EXTENSIONS OF HOPF ALGEBRAS

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ABSTRACT. We investigate the notion of exact sequences of Hopf algebras. We associate to Hopf algebras A and B, and a data consisting of an action of B on A, a cocycle, a coaction of A on B and a co-cocycle, a short exact sequences of Hopf algebras $0 \to A \to C \to B \to 0$. We define cleft short exact sequences of Hopf algebras and prove that their isomorphy classes are in bijective correspondance with the quotient set of datas as above such that the cocycle and the co-cocycle are invertible, modulo a natural action of a subgroup of $\mathrm{Reg}(B, A)$.

§0. **Introduction.** This paper deals with extensions of Hopf algebras. By definition [D1] the category of quantum groups is the dual category to the category of Hopf algebras with bijective antipode. For this reason we can work mainly in this second category and all the results will translate to that of quantum groups in the obvious way. (Some of the results below are true with weaker hypothesis about the antipode). It helps our intuition, however, to keep in mind that a Hopf algebra is the " algebra of functions on a quantum group".

Let us fix, for simplicity, a commutative field $\mathbf{k}$ and let us briefly say "Hopf algebra" for Hopf algebra over $\mathbf{k}$. We shall use the following notation: m, Δ (or δ), $\varepsilon, \mathcal{S}$ mean respectively the multiplication, comultiplication, counit, antipode of a Hopf algebra (or an algebra or a coalgebra), specified with a subscript if necessary. The opposite multiplication or comultiplication are betokened by a superscript "op". We shall also use the following convention: if c is an element of a tensor product $A \otimes B$, then we write $c = c_i \otimes c^i$, omitting the summation symbol. An exception is the case $c = \Delta(x)$, where we use Sweedler's "sigma" notation but dropping again the summatory. The usual transposition $N \otimes N' \to N' \otimes N$ is denoted by $\mathcal{T}$. If $g : N \otimes N \to N \otimes N$ is a morphism of k-modules then $g^{ij} : N^{\otimes m} \to N^{\otimes m}$ has the usual meaning, for example $g^{i,i+1} = \mathrm{id}_{N^{i-1}} \otimes g \otimes \mathrm{id}_{N^{m-i-1}}$. Our main reference for the general theory of Hopf algebras is [Sw].

Key words and phrases. Quantum Groups, Hopf Algebras.

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We consider sequences of morphisms of Hopf algebras of the following sort:

$$(\mathcal{C}) \qquad \mathbf{k} \longrightarrow A \xrightarrow{\iota} C \xrightarrow{\pi} B \longrightarrow \mathbf{k}.$$

We shall say that $(\mathcal{C})$ is exact if

(1) ι is injective. Identify then A with its image.
(2) π is surjective.
(3) $\pi\iota = \varepsilon$.
(4) $\ker\pi = CA^+$. (A^+ is the augmentation ideal, i.e. the kernel of the counit).
(5) $A = \{x \in C : (\pi \otimes \mathrm{id})\Delta(x) = 1 \otimes x\}$.

A "categorical" justification for this definition can be found in Section 1. Indeed, (4) implies that ι is conormal and (5), that π is normal (cf. Definitions 1.1.5 and 1.1.9). That is, this definition enjoys the duality inherent to the theory of Hopf algebras. Clearly, given C and A there exists at most one B (up to isomorphisms) making $(\mathcal{C})$ commutative; and reciprocally, C and B determine A. In fact, given a conormal injective morphism of Hopf algebras $A \xrightarrow{\iota} C$, the unique possible B is the "Hopf cokernel" C/CA^+. Now condition (5) above can be dropped if $A \xrightarrow{\iota} C$ is faithfully flat. In such case, B completes the exact sequence. We do not know if any inclusion of Hopf algebras is faithfully flat. A similar analysis proceeds for a normal, surjective, morphism of Hopf algebras $C \xrightarrow{\pi} B$; but the rôle of the "faithfully flat" requirement is played now by "faithfully coflat" (cf. Proposition 1.2.11).

These questions have already several antecedents in the literature. They were study first in the setting of Hopf algebras graded by non-negative integers, such that the 0 component is isomorphic to $\mathbf{k}$. In this subcategory the case "A commutative, B cocommutative", was treated in [S] (see also [G]). The definition of extension given in [S] is justified by a result from [MM]. In the general situation, a definition of extensions of quantum groups was recently proposed in [PaW]. This definition leds to undeterminacies, see [PaW, 6.3.3] and the Remark after Lemma 1.2.15 below. As stated above, our definition takes care of these problems. After receiving the first version of this paper, S. Montgomery pointed out to the first author that another definition of exact sequences is given in [Sch], remedying the inconvenients in [PaW]. Essentially, the definition in [Sch] requires the axioms (1), ... , (5), but in addition faithful flatness of ι (or equivalently, faithful coflatness of π). See the discussion after Lemma 1.2.15 below.

We wish to answer the following standard questions: given A and B, which additional data produce an exact sequence $(\mathcal{C})$; reciprocally, when an exact sequence $(\mathcal{C})$ can be obtained in such way; how isomorphisms of exact sequences are translated in terms of such data.

Let us discuss the first question. The construction of an *algebra* C out from a Hopf algebra B, an algebra A weakly acted upon by B and a "cocycle" $B \otimes B \to A$ was undertaken in [BCM] and independently in [DT] (see also [Sw2]). With these results at hand, our strategy is simple: first, to obtain a dual statement, i.e. to show how to construct a *coalgebra* C out from a coalgebra B, a Hopf algebra A weakly co-acting on B and a "co-cocycle" $A \to B \otimes B$. Second, to analize whether the algebra and coalgebra structures on the vector space $C = A \otimes B$ provided by the Hopf algebras A, B and the four data (weak action, weak coaction, cocycle, co-cocycle) give rise to a bialgebra, or more precisely to a Hopf algebra. We shall say that the data is *compatible*, in the "bialgebra" case; we say that a compatible data is *Hopf* if the corresponding bialgebra has an antipode. We obtain a complete answer for the bialgebra question (see Theorem 2.20). These considerations are carried out in Section 2. The existence of antipode is proved later, under additional hypothesis (see Lemma 3.2.17).

The isomorphism problem is treated in Section 3. Again, we take profit of what is known in the algebra case [D]; again, we obtain the coalgebra version and then look for the Hopf algebra case. In 3.1, we define an action of the group of invertible morphisms from B to A (with respect to the convolution product) which preserve the unit and counit, on the sets of Hopf and compatible data. One has an application from the quotient sets to isomorphism classes of extensions of Hopf algebras, resp. bialgebras. We ignore if these applications are isomorphisms. In the algebra case [D] this is so if one restricts, on one hand, to data with invertible cocycle, and on the other to *cleft* extensions. An analogous result is true in the coalgebra case. In 3.2, we define cleft extensions of Hopf algebras. In this setting, we have complete answers to the problems stated above: any cleft extension of Hopf algebras is isomorphic to one obtained from a data with invertible cocycle and co-cocycle; the bialgebras constructed from data with invertible cocycle and co-cocycle always have an antipode; and the corresponding quotient set classifies cleft extensions up to isomorphisms.

At various key points (e.g. Lemmas 1.1.12, 1.2.6, 1.2.7), we have used solutions of the Yang-Baxter equation found in [W3]. Thanks to them, any Hopf algebra is generalized commutative and generalized cocommutative. This remark, due to the first author, is completely new and turned out to be very useful (see the Appendix).

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§1. The category of quantum groups. This section is devoted to basic constructions in the category of quantum groups. Our aim is to give a definition of exact sequences, and to prove that it is equivalent to the discussed in the Introduction. We shall freely use the terminology and results of [McL], of common use nowadays. To avoid confussions, all the categoric concepts we will work are thus in the category of Hopf algebras, unless explicitly stated.

§1.1 Kernels. The first point we shall touch is the existence of Hopf kernels, or more generally, of equalizers in the category of Hopf algebras. Let $f, g : A \to B$ be two morphisms of Hopf algebras. Let us consider the following conditions on

elements $x \in A$:

$$(f \otimes \mathrm{id})\Delta(x) = (g \otimes \mathrm{id})\Delta(x) \tag{1.1.0}$$

$$(\mathrm{id} \otimes f)\Delta(x) = (\mathrm{id} \otimes g)\Delta(x) \tag{1.1.1}$$

$$(\mathrm{id} \otimes f \otimes \mathrm{id})(\Delta \otimes \mathrm{id})\Delta(x) = (\mathrm{id} \otimes g \otimes \mathrm{id})(\Delta \otimes \mathrm{id})\Delta(x). \tag{1.1.2}$$

If x satisfies (1.1.0), then $f(x) = g(x)$. Indeed, as f is a morphism of Hopf algebras, $f(x) = (f \otimes \varepsilon)\Delta(x)$. The same is true if x satisfies (1.1.1) or (1.1.2). Clearly, the product of elements satisfying one of these three conditions again does. Let

$$\mathrm{HEqual}(f, g) = \{x \in A : x \text{ satisfies } (1.1.0), (1.1.1) \text{ and } (1.1.2)\}.$$

We shall also consider the algebras

$$\mathrm{LEqual}(f, g) = \{x \in A : x \text{ satisfies } (1.1.0)\},$$
$$\mathrm{REqual}(f, g) = \{x \in A : x \text{ satisfies } (1.1.1)\}.$$

Lemma 1.1.3.

(1) $\mathcal{S}(\mathrm{LEqual}(f, g)) = \mathrm{REqual}(f, g)$ *and* $\mathcal{S}(\mathrm{REqual}(f, g)) = \mathrm{LEqual}(f, g)$.
(2) $\Delta(\mathrm{LEqual}(f, g)) \subseteq \mathrm{LEqual}(f, g) \otimes A$ *and* $\Delta(\mathrm{REqual}(f, g)) \subseteq A \otimes \mathrm{REqual}(f, g)$.
(3) $\mathrm{HEqual}(f, g)$ *is a Hopf subalgebra of* A.
(4) $\mathrm{HEqual}(f, g)$ *is the equalizer of* f *and* g.

Proof. $\mathcal{S}$ and $\mathcal{S}^{-1}$ are anticomultiplicative: this implies (1); (2) follows from the coassociativity of Δ, cf. [Sw, Lemma 16.1.1]. (3) is easy to verify and the condition (1.1.2) is included to guarantee that $\mathrm{HEqual}(f, g)$ is a subcoalgebra.

Let $h : C \to A$ be a morphism of Hopf algebras such that $fh = gh$; then the image of h is contained in $\mathrm{HEqual}(f, g)$ because h is a morphism of Hopf algebras. □

The following is a generalization (and extension) of [BCM, Prop. 1.19].

Lemma 1.1.4. *The following conditions are equivalent:*

(1) $\mathrm{LEqual}(f, g) = \mathrm{HEqual}(f, g)$.
(2) $\mathrm{LEqual}(f, g)$ *is a Hopf subalgebra of* A.
(3) $\mathrm{LEqual}(f, g) = \mathrm{REqual}(f, g)$.
(4) $\mathrm{REqual}(f, g)$ *is a Hopf subalgebra of* A.
(5) $\mathrm{REqual}(f, g) = \mathrm{HEqual}(f, g)$.

Proof. (1) $\Longrightarrow$ (2) is trivial; (2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (4) follow from Lemma 1.1.3; (4) $\Longrightarrow$ (5) is a consequence of universality. Interchanging right by left, we obtain the proof of (5) $\cdots \Longrightarrow \ldots$ (1). □

The base field $\mathbf{k}$ is a zero object in the category of Hopf algebras, i. e. it is initial (the unit $1 : \mathbf{k} \to A$ is the unique morphism of Hopf algebras with such domain and codomain) and final (idem for the counit $\varepsilon : A \to \mathbf{k}$). Thus, the zero morphism between Hopf algebras A and B is $1_B\varepsilon_A$. As usual, the kernel $\mathrm{HKer}\, f$ of a morphism of Hopf algebras is merely the equalizer of it and the zero morphism. Similarly, we have the notions of RKer, LKer.

Remark. The notions of left and right Hopf kernels (sets of elements satisfying (1.1.0) or (1.1.1)) appear at least in [Sw], [BCM]. The condition (1.1.2) was signaled to the first author by B. Enriquez in the course of discussions about [L, 9.2]. The notion of Hopf kernel also appears in [Sch], as was communicated to us by S. Montgomery after reading the first version of this paper.

Definition 1.1.5. A morphism of Hopf algebras $f : A \to B$ is *normal* if the pair f, $g = 1_B\varepsilon_A$ satisfies the equivalent conditions of Lemma 1.1.4.

Example. We extend an example from [BCM]. Let G be a finite group, $H \hookrightarrow G$ a subgroup, A (resp. B) the algebra of functions on G (resp. H) and $R : A \to B$ the restriction morphism. Then it is easy to see that

$$\begin{aligned}
\mathrm{LKer}(R) &= \{f \in A : f \text{ is constant on the left coset } Hx, \quad \forall x \in G\},\\
\mathrm{RKer}(R) &= \{f \in A : f \text{ is constant on the right coset } xH, \quad \forall x \in G\},\\
\mathrm{HKer}(R) &= \{f \in A : f \text{ is constant on the left-right coset } xHy, \quad \forall x, y \in G\}.
\end{aligned}$$

It is then clear that R is a normal morphism if and only if H is a normal subgroup of G.

The category of Hopf algebras is, not abelian but, as in the case of groups, we still have relations between kernels and monomorphisms.

Lemma 1.1.6. *Let $h : A \to B$ be a morphism of Hopf algebras.*
(i) If h is injective, then $\mathrm{HKer}\, h = \mathbf{k}$ and h is normal.
(ii) $\mathrm{HKer}\, h = \mathbf{k}$ if and only if h is a monomorphism of Hopf algebras.

Proof. The first claim is clear: as h is injective, $(h \otimes \mathrm{id})\Delta(x) = 1 \otimes x$ implies $\Delta(x) = 1 \otimes x$ and therefore $x \in \mathbf{k}$. That is, $\mathrm{HKer}\, h = \mathrm{LKer}\, h = \mathbf{k}$.

Assume that $\mathrm{HKer}\, h = \mathbf{k}$ and let $f, g : C \to A$ be morphisms of Hopf algebras such that $hf = hg$. We want to conclude that then $f = g$. We need for this to use the algebra structure on $\hom_{\mathbf{k}}(C, A)$ [Sw]; explicitly

$$(f * g)(c) = f(c_{(1)})g(c_{(2)}).$$

As f is a morphism of Hopf algebras, it is invertible in this algebra and, in fact, $f^{-1} = fS_C$ [Sw, 4.0.4]. Then $h(f^{-1} * g) = \varepsilon_C 1_B$ and then there exists $j : C \to \mathrm{HKer}\, h$ such that $f^{-1} * g = \iota j$, where ι is the inclusion $\mathrm{HKer}\, h \hookrightarrow A$. Thus $f = g$.

Conversely, if h is a monomorphism, as $h\iota = h\varepsilon$, it follows that $\iota_{\mathrm{HKer}\, h, A} = \varepsilon$ and hence $\mathrm{HKer}\, h = \mathbf{k}$. □

In [PaW] another definition of normality (in the category of quantum groups) is proposed (see also [Sch]). We shall see that their definition implies our "conormality" and that both definitions agree under faithful coflatness requirements. The next Lemma is the dual version of the first of the preceding statements.

A quotient Hopf algebra of a Hopf algebra is a quotient vector space provided with a Hopf algebra structure such that the projection is a morphism of Hopf algebras. We shall say that a quotient Hopf algebra is *normal* if the projection is.

Lemma 1.1.7. *Let $\pi : C \to B$ be a projection of Hopf algebras. Suppose that B is a right C-quotient comodule for the right adjoint coaction $\mathrm{ad}(c) = c_{(2)} \otimes \mathcal{S}(bc_{(1)})c_{(3)}$. Then B is a normal quotient Hopf algebra of C.*

Proof. The hypothesis reads: if $\pi(c) = 0$, then $\pi(c_{(2)}) \otimes \mathcal{S}(c_{(1)})c_{(3)} = 0$. Let $c \in \mathrm{RKer}\,\pi$; then $\pi(c_{(2)}) \otimes \mathcal{S}(c_{(1)})c_{(3)} = \varepsilon(c)$. Now $\pi(c_{(1)}) \otimes c_{(2)} = \pi(c_{(3)}) \otimes c_{(1)}\mathcal{S}(c_{(2)})c_{(4)} = 1 \otimes c_{(1)}\varepsilon(c_{(2)})$ because of Lemma 1.1.3 (2). That is, $\mathrm{RKer}\,\pi \subseteq \mathrm{LKer}\,\pi$. By Lemma 1.1.3 (1) we get the equality. □

The definitions of coequalizers and cokernels are rather easy: if $f, g : A \to B$ are two morphisms of Hopf algebras, let J denote the set of the all the elements of the form $f(x) - g(x)$, $x \in A$. Then the quotient of B by the two-sided ideal generated by J is the corresponding coequalizer, denoted $\mathrm{HCoeq}(f, g)$. It is also useful to consider also the quotients

$$\mathrm{LCoeq}(f, g) = B/BJ, \qquad \mathrm{RCoeq}(f, g) = B/JB,$$

which are actually quotient coalgebras of B. Indeed, to see that BJ, JB, BJB are actually coideals of B, one uses a standard trick:

$$\Delta(f(x) - g(x)) = [f(x_{(1)}) - g(x_{(1)})] \otimes f(x_{(2)}) + g(x_{(1)}) \otimes [f(x_{(2)}) - g(x_{(2)})].$$

Lemma 1.1.8. *The following statements are equivalent:*

(1) $BJ = BJB$.
(2) BJ *is a two-sided ideal.*
(3) $BJ = JB$.
(4) JB *is a two-sided ideal.*
(5) $JB = BJB$.

Proof. (1) $\Longrightarrow$ (2) and (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) are obvious. (2) $\Longrightarrow$ (3) because $JB = \mathcal{S}(BJ)$. □

Taking as g the zero morphism of Hopf algebras, we have the definitions of Hopf, left and right cokernels. Observe that the set J in the above definitions specializes in this case to $f(A^+)$, where $A^+ = \{x \in A : \varepsilon(x) = 0\}$ is the augmentation ideal of A.

Definition 1.1.9. We shall say that a morphism $f : A \to B$ of Hopf algebras is *conormal* if $\mathrm{HCoker}\, f = \mathrm{LCoker}\, f$.

Remark. The preceding definition is known; it appears e. g. in [Sch3].

Example. Let us denote $\mathbf{k}\langle X\rangle$ the group-like coalgebra on the set X [Sw, p. 6]. Let us consider a morphism of groups $\phi : G \to H$ and extend it to the group algebras $A = \mathbf{k}\langle G\rangle$, $B = \mathbf{k}\langle H\rangle$. In this case $\mathrm{RCoker}(\phi) = \mathbf{k}\langle \mathrm{Im}(\phi)\backslash H\rangle$, $\mathrm{LCoker}(\phi) = \mathbf{k}\langle H/\mathrm{Im}(\phi)\rangle$ and $\mathrm{HCoker}(\phi) = \mathbf{k}\langle \mathrm{Im}(\phi)\backslash B/\mathrm{Im}(\phi)\rangle$.

Lemma 1.1.10. *Let $h : A \to B$ be a morphism of Hopf algebras.*
(i) If h is surjective, then $\mathrm{HCoker}\, h = \mathbf{k}$ *and h is conormal.*
(ii) $\mathrm{HCoker}\, h = \mathbf{k}$ *if and only if h is an epimorphism of Hopf algebras.*

Proof. It is obvious that h surjective implies $\mathrm{HCoker}\, h = \mathbf{k}$. Now the following formula:

$$bh(a) = b_{(1)}h(a)Sb_{(2)}b_{(3)}$$

implies the conormality of h.

The proof of the second statement is similar to that of 1.1.6. Indeed, if $f, g : B \to C$ are Hopf algebra morphisms such that $fh = gh$ then $(f^{-1} * g)h = \varepsilon 1_C$; thus $f^{-1} * g$ factorizes through $\mathrm{HCoker}\, h$. Conversely, if h is an epimorphism then the projection $B \to \mathrm{HCoker}\, h$ equals the zero morphism and hence $\mathrm{HCoker}\, h = \mathbf{k}$. □

The following Lemma is the dual analogue of Lemma 1.1.7 and relates our definition of conormality with [PaW, 1.5].

We shall say that a Hopf subalgebra is *conormal* if the inclusion is. Let Ad be the well-known right adjoint action of A; one has

$$u\,\mathrm{Ad}(v) = \mathcal{S}(v_{(1)})uv_{(2)}.$$

Lemma 1.1.11. *An Ad-stable Hopf subalgebra is conormal.*

Proof. Suppose that $A \hookrightarrow B$ is Ad-stable. Then, for $a \in A^+, b \in B$, one has $ab = b_{(1)}\mathcal{S}(b_{(2)})ab_{(3)}$, i.e. BA^+ is a two-sided ideal. □

We finish this paragraph with further remarks. A variant of the following Lemma is proved in [Sch, 1.3].

Lemma 1.1.12. *(i) If $h : A \to B$ is normal, then* $\mathrm{HKer}\, h \hookrightarrow A$ *is conormal.*
(ii) If h is conormal, then $B \to \mathrm{HCoker}\, h$ *is normal.*

Proof. (i). We claim first that we always have

$$(\mathrm{LKer}\, h)^+ A = A(\mathrm{LKer}\, h)^+. \tag{1.1.13}$$

It is not difficult to see that $\mathrm{LKer}\, h$ is Ad-stable [BCM]. Thus if $x \in \mathrm{LKer}\, h$ and $a \in A$, we use

$$ax = a_{(1)}x\mathcal{S}(a_{(2)})a_{(3)} = x\,\mathrm{Ad}(a_{(1)})a_{(2)}$$

and $\varepsilon(x\,\mathrm{Ad}\,a) = \varepsilon(x)\varepsilon(a)$ to prove (1.1.13). But now it is obvious that if h is normal, then $\mathrm{HKer}\, h \hookrightarrow A$ is conormal.

(ii). Let $S_0 : B \otimes B \to B \otimes B$ be the map

$$S_0(b \otimes c) = c_{(2)} \otimes b\mathcal{S}(c_{(1)})c_{(3)}. \tag{1.1.14}$$

We claim that there exists a map $\mathbf{S}_0$ making commutative the following diagram:

$$\begin{array}{ccc} B \otimes B & \xrightarrow{S_0} & B \otimes B \\ {\scriptstyle \mathrm{id}\otimes\pi}\downarrow & & \downarrow{\scriptstyle \pi\otimes\mathrm{id}} \\ B \otimes \mathrm{LCoker}\, h & \xrightarrow{\mathbf{S}_0} & \mathrm{LCoker}\, h \otimes B. \end{array}$$

For this, it suffices to check that

$$0 \stackrel{?}{=} (\mathrm{id} \otimes \pi)(c_{(2)} \otimes \mathcal{S}(c_{(1)})c_{(3)}), \qquad \text{if } c = dh(a) \in BJ.$$

But

$$(\mathrm{id} \otimes \pi)\left(d_{(2)}h(a_{(2)}) \otimes \mathcal{S}(h(a_{(1)}))\mathcal{S}(d_{(1)})d_{(3)}h(a_{(3)})\right)$$
$$= d_{(2)}h(a_{(2)}) \otimes \pi(\mathcal{S}(d_{(1)})d_{(3)})\varepsilon(a_{(1)}a_{(3)}).$$

Next we claim that the following diagram commutes:

$$\begin{array}{ccccc} B & \xrightarrow{\Delta} & B \otimes B & \xrightarrow{\mathrm{id} \otimes \pi} & B \otimes \mathrm{LCoeq}\, h \\ \| & & \downarrow{\scriptstyle S_0} & & {\scriptstyle \mathrm{S}_0}\downarrow \\ B & \xrightarrow{\Delta} & B \otimes B & \xrightarrow{\pi \otimes \mathrm{id}} & \mathrm{LCoeq}\, h \otimes B. \end{array}$$

We merely need to show that $\Delta = S_0\Delta$ but this is a routinary computation.

Now assume that h is conormal; and let $b \in \mathrm{LKer}\,\pi$, where $\pi : B \to \mathrm{HCoker}\, h = \mathrm{LCoker}\, h$ is the projection. Then

$$(\pi \otimes \mathrm{id})\Delta(b) = \mathrm{S}(1 \otimes \pi)\Delta(b) = \mathrm{S}(b \otimes 1) = 1 \otimes b$$

and hence $B \to \mathrm{HCoker}\, h$ is normal. □

§*1.2 Exact sequences.* We shall next be concerned with the "image" of a morphism of Hopf algebras $f : A \to B$. Observe first that f factorizes in the following way:

$$\begin{array}{ccccccc} \mathrm{HKer}\, f & \xrightarrow{\iota} & A & \xrightarrow{f} & B & \xrightarrow{\pi} & \mathrm{HCoker}\, f \\ & & \downarrow & & \uparrow & & \\ & & \mathrm{HCoker}(\mathrm{HKer}\, f) & \longrightarrow & \mathrm{HKer}(\mathrm{HCoker}\, f) & & \end{array}$$

This is proved in two steps. First $A(\mathrm{HKer}\, f)^+A \subseteq \ker f$ because $fi = \varepsilon$, and hence f factorizes through $\mathrm{HCoker}(\mathrm{HKer}\, f)$. Second, $\mathrm{Im}\, f \subseteq \mathrm{HKer}(\mathrm{HCoker}\, f)$ because $\pi f = \varepsilon$.

Definition 1.2.0. We shall say that a morphism $f : A \to B$ has a *Hopf image* if the canonical map $\mathrm{HCoker}(\mathrm{HKer}\, f) \to \mathrm{HKer}(\mathrm{HCoker}\, f)$ is an isomorphism, and in such case we shall denote $\mathrm{HIm}\, f = \mathrm{HKer}(\mathrm{HCoker}\, f) \simeq \mathrm{HCoker}(\mathrm{HKer}\, f)$. Thus f has a Hopf image if and only if the following two conditions hold:

(1.2.1) $$\ker f \subseteq A(\mathrm{HKer}\, f)^+A$$

(1.2.2) $$\mathrm{HKer}(\mathrm{HCoker}\, f) \subseteq \mathrm{Im}\, f.$$

Now we are ready to define exact sequences. We shall say that the sequence of morphisms of Hopf algebras

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact if

(1) f is conormal and has a Hopf image,
(2) g is normal and
(3) $\operatorname{HIm} f = \operatorname{HKer} g$.

As always, a sequence

$$\dots A_i \xrightarrow{f_i} A_{i+1} \xrightarrow{f_{i+1}} A_{i+2} \dots$$

is exact if and only if each "piece" $A_i \xrightarrow{f_i} A_{i+1} \xrightarrow{f_{i+1}} A_{i+2}$ is.

Let us consider now a short sequence

$$(\mathcal{C}) \qquad 0 \to A \xrightarrow{\iota} C \xrightarrow{\pi} B \to 0$$

where 0 is of course the Hopf algebra $\mathbf{k}$.

Proposition 1.2.3. *The sequence $(\mathcal{C})$ is exact if and only if the following conditions hold*

(1) *ι is injective. Identify then A with its image.*
(2) *π is surjective.*
(3) $\pi\iota = \varepsilon$.
(4) $\ker \pi = CA^+$.
(5) $A = \{x \in C : (\pi \otimes \mathrm{id})\Delta(x) = 1 \otimes x\}$.

Proof. Let us first assume that $(\mathcal{C})$ is exact.

(i). The exactness of $0 \to A \xrightarrow{\iota} C$ is equivalent to "ι is normal and $\operatorname{HKer} \iota = \mathbf{k}$".

(ii). As ι has a Hopf image, by (1.2.1) $\ker \iota = A(\operatorname{HKer} \iota)^+ A = 0$ and hence ι is injective.

(iii). The exactness of $C \xrightarrow{\pi} B \to 0$ is equivalent to "π is surjective and $\ker \pi \subseteq C(\operatorname{HKer} \pi)^+ C$".

(iv). As $A = \operatorname{HKer} \pi$, one has $\pi\iota = \varepsilon$. But moreover ι is conormal and this together with *(iii)* implies (4). Therefore $C \xrightarrow{\pi} B$ is the cokernel of ι and then $A = \operatorname{HKer}(C \xrightarrow{\pi} B)$, i.e. (5) holds.

(v). Assume now that the conditions (1), … , (5) are true. By Lemma 1.1.6 and *(i)*, we have exactness at $0 \to A \xrightarrow{\iota} C$.

(vi). As A is a Hopf algebra, (5) and Lemma 1.1.6 imply that π is normal. By (2), (4) and *(iii)* we have also exactness at $C \xrightarrow{\pi} B \to 0$.

(vii). (4) also shows that ι is conormal. We show now that ι has a Hopf image: (1.2.1) is clear, by the injectivity of ι. Again, $C \xrightarrow{\pi} B$ is the cokernel of ι and hence (1.2.2) follows from (5). Therefore we have exactness also at $A \xrightarrow{\iota} C \xrightarrow{\pi} B$. □

Remark. Note that the bridge between the Hopf- and set-theoretic conditions is the requirement on the Hopf image. We did not deepened our understanding of this question. But let us mentione a related fact. Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\iota} & C & \xrightarrow{\pi} & B & \longrightarrow & 0 \\ & & {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \Theta}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & \\ 0 & \longrightarrow & A & \xrightarrow{\iota_1} & C_1 & \xrightarrow{\pi_1} & B & \longrightarrow & 0 \end{array}$$

be a morphism of exact sequences. Let us try to prove that Θ is an isomorphism. Let $c \in \mathrm{LKer}\,\Theta$. Then $(\pi \otimes \mathrm{id})\Delta c = (\pi_1 \Theta \otimes \mathrm{id})\Delta c = 1 \otimes c$ and therefore $c \in A$. But then $(\Theta \otimes \mathrm{id})\Delta c = c_{(1)} \otimes c_{(2)} = 1 \otimes c$ and hence $c \in \mathbf{k}$; i.e. Θ is normal and $\mathrm{HKer}\,\Theta = \mathbf{k}$. It is also not difficult to prove that Θ is conormal and $\mathrm{HCoker}\,\Theta = \mathbf{k}$. Thus Θ is a monomorphism and epimorphism of Hopf algebras. We do not know if it is then an isomorphism. We shall prove this if the extension is cleft, see Lemma 3.2.19. We also ignore if a monomorphism (resp., epimorphism) of Hopf algebras is injective (resp., surjective).

It seems that the conditions above are slightly redundant. Here is a result in this sense.

Proposition 1.2.4. *Assume that $A \xrightarrow{\iota} C$ is faithfully flat. Then (C) is exact if and only if (1) to (4) hold.*

Proof. We need to prove that (5) follows from (1), ... , (4) in presence of faithful flatness.

Assume first only (1), ... , (4). Let R be any algebra, and $\gamma : H \to C$ a morphism of Hopf algebras such that $\pi\gamma = \varepsilon$.

Claim. *Let $x, y \in \mathrm{Hom}_{alg}(C, R)$ such that $x\iota = y\iota$. Then $x\gamma = y\gamma$.*

Proof of the Claim. We show first that $x * y^{-1}$ factorizes through B. Indeed, by (4), we need to check that $x * y^{-1}(ca) = 0$, for $c \in C$, $a \in A^+$. But $x * y^{-1}(ca) = x(c_{(1)}a_{(1)})y(\mathcal{S}a_{(2)}\mathcal{S}c_{(2)}) = 0$. Now $x\gamma * (y\gamma)^{-1}(h) = x * y^{-1}(\gamma h) = \varepsilon(h)$. Thus $x\gamma = y\gamma$. □

Let $H = \mathrm{HKer}\,\pi$. By (2), (4) and Lemma 1.12 (ii) (applied to $A \to C$), $H = \mathrm{LKer}\,\pi$; by (3) $A \subseteq H$. Apply now the claim to the inclusion $H \to C$, $R = C \otimes_A C$ (see below), $x : c \mapsto c \otimes 1$, $y :\mapsto 1 \otimes c$. By faithful flatness, $A \supseteq H$. □

Corollary 1.2.5. *Assume that $A \xrightarrow{\iota} C$ is faithfully flat and conormal. Set $B = C/CA^+$ and π the natural projection. Then (C) is exact.*

Remark. We do not know if *any* inclusion of Hopf algebras is faithfully flat. In the commutative setting, this is well-known: a purely algebraic proof is contained in [T]. More generally, it is shown in [Sch] that the inclusion of a central Hopf algebra is faithfully flat. See also [MW].

As A is not central in C, one should be careful with the algebra structure of $C \otimes_A C$.

Lemma 1.2.6. *Let C be a Hopf algebra. The multiplication in $C \otimes C$ defined by*

$$(c \otimes d)(x \otimes y) = cd_{(1)}xSd_{(2)} \otimes d_{(3)}y \tag{1.2.7}$$

is associative with unit $1 \otimes 1$. The applications $c \to c \otimes 1$ and $c \to 1 \otimes c$ are morphisms of algebras, with respect to (1.2.7). Assume in addition that A is an Ad-invariant Hopf subalgebra of C; then $C \otimes_A C$ inherits an algebra structure from (1.2.7).

Proof. We shall check that the kernel of $C \otimes C \to C \otimes_A C$ is a two-sided ideal and leave the rest of the proof to the reader. From the left:

$$\begin{aligned}(c \otimes d)(xa \otimes y - x \otimes ay) &= cd_{(1)}xaSd_{(2)} \otimes d_{(3)}y - cd_{(1)}xSd_{(2)} \otimes d_{(3)}ay \\ &= cd_{(1)}xSd_{(2)}d_{(3)}aSd_{(4)} \otimes d_{(5)}y - cd_{(1)}xSd_{(2)} \otimes d_{(3)}aSd_{(4)}d_{(5)}y.\end{aligned}$$

From the right:

$$\begin{aligned}(xa \otimes y - x \otimes ay)(c \otimes d) &= xay_{(1)}cSy_{(2)} \otimes y_{(3)}d - xa_{(1)}y_{(1)}cSa_{(2)}Sy_{(2)} \otimes a_{(3)}y_{(3)}d \\ &= xa_{(1)}y_{(1)}cSy_{(2)} \otimes Sa_{(2)}a_{(3)}y_{(3)}d - xa_{(1)}y_{(1)}cSa_{(2)}Sy_{(2)} \otimes a_{(3)}y_{(3)}d.\end{aligned}$$

□

Let us recall now that the cotensor product of a right comodule M and a left comodule N over a coalgebra C is

$$M \boxtimes_C N = \ker\left(M \otimes N \xrightarrow{c_M \otimes \mathrm{id} - \mathrm{id} \otimes c_N} M \otimes C \otimes N\right).$$

Here is the dual version of the preceding Lemma.

Lemma 1.2.8. *Let C be a Hopf algebra. The comultiplication in $C \otimes C$ defined by*

$$\tilde{\Delta}(c \otimes d) = c_{(1)} \otimes d_{(2)} \otimes c_{(2)}Sd_{(1)}d_{(3)} \otimes d_{(4)} \tag{1.2.9}$$

is coassociative with counit $1 \otimes 1$. The applications $c \otimes d \to \varepsilon(c)d$ and $c \otimes d \to c\varepsilon(d)$ are morphisms of coalgebras, with respect to (1.2.9). Assume in addition that $C \to B$ is a surjective morphism of Hopf algebras satisfying the following property: there exists $\psi : B \to B \otimes C$ such that $\psi(\pi b) = \pi(b_{(2)}) \otimes Sb_{(1)}b_{(3)}$. Then $C \boxtimes_B C$ inherits a coalgebra structure from (1.2.9).

Proof. Again we prove only the last statement. Assume for simplicity that $a \otimes b \in C \boxtimes_B C$, i.e. that $a \otimes \pi(b_{(1)}) \otimes b_{(2)} = a_{(1)} \otimes \pi(a_{(2)}) \otimes b$. Applying $S_0^{34}m^{24}(\Delta \otimes \psi \otimes \Delta)$ to both sides of this equality we deduce that $\tilde{\Delta}(a \otimes b) \in (C \boxtimes_B C) \otimes C$. (Here S_0 is defined in (1.1.14), the superscript 34 indicates in which copy of $C \otimes C$ S_0 acts, and $m^{24}(x \otimes y \otimes z \otimes u \otimes v) = x \otimes z \otimes yu \otimes v$). Applying instead $S_0^{12}S_0^{34}(\Delta \otimes \mathrm{id} \otimes \Delta)$, we obtain $\tilde{\Delta}(a \otimes b) \in C \otimes (C \boxtimes_B C)$ and we are done. □

Recall that to an epimorphism $C \longrightarrow B$ of coalgebras one can associate an "extension of coefficients" functor $- \boxtimes_B C$ from B- left comodules to C-right comodules. The next Definition and Propositions are known, see [Sch2].

Definition 1.2.10. We shall say that C is *B-coflat* if whenever $M \longrightarrow N$ is an epimorphism of B-comodules then $M \boxtimes_B C \longrightarrow N \boxtimes_B C$ is also an epimorphism of C-comodules.

We shall be interested in a stronger property defined by the three equivalent conditions of the following

Proposition 1.2.11. *Let $C \longrightarrow B$ be coflat. Then the following conditions are equivalent:*

a) *$M \boxtimes_B C \longrightarrow M$ defined by $m \otimes c \longrightarrow m\epsilon(c)$ is surjective for any B comodule M,*
b) *$M \boxtimes_B C = 0$ implies $M = 0$ for any B-comodule M,*
c) *If $M \xrightarrow{\tau} N$ is a morphism of B-comodules and $M \boxtimes_B C \xrightarrow{\tau \boxtimes id} N \boxtimes_B C$ is an epimorphism, then also τ is an epimorphism.*

If these conditions hold, then we say that $C \longrightarrow B$ is faithfully coflat.

Proof. a) $\Longrightarrow$ b) is clear. b) $\Longrightarrow$ c): Let $L = \operatorname{coker} \tau$. Then $N \boxtimes_B C \longrightarrow L \boxtimes_B C$ is an epimorphism by the coflatness of C. The surjectivity of $\tau \boxtimes id$ implies that $L \boxtimes_B C = 0$ and by 2) this means $L = 0$.

c) $\Longrightarrow$ a): By c) it is enough to show that the morphism $M \boxtimes_B C \boxtimes_B C \xrightarrow{\pi} M \boxtimes_B C$ defined by $\pi(m \otimes c \otimes d) = m\epsilon(c) \otimes d$ is surjective, but this follows from the existence of the morphism $\tau : M \boxtimes_B C \longrightarrow M \boxtimes_B C \boxtimes_B C$, $\tau(m \otimes c) \longrightarrow m \otimes c_{(1)} \otimes c_{(2)}$ which is a right inverse of π. □

Proposition 1.2.12. *Let $C \longrightarrow B$ be faithfully coflat. Then for any comodule M, $M \boxtimes_B C \xrightarrow{\rho} M$, $\rho(m \otimes c) = m\epsilon(c)$ is the coequalizer of*

$$M \boxtimes_B C \boxtimes_B C \rightrightarrows M \boxtimes_B C$$

where $p_1(m \otimes c \otimes d) = m \otimes c\epsilon(d)$ and $p_2(m \otimes c \otimes d) m \otimes \epsilon(c) d$.

Proof. Let $M \boxtimes_B C = L$ be the coequalizer. Then there is a comodule epimorphism $L \longrightarrow M$. Let $K = \ker\{L \longrightarrow M\}$. Consider now

$$M \boxtimes_B C \boxtimes_B C \boxtimes_B C \rightrightarrows M \boxtimes_B C \boxtimes_B C.$$

If we can show that the new equalizer is $M \boxtimes_B C$ then by the left exactness of the $\boxtimes_B C$ functor $L \boxtimes_B C = M \boxtimes_B C$ and therefore $K = 0$. The comultiplication Δ_C induces $\tau' : M \boxtimes_B C \boxtimes_B C \longrightarrow M \boxtimes_B C \boxtimes_B C \boxtimes_B C$, $\tau'(m \otimes c \otimes d) = m \otimes c \otimes d_{(1)} \otimes d_{(2)}$. If $x = \sum_i m_i \otimes c_i \otimes d_i \in \ker\{M \boxtimes_B C \xrightarrow{\rho} M\}$ then

$$p_1 \tau(x) = x$$

and

$$p_2 \tau(x) = 0$$

so $x = p_1\tau(x) - p_2\tau(x)$. This shows that $L \boxtimes_B C \longrightarrow M \boxtimes_B C$ is injective. □

Now we present another result concerning the redundancy of the conditions in Proposition 1.2.3.

Proposition 1.2.13. *Consider a sequence $(\mathcal{C})$. Assume that $C \to B$ is faithfully coflat. Then $(\mathcal{C})$ is exact if and only if (1), (2), (3), (5) hold.*

Proof. Assume first only (1), (2), (3), (5). Let R be a coalgebra, and $\gamma : C \to H$ a morphism of Hopf algebras such that $\gamma\iota = \varepsilon$.

Claim. *Let $x, y \in \mathrm{Hom}_{coalg}(R, C)$ such that $\pi x = \pi y$. Then $\gamma x = \gamma y$.*

Proof of the Claim. We show first that $x^{-1} * y$ factorizes through A. Indeed, by (5), we need to check that $(\pi \otimes \mathrm{id})\Delta(x^{-1} * y(r)) = 1 \otimes x^{-1} * y(r)$, for $r \in R$. But the left-hand side is

$$\pi\left(\mathcal{S}(x(r_{(1)})_{(2)})y(r_{(2)})_{(1)}\right) \otimes \mathcal{S}(x(r_{(1)})_{(1)})y(r_{(2)})_{(2)} = \\ \mathcal{S}\pi(x(r_{(2)}))\pi(y(r_{(3)})) \otimes \mathcal{S}(x(r_{(1)}))(y(r_{(4)})) = 1 \otimes \mathcal{S}(x(r_{(1)}))(y(r_{(2)})).$$

Now $(\gamma x)^{-1} * \gamma y = \gamma \circ x^{-1} * y = \varepsilon$. Thus $\gamma x = \gamma y$. □

Let $H = C/CA^+$. By (5), $C \to B$ is normal and hence (by (1) and Lemma 1.1.12 (i)), H is a Hopf algebra. By (3) $C \to B$ factorizes through H. Apply now the claim to the projection $C \to H$, $R = C \boxtimes_B C$, x, y the restrictions of $c \otimes d \mapsto c\varepsilon(d)$, $c \otimes d \mapsto \varepsilon(c)d$. (Here Lemma 1.2.8 applies because $CA^+ = A^+C$). By faithful coflatness, $B \simeq H$, i.e. (4) holds. □

Corollary 1.2.14. *Assume that $C \xrightarrow{\pi} B$ is faithfully coflat and normal. Set $A = \mathrm{HKer}\,\pi$ and ι the inclusion. Then $(\mathcal{C})$ is exact.*

From Corollaries 1.2.5 and 1.2.14 one deduces that the notion discussed in [PaW, 1.5] and our conormality agree, under faithful flatness hypothesis. (See Lemmas 1.1.7 and 1.1.11).

Lemma 1.2.15. *(i) A faithfully flat conormal Hopf subalgebra $A \hookrightarrow C$ is Ad-stable.*

(ii) A faithfully coflat normal quotient Hopf algebra $C \to B$ is a right C-quotient comodule for the right adjoint coaction.

Proof. (i) Let $B = C/A^+C$ and let $\pi : C \to B$ be the canonical projection. By Corollary 1.2.5, $A = \mathrm{LKer}\,\pi$. But if $a \in A$ and $c \in C$, $\mathrm{Ad}(c)a \in \mathrm{LKer}\,\pi$.

(ii) is left to the reader. □

Remark. Another definition of short exact sequences for quantum groups was given in [PaW]. In terms of Hopf algebras, they said that the sequence

$$(\mathcal{C}) \qquad \mathbf{k} \to A \xrightarrow{\iota} C \xrightarrow{\pi} B \to \mathbf{k}$$

is exact if ι is a monomorphism and $B = C/C\iota(A^+)C$, i.e. B is the cokernel of ι. With this definition, two problems arise: given C and B, on one side A is not unique, and on the other it is not proved that such A exists. As for the first, a counterexample is provided in [PaW, 6.3.3]; here is another one. Take a group G with a non-trivial subgroup H such that the union of all the conjugates of H equals G. Let A (resp., C) be the group algebra of H (resp., of G), and let ι be the canonical inclusion. The augmentation ideal of a group algebra is the vector subspace generated by the elements $e_g - 1$ with g non-trivial. Therefore B above is the trivial Hopf algebra. This is an unpleasant situation, which is not possible in our approach. Indeed, if $(\mathcal{C})$ is exact in our sense and B is trivial, then by condition (5) in Proposition 1.2.3 $A = C$.

As for the existence problem in our setting, see Corollaries 1.2.5 and 1.2.14 above.

On the other side, in [Sch] a short sequence of Hopf algebras like $(\mathcal{C})$ is said to be exact if it satisfies either of the following sets of axioms:

(1) A is a conormal faithfully flat Hopf subalgebra of C and $B = \mathrm{HCoker}\,\iota$.
(2) B is a normal faithfully coflat quotient Hopf algebra of C and $A = \mathrm{HKer}\,\pi$.

The equivalence of the preceding requirements is proved using a result from [T3]. By Corollaries 1.2.5 and 1.2.14, the definition in [Sch] agrees with ours, in the faithfully flat case.

If one restricts the attention to the category of finite dimensional Hopf algebras, then the existence problems have both positive answers. Indeed, it is known that finite dimensional Hopf algebras are free over its Hopf subalgebras [NZ] and the dual statement is easy to deduce and well-known. Therefore, it is natural to define a *simple* quantum finite group by any of the two following conditions.

Definition 1.2.16. Let H be a finite dimensional Hopf algebra. We shall say that H is simple if it satisfies any of the following equivalent conditions.

(a) Let K be a normal Hopf quotient algebra of H. Then $K = H$ or $K = k$.
(b) Let K be a conormal Hopf subalgebra of H. Then $K = H$ or $K = k$.

Proof of the equivalence. (a) $\Rightarrow$ (b). Let $B = \mathrm{Hcoker}\,\iota$. Then, by Lemma 1.1.12 ii), the morphism $H \longrightarrow B$ is normal. So by (a), $B = k$ and $K = H$ or $B = H$ and $K = k$.

(b) $\Rightarrow$ (a). Let $C = \mathrm{Hker}\,\pi$. By Lemma 1.1.12 i), $C = H$ and then $K = k$ or $C = k$ and $K = H$. □

Remark. A more general statement than the equivalence between (a) and (b) above is [Sch, Thm. 1.4].

§2. Short exact sequences.

Our purpose now is to study short exact sequences. More precisely, given Hopf algebras A and B we wish (as usual) to study extensions of A by B, i.e. short exact sequences like $(\mathcal{C})$. We recall first some important facts from [BCM], [DT].

Definition 2.0. Let H be a Hopf algebra and A an algebra. A morphism of vector spaces $\rightharpoonup: H \otimes A \longrightarrow A$, $h \otimes a \mapsto h \rightharpoonup a$, is a *weak action* if the following conditions are verified, for all $a, b \in A, h \in H$:

(2.1) $h \rightharpoonup ab = (h_{(1)} \rightharpoonup a)(h_{(2)} \rightharpoonup b)$,
(2.2) $h \rightharpoonup 1 = \varepsilon(h)1$,
(2.3) $1 \rightharpoonup a = a$.

We shall say that a weak action is an *action* if in addition

(2.4) $h \rightharpoonup (\ell \rightharpoonup a) = h\ell \rightharpoonup a$ for all $a \in A, h, \ell \in H$.

Let us fix an algebra A with a weak action of a Hopf algebra H. For each bilinear map $\sigma : H \times H \to A$ we can define a (not necessarily associative) algebra structure on the vector space $A \otimes H$ (denoted $A\#_\sigma H$) as follows:

$$(a \otimes h)(b \otimes \ell) = a(h_{(1)} \rightharpoonup b)\sigma(h_{(2)}, \ell_{(1)}) \otimes h_{(3)}\ell_{(2)}. \tag{2.5}$$

The element $a \otimes h$, when emphasis on the algebra structure is needed, is denoted $a\#h$.

Proposition 2.6. *(i). $1\#1$ is the identity of $A\#_\sigma H$ if and only if*

$$\sigma(1, h) = \sigma(h, 1) = \varepsilon(h)1, \quad \forall h \in H. \tag{2.7}$$

(ii). Assume that $\sigma(h, 1) = \varepsilon(h)1$ for all $h \in H$. Then $A\#_\sigma H$ is associative if and only if, for any $h, l, m \in H$ and $a \in A$, the following conditions hold:

(1) *(cocycle condition)*

$$[h_{(1)} \rightharpoonup \sigma(l_{(1)}, m_{(1)})]\sigma(h_{(2)}, l_{(2)}m_{(2)}) = \sigma(h_{(1)}, l_{(1)})\sigma(h_{(2)}l_{(2)}, m); \tag{2.8}$$

(2) *(twisted module condition)*

$$(h_{(1)} \rightharpoonup (l_{(1)} \rightharpoonup a))\sigma(h_{(2)}, l_{(2)}) = \sigma(h_{(1)}, l_{(1)})(h_{(2)}l_{(2)} \rightharpoonup a). \tag{2.9}$$

Now we pass to the dual notion of the preceding.

Definition 2.10. Let C be a coalgebra with comultiplication δ, H a Hopf algebra. We shall say that a map $\rho : C \to C \otimes H$ is a *weak coaction* if the following conditions are verified:

(2.11) $(\delta \otimes \mathrm{id})\rho = m^{24}(\rho \otimes \rho)\delta$, where $m^{24} : C \otimes H \otimes C \otimes H \to C \otimes C \otimes H$ is the map $c \otimes h \otimes d \otimes k \mapsto c \otimes d \otimes hk$.

(2.12) $(\varepsilon_C \otimes \mathrm{id})\rho = \varepsilon_C \otimes 1$.

(2.13) $(\mathrm{id} \otimes \varepsilon_H)\rho = \mathrm{id}_C$.

As usual, a weak coaction is a *coaction* if in addition

(2.14) $(\mathrm{id} \otimes \Delta)\rho = (\rho \otimes \mathrm{id})\rho$.

Example. The trivial coaction is the map $\rho : C \to C \otimes H$, $\rho(c) = c \otimes 1$.

It is well-known that an extension of groups $H \to G \to K$ with H abelian gives rise in a natural way to an action of K on H. This is still the case for quantum groups. Let

$$1 \to A \xrightarrow{\iota} C \xrightarrow{\pi} B \to 1 \tag{$\mathcal{C}$}$$

be an exact sequence of Hopf algebras and assume that in addition B is cocommutative. Then the adjoint action $\mathrm{ad} : C \to C \otimes C$, $\mathrm{ad}\, c = c_{(2)} \otimes S(c_{(1)})c_{(3)}$, induces $\mathrm{ad}_\pi : C \to B \otimes C$ by composing with $(\pi \otimes 1)$. As $A \xrightarrow{\iota} C$ is conormal, $\mathrm{ad}_\pi(\ker \pi) = \mathrm{ad}_\pi(A^+C) = 0$. So there is a well defined coaction $\gamma : B \to B \otimes C$.

Now the hypothesis "B cocommutative" implies that γ lifts to $\phi : B \to B \otimes A$. Indeed, by condition (5) in Proposition 1.2.3, A is the kernel of $U : C \to B \otimes C$, $U = ((\pi \otimes \mathrm{id}) - (\epsilon \otimes \mathrm{id}))\,\Delta$. Therefore $B \otimes A = \ker(\mathrm{id} \otimes U)$, so we need to show that $(\mathrm{id} \otimes U)\gamma(c) = 0$. Now

$$\begin{aligned}(\mathrm{id} \otimes U)\gamma(c) =& \pi(c_{(3)}) \otimes \pi(\mathcal{S}(c_{(2)})c_{(4)}) \otimes \mathcal{S}(c_{(1)})c_{(5)} \\ J \qquad & - \pi(c_{(3)}) \otimes \epsilon(\mathcal{S}(c_{(2)})c_{(4)}) \otimes \mathcal{S}(c_{(1)})c_{(5)} \\ =& \pi(c_{(2)}) \otimes \pi(\mathcal{S}(c_{(3)})c_{(4)}) \otimes \mathcal{S}(c_{(1)})c_{(5)} - \pi(c_{(2)}) \otimes 1 \otimes \mathcal{S}(c_{(1)})c_{(3)} = 0.\end{aligned}$$

Clearly, ϕ is a coaction.

Examples of extensions with B cocommutative are the cocentral extensions. One says that $\mathcal{C})$ is *cocentral* if the following equivalent conditions hold for any $c \in C$:

(1) $\pi(c_{(1)}) \otimes c_{(2)} = \pi(c_{(2)}) \otimes c_{(1)}$.
(2) $\pi(c_{(2)}) \otimes \mathcal{S}(c_{(1)})c_{(3)} = \pi(c) \otimes 1$.

Proof of the equivalence. Let $x, y : C \to B \otimes C$ be the applications $x(c) = \pi(c) \otimes 1$, $y(c)1 \otimes c$. Consider the usual multiplication in $B \otimes C$. Then (1) reads $x * y = y * x$ and (2), $y^{-1} * x * y = x$. □

Let us fix a coalgebra C with a weak coaction ρ of a Hopf algebra H. For each linear map $\tau : C \to H \otimes H$ we can define a comultiplication (not necessarily coassociative) $\delta^\tau : H\,{}^\tau\#C \to H\,{}^\tau\#C \otimes H\,{}^\tau\#C$ (here $H\,{}^\tau\#C$ denotes, for the sake of differentiating from the usual product coalgebra structure, the vector space $H \otimes C$) as follows:

$$\delta^\tau(h \otimes c) = h_{(1)}\tau(c_{(1)})_j \otimes \rho(c_{(2)})_i \otimes h_{(2)}\tau(c_{(1)})^j\rho(c_{(2)})^i \otimes c_{(3)}. \tag{2.15}$$

Proposition 2.16. *(i). $\varepsilon_{H\,{}^\tau\#C} := \varepsilon_H \otimes \varepsilon_C$ is a counit of $H\,{}^\tau\#C$ if and only if*

$$(\varepsilon_H \otimes \mathrm{id})\tau(c) = \varepsilon_C(c)1_H = (\mathrm{id} \otimes \varepsilon_H)\tau(c). \tag{2.17}$$

(ii). Assume that $\varepsilon_C(c)1_H = (\varepsilon_H \otimes \mathrm{id})\tau(c)$. Then the coproduct δ^τ is coassociative if and only if the following two conditions hold:

(1) *(co-cocycle condition)*

$$m_{H^{\otimes 3}}(\Delta \otimes \mathrm{id} \otimes \tau \otimes \mathrm{id})(\tau \otimes \rho)\delta = (\mathrm{id} \otimes m_{H^{\otimes 2}})(\mathrm{id} \otimes \Delta \otimes \mathrm{id} \otimes \mathrm{id})(\tau \otimes \tau)\delta; \tag{2.18}$$

(2) *(twisted comodule condition)*

$$(\mathrm{id} \otimes m_{H^{\otimes 2}})(\mathrm{id} \otimes \Delta \otimes \mathrm{id} \otimes \mathrm{id})(\rho \otimes \tau)\delta = m^{13}_{H^{\otimes 2}}(\mathrm{id} \otimes \mathrm{id} \otimes \rho \otimes \mathrm{id})(\tau \otimes \rho)\delta, \tag{2.19}$$

where $m^{13}_{H^{\otimes 2}} : H \otimes H \otimes C \otimes H \otimes H \to C \otimes H \otimes H$ sends $h \otimes k \otimes c \otimes \tilde{h} \otimes \tilde{k} \to c \otimes h\tilde{h} \otimes k\tilde{k}$.

The co-cocycle condition reads

$$\begin{aligned}&\left(\tau(c_{(1)})_j\right)_{(1)} \tau\left(\rho(c_{(2)})_i\right)_\ell \otimes \left(\tau(c_{(1)})_j\right)_{(2)} \tau\left(\rho(c_{(2)})_i\right)^\ell \otimes \tau(c_{(1)})^j\rho(c_{(2)})^i \\ &\qquad = \tau(c_{(1)})_j \otimes \left(\tau(c_{(1)})^j\right)_{(1)} \tau(c_{(2)})_h \otimes \left(\tau(c_{(1)})^j\right)_{(2)} \tau(c_{(2)})^h,\end{aligned}$$

and the twisted comodule condition is

$$\begin{aligned}&\rho(c_{(1)})_i \otimes \left(\rho(c_{(1)})^i\right)_{(1)} \tau(c_{(2)})_h \otimes \left(\rho(c_{(1)})^i\right)_{(2)} \tau(c_{(2)})^h \\ &\qquad = \rho\left(\rho(c_{(2)})_i\right)_\ell \otimes \tau(c_{(1)})_j\rho\left(\rho(c_{(2)})_i\right)^\ell \otimes \tau(c_{(1)})^j\rho(c_{(2)})^i.\end{aligned}$$

Proof. We omit the superscript τ in the following.

(i). It is easy to see that $(\mathrm{id}\otimes\varepsilon_{H^\tau\#C})\delta(h\otimes c) = h(\mathrm{id}\otimes\varepsilon)\tau(c_{(1)})\otimes c_{(2)}$ and $(\varepsilon_{H^\tau\#C}\otimes\mathrm{id})\delta(h\otimes c) = h(\varepsilon\otimes\mathrm{id})\tau(c_{(1)})\otimes c_{(2)}$. Thus (2.17) implies that $\varepsilon_{H^\tau\#C}$ is a counit. The reciprocal is trivial.

*(ii).*Let us first assume that (2.18) and (2.19) are satisfied. Clearly, it suffices to check the coassociativity in an element of the form $1\otimes c$. Hence

$$\begin{aligned}(\delta\otimes\mathrm{id})\delta(c) &= [\tau(c_{(1)})_j]_{(1)}\tau\left([\rho(c_{(2)})_i]_{(1)}\right)_\ell \otimes\rho\left([\rho(c_{(2)})_i]_{(2)}\right)_p\otimes[\tau(c_{(1)})_j]_{(2)} \\ &\qquad \tau\left([\rho(c_{(2)})_i]_{(1)}\right)^\ell\rho\left([\rho(c_{(2)})_i]_{(2)}\right)^p\otimes[\rho(c_{(2)})_i]_{(3)}\otimes\tau(c_{(1)})^j\rho(c_{(2)})^i\otimes c_{(3)} \\ &= [\tau(c_{(1)})_j]_{(1)}\tau\left(\rho(c_{(2)})_i\right)_\ell\otimes\rho\left(\rho(c_{(3)})_q\right)_p\otimes[\tau(c_{(1)})_j]_{(2)}\tau\left(\rho(c_{(2)})_i\right)^\ell \\ &\qquad \rho\left(\rho(c_{(3)})_q\right)^p\otimes\rho(c_{(4)})_k\otimes\tau(c_{(1)})^j\rho(c_{(2)})^i\rho(c_{(3)})^q\rho(c_{(4)})^k\otimes c_{(5)} \\ &= \tau(c_{(1)})_j\otimes\rho\left(\rho(c_{(3)})_q\right)_p\otimes[\tau(c_{(1)})^j]_{(1)}\tau(c_{(2)})_h\rho\left(\rho(c_{(3)})_q\right)^p \\ &\qquad \otimes\rho(c_{(4)})_k\otimes[\tau(c_{(1)})^j]_{(2)}\tau(c_{(2)})^h\rho(c_{(3)})^q\rho(c_{(4)})^k\otimes c_{(5)} \\ &= \tau(c_{(1)})_j\otimes\rho(c_{(2)})_i\otimes[\tau(c_{(1)})^j]_{(1)}[\rho(c_{(2)})^i]_{(1)}\tau(c_{(3)})_h \\ &\qquad \otimes\rho(c_{(4)})_k\otimes[\tau(c_{(1)})^j]_{(2)}[\rho(c_{(2)})^i]_{(2)}\tau(c_{(3)})^h\rho(c_{(4)})^k\otimes c_{(5)} \\ &= (\mathrm{id}\otimes\delta)\delta(c)\end{aligned}$$

Here the first and the last equalities follow from the definition; the second, from

$$\begin{aligned}&[\rho(d)_i]_{(1)}\otimes[\rho(d)_i]_{(2)}\otimes[\rho(d)_i]_{(3)}\otimes\rho(d)^i \\ &\qquad\qquad = \rho(d_{(1)})_i\otimes\rho(d_{(2)})_j\otimes\rho(d_{(3)})_k\otimes\rho(d_{(1)})^i\rho(d_{(2)})^j\rho(d_{(3)})^k,\end{aligned}$$

which is an iteration of (2.11); the third, from the cocycle condition (2.18) and the fourth, from the twisted comodule condition (2.19).

Conversely, suppose that the coalgebra $H^\tau\#C$ is coassociative. Applying $(\mathrm{id}\otimes\varepsilon)^{\otimes 3}$ (respectively, $\varepsilon\otimes\mathrm{id}\otimes\mathrm{id}\otimes\varepsilon\otimes\mathrm{id}\otimes\varepsilon$) to both sides of the equality expressing the coassociativity, we get the cocycle condition (resp., the twisted comodule condition; in this case we must use the hypothesis $\varepsilon_C(c)1_H = (\varepsilon_H\otimes\mathrm{id})\tau(c)$). □

We want now to consider the extension problem. First, combining Propositions 2.6 and 2.16, we show how to build extensions of Hopf algebras.

Theorem 2.20. *Let A, B be two Hopf algebras, provided with a weak action $\rightharpoonup: B\otimes A\to A$ and a weak coaction $\rho: B\to B\otimes A$. Let us also fix a cocycle $\sigma: B\otimes B\to A$, and a co-cocycle $\tau: B\to A\otimes A$. Let $C = A^\tau\#_\sigma B$ denote the vector space $A\otimes B$ provided with the multiplication (2.5) and the comultiplication (2.15) (denoted here Δ).*

Then C is a bialgebra if and only if the following conditions hold:

(i) *σ satisfies the unitary condition (2.7), and τ the co-unitary condition (2.17).*
(ii) *σ satisfies the cocycle condition (2.8) and the twisted module condition (2.9).*
(iii) *τ satisfies the co-cocycle condition (2.18) and the twisted comodule condition (2.19).*

(iv) *(compatibility with the unit and counit)* $\rho(1) = \tau(1) = 1 \otimes 1$, $\varepsilon \circ \sigma = \varepsilon \otimes \varepsilon$, $\varepsilon(a \rightharpoonup b) = \varepsilon(a)\varepsilon(b)$.

(v) *(compatibility between the product and the coproduct)*

$$\begin{aligned}(b_{(1)} \rightharpoonup a)_{(1)}\sigma(b_{(2)} \otimes \tilde{b}_{(1)})_{(1)}\tau(b_{(3)}\tilde{b}_{(2)})_j \otimes (b_{(1)} \rightharpoonup a)_{(2)}\sigma(b_{(2)} \otimes \tilde{b}_{(1)})_{(2)}\tau(b_{(3)}\tilde{b}_{(2)})^j \\ = \tau(b_{(1)})_h \left(\rho(b_{(2)})_i \rightharpoonup a_{(1)}\tau(\tilde{b}_{(1)})_p\right) \sigma\left(\rho(b_{(3)})_j \otimes \rho(\tilde{b}_{(2)})_q\right) \\ \otimes \tau(b_{(1)})^h \rho(b_{(2)})^i \rho(b_{(3)})^j (b_{(4)} \rightharpoonup a_{(2)}\tau(\tilde{b}_{(1)})^p \rho(\tilde{b}_{(2)})^q)\sigma(b_{(5)} \otimes \tilde{b}_{(3)})\end{aligned} \tag{2.21}$$

$$\begin{aligned}\rho(b_{(3)}\tilde{b}_{(2)})_i \otimes (b_{(1)} \rightharpoonup a)\sigma(b_{(2)} \otimes \tilde{b}_{(1)})\rho(b_{(3)}\tilde{b}_{(2)})^i \\ = \rho(b_{(1)})_i\rho(\tilde{b}_{(1)})_k \otimes \rho(b_{(1)})^i(b_{(2)} \rightharpoonup a\rho(\tilde{b}_{(1)})^k)\sigma(b_{(3)} \otimes \tilde{b}_{(2)})\end{aligned} \tag{2.22}$$

for all $a, \tilde{a} \in A$, $b, \tilde{b} \in B$.

Assume further that

(vi) *(compatibility with the antipode)*

$$\varepsilon(b) = \mathcal{S}\left(\tau(b)_j\right)\tau(b^j) = \tau(b_j)\mathcal{S}\left(\tau(b)^j\right) \tag{2.23}$$

$$\varepsilon(b) = \sigma\left(b_{(1)} \otimes \mathcal{S}(b_{(2)})\right) = \sigma\left(\mathcal{S}(b_{(1)}) \otimes b_{(2)}\right) \tag{2.24}$$

Then $A^\tau\#_\sigma B$ *is a Hopf algebra, whose antipode is given by*

$$\mathcal{S}(a\#b) = \left\{\left(\mathcal{S}[\rho(b)_{t(2)}] \rightharpoonup \mathcal{S}[\rho(b)^t]\right) \otimes \mathcal{S}[\rho(b)_{t(1)}]\right\} \mathcal{S}(a)\#1. \tag{2.25}$$

Moreover, let $i_A : A \to C$ *and* $p_B : C \to B$ *be the applications* $a \mapsto a \otimes 1$, $a \otimes b \mapsto \varepsilon(a)b$. *Then*

$$0 \xrightarrow{i_A} A \to C \xrightarrow{p_B} B \to 0$$

is an exact sequence.

Proof. It follows from Propositions 2.6 and 2.16 that C is an associative algebra and coassociative coalgebra if and only if *(i)*, *(ii)*, *(iii)* hold. It is clear that *(iv)* means that ε_C is a morphism of algebras and that $\Delta(1) = 1 \otimes 1$. Let us assume now that (2.21) and (2.22) are true. Let $c = a \otimes b$, $\tilde{c} = \tilde{a} \otimes \tilde{b} \in C$. Then

$$\begin{aligned}\Delta(c\tilde{c}) &= a_{(1)}(b_{(1)} \rightharpoonup \tilde{a})_{(1)}\sigma(b_{(2)} \otimes \tilde{b}_{(1)})_{(1)}\tau(b_{(3)}\tilde{b}_{(2)})_j \otimes \rho(b_{(4)}\tilde{b}_{(3)})_i \\ &\quad \otimes a_{(2)}(b_{(1)} \rightharpoonup \tilde{a})_{(2)}\sigma(b_{(2)} \otimes \tilde{b}_{(1)})_{(2)}\tau(b_{(3)}\tilde{b}_{(2)})^j\rho(b_{(4)}\tilde{b}_{(3)})^i \otimes b_{(5)}\tilde{b}_{(4)} \\ &= a_{(1)}\tau(b_{(1)})_h\left(\rho(b_{(2)})_i \rightharpoonup \tilde{a}_{(1)}\tau(\tilde{b}_{(1)})_p\right)\sigma\left(\rho(b_{(3)})_j \otimes \rho(\tilde{b}_{(2)})_q\right) \\ &\quad \otimes \rho(b_{(6)}\tilde{b}_{(4)})_i \otimes a_{(2)}\tau(b_{(1)})^h\rho(b_{(2)})^i\rho(b_{(3)})^j \\ &\quad (b_{(4)} \rightharpoonup \tilde{a}_{(2)}\tau(\tilde{b}_{(1)})^p\rho(\tilde{b}_{(2)})^q)\sigma(b_{(5)} \otimes \tilde{b}_{(3)})\rho(b_{(6)}\tilde{b}_{(4)})^i \otimes b_{(7)}\tilde{b}_{(5)} \\ &= a_{(1)}\tau(b_{(1)})_h\left(\rho(b_{(2)})_i \rightharpoonup a_{(1)}\tau(\tilde{b}_{(1)})_p\right)\sigma\left(\rho(b_{(3)})_j \otimes \rho(\tilde{b}_{(2)})_q\right) \\ &\quad \otimes \rho(b_{(4)})_m\rho(\tilde{b}_{(3)})_t \otimes a_{(2)}\tau(b_{(1)})^h\rho(b_{(2)})^i\rho(b_{(3)})^j\rho(b_{(4)})^m \\ &\quad (b_{(5)} \rightharpoonup \tilde{a}_{(2)}\tau(\tilde{b}_{(1)})^p\rho(\tilde{b}_{(2)})^q\rho(\tilde{b}_{(3)})^t)\sigma(b_{(6)} \otimes \tilde{b}_{(4)}) \otimes b_{(7)}\tilde{b}_{(5)} \\ &= \Delta(c)\Delta(\tilde{c}).\end{aligned}$$

Here the first equality follows from the definitions, the second from (2.21), the third from (2.22) and the last is again by definition, complemented by iteration of (2.1). Conversely, (2.21) (resp., (2.22)) can be deduced from the equality expressing the multiplicative character of Δ by taking $a = 1$ and applying $\mathrm{id} \otimes \varepsilon \otimes \mathrm{id} \otimes \varepsilon$ (resp., $\varepsilon \otimes \mathrm{id} \otimes \mathrm{id} \otimes \varepsilon$).

Assume now that (2.23) and (2.24) hold and let us check that the $\mathcal{S}$ given by (2.25) satisfies the axioms of the antipode. One can suppose that $a = 1$. One has

$$\begin{aligned} m(\mathcal{S} \otimes \mathrm{id})\Delta(1 \otimes b) &= \left\{ \left(\mathcal{S}[\rho(\rho(b_{(2)})_i)_{t(2)}] \rightharpoonup \mathcal{S}[\rho(\rho(b_{(2)})_i)^t] \right) \otimes \mathcal{S}[\rho(\rho(b_{(2)})_i)_{t(1)}] \right\} \\ &\qquad \mathcal{S}[\tau(b_{(1)})_j]\tau(b_{(1)})^j \rho(b_{(2)})^i \otimes b_{(3)} \\ &= \left(\mathcal{S}[\rho(\rho(b_{(2)})_i)_t]_{(1)} \rightharpoonup \mathcal{S}[\rho(\rho(b_{(2)})_i)^t] \mathcal{S}[\tau(b_{(1)})_j]\tau(b_{(1)})^j \rho(b_{(2)})^i \right) \\ &\qquad \sigma\left(\mathcal{S}[\rho(\rho(b_{(2)})_i)_t]_{(2)} \otimes b_{(3)} \right) \otimes \mathcal{S}[\rho(\rho(b_{(2)})_i)_t]_{(3)} b_{(4)} \\ &= \left(\mathcal{S}[\rho(b_{(1)})_j]_{(1)} \rightharpoonup \mathcal{S}[((\rho(b_{(1)})^j)_{(1)} \tau(b_{(2)})_h] ((\rho(b_{(1)})^j)_{(2)} \tau(b_{(2)})^h \right) \\ &\qquad \sigma\left(\mathcal{S}[\rho(b_{(1)})_j]_{(2)} \otimes b_{(3)} \right) \otimes \mathcal{S}[\rho(b_{(1)})_j]_{(3)} b_{(4)} \\ &= \left(\mathcal{S}(b_{(3)}) \rightharpoonup \mathcal{S}[\tau(b_{(4)})_h]\tau(b_{(4)})^h \right) \sigma\left(\mathcal{S}(b_{(2)}) \otimes b_{(3)} \right) \otimes \mathcal{S}(b_{(1)}) b_{(5)} = \varepsilon(b). \end{aligned}$$

Here the first equality follows from the definitions; the second from (2.1) and (2.11); the third from the twisted comodule condition; the fourth from condition (2.13) and the last, from (2.23) and (2.24). In a similar way, one proves $m(\mathrm{id} \otimes \mathcal{S})\Delta(1 \otimes b) = \varepsilon(b)$.

Conditions (1) to (5) of Proposition 1.2.3 are also easy to verify and the Proposition follows. ☐

Remark. Conditions (2.23) and (2.24) are sufficient but not necessary to have an antipode. A more satisfactory answer is given below (Lemma 3.2.17) in the context of cleft extensions. The general question remains however open.

Definition 2.26. Let A, B be two Hopf algebras. A data $\mathcal{D} = (\rightharpoonup, \sigma, \rho, \tau)$ is *compatible* if it satisfes conditions *(i), ... (v)* in the Theorem above. If in addition, $A\,{}^\tau\#_\sigma B$ is a Hopf algebra, then we say that $\mathcal{D}$ is a Hopf data.

We shall show now an example of reconstruction of a Hopf data from an exact sequence. A more general statement will be given in the next section. Let us fix an exact sequence

$$0 \longrightarrow A \xrightarrow{\iota} C \xrightarrow{\pi} B \longrightarrow 0.$$

In particular, we shall consider C as a left A-module and B-comodule in the obvious way. We shall assume in the rest of this section the existence of a linear isomorphism $\mathcal{F} : C \rightarrow A \otimes B$ (whose inverse is denoted $\mathcal{G}$) satisfying the following conditions:

(2.27) $\mathcal{F}$ is a morphism of A-modules, i.e. $\mathcal{F}(ac) = (a \otimes 1)\mathcal{F}(c)$.

(2.28) $\mathcal{F}$ is morphism of B-comodules, i.e. the following diagram commutes

$$\begin{array}{ccc} C & \xrightarrow{(\mathrm{id}\otimes\pi)\Delta} & C\otimes B \\ {\scriptstyle\mathcal{F}}\downarrow & & \downarrow{\scriptstyle\mathcal{F}\otimes\mathrm{id}} \\ A\otimes B & \xrightarrow{\mathrm{id}\otimes\Delta} & A\otimes B\otimes B. \end{array}$$

(2.29) $\mathcal{F}(1) = 1\otimes 1$.

(2.30) $(\varepsilon_A\otimes\varepsilon_B)\mathcal{F} = \varepsilon_C$.

(2.31) $\mathcal{G}(1\otimes b_{(1)})\mathcal{G}(1\otimes Sb_{(2)}) = \varepsilon(b) = \mathcal{G}(1\otimes Sb_{(1)})\mathcal{G}(1\otimes b_{(2)})$.

(2.32) $(p_A\mathcal{F}(c_{(1)})\mathcal{S}(p_A\mathcal{F}(c_{(2)}) = \varepsilon(c) = \mathcal{S}(p_A\mathcal{F}(c_{(1)})(p_A\mathcal{F}(c_{(2)})$.

Here $p_A : A\otimes B \to A$ is the canonical projection, $p_A = \mathrm{id}\otimes\varepsilon_B$; in the same vein, $i_A : A \to A\otimes B$ is the canonical inclusion $a\mapsto a\otimes 1$, and similarly for i_B, p_B. One deduces easily from the first four axioms above that $\mathcal{F}\iota = i_A$, $p_B\mathcal{F} = \pi$.

Thanks to such $\mathcal{F}$, we shall obtain the exact sequence $(\mathcal{C})$ as in the preceding proposition. Let us first introduce $\bar\sigma : B\otimes B \to C$, $\overline{\rightharpoonup} : B\otimes A \to C$, $\bar\tau : C \to A\otimes A$, $\bar\rho : C \to B\otimes A$ by the following formulas:

$$\begin{aligned} \bar\sigma(b\otimes\tilde b) &= \mathcal{G}(1\otimes b_{(1)})\mathcal{G}(1\otimes\tilde b_{(1)})\mathcal{G}(1\otimes S\tilde b_{(2)}Sb_{(2)}) \\ b\overline{\rightharpoonup}a &= \mathcal{G}(1\otimes b_{(1)})a\mathcal{G}(1\otimes Sb_{(2)}), \\ \bar\tau(c) &= \left(Sp_A\mathcal{F}c_{(1)}\right)_{(1)} p_A\mathcal{F}c_{(2)} \otimes \left(Sp_A\mathcal{F}c_{(1)}\right)_{(2)} p_A\mathcal{F}c_{(3)}, \\ \bar\rho(c) &= \pi(c_{(2)})\otimes Sp_A\mathcal{F}c_{(1)}p_A\mathcal{F}c_{(3)}. \end{aligned}$$

Lemma 2.33. *These maps give rise to* $\sigma : B\otimes B \to A$, $\rightharpoonup : B\otimes A \to A$, $\tau : B \to A\otimes A$, $\rho : B \to B\otimes A$.

Proof. It is straightforward. For example, let us show that the image of $\bar\sigma$ is contained in A:

$$\begin{aligned} &(\mathrm{id}\otimes\pi)\Delta\bar\sigma(b\otimes\tilde b) \\ &\quad= (\mathrm{id}\otimes\pi)\Delta\mathcal{G}(1\otimes b_{(1)})(\mathrm{id}\otimes\pi)\Delta\mathcal{G}(1\otimes\tilde b_{(1)})(\mathrm{id}\otimes\pi)\Delta\mathcal{G}(1\otimes S\tilde b_{(2)}Sb_{(2)}) \\ &= \left(\mathcal{G}(1\otimes b_{(1)})\otimes b_{(2)}\right)\left(\mathcal{G}(1\otimes\tilde b_{(1)})\otimes\tilde b_{(2)}\right)\left(\mathcal{G}(1\otimes S\tilde b_{(4)})Sb_{(4)})\otimes S\tilde b_{(3)})Sb_{(3)})\right) \\ &\quad= \mathcal{G}(1\otimes b_{(1)})\mathcal{G}(1\otimes\tilde b_{(1)})\mathcal{G}(1\otimes S\tilde b_{(4)})Sb_{(4)})\otimes b_{(2)}\tilde b_{(2)}S\tilde b_{(3)})Sb_{(3)}) \\ &\qquad= \bar\sigma(b\otimes\tilde b)\otimes 1. \end{aligned}$$

The rest is similar. □

Proposition 2.34. *$\mathcal{F}$ is an isomorphism of Hopf algebras from C onto $A^{\tau}\#_{\sigma}B$.*

Proof. ρ (resp. $\rightharpoonup$) is a weak coaction (resp. action) because of (2.32) (resp. (2.31)). These two axioms also imply $\sigma(b\otimes 1) = \sigma(1\otimes b) = \varepsilon(b)$ and $(\varepsilon\otimes\mathrm{id})\tau = \varepsilon = (\mathrm{id}\otimes\varepsilon)\tau$. From this, we deduce $(a\otimes 1)(1\otimes b) = a\otimes b$.

So let us prove that $\mathcal{G}$ is a morphism of algebras:

$$\mathcal{G}\left((a\otimes b)(\tilde{a}\otimes\tilde{b})\right) = \mathcal{G}\left(a(b_{(1)}\rightharpoonup\tilde{a})\sigma(b_{(2)},\tilde{b}_{(1)})\otimes b_{(3)}\tilde{b}_{(2)}\right) =$$
$$\mathcal{G}\left(a\mathcal{G}(1\otimes b_{(1)})\tilde{a}\mathcal{G}(1\otimes Sb_{(2)})\mathcal{G}(1\otimes b_{(3)})\mathcal{G}(1\otimes\tilde{b}_{(1)})\mathcal{G}(1\otimes S\tilde{b}_{(2)}Sb_{(4)})\otimes b_{(5)}\tilde{b}_{(3)}\right)$$
$$= \mathcal{G}(a\otimes b_{(1)})\varepsilon(b_{(2)})\mathcal{G}(\tilde{a}\otimes\tilde{b}_{(1)})\mathcal{G}(1\otimes S\tilde{b}_{(2)}Sb_{(3)})\mathcal{G}(1\otimes b_{(4)}\tilde{b}_{(3)}) = \mathcal{G}(a\otimes b)\mathcal{G}(\tilde{a}\otimes\tilde{b}).$$

Here the first equality is by definition, the second follows because $\mathcal{G}$ is a morphism of A-modules and the third is a consequence of (2.31).

Now we prove that $\mathcal{F}$ is a morphism of coalgebras:

$$\begin{aligned}\delta^{\tau}(\mathcal{F}(c)) = [p_A\mathcal{F}(c_{(1)})]_{(1)}\tau(\pi c_{(2)})_j\otimes\rho(\pi c_{(3)})_i \\ \otimes[p_A\mathcal{F}(c_{(1)})]_{(2)}\tau(\pi c_{(2)})^j\rho(\pi c_{(3)})^i\otimes\pi c_{(4)} \\ = [p_A\mathcal{F}(c_{(1)})]_{(1)}[Sp_A\mathcal{F}(c_{(2)})]_{(1)}p_A\mathcal{F}(c_{(3)})\otimes\pi c_{(6)} \\ \otimes[p_A\mathcal{F}(c_{(1)})]_{(2)}[Sp_A\mathcal{F}(c_{(2)})]_{(2)}p_A\mathcal{F}(c_{(4)})Sp_A\mathcal{F}(c_{(5)})p_A\mathcal{F}(c_{(7)})\otimes\pi c_{(8)} \\ = p_A\mathcal{F}c_{(1)}\otimes\pi c_{(2)}\otimes p_A\mathcal{F}c_{(3)}\pi c_{(4)} = (\mathcal{F}\otimes\mathcal{F})\Delta(c),\end{aligned}$$

taking into account the formula $\mathcal{F}(c) = p_A\mathcal{F}(c_{(1)})\otimes\pi(c_{(2)})$. □

§3. Isomorphisms of extensions. In this section, we study whether two extensions build from compatible data (cf. Definition 2.26) are isomorphic, in terms of the data. We obtain a complete answer in the case of cleft extensions, see 3.2. Our methods are largely an extension of those in [D]; S. Montgomery communicated us that Blattner and she obtained independently some of the results in [D] (unpublished).

3.1 Rudimentary non-abelian cohomology. Let A, B be two Hopf algebras and let $\mathrm{Reg}(B,A)$ be the group of linear morphisms from B to A which are invertible with respect to the convolution product [Sw]. Let also

$$\begin{aligned}\mathrm{Reg}_1(B,A) &= \{\phi\in\mathrm{Reg}(B,A) : \phi(1) = 1\},\\ \mathrm{Reg}_{\varepsilon}(B,A) &= \{\phi\in\mathrm{Reg}(B,A) : \varepsilon\phi = \varepsilon\},\\ \mathrm{Reg}_{1,\varepsilon}(B,A) &= \mathrm{Reg}_1(B,A)\cap\mathrm{Reg}_{\varepsilon}(B,A);\end{aligned}$$

these are subgroups of $\mathrm{Reg}(B,A)$.

Let $\mathrm{Weak}(B\otimes A, A)$ (resp., $\mathrm{Coweak}(B, B\otimes A)$) be the set of all weak actions of the Hopf algebra B on the algebra A-cf. Definition 2.0 (resp., weak coactions of the Hopf algebra A on the coalgebra B-cf. Definition 2.10).

Lemma 3.1.1. *(i). Let* $\rightharpoonup\, \in \mathrm{Weak}(B\otimes A, A)$, $\sigma \in \hom(B\otimes B, A)$, $\phi \in \mathrm{Reg}(B, A)$. *The formulas*

(3.1.2)
$$b \;{}^{\phi}\!\rightharpoonup a = \phi(b_{(1)})\, b_{(2)} \rightharpoonup a\phi^{-1}(b_{(3)}),$$

(3.1.3)
$${}^{\phi}\sigma(b\otimes d) = \phi(b_{(1)})\left(b_{(2)} \rightharpoonup \phi(d_{(1)})\right)\sigma(b_{(3)}\otimes d_{(2)})\phi^{-1}(b_{(4)}d_{(3)})$$

provide a left action of $\mathrm{Reg}_1(B, A)$ *on* $\mathrm{Weak}(B\otimes A, A)\times \hom(B\otimes B, A)$, *i.e.* $\phi.(\rightharpoonup, \sigma) = ({}^{\phi}\!\rightharpoonup, {}^{\phi}\sigma)$. *(Here A needs to be only an algebra).*

In addition, if σ *is invertible then* ${}^{\phi}\sigma$ *also is; in fact*

$$({}^{\phi}\sigma)^{-1}(b\otimes d) = \phi(b_{(1)}d_{(1)})\sigma^{-1}(b_{(2)}\otimes d_{(2)})\left(b_{(3)} \rightharpoonup \phi^{-1}(d_{(3)})\right)\phi^{-1}(b_{(4)}).$$

(ii). Analogously, let $\rho \in \mathrm{Coweak}(B, B\otimes A)$, $\tau \in \hom(B, A\otimes A)$. *The formulas*

(3.1.4)
$$\rho^{\phi}(b) = \left(1\otimes\phi^{-1}(b_{(1)})\right)\rho(b_{(2)})\left(1\otimes\phi(b_{(3)})\right),$$

(3.1.5)
$$\tau^{\phi}(b) = \Delta\phi^{-1}(b_{(1)})\tau(b_{(2)})(\phi\otimes\mathrm{id})\rho(b_{(3)})\left(1\otimes\phi(b_{(4)})\right)$$

provide a right action of $\mathrm{Reg}_{\varepsilon}(B, A)$ *on* $\mathrm{Coweak}(B, B\otimes A)\times\hom(B, A\otimes A)$, *i.e.* $(\rho,\tau).\phi = (\rho^{\phi}, \tau^{\phi})$. *(Here B needs to be only a coalgebra).*

Moreover, if τ *is invertible then* τ^{ϕ} *also is, and its inverse is*

$$(\tau^{\phi})^{-1}(b) = \left(1\otimes\phi^{-1}(b_{(1)})\right)(\phi^{-1}\otimes\mathrm{id})\rho(b_{(2)})\tau^{-1}(b_{(3)})\Delta\phi(b_{(4)}).$$

Proof. *(i)* is essentially proved in [D], so we prove only the dual statement *(ii)*. Let us first show that ρ^{ϕ} is a weak coaction. Condition (2.12) in Definition 2.10 is obvious and condition (2.13) follows from $\varepsilon\phi = \varepsilon$. Let us proceed with condition (2.11). On one side,

$$(\Delta\otimes\mathrm{id})\rho^{\phi}(b) = \left(1\otimes\phi^{-1}(b_{(1)})\right)(\Delta\otimes\mathrm{id})\rho(b_{(2)})\left(1\otimes\phi(b_{(3)})\right);$$

on the other,

$$\begin{aligned} &m^{24}(\rho^{\phi}\otimes\rho^{\phi})(b_{(1)}\otimes b_{(2)}) \\ &\quad = \rho(b_{(2)})_i\otimes\rho(b_{(5)})_j\otimes\phi^{-1}(b_{(1)})\rho(b_{(2)})^i\phi(b_{(3)})\phi^{-1}(b_{(4)})\rho(b_{(5)})^j\phi(b_{(6)}), \end{aligned}$$

and condition (1) for ρ^{ϕ} holds.

Let us prove now the group action axioms. Let also $\psi \in \mathrm{Reg}_{\varepsilon}(B, A)$. Then

$$\begin{aligned} &(\rho^{\phi})^{\psi}(b) = \left(1\otimes\psi^{-1}(b_{(1)})\right)\rho^{\phi}(b_{(2)})\left(1\otimes\psi(b_{(3)})\right) \\ &= \left(1\otimes\psi^{-1}(b_{(1)})\right)\left(1\otimes\phi^{-1}(b_{(2)})\right)\rho(b_{(3)})\left(1\otimes\phi(b_{(4)})\right)\left(1\otimes\psi(b_{(5)})\right) = \rho(b)^{\phi*\psi}. \end{aligned}$$

On the other hand,

$$\begin{aligned} (\tau^{\phi})^{\psi}(b) &= \Delta\psi^{-1}(b_{(1)})\tau^{\phi}(b_{(2)})(\psi\otimes\mathrm{id})\rho^{\phi}(b_{(3)})\left(1\otimes\psi(b_{(4)})\right) \\ &= \Delta(\phi*\psi)^{-1}(b_{(1)})\tau(b_{(2)})(\phi\otimes\mathrm{id})\rho(b_{(3)})\left(1\otimes\phi(b_{(4)})\right)\left(1\otimes\phi^{-1}(b_{(5)})\right) \\ &\qquad (\psi\otimes\mathrm{id})\rho(b_{(6)})\left(1\otimes\phi(b_{(7)})\right)\left(1\otimes\psi(b_{(8)})\right) = \tau^{\phi*\psi}; \end{aligned}$$

here one uses the axiom (2.11). □

The following simple Lemma is important in what follows.

Lemma 3.1.6. *(i) Let* $(\rightharpoonup, \sigma) \in \mathrm{Weak}(B \otimes A, A) \times \hom(B \otimes B, A)$, $\phi \in \mathrm{Reg}_1(B, A)$. *Then the (non necessarily associative) algebras* $A\#_\sigma B$ *and* $A\#_{\phi\sigma}B$ *are isomorphic.*

(ii) Let $(\rho, \tau) \in \mathrm{Coweak}(B, B \otimes A) \times \hom(B, A \otimes A)$, $\psi \in \mathrm{Reg}_\varepsilon(B, A)$. *Then the (non necessarily coassociative) coalgebras* $A^{\tau}\#B$ *and* $A^{\tau^\psi}\#B$ *are isomorphic.*

Proof. *(i).* Let $\mathcal{F} = \mathcal{F}_\phi : A\#_\sigma B \to A\#_{\phi\sigma}B$ be the application

$$\mathcal{F}(a\#b) = a\phi^{-1}(b_{(1)})\#b_{(2)}.$$

Clearly, $\mathcal{F}_\phi \mathcal{F}_\psi = \mathcal{F}_{\phi * \psi}$. Then on one hand,

$$\begin{aligned}\mathcal{F}\left((a\#b)(c\#d)\right) &= \mathcal{F}\left(a(b_{(1)} \rightharpoonup c)\sigma(b_{(2)} \otimes d_{(1)})\#b_{(3)}d_{(2)}\right)\\ &= a(b_{(1)} \rightharpoonup c)\sigma(b_{(2)} \otimes d_{(1)})\phi^{-1}(b_{(3)}d_{(2)})\#b_{(4)}d_{(3)};\end{aligned}$$

on the other,

$$\begin{aligned}\mathcal{F}(a\#b)\mathcal{F}(c\#d) &= \left(a\phi^{-1}(b_{(1)})\#b_{(2)}\right)\left(c\phi^{-1}(d_{(1)})\#d_{(2)}\right)\\ &= a\phi^{-1}(b_{(1)})(b_{(2)} \,{}^{\phi}\!\rightharpoonup c\phi^{-1}(d_{(1)}))\,{}^{\phi}\sigma(b_{(3)} \otimes d_{(2)})\#b_{(4)}d_{(3)}\\ &= a(b_{(1)} \rightharpoonup c\phi^{-1}(d_{(1)}))(b_{(2)} \rightharpoonup \phi(d_{(2)}))\sigma(b_{(3)} \otimes d_{(3)})\phi^{-1}(b_{(4)}d_{(4)})\#b_{(5)}d_{(5)};\end{aligned}$$

we only used the definitions and (2.1).

(ii). Let $\mathcal{G} : A^{\tau}\#B \to A^{\tau^\psi}\#B$ be the application

$$\mathcal{G}(a\#b) = a\psi(b_{(1)}) \otimes b_{(2)}.$$

Then

$$\begin{aligned}(\mathcal{G} \otimes \mathcal{G})\Delta(a\#b) &= (\mathcal{G} \otimes \mathcal{G})\left(a_{(1)}\tau(b_{(1)})_j \otimes \rho(b_{(2)})_i \otimes a_{(2)}\tau(b_{(1)})^j\rho(b_{(2)})^i \otimes b_{(3)}\right)\\ & a_{(1)}\tau(b_{(1)})_j\psi\left(\rho(b_{(2)})_{i(1)}\right) \otimes \rho(b_{(2)})_{i(2)} \otimes a_{(2)}\tau(b_{(1)})^j\rho(b_{(2)})^i\psi(b_{(3)}) \otimes b_{(4)}\\ &= a_{(1)}\tau(b_{(1)})_j\psi\left(\rho(b_{(2)})_i\right) \otimes \rho(b_{(3)})_h \otimes a_{(2)}\tau(b_{(1)})^j\rho(b_{(2)})^i\rho(b_{(3)})^h\psi(b_{(4)}) \otimes b_{(5)};\end{aligned}$$

and this equals

$$\begin{aligned}\Delta\mathcal{G}(a\#b) &= \Delta\left(a\psi(b_{(1)}) \otimes b_{(2)}\right)\\ &= a_{(1)}\psi(b_{(1)})_{(1)}\tau^\psi(b_{(2)})_j \otimes \rho^\psi(b_{(3)})_i \otimes a_{(2)}\psi(b_{(1)})_{(2)}\tau^\psi(b_{(2)})^j\rho^\psi(b_{(3)})^i \otimes b_{(4)}\end{aligned}$$

by (3.1.4), (3.1.5). □

Observe that, if $A^{\tau}\#_\sigma B$ denotes the "bialgebra" obtained from $(\rightharpoonup, \sigma) \in \mathrm{Weak}(B \otimes A, A) \times \hom(B \otimes B, A)$, $(\rho, \tau) \in \mathrm{Coweak}(B, B \otimes A) \times \hom(B, A \otimes A)$ (without associativity, coassociativity nor compatibility between the multiplication and comultiplication), and $\phi \in \mathrm{Reg}_{1,\varepsilon}(B, A)$, then $A^{\tau}\#_\sigma B$ and $A^{\tau^{\phi^{-1}}}\#_{\phi\sigma}B$ are isomorphic, as follows from the proof of the preceding Lemma.

Let us now introduce

$$
\begin{aligned}
Z^{1,0}(B,A) &= \{(\rightharpoonup,\sigma)\in \mathrm{Weak}(B\otimes A,A)\times \hom(B\otimes B,A): \sigma(h,1)=1,\\
&\qquad (\rightharpoonup,\sigma) \text{ satisfies the cocycle condition (2.4) and the T.M.C. (2.5)}\},\\
Z^{0,1}(B,A) &= \{(\rho,\tau)\in \mathrm{Coweak}(B,B\otimes A)\times \hom(B,A\otimes A): (\varepsilon\otimes \mathrm{id})\tau=\varepsilon,\\
&\qquad (\rho,\tau) \text{ satisfies the co-cocycle condition (2.13) and the T.C-M.C. (2.14)}\},\\
Z^{1}(B,A) &= \{\mathcal{D}=(\rightharpoonup,\sigma,\rho,\tau)\in Z^{1,0}(B,A)\times Z^{0,1}(B,A):\\
&\qquad \mathcal{D} \text{ is a compatible data}\},\\
\mathbf{Z}^{1}(B,A) &= \{\mathcal{D}=(\rightharpoonup,\sigma,\rho,\tau)\in Z^{1,0}(B,A)\times Z^{0,1}(B,A): \mathcal{D} \text{ is a Hopf data}\}.
\end{aligned}
$$

We shall consider the left action of $\mathrm{Reg}_{1,\varepsilon}(B,A)$ on the set of datas $\mathrm{Weak}(B\otimes A,A)\times \hom(B\otimes B,A)\times \mathrm{Coweak}(B,B\otimes A)\times \hom(B,A\otimes A)$ given by $\phi\mathcal{D} = {}^{\phi}\mathcal{D}$, where if $\mathcal{D}=(\rightharpoonup,\sigma,\rho,\tau)$ then ${}^{\phi}\mathcal{D} = ({}^{\phi}\!\rightharpoonup, {}^{\phi}\sigma, \rho^{\phi^{-1}}, \tau^{\phi^{-1}})$.

Proposition 3.1.7. *(i) $Z^{1,0}$ is stable by the action of $\mathrm{Reg}_1(B,A)$ defined in Lemma 3.1 (i). Let $H^{1,0}(B,A)$ be the quotient of $Z^{1,0}$ by the action of $\mathrm{Reg}_1(B,A)$. Then $(\rightharpoonup,\sigma)\mapsto A\#_\sigma B$ induces an application from $H^{1,0}(B,A)$ to the set of isomorphism classes of B-extensions of algebras of A (see 3.2 below).*

(ii) $Z^{0,1}$ is stable by the action of $\mathrm{Reg}_\varepsilon(B,A)$ defined in Lemma 3.1 (ii). Let $H^{0,1}(B,A)$ be the quotient of $Z^{0,1}(B,A)$ by of $\mathrm{Reg}_\varepsilon(B,A)$. Then $(\rho,\tau)\mapsto A^{\tau}\#B$ gives rise to an application from $H^{0,1}(B,A)$ to the set of isomorphism classes of A-extensions of coalgebras of B.

(iii) Z^1 is stable by the action of $\mathrm{Reg}_{1,\varepsilon}(B,A)$ defined above. Let $H^1(B,A)$ be the quotient of $Z^1(B,A)$ by of $\mathrm{Reg}_{1,\varepsilon}(B,A)$. Then $(\rightharpoonup,\sigma,\rho,\tau)\mapsto A^{\tau}\#_\sigma B$ gives rise to an application from $H^1(B,A)$ to the set of isomorphism classes of B-extensions of bialgebras of A.

(iv) The statement obtained from (iii) by replacing "compatible" by "Hopf" is still true.

Proof. *(i)* follows from Lemma 3.1.6 (i) and Proposition 2.6; in turn *(ii)* follows from Lemma 3.1.6 (ii) and Proposition 2.16. *(iii)* follows from the preceding and 2.20, and *(iv)* from *(iii)* and [Sw, 4.0]. □

3.2 Cleft extensions. In this subsection, we first recall some facts about cleft extensions of algebras from [D] and then state their dual analogues.

Let B be a Hopf algebra. A B-comodule algebra is an algebra C, which is simultaneously a B-comodule and whose structural morphism $\gamma: C\to C\otimes B$ is a morphism of algebras. The subalgebra of invariants is

$$\{c\in C:\gamma(c)=c\otimes 1\}.$$

Let A be the subalgebra of invariants: then one says that C is a B-extension of A (more precisely, an extension of algebras), and denotes C/A. A morphism of extensions preserves, by definition, the algebra and the comodule structures, and induces the identity on the algebra of invariants. An extension is *cleft* if there exists $\chi \in \mathrm{Reg}_1(B, C)$ such that

$$\gamma\chi = (\chi \otimes \mathrm{id})\Delta; \tag{3.2.0}$$

such χ is called a section. Notice that [D]

$$\gamma\chi^{-1}(b) = \chi^{-1}(b_{(2)}) \otimes b_{(1)}.$$

Assume that B acts weakly on an algebra A and let σ be a cocycle satisfying (2.3), (2.4), (2.5). Then $C = A\#_\sigma B$ is a B-extension of A, up to identifying the last with the subalgebra of invariants via the map $a \mapsto a\#1$. Here, the comodule map is

$$a\#b \mapsto a\#b_{(1)} \otimes b_{(2)}.$$

Moreover, the morphism $\chi : B \to C$, $\chi(b) = 1\#b$ is a section if σ belongs to $\mathrm{Reg}(B \otimes B, A)$; the only non-trivial part follows from [BM, Prop.1.8]. Let us recall from *loc. cit.* that

$$\chi^{-1}(b) = \sigma^{-1}(Sb_{(2)} \otimes b_{(3)}) \otimes Sb_{(1)}.$$

Thus, in such case $A\#_\sigma B$ is a cleft B-extension of A. Conversely, one has the following important fact:

Theorem 3.2.1 ([DT, Th. 11]). *Let C be a cleft B-extension of A, $\chi : B \to C$ a section. Define $\rightharpoonup: B \otimes A \to A$, $\sigma : B \otimes B \to A$, by*

$$b \rightharpoonup a = \chi(b_{(1)})a\chi^{-1}(b_{(2)}), \tag{3.2.2}$$

$$\sigma(b \otimes \tilde{b}) = \chi(b_{(1)})\chi(\tilde{b}_{(1)})\chi^{-1}(b_{(2)}\tilde{b}_{(2)}). \tag{3.2.3}$$

Then $\rightharpoonup$ is a weak action, σ is inversible, the algebra $A\#_\sigma B$ is associative and the cleft extensions C/A and $A\#_\sigma B/A$ are isomorphic via

$$a\#b \mapsto a\chi(b), \qquad c \mapsto c_{(0)}\chi^{-1}(c_{(1)})\#c_{(2)}.$$

For obvious reasons, we change now our notation. Let A be a Hopf algebra. An A-module coalgebra is a coalgebra C, which is simultaneously an A-module and whose structural morphism $\mu : A \otimes C \to C$ is a morphism of coalgebras. The coalgebra of covariants is C/A^+C. If B is the coalgebra of covariants, then one says that C is an A-extension of B (of coalgebras) and denotes $C\backslash B$. We shall denote by π the canonical morphism $C \to B$. Morphisms of extensions of coalgebras preserve both the module and the coalgebra structures and induce the identity on the coalgebra of covariants. An extension is *cleft* if there exists $\xi \in \mathrm{Reg}_\varepsilon(C, A)$ such that

$$\xi(ac) = a\xi(c), \quad \forall a \in A, c \in C; \tag{3.2.4}$$

ξ is then called a retraction. Notice that

$$\xi^{-1}(ac) = \xi^{-1}(c)S(a).$$

Assume now that B is a coalgebra provided with a weak coaction $\rho : B \to B\otimes A$. Let us also fix a co-cocycle $\tau : B \to A \otimes A$ satisfying $(\varepsilon \otimes \mathrm{id})\tau = \varepsilon$, (2.13), (2.14). Then $C = A^{\tau}\#B$ is an A-extension of B, up to identifying the last with the coalgebra of covariants via the map $a\#b \mapsto \varepsilon(a)b$. A acts by $a.(\tilde{a}\#b) = a\tilde{a}\#b$. Moreover, the morphism $\xi : C \to A$, $\xi(a\#b) = a\varepsilon(b)$ is a retraction if τ belongs to $\mathrm{Reg}(B, A \otimes A)$, as follows from the following Lemma whose proof runs as that of [BM, Prop.1.8].

Lemma 3.2.5. *ξ is invertible if and only if τ is.*

Proof. Assume that ξ is invertible. One proves easily that

$$\tau(\pi c) = \Delta(\xi^{-1}c_{(1)})\xi(c_{(2)}) \otimes \xi(c_{(3)});$$

from this one finds the following expression

$$\tau^{-1}(\pi c) = \xi^{-1}(c_{(2)}) \otimes \xi^{-1}(c_{(1)})\, \Delta(\xi c_{(3)}).$$

Conversely, assume that τ is invertible. Let

$$\eta(a\#b) = \tau^{-1}(b)_k S(a\tau^{-1}(b)^k).$$

One proves easily that $\xi * \eta$ is the trivial morphism. The other multiplication is the only non-trivial point of the Lemma. One has

$$\eta*\xi(a\#b) = \varepsilon(a)\tau^{-1}\left(\rho(b_{(2)})_i\right)_k S\left(\tau(b_{(1)})_j\tau^{-1}\left(\rho(b_{(2)})_i\right)^k\right)\tau(b_{(1)})^j\rho(b_{(2)})^i. \tag{3.2.6}$$

Multiplying the co-cocycle condition on $b_{(2)}$ by $(\Delta \otimes \mathrm{id})\tau^{-1}(b_{(1)})$, one gets

$$\begin{aligned}(\tau \otimes \mathrm{id})\rho(b) &= \left[\tau^{-1}(b_{(1)})_k\right]_{(1)}\tau(b_{(2)})_j \\ &\otimes \left[\tau^{-1}(b_{(1)})_k\right]_{(2)}\left[\tau(b_{(2)})^j\right]_{(1)}\tau(b_{(3)})_h \otimes \tau^{-1}(b_{(1)})^k\left[\tau(b_{(2)})^j\right]_{(2)}\tau(b_{(3)})^h\end{aligned} \tag{3.2.7}$$

On the other hand, as ρ is a weak coaction, the inverse of the application $b \mapsto (\tau \otimes \mathrm{id})\rho(b)$ is the application $b \mapsto (\tau^{-1} \otimes \mathrm{id})\rho(b)$. One deduces therefore from (3.2.7) that

$$\begin{aligned}(\tau^{-1} \otimes \mathrm{id})\rho(b) &= \tau^{-1}(b_{(2)})_j\left[\tau(b_{(3)})_k\right]_{(1)} \\ &\otimes \tau^{-1}(b_{(1)})_h\left[\tau^{-1}(b_{(2)})^j\right]_{(1)}\left[\tau(b_{(3)})_k\right]_{(2)} \otimes \tau^{-1}(b_{(1)})^h\left[\tau^{-1}(b_{(2)})^j\right]_{(2)}\tau(b_{(3)})^k\end{aligned} \tag{3.2.8}$$

Now one can proceed with (3.2.6), thanks to (3.2.8):

$$\begin{aligned}&\tau^{-1}\left(\rho(b_{(2)})_i\right)_k S\left(\tau(b_{(1)})_j\tau^{-1}\left(\rho(b_{(2)})_i\right)^k\right)\tau(b_{(1)})^j\rho(b_{(2)})^i \\ &= \tau^{-1}(b_{(3)})_h\left[\tau(b_{(4)})_q\right]_{(1)} S\left(\tau(b_{(1)})_j\tau^{-1}(b_{(2)})_k\left[\tau^{-1}(b_{(3)})^h\right]_{(1)}\left[\tau(b_{(4)})_q\right]_{(2)}\right) \\ &\qquad \tau(b_{(1)})^j\tau^{-1}(b_{(2)})^k\left[\tau^{-1}(b_{(3)})^h\right]_{(2)}\tau(b_{(4)})^q \\ &= \tau^{-1}(b)_h\varepsilon(\tau^{-1}(b)^h) = (\mathrm{id}\otimes\varepsilon)\tau^{-1}(b) = \varepsilon(b).\end{aligned}$$

This proves that $\eta * \xi$ is the trivial morphism and hence that ξ is invertible. □

Now we prove the dual version of Theorem 3.2.1.

Proposition 3.2.9. *Let C be a cleft A-extension of a coalgebra B and let $\xi \in \mathrm{Reg}_\varepsilon(C, A)$ be a retraction. Define $\bar{\rho} : C \to B \otimes A$, $\bar{\tau} : C \to A \otimes A$ by*

$$\bar{\rho}(c) = \pi(c_{(2)}) \otimes \xi^{-1}(c_{(1)})\xi(c_{(3)}), \tag{3.2.10}$$

$$\bar{\tau}(c) = \Delta\xi^{-1}(c_{(1)})\,\xi(c_{(2)}) \otimes \xi(c_{(3)}). \tag{3.2.11}$$

Then $\bar{\rho}$, $\bar{\tau}$ give rise to $\rho : B \to B \otimes A$, $\tau : B \to A \otimes A$; ρ is a weak coaction; (ρ, τ) belongs to $Z^{0,1}(B, A)$ and hence $A^\tau \# B$ is a cleft A-extension of B. Moreover, the application $C \to A^\tau \# B$ given by

$$c \mapsto \xi(c_{(1)}) \otimes \pi(c_{(2)})$$

is an isomorphism of A-extensions of B, whose inverse is induced by the map $A \otimes C \to C$,

$$a \otimes c \mapsto a\xi^{-1}(c_{(1)})c_{(2)}.$$

Proof. It is straightforward; for example one proves that both sides of the co-cocycle condition equal $(\Delta \otimes \mathrm{id})\Delta\xi^{-1}(c_{(1)})(\xi^{\otimes 3}\Delta(c_{(2)}))$; and both sides of the twisted comodule condition equal $\pi(c_{(2)}) \otimes \Delta\xi^{-1}(c_{(1)})(\xi^{\otimes 2}\Delta(c_{(3)}))$. We leave the details to the reader. □

Let us now consider

$$Z_*^{1,0}(B, A) = \{(\rightharpoonup, \sigma) \in Z^{1,0}(B, A) : \sigma \text{ is invertible}\},$$
$$Z_*^{0,1}(B, A) = \{(\rho, \tau) \in Z^{0,1}(B, A) : \tau \text{ is invertible}\}.$$

Let $H_*^{1,0}$ (resp., $H_*^{0,1}$) be the quotient of $Z_*^{1,0}$ (resp., $Z_*^{0,1}$) by the action of Reg_1 (resp., Reg_ε).

Proposition 3.2.12. *The applications defined in Proposition 3.1.7 give rise to bijections*

$$H_*^{1,0}(B, A) \simeq \{\textit{isomorphy classes of cleft extensions of algebras } C/A\},$$
$$H_*^{0,1}(B, A) \simeq \{\textit{isomorphy classes of cleft extensions of coalgebras } C\backslash B\}.$$

Proof. The first is proved in [D]. As for the second, surjectivity follows from Proposition 3.2.9. Let us prove injectivity. Let $(\rho, \tau), (\rho_1, \tau_1) \in Z_*^{0,1}(B, A)$ and let $\eta : C = A^\tau \# B \to A^{\tau_1} \# B = C_1$ be an isomorphism of extensions. Let $\xi : C \to A$, $\xi_1 : C_1 \to A$ be the retractions defined above and let $\bar{\xi} = \xi_1\eta$, $\bar{\nu} = \xi^{-1} * \bar{\xi}$. Then $\bar{\nu}$ factorizes through $\nu \in \mathrm{Reg}_\varepsilon(B, A)$; indeed $\bar{\nu}(ac) = \xi^{-1}(a_{(1)}c_{(1)})\bar{\xi}(a_{(2)}c_{(2)}) = \xi^{-1}(c_{(1)})\mathcal{S}(a_{(1)})a_{(2)}\bar{\xi}(c_{(2)}) = \varepsilon(a)\bar{\nu}(c)$. Now

$$\begin{aligned}\rho^\nu(\pi c) &= (1 \otimes \nu^{-1}\pi(c_{(1)}))\rho(\pi c_{(2)})(1 \otimes \nu\pi(c_{(3)}))\\ &= \left(1 \otimes (\bar{\xi})^{-1}(c_{(1)})\xi(c_{(2)})\right)\left(\pi c_{(4)} \otimes \xi^{-1}(c_{(3)})\xi(c_{(5)})\right)\left(1 \otimes \xi^{-1}(c_{(6)})\bar{\xi}(c_{(7)})\right)\\ &= \pi c_{(2)} \otimes (\xi_1)^{-1}(\eta c_{(1)})\xi_1(\eta c_{(3)}) = \rho_1(\pi c).\end{aligned}$$

Here one uses all the requirements to a morphism of extensions. Similarly, $\tau^\nu = \tau_1$ and injectivity follows. □

Definition 3.2.13. Let

$$(\mathcal{C}) \qquad 0 \to A \xrightarrow{\iota} C \xrightarrow{\pi} B \to 0$$

be an exact sequence of Hopf algebras. We shall say that $(\mathcal{C})$ is cleft, or else that C is a cleft extension of the Hopf algebra A by the Hopf algebra B, if there exist a section $\chi \in \mathrm{Reg}_1(B, C)$ of the algebra extension and a retraction $\xi \in \mathrm{Reg}_\varepsilon(C, A)$ of the coalgebra extension, satisfying the following equivalent conditions:

(1) $\chi^{-1}(\pi c) = \mathcal{S}(c_{(1)})\xi(c_{(2)})$.
(2) $\chi(\pi c) = \xi^{-1}(c_{(1)})c_{(2)}$.
(3) $\xi^{-1}(c) = \chi(\pi c_{(1)})\mathcal{S}(c_{(2)})$.
(4) $\xi(c) = c_{(1)}\chi^{-1}(\pi c_{(2)})$.
(5) $\xi\chi = \varepsilon_B 1_A$.

Proof of the equivalence. (1) $\iff \cdots \iff$ (4) $\implies$ (5) is easy. (5) $\implies$ (1): Let $\eta(\pi c) = \mathcal{S}(c_{(1)})\xi(c_{(2)})$. As χ is a section one shows easily that $\chi(\pi c_{(1)})\mathcal{S}(c_{(2)}) \in A$. Then $\chi * \eta(\pi c) = \chi(\pi c_{(1)})\mathcal{S}(c_{(2)})\xi(c_{(3)}) = \xi\left(\chi(\pi c_{(1)})\mathcal{S}(c_{(2)})c_{(3)}\right) = \varepsilon(c)$, by hypothesis. As χ is invertible, this implies that $\eta = \chi^{-1}$. □

Now we are ready to present the main result of this section. Let

$$Z^1_*(B, A) = \{(\rightharpoonup, \sigma, \rho, \tau) \in Z^1(B, A) : \sigma \text{ and } \tau \text{ are invertible}\}.$$

We shall see (cf. Lemma 3.2.17 below) that the bialgebra $A^\tau \#_\sigma B$ is actually a Hopf algebra if $(\rightharpoonup, \sigma, \rho, \tau) \in Z^1_*(B, A)$. Let H^1_* be the quotient of Z^1_* by the action of $\mathrm{Reg}_{1,\varepsilon}$ defined before Proposition 3.1.7.

Theorem 3.2.14. *$H^1_*(B, A)$ classifies cleft extensions $0 \to A \xrightarrow{\iota} C \xrightarrow{\pi} B \to 0$ up to isomorphisms.*

Proof. By [BM, Prop. 1.8], Lemma 3.2.5 and condition (5) in Definition 3.2.13, the application considered in Proposition 3.1.7 gives rise to a map from $H^1_*(B, A)$ to the set of isomorphy classes of cleft extensions of Hopf algebras. Let us prove that it is surjective. Let $(\mathcal{C})$ be a cleft exact sequence of Hopf algebras, with a section χ and a retraction ξ as in Definition 3.2.13. Let $\rightharpoonup, \sigma, \rho, \tau$ be defined by (3.2.2), (3.2.3), (3.2.10), (3.2.11). Let $\mathcal{F} : A^\tau \#_\sigma B \to C$ be defined by $\mathcal{F}(a\#b) = a\chi(b)$; $\mathcal{F}$ is an isomorphism of extensions of algebras (Theorem 3.2.1). But $\mathcal{F}(a\#\pi c) = a\xi^{-1}(c_{(1)})c_{(2)}$ (by (2) in 3.1.13) and hence $\mathcal{F}$ is also an isomorphism of extensions of coalgebras (Proposition 3.2.9). This implies the surjectivity.

Let us proceed then with injectivity. Let $(\rightharpoonup, \sigma, \rho, \tau), (\rightharpoonup_1, \sigma_1, \rho_1, \tau_1) \in Z^1_*(B, A)$. Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\iota} & A^\tau \#_\sigma B & \xrightarrow{\pi} & B & \longrightarrow & 0 \\ & & \downarrow \mathrm{id} & & \downarrow \Theta & & \downarrow \mathrm{id} & & \\ 0 & \longrightarrow & A & \xrightarrow{\iota_1} & A^{\tau_1} \#_{\sigma_1} B & \xrightarrow{\pi_1} & B & \longrightarrow & 0 \end{array}$$

be an isomorphism of (cleft) exact sequences of Hopf algebras and let χ, ξ, χ_1, ξ_1 be the corresponding sections and retractions. Let $\nu \in \mathrm{Reg}_\varepsilon(B, A)$ such that $\nu\pi =$

$\xi^{-1} * (\xi_1\Theta)$ and let $\mu \in \mathrm{Reg}_1(B, A)$ such that $\iota\mu = (\Upsilon\chi_1) * \chi^{-1}$. We know that $(\rightharpoonup_1, \sigma_1) = \mu.(\rightharpoonup, \sigma)$ and $(\rho_1, \tau_1) = (\rho, \tau).\nu$ ([D, Lemma 2.1] and Prop. 3.2.12). So we only need to prove that

$$\mu \stackrel{?}{=} \nu^{-1}. \tag{3.2.15}$$

Let $c = a\#b \in A^\tau\#_\sigma B$. Then

$$\begin{aligned}\nu(\pi c) &= \xi^{-1}\left(a_{(1)}\tau(b_{(1)})_j\#\rho(b_{(2)})_i\right)\xi_1\Theta\left(a_{(2)}\tau(b_{(1)})^j\rho(b_{(2)})^i\#b_{(3)}\right)\\ &= \tau^{-1}\left(\rho(b_{(2)})_i\right)_k \mathcal{S}\left(a_{(1)}\tau(b_{(1)})_j\tau^{-1}\left(\rho(b_{(2)})_i\right)^k\right)a_{(2)}\tau(b_{(1)})^j\rho(b_{(2)})^i\xi_1\Theta(1\#b_{(3)})\\ &= \varepsilon(a)\xi_1\Theta(1\#b) = \xi_1\Theta(1\#\pi c).\end{aligned}$$

Here the first equality is by definition, the second uses the formula in Lemma 3.2.4 and that both ξ_1 and Θ are morphisms of A-modules; the third follows because (3.2.6) is equal to the trivial morphism. On the other hand, from $c = \xi(c_{(1)})\otimes\pi(c_{(2)})$ it follows that

$$\Upsilon(1\#b) = \xi\Upsilon(1\#b_{(1)})\#b_{(2)} \tag{3.2.16}$$

and hence

$$\begin{aligned}&\iota\mu(b) = \Upsilon(1\#b_{(1)})\sigma^{-1}(\mathcal{S}b_{(3)}\otimes b_{(4)})\#\mathcal{S}(b_{(2)})\\ &= \xi\Upsilon(1\#b_{(1)})\left(b_{(2)} \rightharpoonup \sigma^{-1}(\mathcal{S}b_{(7)}\otimes b_{(8)})\right)\sigma(b_{(3)}\otimes\mathcal{S}b_{(6)})\#b_{(4)}\mathcal{S}b_{(5)} = \xi\Upsilon(1\#b)\#1,\end{aligned}$$

where we have used the formulas in the proof of [BM, Prop. 1.8]. Applying now Θ to (3.2.16), we obtain

$$1\#b = \xi\Upsilon(1\#b_{(1)})\Theta(1\#b_{(2)}) = \xi\Upsilon(1\#b_{(1)})\xi_1\Theta(1\#b_{(2)})\#b_{(3)}$$

and hence $\pi * \nu$ is the trivial morphism, i.e. (3.2.15) holds. □

Lemma 3.2.17. *Let $(\rightharpoonup, \sigma, \rho, \tau) \in Z^1_*(B, A)$. Then the bialgebra $A^\tau\#_\sigma B$ is a Hopf algebra and its antipode is defined by*

$$\begin{aligned}\mathcal{S}(a\#b) = &\left[\left(\sigma^{-1}\left(\mathcal{S}\rho(b_{(2)})_h\otimes\rho(b_{(3)})_j\right)\otimes\mathcal{S}\rho(b_{(1)})_i\right]\right.\\ &\left[\tau^{-1}(b_{(4)})_k\mathcal{S}\left(a\rho(b_{(1)})^i\rho(b_{(2)})^h\rho(b_{(3)})^j\tau^{-1}(b_{(4)})^k\right)\otimes 1\right]\end{aligned} \tag{3.2.18}$$

Proof. Let ξ, χ be the "canonical" retraction and section of $A^\tau\#_\sigma B$. Then the equality $c = \xi(c_{(1)})\otimes\pi(c_{(2)})$ can be rephrased as $\mathrm{id}_C = (\iota\xi) * (\chi\pi)$ in the algebra $\mathrm{End}(C)$. But we know that ξ and χ are invertible ([BM, Prop. 1.8] and Lemma 3.2.5). It follows that id_C is invertible and $\mathcal{S}_C = {\mathrm{id}_C}^{-1} = (\chi^{-1}\pi) * (\iota\xi^{-1})$. From the expressions for χ^{-1} and ξ^{-1} (3.2.18) follows. □

Cleft extensions have another pleasant properties.

Lemma 3.2.19. *Let us a consider a morphism of exact sequences of Hopf algebras*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\iota} & C & \xrightarrow{\pi} & B & \longrightarrow & 0 \\ & & {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \Theta}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & \\ 0 & \longrightarrow & A & \xrightarrow{\iota_1} & C_1 & \xrightarrow{\pi_1} & B & \longrightarrow & 0 \end{array}$$

where the top exact sequence is cleft. Then the bottom is also cleft and Θ is an isomorphism.

Proof. Let χ, ξ be as in Definition 3.2.13. Let $\chi_1 = \Theta\chi$; clearly, χ_1 is a retraction of C_1. Let $d \in C_1$; one has $d = d_{(0)}\chi_1^{-1}(d_{(1)})\chi_1(d_{(2)})$. But $d_{(0)}\chi_1^{-1} \in A \subset \operatorname{Im}\Theta$ and hence Θ is surjective. Next we claim that $\xi_1(d) = \xi(c)$ if $\Theta(c) = d$ is well defined. For, if $\Theta(c) = 0$, then $\xi(c) = c_{(1)}\chi^{-1}(\pi c_{(2)}) = \Theta(c_{(1)})\chi_1^{-1}\pi_1\Theta(c_{(2)}) = 0$. Moreover, ξ_1 is a section and the bottom exact sequence is cleft. Now assume again that $\Theta(c) = 0$. Then $0 = (\xi_1 \otimes \pi_1)(\Theta \otimes \Theta)\Delta(c) = (\xi \otimes \pi)\Delta(c)$ and hence $c = c_{(1)}\chi^{-1}(\pi c_{(2)})\chi(\pi c_{(3)}) = \xi(c_{(1)})\chi(\pi c_{(2)}) = 0$. □

APPENDIX. BICOVARIANT BIMODULES

NICOLÁS ANDRUSKIEWITSCH

The notion of bicovariant bimodule was introduced in [W], see also [W3]. A crucial feature is that each bicovariant bimodule comes equipped with a solution of the braided (or Yang-Baxter) equation. According to [PW, p. 411], Connes conjectured in 1986 that "bicovariant bimodules over the algebra of smooth functions on a quantum group are (in natural way) labeled by representations of another quantum group". This was solved (affirmatively) in [PW], by introducing the quantum double, in a dual way to [D1]. In [T2], it was given an alternatively description of the quantum double. It turns out [T2] that the representations of the quantum double are exactly the crossed bimodules for the original algebra. (This was also previously observed in [M] under a finiteness hypothesis.) Crossed bimodules were introduced in [Y] and it was proved there that their category is *braided*.

In this appendix, we review briefly these facts and complete this circle of ideas. We show that the space of "left invariants" is in fact a crossed bimodule and that there is a one-to-one correspondence beteween crossed bimodules and bicovariant bimodules. (This is merely a translation of some facts in [W] to a coordinate-free language.) Moreover, the category of such bimodules is quasitensorial [D2], hence braided, and the corresponding solution of the quantum Yang-Baxter equation (found in [Y]) is the same that of [W].

Interesting examples of crossed bimodules are the right adjoint corepresentation with the right multiplication, or (dually) the right adjoint representation with the right comultiplication. The solutions of the QYBE they give rise were first presented in [W2], by a direct computation. (See Corollary A.3) Moreover, by means of these solutions, any Hopf algebra is *generalized commutative* and *generalized cocommutative*, in the sense of[GRR], [C], [Mn].

A.1 Left covariant bimodules. We preserve the notation of the paper. The structural morphism of a comodule will be usually denoted c; when one space carries two different comodule structures, we shall write c_r (resp., c_ℓ or simply c) for the right (resp., left) one.

A left covariant bimodule M has, by definition, a bimodule structure and a left comodule structure, over A, both related by imposing the comodule structural morphism $c : M \to A \otimes M$ to be a morphism of bimodules; here as always we use the comultiplication to endow $A \otimes M$ with a bimodule action of A.

Let N be a right A-module. Then $M := A \otimes N$ is a left covariant bimodule via the following formulae:

$$\text{(A.1)} \quad a(b \otimes n) = ab \otimes n; \quad (b \otimes n)a = ba_{(1)} \otimes na_{(2)}; \quad c(b \otimes n) = \Delta(b)(1 \otimes 1 \otimes n).$$

Moreover, any left covariant bimodule arises in this way. Indeed, let M be a left covariant bimodule and let

$$M_{inv} = \{m \in M : c(m) = 1 \otimes m\}.$$

Let also $P : M \to M$, $P(m) = S(m_{(1)})m_{(2)}$.

Proposition A.1. *P is a projector whose image is M_{inv}, and the latter acquires a right module structure by $n.a = P(na)$; let us denote it by N. Then M is isomorphic, as a left covariant bimodule, to $A \otimes N$ with the structure explained above.*

Sketch of Proof. (See [W, Lemma 2.2, Theorem 2.1].) One shows first easily that $c(P(m)) = 1 \otimes P(m)$ and then, that $\operatorname{Im} P = M_{inv}$, $P^2 = P$. It is also obvious that $P(am) = \varepsilon(a)P(m)$ and $m = m_{(1)}P(m_{(2)})$. In particular, the restriction of the multiplication is an epimorphism $\varphi : A \otimes N \to M$. Suppose that there exist $a_i \in A$, $m_i \in N$ such that $0 = \sum a_i m_i$. Applying $(\mathrm{id} \otimes P)c$, we get $0 = \sum a_i \otimes m_i$ and hence φ is a linear isomorphism. N is a right module with the action defined above, as follows immediately from the formula $P(na) = S(a_{(1)})na_{(2)}$, $n \in N$. Consider N as a left covariant bimodule via (A.1); obviously, φ preserves the left action of A and the comodule structure. It is also easy to show that φ preserves the right action: $\varphi((1 \otimes n)b) = b_{(1)}P(nb_{(2)}) = b_{(1)}S(b_{(2)})nb_{(3)} = nb$. □

Remark. Compare the preceding with [Sw, 4.1].

A.2 Bicovariant bimodules. The notion of "left covariant bimodule" has an immediate translation to "right covariant bimodule". Now let us recall the definition of a bicovariant bimodule. This is a bimodule M, which is in addition left covariant, with structural morphism $c_\ell : M \to A \otimes M$, and right covariant, via $c_r : M \to M \otimes A$; moreover, the following diagram must commute:

$$\begin{array}{ccc} M & \xrightarrow{c_\ell} & A \otimes M \\ {\scriptstyle c_r}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \otimes c_r} \\ M \otimes A & \xrightarrow{c_\ell \otimes \mathrm{id}} & A \otimes M \otimes A. \end{array}$$

Let now N be a rigth A-module and a right A-comodule, such that

$$(1 \otimes a_{(1)})c_r(na_{(2)}) = c_r(n)\Delta(a), \quad n \in N,\, a \in A. \tag{A.2}$$

(That is, N is a "right crossed bimodule", in the terminology of [Y].)

Let $M = A \otimes N$ provided with the left covariant bimodule structure explained in *A.1* and extend c_r to M in the following way:

$$c_r(a \otimes n) = \Delta(a)c_r(n) = a_{(1)} \otimes n_{(0)} \otimes a_{(2)}n_{(1)}. \tag{A.3}$$

It is easy to see that M becomes in this way a bicovariant bimodule. In fact, (A.2) guarantees that c_r is a morphism of right modules. Moreover, any bicovariant bimodule is obtained in this guise. Finally, it is obvious that a morphism of rigth modules and right comodules $f : N \to N'$ gives rise to a morphism of bicovariant bimodules $\mathrm{id} \otimes f : M \to M'$, and that any such morphism has this form.

Example A.1. Consider A as a right module via the right multiplication and as a right comodule via the adjoint; recall that $\mathrm{ad} : A \to A \otimes A$ is defined by $\mathrm{ad}(b) = b_{(2)} \otimes S(b_{(1)})b_{(3)}$.

We claim that the preceding data satisfies (A.2). Indeed,

$$\begin{aligned}(1 \otimes a_{(1)})\,\mathrm{ad}(ba_{(2)}) &= (ba_{(2)})_{(2)} \otimes a_{(1)}S((ba_{(2)})_{(1)})(ba_{(2)})_{(3)} \\ &= b_{(2)}a_{(3)} \otimes a_{(1)}S(a_{(2)})S(b_{(1)})b_{(3)}a_{(4)} = \mathrm{ad}(b)\Delta(a).\end{aligned}$$

Observe also that $\ker \varepsilon$ is a right subcomodule for the adjoint. In fact, one has $(\varepsilon \otimes \mathrm{id})\,\mathrm{ad} = \varepsilon$, and also $(\mathrm{id} \otimes \varepsilon)\,\mathrm{ad} = \mathrm{id}$.

Example A.2. Consider now A as a right comodule via the comultiplication and as a right module via the Adjoint, that is, via the following formula: $u\,\mathrm{Ad}(v) = S(v_{(1)})uv_{(2)}$. Again (A.2) is fulfilled:

$$\begin{aligned}(1 \otimes a_{(1)})\Delta(b\,\mathrm{Ad}(a_{(2)})) &= (Sa_{(2)}ba_{(3)})_{(1)} \otimes a_{(1)}(Sa_{(2)}ba_{(3)})_{(2)} = \\ &Sa_{(3)}b_{(1)}a_{(4)} \otimes a_{(1)}Sa_{(2)}b_{(2)}a_{(5)} = S(a_{(1)})b_{(1)}a_{(2)} \otimes b_{(2)}a_{(3)}.\end{aligned}$$

Remark. Let us now assume that U is a Hopf algebra dual to A. The left U-module structures on A provided respectively by $\langle u.a, v\rangle = \langle a, v.\,\mathrm{Ad}(u)\rangle$ and (A.16) below applied to (A, ad) coincide. In fact, one has $\langle v \otimes u, \mathrm{ad}\, a\rangle = \langle v\,\mathrm{Ad}(u), a\rangle$.

Example A.3. Consider a right comodule N as a trivial A-module, i.e. $na = \varepsilon(a)n$. Then the compatibility condition (A.2) reads $n_{(0)} \otimes an_{(1)} = n_{(0)} \otimes n_{(1)}a$, which is fulfilled if A is commutative.

A.3 The quantum Yang-Baxter equation. Let *Bicov* ($Bicov_A$ if necessary) be the category of all rigth modules and right comodules satisfying the compatibility condition (A.2); the morphisms must of course preserve both structures. Let N, N' be objects of *Bicov* and let $R_{N,N'}$ be defined by the commutativity of the following diagram:

(A.4a)
$$\begin{array}{ccc} N\otimes N' & \xrightarrow{R_{N,N'}} & N\otimes N' \\ {\scriptstyle \mathrm{id}\otimes c}\downarrow & & \uparrow{\scriptstyle \mu\otimes \mathrm{id}} \\ N\otimes N'\otimes A & \xrightarrow{\mathcal{T}^{23}} & N\otimes A\otimes N'; \end{array}$$

let $S_{N,N'} : N\otimes N' \to N'\otimes N$ be defined by

(A.4 b)
$$S_{N,N'} = \mathcal{T} R_{N,N'}.$$

(We shall omit the subscripts whenever no danger of confusion is present.)

Proposition A.2. *(i) Bicov is a quasitensor category (cf. [D2]), whose associativity constraint is the usual one and whose "commutativity constraint" is* $S_{N,N'}$.

(ii) S satisfies the quantum Yang-Baxter equation (QYBE for short); that is, if N, N', N'' *are objects of Bicov, then*

(A.5)
$$(S_{N',N''}\otimes \mathrm{id})(\mathrm{id}\otimes S_{N,N''})(S_{N,N'}\otimes \mathrm{id}) = (\mathrm{id}\otimes S_{N,N'})(S_{N,N''}\otimes \mathrm{id})(\mathrm{id}\otimes S_{N',N''}).$$

(iii) S is invertible and its inverse is given by

(A.6)
$$\begin{array}{ccc} N\otimes N' & \xrightarrow{S^{-1}} & N'\otimes N \\ {\scriptstyle c\otimes \mathrm{id}}\downarrow & & \uparrow{\scriptstyle \mathcal{T}} \\ N\otimes A\otimes N' & \xrightarrow{\mathrm{id}\otimes \mu_{S^{-1}}} & N\otimes N'. \end{array}$$

Here $\mu_{S^{-1}} : A\otimes N' \to N'$ *is the left module structure given by* $\mu_{S^{-1}}(a\otimes n') = n'S^{-1}(a)$*, using that* S^{-1} *is an antihomomorphism of algebras.*

In other words, $S_{N,N''}$ is a morphism in *Bicov*, in fact a natural transformation, and the following diagrams must commute:

(A.7a)
$$\begin{array}{ccccc} (N_1\otimes N_2)\otimes N_3 & \xrightarrow{S} & N_3\otimes (N_1\otimes N_2) & \xrightarrow{\sim} & (N_3\otimes N_1)\otimes N_2 \\ \downarrow{\scriptstyle \wr} & & & & \uparrow{\scriptstyle S\otimes \mathrm{id}} \\ N_1\otimes (N_2\otimes N_3) & \xrightarrow{\mathrm{id}\otimes S} & N_1\otimes (N_3\otimes N_2) & \xrightarrow{\sim} & (N_1\otimes N_3)\otimes N_2, \end{array}$$

(A.7b)
$$\begin{array}{ccccc} N_1\otimes (N_2\otimes N_3) & \xrightarrow{S} & (N_2\otimes N_3)\otimes N_1 & \xrightarrow{\sim} & N_2\otimes (N_3\otimes N_1) \\ \downarrow{\scriptstyle \wr} & & & & \uparrow{\scriptstyle \mathrm{id}\otimes S} \\ (N_1\otimes N_2)\otimes N_3 & \xrightarrow{S\otimes \mathrm{id}} & (N_2\otimes N_1)\otimes N_3 & \xrightarrow{\sim} & N_2\otimes (N_1\otimes N_3). \end{array}$$

Proof. *(i)* Let us first prove that $S_{N,N''}$ is a morphism of A-modules. Let $a \in A$, $n \in N$, $n' \in N'$. Then

$$\begin{aligned} S((n \otimes n')a) &= S(na_{(1)} \otimes n'a_{(2)}) \\ = (n'a_{(2)})_{(0)} \otimes na_{(1)}(n'a_{(2)})_{(1)} &= n'_{(0)}a_{(1)} \otimes nn'_{(1)}a_{(2)} \\ &= S(n \otimes n')a, \end{aligned}$$

thanks to (A.2).

Now let us show that it is a morphism of comodules. Using the compatibility condition (A.2) and the first axiom of comodules, we have:

$$\begin{aligned} (S \otimes \mathrm{id})c(n \otimes n') = (S \otimes \mathrm{id})(n_{(0)} \otimes n'_{(0)} \otimes n_{(1)}n'_{(1)}) &= n'_{(0)} \otimes n_{(0)}n'_{(1)} \otimes n_{(1)}n'_{(2)} \\ = n'_{(0)} \otimes (nn'_{(2)})_{(0)} \otimes n'_{(1)}(nn'_{(2)})_{(1)} &= c(S(n \otimes n')). \end{aligned}$$

The commutativity of (A.7 a) (resp., (A.7 b)) follows from the coassociativity in the definition of comodule (resp., the definition of tensor product of comodules).

If $f : N \to P$ (resp., $g : N' \to P'$) is a morphism of modules (resp., of comodules) then $S_{P,P'}(f \otimes g) = (f \otimes g)S_{N,N'}$. In particular, S is a natural transformation.

(ii) follows from *(i)*, see [D2, Remark 4 before Prop. 3.1]. (A direct proof is straightforward.)

(iii): use that $\mathcal{S}^{-1}$ is the antipode for the opposite comultiplication and the same multiplication. □

Remark. Proposition A.2 is a generalization of [W, Prop. 3.1]. Indeed, our formula (A.4), in the case $N = N' = N''$, coincides with [W, (3.5)]; this follows from [W, (3.15), (2.35) and (2.13)]. On the other hand, Proposition A.2 (ii), (iii) were first proved in [Y].

Corollary A.3. *(i) Let $S_0 : A \otimes A \to A \otimes A$ be defined by*

$$S_0(a \otimes b) = b_{(2)} \otimes a\mathcal{S}(b_{(1)})b_{(3)}. \tag{A.8}$$

Then S_0 satisfies the quantum Yang-Baxter equation. Moreover, $S_0(a \otimes b\,\Delta(c)) = S_0(a \otimes b)\Delta(c)$ and $(S_0 \otimes \mathrm{id})\,\mathrm{ad}^{\otimes 2} = \mathrm{ad}^{\otimes 2} S_0$.

(ii) Let $S_1 : A \otimes A \to A \otimes A$ be defined by

$$S_1(a \otimes b) = b_{(1)} \otimes \mathcal{S}(b_{(2)})ab_{(3)}. \tag{A.9}$$

Then S_1 is also a solution of the QYBE.

Proof. Apply the Proposition to A within the setting of Example A.1 (resp. A.2) above. Note that if U is as in the Remark following Example A.2, then

$$\langle S_1(u \otimes v), a \otimes b\rangle = \langle u \otimes v, S_0(a \otimes b)\rangle, \quad a, b \in A, \ u, v \in U.$$

□

Now let us consider the quantum double $\mathcal{E}$ of A, as defined in [T2]: as a vector space, $\mathcal{E} = \mathrm{End}(A)$; the multiplication is given by $(T_1T_2)(a) = T_1(a_{(1)})T_2(a_{(2)})$; the comultiplication, by

$$\tilde{\Delta}(T)(x \otimes y) = (1 \otimes x_{(1)})\Delta(T(yx_{(2)})(1 \otimes \mathcal{S}(x_{(3)}));$$

cf *loc cit* for the remaining definitions.

Proposition A.4. *(i) Let V be a right $\mathcal{E}$-comodule, with structural morphism $c_{\mathcal{E}}$. Then V is an object of $Bicov_A$, with action and coaction defined by*

$$v.a = v_{(1)}\langle \varepsilon, v_{(2)}(a)\rangle, \quad c(v) = v_{(1)} \otimes v_{(2)}(1).$$

Conversely, any object V of $Bicov_A$ is an $\mathcal{E}$-comodule via the application $c_{\mathcal{E}} : V \to V\hat{\otimes}\mathcal{E} \simeq \mathrm{Hom}(A, V \otimes A)$ defined by $c_{\mathcal{E}}(v)(a) = c(va)$. These assignements are inverse of each other and hence the categories $Bicov_A$ and $Comod_{\mathcal{E}}$ are equivalent.

(ii) Let $B : \mathcal{E} \times \mathcal{E} \to k$ be the bilinear form defined by

$$B(F, G) = \langle \varepsilon, FG(1)\rangle \tag{A.10}$$

Let V, V' be two $\mathcal{E}$-comodules, let $R_{V,V'}$ be defined by the commutativity of the following diagram:

$$\begin{array}{ccc} V \otimes V' & \xrightarrow{R_{V,V'}} & V \otimes V' \\ {\scriptstyle c_V \otimes c_{V'}}\downarrow & & \uparrow{\scriptstyle \mathrm{id} \otimes B} \\ V \otimes \mathcal{E} \otimes V' \otimes \mathcal{E} & \xrightarrow{\tau^{23}} & V \otimes V' \otimes \mathcal{E} \otimes \mathcal{E}; \end{array} \tag{A.11a}$$

and let $S_{V,V'}$ be given by

$$S_{V,V'} = \tau R_{V,V'}. \tag{A.11b}$$

Then $Comod_{\mathcal{E}}$ is a quasitensor category whose commutativity constraint is $S_{V,V'}$ and the equivalence stated in (i) preserves this additional structure.

Proof. *(i)* See [T2]. *(ii)* is left to the reader.

Part *(ii)* of the Proposition above is a particular case of the following fact, a slight generalization of [Ly, Th. 2.3.3]:

Proposition A.5. *Let $\rho : A \otimes A \to k$ be a non-degenerate bilinear form satisfying*

$$\begin{aligned} f_{(1)}g_{(1)}\rho(g_{(2)}, f_{(2)}) &= \rho(g_{(1)}, f_{(1)})g_{(2)}f_{(2)}, \\ \rho(fg, h) &= \rho(f, h_{(1)})\rho(g, h_{(2)}) \\ \rho(h, gf) &= \rho(h_{(1)}, f)\rho(h_{(2)}, g) \end{aligned}$$

for any $f, g, h \in A$. Then the category of right A-comodules is quasitensorial and the commutativity constraint $S_{M,N} : M \otimes N \to N \otimes M$ is given by

$$S_{M,N}(m \otimes n) = n_{(1)} \otimes m_{(1)}\rho(m_{(2)}, n_{(2)}).$$

The data (A, ρ) is the dual version of a quasitriangular Hopf algebra, see below.

A.4 Quasitriangular Hopf algebras.

Let us assume the existence of a Hopf algebra U such that $A \hookrightarrow U^*$ is a Hopf algebra dual to U. Let us suppose first that U is quasitriangular [D1], i.e. there exists $R \in U \otimes U$ invertible satisfying

$$\Delta'(u) = R\Delta(u)R^{-1}, \quad u \in U, \tag{A.12}$$

$$(\Delta \otimes \mathrm{id})(R) = R^{13}R^{23}, \quad (\mathrm{id}\otimes\Delta)(R) = R^{13}R^{12}, \tag{A.13}$$

where Δ' is the opposite comultiplication. (A.13) implies that the application $A \to U$ given by $a \mapsto \mathrm{id}\otimes\langle a, \_\rangle(R)$ (resp., $a \mapsto \langle a, \_\rangle \otimes \mathrm{id}(R)$) is an antihomomorphism (resp., a homomorphism) of algebras. Notice that:

(A.14) If U^{cop} denotes U as Hopf algebra with the opposite comultiplication, then (U^{cop}, R^{-1}) is also a quasitriangular Hopf algebra, as well as (U^{op}, R^{-1}).

(A.15) If (V, T) is another quasitriangular Hopf algebra, then $(U \otimes V, \tau^{23}(R \otimes S))$ is also one.

Let now N be a right A-comodule, hence a left U-module with the action defined by the commutativity of the following diagram:

$$\begin{array}{ccc} U \otimes N & \longrightarrow & N \\ \tau\big\downarrow & & \big\uparrow \mathrm{id}\otimes\langle,\rangle \\ N \otimes U & \longrightarrow & N \otimes A \otimes U. \end{array} \tag{A.16}$$

We shall consider N as a right A-module by composing the preceding with the antihomomorphism $a \mapsto \mathrm{id}\otimes\langle a, \_\rangle(R)$. In concrete terms,

$$na = n_{(1)}\langle n_{(2)} \otimes a, R\rangle, \quad n \in N,\ a \in A. \tag{A.17}$$

To insure the pertenence of N to *Bicov*, we need to check the compatibility condition (A.2). Notice first that (A.12) implies, for any $u \in U$, $a, b \in A$, the equality $\langle a_{(1)} \otimes b_{(1)} \otimes a_{(2)} \otimes b_{(2)}, \Delta'(u) \otimes R\rangle = \langle a_{(1)} \otimes b_{(1)} \otimes a_{(2)} \otimes b_{(2)}, R \otimes \Delta(u)\rangle$ and thus

$$\langle a_{(2)} \otimes b_{(2)}, R\rangle b_{(1)}a_{(1)} = \langle a_{(1)} \otimes b_{(1)}, R\rangle a_{(2)}b_{(2)}. \tag{A.18}$$

Taking into account (A.17), the left hand side of (A.2) is

$$\langle n_{(2)} \otimes a_{(2)}, R\rangle n_{(0)} \otimes a_{(1)}n_{(1)};$$

the right one is

$$\langle n_{(1)} \otimes a_{(1)}, R\rangle n_{(0)} \otimes n_{(2)}a_{(2)}$$

and the equality follows from (A.18) (that is, from (A.12)).

On the other hand, let N' be another A-comodule, $n \in N$, $n' \in N'$. Then the application (A.4) gives in this case, thanks to (A.17),

$$n \otimes n' \mapsto \langle n_{(1)} \otimes n'_{(1)}, R\rangle n_{(0)} \otimes n'_{(0)}$$

which is the same as the action of R on the $U \otimes U$-module $N \otimes N'$. We have therefore proved the following fact, essentially due to Rosso [R]:

Proposition A.6. *If U is quasitriangular, then any A-comodule belongs to Bicov in a "canonical" way; moreover, the solutions of the quantum Yang-Baxter equation provided by (A.4) and R coincide.*

Let us consider A as $A \otimes A^{cop}$-comodule via $a \mapsto a_{(2)} \otimes a_{(3)} \otimes a_{(1)}$. It follows from (A.14) and (A.15) that the application $S_2 : A \otimes A \to A \otimes A$ given by

$$S_2(a \otimes b) = \langle a_{(3)} \otimes a_{(1)}, R \rangle \langle b_{(3)} \otimes b_{(1)}, R^{-1} \rangle b_{(2)} \otimes a_{(2)} \tag{A.19}$$

satisfies the Yang-Baxter equation.

Let us now recall that a *generalized commutative* algebra is a pair $(B, \mathbf{S})$, where B is an algebra with multiplication m and $\mathbf{S} : B \otimes B \to B \otimes B$ is a solution of the QYBE such that

$$m\mathbf{S} = m, \tag{A.20a}$$

$$\mathbf{S}(b \otimes 1) = 1 \otimes b, \quad \mathbf{S}(1 \otimes b) = b \otimes 1, \tag{A.20b}$$

$$\mathbf{S}(m \otimes \mathrm{id}) = (\mathrm{id} \otimes m)\mathbf{S}^{12}\mathbf{S}^{23}, \quad \mathbf{S}(\mathrm{id} \otimes m) = (m \otimes \mathrm{id})\mathbf{S}^{23}\mathbf{S}^{12}. \tag{A.20c}$$

The following fact is well-known.

Proposition A.7. *(A, S_2) is a generalized commutative algebra.*

Proof. We already showed that S_2 satisfies the quantum Yang-Baxter equation. (A.20a and c) are direct consequences of (A.12 and 13), respectively, whereas to prove (A.20b) one uses the following equalities (cf. [D3, Prop. 3.1]):

$$(\varepsilon \otimes \mathrm{id})(R) = 1 = (\mathrm{id} \otimes \varepsilon)(R). \tag{A.21}$$

Now we show that any Hopf algebra is generalized commutative.

Proposition A.8. *(A, S_1) is a generalized commutative algebra.*

Proof. It is straightforward. For example,

$$mS_1(a \otimes b) = b_{(1)}S(b_{(2)})ab_{(3)} = ab;$$

$$(\mathrm{id} \otimes m)S^{12}S^{23}(a \otimes b \otimes c) = c_{(1)} \otimes S(c_{(2)})ac_{(3)}S(c_{(4)})bc_{(5)}$$
$$= c_{(1)} \otimes S(c_{(2)})abc_{(3)} = S_1(m \otimes \mathrm{id})(a \otimes b \otimes c);$$

and the rest is similar. □

In the same vein, one defines *generalized cocommutative* coalgebras and proves that, if A is any Hopf algebra, then (A, S_0) is generalized cocommutative.

References

[BCM] R. J. Blattner, M. Cohen and S. Montgomery, *Crossed products and inner actions of Hopf algebras*, Trans. Amer. Math. Soc. **298** (1986), 671–711.

[BM] R. J. Blattner and S. Montgomery, *Crossed products and Galois extensions of Hopf algebras*, Pacific J. of Math. **137** (1989), 37–54.

[C] P. Cartier, *Exposés au Seminaire "groupes quantiques", fevrier 1992; communication personnelle.*.

[D] Y. Doi, *Equivalent crossed products for a Hopf algebra*, Commun. Algebra **17** (1989), no. 12, 3053–3085.

[DT] Y. Doi and M. Takeuchi, *Cleft comodule algebras for a bialgebra*, Commun. Algebra **14** (1986), 801–818.

[D1] V. G. Drinfeld, *Quantum groups*, Proc. of the ICM, Berkeley (1987), 798–820.

[D2] ———, *On almost cocommutative Hopf algebras*, Leningrad Math. J. **1** (1990), no. 2, 1419–1457.

[D3] ———, *Quasi-Hopf algebras*, Leningrad Math. J. **1** (1990), no. 6, 1419–1457.

[G] V. Gugenheim, *On extensions of algebras, coalgebras, and Hopf algebras I*, Amer. J. Math. **84** (1962), 349–385.

[GRR] D. I. Gurevich, A. Radul and V. Rubtsov, *Non-commutative differential geometry and the Yang-Baxter equation*, Preprint IHES.

[L] G. Lusztig, *Introduction to quantized enveloping algebras*, Proceedings of the Workshop on Representation theory of reductive Lie groups (Carlos Paz), Birkhäuser, 1992.

[Ly] V. V. Lyubashenko, *Vectorsymmetries*, Seminar on supermanifolds N! 15, Rep.,Dep. Math., Univ. Stockh. **19** (1987).

[McL] S. Mac Lane, *Categories for the Working Mathematician*, Springer-Verlag, 1971.

[M] S. Majid, *Doubles of quasitriangular Hopf algebras*, preprint.

[Mn] Y. I. Manin, *Quantum groups and non-commutative geometry*, preprint Montreal University CRM-1561 (1988).

[MW] A. Masuoka and D. Wigner, *Faithful flatness of Hopf algebras*, preprint (1992).

[MM] J. Milnor and J.C. Moore, *On the structure of Hopf algebras*, Ann. of Math. **81** (1965), 211–264.

[NZ] W.D. Nichols and M.B. Zoeller, *A Hopf algebra freeness theorem*, Amer.J. of Math. **111** (1989), 381–385.

[PaW] B. Parshall and J.P. Wang, *Quantum linear groups*, Memoirs of the AMS **439** (1991).

[PW] P. Podles and S. L. Woronowicz, *Quantum Deformation of Lorentz group*, Comm. in Math. Phys. **130** (1990), 381–431.

[R] M. Rosso, *Algèbres enveloppantes quantifiées, groupes quantiques compacts de matrices et calcul différentiel non commutatif*, Duke Math. J. **61** (1990), 11–39.

[S] W. Singer, *Extension theory for connected Hopf algebras*, J. of Algebra **21** (1972), 1–16.

[Sch] H.-J. Schneider, *Some remarks on exact sequences of quantum groups*, preprint (1992).

[Sch2] ———, *Principal homogeneous spaces for arbitrary Hopf algebras*, Israel J. of Maths. **72** (1990), 167–195.

[Sch3] ———, *Representation theory of Hopf Galois extensions*, Israel J. of Maths. **72** (1990), 196–231.

[Sw] Sweedler, M., *Hopf algebras*, Benjamin, New York, 1969.

[Sw2] ———, *Cohomology of algebras over Hopf algebras*, Trans. Amer. Math. Soc. **133** (1968), 205–239.

[T] M. Takeuchi, *A correspondance between Hopf ideals and sub-Hopf ideals*, Manuscr. Math. **7** (1972), 251–270.

[T2] ———, *Finite dimensional representations of the quantum Lorentz group*, Comm. in Math. Phys. **144** (1992), 557–580.

[T3] ———, *Relative Hopf modules - equivalences and freeness criteria*, J. Algebra **60** (1979), 452–471.

[W] S. L. Woronowicz, *Differential Calculus on Compact Matrix Pseudogroups (Quantum Groups)*, Comm. in Math. Phys. **122** (1989), 125–170.

[W2] ______, *Solutions of the Braid Equation related to a Hopf Algebra*, Lett. Math. Phys. **23** (1991), 143–145.

[W3] ______, *Twisted SU(2) group. An example of noncommutative differential calculus*, Publ. Res. Inst. Math. Sci. **23** (1987), 117–181.

[Y] D. N. Yetter, *Quantum groups and representations of monoidal categories*, Math. Proc. Camb. Phil. Soc. **108** (1990), 261–290.

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