INTEGRAL REPRESENTATIONS OF N–POINT CONFORMAL CORRELATORS IN THE WZW MODEL

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Explicit integral formulas are presented for n–point genus 0 conformal correlators of all primary operators in the conformally invariant two–dimensional Wess–Zumino–Witten models with an arbitrary simple gauge group. Moreover, these formulas provide the solution of analogues of Knizhnik–Zamoldchikov equations for the case when gauge Lie algebra is an arbitrary Kac–Moody Lie algebra.

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1. Introduction

In the remarkable papers [1] (based on the earlier work [2]) Dotsenko and Fateev have shown that all genus 0 conformal correlators[1)] of Belavin–Polyakov–Zamolodchikov minimal models of conformal field theory [3] can be expressed as certain integrals of hypergeometric type. These integrals provide a convenient tool for solving the questions connected with the mondromy of conformal correlators. Since that time their results where extended to other models of CFT, and now it is more or less clear that any such a model should admit Dotsenko–Fateev integral representations. For the WZW–model first studied by Knizhnik and Zamoldchikov, [4], such integral representations where constructed in [5–7]. However, in the above papers the explicit integral expressions for conformal correlators were written down only for some particular cases.

The aim of the present paper is to fill this gap and give explicit integral expressions for all n–point (genus 0) conformal correlators of WZW model with an arbitrary simple gauge group G. Conformal correlators of this model are characterized by the property of being solutions of the remarkable integrable system of differential equations derived by Knizhnik and Zamoldchikov [4]. So our approach is quite direct: We write down some integrals and verify that they satisfy Knizhnik–Zamolodchikov equations. Our starting point was the paper [6] , were this was first done in a particular case. It turned out that the formulas depend only on the Cartan matrix of the Lie algebra of a gauge group. This allows to generalize both KZ equations and the solutions to the case when $\mathfrak{g}$ is an arbitrary Kac–Moody Lie algebra associated with a symmetriable Cartan matrix, [8].

1) We'll follow the terminology of [1] and call conformal correlators holomorphic parts of physical correlation functions.

Note that integrals of Dotsenko–Fateev type were independently studied by mathematicians, [9–11]; especially the point of view of [9] is very close to ours. We give here mostly the formulations; the main results are Theorems 1 and 2. The detailed proofs will appear elsewhere.

2. The Knizhnik–Zamoldchikov equations

In this section we recall some definitions from Kac's book [8] and introduce the KZ equations in a more general then original setting.

2.1. Let $A = (A_{ij})$ be a generalized Cartan matrix, i.e. an $r \times r$ matrix such that $a_{ii} = 2$, a_{ij} are non–positive integers for $i \neq j$, and $a_{ij} = 0$ implies $a_{ji} = 0$. We'll suppose that A is symmetrisable and fix a symmetrisation, i.e. a decomposition $A = DB$, where $D = (d_{ij})$ is diagonal $d_{ij} = \delta_{ij}\epsilon_i$, all $\epsilon_i \neq 0$, and $B = (b_{ij})$ is symmetric, $b_{ij} = b_{ji}$.

Let $(\mathfrak{h}, M, M^V)$ be a realization of A, i.e. a complex vector space $\mathfrak{h}$ of dimension $2r - \ell$, where $\ell = \operatorname{rank} A$, together with two sets of linearly independent vectors $M = \{\alpha_1, \ldots, \alpha_r\} \subset \mathfrak{h}^*$ (the dual space); $M^V = \{h_1, \ldots, h_r\} \subset \mathfrak{h}$ such that $\langle h_i, \alpha_j \rangle = a_{ij}$.

The Kac–Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ is by definition a Lie algebra with the generators $e_1, \ldots, e_r$, $f_1, \ldots, f_r$, $h \in \mathfrak{h}$, subject to the relations

$$
\begin{aligned}
&[e_i, f_j] = \delta_{ij} h, \quad [h, h'] = 0, \\
&[h, e_i] = \langle h, \alpha_i \rangle e_i; \ [H, f_i] = -\langle h, \alpha_i \rangle f_i, \\
&\qquad\qquad (h, h' \in \mathfrak{h}, \ i, j = 1, \ldots, r);
\end{aligned} \tag{2.1}
$$

the Chevalley–Serre relations:

$$
(\operatorname{ad} e_j)^{1-a_{ij}}(e_j) = 0; \ (\operatorname{ad} f_i)^{1-a_{ij}}(f_j) = 0 \ \text{ if } i \neq j. \tag{2.2}
$$

One has the root decomposition $\mathfrak{g} = (\bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_{-\alpha}) \oplus \mathfrak{f} \oplus (\bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_\alpha)$ where Δ_+ denotes the set of positive roots.

The symmetrisation of A defines a certain invariant non–degenerate symmetric bilinear form $(,)$ on $\mathfrak{g}$, see [8, Ch. 2]. It induces the isomorphism $\nu : \mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$, and one has

$$\nu(h_i) = \epsilon_i \alpha_i \tag{2.3}$$

We'll use the same notation $(,)$ for the induced form on $\mathfrak{h}^*$.

2.2. Let $\Lambda \in \mathfrak{h}^*$. A Verma module $M(\Lambda)$ over $\mathfrak{g}$ is generated by one (vacuum) vector v subject to the relations $e_i v = 0$, $\forall i$, $hv = \langle h,\Lambda\rangle v$, $h \in \mathfrak{h}$. It contains a unique maximal proper submodule $M'(\Lambda)$; the quotient $L(\Lambda) = M(\Lambda)/M'(\Lambda)$ is irreducible.

There exists a unique symmetric bilinear form $S(,)$ on $M(\Lambda)$, called Shapovalov or contravariant form [8, 9.4, 12] characterized by the properties

$$S(v,v) = 1; \; S(e_i x,y) = S(x,f_i y) .$$

$M'(\Lambda)$ coincides with the kernel of S, so S is non–degenerate on $L(\Lambda)$.

One has the weight decomposition $M(\Lambda) = \oplus M(\Lambda)_\lambda$, $\lambda \in \Lambda - P_+ \subset \mathfrak{h}^*$, where $P_+ = \{ \sum_{i=1}^{r} k_i \alpha_i$, k_i are non–negative integers$\}$, is the set of dominant integral weights.
An analogous weight decomposition is induced on $L(\Lambda)$.

2.3 Now suppose that several $\Lambda^1,\ldots,\Lambda^n \in \mathfrak{h}^*$ are given. Put $M = M(\Lambda^1)\otimes\ldots\otimes M(\Lambda^n)$; $L = L(\Lambda^1)\otimes\ldots\otimes L(\Lambda^n)$ $\Lambda = \Sigma\Lambda^i$. As in 2.2 one has weight decompositions $M = \oplus M_\lambda$, $L = \oplus L_\lambda$, $\lambda \in \Lambda - P_+$. The Shapovalov form may be extended to M by the rule $S(x_1\otimes\ldots\otimes x_n, y_1\otimes\ldots\otimes y_n) = \prod_{i=1}^{n} S(x_i,y_i)$. As in 2.2, it becomes non degenerate on L.

Let $V_\lambda \subset L_\lambda$ denote the "vacuum subspace", i.e. $V_\lambda = \{x \in L_\lambda \mid e_i x = 0 \text{ for all } i\}$. One has the natural map

(2.4) $$\bigoplus_\lambda (L(\lambda)\otimes V_\lambda) \longrightarrow L$$

which is an isomorphism when $\mathfrak{g}$ is finite dimensional.

Analogously, define "covacuum spaces" W_λ as quotients $W_\lambda = L_\lambda / \sum_{i=1}^{r} f_i(L_{\lambda+\alpha_i})$. The Shapovalov form establishes the isomorphism

(2.5) $$S : V_\lambda \xrightarrow{\sim} W_\lambda^*$$

2.4 Define a bilinear Kazimir element Ω as follows. Choose some dual bases $x_i, x^i \in \mathfrak{h}$, $(x_i,x^i) = \delta_{ij}$. Further $(\mathfrak{g}_\alpha,\mathfrak{g}_{-\beta}) = 0$ for $\alpha \neq \beta$, $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are orthogonal complements of each other. Choose dual bases $f_\alpha^i \in \mathfrak{g}_{-\alpha}^i$; $e_\alpha^i \in \mathfrak{g}_\alpha$, $(e_\alpha^i,f_\alpha^j) = \delta_{ij}$. Put

(2.6) $$\Omega = \sum_i x_i \otimes x^i + \sum_{\alpha\in\Delta_+} \sum_i (e_\alpha^i \otimes f_\alpha^i + f_\alpha^i \otimes e_\alpha^i) \in \mathfrak{g}\,\hat{\otimes}\,\mathfrak{g}\,.$$

The cap ˆ over $\otimes$ means that we use infinite sum (over $\alpha \in \Delta_+$) in the definition. This definition does not depend on the choice of bases. Of course, if $\mathfrak{g}$ is finite dimensional, we return to the usual definition. Ω has the following crucial property: for any $x \in \mathfrak{g}$

$$[x \otimes 1 + 1 \otimes x, \Omega] = 0 \tag{2.7}$$

([8, Lemma 2.4]).

2.5 More generally, for any $n \geq 2$, $1 \leq i < j \leq n$, put $\Omega_{ij} = \varphi_{ij}(\Omega)$, where $\varphi_{ij} : \mathfrak{g} \hat{\otimes} \mathfrak{g} \longrightarrow \underbrace{U\mathfrak{g} \hat{\otimes} \ldots \hat{\otimes} U\mathfrak{g}}_{n\text{-times}}$, $\varphi_{ij}(x \otimes y) = 1 \otimes \ldots 1 \otimes \underset{i}{x} \otimes 1 \ldots 1 \otimes \underset{j}{y} \otimes 1 \otimes \ldots \otimes 1$, where $U\mathfrak{g}$ is the universal enveloping algebra.

Suppose that n representations $X^1, \ldots, X^n$ of $\mathfrak{g}$ are given, each having the property that for any vector $x \in X$ $\mathfrak{g}_\alpha x = 0$ for sufficiently large α. (For example $M(\Lambda)$, $L(\Lambda)$ have such property). Then elements Ω_{ij} act on $X^1 \otimes \ldots \otimes X^n$.

The system of differential equations on $X^1 \otimes \ldots \otimes X^n$–valued function $\varphi(z_1, \ldots, z_n)$, $z_i \in \mathbb{C}$, $z_i \neq z_j$ for $i \neq j$,

$$\frac{\partial \varphi}{\partial z_i} = \frac{1}{\kappa} \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} \cdot \varphi, \quad i = 1, \ldots, n, \tag{2.8}$$

is called the <u>Knizhnik–Zamoldchikov system</u>, [4]. Here κ is a complex number, [4]. The basic fact is that <u>this system is integrable</u>, i.e. the connection with potential $\sum_{1 \leq i < j \leq n} \Omega_{ij} d\log(z_i - z_j)$ is flat. This fact is essentially equivalent to (2.7) (cf. [13]).

We will be mostly interested in the case were representations X^i are irreducible highest weight representations $X^i = L(\Lambda^i), \Lambda^i \in \mathfrak{h}^*$. In this case (2.8) leaves invariant vacuum subspaces V_λ. In the next sections we'll solve (2.8) in each subspace V_λ separately. Using the map (2.4), we get the whole solution.

3. <u>Solutions: the first construction</u>

3.1. Fix a set of weights $\Lambda^1,\dots,\Lambda^n \in \mathfrak{h}^*$ and a dominant integral weight $\alpha = \sum_{i=1}^{r} k_i\alpha_i$. Put $\Lambda = \sum_{i=1}^{n} \Lambda^i$, $\lambda = \Lambda - \alpha$, $k = \sum_{i=1}^{r} k_i$. Put $L = L(\Lambda^1)\otimes\dots\otimes L(\Lambda^n)$, $L = \oplus L_\mu$. The aim of this section is to write down the solutions of KZ equations taking values in $V_\lambda \subset L_\lambda$. In the next section we'll present the solutions with values in W_λ^*; the Shapovalov form will establish the isomorphism between the two constructions. It seems that both ways, as well as the connection between them, are of an interest.

First introduce the affine complex space $\mathbb{C}^k$ whose coordinates we'll denote $t_i(j)$, $j = 1,\dots,k_i$; $i = 1,\dots,r$. So, if some $k_i = 0$ then there is no $t_i(j)$. On this space the product of symmetric groups $S_{k_1}\times\dots\times S_{k_r}$ acts by the rule $(\sigma_1,\dots,\sigma_r)(t_i(j)) = t_i(\sigma_i(j))$. Suppose that n distinct complex numbers $z_1,\dots,z_n$ are given. The expression that follow will depend on z_i as on parameters. Define a collection of affine hyperplanes in $\mathbb{C}^k$ by the equations $t_i(j) - z_m = 0$, $t_i(j) - t_{i'}(j') = 0$, i,i',j,j',m take all possible values. Denote by $X_\lambda(z)$ the complement in $\mathbb{C}^k$ of the union of these hyperplanes. Next, associate to every hyperplane a number (<u>its exponent</u>): to $(t_i(j) - z_m) - (-\alpha_i,\Lambda^m)$; to $(t_i(j) - t_{i'}(j')) - (-\alpha_i,-\alpha_j)$. Here the scalar product is taken in $\mathfrak{h}^*$, see the end of 2.1. This collection of hyperplanes and exponents we'll call configuration $C(\lambda,z)$. Define a multivaled function on $X_\lambda(z)$.

$$(3.1)\ \ell_\lambda = \ell_\lambda(t;z) = \prod_{i,j,m}(t_i(j) - z_m)^{(-\alpha_i,\Lambda^m)/\kappa} \prod_{\substack{i<i'\\ j,j'}}(t_i(j) - t_{i'}(j'))^{(-\alpha_i,-\alpha_{i'})/\kappa}$$

3.2. Let $\mathscr{P}(\lambda) = \mathscr{P}(\lambda;n)$ denote the set of all sequences of the form

$$I = \{i^1_1,\dots,i^1_{k^1};i^2_1,\dots,i^2_{k^2};\dots;i^n_1,\dots,i^n_{k^n}\}$$

where i^p_q are integers, $1 \le i^p_q \le r$, $k^m \ge 0$; $\sum_{m=1}^{n} k^m = k$, such that among i^m_p there are exactly k_j numbers j, for all $j = 1,\dots,r$. To every such sequence corresponds an element

$$f_I = f_{i^1_1} \cdot f_{i^1_{k^1}} \otimes \dots \otimes f_{i^n_1} \cdot f_{i^n_2} \cdot \dots \cdot f_{i^n_{k^n}} \tag{3.2}$$

from L_λ, and all f_I generate L_λ.

Now to every I we wish to associate a multivaluate differential k–form $\eta(I) = \eta(I;z)$ on $X_\lambda(z)$. First suppose that $k_i = 1$ for all i. Denote $t_{i^m_p}(1)$ by t^m_p, and put

$$\eta(I) = \prod{}^{-1}\ \ell_\lambda dt_1(1)\Lambda\dots\Lambda dt_k(1) \tag{3.3}$$

where

$$\prod = \prod_{m=1}^{n} [(\prod_{p=1}^{k^m-1} (t^m_p - t^m_{p+1}))(t^m_{k^m} - z_m)] \tag{3.4}$$

<u>Example.</u>

$$\eta(f_2 \otimes f_1) = (t_1(1) - z_1)^{(-\alpha_1,\Lambda^1)/\kappa} (t_1(1) - z_2)^{((-\alpha_1,\Lambda^2)/\kappa)-1} (t_2(1) - z_1)^{((-\alpha_2,\Lambda^1)/\kappa)-1}.$$

$$\cdot\ (t_2(1)-z_2)^{(-\alpha_2,\Lambda^2)/\kappa}\,dt_1(1)\wedge dt_2(1)$$

Now suppose that all k_i are arbitrary positive. Then we apply a symmetrisation procedure. Let t_p^m denote the variable $t_i(j)$, where $i=i_p^m$, and the number i appears in I in j–th time (counting from the left). Define $\prod$ by (3.4). Now on variables $t_i(j)$ the group $S_{k_1}\times\ldots\times S_{k_r}$ acts (see 3.1). Put

$$\eta(I)=\sum_\sigma \sigma(\textstyle\prod)^{-1}\ell_\lambda dt_1(1)\wedge\ldots\wedge dt_r(k_r)\,, \tag{3.5}$$

where the sum is taken over all $\sigma\in S_{k_1}\times\ldots\times S_{k_r}$; $dt_i(j)$ are ordered by the lexicographic order.

<u>Example</u> $\eta(f_1^2)=[((t_1(1)-t_1(2))(t_1(2)-z_1))^{-1}+$

$$+((t_1(2)-t_1(1))(t_1(1)-z_1))^{-1}](t_1(1)-z_1)^{(-\alpha_1,\Lambda^1)/\kappa}(t_1(2)-z_1)^{(-\alpha_1,\Lambda^1)/\kappa}\ \cdot$$

$$\cdot\ (t_1(1)-t_1(2))^{(-\alpha_1,-\alpha_1)/\kappa}\,dt_1(1)\wedge dt_1(2)$$

3.3 Now let us say a few words about the integration of our forms. Our collection of exponents defines over $X_\lambda(z)$ a flat line bundle $\mathscr{L}_\lambda$ whose horizontal section is ℓ_λ (3.1). Denote by S_λ the corresponding local system over $X_\lambda(z)$. Let $\Omega'(\mathscr{L}_\lambda)$ be the De Rham complex of $\mathscr{L}_\lambda$, with the differential d_λ induced by the flat connection, i.e. $d_\lambda\omega=d\omega+d(\log\ell_\lambda)$. The cohomology $H^\cdot(\Omega^\cdot(\mathscr{L}_\lambda))$ is equal to $H^\cdot(X_\lambda(z);S_\lambda)$. Our forms $\eta(f_I)$ lie in $\Omega^k(\mathscr{L}_\lambda)$, so they represent classes in $H^k(X(z);S_\lambda)$. The last space is by Poincaré duality dual to the homology space $H_k(X_\lambda(z);S_\lambda^*)$, see [14]. This space is generated by certain k–dimensional cycles of $C(\lambda;z)$, and the pairing between H^k and H_k is given by the integration of a form over a cycle.

If, for example, all exponents of $C(\lambda;z)$ have real parts >1, then we can integrate our forms over relatively compact chain Δ in $\mathbb{C}^k$ with the boundary on hyperplanes, [14]. Otherwise one needs some regularization, for example, an analytic continuation with respect to exponents or higher–dimensional analogous of "double loops" (see [15], [16]).

Anyway, in the following by the word "cycle" of $C(\lambda,z)$ we can have in mind an element of $H_k^{\ell f}(X_\lambda(z);S_\lambda^*)$.

3.4 Now consider the expression

$$\eta_\lambda = \sum_{I \in \mathscr{P}(\lambda)} \eta(I) \cdot f_I$$

lying in $\Omega^k(\mathscr{L}_\lambda) \otimes L_\lambda$.

<u>Proposition 1.</u> Fix some i, $1 \leq i \leq r$. One can associate with every $J \in \mathscr{P}(\lambda + \alpha_i)$, a $(k-1)$–form $\eta(J) \in \Omega^{k-1}(\mathscr{L}_\lambda)$ such that

$$e_i \eta_\lambda := \sum_I \eta(I) \cdot e_i f_I = \sum_J d\eta(J) \cdot f_J.$$

Now, if we have a cycle, we can consider an element

$$\eta_\lambda(\Delta) = \sum_I \int_\Delta \eta(I) f_I \tag{3.6}$$

from L_λ.

Corollary. All $\eta_\lambda(\Delta)$ lie in the vacuum subspace V_λ, i.e. $e_i\eta_\lambda(\Delta) = 0$ for all i.

3.5 If some cycle Δ^0 of $X_\lambda(z^0)$ is given, then we can in a unique way continuously deform it along any path from z^0 to arbitrary z, and obtain a (multivaluate in general) family of cycles $\Delta = \{\Delta(z)\}$, $\Delta(z)$ being a cycle of $X_\lambda(z)$. This family constitutes a horizontal section of the Gauss–Manin connection in homology.

Now consider the function

$$(3.7) \qquad \varphi_\lambda(\Delta;z) = \prod_{1\le i<j\le n} (z_i - z_j)^{(\Lambda^i,\Lambda^j)/\kappa} \cdot \eta_\lambda(\Delta(z))$$

Theorem 1 $\varphi_\lambda(\Delta;Z)$ satisfies KZ equations (2.8). So, we have constructed solutions of KZ equations with values in V_λ. (cf. discussion below).

(3.6) Example. Let $g = sl(2)$ with the scalar product $(x,y) = tr(xy)$; e,f,h be standard generators, $\alpha \in \mathfrak{h}^*$ the unique root. Put $m_i = \langle h,\Lambda^i\rangle$, $i = 1,\ldots,n$; $\lambda = \sum_{i=1}^{n} \Lambda^i - k\alpha$.

We have

$$(3.8) \quad \ell_\lambda = \ell_\lambda(t_1,\ldots,t_k;z_1,\ldots,z_n\} = \prod_{i=1}^{k} \Big(\prod_{j=1}^{n} (t_i - z_j)^{-m_j/\kappa} \Big) \cdot \prod_{i<j} (t_i - t_j)^{2/\kappa}$$

Now let $f = f^{k^1} \otimes \ldots \otimes f^{k^n}$ be an element of L_λ, $\sum_{i=1}^{n} k^i = k$.

Then we have

$$\eta_\lambda = \sum \eta(k^1,\ldots,k^n) \cdot (f^{k^1} \otimes \ldots \otimes f^{k^n}) ,$$

the summing is over all $(k^1,\ldots,k^n)$ such that $\sum_{i=1}^{n} k^i = k$. Forms $\eta(k^1,\ldots,k^n)$ are defined as follows. For a given $\vec{k} = \{k^1,\ldots,k^n\}$ introduce the sequence of k integers $J(\vec{k}) = \{j_1,\ldots,j_k\}$, setting the first k^1 elements to be equal to 1, the next k^2 – to 2,... the last k^n – to n. Then we have

$$\eta(\vec{k}) = k_1! \cdot \ldots \cdot k_n! \sum_{\sigma \in S_k} \frac{\ell_\lambda dt_1 \wedge \ldots \wedge dt_k}{(t_1 - z_{j_{\sigma(1)}}) \cdot (t_2 - z_{j_{\sigma(2)}}) \cdot \ldots \cdot (t_k - z_{j_{\sigma(k)}})} \tag{3.9}$$

For $n = 4$ we get a variant of formulas from [6].

4. Solutions: the second construction

We'll save the notations of the preceding section.

4.1 Now we wish to associate with every $I \in \mathscr{P}(\lambda)$ another differential k–form, $\omega(I) \in \Omega^k(\mathscr{L}_\lambda)$. We proceed as in 3.2.

First suppose that all $k_i = 1$. Then put

$$\omega(I) = \pi(I)\ell_\lambda dt_1(1)\wedge\ldots\wedge dt_k(1), \tag{4.1}$$

where

$$\pi(I) = \prod_{m=1}^{n} \prod_{j=1}^{k^m} \left(\frac{(-\alpha_j^m, \Lambda^m)}{t_j^m - z_m} + \sum_{p=j+1}^{k^m} \frac{(-\alpha_j^m, -\alpha_p^m)}{t_j^m - t_p^m}\right) \tag{4.2}$$

where, we put $t_j^m = t_{i_j^m}(1)$; $\alpha_j^m = \alpha_{i_j^m}$ (cf. 3.2).

Now suppose all k_i are arbitrary positive. Introduce t_j^m by the same rule as in 3.2, and again define $\pi(I)$ by (4.2). Put

$$\omega(I) = \sum_\sigma \sigma(\pi(I))\ell_\lambda dt_1(1)\wedge\ldots\wedge dt_r(k_r) \tag{4.3}$$

where the summing is the same as in (3.5).

Remark. The forms $\omega(I)$ are nothing but flag forms associated with configuration $C(\lambda;n)$ which were studied in [11].

Proposition 2. For any $I \in \mathcal{P}(\lambda)$

$$\omega(I) = (-1)^k \epsilon(\lambda)^{-1} \sum_{J \in \mathcal{P}(\lambda)} S(f_I, f_J)\eta(J) \tag{4.4}$$

where S is the Shapovalov form,

$$\epsilon(\lambda) = \prod_{i=1}^{r} \epsilon_i^{k_i}, \tag{4.5}$$

ϵ_i being as in (2.3).

4.2. Denote by $\tilde{\mathfrak{g}}(A)$ a Lie algebra with generators $\tilde{e}_i, \tilde{f}_i$, $h \in \mathfrak{h}$, subject only to the relations (2.1) but not (2.2). Let $\tilde{M}(\Lambda^i)$ denote Verma modules over $\tilde{g}(A)$ defined as in 2.2, etc. One has the weight decomposition $\tilde{M}(\Lambda) = \oplus \tilde{M}(\Lambda)_\lambda$, and monomials $\tilde{f}_I$, $I \in \mathcal{P}(\lambda)$ constitute a basis of $\tilde{M}(\Lambda)_\lambda$, so we can define by linearity $\omega(x)$ for every $x \in \tilde{M}(\Lambda)_\lambda$. Here is maybe the most interesting property of forms $\omega(I)$:

Proposition 3. Forms $\omega(I)$ satisfy the Chevalley–Serre relations (2.2). This means that for any $x = x_1 \otimes \ldots \otimes x_n \in \tilde{M}(\Lambda)_\lambda$ such that for some m, $1 \le m \le n$, x_m has the form $y[(\mathrm{ad}\tilde{f}_i)^{1-a_{ij}}(\tilde{f}_j)]z$, $y \in \tilde{g}(A)$, $z \in \tilde{M}(\Lambda^m)$, one has $\omega(x) = 0$.

Proposition 4. Let $x = x_1 \otimes \ldots \otimes x_n \in \tilde{M}(\Lambda)_\lambda$ be such that some x_m lies in the kernel of the Shapovalov form. Then $\omega(x) = 0$.

This follows from (4.4). By proposition 3 and 4, $\omega(x)$ depend only on the projection of $x \in \tilde{M}(\Lambda)_\lambda$ to $L(\Lambda)_\lambda$, so we get the mapping

$$\omega : L(\Lambda)_\lambda \longrightarrow \Omega^k(\mathscr{L}_\lambda)\,.$$

Proposition 5. For any i, $1 \le i \le r$, one can define the canonical mapping $\omega' : L(\Lambda)_{\lambda-\alpha_i} \longrightarrow \Omega^{k-1}(\mathscr{L}_\lambda)$ such that

(4.6) $$\omega(f_i x) = d\omega'(x)$$

for any x from $L(\Lambda)_\lambda$. (Cf. prop. 1). So, ω defines the map

$$\omega_\lambda : W(\Lambda)_\lambda \longrightarrow H^k(X_\lambda(z);S_\lambda)$$

where $W(\Lambda)_\lambda$ is as in 2.3.

4.3. Now, if we have a cycle $\Delta(z))$ in $X(\lambda;z)$, integration over $\Delta(z)$ gives the functional $\int_{\Delta(z)} \omega_\lambda \in W(\Lambda)^*_\lambda = \omega_\lambda(\Delta(z))$. Let $\Delta = \Delta(z)$ be a covariantly constant family of cycles . Consider the $W(\Lambda)^*_\lambda$–valued function

(4.7) $$\psi_\lambda(\Delta;z) = \prod_{1\le i\le j\le n} (z_i - z_j)^{(\Lambda^i,\Lambda^j)/\kappa} \cdot \omega_\lambda(\Delta(z))$$

(cf. (3.7)).

Theorem 2. $\psi_\lambda(\Delta;z)$ satisfies the KZ equation with values in W^*_λ.

The Shapovalov isomorphism (2.5) takes $\varphi_\lambda(\Delta;z)$ to $(-1)^k\epsilon(\lambda)\psi_\lambda(\Delta;z)$.

4. Discussion

We have constructed a set of solutions of KZ equation in Clebsch–Gordan spaces V_λ, parametrized by cycles Δ lying in homology spaces $H_k = H_k(X_\lambda(z_0),S_\lambda)$. One should expect that when Δ runs through the whole space H_k, we get the complete set of solutions. For generic values of x (cf. [6]) all other homology spaces H_i, $i \neq k$, vanish, [14]. It is an important problem to study homology spaces H_i when some of $(\alpha_j,\Lambda^m)/\kappa$ becomes integer. In this case nonzero H_i, $i \neq k$ may appear and the rank of H_k may switch. This phenomenon is intimately connected with the "truncation" of operator algebras, [18].

By the result of Kohno [13], [17] the mondromy of KZ equations may be described by constant R–matrices, and provides representations of Hecke algebras. These results present an example of recently discovered connection between CFT and quantum groups, [17], [18]. The integral formulas give a tool for the explicit calculation of mondromy representations. The above results may be viewed as a geometric construction of such representations. It is interesting to compare it with Springer and Ginsburg constructions of representations of Weil groups and Hecke algebras [19].

Finally, maybe it is of some interest to compare the generalization to infinite–dimensional Lie algebras with the discussion of [20].

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