bi–algebra structure conditions.** By the above, in any strict monoidal category with finite decomposition $\mathscr{B}$ has a unital product and a co–unital co–product. However, the compatibility axioms of a bi–algebra *do not hold in general*. For instance, one needs to check

$$\Delta \circ \mu = (\mu \otimes \mu) \circ \pi_{2,3} \circ (\Delta \otimes \Delta)$$

where $\pi_{2,3}$ switches the 2nd and 3rd tensor factors. Each side of the equation is represented by a sum over diagrams.

For $\Delta \circ \mu$ the sum is over diagrams of the type

$$\begin{tikzcd} X \otimes X' \arrow[rr, "\Phi=\phi\otimes\psi"] \arrow[dr, "\Phi_1"'] & & Z \otimes Z' \\ & Y \arrow[ur, "\Phi_0"'] & \end{tikzcd} \tag{5.9}$$

where $\Phi = \Phi_0 \circ \Phi_1$.

When considering $(\mu \otimes \mu) \circ \pi_{23} \circ (\Delta \otimes \Delta)$ the diagrams are of the type

$$\begin{tikzcd} X \otimes X' \arrow[rr, "\phi\otimes\psi"] \arrow[dr, "\phi_0\otimes\psi_0"'] & & Z \otimes Z' \\ & Y \otimes Y' \arrow[ur, "\phi_1\otimes\psi_1"'] & \end{tikzcd} \tag{5.10}$$

where $\phi = \phi_0 \circ \phi_1$ and $\psi = \psi_0 \circ \psi_1$. And there is no reason for there to be a bijection of such diagrams.

The compatibility *does hold* when dealing with Feynman categories as we now show.

5.3. **Feynman categories and bi–algebra structures.** Here we give the definition of the various Feynman categories and prove the theorems previewed above. Examples can be found in §7.

5.3.1. **Definition of a Feynman category.** Consider the following data:

(1) $\mathcal{V}$ a groupoid, with $\mathcal{V}^{\otimes}$ the free symmetric monoidal category on $\mathcal{V}$.
(2) $\mathcal{F}$ a symmetric monoidal category, with monoidal structure denoted by $\otimes$.
(3) $\imath : \mathcal{V} \to \mathcal{F}$ a functor, which by freeness extends to a monoidal functor $\iota^{\otimes}$ on $\mathcal{V}^{\otimes}$,

$$\begin{tikzcd} \mathcal{V} \arrow[r, "\imath"] \arrow[d, "\jmath"'] & \mathcal{F} \\ \mathcal{V}^{\otimes} \arrow[ur, "\imath^{\otimes}"'] \arrow[r] & Iso(\mathcal{F}) \arrow[u, hook] \end{tikzcd}$$

where $Iso(\mathcal{F})$ is the maximal (symmetric monoidal) subgroupoid of $\mathcal{F}$.

Consider the comma categories $(\mathcal{F} \downarrow \mathcal{F})$ and $(\mathcal{F} \downarrow \mathcal{V})$ defined by $(id_{\mathcal{F}}, id_{\mathcal{F}})$ and $(id_{\mathcal{F}}, \imath)$.

Definition 5.17. A triple $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \imath)$ as above is called a *Feynman category* if

(i) $\imath^{\otimes}$ induces an equivalence of symmetric monoidal groupoids between $\mathcal{V}^{\otimes}$ and $Iso(\mathcal{F})$.
(ii) $\imath$ and $\imath^{\otimes}$ induce an equivalence of symmetric monoidal groupoids $Iso(\mathcal{F} \downarrow \mathcal{V})^{\otimes}$ and $Iso(\mathcal{F} \downarrow \mathcal{F})$.
(iii) For any object $*_v$ of $\mathcal{V}$, $(\mathcal{F} \downarrow *_v)$ is essentially small.

The first condition says that $\mathcal{V}$ knows all about the isomorphisms. The third condition is technical to guarantee that certain colimits exist. The second condition, also called the *hereditary condition*, is the key condition. It can be understood as follows: any morphism in $\mathcal{F}$ is isomorphic, up to unique isomorphism, to a tensor product of basic morphisms, which are those in $(\mathcal{F} \downarrow \mathcal{V})$ (aka. one-comma generators). Viz.:

(1) For any morphism $\phi : X \to X'$, if we choose $X' \simeq \bigotimes_{v\in I} \imath(*_v)$ by (i), there are X_v and $\phi_v : X_v \to \iota(*_v)$ in $\mathcal{F}$ such that ϕ is isomorphic to $\bigotimes_{v\in I} \phi_v$,

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "\simeq"'] & X' \arrow[d, "\simeq"] \\ \bigotimes_{v\in I} X_v \arrow[r, "\bigotimes_{v\in I} \phi_v"] & \bigotimes_{v\in I} \imath(*_v). \end{tikzcd} \tag{5.11}$$

(2) For any two such decompositions $\bigotimes_{v\in I} \phi_v$ and $\bigotimes_{v'\in I'} \phi'_{v'}$ there is a bijection $\psi : I \to I'$ and isomorphisms $\sigma_v : X_v \to X'_{\psi(v)}$ such

that $P \circ \bigotimes_v \phi_v = \bigotimes_v (\phi'_{\psi(v)} \circ \sigma_v)$ where P is the permutation corresponding to ψ.

(3) These are the only isomorphisms between morphisms.

5.3.2. **Non-symmetric version.** Now let $(\mathcal{V}, \mathcal{F}, \imath)$ be as above with the exception that $\mathcal{F}$ is only a monoidal category, $\mathcal{V}^{\otimes}$ the free monoidal category, and $\imath^{\otimes}$ is the corresponding morphism of monoidal groupoids.

Definition 5.18. A triple $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \imath)$ as above is called a *non-*Σ *Feynman category* if

(i) $\imath^{\otimes}$ induces an equivalence of monoidal groupoids between $\mathcal{V}^{\otimes}$ and $Iso(\mathcal{F})$.

(ii) $\imath$ and $\imath^{\otimes}$ induce an equivalence of monoidal groupoids $Iso(\mathcal{F} \downarrow \mathcal{V})^{\otimes}$ and $Iso(\mathcal{F} \downarrow \mathcal{F})$.

(iii) For any object $*_v$ in $\mathcal{V}$, $(\mathcal{F} \downarrow *_v)$ is essentially small.

5.3.3. **Strict Feynman categories.** We call a Feynman category *strict* if the monoidal structure on $\mathcal{F}$ is strict, ι is an inclusion, and $\mathcal{V}^{\otimes} = Iso(\mathcal{F})$ where we insist on using the strict free monoidal category, see e.g. [Kau17] for a thorough discussion. Up to equivalence in $\mathcal{V}$, $\mathcal{F}$ and in $\mathfrak{F}$ this can always be achieved.

Note that in the strict case one can assume the right vertical arrow in (5.11) is an identity, thus for any morphism ϕ we have $\phi = \bigotimes \phi_v \circ P$. Here $\phi_v : X_v \to \iota(*_v)$ and P is an isomorphism in $\mathcal{V}^{\otimes} = Iso(\mathcal{F})$, which we can fix to be simply a permutation $P : X \xrightarrow{\sim} \bigotimes_v X_v$ after absorbing possible isomorphisms $\tau_v : X_v \xrightarrow{\sim} X'_v$ into the ϕ_v by pre-composition. The permutation is by definition trivial in the non-Σ case.

5.3.4. **Bi–algebra structure for non–Σ Feynman categories.**

Lemma 5.19. *In a strict decomposition finite Feynman category $\Delta(id_{\mathbb{1}})$ is group like, i.e.: $\Delta(id_{\mathbb{1}}) = id_{\mathbb{1}} \otimes id_{\mathbb{1}}$*

Proof. By (5.4) $\Delta(id_{\mathbb{1}}) = id_{\mathbb{1}} \otimes id_{\mathbb{1}} + \sum \phi_L \otimes \phi_R$ with $\phi_R : \mathbb{1} \to X$ and $\phi_L : X \to \mathbb{1}$ with $\phi_L \circ \phi_R = id_{\mathbb{1}}$. It follows from axiom (ii) that there are only morphisms $X \to \mathbb{1}$ for $X = \mathbb{1}$ and thus $\phi_L, \phi_R : \mathbb{1} \to \mathbb{1}$. By Lemma 5.11 they have to be isomorphisms, $\mathbb{1} \to \mathbb{1}$ and thus by axiom (i) $\phi_L = \phi_R = id_{\mathbb{1}}$. Hence $\Delta(id_{\mathbb{1}})$ only has one summand corresponding to $id_{\mathbb{1}} \otimes id_{\mathbb{1}}$. $\square$

Theorem 5.20. *For any strictly monoidal, finite decomposition, non-Σ Feynman category $\mathfrak{F}$ the tuple $(\mathscr{B}, \otimes, \Delta, \epsilon, \eta)$ defines a bialgebra over $\mathbb{Z}$.*

Proof. We check the compatibility axioms:

(1) The co–unit is multiplicative $\epsilon(\phi \otimes \psi) = \epsilon(\phi)\epsilon(\psi)$. First, $id_X \otimes id_Y = id_{X \otimes Y}$, since $\mathcal{F}$ is strict monoidal. Because of axiom (i) this is then the unique decomposition of $id_{X \otimes Y}$, and hence both sides are either zero or $\phi = n\, id_X$ and $\psi = m\, id_Y$ in which case both sides equal to nm.

(2) The unit is co–multiplicative: by Lemma 5.19, $\Delta(id_{\mathbb{1}}) = \mathrm{id}_{\mathbb{1}} \otimes id_{\mathbb{1}}$, so $\Delta \circ \eta = \eta \otimes \eta$.

(3) Compatibility of unit and co–unit: $\epsilon(1) = \epsilon(id_{\mathbb{1}}) = 1$ and hence $\epsilon \circ \eta = id$.

(4) Bi–algebra equation: In order to prove that Δ is an algebra morphism, we consider the two sums over the diagrams (5.9) and (5.10) above and show that they coincide. First, it is clear that all diagrams of the second type appear in the first sum. Vice–versa, given a diagram of the first type, we know that $Y \simeq \hat{Y} \otimes \hat{Y}'$, since Φ_1 has to factor by axiom (ii) and the Feynman category is strict. Then again by axiom (ii) Φ_0 must factor. We see that we obtain a diagram:

$$\begin{tikzcd}
\hat{X} \otimes \hat{X}' \arrow[r, "\simeq\sigma", "=\sigma_1\otimes\sigma_2"'] \arrow[ddrr, "\hat{\phi}_0\otimes\hat{\psi}_0"'] & X \otimes X' \arrow[rr, "\Phi=\phi\otimes\psi"] \arrow[dr, "\Phi_0"] & & Z \otimes Z' \\
& & Y = Y' \otimes Y'' \arrow[ur, "\Phi_1"] \arrow[d, "\sigma'\simeq"', "=\sigma'_1\otimes\sigma'_2"] & \\
& & \hat{Y} \otimes \hat{Y}' \arrow[uur, "\hat{\phi}_1\otimes\hat{\psi}_1"'] &
\end{tikzcd} \tag{5.12}$$

Now since the Feynman category is strict and non-symmetric, the two isomorphisms also decompose as $\sigma = \sigma_1 \otimes \sigma_2$, and $\sigma' = \sigma'_1 \otimes \sigma'_2$, for a splitting $Y = Y' \otimes Y''$ so that $\Phi_0 = \sigma_1'^{-1} \circ \hat{\phi}_0 \circ \sigma_1^{-1} \otimes \sigma_2'^{-1} \circ \hat{\psi}_0 \circ \sigma_2^{-1}$ and $\Phi_1 = \hat{\phi}_1 \circ \sigma'_1 \otimes \hat{\psi}_2 \circ \sigma'_2 : Y = Y' \otimes Y'' \to Z \otimes Z'$ and one obtains that both diagram sums agree.

□

Examples are discussed in detail in §7.

5.3.5. **Bi–algebra structure on $\mathscr{B}^{iso}$ for a Feynman category.**

Theorem 5.21. *Given a factorization finite Feynman category $\mathfrak{F}$, $(\mathscr{B}^{iso}, \otimes, \eta, \Delta^{iso}, \epsilon^{iso})$ is a bi–algebra, both in the symmetric and the non–Σ case.*

Proof. We can retrace the steps in the proof of Theorem 5.20, up until the decomposition of σ and σ' into tensor products. Even without this assumption, the diagram (5.12) clearly shows that $[(\Phi_0, \Phi_1)] = [(\hat{\phi}_0 \otimes \hat{\psi}_0), (\hat{\phi}_1 \otimes \hat{\psi}_1)]$, so that there is indeed a bijection of the equivalence classes and hence the bi–algebra equation holds. The compatibilities for the unit and co–unit are simple computations along the lines of the proof of Theorem 5.20. ☐

Remark 5.22.

(1) If $\mathcal{V}$ is discrete in the non–Σ case, then $\mathscr{B}^{iso} = \mathscr{B}$.
(2) In the symmetric case, there is a difference in the count of diagrams in $\mathscr{B}$, which is controlled by the action $\bar{d}$ and the symmetric group actions. This is made precise in §6.
(3) We have so far considered Feynman categories over $\mathcal{S}et$. The theorems also hold in the case of enriched Feynman categories such as $\mathfrak{F}_{\mathcal{O}}$, see (§7.8.1). The enrichment can be over a tensor category $\mathcal{E}$ which has a faithful functor to $\mathcal{A}b$, e.g. k-Vect. In this case one should work over the ring $K = Hom_{\mathcal{F}}(\mathbb{1}, \mathbb{1})$, see §7.8.1 and [KW17, §4] for more details.
(4) This coproduct actually corresponds to the category $\mathfrak{F}'_{\mathcal{V}'}$ of universal operations [KW17, §6]. Here all channels with $[\phi_1] = [\psi]$ corresponds to the class of morphisms in $Hom_{\mathcal{F}'}(\phi, \psi)$. That means that each class of such a morphism under isomorphism corresponds to a channel and contributes a term to the sum. The associativity of the co–product is then just the associativity of the composition in $\mathfrak{F}'_{\mathcal{V}}$.

5.4. **Co–module structure.** Let $\mathscr{B}_1 = \mathscr{B}_1 = \mathbb{Z}[Ob(\imath^{\otimes} \downarrow \imath)]$ be the free Abelian group on the basic morphisms —see also §6.1 below.

Proposition 5.23. *For a decomposition finite (non–Σ) Feynman category the set of basic morphisms, that is objects of $(\mathcal{F} \downarrow \mathcal{V})$ form a co–module, viz. $\rho := \Delta|_{\mathscr{B}_1} : \mathscr{B}_1 \to \mathscr{B}_1 \otimes \mathscr{B}$ is a co–module for Δ. For a factorization finite Feynman category the analogous statement holds true for $\mathscr{B}_1^{iso} = \mathscr{B}_1/\sim$ and $\mathscr{B}^{iso}$.*

Proof. , we see that if $\phi \in Ob(\imath^{\otimes} \downarrow \imath)$, then $\Delta(\phi) \in \mathscr{B}_1 \otimes \mathscr{B}$, since the target of ϕ_0 will be an object of $\mathscr{B}$ for any factorization $\phi = \phi_0 \circ \phi_1$. ☐

See §6.1 for more details on this point of view.

5.4.1. B_+ **operator.** The definition of B_+ operators in general is quite involved, see [KW17, §3.2.1]. Functorially, such an operator gives a morphism $\mathscr{B} \to \mathscr{B}_1$. Without going into a full analysis, which will be done in [Kau19], we simply make the following:

Definition 5.24. A B_+ multiplication for $\mathfrak{F}$ is a morphism $B_+ : \mathscr{B} \to \mathscr{B}_1$ such that $\Delta_1 := (id \otimes B_+) \circ \Delta : \mathscr{B}_1 \to \mathscr{B}_1 \otimes \mathscr{B}_1$ and $\mu_1 := \mathscr{B}_+ \circ \mu_\otimes$ together with the unit and co–unit, yield a unital, co–unital bi–algebra structure on $\mathscr{B}_1$.

The multiplication for a co–operad with multiplication of §3 is an example, see §7.17.

5.5. **Opposite Feynman category yields the co-opposite bi–algebra.** Notice that usually the opposite category of a Feynman category is not a Feynman category, but it still defines a bi–algebra. Namely, the constructions above just yield the co-opposite bi–algebra structure $\mathscr{B}^{co-op}$. This means, the multiplication is unchanged but the co–multiplication is switched. That is $\Delta(\phi^{op}) = \sum_{\phi_1 \circ \phi^0 = \phi} \phi_1^{op} \otimes \phi_0^{op}$.

The same holds for quotient and Hopf algebra structures discussed below, i.e. $\mathscr{H}$ is replaced by $\mathscr{H}^{co-op}$.

5.6. **Hopf algebras from Feynman categories.** The above bi–algebras are usually not connected. There are several obstructions. Each isomorphism class of an object X potentially gives a unit and, unless $\mathcal{V}$ is discrete, there are isomorphisms which are not co–nilpotent and which prevent the identities of the different X from being group-like elements. There are several other obstructions to co–nilpotence, which one has to grapple with in the general case. We will now formalize this and give checkable criteria that are met by the main examples.

5.6.1. **Almost group–like identities and the putative Hopf quotient.** A Feynman category has *almost group–like* identities if each of the ϕ_L and hence each of the ϕ_R appearing in a co–product of any id_X (5.4) is an isomorphism.

Example 5.25. A counter–example, that is a Feynman category that does not have group–like identities, is $\mathcal{F}in\mathcal{S}et_<$ or its skeleton Δ_+. In this case, the category is also not decomposition finite. The reason is that each $id : \underline{n} \to \underline{n}$ factors as $\underline{n} \hookrightarrow \underline{m} \twoheadrightarrow \underline{n}$ for all $m \geq n$. Both $\mathfrak{F}\mathfrak{S}$ and $\mathfrak{F}\mathfrak{I}$ as well as all the graphical examples have group like identities.

The assumption of almost group like identities is, however, very natural and is often automatic. The example above is symptomatic.

Lemma 5.26. *If $\mathfrak{F}$ is decomposition finite and has almost group like identities then both in the symmetric and non–Σ case:*

(1) *the classes $[id_X]$ are group–like in $\mathscr{B}^{iso}$ that is $\Delta^{iso}([\mathrm{id}_X]) = [id_X] \otimes [id_X]$.*
(2) *the two–sided ideal $\mathcal{I} = \langle [id_X] - [id_Y] \rangle$ in $\mathscr{B}^{iso}$ is also a co–ideal.*

Proof. Under the assumption of almost group like identities:

$$\Delta(id_X) = \sum_{X',\sigma\in Iso(\mathcal{F})(X,X')} \sigma\otimes\sigma^{-1} \tag{5.13}$$

thus there is only one decomposition channel with multiplicity 1, since $[(\sigma,\sigma^{-1})] = [(id_X\otimes id_X)]$.

Using (i) $\Delta^{iso}([id_X]-[id_Y]) = [id_X]\otimes[id_X]-[id_Y]\otimes[id_Y] = ([id_X]-[id_Y])\otimes[id_X]+[id_Y]\otimes([id_X]-[id_Y])$ and $\epsilon^{quot}([id_X]-[id_Y]) = 1-1 = 0$, so that $\mathcal{I}$ is a coideal. □

Definition 5.27. If $\mathfrak{F}$ is factorization finite and has almost group like identities then both in the symmetric and non–Σ case, we set $\mathscr{H} = \mathscr{B}^{iso}/\mathcal{I}$. We call $\mathfrak{F}$ Hopf, if it satisfies the stated conditions and the bi–algebra $\mathscr{H}$ has an antipode, i.e. $\mathscr{H}$ is a Hopf algebra.

Theorem 5.28. *A Hopf Feynman category yields a Hopf algebra $\mathscr{H} := \mathscr{B}^{iso}/\mathcal{I}$, both in the symmetric and non–symmetric case. $\mathscr{H}$ is in general not co–commutative. It is commutative in the symmetric case and not necessarily commutative in the non–symmetric case.*

Proof. The only new claim is the commutativity in the symmetric case. This is due to the fact that the commutativity constraints are isomorphisms and these become identities already in $\mathscr{B}^{iso}$. □

In general, the existence of an antipode is complicated. We do know that for graded connected bi–algebras an antipode exists. In terms of Feynman categories this situation can be achieved by looking at definite Feynman categories.

5.6.2. **Graded Feynman categories.** One thing that helps to check connectedness and co–nilpotence is a grading. Each Feynman category has a *native length* for objects and morphisms. Due to condition (i) for a Feynman catgory every object X has a unique length $|X|$ given by the tensor word length of any object of $\mathcal{V}^{\otimes}$ representing it. We define the length decrease (or just length) of a morphism $\phi: X\to Y$ as $|\phi| = |X|-|Y|$. This is additive under composition and tensor. Isomorphic objects have the same length, so isomorphisms have length zero. Morphisms can also increase length, that is, have negative length (decrease), as one may have a morphism $\mathbb{1}\to\imath(*)$ which increases length by one and hence has length -1, see [KW17].

An *integer degree function* for a Feynman category is a function $\deg : Mor(\mathcal{F})\to\mathbb{Z}$ which is additive under composition and tensor product: $\deg(\phi\circ\psi) = \deg(\phi\otimes\psi) = \deg(\phi)+\deg(\phi)$, with the additional condition that isomorphisms have degree 0. Thus the length function $|\,.\,|$ is a degree function.

A graded Feynman category with an integer degree function is *non-negative* or *non-positive* if all morphisms have non-negative or non-positive degree respectively. We call a graded Feyman category *definite* if it is non–positive or non–negative. Of course by changing deg to $-$deg, one can change from non–positive to non–negative. One has extra structure in the definite case, which allows one to define the condition of almost connected, see Definition 5.35 and Lemma 5.32. All the main examples are definite.

Remark 5.29. In [KW17, Definition 7.2.1], similar notions were introduced: a *degree function* has the two additional conditions: (1) to have positive values and (2) all the morphisms are generated by degree 0 and 1 morphisms by composition and tensor product. It is called a *proper*, if all morphisms of degree 0 are isomorphisms. Many, but not all, Feynman categories have a proper degree function. Proper implies definite.

Example 5.30.

(1) In the set based examples: for $\mathfrak{Fin}\mathfrak{S}et$, $|\phi|$ is a proper integer degree function. On $\mathfrak{FS}$, $|\phi|$ is a proper degree function and on $\mathfrak{FI}$, $\deg(\phi) = -|\phi|$ is a proper degree function.

(2) In the case of graphs of higher genus ($b_1 > 0$), loop contractions are of native length 0. It is more natural, to have a different grading, in which both loop and edge contractions have degree 1 and mergers have degree 0. This makes the relations homogeneous, cf. [KW17, §5.1]. For $\mathcal{A}gg^{ctd}$ this is a proper degree function. The degree of a morphism ϕ is the number of edges of $\mathbb{\Gamma}(\phi)$.

In most practical examples, mergers are excluded, making life simple. This includes the Feynman categories for operads, colored operads, modular operads, etc., however this excludes PROPs and other "disconnected" types. In all of $\mathcal{A}gg$ it actually suffices to have the generators (a) isomorphisms, (c) simple loop contractions and (d) mergers. In this setting a proper degree function is given by assigning isomorphisms degree 0, and loop contractions and mergers degree 1.

(3) All three main examples are definite. The Feynman categories $\mathfrak{FS}_{\mathcal{O}}$ for algebras over operads, see §7.8.1 and [KW17, §4] are precisely non–negative, if there is no $\mathcal{O}(0)$; the length of elements of $\mathcal{O}(n)$ is $n-1$. They are proper if $\mathcal{O}(1) = \mathbb{1}$ is reduced. If $O^{red}(1)$ has no invertible morphisms, they are have group like identities. Surjections are also non–negative. Dually,

regarding only injections is an example of a non–positive Feynman category. All graph examples —without extra morphisms, see [KW17]— are also non–negative.

Proposition 5.31. *Given a factorization $id_X : X \stackrel{\phi_R}{\to} Y \stackrel{\phi_L}{\to} X$ it follows that*

1. *For any integer degree function* $\deg(\phi_R) = -\deg(\phi_L)$.
2. $|\phi_R| \leq 0$ *and* $|\phi_L| \geq 0$.
3. *If $\mathfrak{F}$ has a definite integer degree function then* $\deg(\phi_R) = \deg(\phi_L) = 0$. *I.e. any morphism with a left or right inverse has degree* 0.
4. *If $\mathfrak{F}$ is definite and if the only morphisms of $\mathfrak{F}$ with length 0, which have one–sided inverses are isomorphisms, then $\mathfrak{F}$ has group like identities.*
5. *If $\mathfrak{F}$ has a proper degree function then $\mathfrak{F}$ has almost group like units.*
6. *If $\mathfrak{F}$ is decomposition finite, then the identity of any object X does not have a factorization $id_X : X \stackrel{\phi_R}{\to} X \otimes Y \stackrel{\phi_L}{\to} X$ with $|\phi_R| < 0$.*

Proof. (1): $\deg(\phi_L) + \deg(\phi_R) = \deg(\phi_L \circ \phi_R) = \deg(\mathrm{id}_X) = 0$. (2): Decomposing the morphisms for $X = \bigotimes_v *_v$ according to (ii) we end up with sequences

$$*_v \stackrel{\phi_{R,v}}{\to} Y_v \stackrel{\phi_{L,v}}{\to} *_v$$

with $\phi_{L,v} \circ \phi_{R,v} = id_{*_v}$. This follows from decomposing ϕ_L and ϕ_R and then comparing to the decomposition of the isomorphism ϕ. We see that $|Y_v| \geq 1$ since there are no morphisms from any X of length greater or equal to one to $\mathbb{1}$. Thus $|\phi_{R_v}| \leq 0$ and hence $|\phi_R| = \sum_v |\phi_{R_v}| \leq 0$. (3) follows from (2) and (4) and (5) follow from (3).

(6): Define $\phi_R^1 = \phi_R$ and for $n \geq 2$: $\phi_R^n = \phi_R \otimes id \circ \phi_R^{n-1} : X \to X \otimes Y^{\otimes n}$ and likewise set $\phi_L^1 = \phi_L$ and for $n \geq 1$: $\phi_L^n = \phi_L^{\otimes n-1} \circ \phi_L \otimes id : X \otimes Y^n \to X$. These satisfy $\phi_L^n \circ \phi_R^n = id_X$ and there will be infinitely many possible decompositions of id_X, one for each n and hence we arrive at a contradiction. □

5.6.3. **Morphisms of degree 0 and almost–connectedness in the definite case.** We can reduce the question of the existence of an antipode further in the case of a definite Feynman category to the connectedness of the degree 0 morphisms. Let $\mathscr{B}_0$ be the span of the morphisms of degree 0 and set $\mathscr{B}_{\mathcal{V}} = \mathbb{Z}[Hom_{\mathcal{F}}(\imath(\mathcal{V}), \imath(\mathcal{V}))]$.

Lemma 5.32. *Assume that $\mathfrak{F}$ is decomposition finite, strict and definite w.r.t.* deg, *then*

(1) *$\mathscr{B}_0$ together with the restriction of the counit $\epsilon|_{\mathscr{B}_0}$ are a sub–co–algebra of $\mathscr{B}$. Together with $\otimes$ the unit η, $\mathscr{B}_0$ is a sub–algebra.*
(2) *$\mathscr{B}_0$ is isomorphic to the symmetric tensor algebra on morphisms $\phi_v : X \to \imath(*_v)$ of degree 0.*

If the length function $|\ .\ |$ is definite, then

(3) *$\mathscr{B}_{\mathcal{V}}$ together with the counit $\epsilon|_{\mathscr{B}_{\mathcal{V}}}$ and the unit η form a pointed coalgebra.*
(4) *$\mathscr{B}_{\mathcal{V}} = Hom_{\mathcal{F}}(\iota(\mathcal{V}), \iota(\mathcal{V}))^{\otimes} = \mathscr{B}_{\mathcal{V}}^{\otimes}$. Thus any morphism of length 0 has a decomposition into morphisms of $\mathscr{B}_{\mathcal{V}}$ up to permutations in the symmetric case.*

Proof. Suppose $\phi : X \overset{\phi_R}{\to} Z \overset{\phi_L}{\to} Y$ has degree 0, then $\deg(\phi_R) + \deg(\phi_L) = \deg(\phi) = 0$. In the definite case this implies $\deg(\phi_R) = \deg(\phi_L) = 0$, which shows that $\mathscr{B}_0$ is a subcoalgebra and since $\otimes$ has is additive in degtree, $\mathscr{B}_0$ is also subalgebra. The co-unit restricts and the unit is of length 0. Also if $\deg(\phi) = 0$ as $\phi \simeq \bigotimes_{v\in V} \phi_v$ with $\deg(\phi_v) \geq 0$ (or ≤ 0) and $\sum_{v\in V} |\phi_v| = 0$, which means that $\forall v \in V : \deg(\phi_v) = 0$. In particular, if $|Y| = 1$, we see that $|X| = 1$ as $|\phi| = 0$ and since $|\phi_L| = |\phi_R| = 0$, also $|Z| = 1$ so that $\mathscr{B}_{\mathcal{V}}$ is a sub–co–algebra. The image of η is in $\mathscr{B}_{\mathcal{V}}$ and ϵ restricts as the $id_X \subset \mathscr{B}_V$. The last statement follows from (2) by the observation that if $|\phi_v| = 0$, then $|X_v| = 1$ and hence $X_v = \imath(*_v)$.

□

Remark 5.33. The elements of $\mathscr{B}_{\mathcal{V}}$ split according to whether they are isomorphisms or not. That is, whether or not they lie in $Mor(\mathcal{V})$.

By induction, one can see that what can keep things from being connected is $\mathscr{B}_0$ or in the case of deg $= |\ .\ |$ being definite $\mathscr{B}_{\mathcal{V}}$. This is analogous to the situation for co–operads with multiplication, where, $\mathcal{V}$ is trivial and $\mathscr{B}_{\mathcal{V}} = \mathcal{O}(1)$ is the pointed co–algebra as in Definition 5.35.

Corollary 5.34. *Assume that $\mathfrak{F}$ has almost group–like identities. If $\mathfrak{F}$ has a definite degree function then $\mathscr{B}_0^{iso} := (\mathscr{B}_0/\sim)$ is a sub–bi–algebra with the induced unit and co–unit. $\mathcal{I}_0$ restriction to of the ideal and co–ideal $\mathcal{I} = \langle [id_X] - [id_Y] \rangle$to $\mathscr{B}_0$ then $(\mathscr{H}_0 = \mathscr{B}_0^{iso}, \eta, \epsilon)$ is a sub–bi–algebra of $\mathscr{H}$ Similarly if $|\ .\ |$ is a definite degree function then $\mathscr{B}_{\mathcal{V}}^{iso} := (\mathscr{B}_{\mathcal{V}}/\sim)$ is a sub–bi–alegbra with the induced unit and co–unit. Let $\mathcal{I}_{\mathcal{V}}$ be $\mathcal{I}$ restricted to $\mathscr{B}_{\mathcal{V}}^{iso}$ then $(\mathscr{H}_V,]\epsilon, \eta)$ is a sub–bi–algebra of $\mathscr{H}$.*

Proof. Immediate from the above. □

Definition 5.35. We call $\mathfrak{F}$ almost connected with respect to a given definite degree function if

(i) $\mathfrak{F}$ is factorization finite.
(ii) $\mathfrak{F}$ has almost group like identities.
(iii) $(\mathscr{H}_0^{iso}, \Delta^{iso}, \epsilon, \eta)$ is connected as a pointed co–algbera

Lemma 5.36. *Assume $\mathfrak{F}$ is factorization finite, has almost group like identities and $|\ .\ |$ is a definite degree function. Then: if $(\mathscr{H}_V, \epsilon, \eta)$ is almost connected, $(\mathscr{H}_0, \epsilon, \eta)$ is as well and hence $\mathfrak{F}$ is almost connected w.r.t. $|\ .\ |$.*

Proof. This follows from Lemma 5.32 (4) by applying the bialgbra equation. □

Theorem 5.37. *If $\mathfrak{F}$ is almost connected then $\mathfrak{F}$ is Hopf.*

Proof. We show that $\mathscr{H}$ is conilpotent and hence connected. WLOG we assume deg is non–negative. Any decomposition of a morphisms ϕ into (ϕ_0, ϕ_1) for which $\deg(\phi_0), \deg(\phi_1) \neq 0$ has $\deg(\phi_0), \deg(\phi_1) < \deg(\phi)$ due to the additivity of deg. These terms of Δ^{iso} of lesser degree are taken care of by induction. The terms with degree 0 factors are taken care of by the almost connectedness of $\mathscr{B}_0$ and co–associativity. □

Proposition 5.38. *If* deg *is a proper degree function for a factorization finite $\mathfrak{F}$, then $\mathfrak{F}$ is almost connected and hence Hopf.*

Proof. By Proposition 5.31 (5) $\mathfrak{F}$ has almost group like identities. $\mathscr{H}_0 = [id_{\mathbb{1}}]$ and is connected. □

Remark 5.39. Any morphism $\phi : X \to Y$ satisfies $\Delta(\phi) = id_X \otimes \phi + \phi \otimes id_Y + \dots$, id_X (suitably normalized) are group like elements in $\mathscr{B}^{iso}$. Hence it is interesting to study the co–radical filtration and the $([id_X], [id_Y])$–primitive elements in $\mathscr{B}$. They correspond to the generators for morphisms in Feynman categories [KW17]. In the main examples they are all tensors of elements of length 1.

5.7. **Functoriality.** Let $\mathfrak{f} : \mathfrak{F} \to \mathfrak{F}'$ be a morphism of Feynman categories. In the strict case, this is a pair of functors $\mathfrak{f} = (v, f)$: $v : \mathcal{V} \to \mathcal{V}$ and $f : \mathcal{F} \to \mathcal{F}'$, strict symmetric monoidal, compatible with all the structures, see [KW17, Chapter 1.5]. In general, one allows strong monoidal functors. In the non-Σ case, the functor f is required to be strict, resp. strong monoidal. For a morphism $\phi \in Mor(\mathcal{F}')$ thought of as a characteristic function $\phi(\psi) = \delta_{\phi,\psi}$ one calculates

$$f^*(\phi) := \phi \circ f = \sum_{\hat{\phi} \in Mor(\mathcal{F}) : f(\hat{\phi}) = \phi} \hat{\phi} \tag{5.14}$$

This induces a pull–back operation under $\mathfrak{f}$. The pull–back descends to isomorphism classes. We set $[\phi]([\psi]) = 1$ if ϕ and ψ are in the same class and 0 otherwise. The lift is defined by $f^*([\phi])([\hat{\psi}]) := [\phi]([f(\hat{\psi})])$.

Proposition 5.40. *For non–Σ Feynman categories: Via f^*, $\mathfrak{f}$ induces a morphism of unital algebras $\mathscr{B}_{\mathfrak{F}'} \to \mathscr{B}_{\mathfrak{F}}$. If f is injective on objects, then f^* induces a morphism of co–algebras $\mathscr{B}_{\mathfrak{F}'} \to \mathscr{B}_{\mathfrak{F}}$. If f^* is bijective on objects, it induces a morphism of counital coalgebras $\mathscr{B}_{\mathfrak{F}'} \to \mathscr{B}_{\mathfrak{F}}$.*

For Feynman categories: Via f^*, $\mathfrak{f}$ induces a morphism of unital algebras $\mathscr{B}^{iso}_{\mathfrak{F}'} \to \mathscr{B}^{iso}_{\mathfrak{F}}$. If f is essentially injective on objects, then f^* induces a morphism of co–algebras $\mathscr{B}^{iso}_{\mathfrak{F}'} \to \mathscr{B}^{iso}_{\mathfrak{F}}$. If f^* is essentially bijective on objects, it induces a morphism of co–unital co–algebras $\mathscr{B}^{iso}_{\mathfrak{F}'} \to \mathscr{B}^{iso}_{\mathfrak{F}}$.

Proof. In the non–Σ case: For a strictly monoidal $f : \mathcal{F} \to \mathcal{F}'$, f^* is a functorial for the algebra structure: using (5.14). Consider $\phi : X \to Y, \psi : X' \to Y'$, then $(f^* \otimes f^*)(\phi \otimes \psi) = (\phi \circ f) \otimes (\psi \circ f) = (\phi \otimes \psi) \circ f = f^*(\phi \otimes \psi)$. Here for the penultimate equality: let $\hat{X}, \hat{X}', \hat{Y}$ and $\hat{Y}'$ be lifts of X, X', Y and Y' and let $\hat{\Phi} : \hat{X} \otimes \hat{X}' \to \hat{Y} \otimes \hat{Y}'$ then $\hat{\Phi}$ decomposes as $\hat{\Phi}^1 \otimes \hat{\Phi}^2$, $\hat{\Phi}^1 : \hat{X} \to \hat{Y}$, $\hat{\Phi}^2 : \hat{X}' \to \hat{Y}'$, since we are in the non–Σ case, and thus $f(\hat{\Phi}) = f(\hat{\Phi}^1 \otimes \Phi^2) = f(\hat{\Phi}^1) \otimes f(\Phi^2)$.

For the co–product one calculates:

$$\begin{aligned} \Delta(f^*\phi) &= \sum_{\hat{\phi} \in Mor(\mathcal{F}): f(\hat{\phi}) = \phi} \; \sum_{(\hat{\phi}_0, \hat{\phi}_1): \hat{\phi}_1 \circ \hat{\phi}_0 = \hat{\phi}} \hat{\phi}_0 \otimes \hat{\phi}_1 \qquad (5.15) \\ (f^* \otimes f^*)\Delta(\phi) &= \sum_{(\phi_0, \phi_1): \phi_1 \circ \phi_0 = \phi} \; \sum_{\hat{\phi}_0, \hat{\phi}_1 \in Mor(\mathcal{F}): f(\hat{\phi}_0) = \phi_0, f(\hat{\phi}_1) = \phi_1} \hat{\phi}_0 \otimes \hat{\phi}_1 \end{aligned}$$

We now check that the sums coincide. Certainly for any term in the first sum corresponding to decomposition $\hat{\phi} = \hat{\phi}_1 \circ \hat{\phi}_0$ appears in the second sum, since f is a functor: $f(\hat{\phi}_1) \circ f(\hat{\phi}_0) = f(\hat{\phi}_0 \circ \hat{\phi}_1) = f(\hat{\phi}) = \phi$. The second sum might be larger, since the lifts need not be composable. If, however, f is injective on objects, then all lifts of a composition are composable and the two sums agree. The unit agrees, because of the injectivity and uniqueness of the unit object and the triviality of $Hom(\mathbb{1}, \mathbb{1})$. For the co–unit, we need bijectivity. Namely, $1 = \epsilon(id_X)$, but if f is not surjective, then some $f^*(id_X) = 0$ and $\epsilon(f^*(id_X)) = 0 \neq 1$. If f is not injective, then as all the $f(id_{\hat{X}}) = id_X$ for all $\hat{X} : f(\hat{X}) = X$, $\epsilon(f^*(id_X) = \sum_{\hat{X}: f(\hat{X}) = X}$ and the sum is > 2 for some X. Thus the condition is necessary. It is also sufficient. If f is bijective on objects, then, $\widehat{id_X} = id_{\hat{X}} + T$, with $\epsilon(T) = 0$. This implies that $\epsilon(f^*(id_X)) = \epsilon(id_{\hat{X}}) = 1$ and $\epsilon(f^*(\phi)) = 0$ if $\phi \neq id_{\hat{X}}$. as there is no $id_{\hat{X}}$ in the fiber over ϕ if ϕ is not an identity. If the functor is not injective, we might have more objects in the fiber and if it is not surjective $f^*(id_X)$ can be 0.

In the symmetric situation, the arguments are analogous using isomorphism classes. Although one cannot guarantee the decomposition of Φ as above, there is a decomposition up to isomorphism $\hat{\Phi} = \hat{\Phi}^1 \otimes \hat{\Phi}^2 \circ \sigma$. Likewise, the essential injectivity ensures that the lifts are composable as classes and the essential surjectivity is needed to preserve the co–unit. $\square$

Definition 5.41. We call a functor $\mathfrak{f}$ as above Hopf compatible if it is essentially bijective and $f^*(\mathcal{I}_{\mathfrak{F}'}) \subset I_{\mathfrak{F}}$.

The following is straightforward.

Proposition 5.42. *If $\mathfrak{F}$ and $\mathfrak{F}'$ are Hopf, a Hopf compatible functor induces a morphism of Hopf algebras $\mathscr{H}_{\mathfrak{F}'} \to \mathscr{H}_{\mathfrak{F}}$.* $\square$

The following is a useful criterion:

Proposition 5.43. *If in addition to being essentialy bijective f does not send any non–invertible elements of $Mor(\mathcal{F})$ to invertible elements in $Mor(\mathcal{F}')$, then $\mathfrak{f}$ is Hopf compatible.*

Proof. That the condition is necessary is clear. Fix X, then up to isomorphism there is a unique lift $\hat{X}$ of X. Any lift of id_X is then an isomorphism $\hat{\phi}: \hat{Y} \to \hat{Y}'$ with both $\hat{Y}$ and $\hat{Y}'$ being isomorphic to $\hat{X}$ which means that $[\hat{\phi}] = [\mathrm{id}_{\hat{X}}]$ and $f^*[id_X] = [id_{\hat{X}}]$. $\square$

These criteria reflect that Hopf algebras are very sensitive to invertible elements. It says that that we can identify isomorphisms and are allowed to identify morphisms, but only in each class separately.

Example 5.44. An example is provided by the map of operads: rooted 3–regular forests $\to$ rooted corollas. This give a functor of Feynman categories enriching $\mathfrak{F}\mathfrak{S}$ or in the planar version of $\mathfrak{F}\mathfrak{S}_<$, see 7.8.1. This functor is Hopf compatible thus induces a map of Hopf algebras which is the morphism considered by Goncharov in [Gon05].

Example 5.45. Another example is given by the map of rooted forests with no binary vertices $\to$ corollas. The corresponding morphisms of Feynman categories is again Hopf compatible.

However, if we consider the functor of Feynman categories induced by rooted trees $\to$ rooted corollas is not Hopf compatible. It sends all morphisms corresponding to binary trees to the identity morphism of the corolla with one input. Thus is maps non–invertible elements to invertible elements. The presence of these extra morphisms in $\mathscr{H}_{CK}$ is what makes it especially interesting. They also correspond to a universal property [Moe01] and Example 2.50.

6. Variations on the bi– and Hopf algebra structures

Here we will give some variations of the structures above. The first is an analysis of the role of basic morphisms and the second is the possibility to modify the bi–algebra structure and how to twist by co–cycles. There are two relevant constructions. The first involves quotienting by isomorphisms and the second uses co–cycles to twist the co–mulitplication. This possibly changes the multiplicities. In the symmetric case, with the symmetries the bi–algebra equation fails on the level of morphisms, i.e. without passing to the isomorphism classes. The reason for this is that $Aut(X) \times Aut(X') \subset Aut(X \otimes X')$ is a proper subset due to the permutation symmetries. To remedy this one can twist in certain situations, for example if $\bar{d}$ is a free action.

There is a third alternative, which is to use representations, in the spirit they appear in fusion rules in physics, but we will not delve into this further technical complication at this point.

6.1. **Bi-algebra structure induced from indecomposables.** For a strict Feynman category $Mor(\mathcal{F}) = Obj(\imath^{\otimes} \downarrow \imath)^{\otimes}$ and hence $\mathscr{B}$ is the strictly associative free monoid on $\mathscr{B}_1 = \mathbb{Z}[Ob(\imath^{\otimes} \downarrow \imath)] \subset \mathscr{B}$ with additional symmetries possibly given by the commutativity constraints induced by $\mathcal{F}$.

Lemma 6.1. *If $\mathcal{F}$ is strict and non–Σ, $\mathscr{B}_1$ is the set of indecomposables.*

Proof. By axiom (ii) any morphism with target of length greater or equal to 2 is decomposable. If the target of a morphism ϕ has length 1, it can only decompose as $\phi = \hat{\phi} \otimes_{\mathbb{Z}} \lambda$ with $\lambda \in \mathbb{Z}[Hom(\mathbb{1}, \mathbb{1})] = \mathbb{Z}id_{\mathbb{1}}$, since the only object of length 0 is unit $\mathbb{1}$ and $\mathfrak{F}$ was taken to be strict. Hence $\lambda = \pm id_{\mathbb{1}}$ is itself a unit in the algebra and $\phi = \pm\hat{\phi}$. □

We now suppose that $\mathscr{B}_1$ is decomposition finite, which means that the sum in (6.1) is finite. Consider the one–comma generators $\mathscr{B}_1$ and define

$$\Delta_{indec}(\phi) = \sum_{\{(\phi_0, \phi_1): \phi = \phi_0 \circ \phi_1\}} \phi_0 \otimes \phi_1 \tag{6.1}$$

here $\phi_0 \in \mathscr{B}_1$ and $\phi_1 = \bigotimes_{v \in V} \phi_v$ for $\phi_v \in \mathscr{B}_1$. We extend the definition of Δ_{indec} to all of $\mathscr{B}$ via the bi-algebra equation.

$$\Delta_{indec}(\phi \otimes \psi) := \sum (\phi_0 \otimes \psi_0) \otimes (\phi_1 \otimes \psi_1) \tag{6.2}$$

where we used Sweedler notation.[5]

$$\epsilon(\phi)=\begin{cases}1 & \text{if } \phi = id_X\\ 0 & \text{else}\end{cases}$$

In this case there is a direct proof of the bi–algebra structure. A posteriori using Lemma 6.1 it follows that this bialgebra structure coincides with the decomposition bialgebra structure.

Proposition 6.2. *With the assumptions on $\mathcal{F}$ as above and that $\mathscr{B}_1$ is decomposition finite, the tuple $(\mathscr{B}, \otimes_{\mathcal{F}}, \Delta_{indec}, \mathbb{1}, \epsilon)$ is a bi–algebra. A posteriori $\Delta = \Delta_{indec}$.*

Proof. The multiplication is unital and associative. That the co–product is co–associative and ϵ is a co-unit is a straightforward check. The latter follows from the decomposition $id_X = \otimes_v id_{*_v}$ if $X = \otimes_v *_v$. The fact that the bi–algebra equation holds, follows from the fact that all elements in $\mathscr{B}_1$ are indecompsable with respect to this product. For the co–associativity, we notice that in both iterations we get sum over decomposition diagrams $\phi = \phi''' \circ \phi'' \circ \phi'$.

$$\begin{tikzcd}
X = \bigotimes_v \bigotimes_{w\in V_v} X_w \arrow[r, equal] \arrow[d, "\phi'=\otimes_w \phi_w"'] & \bigotimes_v \bigotimes_{w\in V_v} \bigotimes_{u\in V_w} *_u \arrow[rr, "\phi=\bigotimes_u \phi_u"] & & * \\
Z_1 = \bigotimes_v \bigotimes_{w\in V_v} *_w \arrow[r, equal] & \bigotimes_v Z_v \arrow[rr, "\phi''=\bigotimes_v \phi''_v"] & & Z_2 = \bigotimes_v *_v \arrow[u, "\phi'''=\bigotimes \phi'''_v"']
\end{tikzcd} \tag{6.3}$$

where the order of the factors is fixed and the sum is over the possible morphisms and bracketings. That $\Delta = \Delta_{basic}$ follows from the equality of the co–products on indecomposables for the bi–algebra which by Lemma 6.1 are precisely $\mathscr{B}_1$. $\square$

Remark 6.3. This two step process corresponds to the free construction $\check{\mathcal{O}}^{nc}$ in Chapter 1. A prime example is the bi–algebra of rooted planar trees aka. bialgebra of forests of Connes and Kreimer [CK98]. The usual way this is defined is to give the co–product on indecomposable, viz. trees, and then extend using the bi–algebra equation.

6.2. **Isomorphisms, quotients and twists.** We collect more precise information about the isomorphisms and their role in order to make the more specialized constructions. The first is a quotient by the co–ideal of isomorphisms in the non–Σ case. In the symmetric case, although we have a co–ideal to divide by there is a problem with the bi–algebra

[5]If there is a non–trivial commutativity constraint, we take this to mean $\sigma_{23} \circ \Delta \otimes \Delta$

equation already on the level of the morphisms. Note, we are not taking isomorphisms yet. To remedy the situation, one can introduce twists in certain situations.

6.2.1. **Iso– and Automorphisms.** By the conditions of a Feynman category for $X = \bigotimes_{i=1}^{k} *_i$. In the non–symmetric case, any automorphism factors, so

$$Aut(X) \simeq Aut(*_1) \times \cdots \times Aut(*_k) \text{ in the non–symmetric case.}$$

In the symmetric case its automorphisms group is the wreath product

$$Aut(X) \simeq (Aut(*_1) \times \cdots \times Aut(*_k)) \wr \mathbb{S}_k \text{ in the non–symmetric case.}$$

6.2.2. **The co–ideal generated by the isomorphisms relation.** Recall that $f \sim g$ if they are isomorphic, c.f. §5.2.1.

Proposition 6.4. *Let $\mathcal{C}$ be the ideal generated by elements $f - g$ with $f \sim g$. Then*

$$\Delta(\mathcal{C}) \subset \mathscr{B} \otimes \mathcal{C} + \mathcal{C} \otimes \mathscr{B} \tag{6.4}$$

and hence $\mathscr{B}/\mathcal{C}$ is a unital algebra and (non–co–unital) co–algebra.

Extending scalars to $\mathbb{Q}$, there is a co–unit on $\mathscr{B}^{quot}_{\mathbb{Q}} = \mathscr{B}/\mathcal{C} \otimes_{\mathbb{Z}} \mathbb{Q}$

$$\epsilon^{quot}([f]) := \begin{cases} \frac{1}{|Iso(X)||Aut(X)|} & \text{if } [f] = [id_X] \\ 0 & \text{else} \end{cases} \tag{6.5}$$

Proof. To compute the co–product, we break up the sum over the factorizations of f and g with $f \sim g$ into the pieces that correspond to a factorization through a fixed space Z.

$$\begin{tikzcd}
& Z \arrow[dr, "f_2"] & \\
X \arrow[ur, "f_1"] \arrow[rr, "f"] \arrow[d, "\simeq\sigma'"'] & & Y \arrow[d, "\simeq\sigma"] \\
X' \arrow[rr, "g"] \arrow[dr, "g_1"'] & & Y' \\
& Z \arrow[ur, "g_2"'] &
\end{tikzcd} \tag{6.6}$$

Now the term in Δ_{f-g} corresponding to Z is $\sum_i f_2^i \otimes f_1^i - \sum_j g_2^j \otimes g_1^j$. Re–summing using the identification $g_1^i := f_1^i \circ \sigma'^{-1}$ and $g_2^i := \sigma \circ f_2^i$ this equals to
$\sum_i (f_2^i \otimes f_1^i - g_2^i \otimes g_1^i) = \sum_i (f_2^i - g_2^i) \otimes g_1^i + \sum_i f_2^i \otimes (f_1^i - g_1^i)$.

For the co–unit, notice that $\Delta([f]) = [\Delta(f)]$ is a sum of terms factoring through an intermediate space Z. If $Z \not\simeq X, Y$ then these terms are killed by ϵ^{quot} on either side, since there will be no isomorphism in the decomposition. If $Z \simeq X$, then any factorization $f \circ \sigma^{-1} \otimes \sigma$ with $\sigma \in Iso(X, Z)$ descends to $[f \circ \sigma^{-1}] \otimes [\sigma] = [f] \otimes [id_X]$. Since $Iso(X, Z)$ is a left $Aut(X)$ torsor, there are exactly $|Aut(X)||Iso(X)|$ of these terms and $\epsilon^{quot} \otimes id$ evaluates to $1 \otimes [f]$ on their sum. By Lemma 5.4, all other decompositions will evaluate to 0 and we obtain that ϵ^{quot} is a left co–unit. Likewise ϵ^{quot} is a right co–unit by considering the terms which factor through $Y' \in Iso(Y)$.

□

Remark 6.5.

(1) Note that $\mathcal{C}$ is not a co–ideal in general, since for any automorphism $\sigma_X \in Aut(X) : [\sigma_X] = [id_X]$ and hence $\epsilon(\mathcal{C}) \not\subseteq ker(\epsilon)$. Likewise if $X \simeq Y \overset{\sim\phi}{\to} Y'$ then $[id_X] = [\phi]$ from Lemma 5.4. This is why we need a new definition for the co–unit. If there are no automorphisms and the underlying category is skeletal, then ϵ descends as claimed in [JR79].

(2) The equivalence relation $\sim$ is coarser than the equivalence studied in [JR79] for the standard reduced incidence category.

(3) Extending scalars from $\mathbb{Z}$ all the way to $\mathbb{Q}$ may not be necessary; we only need that $|Iso(X)|$ and $|Aut(X)|$ are invertible for all X. Although in the symmetric case, the automorphisms groups will contain all $\mathbb{S}_n$ and hence $\mathbb{Q}$ is necessary.

(4) One can get rid of the terms $X' \in Iso(X)$ in $\Delta(id_X)$ and the factor $|Iso(X)|$ by considering a skeletal version. Recall that skeletal means that there is only one object per isomorphism class.

(5) Although in the symmetric case, the bi–algebra equation does not hold on $\mathscr{B}$, it does on a non–Σ Feynman category. The difference is due to §6.2.1. The failure in the symmetric case is analyzed in detail in §6.2.3 below.

Theorem 6.6. *Let $\mathcal{F}$ be a decomposition finite non–Σ Feynman category set $\mathscr{B}^{quot}_{\mathbb{Q}}$ with the induced product, unit, co–product and co–unit ϵ^{quot} is a bialgebra.*

Proof. In the non–symmetric case, the compatibility of product and co–product descend as does the compatibility of the unit. For the co–unit, we notice that $\epsilon^{quot}([\phi \otimes \psi])$ as well as $\epsilon^{quot}([\phi])\epsilon^{quot}([\psi])$ are 0 unless $[\phi] = \lambda[id_X]$ and $[\psi] = \mu[id_Y]$. If this is satisfied, by the conditions of a non–symmetric Feynman category $|Aut(X)||Aut(Y)| = |Aut(X \otimes Y)|$ as well as $|Iso(X)||Iso(Y)| = |Iso(X \otimes Y)|$ so that $\epsilon^{quot}([id_X] \otimes [id_Y]) = \epsilon^{quot}([id_X])\epsilon^{quot}([id_Y])$. □

We define the ideal $\bar{\mathscr{J}} = \langle |Aut(X)||Iso(X)|id_X - |Aut(Y)||Iso(Y)|id_Y\rangle$ of $\mathscr{B}^{quot}_{\mathbb{Q}}$, and then consider $\mathscr{H}^{quot}_{\mathbb{Q}} = \mathscr{B}^{quot}_{\mathbb{Q}}/\bar{\mathscr{J}}$.

Theorem 6.7. *Assume that $\mathfrak{F}$ is decomposition finite non–Σ and has almost group like identities, then, $\bar{\mathscr{J}}$ is a co–ideal in $\mathscr{B}^{quot}_{\mathbb{Q}}$ and $\mathscr{H}^{quot}_{\mathbb{Q}} = \mathscr{B}^{quot}_{\mathbb{Q}}/\bar{\mathscr{J}}$ is a bialgebra with co–unit induced by ϵ^{quot} and unit $\eta_{\mathscr{H}^{quot}_{\mathbb{Q}}}(1) = [\mathrm{id}_{\mathbb{1}_{\mathcal{F}}}]$. If $\mathscr{H}^{quot}_{\mathbb{Q}}$ is connected, then it is a Hopf algebra.*

Proof. In $\mathscr{B}^{quot}_{\mathbb{Q}}$, (5.13) reads $\Delta([id_X]) = |Aut(X)||Iso(X)|[id_X]\otimes[id_X]$, so that

$$\begin{aligned}
&\Delta(|Aut(X)||Iso(X)|[id_X]) - |Aut(Y)||Iso(Y)|[id_Y] \\
&\quad = (|Aut(X)||Iso(X)|)^2[id_X]\otimes[id_X] - (|Aut(Y)||Iso(Y)|)^2[id_Y] \\
&= (|Aut(X)||Iso(X)|[id_X] - |Aut(Y)||Iso(Y)|[id_Y]\otimes|Aut(X)||Iso(X)|[id_X] + \\
&|Aut(Y)||Iso(Y)|[id_Y]\otimes(|Aut(X)||Iso(X)|[id_X] - |Aut(Y)||Iso(Y)|[id_Y])
\end{aligned}$$

Hence, the ideal $\bar{\mathscr{J}}$ is generated by elements $|Aut(X)||Iso(X)|[id_X] - |Aut(Y)||Iso(Y)|[id_Y]$ is also a co–ideal, as these also satisfy

$$\epsilon^{quot}(|Aut(X)||Iso(X)|[id_X] - |Aut(Y)||Iso(Y)|[id_Y]) = 1 - 1 = 0$$

It is easy to check that $\eta_{\mathscr{H}^{quot}_{\mathbb{Q}}}$ yields a split co–unit. $\square$

Remark 6.8.

(1) One can use a notions of grading and almost connectedness here as in previous analysis of connectedness. This is entirely analogous to §2.5.1.

(2) If $\mathcal{V}$ is also discrete and hence $\mathcal{F}$ skeletal $\bar{\mathscr{J}} = \langle [id_X] - [id_Y]\rangle$ and $\mathscr{B}^{quot}_{\mathbb{Q}} = \mathscr{B}^{iso}\otimes_{\mathbb{Z}}\mathbb{Q}$. This is the case for the non–Σ operads, see §7.8.

6.2.3. **The symmetric case: a careful analysis of the two sides of the bi–algebra equation.** The following proposition a finer version of Proposition 5.20 which also holds in the symmetric case.

Proposition 6.9. *For any factorization of $\Phi = \phi\otimes\psi : X\times X' \to Z\otimes Z'$ as $\Phi_0\circ\Phi_1 : X\times X' \to Y \to Z\otimes Z'$ there exists a decomposition $\sigma' : Y \simeq \hat{Y}\otimes\hat{Y}'$ and a factorization $(\phi_0\otimes\psi_0, \phi_1\otimes\psi_1)$ factoring through $\hat{Y}\otimes\hat{Y}'$ such that $(\Phi_0,\Phi_1) = \bar{d}(\sigma')(\phi_0\otimes\psi_0, \phi_1\otimes\psi_1) = (\phi_0\otimes\psi_0\circ\sigma'^{-1}, \sigma'\circ\phi_1\otimes\psi_1)$. Furthermore, all such factorizations are in 1–1 correspondence with the cosets $Iso(Y, \hat{Y}\otimes\hat{Y}')/Aut(\hat{Y})\times Aut(\hat{Y}')$.*

Proof. Given a decomposition of Φ as (Φ_0,Φ_1), we can follow the argument of the proof of Theorem 5.20 up until the discussion of the isomorphisms σ and σ'.

In the symmetric case, *a priori* there could be permutations involved for σ and σ'. This is, however, not the case for σ, and we can absorb it to get decompositions of Φ. More precisely, the isomorphism σ has to be a block isomorphism as axiom (ii) applies to the two decompositions $\Phi = \phi\otimes\psi$ and $\Phi \simeq \hat{\phi}_0\circ\hat{\psi}_0\otimes\hat{\phi}_1\circ\hat{\psi}_1$. This means that σ in (5.12) is uniquely a tensor product of isomorphisms $\sigma = \sigma_1\otimes\sigma_2$, since both decompositions have the same target decomposition $Z\otimes Z'$. By pre-composing, we get the tensor decomposition $\Phi = (\hat{\phi}_0\otimes\hat{\psi}_0)\circ(\hat{\phi}_1\otimes\hat{\psi}_1)\circ(\sigma_1^{-1}\otimes\sigma_2^{-1})$.

Continuing with the decomposition of this form, we turn to σ'. We know that by (ii) that σ' can be written as a tensor product decomposition preceded by a permutation. If $\sigma' = \sigma'_1\otimes\sigma'_2$, we have that $Y = Y'\otimes Y''$ and (Φ_0,Φ_1) appears as a tensor product. Again absorbing the tensor decomposition means that the remaining terms corresponding to non–tensor decomposable permutations, and hence to a sum over the respective cosets. ☐

Notice that fixing any isomorphism in $Iso(Y,\hat{Y}\otimes\hat{Y}')$ identifies it with $Aut(\hat{Y}\otimes\hat{Y}')$ so that the quotient group $Iso(Y,\hat{Y}\otimes\hat{Y}')/[Aut(\hat{Y})\times Aut(\hat{Y}')]$ becomes identified with $Aut(\hat{Y}\otimes\hat{Y}')/[Aut(\hat{Y})\times Aut(\hat{Y}')] \simeq Aut(Y)/(Aut(Y')\times Aut(Y))$. Using this identification, we can see that if $\mathcal{F}$ is a Feynman category, then in the proof of Theorem 5.20 the sets of diagrams agree up to a choice of cosets of isomorphisms of σ' in (5.12), that is the difference in the count of diagrams will result from the cosets $Aut(\hat{Y}\otimes\hat{Y}')/(Aut(\hat{Y})\times Aut(\hat{Y}'))$. More precisely:

Corollary 6.10. *Splitting the sum $\Delta\circ\mu$ into subsums over a fixed decomposition of $Y = Y'\otimes Y''$, $\Delta\circ\mu = \sum_Y(\Delta\circ\mu)_Y$, we have*

$$\sum_Y(\Delta\circ\mu)_Y = \sum_{Y=Y'\otimes Y''}\ \sum_{[\sigma']\in Aut(Y)/(Aut(Y')\times Aut(Y''))} \bar{d}(\sigma')\,(\mu\otimes\mu\circ\pi_{23}\circ\Delta\otimes\Delta)_{\hat{Y}\otimes\hat{Y}'} \tag{6.7}$$

where we have fixed a decomposition $\hat{Y}\otimes\hat{Y}'\simeq Y$ and used the identification above.

In the non–Σ case, $Aut(Y)\simeq(Aut(Y')\times Aut(Y''))$, so that $Aut(Y)/(Aut(Y')\times Aut(Y''))$ is trivial and we recover Theorem 5.20.

In the symmtric case, usually the bi–agebra equation fails on $\mathscr{B}$. An interesting aspect is the possiblility to twist the co–multiplication by a co–cycle, to make it hold on $\mathscr{B}/\mathcal{C}$, which In certain cases this leads to a bi–algebra structure.

Example 6.11. In the case of trivial $\mathcal{V}$, in the symmetric case, we have $Aut(n)\times Aut(m) = \mathbb{S}_n\times\mathbb{S}_m\subset\mathbb{S}_{n+m} = Aut(n+m)$ in $\mathcal{V}^{\otimes}$. Let us consider the trivial Feynman cateogry with trivial $\mathcal{V}$, that is $\mathcal{F} = \mathbb{S}$, the skeletal version of $\mathcal{V}^{\otimes}$, which has the natural numbers as objects and only isomorphisms as morphisms, where $Hom(n,n) = Aut(n,n) = \mathbb{S}_n$. We will consider $\Delta(id_n \otimes id_m) = \Delta(id_{n+m}) = \sum_{\sigma \in \mathbb{S}_{n+m}} \sigma \otimes \sigma^{-1}$. We analyze the possible diagrams (5.12) for the summand $\sigma \otimes \sigma^{-1}$ in the proof of Theorem 5.20.

$$\begin{tikzcd}
n \otimes m \arrow[r, "\sigma'"] \arrow[ddrr, "\hat{\sigma}_n \otimes \hat{\sigma}_m"'] & n \otimes m = n+m \arrow[rr, "id_{n+m}=id_n \otimes id_m"] \arrow[dr, "\sigma"] & & n+m \\
& & n \otimes m \arrow[ur, "\sigma^{-1}"] \arrow[d, "\sigma_n \otimes \sigma_m \circ \sigma^{-1}"'] & \\
& & n \otimes m \arrow[uur, "\sigma_n^{-1} \otimes \sigma_m^{-1}"'] &
\end{tikzcd} \tag{6.8}$$

And we see that $\sigma' = \sigma_n^{-1} \otimes \sigma_m^{-1} \circ \hat{\sigma}_n \otimes \hat{\sigma}_m = \sigma_n^{-1} \circ \hat{\sigma}_n \otimes \sigma_m^{-1} \circ \hat{\sigma}_m$ absorbing this block isomorphism into $\hat{\sigma}_n \otimes \hat{\sigma}_m$, we get the diagram.

$$\begin{tikzcd}
n \otimes m = n+m \arrow[rr, "id_{n+m}=id_n \otimes id_m"] \arrow[dr, "\sigma"] \arrow[ddr, "\sigma_n \otimes \sigma_m"'] & & n+m \\
& n \otimes m \arrow[ur, "\sigma^{-1}"] \arrow[d, "\sigma_n \otimes \sigma_m \circ \sigma^{-1}"'] & \\
& n \otimes m \arrow[uur, "\sigma_n^{-1} \otimes \sigma_m^{-1}"'] &
\end{tikzcd} \tag{6.9}$$

If σ is of the form $\sigma_n \otimes \sigma_m$, then the term appears in $\Delta(id_n) \otimes \Delta(id_m)$. Otherwise, the action of $Aut(Y)$ on $Hom(X,Y) \otimes Hom(Y,Z)$ with $X = Y = Z = n+m$, on the decompositions appearing in $\Delta(id_n) \otimes \Delta(id_m)$ and moreover, picking representatives σ^r of $Aut(n+m)/(Aut(n) \times Aut(m))$ and summing over their action, we get an equality

$$\Delta(id_n \otimes id_m) = \sum_{\sigma^r \in \mathbb{S}_{n+m}/(\mathbb{S}_n \otimes \mathbb{S}_m)} \rho(\sigma^r) \Delta(id_n) \otimes \Delta(id_m)$$

In particular for equivalence classes in $\mathscr{B}/C$, we get

$$\Delta([id_n] \otimes [id_m]) = \frac{(n+m)!}{n!m!} \Delta([id_n]) \otimes \Delta([id_m])$$

which shows the failure of the bi–algebra equation.

However, the difference can be absorbed by a co–cycle: Set $\beta(\sigma_n, \sigma_n^{-1}) = \frac{1}{|Aut(n)|} = \frac{1}{n!}$. Define a new co–multiplication: $\Delta_\beta(id_n) = \beta(\sigma_n, \sigma_n^{-1})\, \sigma_n \otimes \sigma_n^{-1}$ then $\otimes$ and Δ_β on $\mathscr{B}^{quot}_{\mathbb{Q}}$ satisfy the bialgebra equation.

6.2.4. **Actions and cocycles.** Recall that there is an $Aut(Z)$ on $Hom(Z,Y) \times Hom(X,Z)$ given by $\bar{d}(\sigma)(\phi_0, \phi_1) = (\phi_0 \circ \sigma^{-1}, \sigma \circ \phi_1)$.

By a twisting co–cycle for the co–product, we mean a morphism $\mathscr{B} \to Hom(\mathscr{B} \otimes \mathscr{B}, K)$ that is a linear collection of bilinear morphisms β_ϕ, s.t. $\Delta_\beta(\phi) = \sum_{(\phi_0,\phi_1)} \beta_\phi(\phi_0,\phi_1)\phi_0 \otimes \phi_1$ is still co–associative. Such a co–cycle is called multiplicative if $\beta_{\phi\otimes\psi} = \beta_\phi\beta_\psi$ on decomposables. β is called co–unital, if there exists a co–unit ϵ_β.

Proposition 6.12. *Assuming for simplicity that we are in the skeletal case. If the $Aut(Z)$ action is free on all decompositions, then we can define define a modified co–product Δ_β on $\mathscr{B}$, defined the multiplicative co–cycle, which is given by a co–cycle $\beta(\phi_0,\phi_1) = \frac{1}{|Aut(Z)|}$ for a factorization $\phi : X \xrightarrow{\phi_1} Z \xrightarrow{\phi_0} Y$. In the non-$\Sigma$ case the co–cycle is multiplicative, and, if the identities are almost group–like, co–unital.*

This co–algebra structure descends to $\mathscr{B}/\mathcal{C}$ furnishing a bi–algebra structure:

$$\Delta^{red}([\phi]) := [\Delta_\beta](\phi) = \sum_Z \sum_{i_r} [\phi_1^{i_r}] \otimes [\phi_0^{i_r}] \tag{6.10}$$

where the sum runs over representatives of the $Aut(Z)$ action. There is a co–unit

$$\epsilon^{red}([\phi]) = \begin{cases} 1 & \text{if } [\phi] = [id_X] \\ 0 & \text{else} \end{cases} \tag{6.11}$$

This is true, both in the non–Σ and the symmetric case.

Proof. The fact that this this is co–associative is a straightforward calculation given that action is free and the $Aut(Z_1)$ and $Aut(Z_2)$ actions on decompositions $X \to Z_1 \to Z_2 \to Y$ commute. The co–unit in the skeletal case is simply $\epsilon_\beta(\phi) = 1$ if $\phi = id_X$ and 0 else. The multiplicativaty in the non–Σ case corresponds to the fact that $Aut(Y \otimes Y') \simeq Aut(Y) \otimes Aut(Y')$.

On $\mathscr{B}/\mathcal{C}$ one calculates:

$$\begin{aligned} \Delta^{red}[(\phi]) &= [\Delta_\beta(\phi)] = \sum_Z \sum_i \beta(\phi_0,\phi_1)[\phi_0^i \otimes \phi_1^i] \\ &= \sum_Z \sum_{i_r} \sum_{\sigma\in Aut(Z)} \frac{1}{|Aut(Z)|}[\phi_0^{i_r} \circ \sigma^{-1}] \otimes [\sigma \circ \phi_1^{i_r}] \\ &= \sum_Z \sum_{i_r} [\phi_0^{i_r}] \otimes [\phi_1^{i_r}] \end{aligned}$$

For the bi–algebra equation in the symmetric case: Inspecting the proof of Corollary 6.10, we get an additional factor of $\frac{1}{|Aut(Y)|}$ for each summand in $\Delta\circ\mu$, while on the other side of the equation the factor is $\frac{1}{|Aut(\hat{Y})||Aut(\hat{Y}')|}$ which cancel with the additional factor of $\frac{|Aut(Y)|}{|Aut(\hat{Y})||Aut(\hat{Y}')|}$ in (6.7). □

6.2.5. **Balanced actions.** More generally, one could define the putative co–cycle $\beta(\phi_1^i,\phi_0^i)=\frac{1}{|Or(\phi_1,\phi_0)|}$ where $Or(\phi_0,\phi_1)$ is the orbit under the $Aut(Z)$ action. If this is indeed a co–cycle then we say that $\mathfrak{F}$ has a *balanced* action by automorphisms. The trivial and free actions are balanced. We conjecture that this is always the case, but leave the analysis for the future.

Proposition 6.13. *If $\mathfrak{F}$ is non–symmetric, skeletal in the above sense, and decomposition finite with balanced actions as above then tuple $(\mathscr{B},\otimes,\Delta_\beta,\eta,\epsilon_\beta)$ is also a bialgebra.*

Proof. The fact that we have an algebra remains unchanged. For the co–algebra, we have to check co–associativity, which is guaranteed by the assumtption that the action is balanced. The bi–algebra equation still holds, since the co–cycle is multiplicative: $\beta(\phi_1\otimes\psi_1,\phi_0\otimes\psi_0)=\beta(\phi_1,\psi_1)\beta(\phi_0,\psi_0)$. This follows from the fact that in the non–Σ case: $Aut(Z\otimes Z')=Aut(Z)\otimes Aut(Z')$. □

Remark 6.14.

(1) Note, this reduced structure is available for the non–skeletal version. Here, for instance in the free action case, one obtains factors $|Iso(Z)||Aut(Z)|$ which again constitutes a multiplicative co–cycle.
(2) In the symmetric case, there is are the additional problems that the bi–algebra equation does not hold and that the co–cycle above is not multiplicative. It turns out that these two deficiencies cancel each other out in the free case. We conjecture that this is true in the balanced case and even in general.
(3) *A priori* It seems that the two bi–algebra structures Δ_β for a balanced action and Δ^{iso} may differ. We conjecture that this is the case for all Feynman categories of crossed type [KW17, §5.2].

6.2.6. **Summary.** Since there are many constructions at work here, we will collect the results for the bialgebras in an overview theorem:

Theorem 6.15. *Fix a composition finite Feynman category $\mathfrak{F}$, let $\mathscr{B}=\mathbb{Z}$ and $\mathscr{B}^{sk}:=\mathscr{B}_{\mathcal{F}_{sk}}$ based on the skeletal version of $\mathcal{F}$. Let $\mathcal{C}$ be the ideal generated by $\sim$ in $\mathscr{B}$ and $\mathcal{C}^{sk}$ the respective ideal in $\mathscr{B}^{sk}$. Set $\mathscr{B}^{iso}=\mathscr{B}/\mathcal{C}$, $\mathscr{B}^{quot}_{\mathbb{Q}}=\mathscr{B}^{iso}\otimes_{\mathbb{Z}}\mathbb{Q}$.*

(1) *Both $\mathscr{B}$ and $\mathscr{B}^{sk}$ are unital algebras with $\otimes$ as product and $id_{\mathbb{1}}$ as the unit. They are Morita equivalent as algebras*
(2) *Both $\mathscr{B}$ and $\mathscr{B}^{sk}$ are co–unital co–algebras with respect to the de–concatenation co–product with co–unit ϵ.*
(3) *If $\mathfrak{F}$ is a non–Σ Feynman category: $\mathscr{B}$ and $\mathscr{B}^{sk}$ are unital, co–unital bialgebras.*
(4) *$\mathscr{B}^{iso} \simeq \mathscr{B}^{sk}/C^{sk}$ as algebras and there is a bi–algebra structure $(\mathscr{B}^{iso}, \otimes, \eta^{iso}, \Delta^{iso}, \epsilon^{iso})$ defined via co–invariants as in Theorem 5.21.*
(5) *If $\mathfrak{F}$ is non–Σ then there is a unital, co–unital quotient bi–algebra $(\mathscr{B}^{quot}_{\mathbb{Q}}, \otimes, \eta^{quot}, \Delta^{quot}, \epsilon^{quot})$ as defined in §6.2.*
(6) *If the action of $Aut(Z)$ on $Hom(X, Z) \times Hom(Z, Y)$ is free for all X, Y, Z, then the twisted $\mathscr{B}$ descends to a bi–algebra $(\mathscr{B}^{quot}_{\mathbb{Q}}, \otimes, \eta^{quot}, \Delta^{red}, \epsilon^{red})$*
(7) *All the structures above are graded by the length of a morphism or the degree of a morphism if there is an integer degree function.*

□

6.3. **Feynman categories, groupoids and de–compositions.** The co–nilpotence of the de–concatenation is related to iterated factorizations, which appear in [KW17, §3.3] in the form of iterated Feynman categories $\mathfrak{F}', \dots \mathfrak{F}^{(n)}, \dots$. The associated maximal subgroupoids $\mathcal{V}^{(n)\otimes}, \dots$ form a simplicial groupoid: objects at level n are factorizations of morphisms into n chains, with the isomorphisms between these chains. In operad theory this type of groupoid explicitly appeared earlier in [GK98] in the context of (twisted) modular operads, cf. also [MSS02].

More explicitly, consider the 'fat nerve' $\mathcal{X} = \mathcal{X}(\mathcal{F})$ of any category $\mathcal{F}$, the simplicial groupoid with $\mathcal{X}_n$ the groupoid of n-chains

$$\alpha_n = (X_0 \to X_1 \to \cdots \to X_n) \text{ in } \mathcal{F}$$

and the isomorphisms between such chains, and $\mathcal{X}_0 = Iso(\mathcal{F})$. The simplicial operator $d_1 : \mathcal{X}_2 \to \mathcal{X}_1$ is composition in $\mathcal{F}$. Its homotopy fiber over an object $\phi : X \to X'$ in $\mathcal{X}_1$ is thus the groupoid $\mathrm{Fact}(\phi)$ of factorizations $\phi \simeq \phi_1 \circ \phi_2$.

In a special situation, one can use the theory of decompositions which was developed after [KW17] and the beginning of this paper, cf. [KW13, §3.3].

In the transition to decomposition spaces, one however looses the simplicity that the co–product was initially just the dual of the composition.

Suppose $\mathcal{F}$ is any Feynman category such that the factorisations of the identity on the monoidal unit form a contractible groupoid. Then it can be shown that in fact $\mathcal{X}(\mathcal{F})$ is a *symmetric monoidal decomposition groupoid* [GCKT15a, §9]. The tensor and unit of $\mathcal{F}$ clearly define $\eta : * \to \mathcal{X}$, $\mu : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$, but it is the key hereditary condition of a Feynman category that shows that tensor and composition are compatible: they form a homotopy pullback square

$$\begin{tikzcd}
\mathrm{Fact}(\phi) \times \mathrm{Fact}(\phi') \arrow[r] \arrow[d, "\otimes"', "\simeq"] & \mathcal{X}_2 \times \mathcal{X}_2 \arrow[r, "\circ\times\circ"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{X}_1 \times \mathcal{X}_1 \arrow[d, "\otimes"] & \ni (\phi, \phi') \\
\mathrm{Fact}(\phi \otimes \phi') \arrow[r] & \mathcal{X}_2 \arrow[r, "\circ"'] & \mathcal{X}_1 & \ni \phi \otimes \phi',
\end{tikzcd}$$

for all $\phi : X \to Y$ and $\phi' : X' \to Y'$, that is, $\otimes : \mathrm{Fact}(\phi) \times \mathrm{Fact}(\phi') \to \mathrm{Fact}(\phi \otimes \phi')$ is a groupoid equivalence.

From [GCKT15a, Theorem 7.2 and §9] we see that $\mathcal{X}(\mathcal{F})$ induces a bialgebra in the symmetric monoidal category of comma categories of groupoids and linear functors between them, and in [GCKT15b] the finiteness conditions necessary and sufficient to pass to bialgebras in the category of $\mathbb{Q}$-vector spaces are studied.

7. Constructions and Examples

The main examples are already directly accessible via the formalism above. However, more context is provided, by using several universal constructions on Feynman categories from [KW17].

We will go through the examples starting with the basic ones, which contain the three main examples, and then introduce further complexity to provide better insight and further examples.

7.1. **Examples with trivial $\mathcal{V}$ a.k.a. Operads and the three main examples.** Let $\mathcal{V} = \underline{*}$ be the trivial category with one object $*$ and its identity morphism id_*. In the *non–symmetric case*, there is an equivalence $\mathcal{V}^{\otimes} \simeq \mathbf{N}_0$ with the discrete category whose objects are the natural numbers, with n representing $*^{\otimes n}$. The monoidal structure is given by addition. Here $0 = *^{\otimes 0} = \varnothing$. In the *symmetric monoidal* case there is an equivalence $\mathcal{V}^{\otimes} = \mathbb{S}$, which has again has the natural numbers as objects, but with $Hom_{\mathbb{S}}(n, m) = \varnothing$ for $n \neq m$ and $Hom_{\mathbb{S}}(n, n) = \mathbb{S}_n$, the symmetric group. This category is sometimes also denoted by Σ and it is the skeleton of $Iso(\mathcal{F}in\mathcal{S}et)$, where $\mathcal{F}in\mathcal{S}et$ is the category of finite sets with set maps. For more details, see [Kau17], especially §2.4 of *loc. cit.*.

Consider a strict Feynman category $\mathfrak{F} = (\underline{*}, \mathcal{F}, \imath)$ with $Obj(\mathcal{F}) = \mathbb{N}_0$. The monoidal unit is $\mathbb{1} = 0$. The basic morphisms will be

$\mathcal{F}(n,1) := \mathcal{O}(n)$. Since $\mathrm{Hom}_{\mathcal{F}}(n,n) = \mathbb{S}_n$, the collection $\mathcal{O}(n)$ has an action of $\mathbb{S}_n$ in the symmetric case. By the hereditary condition (ii) $\mathrm{Hom}_{\mathcal{F}}(n,k) = \amalg_{(n_1,\dots,n_k:\sum n_i=n}\mathrm{Hom}_{\mathcal{F}}(n_1,1)\amalg\cdots\amalg\mathrm{Hom}_{\mathcal{F}}(n_k,1) = \amalg_{(n_1,\dots,n_k:\sum n_i=n}\mathcal{O}(n_1)\amalg\cdots\amalg\mathcal{O}(n_k)$ and the composition is a morphism $\circ : \mathrm{Hom}_{\mathcal{F}}(k,1)\times\mathrm{Hom}_{\mathcal{F}}(n,k)\to\mathrm{Hom}_{\mathcal{F}}(n,1)$ will be given by

$$\circ : \mathcal{O}(k)\times(\amalg_{(n_1,\dots,n_k:\sum n_i=n}\mathcal{O}(n_1)\amalg\cdots\amalg\mathcal{O}(n_k))\to\mathcal{O}(n) \tag{7.1}$$

or in components by

$$\gamma_{k;n_1,\dots,n_k} : \mathcal{O}(k)\times(\mathcal{O}(n_1)\amalg\cdots\amalg\mathcal{O}(n_k))\to\mathcal{O}(n) \tag{7.2}$$

The fact that $\circ$ is associative and the properties of id_1 implies that the γ give the collection $\mathcal{O}(n)$ the structure of an operad with unit $u = id_1$.

Furthermore, because of axiom (i), we see that $Aut(1) = id_1$, so that $\mathcal{O}(1)$ only has id_1 as an invertible element. In principle, there can be morphisms in $\mathrm{Hom}_{\mathcal{F}}(0,1) = \mathcal{O}(0)$. The length of a morphisms in $\mathrm{Hom}_{\mathcal{F}}(n,k) = n-k$.

This recovers §2.3 for the duals of operads in $\mathcal{S}et$, which contains the examples of rooted trees. For operads in other categories, see §7.8.1.

Proposition 7.1. *The strict Feynman categories whose underlying $\mathcal{V}$ is trivial are in 1–1 correspondence with set–operads, whose $\mathcal{O}(1)$ splits as $\mathcal{O}(1) = id_1 \amalg \mathcal{O}(1)^{red}$ where no element in $\mathcal{O}^{red}(1)$ is invertible. They are non–negative with respect to length, if $\mathcal{O}(0) = \varnothing$ and are non–positive w.r.t. length, if $\mathcal{O}(i) = \varnothing$ for $i > 0$.*

The construction of bi–algebras and conditions for Hopf algebras coincide in both formulations under this translation.

□

7.2. **Connes–Kreimer tree algebras.** Let $\mathfrak{F}_{CK}$ be the Feynman category with trivial $\mathcal{V}$, $\mathcal{F}$ having objects $\mathbb{N}_0$ and morphisms given by rooted forests: $Hom(\underline{n},\underline{m})$ is the set of n-labelled rooted forests with m roots. The composition is given by gluing the roots to the leaves. This is the twist of $\mathfrak{F}\mathfrak{S}$ by the operad of leaf-labelled rooted trees, see 7.8.1.

In the non-Σ version, one uses planar forests/trees and omits labels or equivalently uses orders on the sets of labels. Here this is the twist by the non–sigma operad of planar forests of $\mathfrak{F}\mathfrak{S}_<$.

Here there is non–trivial $\mathcal{O}(1)$. This is basically the difference of the + and the *hyp* construction, see §7.8.1. The grading $n-p$ is the native grading and the coradical length is the word length of a morphism and is given by the number of vertices.

7.2.1. **Leaf labelled and planar version of Connes–Kreimer.** We now give complete details. Let $\mathcal{O}$ be the operad of leaf labelled rooted trees or planar planted trees. Here $\mathcal{O}(1)$ has two generators id_1 which we denote by $|$ and ϕ, the rooted tree with one binary non–root vertex. Now composing ϕ with itself will result in ϕn, the rooted tree with n binary non–root vertices, aka. a ladder. We also identify $\phi 0 = |$. Taking the dual, either as the free Abelian group of morphisms, or simply the dual as a co–operad, we obtain a co–operad and the multiplication is either $\otimes$ from the Feynman category or $\otimes$ from the free construction. That these two coincide follows from condition (ii) of a Feynman category. η is given by $| = id_1$. The Feynman category and the co–operad are almost connected, since $\Delta(\phi n) = \sum_{(n_1,n_2):n_1,n_1\geq 0,n_1+n_2=n} \phi n_1 \otimes \phi n_2$ and hence the reduced co-product is given by $\bar{\Delta}(\phi n) = \sum_{(n_1,n_2):n_1,n_1\geq 1,n_1+n_2=n} \phi n_1 \otimes \phi n_2$ whence $\check{\mathcal{O}}(1)$ is nilpotent.

If we take planar trees, there are no automorphisms and we obtain the first Hopf algebra of planted planar labelled forests. Notice that in the quotient $[|] = [||\dots|] = [1]$ which says that there is only one empty forest.

If we are in the non–planar case, we obtain a Hopf algebra of rooted forests, with labelled leaves. One uses $\mathcal{V}$ as finite subsets of $\mathbb{N}$ with isomorphisms.

These structures are also discussed in [Foi02b], [Foi02a] and [EFK05].

7.2.2. **Algebra over the operad description for Connes–Kreimer.** If one considers the algebras over the operad $\mathcal{O}$, then for a given algebra (ρ, V), $\rho(\phi) \in Hom(V,V)$ is a "marked" endomorphism. This is the basis of the constructions of [Moe01]. One can also add more extra morphisms, say ϕc for $c \in C$ where C is some indexing set of colors. This was considered in [vdLM06b]. In general one can include such marked morphisms into Feynman categories (see [KW17][2.7]) as morphisms of $\varnothing \to *_{[1]}$.

7.2.3. **Unlabelled and symmetric version.** In the non–planar case, we have the action of the symmetric groups as $Aut(n)$. The bi–algebra on the co–invariants and the Hopf quotient of Theorem 5.21 and yields the same results as the constructions §2 in the symmetric case. The result is the commutative Hopf algebra of rooted forests with unlabelled tails.

The action of the automorphisms is free and hence there is also the reduced version of the co– and Hopf algebras.

7.2.4. **No tail version.** For this particular operad, there is the construction of forgetting tails and we can use the construction of §2.10.

In this case, we obtain the Hopf algebras of planted planar forests without tails or the commutative Hopf algebra of rooted forests, which is called $\mathscr{H}_{CK}$. On the Feynman category level, this construction is done using universal operations of §7.9 applied to the decorated Feynman categories, see §7.5, $\mathfrak{FS}_{dec\,\mathcal{O}}$ and $\mathfrak{FS}_{<,dec\,\mathcal{O}}$ for the (non–Σ) operad of leaf labelled trees.

7.3. **Colored operads and their dual co–operads.** Colored operads are partial operads, where the compositions are allowed if the colors match. More precisely, fix a set of colors C then, a colored operad is a collection $\mathcal{O}(c_1, \dots, c_n, c)$ with $c, c_i \in C$ and there is a composition $\gamma : \mathcal{O}(c_1, \dots, c_n, c) \otimes \mathcal{O}(c_1^1, \dots, c_n^1, c_1) \otimes \dots \otimes \mathcal{O}(c_1^n, \dots, c_n^n, c_n) \to \mathcal{O}(c_1^1, \dots, c_n^1, \dots, c_1^n, \dots, c_n^n, c_n, c)$.

Remark 7.2. The dual of a colored operad is a co–operad. Indeed, one only decomposes into factors that are *a priori* composable.

In the Feynman category terms, cf. [KW17, §2.5], these are $\mathcal{O}ps$ for a Feynman category whose vertices are rooted corollas together with a morphisms of the flags to C. This is technically a decoration, see §7.5. One then restricts to those morphisms whose underlying ghost graphs have the property that both flags of any ghost edge have the same color, see §7.4. Coloring is a form of decoration and restriction as discussed in [KL16, §6.4]. Such a colored operad also furnishes an (enriched) Feynman category whose vertices are $c \in C$ and whose basic morphisms are given by the $\mathcal{O}(c_1, \dots, c_n; c) : \amalg_{i=1}^n c_i \to c$. The c_i are called input colors and c is the output color.

Proposition 7.3. *The strict Feynman categories based on colored operads as above are are non–negative with respect to length, if $\mathcal{O}(\varnothing, c) = \varnothing$ and are non–positive w.r.t. length, if $\mathcal{O}(c_1, \dots, c_n, c) = \varnothing$ for $n > 0$.*

The construction of bi–algebras and conditions for Hopf algebras coincide in both formulations under this translation to the bi–algebras and Hopf algebras obtained from the dual co–operads.

□

This includes the examples of Goncharov and Baues in their form §2.24.

Remark 7.4. If the co–operads are not in $\mathcal{S}et$ the construction and statement are analogous, see §7.8.1 below.

7.3.1. **Bi– and Hopf algebras from categories, sequences and Goncharov's Hopf algebra.**

Proposition 7.5. *Every category defines a colored operad and thus we obtain an associated bi–algebra and possibly a Hopf algebra from any category.*

This recovers the Hopf algebra of Goncharov's and Baues' construction when considering a complete groupoid.

Proof. Consider $X_n = N_n(\mathcal{C})$ the simplicial object given by the nerve of a category. Let $C = N_1(\mathcal{C}) = Mor(\mathcal{C})$ be the set of colors. Then there is a colored operad defined by $\mathcal{O}(\phi_1, \dots, \phi_n, \phi) = \{X_0 \stackrel{\phi_1}{\to} \cdots \stackrel{\phi_n}{\to} X_n \in N_n(\mathcal{C}) : \phi = \phi_n \circ \cdots \circ \phi_n\}$. If $X_0 \stackrel{\phi_1}{\to} \cdots \stackrel{\phi_n}{\to} X_n$ is an n simplex and $X_{i-1} = Y_0 \stackrel{\psi_1}{\to} \cdots \stackrel{\psi_m}{\to} Y_m = X_i$ is an m simplex, with $\psi_m \circ \cdots \circ \psi_1 = \phi_i$, then we can compose to

$$X_0 \stackrel{\phi_1}{\to} \cdots \stackrel{\phi_{i-1}}{\to} X_{i-1} = Y_0 \stackrel{\psi_1}{\to} \cdots \stackrel{\psi_m}{\to} Y_m = X_{i+1} \stackrel{\phi_{i+1}}{\to} \cdots \stackrel{\phi_n}{\to} X_n$$

.

If the underlying category is a complete groupoid, so that there is exactly one morphism per pair of objects, then any n–simplex can simply be replaced by the word $X_0 \cdots X_n$ of its sources and targets. ☐

Notice that in the complete groupoid case $\mathcal{V} = \{X_0X_1\}$ is the set of words of length 2 not 1. This explains the constructions of Goncharov involving multiple zeta values, but also polylogarithms [Gon05], and the subsequent construction of Brown. This matches our discussion in §4 and §7.5.1.

7.3.2. **Marking angles by morphisms.** Considering the simplicial object given by the nerve of a category $N_\bullet(\mathcal{C})$ yields a particularly nice example of the duality between marking angles vs. marking tails. An n–simplex $X_0 \stackrel{\phi_1}{\to} X_1 \cdots \stackrel{\phi_n}{\to} X_n$ naturally gives rise to a decorated corolla, where the angles are decorated by the objects and the leaves are decorated by the morphisms, viz. the colors, see Figure 11. The operation that the corolla represents is the the composition of all of the morphisms to get a morphism $\phi = \phi_n \circ \cdots \circ \phi_0 : X_0 \to X_n$, viz. the output color. If there is a single morphism between any two objects either one of the markings, tail or angle, will suffice to give a simplex. In the general case, one actually needs both the markings. The angle/tail duality is related to Joyal duality defined in Appendix C.1; see also §7.6 below.

7.4. **Graph examples.** The basic graph Feynman category is $\mathfrak{G} = (\mathcal{C}rl, \mathcal{A}gg, \imath)$, defined in detail in [KW17, §2.1], see also Appendix A. The notion of graph that is used is that of [BM08]. The BM–graphs

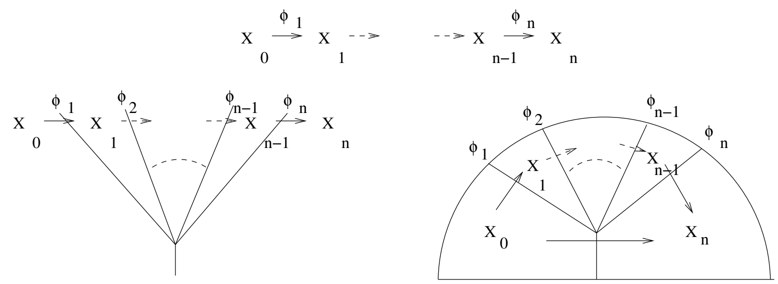

FIGURE 11. Marking a corolla by a simplex in $N_\bullet(\mathcal{C})$. The morphisms decorate the ends of the tree, while the objects decorate the angles which correspond to the marks on the half circle

from a category, and $\mathcal{A}gg$ is the full subcategory whose objects are aggregates of corollas. A corolla is a graph with one vertex and no edges, and an aggregate is a disjoint union of these. $\mathcal{C}rl$ is the groupoid of corollas and their isomorphisms, and $\imath$ is inclusion. To each BM–morphisms $\phi : X \to Y$ between two aggregates X and Y, one can associate a ghost graph $\mathbb{\Gamma}(\phi)$, see Appendix §A.1.4. A morphism ϕ is roughly a graph $\mathbb{\Gamma}(\phi)$, together with an identification of the vertices of $\mathbb{\Gamma}(\phi)$ with the source aggregate and an identification of $\mathbb{\Gamma}(\phi)/E_{\mathbb{\Gamma}(\phi)}$ with the aggregate, see [KW17, §2.1] and the appendix for details. Different varieties of graph based Feynman categories are then given by restricting or decorating graphs in a manner respected by composition (see the appendix and the examples in §5). A first new example is that of collections of 1-PI graphs, which we call the Broadhurst–Connes–Kreimer Feynman category.

Without going into all the details, we wish to note the following facts, cf. [KW17, §2.1, §5 and Appendix A].

(1) The morphisms of $\mathcal{A}gg$ are generated by (a) isomorphisms, (b) simple edge contractions, (c) simple loop contractions, (d) simple mergers.

A simple edge contraction glues two flags from two different corollas together to form an edge and then contracts the edge leaving a corolla. A simple loop contraction does the same with the exception that the two flags come from the same corolla. A simple merger identifies two distinct corollas by identifying their vertices and keeping all flags. The ghost graph keeps track of which flags have been glued together to form edges that are subsequently contracted.

(2) The subcategory generated by only the first three classes defines the wide subcategory $\mathcal{A}gg^{ctd}$ of $\mathcal{A}gg$ and the Feynman category $\mathfrak{G}^{ctd} = (\mathcal{C}rl, \mathcal{A}gg^{ctd}, \imath)$. The ghost graphs of morphisms in $(\mathcal{A}gg \downarrow \mathcal{C}rl)$ are connected.

(3) A ghost graph does not define a morphism uniquely, but the isomorphisms class $[\phi]$ for $\phi \in \mathcal{A}gg^{ctd}$ is fixed by the ghost graph $\mathbb{\Gamma}(\phi)$. In $\mathcal{A}gg$ the same is true for the morphisms in $(\mathcal{A}gg \downarrow \mathcal{C}rl)$. The ghost graph also fixes the source of a morphism and the target up to isomorphism.

(4) Composition of morphisms corresponds to graph insertion. In particular in $\mathcal{A}gg^{ctd}$, $\mathbb{\Gamma}(\phi \circ \psi) = \mathbb{\Gamma}(\phi) \circ \mathbb{\Gamma}(\psi)$ where $\mathbb{\Gamma}(\phi)$ has connected components corresponding to the vertices of $\mathbb{\Gamma}(\psi)$: $\mathbb{\Gamma}(\phi) = \amalg_{v \in V(\mathbb{\Gamma}(\psi))} \mathbb{\Gamma}_v(\phi)$. The insertion inserts $\mathbb{\Gamma}_v(\phi)$ into the vertex v of $\mathbb{\Gamma}(\psi)$ – using extra data provided by the morphisms to identify the flags aka. half–edges adjacent to v with the tails aka. external legs of $\mathbb{\Gamma}_v(\phi)$. An example is given in Figure 12.

(5) If $\phi = \phi_0 \circ \phi_1$ then (a) $\mathbb{\Gamma}(\phi) = \mathbb{\Gamma}(\phi_0) \circ \mathbb{\Gamma}(\phi_1)$ as above, but also (b) $\mathbb{\Gamma}(\phi_1) \subset \mathbb{\Gamma}(\phi)$ is (not necessarily connected) subgraph and $\mathbb{\Gamma}(\phi_0) \simeq \mathbb{\Gamma}(\phi)/\mathbb{\Gamma}(\phi_1)$. The corresponding factorization of a morphism in $(\mathcal{F} \downarrow \mathcal{V})$ is

$$\begin{tikzcd} X \arrow[r, "\phi{,}\mathbb{\Gamma}(\phi)"] \arrow[d, "\phi_1{,}\mathbb{\Gamma}(\phi_1)"'] & * \\ Y \arrow[ur, "\phi_0{,}\mathbb{\Gamma}(\phi_0)=\mathbb{\Gamma}(\phi)/\mathbb{\Gamma}(\phi_1"'] & \end{tikzcd} \tag{7.3}$$

where $\mathbb{\Gamma}(\phi_1)a$ is a subgraph, $\mathbb{\Gamma}(\phi)/\mathbb{\Gamma}(\phi_0)$ is sometimes called the co–graph and $*$ is the residue in the physics nomenclature.

Lemma 7.6. *In $\mathcal{A}gg^{ctd}$ the action of $Aut(Y)$ on $Hom(X, Y)$ is free.*

Proof. We use the terminology and formalism of Appendix A. A morphism is given by $\phi = (\phi_V, \phi^F, \imath_\phi)$ the action of $\sigma = (\sigma_V, \sigma^F, id)$ with both σ^F and σ_V bijections. Now $(\sigma \circ \phi)^F = \phi^F \circ \sigma^F$, which already implies the result as σ^F is an injection. □

Corollary 7.7. *In $\mathcal{A}gg^{ctd}$ the action on the middle space is a free action on the decompositions.*

Proposition 7.8. *On isomorphism classes $\mathbb{\Gamma}$ in $\mathcal{A}gg^{ctd}$.*

$$\Delta^{iso}(\mathbb{\Gamma}) = \sum_{\mathbb{\Gamma}_1 \subset \mathbb{\Gamma}} \mathbb{\Gamma}/\mathbb{\Gamma}_1 \otimes \mathbb{\Gamma}_1 = \sum_{\mathbb{\Gamma}_1 \subset \mathbb{\Gamma}} \mathbb{\Gamma}_0 \otimes \mathbb{\Gamma}_1 \tag{7.4}$$

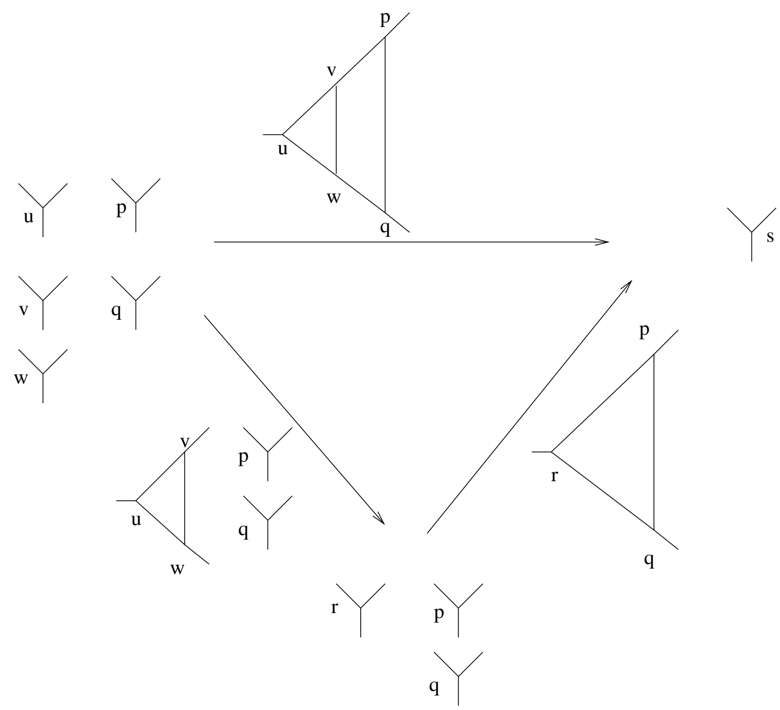

FIGURE 12. An example of a factorization in three–valent graphs aka. ϕ^3. Alternatively the top graph $\mathbb{\Gamma}$ results from inserting the left graph $\mathbb{\Gamma}_1$ into the right graph according $\mathbb{\Gamma}_0$ to the vertices, viz. $\mathbb{\Gamma} = \mathbb{\Gamma}_0 \circ \mathbb{\Gamma}_1$, or the left graph is a subgraph of the top graph $\mathbb{\Gamma}_1 \subset \mathbb{\Gamma}$ and the right graph $\mathbb{\Gamma}_0$ is the quotient graph. $\mathbb{\Gamma}_0 = \mathbb{\Gamma}/\mathbb{\Gamma}_1$

Here $\mathbb{\Gamma}$ is the isomorphism class $\mathbb{\Gamma} = [\phi] = \mathbb{\Gamma}(\phi)$ and $\mathbb{\Gamma}_1 = \mathbb{\Gamma}(\phi_1)$ is a subgraph, which corresponds to the isomorphism class of a decomposition $[(\phi_0, \phi_1)]$ where then necessarily $\mathbb{\Gamma}(\phi_0) = \mathbb{\Gamma}(\phi)/\mathbb{\Gamma}_1$. Moreover if $\mathbb{\Gamma}$ is connected, so is $\mathbb{\Gamma}_0$. — both are isomorphism classes in $(\mathcal{A}gg^{ctd} \downarrow \mathcal{C}rl)$.

Proof. Given ϕ its isomorphism type is fixed by $\mathbb{\Gamma}(\phi)$. We can choose a representative for ϕ. The claim is then, that the factorizations up to the action on the middle space are given precisely by the subgraphs. Indeed, given any subgraph, there is surely a factorization. We have to show that there is exactly one term per sub–graph For this, we "enumerate everything". That is the flags, vertices, ghost edges etc. to fix the morphism. For a given subgraph there is a putative morphism, whose source is fixed and whose target is fixed up to isomorphism. This ambiguity is exactly compensated by the action on the middle space. This actions is free, on the decompositions and does not change

FIGURE 13. The coproduct of a graph. The factor of 2 is there, since there are two distinct subgraphs —given by the two distinct edges— which give rise to two factorizations whose abstract graphs coincide

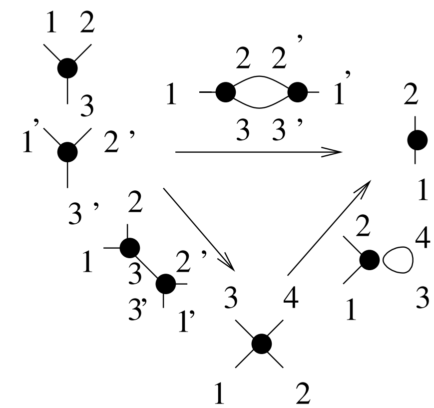

FIGURE 14. One decomposition. To fix ϕ we specify $\phi^F(1) = 1, \phi^F(2) = 1'$, to fix ϕ_1, we set $\phi_1^F(1) = 1, \phi_1^F(2) = 1, \phi_1^F(3) = 1', \phi_1^F(4) = 2'$ and to fix ϕ_0 we fix $\phi_0^F(1) = 1, \phi_0^F(2) = 2$. There is no choice for the vertex maps and the involution is the one given by the ghost graph.

the subgraph and hence every subgraph appears exactly once in the sum. □

Note that the multiplicities of the graphs appearing on the right side can be higher than one as the same graph may appear in several ways yielding different subgraphs, but isomorphic quotient graphs.

Example 7.9. We consider the morphism of Figure 13. Each edge leads to a factorization. One such factorization is given in Figure 14. If we write $\phi = \phi_0 \circ \phi_1$, we note that $im(\phi_1^F) = \{1, 1', 2, 2'\}$. If $(\hat{\phi}_0, \hat{\phi}_1)$ is the decomposition with respect to the other edge $\{2, 2'\}$, then $im(\phi_1^F) = \{1, 1', 3, 3'\}$ and since this invariant under the $Aut*_{1,2,3,4}$ action . Thus $(\hat{\phi}_0, \hat{\phi}_1)$ (ϕ_1, ϕ_0) this is not equivalent under this action. But the abstract one edge graphs are the same. $\mathbb{\Gamma}(\hat{\phi}_i) = \mathbb{\Gamma}(\phi_i) : i = 0, 1$. To be clear, different subgraphs, same underlying graph.

7.4.1. **Graph based Feynman categories and Connes–Kreimer Hopf algebras.** If we look at the Feynman category $\mathfrak{G} = (\mathcal{C}rl, \mathcal{A}gg, \imath)$

then, we obtain the core Hopf algebra of graphs of Connes and Kreimer [CK98]. The standard "refined" grading is as follows. Usually there will be no mergers involved, and edge contractions and loop contractions are assigned degree 1. The co-radical grading is by word length in the elementary morphisms, that is the grading above, which coincides with the number of edges.

There are several restrictions and decoration that one can put on the graphs to obtain sub–categories indexed over the category $\mathfrak{G}$. Here indexing means that there is a functor surjective on objects, cf. [KW17, §1.2.7]. Decoration is used in the technical sense described below §7.5; see [KL16, §6.4] for standard decorations of graphs.

The key thing is that the extra structures and restrictions respect the concatenation of morphisms, which boils down to plugging graphs into vertices. Examples of this type furnish bi– and Hopf algebras of of modular graphs, non–Σ modular graphs, trees, planar trees, etc..

7.4.2. **1–PI graph version.** A not so standard example, at least for mathematicians, are 1–PI graphs. Recall that a connected 1–PI graph is a connected graph that stays connected, when one severs any edge and in general a 1–PI graph is a graph whose every component is 1–PI. A nice way to write this is as follows [Bro17]. Let $b_1(\Gamma)$ be the first Betti number of the graph Γ. Then a graph is 1–PI if for any proper subgraph $\gamma \subsetneq \Gamma$: $b_1(\gamma) < b_1(\Gamma)$. This means that 1-PI for non-connected graphs any edge cut decreases the first Betti (or loop) number by one.

It is easy to see that the property of being 1–PI is preserved under composition in $\mathfrak{G}$, namely, blowing up a vertex of a 1-PI graphs into a 1-PI graph leaves the defining property (namely connectivity) invariant. Hence, we obtain a bi–algebra of 1–PI graphs. It is almost connected and after amputation, one obtains the Hopf algebra used in physics.

A decorated version of this is Brown's Hopf algebra of motic graphs, see below §7.5.1.

7.5. **Decoration: $\mathfrak{F}_{dec\mathcal{O}}$.** This type of modification was defined in [KL16] and further analyzed in the set–based case in [BK19]. It gives a new Feynman category $\mathfrak{F}_{dec\mathcal{O}}$ from a pair $(\mathfrak{F}, \mathcal{O})$ of a Feynman category $\mathfrak{F}$ and a strong monoidal functor $\mathcal{O} : \mathcal{F} \to \mathcal{C}$. The objects of $\mathfrak{F}_{dec\mathcal{O}}$ are pairs $(X, a_X), a_X \in \mathcal{O}(X)$ $(a_X \in Hom_{\mathcal{E}}(\mathbb{1}, \mathcal{O}(X))$ in the general enriched case). The morphisms from (X, a_X) to (Y, a_Y) are those $\phi \in Hom_{\mathcal{F}}(X, Y)$ for which $\mathcal{O}(\phi)(a_X) = a_Y$. For a morphism ϕ, we let $s(\phi)$ and $t(\phi)$ be the source and target of ϕ .

Lemma 7.10. *The morphism of* $\mathfrak{F}_{dec\mathcal{O}}$ *are pairs* $(\phi, a_{s(\phi)}), a_{s(\phi)} \in \mathcal{O}(s(\phi))$. *If* $\mathfrak{F}$ *is decomposition finite, then so is* $\mathfrak{F}_{dec\mathcal{O}}$. *If* $\mathfrak{F}$ *is Hopf, then so is* $\mathfrak{F}_{dec\mathcal{O}}$.

Proof. By descriptions, any morphism $(X, a_X) \to (Y, a_Y)$ is a lift of a morphism $\phi : X \to Y$. Such a lift exists if $a_Y = \mathcal{O}(a_X)$. Thus fixing $\phi : X \to Y$ and $a_X \in \mathcal{O}(X)$, there is a unique morphism $(\phi, a_X) : (X, a_X) \to (Y, \mathcal{O}(\phi(a_X))$ and these are all the morphisms. Since the source and ϕ fix the target:

$$\Delta((\phi, a_X) = \sum_{(\phi_0,\phi_1):\phi=\phi_0\circ\phi_1} (\phi_0, \mathcal{O}(\phi_1(a_X)) \otimes (\phi_1, a_X) \tag{7.5}$$

This equation also shows that the Hopf property is preserved. □

7.5.1. **Brown's motic Hopf algebras.** In [Bro17] a generalization of 1–PI graphs is given. In this case there are the decorations of (ghost) edges of the morphisms by masses and the momenta; that is, maps $m : E(\Gamma) \to \mathbb{R}$ and $q : T(\Gamma) \to \mathbb{R}^d \cup \{\varnothing\}$. Notice that these are decorations in the technical sense of [KL16] as well. For this, we set look at the decoration operad $\mathcal{O}(*_S) = \{S \mapsto \mathbb{R}^d \amalg \mathbb{R}\}$, so that each flag is either decorated by a momentum, or a mass. As a functor, under edge/loop contractions, we just forget the decoration on the flags. This gives a decoration of all the flags of the ghost graph. It is not the end result, but we further to restrict to those morphisms whose ghost graphs have the *same* decoration for any two flags that make up a ghost edge. This is the standard procedure, cf. [KL16, §6.4]. This results in the ghost edges being decorated by masses. The masses carry over onto the new edges upon insertion. Note that the flags that carry momenta are never glued

A subgraph γ of a graph Γ is called momentum and mass spanning (m.m.) if it contains all the tails and all the edges with non–zero mass. This means that as a ghost graph its target has corollas, whose flags are labelled with 0 mass except possibly one corolla whose flags are labelled with all the external momenta. A graph Γ is called motic if for any m.m. subgraph γ: $b_1(\gamma) < b_1(\Gamma)$. This condition invented by Brown generalizes 1–PI. It is again stable under composition, i.e. gluing graphs into vertices as can be readily verified, see [Bro17, Theorem 3.6].

After taking the quotient and amputating all tails marked by momenta, we see that the one vertex ghost graph becomes identified with the empty graph and we obtain the Hopf algebra structure of [Bro17, Theorem 4.2].

7.6. **Simplicial structures and Feynman categories.** In this section, we consolidate and expand the construction of §4 in the setting of Feynman categories.

7.6.1. **The Feynman category $\mathfrak{Fin}\mathfrak{S}et$ and variations.** The basic non–trivial Feynman category with trivial $\mathcal{V}$, is $\mathfrak{Fin}\mathfrak{S}et = (\underline{*}, \mathcal{F}in\mathcal{S}et, \imath)$ where $\mathcal{F}in\mathcal{S}et$, the category of finite sets and set maps with monoidal structure given by the disjoint union $\amalg$. The functor $\imath$ is given by sending $*$ to the atom $\{*\}$. The equivalence between $\mathbb{S}$ and $Iso(\mathcal{F}in\mathcal{S}et)$ is clear as $\mathbb{S}$ is the skeleton of $Iso(FinSet)$. Condition (iii) holds as well. Given any morphisms $S \to T$ between finite sets, we can decompose it using fibers as.

$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[d, "="'] & & T \arrow[d, "="] \\ \amalg_{t\in T} f^{-1}(t) \arrow[rr, "\amalg f_t"] & & \amalg_{t\in T}\{*\} \end{tikzcd} \tag{7.6}$$

where f_t is the unique map $f^{-1}(t) \to \{*\}$. Note that this map exists even if $f^{-1}(t) = \varnothing$. This shows the condition (ii), since any isomorphisms of this decomposition must preserve the fibers. The skeleton of Feynman category is the strict Feynman cagtegory $(\underline{*}, \Delta_+S, \imath)$, where Δ_+S is the augmented simplicial category and $\imath(*) = [0])$.

$\mathfrak{Fin}\mathfrak{S}et$ has the Feynman subcategories $\mathfrak{F}\mathfrak{S} = (\underline{*}, FS, \imath)$ and $\mathfrak{F}\mathfrak{I} = (\underline{*}, FI, \imath)$, where the maps are restricted to be surjections resp. injections. This means that none of the fibers are empty or all of the fibers are empty.

In the non–Σ case, a basic example is $\mathfrak{Fin}\mathfrak{S}et_< = (\underline{*}, \mathcal{F}in\mathcal{S}et_<, \imath)$, where $\mathcal{F}in\mathcal{S}et_<$ is the category of ordered finite sets with order preserving maps has as $\mathcal{F}$ the category of and with $\amalg$ as monoidal structure. The order of $S \amalg T$ is lexicographic, S before T. The functor $\imath$ is given by sending $*$ to the atom $\{*\}$. Viewing an order on S as a bijection to $\{1, \dots, |S|\}$, we see that $\mathbf{N}_0$ is the skeleton of $Iso(\mathcal{F}in\mathcal{S}et_<)$. The diagram (7.6) translates to this situation and we obtain a non–Σ Feynman category. The skeleton of Feynman category is the strict Feynman cagtegory $(\underline{*}, \Delta_+, \imath)$, where Δ is the augmented simplicial category and $\imath(*) = [0])$. Restricting to order preserving surjections and injections, we obtain the Feynman subcategories $\mathfrak{F}\mathfrak{S}_< = (\underline{*}, OS, \imath)$ and $\mathfrak{F}\mathfrak{I}_< = (\underline{*}, OI, \imath)$. We can also restrict the skeleton of $\mathcal{F}in\mathcal{S}et_<$ given by Δ_+ and the subcategory of order preserving surjections and injections. See Tables 1 and 2. In Δ_+ the image of $*^{\otimes n}$ under $\imath^{\otimes}$ will be the set $\underline{n}$ with its natural order.

$\mathfrak{F}$	$\mathcal{F}$	definition
$\mathfrak{FinSet}$	$\mathcal{F}in\mathcal{S}et$	Finite sets and set maps
$\mathfrak{FS}$	$\mathcal{S}urj$	Finite sets and surjections
$\mathfrak{FI}$	$\mathcal{I}nj$	Finite sets and injections

TABLE 1. Set based Feynman categories Feynman categories. $\mathcal{V} = \underline{*}$ is trivial.

non-Σ $\mathfrak{F}$	$\mathcal{F}$	definition
$\mathfrak{FinSet}_<$	$\mathcal{F}in\mathcal{S}et_<$	Finite sets and order preserving maps.
$\mathfrak{FS}_<$,	OS	Ordered finite sets and ordered preserving surjections
$\mathfrak{FI}_<$	OI	Ordered finite sets and order preserving injections
Δ_+	Δ_+	Augmented Simplicial category, Skeleton of $\mathcal{F}in\mathcal{S}et_<$
$\mathfrak{FI}_{*,*}$	$OI_{*,*}$	Subcategory of Δ_+ of double base–point preserving injections

TABLE 2. Set based non-Σ Feynman categories. $\mathcal{V} = \underline{*}$ is trivial.

Example 7.11 (Bi– and Hopf–algebra structures)**.** $\mathcal{F}in\mathcal{S}et$ and $\mathcal{F}in\mathcal{S}et_<$ are not decomposition finite, but the restrictions to injections and surjections in the skeletal version are. The bi–algebra structure on surjections is as follows: the basic morphisms are surjections $\pi_n : n \twoheadrightarrow 1$ which can be alternatively viewed as corollas with n inputs. In the non–sigma case, $\mathcal{V}$ is discrete and $\mathscr{B} = \mathscr{B}^{iso}$. We get

$$\Delta(\pi_n) = \sum_{1\leq k\leq n, f:(n,<)\twoheadrightarrow(k,<)} \pi_k \otimes f = \sum_{1\leq k\leq n,(n_1,\dots,n_k):n_1\geq 1,\sum n_i=n} \pi_k\otimes(\pi_{n_1}\otimes\cdots\otimes\pi_{n_k}) \tag{7.7}$$

since an order preserving surjection is uniquely determined by the cardinalities of its ordered set of fibers. In the Hopf algebra, we get

$$\Delta^{\mathscr{H}}(\pi_n) = \pi_n\otimes 1 + 1\otimes\pi_n + \sum_{1<k<n,(n_1,\dots,n_k):n_i>1,1<\sum n_i<n} \pi_k\otimes(\pi_{n_1}\otimes\cdots\otimes\pi_{n_k}) \tag{7.8}$$

as in the quotient $[id_1] = [1 \twoheadrightarrow 1] = 1$ as well as its products. This is the answer for the example of corollas Example 2.54.

For the case of $\mathfrak{FS}$, we can use a skeleton for the isomorphism classes. The bi–algebra is then

$$\Delta([\pi_n]) = \sum_{1\leq k\leq n,[f]:f:\twoheadrightarrow(k,<)} [\pi_k]\otimes[f] = \sum_{1\leq k\leq n,\{n_1,\dots,n_k\}:n_1\geq 1,\sum n_i=n} \pi_k\otimes[\pi_{n_1}]\cdots[\pi_{n_k}] \tag{7.9}$$

$$\Delta^{\mathscr{H}}(\pi_n) = [\pi_n]\otimes 1 + 1\otimes[\pi_n] + \sum_{1<k<n,\{n_1,\dots,n_k\}:n_i>1, 1<\sum n_i<n} [\pi_k]\otimes[\pi_{n_1}]\cdots[\pi_{n_k}] \tag{7.10}$$

Note that this gives the same multiplicities as in Example 2.76.

7.6.2. **The Feynman category of simplices, Intervals and the Joyal dual of $\mathfrak{F}\mathfrak{S}_<$.** As stated previously, there is a very interesting and useful contravariant duality [Joy97] of subcategories of Δ_+ between Δ and the category $\Delta_{*,*}$, which are the endpoint preserving morphisms in Δ_+. It maps surjections OS in Δ to double base point preserving injections $OI_{*,*}$, see Appendix C.1.

Thus the category $\mathfrak{F}\mathfrak{I}_{*,*}^{op}$ is also a non–Σ Feynman category with trivial $\mathcal{V}$. One has to be careful with the monoidal stucture: while in Δ the monoidal structure is disjoint union of small categories, for which $[n]\otimes[m] = [n+m+1]$, with unit $\varnothing = [-1]$. The monoidal structure on $\Delta_{*,*}$ is the one defined in Definition 4.9, whose unit is $[0]$. We will denote this tensor product by ${}_*\otimes_*$, so that $[n]{}_*\otimes_*[m] = [n+m]$ by identifying n and 0.

Surprisingly, $\mathfrak{F}\mathfrak{I}_{*,*} = (\underline{*}, OI_{*,*}, {}_*\otimes_*), \imath)$ is also a subcategory of the non–Σ Feynman $\mathfrak{F}\mathfrak{I}_<$ category itself. The underlying objects of $\mathcal{F}$ are the natural numbers. To each n we associate $[n]$, technically $\imath(*) = [1]$. For the morphisms, we have the identity $id_{[1]}$ in $Hom([1],[1])$, and one can check that indeed $id_{[1]}^{{}_*\otimes_* n} = id_{[n]}$.

To get injections in Δ_+, we only need to add one morphism in $p : [0] \to [1])$ which we will call special. This generates all injections, c.f. [KW17, §2.10.3]. Any double–base point preserving injection from $[n+1]$ to $[m+1]$ in Δ_+ is then represented by a tensor product of identities and special maps for the tensor product $\otimes$. This can be used to give a representation of the Feynman category $\mathfrak{F}\mathfrak{I}_{*,*}$ in terms of generators and relations as defined in [KW17, §5]. In particular, any double base point preserving injection can be written as $id \otimes p^{n_1-1} \otimes id \otimes p^{n_2-1} \otimes \cdots \otimes p^{n_d-1} \otimes id : [d] \to [N]$, where $N = \sum_{i=1}^{d} n_i$ is the operadic degree, the length is $N-1$. Let us introduce the notation $(1; 0^{n_1-1}, 1, 0^{n_2-1}, \dots, 1, 0^{n_d-1}; 1)$ for this morphism. Where we think of $0^{n-1} = 0, 0, \dots, 0$ as $n-1$ occurrences of 0 indicating the elements in the target that are not hit.

Just as the surjections are generated by the unique maps $\underline{n} \twoheadrightarrow \underline{1}$ so dually are the double base point preserving injections by the unique maps $[1] \to [n] \in Hom_{*,*}([1],[n])$. These are the basic morphisms in the notation above the unique double base point preserving injection $[1] \to [n]$ is $(1; 0^{n-1}; 1) = (1; 0, \dots, 0; 1)$ with $n-1$ copies of 0. It is given by $id \otimes p^{\otimes n-1} \otimes id$. For example: $(1; 0^{n-1}, 1)_* \otimes_* (1; 0^{m-1}, 1) = (1; 0^{n-1}, 1, 0^{m-1}; 1) = id \otimes p^{\otimes n-1} \otimes id \otimes p^{\otimes m-1} \otimes id : [1]_* \otimes_* [1] = [2] \to [n]_* \otimes_* [m] = [n+m]$ is the morphism that sends $0 \mapsto 0, 1 \mapsto n, 2 \mapsto n+m$.

In general

$$(1; 0^{n_1-1}, 1, 0^{n_2-1}, \dots, 1, 0^{n_d-1}; 1) = \\ (1; 0^{n_1-1}; 1)_* \otimes_* (1; 0^{n_2-1}; 1)_* \otimes_* \cdots {}_* \otimes_* (1; 0^{n_d-1}; 1)$$

The factorizations dual to the surjections $\underline{n} \twoheadrightarrow \underline{k} \twoheadrightarrow \underline{1}$, i.e. $[0] \to [k] \to [n]$ yields the co–product

$$\Delta(1; 0^{n-1}; 1) = \sum_{\substack{k \geq 0 \\ (n_1, \dots, n_k): \sum n_i = n)}} (1; 0^{k-1}; 1) \otimes (1; 0^{n_1-1}, 1, \dots, 0^{n_k-1}; 1) = \\ \sum_{\substack{k \geq 0 \\ (n_1, \dots, n_k): \sum n_i = n)}} (1; 0^{k-1}; 1) \otimes \big((1; 0^{n_1-1}; 1)_* \otimes_* (1; 0^{n_2-1}; 1)_* \otimes_* \cdots {}_* \otimes_* (1; 0^{n_k-1}; 1)\big) \tag{7.11}$$

The Hopf quotient is then given by setting $id_1 = (1; \varnothing; 1) = 1 = id_{\mathbb{1}}$.

Remark 7.12. In terms of §3 a multiplication is given by sending free tensor product $\boxtimes$ to ${}_* \otimes_*$ —and evaluating. See §7.6.3 for pictorial representations. This corresponds to the equivalence in axiom (ii) for Feynman categories by picking a functor from the free monoidal category realizing the equivalence. Identifying $\boxtimes$ with $\otimes$ explains the appearance of (op)-lax monoidal functors, see §7.7 and Proposition 4.10.

Remark 7.13. Note that the depth is the number of 1s. Except for the interpretation as a lax monoidal functor, it is not clear how this is exactly related to the multi–zeta values and will be a field of further study. A different encoding would be to use the symbol $(0; 1, \dots, n-1; n)$ for the unique double base point preserving injection $[1] \to [n]$. Then the formula becomes.

$$\Delta(0; 1, \dots, n-1; n) = \sum_{\substack{k \geq 0 \\ (j_1, \dots, j_k): \sum j_i = n)}} (0; 1, \dots, k-1; k) \otimes \\ ((0; 1, \dots, j_1 - 1; j_1)_* \otimes_* (0; 1, \dots, j_2 - 1; j_2)_* \otimes_* \cdots {}_* \otimes_* (0; 1, \dots, j_k - 1; j_k)) \tag{7.12}$$

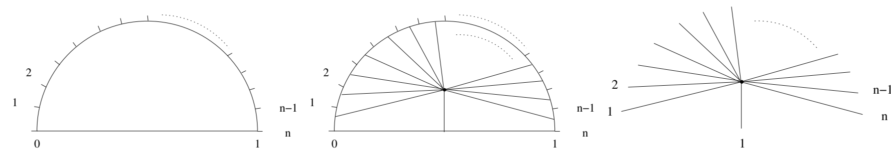

FIGURE 15. The interval injection $[1] \to [n]$ on the left, the surjection $\underline{n} \to \underline{1}$ on the right and and Joyal duality in the middle. Here reading the morphism upwards yields the double base point preserving injection, while reading it downward the surjection.

This is the basic structure of (1.4), which only needs one more step of decoration, see §7.5. In this particular case these are angle markings, see §7.3.2. Further connections are given in Example 7.23 and §7.7.1.

7.6.3. **Pictorial representation.** Pictorially the surjection is naturally depicted by a corolla while the injection is nicely captured by drawing an injection as a half circle. The use of half circles goes back to Goncharov, albeit he did not associate them to double base point preserving injections. Joyal duality can then be seen by superimposing the two graphical images. The superposition goes back to [GGL09]. The connection to Joyal duality is new. This duality is also that of dual graphs on bordered surfaces. This is summarized in Figure 15. Notice that in this duality, the elements of $[n]$ correspond to the angles of the corolla and the elements of $\underline{n}$ label the leaves of the corolla.

This also explains the adding and subtraction of 1 in the correspondence (C.1).

For general surjections, the picture is the a forest of corollas and a collection of half circles. The composition then is given by composing corollas to corollas and by gluing on the half circles to the half circles by identifying the beginning and endpoints. This is exactly the map of combining simplicial strings. The prevalent picture for this in the literature on multi–zetas and polylogs is by adding line segments as the base for the arc segments. This is pictured in Figure 16. The composition is then given by contracting the internal edges or dually erasing the internal lines. This is depicted in Figure 17.

We have chosen here the traditional way of using half circles. Another equivalent way would be to use polygons with a fixed base side. Finally, if one includes both the tree and the half circle, one can modify the picture into a more pleasing aesthetic by deforming the line segments into arcs as is done in §4, where also an explicit composition is given in all details, see Figure 8.

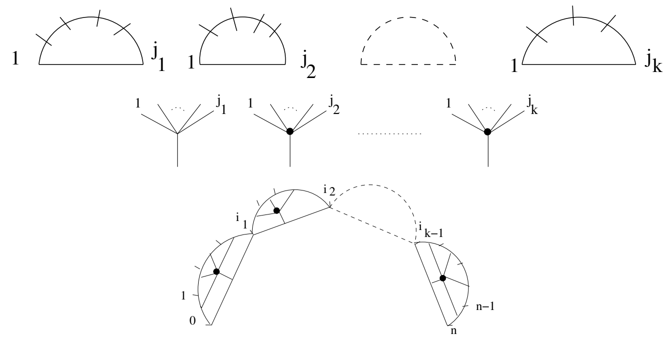

FIGURE 16. The first step of the composition is to assemble a collection of half discs or a forest into one morphism. This is pictured on the right. The j and i are related by $i_l = j_1 + \dots j_k$. Notice that in the half disc assembly is glued at the i_l essentially repeating them, while the forest assembly does not repeat. This also corresponds to an iterated cup product.

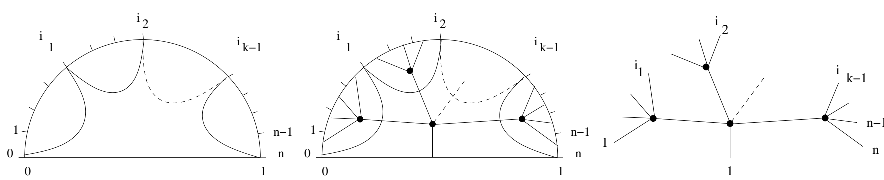

FIGURE 17. The second step of composition. For half circles on the left, where we have deformed the half circles such that the outer boundary is now a half circle, corollas on the right and the duality in the middle. is done in Figure 16. The result of the composition is after the third step, which erases the inner curves or sements and in the corrola picture contracts the edges. The result is in Figure 15.

7.6.4. **Joyal duality in formulas.** In this formulation Joyal duality is also easy to grasp. A double base point preserving injection is given by the symbol $(1; 0^{n_1-1}, 1, \dots, 0^{n_d-1}; 1) = (1; w; 1) : [d] \to [N]$ as above. Where 1 stands for id, 0 for p and w is a word in these letters. Now, the word w in the middle is uniquely fixed by knowing the n_i. Vice–versa, given the n_i there is a unique order preserving surjection $[N-1] \twoheadrightarrow [d-1]$ whose fibers have cardinalities $n_1, \dots n_d$, that is $\pi_{n_1} \amalg \dots \amalg \pi_{n_d}$. This gives half of Joyal duality $OI_{*,*}([n+1], [m+1]) \simeq OS([m], [n])$, where the bijections are natural. One can think of mapping the intervals in Figures 15 and 17 surjectively from the top to the bottom.

To get the other direction note that any injection is given uniquely by a word w as above. This will be a morphism $[d-2] \to [N-2]$. The corresponding surjection is a map $[N-1] \to [d-1]$. Now since $OS = OS_{*,*}$ since all order preserving surjections have to preserve the base points, we have the second part of Joyal duality given by $OI(n,m) \simeq OS_{*,*}(m+1, n+1)$. We also see the different monoidal structures. In the surjections, the monoidal structure is just $\amalg$; for the half–circles, intervals, dually this means that they have to be joined at the base points, see Figure 17.

Remark 7.14. Using this logic, we also see that $OI(n,m) \simeq OS_{*,*}(m+1, n+1) = OS(m+1, n+1) \simeq OI_{*,*}(n+2, m+2)$,, where the bijections are natural. This is just the isomorphism which sends w to $(1; w; 1)$. In OI, we just have the concatenation of words: $w_1 \otimes w_2 = w_1 w_2$. Thus to get the right monoidal structure on $OI_{*,*}$, we have to use ${}_*\otimes_* : (1; w_1; 1)\,{}_*\otimes_*(1; w_2; 1) = (1; w_1w_2; 1)$. Dually, we see that when combining the words w_1w_2 if there are occurrences of 0 in the middle they will add as $0^{n_d-1}0^{m_1-1} = 0^{n_d-1+m_1-1}$ which means that the two surjections will be merged using ${}_*\otimes_*$

7.6.5. **Decorating with sequences.** Consider the Feynman category Δ_+ and fix a set S. The contravaiant functor $Seq : \Delta_+^{op} \to \mathcal{S}et$: $[n] \to Hom([n], S)$ associates to $[n]$ the set of sequences $\{(a_0, \dots, a_n) : a_i \in S\}$ in S. Injections act as restrictions and surjections as repetitions. The usual tensor product which takes the ordered sets $([n], [m])$ to the ordered set $[n+m+1]$ concatenates two sequences. $(a_0, \dots, a_n) \amalg (b_0, \dots, a_m) = (a_0, \dots, a_n, b_0, \dots, b_n)$ thus making Seq into a monoidal functor. For the Feynman category version, we can consider $Seq : \Delta_+ \to \mathcal{S}et^{op}$. In the decorated version, we have objects $([n], (a_0, \dots, a_n))$ which one can view as an interval with $n-1$ marked internal points (only their order matters), where the i–th point, counting both internal and boundary points, is marked by a_i.

Restricting to $\mathfrak{F}\mathfrak{I}^{op}_{*,*} \simeq \mathfrak{F}\mathfrak{S}$, we see that alternatively, $Seq : \mathfrak{F}\mathfrak{S} \to \mathcal{S}et$. In this setting is is more natural to set the image of $[n]$ to be $\underline{n} = \{1, \dots, n\}$. Now, the decoration of $\underline{n}$ is by $(a_0, \dots, a_n)$, that is $n+1$ elements, which we can take as an angle decorations. The morphism $\pi_n := \underline{n} \twoheadrightarrow \underline{1}$ dual to $(1; 0^{n-1}; 1) : [1] \to [n]$ sends a decoration $(a_0, \dots, a_n)$ to (a_0, a_1), that is the two outer angle markings. The graphical depiction of the morphism π_n is a planar corolla as previously discussed, and the decoration by $(a_0, \dots, a_n)$ then naturally is carried by the angles, see Figure 7.3.2..

This gives rise to the colored operad structure of §7.3 in the context of Goncharov, see Example 2.24 as the decorations need to match and the category splits into the final objects $(\underline{1},(a_0,a_1))$. The monoidal structure in this setting on $\underline{1}$ is addition while the monoidal structure on the decorations is ${}_*\otimes_*$ due to the use of Joyal duality.

Remark 7.15. One can view the tensor product ${}_*\otimes_*$ as a partial product, whose dual co–product is the reason for the op–lax structure, namely the dual to the partial multiplication given by ${}_*\otimes_*$.

$$\Delta((a_0,\dots,a_n))=\sum_{i=1}^{n}(a_0,\dots,a_i)_*\otimes_*(a_i,\dots,a_n) \tag{7.13}$$

which is also the co–derivation discussed in §2.7 and an instance of the Alexander–Whitney map; see the next paragraph.

7.6.6. **Sequences as (Semi)–simplicial objects.** In general, we can decorate $\mathfrak{F}\mathfrak{S}_<^{\boxtimes}$ with the semi–simplical set $X_\bullet$, and then regard the decorated $\mathfrak{F}\mathfrak{S}^{\boxtimes}_{<,decX_\bullet}$. By definition, the objects will be $(\underline{n}_1\boxtimes\cdots\boxtimes\underline{n}_k, x_1\otimes\cdots\otimes x_k\in X_{n_1}\otimes\cdots\otimes X_{n_k})$. Using the B_+ operator given by $\boxtimes\mapsto\otimes$ and the Alexander Whitney map, we re–obtain the simplicial results of §4.

In order to read off the structure for Baues, we see that under the tensor product, we are looking at the tensor algebra on the simplicial objects $C_\bullet$, which is the underlying space of the bar–transform, when we regard everything as graded and use the usual shift $B(C_\bullet)=TC_\bullet[1]$.

Such a transition to the tensor algebra is also known as second quantization, cf. e.g. [Kau04b].

Example 7.16. The decoration above can be viewed as a decoration by (semi)–simple objects. For this, we just consider S to be the vertex set of an abstract simplicial complex $\mathscr{S}$. Then the sequences are simply the ordered simplices of S. Their linearization is $C_*^{\mathrm{ord}}(S)$ the ordered simplicial chain complex. In this setting, we have a different tensor product. It corresponds to the tensor product of chain complexes, so that $(a_0,\dots,a_n)\otimes(b_0,\dots,b_m)\in C_n^{ord}(\mathscr{S})\otimes C_m^{ord}(\mathscr{S})$. This gives rise to the construction of Goncharov if we regard the C_n as ungraded objects and use Joyal duality as in the previous paragraph. In this context, the shuffle product (1.5) appears naturally, as the Eilenberg–Zilber map $C_n(\mathscr{S})\otimes C_m(\mathscr{S})\to C_{n+m}(\mathscr{S})$.

7.6.7. **Bi– and Hopf–algebra from the decoration by the algebra of co–chains.** As $\mathfrak{F}\mathfrak{S}$-$\mathcal{O}ps_{\mathcal{C}}$ are algebras in $\mathcal{C}$, we can decorate by any algebra.

Given a semi–simplicial set $X_\bullet$ then $C^*(X_\bullet)$ can be made into a functor from $\mathfrak{F}\mathfrak{S}_<$, since it is an algebra. Namely, we assign to each n the set $C^*(X_\bullet)^{\otimes n}\simeq C^*(X_\bullet^{\times n})$ and to the unique map $n\to 1$ the iterated cup product $\cup^{n-1}$. After decorating, the objects become collections of co–chains, and there is a unique map with source an n–collection of co–chains and target a single cochain, which is the iterated cup product. Thus, one can identify the morphisms of this type with the objects. Furthermore, the set of morphisms then possesses a natural structure of Abelian group. Dualizing this Abelian group, we get the co–operad structure on $C_*(X_\bullet)$ and the co–operad structure with multiplication on $C_*(X_\bullet)^{\otimes}$ that coincides with the one considered in chapter §4.

The bi–algebra is almost connected if the 1–skeleton of $X_\bullet$ is connected. And after quotienting we obtain the same Hopf algebra structure from both constructions.

7.6.8. **Decorating with the bar/cobar complex.** Given an algebra A, we can decorate $\mathfrak{F}\mathfrak{S}_<$ directly. Alternatively, we can decorate $\mathfrak{F}\mathfrak{S}_<$ with BA as an *op* decoration. $OS \to \mathcal{C}^{op}$. Conversely given a co–algebra C, we can decorate with the algebra ΩC. This leads to the construction of Baues.

7.6.9. **Relation to $\cup_i$ products.** It is here that we find the similarity to the $\cup_i$ products also noticed by JDS Jones. Namely, in order to apply $\cup^{n-1}$ to a simplex, we first use the Joyal dual map $[1] \to [n]$ on the simplex. This is the map that is also used for the $\cup_i$ product. The only difference is that instead of using n cochains, one only uses two. To formalize this one needs a surjection that is not in Δ, but uses a permutation, and hence lives in $S\Delta_+$. Here the surjection $\mathfrak{F}\mathfrak{S}$ gives rise to what is alternatively called the sequence operad. Joyal duality is then the fact that one uses sequences and overlapping sequences in the language of [MS03]. The pictorial realizations and associated representations can be found in [Kau08] and [Kau09]. This is also related to the notion of discs in Joyal [Joy97]. This connection will be investigated in the future.

In the Hopf algebra situation, we see that the terms of the iterated $\cup_1$ product coincide with the second factor of the coproduct Δ. Compare Figure 16.

7.7. **Non–connected and free Feynman categories, simplicial objects and strings.** Given a Feynman category $\mathfrak{F}$ there are two associated Feynman categories $\mathfrak{F}^{\boxtimes}, \mathfrak{F}^{nc}$ (nc stands for non–connected), which have the properties

$$Fun_\otimes(\mathcal{F}^{\boxtimes},\mathcal{C}) = Fun(\mathfrak{F},\mathcal{C}) \text{ and } Fun_\otimes(\mathcal{F}^{nc},\mathcal{C}) = Fun_{lax-\otimes}(\mathcal{F},\mathcal{C}) \tag{7.14}$$

see [KW17, §3.1,3.2].

Remark 7.17 (Co–operads with multiplication as an example of a B_+ operator)**.** Using §5.4.1 in the particular case of $\mathfrak{FS}_{<,\mathcal{O}}$, $\mu = B_+ : \check{\mathcal{O}}^{nc} \to \check{\mathcal{O}}$ is precisely which satisfies the compatibility equations for a co–operad with multiplication and the conditions for the unit and co–unit. This allows us to understand the constructions of §3 which become natural in this definition.

7.7.1. **Simplicial objects and links to Chapter 4.** By definition a simplicial object in $\mathcal{C}$ is(1) a functor $X_\bullet : \Delta^{op} \to \mathcal{C}$, and rewriting this, we see that this is equivalent either (2) to a functor $X_\bullet^{op} : \Delta \to \mathcal{C}^{op}$ or (3) to a functor $X_\bullet^{Joy} : \Delta_{*,*} \to \mathcal{C}$. The second and third descriptions open this up for a description in terms of Feynman categories and our constructions of §4 mostly work with the last interpretation.

For (2) and (3) notice that in this interpretation $X_\bullet^{op}$ can be extended to a functor from Δ_+, but it is not monoidal. However, it does give rise to an functor $X_\bullet^{op,\boxtimes} \in \Delta_+^{\boxtimes}\text{-}\mathcal{O}ps_{\mathcal{C}^{op}}$, or an element in $X_\bullet^{Joy,\boxtimes} \in \Delta_{*,*}^{\boxtimes}\text{-}\mathcal{O}ps_{\mathcal{C}}$.

In particular, the relevant constructions are on semi-simplicial objects in $\mathcal{C}$ which again described as (1) a functor $X_\bullet : \mathfrak{FS}_<^{op} \to \mathcal{C}$, (2) to a functor $X_\bullet^{op} : \mathfrak{FS}_< \to \mathcal{C}^{op}$, equivalently $X_\bullet^{op,\boxtimes} \in \mathfrak{FS}_<^{\boxtimes}\text{-}\mathcal{O}ps_{\mathcal{C}^{op}}$, or (3) a functor $X_\bullet^{Joy} : \Delta_{*,*} \to \mathcal{C}$, equivalently an element $X_\bullet^{Joy,\boxtimes} \in \Delta_{*,*}^{\boxtimes}\text{-}\mathcal{O}ps_{\mathcal{C}}$.

There is one more level of sophistication given by Proposition 4.10 which one can rephrase as:

$$\Delta_+^{\boxtimes} = \Omega\Delta \text{ and } \Delta_{*,*}^{nc} = \Omega\Delta \tag{7.15}$$

which identifies simplicial strings as the free, receptively n.c. construction by using that in the correspondence $Fun(\mathcal{F}^{op}, \mathcal{C}) \overset{1-1}{\leftrightarrow} Fun(\mathcal{F}, \mathcal{C}^{op})$ an oplax monoidal functors map to lax monoidal functors. What is intriguing is that although in (i) the original tensor product $\otimes$ is basically forgotten, in (ii) the dual tensor product already is weakly respected by the functor and hence Joyal duality furnishes an intermediate step. That is one only has to add the the oplax monoidal structure §4.3, induced by the Alexander–Whitney map $X_{p+q} \to X_p \times X_q$, which is also represented in the monodial structure of Joyal duality, as explained above, see also §7.6.5 for a concrete example.

The cubical realization of this using the functors L of §4.3 In the more general context of $\mathfrak{F}^{\otimes}$ and $\mathfrak{F}^{nc}$ will be the subject of further investigation.

7.8. **Enrichment and operad based Feynman categories.**

7.8.1. **Enrichments, plus construction and hyper category $\mathfrak{F}^{hyp}$.** The first construction is the plus construction $\mathfrak{F}^+$ and its quotient $\mathfrak{F}^{hyp}$ and its equivalent reduced version $\mathfrak{F}^{hyp,rd}$, see [KW17]. The main result of [KW17, Lemma 4.5] says that for any Feynman category $\mathfrak{F}$ there exists a Feynman category $\mathfrak{F}^{hyp}$ and the set of monoidal functors $\mathcal{O} : \mathcal{F}^{hyp} \to \mathcal{E}$ is in 1–1 correspondence with indexed enrichments $\mathcal{F}_{\mathcal{O}}$ of $\mathcal{F}$ over $\mathcal{E}$.

For such an enrichment, one has $Obj(\mathcal{F}_{\mathcal{O}}) = Obj(\mathcal{F})$ and

$$Hom_{\mathcal{F}_{\mathcal{O}}}(X,Y) = \coprod_{\phi \in Hom_{\mathcal{F}}(X,Y)} \mathcal{O}(\phi) \tag{7.16}$$

And the additional condition that if ϕ is an isomorphism, then $\mathcal{O}(\phi) \simeq \mathbb{1}_{\mathcal{E}}$ This generalizes the notion of hyperoperads of [GK98], whence the superscript *hyp*.

The compositions in $\mathcal{F}$ then give rise to compositions in $\mathcal{F}_{\mathcal{O}}$ for instance for the composition $\phi = \phi_1 \circ \phi_0$, we get:

$$\mathcal{O}(\phi_1) \otimes \mathcal{O}(\phi_0) \to \mathcal{O}(\phi) \tag{7.17}$$

The extra condition guarantees that one does not have to enlarge $\mathcal{V}$. A slightly less strict restriction is that one regards $\mathcal{O} : \mathcal{F}^+ \to \mathcal{E}$, such that $\mathfrak{F}_{\mathcal{O}} = (\mathcal{V}, \mathcal{F}_{\mathcal{O}}, \imath)$ with $\mathcal{F}_{\mathcal{O}}$ as defined as above is still a Feynman category. The largest possible enrichment is given for any $O : \mathcal{F}^+ \to \mathcal{E}$, such that $\mathcal{O}(\phi)$ is free. In this case one can enlarge $\mathcal{V}$ to include any invertible generators. In all of the cases $\mathfrak{F}_{\mathcal{O}}$ is a weak Feynman category [KW17, Definition 1.9.1].

7.8.2. **Bootstrap.** There is the following nice observation. The simplest Feynman category is given by $\mathfrak{F}_{triv} = (\mathcal{V} = triv, \mathcal{F} = \mathcal{V}^{\otimes}, \imath)$ and $\mathfrak{F}^+_{triv} = \mathfrak{F}_{surj}$ [KW17, Example 3.6.1]. Going further, $\mathfrak{F}^+_{surj} = \mathfrak{F}_{May\ operads}$ [Example 3.6.2]. Adding units gives $\mathfrak{F}_{operads}$ and then $\mathfrak{F}_{\mathcal{V}}$ gives $\mathfrak{F}_{surj, \mathcal{O}=leaf\ labelled\ trees}$. Decorating by simplicial sets, we obtain the three original examples from these constructions.

7.8.3. **Bi– and Hopf algebras in the enriched case.** The bi– and Hopf algebras in the enriched case use the formulation of the hereditary condition in the enriched setting. We refer the reader to [KW17, §4] for the rather technical details. In the enriched setting, we will already postulate that the Hom spaces are Abelian groups. This means that the category $\mathcal{E}$ over which $\mathcal{F}$ is enriched, has a faithful functor to the category of Abelian groups. In this case, we say $\mathcal{F}_{\mathcal{O}}$ is $\mathcal{A}b$ enriched over $\mathcal{E}$. We also assume that $\mathcal{E}$ has internal homs and regard it as enriched over itself. A basic example is $\mathcal{E} = dg\mathcal{V}ect$. Assume that $sk(\mathcal{F})$ is small, $\mathcal{F}$ is strict. In this case, we set $\mathscr{B} = \bigoplus_{X,Y} Hom^{\vee}_{sk(\mathcal{F}_{\mathcal{O}})}(X,Y)$, where $\vee$ is the dual in $\mathcal{E}$ given by $\check{V} = \underline{Hom}(V, \mathbb{1})$ and define the multiplication on $\mathscr{B}$ by $\otimes$. The unit is again $\mathrm{id}_{\mathbb{1}}$. For the co–multiplication Δ, we take the dual of the composition $\circ$

$$\circ : \underline{Hom}_{\mathcal{F}_O}(Y,Z) \otimes \underline{Hom}_{\mathcal{F}_O}(X,Z) \to \underline{Hom}_{\mathcal{F}_O}(X,Y) \tag{7.18}$$

as a morphism in $\mathcal{E}$.

$$\Delta : \underline{Hom}^{\vee}_{\mathcal{F}_O}(X,Y) \to \underline{Hom}^{\vee}_{\mathcal{F}_O}(Y,Z) \otimes \underline{Hom}^{\vee}_{\mathcal{F}_O}(X,Z) \tag{7.19}$$

Again it is clear that $\epsilon(\phi) = 1$ if $\phi = id_X$ and $\epsilon(\phi) = 0$ if ϕ is not in a component $\mathbb{1}$ corresponding to id_X is a co–unit. Similarly to §5, assuming that the we can define $\mathscr{B}^{iso}$ by using co–invariants, assuming that these exist.

Theorem 7.18. *Let $\mathfrak{F}_{\mathcal{O}}$ be an indexed enriched Feynman category or more generally a weak Feynman category $\mathcal{A}b$ enriched over a co–complete $\mathcal{E}$, which is enriched over $\mathcal{A}b$, and $\mathcal{F}$ is factorization finite, then $\mathscr{B}^{iso}$ is a bi–algebra in $\mathcal{E}$. In the non–Σ case, already $\mathscr{B}$ is a bi–algebra.*

Proof. The co–associativity and well–definedness of Δ follows from the condition the the underlying $\mathcal{F}$ is factorization finite. The hereditary condition (ii) is replaced by a a co–end formula which can be written as, cf. [KW17, Proposition 1.8.8,§4]:

$$\begin{aligned} &Hom_{\mathcal{F}}(\imath^{\otimes}\cdot, X \otimes Y) = \\ &\int^{Z,Z'} Hom_{\mathcal{F}}(\imath^{\otimes} Z, X) \otimes Hom_{\mathcal{F}}(\imath^{\otimes} Z', Y) \otimes Hom_{\mathcal{V}^{\otimes}}(\,\cdot\,, Z \otimes Z') \end{aligned} \tag{7.20}$$

This formula precisely states that the space of morphisms into a product coincides with the product of the space of morphisms, up to natural isomorphisms changing the intermediate $Z \otimes Z'$.

$$\begin{tikzcd} W \arrow[rr, "\phi"] \arrow[dr, "\simeq"'] & & X \otimes Y \\ & Z \otimes Z' \arrow[ur, "\phi_1 \otimes \phi_2"'] & \end{tikzcd} \tag{7.21}$$

This directly implies that the bi–algebra equation holds on the level of isomorphism classes.

In the non–Σ case, the isomorphism between W and $Z \otimes Z'$ must be a product as well, as $Hom_{\mathcal{V}^{\otimes}}(W, Z \otimes Z') = Hom_{\mathcal{V}^{\otimes}}(W,Z) \otimes Hom_{\mathcal{V}^{\otimes}}(W,Z')$ so that the bi–algebra equation already holds on the level of morphism spaces.

□

Again, define $\mathcal{I} = \langle [id_X] - [id_Y] \rangle$ and $\mathscr{H} = \mathscr{B}^{iso}/I$, then $\mathscr{H}$ is a bi–algebra which may or not be Hopf.

Definition 7.19. We call $\mathcal{F}_{\mathcal{O}}$ as above Hopf, if $\mathscr{H}$ has an anti–pode.

The discussion of criteria is analogous to that of the non–enriched case, by lifting all the notions from $\mathcal{F}$ to $\mathcal{F}_{\mathcal{O}}$. This is straight–forward and will be omitted here.

Example 7.20. The relevant example is that $\mathfrak{FS}^{hyp,rd} \simeq \mathfrak{F}_{operads,0}$ that is, operads whose $\mathcal{O}(1)$ is reduced.

Thus any such operad $\mathcal{O} : \mathcal{F}_{operads,0} \to \mathcal{E}$ gives rise to a Feynman category $\mathfrak{FS}_{\mathcal{O}}$ whose morphisms are determined by

$$Hom_{\mathfrak{FS}_{\mathcal{O}}}(n,1) = \mathcal{O}(n) \tag{7.22}$$

In particular, if $f : S \twoheadrightarrow T$ then $\mathcal{O}(f) = \bigotimes_{t \in T} \mathcal{O}(f^{-1}(t))$ since f decomposes as one–comma generators $f_t : f^{-1}(t) \twoheadrightarrow \{t\}$.

Remark 7.21. For operads with not necessarily reduced $\mathcal{O}(1)$, one can use the $\mathfrak{FS}^+ = \mathfrak{F}_{May}$, and restrict to those functors whose $\mathcal{O}(1)$ is split unital. See also §7.1 and [KW17, 4.3.1.].

7.8.4. **Bi– and Hopf algebras.** For concreteness, we will provide the details for the framework of twisted Feynman categories, in the sepcific case $\mathfrak{FS}_{\mathcal{O}}$.

In this language, the diagrams (2.46) identify certain summands in the co–product and on the coinvariants one is left with the channels.

Indeed in $\mathfrak{FS}$ decomposing $\pi_S : S \twoheadrightarrow \{*\}$ yields the sum $S \overset{f}{\twoheadrightarrow} T \overset{\pi_T}{\twoheadrightarrow} \{*\}$. This is a typical morphism in $\mathfrak{FS}'$ from π_S to π_T.

The composition operation on the twisted $\mathfrak{FS}_{\mathcal{O}}$: $\gamma_f : \mathcal{O}(f) \otimes \mathcal{O}(T) \to \mathcal{O}(S)$, corresponding to the composition $\pi_T \circ f = \pi_S$ cf. 7.8.1. Dually, there is one summand of this type $\check{\gamma}_f$ in the co-product. We identify two such summands in the co–product under the action of the automorphism groups. This corresponds to the diagrams 2.43 which are the isomorphisms in $\mathfrak{FS}'$. Effectively, this means that fixing the size of S and T there is only one channel per partition of $S = S_1 \amalg \cdots \amalg S_k$ into fibers of f.

If on would like to include $\mathcal{O}(1)$ has more invertible elements, one has to enlarge $\mathfrak{FS}$ by choosing the appropriate $\mathcal{V}$. In the case of Cartesian $\mathcal{E}$ this is $Hom_{\mathcal{V}'}(1,1) = \mathcal{O}(1)^{\times}$. This gives rise to extra isomorphisms and/or a K–collection, see [KW17, 2.6.4].

This means in particular that any operad gives rise to an enriched Feynman category whose morphisms are this operad. The dual of the morphisms are then co–operads and the co–operadic and Feynman categorical construction coincide.

The non-Σ case is similar. For this one uses $\mathfrak{F}\mathfrak{S}_<$ and then obtains enrichments by non–Σ operads. Thus again the co–operadic methods apply and yield the same results as the Feynman category constructions.

In this case, we see that $\mathscr{B}$ is the free tensor algebra on the basic morphisms, that is $\mathscr{B} = T\check{\mathcal{O}}(n)$ as in §2 and we obtain the following theorem, recovering all of §2.

Theorem 7.22. *In both the symmetric case $\mathfrak{F}\mathfrak{S}_{\mathcal{O}}$ and non–symmetric case $\mathfrak{F}\mathfrak{S}_{<\mathcal{O}}$, we obtain unital, co–unital biagebras $\mathscr{B}^{iso}$ respectively $\mathscr{B}$. If the quotient by the ideal $\mathcal{I} = \langle id_1 - id_{\mathbb{1}}\rangle$ is connected, we obtain a Hopf algebra. The latter is the case if there (a) there is no $\mathcal{O}(0)$ or (b) there is no $\mathcal{O}(i) : i > 1$, and $(\mathcal{O}(1), \mathrm{id}_1, \epsilon)$ is connected.* □

7.8.5. **Enrichment over $\mathcal{C}^{op}$ and opposite Feynman category.** Notice that we can regard functors $\mathfrak{F} \to \mathcal{C}^{op}$ as co–versions of operads, etc.. In particular if we have a functor $\mathfrak{F}^{hyp} \to \mathcal{C}^{op}$, we get a Feynman category $\mathfrak{F}_{\mathcal{O}}$ enriched over $\mathcal{C}^{op}$. This means that $\mathfrak{F}^{op}_{\mathcal{O}}$ is enriched over $\mathcal{C}$.

Example 7.23. In particular, if $\mathcal{O} : \mathfrak{F}\mathfrak{S}^{hyp} = \mathfrak{F}_{operads,0} \to \mathcal{C}^{op}$ that is dually a pointed almost connected co–operad in $\mathcal{C}$. Then twisting with $\mathcal{O}$ gives us $\mathfrak{F}\mathfrak{S}_{<,\mathcal{O}}$ which is enriched in $\mathcal{C}^{op}$. Taking the opposite we get $\mathfrak{F}\mathfrak{S}^{op}_{<,\mathcal{O}}$. The underlying category is $\mathfrak{F}\mathfrak{I}_{*,*}$ enriched by $\check{\mathcal{O}}$, where $\check{\mathcal{O}}$ is the co–operad in $\mathcal{C}$ corresponding to the operad in $\mathcal{C}^{op}$. This means that the objects are the natural numbers n and the morphisms are $Hom(1,n) = \check{\mathcal{O}}(n)$. This is the enrichment in which the unique map in $Hom_{\mathfrak{F}\mathfrak{I}_{*,*}}([1],[n])$ is assigned $\check{\mathcal{O}}(n)$ in the overlying enriched category $(\mathfrak{F}\mathfrak{I}_{*,*})_{\check{\mathcal{O}}}$.

Putting all the pieces together then yields the following:

Theorem 7.24. *Given a co–operad $\check{\mathcal{O}}$ that is given by a functor $\mathcal{O} : \mathfrak{F}_{operads,0} \to \mathcal{C}^{op}$. Let $\mathscr{B}_{\check{\mathcal{O}}^{nc}}$ be the bialgebra of Example 2. And let $\mathscr{B}_{\mathfrak{F}\mathfrak{S}^{op}_{<,\mathcal{O}}}$ be the bialgebra of the Feynman category discussed above then these two bialgebra coincide.*

Moreover if $\mathfrak{F}\mathfrak{S}_{<,\mathcal{O}}$ is almost connected, the so is $\check{\mathcal{O}}$ and the corresponding Hopf algebras coincide. □

This is another explanation of the relation between Joyal duality and the dual co–operad structure to a colored operad structure.

7.9. **Universal operations.** It is shown that $\mathfrak{F}_{\mathcal{V}}$, which is given by $\mathcal{F}_{\mathcal{V}} = \mathrm{colim}_{\mathcal{V}}\imath$, yields a Feynman category with trivial groupoind $\mathcal{V}_{\mathcal{V}} \simeq \underline{*}$. This generalizes the Meta–Operad structure of [Kau07]. The result is again a Feynman category whose morphisms define an operad and hence the free Abelian group yields a co–operad.

Moreover in many situations, the morphisms of the category are weakly generated [KW17, §6.4] by a simple Feynman category obtained by "forgetting tails". The action is then via a foliation operator as introduced in [Kau07]. In fact there is a poly–simplicial structure here, see also [BB09]. In order to establish this, we recall that any operad under the equivalence established in [KW17][Example 4.12] can be thought of either an enrichment of the Feynman category of sets and surjections or as a functor from the Feynman category for operads to a target category, see also §7.8. As the latter, we obtain universal operations through colimits, see paragraph §6 of [KW17]. On the other hand, we obtain the colimits, in the same form as here, via the construction in paragraph §5 below.

Example 7.25. For the operad of leaf labelled trees,, one can effectively amputate the tails using this construction. One obtains the co–operad dual to the pre–Lie operad [CL01, Kau07]. That is $\mathscr{H}_{amp}$ is realized naturally from a weakly generating suboperad.

8. Summary and outlook

8.1. **Constructions.** We have shown that one can construct Bi–algebras that under checkable conditions yield Hopf algebras in the following related constructions, all of which exist in a symmetric and a non-Σ version.

(i) From a locally finite (unital) operad.
(ii) From a locally finite co–operad.
(iii) From a locally finite co–operad with multiplication.
(iv) From a simplicial object.
(v) From a suitable Feynman category.
(vi) From a suitable Feynman category with a B_+ operator.

Here the transition from (i) to (ii) is dualization. The construction (iii) replaces the free product with a chosen compatible one. Construction (i) and (ii)and (iv) are the special cases of (v) that appear as enriched Feynman categories, in particular enrichments of the Feynman categories of surjections or ordered surjections. The construction (iii) is a special case of the nc construction together with a B_+ operator. The construction (iv) can be seen as a special case of (i) and (ii), but there is an additional structure coming from the simplicial category and Joyal duality.

We also gave criteria when these constructions are functorial. Furthermore, there are infinitesimal versions, which yield Brown's derivations in the (co)–operad case and are related to the generators for the Feynman categories and hence to master equations, cf. [KW17, KWZ12].

8.1.1. **Main Results.** The main upshot is that in all these cases and the classical examples the co–algebra structure is simply the dualization of a partial product structure provided by concatenation in a category. Furthermore, the bi–algebra equation in a general monoidal category is non–trivial and the conditions for Feynman categories are a sufficient condition for it to hold. The Hopf algebras of interest are connected and they are quotients of the natural bi–algebras. The quotient effectively identifies all the objects of mentioned categories.

8.1.2. **Further results and constructions.** Further results and constructions concern deformations, co–module structures, derivations/ infinitesimal structures and a detailed analysis of Joyal duality and its consequences among others.

8.2. **Connes–Kreimer.** There are several types of Connes–Kreimer Hopf algebras which appear as special examples. The tree–type Hopf algebras stem from the construction (i) while the graph–type algebras are examples of (iii).

8.2.1. **CK–forests.** The CK–forrests in the planar and non–planar version can be viewed as coming from construction (i) for the (non-Σ) operads of leaf–labelled and leaf–labelled planar trees. These are alternatively constructed using set–based Feynman categories with trivial $\mathcal{V}$, which can be thought of as indexed enrichments. The amputated versions can be thought of as co–limits, either over a semi–simplicial system of maps, or via the universal operations in Feynman categories.

8.2.2. **Decorated/motic versions.** Using decorations and restrictions, one can obtain other versions, such as the motic versions from Brown, a 1-PI version and more generally colored and weighted versions.

8.2.3. **CK-graphs.** The full graph algebra is the basic example coming from a graphical Feynman category, i.e. one that is indexed over the Feynman category $\mathfrak{G}$, which is a full subcategory of the Borisov–Manin category of graphs. A main ingredient is that the ghost graph of a morphism fixes its isomorphism class.

Restricting and decorating allows one to give the “core” versions and the “renomalization” versions.

8.3. **Goncharov/Baues.** The Hopf algebra of Goncharov and its graded analogue that of Baues can be analyzed in the settings (i), (ii) and (v). In terms of (i) one is using a colored opread. There is an additional structure provided by Joyal duality, which we discussed and which links the constructions to lax–monoidal functors and the nc–construction. This duality also gives rise to the colored operad structure and explains the corolla vs. semi–circle representations. Furthermore, using the cup product, there is a direct link to the decoration by an algebra.

We also found re–interpretations of the additional structures and restrictions of Goncharov and Baues.

8.3.1. **Goncharov multiple zeta values and polylogarithms.** In terms of (iv) taking the contractible groupoid on $0, 1$ we obtain the construction of $\mathscr{H}_{Gon}$ for the multi–zeta values. If we take that with objects z_i, we obtain Goncharov's Hopf algebra for polylogarithms [Gon05].

8.3.2. **Baues.** This is the case of a general simplicial set, which however is 1-connected. We note that since we are dealing with graded objects, one has to specify that one is in the usual monoidal category of graded $\mathbb{Z}$–modules whose tensor product is given by the Koszul or super sign. The 1–connectedness is needed for the bi–algebra quotient to be Hopf. To obtain the connection to double loop spaces, we furthermore need 2–connectedness.

8.4. **Simplicial.** In general, in the simplicial setting, we provided a bi–algebra structure which is Hopf if the simplicial set is 1–connected. We could explain these constructions on several levels.

(1) as derived from the fact that simplices form an operad.
(2) through monoidal and lax–monoidal functors.
(3) using Joyal duality.
(4) using the fact that the simplicial category is a Feynman category.
(5) As an operadic enriched Feynman category.
(6) As a decorated Feynman using the $\cup$ product as an algebra structure. This gives the relationship to the iterated $\cup$ product. The symmetric version also give the relationship to iterated $\cup_i$ products.

8.5. **Outlook.** We expect these results to be the basis of further work. There will be a closer analysis of the role of the B_+ operator and its use inside the theory of Feymnan categories as well as its Hopf-theoretic nature [Kau19]. It will also play a role in the truncation/blow up of moduli spaces and outer space cells [BK19] its sequel and [KZ19]. There are further applications to the theory of Feynman categories, theoretical physics, number theory and algebraic geometry along the basic examples of this paper and *loc. cit.*. In particular, we will analyze and build upon the combinatorial invariants and analysis of Feynman graphs as put forth by the Kreimer group. Here the next steps are applying our general cubical structures [KW17, BK19] to the understanding of the Cutkosky rules.

Appendix

Appendix A. Graph Glossary

A.1. **The category of graphs.** Interesting examples of Feynman categories used in operad–like theories are indexed over a Feynman category built from graphs. It is important to note that although we will first introduce a category of graphs $\mathcal{G}raphs$, the relevant Feynman category is given by a full subcategory $\mathcal{A}gg$ whose objects are disjoint unions or aggregates of corollas. The corollas themselves play the role of $\mathcal{V}$.

Before giving more examples in terms of graphs it will be useful to recall some terminology. A very useful presentation is given in [BM08] which we follow here.

A.1.1. **Abstract graphs.** An abstract graph Γ is a quadruple $(V_\Gamma, F_\Gamma, i_\Gamma, \partial_\Gamma)$ of a finite set of vertices V_Γ, a finite set of half edges or flags F_Γ, an involution on flags $i_\Gamma\colon F_\Gamma \to F_\Gamma; i_\Gamma^2 = id$ and a map $\partial_\Gamma\colon F_\Gamma \to V_\Gamma$. We will omit the subscript Γ if no confusion arises.

Since the map i is an involution, it has orbits of order one or two. We will call the flags in an orbit of order one *tails* and denote the set of tails by T_Γ. We will call an orbit of order two an *edge* and denote the set of edges by E_Γ. The flags of an edge are its elements. The function ∂ gives the vertex a flag is incident to. It is clear that the set of vertices and edges form a 1-dimensional CW complex. The realization of a graph is the realization of this CW complex.

A graph is (simply) connected if and only if its realization is. Notice that the graphs do not need to be connected. Lone vertices, that is, vertices with no incident flags, are also possible.

We also allow the empty graph $\mathbb{1}_\varnothing$, that is, the unique graph with $V = \varnothing$. It will serve as the monoidal unit.

Example A.1. A graph with one vertex and no edges is called a *corolla*. Such a graph only has tails. For any set S the corolla $*_{p,S}$ is the unique graph with $V = \{p\}$ a singleton and $F = S$.

We fix the short hand notation $*_S$ for the corolla with $V = \{*\}$ and $F = S$.

Given a vertex v of a graph, we set $F_v = \partial^{-1}(v)$ and call it *the flags incident to* v. This set naturally gives rise to a corolla. The *tails* at v is the subset of tails of F_v.

As remarked above, F_v defines a corolla $*_v = *_{\{v\},F_v}$.

Remark A.2. The way things are set up, we are talking about (finite) sets, so changing the sets even by bijection changes the graphs.

Remark A.3. As the graphs do not need to be connected, given two graphs Γ and Γ' we can form their disjoint union:

$$\Gamma \sqcup \Gamma' = (F_\Gamma \sqcup F_{\Gamma'}, V_\Gamma \sqcup V_{\Gamma'}, i_\Gamma \sqcup i_{\Gamma'}, \partial_\Gamma \sqcup \partial_{\Gamma'})$$

One actually needs to be a bit careful about how disjoint unions are defined. Although one tends to think that the disjoint union $X \sqcup Y$ is strictly symmetric, this is not the case. This becomes apparent if $X \cap Y \neq \varnothing$. Of course there is a bijection $X \sqcup Y \overset{1-1}{\longleftrightarrow} Y \sqcup X$. Thus the categories here are symmetric monoidal, but not strict symmetric monoidal. This is important, since we consider functors into other not necessarily strict monoidal categories.

Using MacLane's theorem it is however possible to make a technical construction that makes the monoidal structure (on both sides) into a strict symmetric monoidal structure

Example A.4. An *aggregate of corollas* or aggregate for short is a finite disjoint union of corollas, that is, a graph with no edges.

Notice that if one looks at $X = \bigsqcup_{v \in I} *_{S_v}$ for some finite index set I and some finite sets of flags S_v, then the set of flags is automatically the disjoint union of the sets S_v. We will just say just say $s \in F_X$ if s is in some S_v.

A.1.2. **Category structure; Morphisms of Graphs.**

Definition A.5. [BM08] Given two graphs Γ and Γ', consider a triple (ϕ^F, ϕ_V, i_ϕ) where

(i) $\phi^F\colon F_{\Gamma'} \hookrightarrow F_\Gamma$ is an injection,
(ii) $\phi_V\colon V_\Gamma \twoheadrightarrow V_{\Gamma'}$ and i_ϕ is a surjection and
(iii) i_ϕ is a fixed point free involution on the tails of Γ not in the image of ϕ^F.

One calls the edges and flags that are not in the image of ϕ the contracted edges and flags. The orbits of i_ϕ are called *ghost edges* and denoted by $E_{ghost}(\phi)$.

Such a triple is *a morphism of graphs* $\phi\colon \Gamma \to \Gamma'$ if

(1) The involutions are compatible:
  (a) An edge of Γ is either a subset of the image of ϕ^F or not contained in it.
  (b) If an edge is in the image of ϕ^F then its pre–image is also an edge.

(2) ϕ^F and ϕ_V are compatible with the maps ∂:
  (a) Compatibility with ∂ on the image of ϕ^F:
  If $f = \phi^F(f')$ then $\phi_V(\partial f) = \partial f'$
  (b) Compatibility with ∂ on the complement of the image of ϕ^F:
  The two vertices of a ghost edge in Γ map to the same vertex in Γ' under ϕ_V.

If the image of an edge under ϕ^F is not an edge, we say that ϕ grafts the two flags.

The composition $\phi' \circ \phi \colon \Gamma \to \Gamma''$ of two morphisms $\phi \colon \Gamma \to \Gamma'$ and $\phi' \colon \Gamma' \to \Gamma''$ is defined to be $(\phi^F \circ \phi'^F, \phi'_V \circ \phi_V, i)$ where i is defined by its orbits viz. the ghost edges. Both maps ϕ^F and ϕ'^F are injective, so that the complement of their concatenation is in bijection with the disjoint union of the complements of the two maps. We take i to be the involution whose orbits are the union of the ghost edges of ϕ and ϕ' under this identification.

Remark A.6. A *naïve morphism* of graphs $\psi \colon \Gamma \to \Gamma'$ is given by a pair of maps $(\psi_F \colon F_\Gamma \to F_{\Gamma'}, \psi_V \colon V_\Gamma \to V_{\Gamma'})$ compatible with the maps i and ∂ in the obvious fashion. This notion is good to define subgraphs and automorphisms.

It turns out that this data *is not enough* to capture all the needed aspects for composing along graphs. For instance it is not possible to contract edges with such a map or graft two flags into one edge. The basic operations of composition in an operad viewed in graphs is however exactly grafting two flags and then contracting.

For this and other more subtle aspects one needs the more involved definition above which we will use.

Definition A.7. We let $\mathcal{G}raphs$ be the category whose objects are abstract graphs and whose morphisms are the morphisms described in Definition A.5. We consider it to be a monoidal category with monoidal product $\sqcup$ (see Remark A.3).

A.1.3. **Decomposition of morphisms.** Given a morphism $\phi \colon X \to Y$ where $X = \bigsqcup_{w \in V_X} *_w$ and $Y = \bigsqcup_{v \in V_Y} *_v$ are two aggregates, we can decompose $\phi = \bigsqcup \phi_v$ with $\phi_v \colon X_v \to *_v$ where X_v is the subaggregate $\bigsqcup_{\phi_V(w)=v} *_w$, and $\bigsqcup_v X_v = X$. Here $(\phi_v)_V$ is the restriction of ϕ_V

to V_{X_v}. Likewise ϕ_v^F is the restriction of ϕ^F to $(\phi^F)^{-1}(F_{X_v} \cap \phi^F(F_Y))$. This is still injective. Finally i_{ϕ_v} is the restriction of i_ϕ to $F_{X_v} \setminus \phi^F(F_Y)$. These restrictions are possible due to the condition (2) above.

A.1.4. **Ghost graph of a morphism.** The following definition from [KW17] is essential. The underlying ghost graph of a morphism of graphs $\phi\colon \Gamma \to \Gamma'$ is the graph $\mathbb{\Gamma}(\phi) = (V(\Gamma), F_\Gamma, \hat{\imath}_\phi)$ where $\hat{\imath}_\phi$ is i_ϕ on the complement of $\phi^F(\Gamma')$ and identity on the image of flags of Γ' under ϕ^F. The edges of $\mathbb{\Gamma}(\phi)$ are called the ghost edges of ϕ.

A.2. **Extra structures.**

A.2.1. **Glossary.** This section is intended as a reference section. All the following definitions are standard.

Recall that an order of a finite set S is a bijection $S \to \{1, \dots, |S|\}$. Thus the group $\mathbb{S}_{|S|} = Aut\{1, \dots, n\}$ acts on all orders. An orientation of a finite set S is an equivalence class of orders, where two orders are equivalent if they are obtained from each other by an even permutation.

A tree	is a connected, simply connected graph.
A directed graph Γ	is a graph together with a map $F_\Gamma \to \{in, out\}$ such that the two flags of each edge are mapped to different values.
A rooted tree	is a directed tree such that each vertex has exactly one "out" flag.
A ribbon or fat graph	is a graph together with a cyclic order on each of the sets F_v.
A planar graph	is a a ribbon graph that can be embedded into the plane such that the induced cyclic orders of the sets F_v from the orientation of the plane coincide with the chosen cyclic orders.
A planted planar tree	is a rooted planar tree together with a linear order on the set of flags incident to the root.
An oriented graph	is a graph with an orientation on the set of its edges.
An ordered graph	is a graph with an order on the set of its edges.
A γ labelled graph	is a graph together with a map $\gamma : V_\Gamma \to \mathbb{N}_0$.
A b/w graph	is a graph Γ with a map $V_\Gamma \to \{black, white\}$.
A bipartite graph	is a b/w graph whose edges connect only black to white vertices.
A c colored graph	for a set c is a graph Γ together with a map $F_\Gamma \to c$ s.t. each edge has flags of the same color.
A connected 1–PI graph	is a connected graph that stays connected, when one severs any edge.
A 1–PI graph	is a graph whose every component is 1–PI.

A.2.2. **Remarks and language.**

(1) In a directed graph one speaks about the "in" and the "out" edges, flags or tails at a vertex. For the edges this means the one flag of the edges is an "in" flag at the vertex. In pictorial versions the direction is indicated by an arrow. A flag is an "in" flag if the arrow points to the vertex.

(2) As usual there are edge paths on a graph and the natural notion of an oriented edge path. An edge path is a (oriented) cycle if it starts and stops at the same vertex and all the edges are pairwise distinct. It is called simple if each vertex on the cycle has exactly one incoming flag and one outgoing flag belonging to the cycle. An oriented simple cycle will be called a *wheel*. An edge whose two vertices coincide is called a *(small) loop*.

(3) There is a notion of a the genus of a graph, which is the minimal dimension of the surface it can be embedded on. A ribbon graph is planar if this genus is 0.

(4) For any graph, its Euler characteristic is given by

$$\chi(\Gamma) = b_0(\Gamma) - b_1(\Gamma) = |V_\Gamma| - |E_\Gamma|;$$

where b_0, b_1 are the Betti numbers of the (realization of) Γ. Given a γ labelled graph, we define the total γ as

$$\gamma(\Gamma) = 1 - \chi(\Gamma) + \sum_{v \text{ vertex of } \Gamma} \gamma(v) \tag{A.1}$$

If Γ is *connected*, that is $b_0(\Gamma) = 1$ then a γ labeled graph is traditionally called a genus labeled graph and

$$\gamma(\Gamma) = \sum_{v \in V_\Gamma} \gamma(v) + b_1(\Gamma) \tag{A.2}$$

is called the genus of Γ. This is actually not the genus of the underlying graph, but the genus of a connected Riemann surface with possible double points whose dual graph is the genus labelled graph.

A genus labelled graph is called *stable* if each vertex with genus labeling 0 has at least 3 flags and each vertex with genus label 1 has at leas one edge.

(5) A planted planar tree induces a linear order on all sets F_v, by declaring the first flag to be the unique outgoing one. Moreover, there is a natural order on the edges, vertices and flags given by its planar embedding.

(6) A rooted tree is usually taken to be a tree with a marked vertex. Note that necessarily a rooted tree as described above has

exactly one "out" tail. The unique vertex whose "out" flag is not a part of an edge is the root vertex. The usual picture is obtained by deleting this unique "out" tail.

A.2.3. **Category of directed/ordered/oriented graphs.**

(1) Define the category of directed graphs $\mathcal{G}raphs^{dir}$ to be the category whose objects are directed graphs. Morphisms are morphisms ϕ of the underlying graphs, which additionally satisfy that ϕ^F preserves orientation of the flags and the i_ϕ also only has orbits consisting of one "in" and one "out" flag, that is the ghost graph is also directed.
(2) The category of edge ordered graphs $\mathcal{G}raphs^{or}$ has as objects graphs with an order on the edges. A morphism is a morphism together with an order *ord* on all of the edges of the ghost graph.
    The composition of orders on the ghost edges is as follows. $(\phi, ord)\circ\coprod_{v\in V}(\phi_v, ord_v) := (\phi\circ\coprod_{v\in V}\phi_v, ord\circ\coprod_{v\in V} ord_v)$ where the order on the set of all ghost edges, that is $E_{ghost}(\phi)\sqcup\coprod_v E_{ghost}(\phi_v)$, is given by first enumerating the elements of $E_{ghost}(\phi_v)$ in the order ord_v where the order of the sets $E(\phi_v)$ is given by the order on V, i.e. given by the explicit ordering of the tensor product in $Y=\coprod_v *_v$.[6] and then enumerating the edges of $E_{ghost}(\phi)$ in their order *ord*.
(3) The oriented version $\mathcal{G}raphs^{or}$ is then obtained by passing from orders to equivalence classes.

A.2.4. **Category of planar aggregates and tree morphisms.** Although it is hard to write down a consistent theory of planar graphs with planar morphisms, if not impossible, there does exist a planar version of special subcategory of $\mathcal{G}raphs$.

We let $\mathcal{C}rl^{pl}$ have as objects planar corollas — which simply means that there is a cyclic order on the flags — and as morphisms isomorphisms of these, that is isomorphisms of graphs, which preserve the cyclic order. The automorphisms of a corolla $*_S$ are then isomorphic to $C_{|S|}$, the cyclic group of order $|S|$. Let $\mathfrak{C}^{pl}$ be the full subcategory of aggregates of planar corollas whose morphisms are morphisms of the underlying corollas, for which the ghost graphs in their planar structure induced by the source is compatible with the planar structure on the target via ϕ^F. For this we use the fact that the tails of a planar tree have a cyclic order.

[6]Now we are working with ordered tensor products. Alternatively one can just index the outer order by the set V by using [Del90]

Let $\mathcal{C}rl^{pl,dir}$ be directed planar corollas with one output and let $\mathfrak{O}^{pl}$ be the subcategory of $\mathcal{A}gg^{pl,dir}$ of aggregates of corollas of the type just mentioned, whose morphisms are morphisms of the underlying directed corollas such that their associated ghost graphs are compatible with the planar structures as above.

A.3. **Flag killing and leaf operators; insertion operations.**

A.3.1. **Killing tails.** We define the operator $trun$, which removes all tails from a graph. Technically, $trun(\Gamma) = (V_\Gamma, F_\Gamma \backslash T_\Gamma, \partial_\Gamma|_{F_\Gamma\backslash T_\Gamma}, \imath_\Gamma|_{F_\Gamma\backslash T_\Gamma})$.

A.3.2. **Adding tails.** Inversely, we define the formal expression $leaf$ which associates to each Γ without tails the formal sum $\sum_n \sum_{\Gamma':trun(\Gamma')=\Gamma; F(\Gamma')=F(\Gamma')\sqcup\underline{n}} \Gamma'$, that is all possible additions of tails where these tails are a standard set, to avoid isomorphic duplication. To make this well defined, we can consider the series as a power series in t: $leaf(\Gamma) = \sum_n \sum_{\Gamma':trun(\Gamma')=\Gamma; F(\Gamma')=F(\Gamma')\sqcup\bar{n}} \Gamma' t^n$

This is the foliage operator of [KS00, Kau07] which was rediscovered in [BBM13].

A.3.3. **Insertion.** Given graphs, Γ,Γ', a vertex $v \in V_\Gamma$ and an isomorphism ϕ: $F_v \mapsto T_{\Gamma'}$ we define $\Gamma \circ_v \Gamma'$ to be the graph obtained by deleting v and identifying the flags of v with the tails of Γ' via ϕ. Notice that if Γ and Γ' are ghost graphs of a morphism then it is just the composition of ghost graphs, with the morphisms at the other vertices being the identity.

A.3.4. **Unlabelled insertion.** If we are considering graphs with unlabelled tails, that is, classes $[\Gamma]$ and $[\Gamma']$ of coinvariants under the action of permutation of tails. The insertion naturally lifts as $[\Gamma] \circ [\Gamma'] := [\sum_\phi \Gamma \circ_v \Gamma']$ where ϕ runs through all the possible isomorphisms of two fixed lifts.

A.3.5. **No–tail insertion.** If Γ and Γ' are graphs without tails and v a vertex of v, then we define $\Gamma \circ_v \Gamma' = \Gamma \circ_v \mathrm{coeff}(leaf(\Gamma'), t^{|F_v|})$, the (formal) sum of graphs where ϕ is one fixed identification of F_v with $\overline{|F_v|}$. In other words one deletes v and grafts all the tails to all possible positions on Γ'. Alternatively one can sum over all $\partial : F_\Gamma \sqcup F_{\Gamma'} \to V_\Gamma \setminus v \sqcup V_{\Gamma'}$ where ∂ is ∂_G when restricted to $F_w, w \in V_\Gamma$ and $\partial_{\Gamma'}$ when restricted to $F_{v'}, v' \in V_{\Gamma'}$.

A.3.6. **Compatibility.** Let Γ and Γ' be two graphs without flags, then for any vertex v of Γ $leaf(\Gamma \circ_v \Gamma') = leaf(\Gamma) \circ_v leaf(\Gamma')$.

A.4. **Graphs with tails and without tails.** the leaf adding is adjoint to σ_j that is $\delta_\tau \to \delta_{\sigma_j}\tau$ There are two equivalent pictures one can use for the (co–)operad structure underlying the Connes–Kreimer Hopf algebra of rooted trees. One can either work with tails that are flags, or with tail vertices. These two concepts are of course equivalent in the setting where if one allows flag tails, disallows vertices with valence one and vice–versa if one disallows tails, one allows one–valenced vertices called tail vertices. In [CK98] graphs without tails are used. Here we collect some combinatorial facts which represent this equivalence as a useful dictionary.

There are the obvious two maps which either add a vertex at each the end of each tail, or, in the other direction, simply delete each valence one vertex and its unique incident flag, but what is relevant for the Connes–Kreimer example is another set of maps. The first takes a graph with no flag tails to the tree which to every vertex, we *add* a tail, we will denote this map by $\sharp$ and we add one extra (outgoing) flag to the root, which will be called the root flag.

The second map $\flat$ simply deletes all tails. We see that $\flat \circ \sharp = id$. But $\flat$ is not the double sided inverse, since $\sharp \circ \flat$ replaces any number of tails at a given vertex by one tail. It is the identity on the image of $\sharp$, which we call single tail graphs.

Notice that $\sharp$ is well defined on leaf labelled trees by just transfering the labels as sets. Likewise $\flat$ is well defined on single tail trees again by transfering the labels. This means that each vertex will be labelled.

There are the following degenerate graphs which are allowed in the two setups: the empty graph $\varnothing$ and the graph with one flag and no vertices $|$. We declare that

$$\varnothing^\sharp = | \text{ and vice–versa } |^\flat = \varnothing \tag{A.3}$$

A.4.1. **Planted vs. rooted.** A planted tree is a rooted tree whose root has valence 1. One can plant a rooted tree τ to obtain a new planted rooted tree $\tau^\downarrow$, by adding a new vertex which will be the root of $\tau^\downarrow$ and adding one edge between the new vertex and the old root. Vice–versa, given a planted rooted tree τ, we let $\tau^\uparrow$ be the uprooted tree that is obtained from τ by deleting the root vertex and its unique incident edge, while declaring the other vertex of that edge to be the root.

A.5. **Operad structures on rooted/planted trees.** There are several operad structures on leaf–labelled trees which appear.

For rooted trees without tails and labelled vertices, we define

(1) $\tau \circ_i \tau'$ is the tree where the i-th vertex of τ is identified with the root of τ'. The root of the resulting tree being the image of the root of τ.
(2) $\tau \circ_i^+ \tau'$ is the tree where the i-th vertex of τ is joined to the root of τ' by a new edge, with the root of the resulting tree is then the image of the root of τ.

It is actually the second operad structure that underlies the Connes-Kreimer Hopf algebra.

One can now easily check that

$$\tau \circ_i^+ \tau' = \tau \circ_i \tau'^{\downarrow} = (\tau^{\downarrow} \circ_i \tau'^{\downarrow})^{\uparrow} \tag{A.4}$$

These constructions also allow us to relate the compositions of trees with and without tails as follows

$$(\tau^{\sharp} \circ_i \tau'^{\sharp})^{\flat} = \tau \circ_i^+ \tau' \tag{A.5}$$

where the $\circ_i$ operation on the left is the one connecting the ith flag to the root flag.

A.5.1. **Planar case: marking angles.** In the case of planar trees, we have to redefine $\sharp$ by adding a flag to every *angle* of a planar tree. The labels are then not on the vertices, but rather the angles. The analogous equations hold as above. Notice that to give a root to a planar tree actually means to specify a vertex and an angle on it. Planting it connects a new vertex into that angle.

This angle marking is directly to the angle marking in Joyal duality, see below and Figures 15 and 11. This also explains the appearance of angle markings in [Gon05].

Appendix B. Coalgebras and Hopf algebras

A good source for this material is [Car07].

Definition B.1. A coalgebra with a split counit is a triple $(\mathscr{H}, \epsilon, \eta)$, where $(\mathscr{H}, \epsilon)$ is a cogebra and $\eta : \mathbb{1} \to \mathscr{H}$ is a section of η, such that if $| := \eta(1)$, $\Delta(|) = | \otimes |$.

Using η, we split $\mathscr{H} = \mathbb{1} \oplus \bar{\mathscr{H}}$ where $\bar{\mathscr{H}} := ker(\epsilon)$

Following Quillen [Qui67], one defines $\bar{\Delta}(a) := \Delta(a) - | \otimes a - a \otimes |$ where $| := \eta(1)$

If $(\mathscr{H}, \mu, \Delta, \eta, \epsilon)$ is a bialgebra then the restriction $(\mathscr{H}, \Delta, \epsilon)$ is a coalgebra with split counit.

A coalgebra with split counit $\mathscr{H}$ is said to be conilpotent if for all $a \in \bar{\mathscr{H}}$ there is an n such that $\bar{\Delta}^n(a) = 0$ or equivalently if for some $m : a \in ker(pr^{\otimes m+1} \circ \Delta^m)$.

Quillen defined the following filtered object.

$$F^0 = \mathbb{1}; F^m = \{a : \bar{\Delta}a \in F^{m-1} \otimes F^{m-1}\} \tag{B.1}$$

$\mathscr{H}$ is said to be connected, if $\mathscr{H} = \bigcup_m F^m$. If $\mathscr{H}$ is connected, then it is nilpotent, and conversely if taking kernels and the tensor product commute then conilpotent implies connected where $F^m = ker(pr^{\otimes m+1} \circ \Delta^m)$.

For a conilpotent bialgebra algebra there is a unique formula for a possible antipode given by:

$$S(x) = \sum_{n \geq 0} (-1)^{n+1} \mu^n \circ \bar{\Delta}^n(x) \tag{B.2}$$

where $\bar{\Delta}^n : \mathscr{H} \to \mathscr{H}^{\otimes n}$ is the $n-1$-st iterate of $\bar{\Delta}$ that is unique due to coassociativity and $\mu^n : \mathscr{H}^n \to \mathscr{H}$ is the $n-1$-st iterate of the multiplication μ that is unique due to associativity.

Appendix C. Joyal duality, surjections, injections and leaf vs. angle markings

C.1. **Joyal duality.** There is a well known duality [Joy97] of two subcategories of Δ_+. This history of this duality can be traced back to [Str80]. Here we review this operation and show how it can be graphically interpreted. The graphical notation we present in turn connects to the graphical notation in [Gon05] and [GGL09].

The first of the two subcategories of Δ_+ is Δ and the second is the category of intervals. Since we will be dealing with both Δ and Δ_+, we will use the notation $\underline{n}$ for the small category $1 \to \cdots \to n$ in Δ and $[n]$ for $0 \to 1 \to \cdots \to n$ in Δ_+. The subcategory of intervals is then the wide subcategory of Δ_+ whose morphisms preserve both the beginning and the end point. We will denote these maps by $Hom_{*,*}([m],[n])$. Explicitly $\phi \in Hom_{*,*}([m],[n])$ is $\phi(0) = 0$ and $\phi(m) = n$.

The contravariant duality is then given by the association $Hom_{*,*}([m],[n]) \simeq Hom(\underline{n}, \underline{m})$ defined by $\phi \overset{1-1}{\leftrightarrow} \psi$ given by

$$\psi(i) = \min\{j : \phi(j) \geq i\} - 1, \quad \phi(j) = \max\{i : \phi(i) < j\} + 1.$$

This identification is contravariant.

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