Workshop “Elliptic Cohomology”

11.-13. Juni 1995

Max-Planck-Institut
für Mathematik
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MPI/95-60

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MAX-PLANCK-INSTITUT FÜR MATHEMATIK

Workshop "Elliptic Cohomology"

Program
Sunday, 11.06.1995

10.00-11.00	**Charles Thomas:**	**Report on the prize question in the book of Hirzebruch-Berger-Jung**
11.45-12.45	**Stephan Stolz:**	**Conjecture concerning positive Ricci curvature and the Witten genus**

15.00-16.00	**Nicholas Kuhn:**	**Generalized characters for complex oriented theories**
16.45-17.45	**Stephan Klaus:**	**The Ochanine k-invariant is a Brown-Kervaire invariant**

MAX-PLANCK-INSTITUT FÜR MATHEMATIK

Workshop "Elliptic Cohomology"

Program
Monday, 12.06.1995

10.00-11.00	**Remi Léandre:**	*Brownian motion and elliptic genera of Level N*
11.45-12.45	**Anand Dessai:**	*Rigidity theorems for spin-c manifolds*

15.00-16.00	**Andrew Baker:**	*Generalized modular forms, Hecke algebras, and operations in elliptic cohomology*
16.45-17.45	**Geoffrey Mason:**	*CFT and LBG*

MAX-PLANCK-INSTITUT FÜR MATHEMATIK

Workshop "Elliptic Cohomology"

Program

Tuesday, 13.06.1995

10.00-11.00	**Gerald Höhn:**	*A moonshine module for the baby monster*
11.45-12.45	**Hirotaka Tamanoi:**	*Elliptic genera and Vertex operator algebras*

15.00-16.00	**Kefeng Liu:**	*Applications of modular invariance in topology*
16.45-17.45	**Rainer Jung:**	*Kreck-Stolz type elliptic homology theories and applications*

Titel: Report on the prize question in the book of Hirzebruch-Berger-Jung
Autor: Charles Thomas
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As part of his discussion of the Witten genus in [H] Hirzebruch asks the following

Prize Question: Does there exist a 24-dimensional compact, orientable, smooth manifold with $p_1(X) = 0$, $w_2(X) = 0$, $\hat{A}(X) = 1$ and $\hat{A}(X, T_{\mathbb{C}}) = 0$?

If such a manifold exists it is also reasonable to ask if it admits a smooth action by the monster simple group $\mathbb{M}$. The aim of this lecture is (a) to outline a recent argument of M. Hopkins & M. Mahowald providing a positive answer to this prize question, and (b) to advertise a technique developed by M. Aschbacher for constructing manifold-like objects admitting $\mathbb{M}$-actions. With the link between free actions on spheres and groups with periodic cohomology in mind we will also list what we know about the integral cohomology $H^*(\mathbb{M}, \mathbb{Z})$.

1. Preliminaries

Recall that a genus is a homomorphism from some cobordism ring (usually Ω_*^{SO}) into a $\mathbb{Q}$-algebra Λ, and that the genus is said to be elliptic if it is multiplicative for fibre bundles fibred by $\mathbb{C}P^{2n-1}$. As a map from $\Omega_*^{SO} \otimes \mathbb{Q}$ such a map is determined by its values on $\mathbb{C}P^2$ and $\mathbb{H}P^2$, and in the special case when these are $-1/8$ and 0 respectively, we obtain the genus $\hat{A}$. This can also be defined via the Pontrjagin classes of the tangent bundle T, and when these are "twisted" by means of the Chern character of the complexification $T_{\mathbb{C}}$ we obtain the modified genus $\hat{A}(\,,T_{\mathbb{C}})$.

More generally we define the Witten genus by

$$\varphi_W(X^{4k}) = q^{-4k/24}\,\hat{A}\big(X, \bigotimes_{n=1}^{\infty} S_{q^n} T_{\mathbb{C}}\big)\,\Delta^{4k/24},$$

where $\Delta = \frac{1}{1728}(E_4^3 - E_6^2) = q - 24q^2 + 252q^3 - \dots$ is the familiar modular form of weight 12. The genus φ_W is itself modular with integral Fourier coefficients provided that $w_2(X) = \frac{p_1}{2}(X) = 0$, i.e. provided that the tangential classifying map $X \to BO$ lifts to the 7-connected covering space $BO\langle 8\rangle$. Such manifolds X will be called $MO\langle 8\rangle$-manifolds.

In dimension 24 it is easy to see that $\varphi_W(X) = \hat{A}(X, T_{\mathbb{C}})\Delta + \hat{A}(X)\bar{\Delta}$, with $\bar{\Delta} = (E_4^3 - 744\Delta)$, and that if the conditions of the prize question are satisfied then

$$\frac{\varphi_W(X)}{1 \cancel{\hat{A}(X)}} = \bar{\Delta}.$$ Dividing by Δ we would then have

$$j - 744 = q^{-1}\cdot \hat{A}\big(X, \bigotimes_{n=1}^{\infty} S_{q^n} T_{\mathbb{C}}\big) = q^{-1} + 196884\cdot q + 21493760\cdot q^2 + \dots$$

Remark: Following a suggestion of Harada 4th. August 1968 may be referred to as the "official birthday" of monstrous moonshine.

Given the relation between the coefficients of $j - 744$ and the degrees of irreducible representations of $\mathbb{M}$, the interest of the "prizequestion" is obvious.

2. Existence of the required manifold X^{24}.

In much of what follows we think of a genus such as $\hat{A}$ as being defined in terms of a cohomology class K, whose components are the polynomials $K_k(p_1, p_2, \dots)$ defining the genus. If (X, γ) is a pair consisting of a bundle γ over the space X, and $a \in H_*(X)$ we can form $\langle \gamma^*(K), a\rangle$, obtaining the original genus in the special case $\gamma = TX$, $a = [M]$.

The answer to our abstract existence question depends on constructing a suitable element in $\pi_{24}(MO\langle 8\rangle)$. We start with a fibration

$$\begin{array}{ccl} S^9 \to X & & \text{classified by an element} \\ \downarrow & & 2\nu \text{ of order 12 in } \pi_{12}(S^9). \\ S^{13} & & \end{array}$$

It is easy to see that $H_*(\Omega X, \mathbb{Z}) = \mathbb{Z}[a_8, a_{12}]$ and that there is a map $\gamma : \Omega X \to BO$ mapping 8-dimensional generator to 8-dim. generator in homology. Let $\overline{X}$ be the spectrum defined by the Thom space of the bundle γ over ΩX.

BASIC RESULT: For every $4k$-dimensional $MO\langle 8\rangle$-manifold M there is a manifold N such that the normal bundle for N is classified by the composite $N \to \Omega X \to BO\langle 8\rangle$, <u>and</u> such that $\varphi_W(M) = \varphi_W(N)$.

Exercise: With φ_W replaced by $\hat{A}$ and $BO\langle 8\rangle$ by $BSpin$, what space replaces ΩX above?

Restricting to dimension 24 our problem now is to specify a map $S^{24} \to \overline{X}$ whose image in homology gives (some multiple of) $\overline{\Delta}$ when evaluated under the class K_W corresponding to φ_W. We do this by constructing two bundles, according to the scheme below.

$S^9 \longrightarrow X$ degree $= 240$	$S^{13} \longrightarrow X$ degree $= 3024$
$\gamma_1 : \Omega S^9 \longrightarrow \Omega X$	$\gamma_2 : \Omega S^{13} \longrightarrow \Omega X$
$\langle \hat{A}(\gamma_1), \overline{a}_8^{\,3} \rangle = -1$	$\langle \hat{A}(\gamma_2), \overline{a}_{12}^{\,2} \rangle = +1$
$\langle \hat{A}(\gamma_1, \gamma_1 \otimes \mathbb{C}), \overline{a}_8^{\,3} \rangle = -744$	$\langle \hat{A}(\gamma_2, \gamma_2 \otimes \mathbb{C}), \overline{a}_{12}^{\,2} \rangle = -984$
Describing $(\Omega S^9)^{\gamma_1}$ as a wedge of spheres $\bigvee_{i \geq 0} S^{8i}$ gives rise to a family of manifolds for which $\varphi_W =$ power of E_4	For this bundle there is a map $\bigvee_{i \geq 0} S^{24i} \longrightarrow (\Omega S^{13})^{\gamma_2}$ of degree 1, whose restriction to the sphere S^{4i} gives rise to a manifold having $\varphi_W = E_6^{2i}$

Combining these two approaches we are now in a position to construct a Poincaré complex / bundle pair realising $\varphi_W = -24\overline{\Delta}$. The corresponding homology class is $24((20a_8)^3 + 3(42\, a_{12})^2)$

$$= 3(40a_8)^3 + 2(252\, a_{12})^2$$

In order to obtain the corresponding bundle we start with a map

$$S^9 \underset{84\nu}{\cup} e^{13} \longrightarrow S^9 \underset{2\nu}{\cup} e^{13}$$

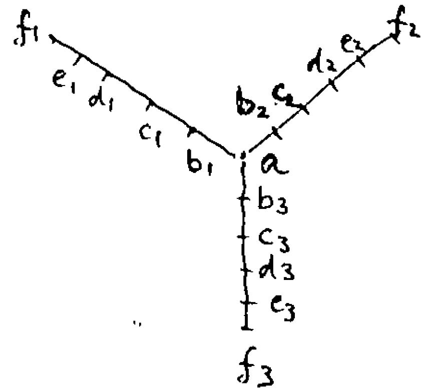
which has degree 40 in dimension 9 and degree 252 in dimension 13. If X denotes the original S^9-fibration (see previous page) and Y is similarly constructed using 84ν (which is stably trivial), there is a map $Y \to X$ restricting to the 13-skeleton as above. Take loops and write $\gamma_3 : \Omega Y \to \Omega X$. If b_i plays the role of a_i in $H_*\Omega Y$, evaluation of γ_3 on $2b_{12}^2 + 3b_8^3$ gives $\varphi_W = -24\bar{\Delta}$.

An interesting complement to this argument, which uses little more than linear algebra, combined with the work of S. Stolz (see these proceedings) shows:

* There exists a smooth 24-dimensional, simply connected manifold admitting a metric of positive scalar curvature, which is NOT of positive Ricci curvature. *

3. Actions of the Monster simple group $\mathbb{M}$.

Aschbacher's idea is to construct actions of $\mathbb{M}$ by restricting actions of the "Bimonster" Y_{555}, isomorphic to the wreath product $\mathbb{M} \wr (2)$. The group Y_{555} is easier to consider, since it is of Coxeter type, i.e. is associated to the diagram



Here each vertex is associated with an element of order 2, and the product of two generators has order 2 (3) if the corresponding vertices are unjoined (joined) by a single unlabelled edge. We have an additional relator $(ab_1c_1\,ab_2c_2\,ab_3c_3)^{10}$. See for example [N].

If Y_{555} were a genuine Coxeter group we could apply the following theorem of M. Davis:

Let K^n be a homology manifold and $G \le \mathrm{Aut}(K)$ a strongly admissible subgroup which is transitive on n-simplexes. Then G is the image of a Coxeter group whose diagram is spherical, affine or hyperbolic. Furthermore K is the coset complex of the parabolic subgroups of G.

Technical condition: G is strongly admissible if the fixing of a simplex by $g \in G$ implies the fixing of its vertices.

Y_{555} is not close enough to a Coxeter group for this theorem to apply directly, but the method applies to prove that Y_{555} (and hence $\mathbb{M}$) can act on a more general class of CW-complexes. [Even if one of these fails in the first instance to be a manifold, it seems to satisfy Poincaré duality, and is hence amenable to the methods of surgery.] Specific examples are known in dimension 9, and hence are of no immediate application to the prize question. But it is the method rather than the actual complex which is important ...

4. Cohomology of $\mathbb{M}$.

Since the cohomology groups of G (G = arbitrary discrete group) are defined in terms of a G-action on a contractible CW-complex, there are necessary homological conditions which have to be satisfied if G is to act on a space of preassigned type. Familiar example: if G acts freely on S^{2n-1} then $H^*(G, \mathbb{Z})$ is periodic with period dividing $2n$. In the present case obvious restrictions are imposed on the isotropy subgroup structure by the fact that the dimension of the smallest non-trivial representation of $\mathbb{M}$ (196883) is so much larger than 24.

Here is a summary of what we know – further details will appear in [G].

$|\mathbb{M}| = 2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71.$

Hence for all $p \geq 17$ $H^*(\mathbb{M}, \mathbb{Z})_{(p)}$ is periodic with period $2d(p)$, and we have the following table:

$p =$	17	19	23	29	31	41	47	59	71
$2d(p) =$	32	36	22	56	30	80	46	58	70

In each case the $(p-1)$st. Chern class of (196883) provides either a maximal generator ($p = 17, 19, 29, 41$) or the square of a maximal generator ($p = 23, 31, 47, 59, 71$).

At the prime 11 H^{even} is generated in degrees 40, 60 (2 gens.), 80 (2 gens.), 160, 240. This contains the Chern subring, generated by classes c_{110} and c_{120}, properly.

At the prime 13 the situation is a little more complicated, since (in the notation of the "Atlas") $\mathbb{M}_{13} \cong 13_+^{1+2}$, and $H^{even}(\mathbb{M}, \mathbb{Z})_{(13)}$ is properly contained in the subring of $H^{even}(13_+^{1+2}, \mathbb{Z})$ invariant under the action of the normaliser. Again the classes $c_{p(p-1)}$ and $c_{(p-1)(p+1)}$ play an important rôle – as when $p = 11$ they correspond to Dickson invariants.

At $p = 7$ the situation is not as hopeless as it might appear, given that $N(\mathbb{M}_7) \cong 7_+^{1+4} : (3 \times 2S_7)$, and that recent work of several authors has thrown a great deal of light on the Chern subring of extraspecial p-groups.

References

* M. Aschbacher; Constructing representations of finite groups on manifolds
D. Green, C.B. Thomas, ...; Cohomological supplement to the "Atlas" (in preparation)
F. Hirzebruch et al.; Manifolds & modular forms, Vieweg (1992)
* M. Hopkins & M. Mahowald; The Witten genus in homotopy
S. Norton; Constructing the Monster, LMS Lecture Notes #165.

* = recent pre-preprint.

Bonn, June 1995.

Titel: A conjecture concerning positive Ricci curvature and the Witten genus

Autor: Stephan Stolz

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Conjecture: Let M be a $4k$-dimensional spin manifold with $\frac{p_1}{2}(M)=0$. If M admits a metric of positive Ricci curvature then the Witten genus $\Phi_W(M)$ vanishes.

Evidence: the conjecture is true for

i) complete intersections

ii) homogeneous spaces G/H, G compact, semi-simple

iii) total spaces of G/H-bundle with structure group G

The above conjecture is analogous to the following result:

Theorem (Lichnerowicz, 1963) Let M be a $4k$-dimensional spin manifold. If M admits a metric of positive scalar curvature, then $\hat{A}(M)=0$.

We recall that the proof of this theorem follows by interpreting $\hat{A}(M)$ as the index of the Dirac operator on M via the index theorem and by using the Weitzenböck formula to argue that positive scalar curvature implies that the Dirac operator is invertible.

A heuristic proof of the above conjecture results ~~for~~ by applying the same arguments to the free loop space of M. A conjectured "index theorem" (due to Witten) and a conjectural "Weitzenböck formula" (yet to be formulated) imply to conjecture. (Preprint available: Stolz.1@ND.EDU)

Titel: Generalized characters for complex oriented theories

Autor: Nicholas Kuhn

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This is a survey of joint work with Mike Hopkins and Doug Ravenel (ongoing since 1986). 1

I. Motivation E^* = generalized cohomology theory (with products)
G = finite group.

Study $G \longmapsto E^*(BG)$.

A good group theoretic description of $E^*(BG)$ tends to reveal interesting things about E^* (e.g. homology operations) + info. about E^*-char. classes for G-bundles.

Example (Atiyah) $K^0(BG) \simeq R(G)^\wedge$ $(K^1(BG) = 0)$

$R(G)$ is studied via characters:

$$\chi : R(G) \longrightarrow \{\text{Ring of class functions } G \longrightarrow \mathbb{C}\}$$

K^* is an example of a complex oriented theory.

Basic idea: One can set up charaters for $E^*(BG)$ for lots of complex oriented theories where the target for the character map is a ring of class functions that depend on the formal group law associated to E^*.

II. Main Theorem

Assumptions: $E^*(\ )$ cx. oriented, with ring of coefficients E^* graded, local, complete, E^*/m of char p, p not a 0 divisor in E^*

F_E = f.g.l. for E with $F_{E/m}$ of height n.

Now given $E^* \longrightarrow \mathcal{O} \subseteq L$ $\mathcal{O}$ = integers in complete graded algebraically closed field L. Then, we can construct, natural in G,

$$\chi : E^*(BG) \longrightarrow \mathrm{Map}_G(\underbrace{\mathrm{Hom}(\mathbb{Z}_p^n, G)}_{\text{commuting } n \text{ tuples of order a power of } p.}, L)$$

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L^2

Theorem! χ extends to an isomorphism

$$\chi : E^*(BG) \otimes_{E^*} L \xrightarrow{\sim} \mathrm{Map}_G(\mathrm{Hom}(\mathbb{Z}_p^n, G), L)$$

Remarks (1) Have a good formula for induction: χ is an iso. of Mackey functors.

(2) Extends to a result for $E^*(EG \times_G X)$, X finite G C.W. complex.

III. Input

#1. Stable equiv. homotopy theory.

$X \longmapsto E^*(EG \times_G X) \otimes \mathbb{Z}[\frac{1}{|G|}]$ is an equivariant coh. theory.

Lemma $\{X, Y\}_{G\text{-spectra}} \longrightarrow \prod_{(H)} \{X^H, Y^H\}_{\text{spectra}}^{W_G(H)}$

becomes iso after $\otimes \mathbb{Z}[\frac{1}{|G|}]$.

Consequence One expects a formula for $\frac{1}{|G|}E^*(EG \times_G X)$ in terms of $E^*(X^H)$'s.

#2. Splitting principle

Let $G \leq U(n)$. Let $F = U(n)/T^n$ (Flag manifold)

$p : EG \times_G F \to BG$ (or $EG \times_G (X \times F) \longrightarrow EG \times_G X$)

induces $p^* : E^*(BG) \longrightarrow E^*(EG \times_G F)$.

E^* complex oriented $\Longrightarrow$ p^* makes $E^*(EG \times_G F)$ into a free rank $n!$ $E^*(BG)$-module.

Note: $F^H = \emptyset$ if H not abelian.

Consequence The formula for $\frac{1}{|G|} E^*(EG \times_G X)$ should only involve $E^*(X^A)$, $A < G$ abelian.

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Adresse:

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#3. Group theory: Mackey Functors

$H<G$ have $\mathrm{Ind}: E^*(BH) \to E^*(BG)$, making $G \mapsto E^*(BG)$ into a Mackey Functor. One can exploit ideas of Green, Dress...

Consequence of #1 – #3: an "Artin's Theorem":

Theorem 2 E^* complex oriented.

$$\tfrac{1}{|G|} E^*(BG) \xrightarrow[\sim]{} \varprojlim_{\substack{A<G \\ \text{abelian}}} \tfrac{1}{|G|} E^*(BA)$$

$$\left(\tfrac{1}{|G|} E^*(EG\times_G X) \xrightarrow[\sim]{} \varprojlim_{A} \tfrac{1}{|G|} E^*(EG\times_A X) \right)$$

Remark If $E^{odd}(BA) = 0$ $\forall$ abelian A (this often happens), then $E^{odd}(BG)$ will all be $\frac{1}{|G|}$-torsion.

Example Recent result of Kriz suggests this might happen for $Ell^*(BG)$.

#4 Formal Group Law Theory

$\mu: \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ $\quad \mu^*(\text{orientation}) \in E^*(\mathbb{C}P^\infty \wedge \mathbb{C}P^\infty) = E^*[[x,y]]$.

denoted $x +_F y$.

$[k](x) \equiv \underbrace{x +_F x +_F x + \cdots}_{k}$ (k series)

Under our hypotheses, $E^*(BA)$ is determined by the f.g. law.

E.G. $E^*(B\mathbb{Z}/p^r) = E^*[[x]]/([p^r](x))$ = free E^*-module with basis $\{1, \ldots, x^{p^{rn}-1}\}$.

Lubin–Tate: solutions to $[p^r](x) = 0$ in L form an abelian group, under $x +_F y$, $\simeq (\mathbb{Z}/p^r)^n$. Thus get, e.g.

(*) $\quad E^*(B\mathbb{Z}/p) \otimes_{E^*} L \xrightarrow[\sim]{} \prod_{a \in (\mathbb{Z}/p)^n \subset L} L$, where "$a^{th}$" component sends x to the solution "a".

(*) is the building block for our characters.

Titel: The Ochanine k-invariant is a Brown-Kervaire invariant
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1. Motivation for the problem:

We first consider closed differentiable oriented manifolds of dimension $4m$. One can define two invariants: a) the signature of the intersection form on $H^{2m}(M;\mathbb{Q})$, which gives a homotopy-theoretical bordism invariant

$$\mathrm{sign}: \Omega^{SO}_{4m} \longrightarrow \mathbb{Z},$$

b) the L-genus defined by the characteristic power series $Q(x) = \frac{x}{\tanh x}$, which gives a differential-topological bordism invariant

$$L: \Omega^{SO}_{4m} \longrightarrow \mathbb{Z}.$$

The Hirzebruch signature theorem says, that $\mathrm{sign} = L$. In particular, sign is a characteristic number, and L is an invariant of the oriented bordism type, which vanishes for $H^{2m}(M;\mathbb{Q}) = 0$.

Remark: By the Atiyah-Singer index theorem, L has also an analytical expression as the index of the signature operator.

2. Brown-Kervaire invariants and the Ochanine k-invariant:

Now, we consider closed differentiable Spin-manifolds of dimension $8m+2$. Again, we can define two types of invariants: a) the Brown-Kervaire invariants

$$K_h: \Omega^{Spin}_{8m+2} \longrightarrow \mathbb{Z}/2,$$

which are given as the Arf invariant of certain quadratic refinements $q_h: H^{4m+1}(M;\mathbb{Z}/2) \to \mathbb{Z}/2$ of the $\mathbb{Z}/2$-intersection form. These are homotopy-theoretical invariants, where one has to choose a parameter h in a certain finite affine space Q^{Spin}_{8m+2} over $\mathbb{Z}/2$ in order to define q_h and K_h.

b) The Ochanine k-invariant

$$k: \Omega^{Spin}_{8m+2} \longrightarrow \mathbb{Z}/2.$$

We do not consider its original definition, but use a result of Ochanine to describe it as the highest coefficient a_m in the $\bar{\varepsilon}$-expansion of the Ochanine elliptic genus $\beta: \Omega^{Spin}_{*} \longrightarrow KO_*[[q]]$.

Here, β is defined by $\beta(M^n) := \langle \bigotimes_{k\geq 1} S_{q^{2k}}(TM^n - n) \otimes \Lambda_{-q^{2k-1}}(TM^n - n), [M^n]_{KO} \rangle$ and has for $n = 8m+2$ an expansion $\beta(M^n) = a_0 + a_1\bar{\varepsilon} + \dots + a_m\bar{\varepsilon}^m$ with $\bar{\varepsilon} := \sum_{k\geq 0} q^{(2k+1)^2} \in \mathbb{Z}/2[[q]]$. The constant term of β is given by the Atiyah α-invariant $\alpha : \Omega_*^{Spin} \to KO_*$, $\alpha(M) := \langle 1, [M^n]_{KO} \rangle$.

By definition, β and thus also k is a KO-characteristic number.

<u>Remark</u>: By the Atiyah-Singer real family index theorem, k has thus also an analytical expression as the mod-2-index of a twisted Dirac-operator.

It is now natural to ask for the connection between the invariants K_h and k.

In my thesis I proved the following result:

<u>Theorem</u>: In each dimension $8m+2$, there exists a parameter $h \in \mathbb{Q}^{Spin}_{8m+2}$ such that $K_h = k$.

As a Corollary, we obtain that this K_h is KO-characteristic number, and k is an invariant of the Spin-homotopy type, which vanishes for $H^{4m+1}(M; \mathbb{Z}/2) = 0$.

We give now an outline of the proof. We use the coefficients of integral elliptic homology of Kreck and Stolz to characterize k by multiplicativity in $\mathbb{H}P^2$-bundles. Then we show the existence of a K_h with this property, using the theory of Kristensen on cochain operations in order to compute a product formula for a certain secondary cohomology operation.

<u>3. Characterization of k by integral elliptic homology</u>:

According to Stolz one considers in Ω_*^{Spin} fibre bundles with fibre $\mathbb{H}P^2$ and structure group $PSp(3)$. One has then a transfer map

$$\Psi : \Omega_*^{Spin}(BPSp(3)) \longrightarrow \Omega_{*+8}^{Spin}$$

which maps $[M^n \to BPSp(3)]$ to $\bar{N}^{n+8}$, the total-space of the associated $\mathbb{H}P^2$-bundle. We denote the forgetful map $[M^n \to BPSp(3)] \mapsto [M^n] \in \Omega_n^{Spin}$ by p.

<u>Theorem</u> (Stolz): $\operatorname{Ker}\alpha = \operatorname{im}\Psi$

<u>Theorem</u> (Kreck, Stolz): $\operatorname{Ker}\beta = \Psi(\operatorname{Ker} p)$, and

$$\Omega_*^{Spin} / \operatorname{Ker}\beta = \mathbb{Z}[\bar{S}^1, K^4, \mathbb{H}P^2, B^8] \Big/ \begin{array}{l} 2\bar{S}^1 = \bar{S}^{1\,2} = \bar{S}^1 K^4 = 0 \\ (K^4)^2 = 4B^8 + 256\,\mathbb{H}P^2 \end{array}$$

where $\bar{S}^1, K^4, \mathbb{H}P^2, B^8$ denote the (bordism classes of) circle with the nontrivial Spin-structure, the Kummer surface, $\mathbb{H}P^2$, and the Bott manifold, respectively.

As a corollary, one obtains that k is characterized by
i) k is multiplicative in $\mathbb{HP}^2$-bundles
ii) $k(M^{8m} \times \bar{S}^1 \times \bar{S}^1) \equiv \operatorname{sign}(M^{8m}) \bmod 2$

4. Brown-Kervaire invariants in $\mathbb{HP}^2$-bundles:

By a product formula of Brown, all K_k satisfy ii). In order to check i), we restrict us to special Brown-Kervaire invariants, where the quadratic form q_k is given by a secondary cohomology operation. These are constructed by the decomposition $Sq^{4m+2} = Sq^2 Sq^{4m} + Sq^1 Sq^{4m} Sq^1$, and Brown and Peterson proved that the corresponding unstable operations

$$\Phi_m : \operatorname{Ker}(Sq^{4m}, Sq^{4m}Sq^1) \left(\subseteq H^{4m+1}(X;\mathbb{Z}/2)\right) \longrightarrow \operatorname{coker}(Sq^2 + Sq^1) \left(\leftarrow H^{8m+2}(X;\mathbb{Z}/2)\right)$$

satisfy $\Phi_m(x+y) = \Phi_m(x) + \Phi_m(y) + xy$, giving thus quadratic forms q_k for 1-connected Spin-manifolds. Now we have to compute Φ_{m+1} on $H^{4m+5}(N;\mathbb{Z}/2)$ for an $\mathbb{HP}^2$-bundle $N^{8m+10} \to M^{8m+2}$. By the Leray-Hirsch theorem, we have

$$H^{4m+5}N = H^{4m+5}M \oplus x \cdot H^{4m+1}M \oplus x^2 \cdot H^{4m-3}M ,$$

where $x \in H^4N$ is the Leray-Hirsch generator and all cohomology groups are with $\mathbb{Z}/2$-coefficients. As Φ_{m+1} vanishes on $H^{4m+5}M$, we get by the sublagrangian lemma for quadratic forms that $\operatorname{Arf} \Phi_{m+1} = \operatorname{Arf} \Phi_{m+1}|_{x \cdot H^{4m+1}M}$.
Thus one has to compute $\Phi_{m+1}(x \cdot y)$ with $y \in H^{4m+1}M$. Standard-homotopy theory shows that $\Phi_{m+1}(x\cdot y) = x^2 \cdot \Phi_m(y) + x^2 \varepsilon(y)$, where ε is given by primary cohomology operations and $BPSp(3)$-characteristic classes, depending on the choice of secondary operations Φ_m, Φ_{m+1}.
Now, property ii) is satisfied (and the theorem is proven) as soon as one can show that $\varepsilon = 0$ for a good choice of Φ_m, Φ_{m+1}. We showed this using Kristensen's theory of cochain operations, which gives in particular Cartan-formulas for secondary cohomology operations with smallest possible undeterminacy. These computations are lengthy and somewhat involved, so we cannot go here into further details by the lack of space.

Titel: BROWNIAN MOTION AND ELLIPTIC GENERA OF LEVEL N.
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The Witten genus should be the equivariant index of the Dirac operator over the free loop space. F. Hirzebruch has introduced a genera which should be the equivariant Index of the free loop space of a complex manifold.

The purpose of this talk is to construct a regularized Dolbeault operator over the free loop space. We use the fact that there is a measure over the free loop space, which is invariant by rotation and which lives over continuous loop: it is the brownian bridge measure. Bismut has defined (1989) an Hilbert tangent space of a stochastic loop which allows to get integration by parts. If the manifold is Kähler there is a natural complex structure over the stochastic loop space.

This allows us to define Hilbert spaces of sections of $T^{1,0}$ or $T^{0,1}$ forms by using the Fermionic Fock space. Moreover, a regularized $\bar{\partial}$ operator is constructed, by using a connection invariant by rotation, over a dense set of sections. Its adjoint is computed $\bar{\partial}^*$, such that we can define an extension of $\bar{\partial} + \bar{\partial}^*$. Its equivariant index should be, after regularization, the elliptic genus of level N.

We motivate this conjecture (Hirzebruch Witten conjecture) by using the brownian bridge in small time. We analyze the operator in small time and we get at the limit operator in the manner of Tauber: the probabilistic model is constituted of the family of Brownian bridges in the family of tangent spaces. Over the limit model, we can perform the exact computations. The conjecture is justified because the index should be stable under deformation

Titel: Rigidity Theorems for Spin^c-manifolds

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I) Let M^{2m} be always a smooth connected closed manifold and $Spin^c(2m) \hookrightarrow P \to M$ a $Spin^c$-structure on M. Let $c := c_1(P/Spin(2m) \xrightarrow{U(1)} M)$ be the first Chern class of P. Since M has a K-theory orientation there is a push-forward $\pi_!^M : K(M) \to K(*) \cong \mathbb{Z}$ for $\pi^M : M \to *$, which can be expressed in cohomology

$$E \mapsto \langle \hat{A}(M) e^{c/2} \cdot ch(E), [M] \rangle .$$

Here $\hat{A}(M) = \pi \frac{x_i}{e^{x_i/2} - e^{-x_i/2}}$ and $\pm x_i$ are the roots of M.

From now on we assume $V \to M$ is a complex bdl of complex dimension s with $c_1(V) = c$ and W is a $Spin(2t)$-vectorbdl. over M with associated principal bdl. P_W.

Definition 1: (i) $F(M; V, W) :=$

$$\bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{TM_{\mathbb{C}}}) \otimes \Lambda_{-1}(V^*) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n}(\widetilde{V_{\mathbb{C}}}) \otimes \Delta(\widetilde{W}) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n}(\widetilde{W_{\mathbb{C}}})$$

(ii) $ind(\partial_c(M; V, W))(\tau) := \pi_!^M(F(M; V, W)) \in \mathbb{Q}[[q]]$.

Here $\widetilde{E} = E - \dim E$ and $\Delta(W) := P_W \times_{Spin} (\Delta^+_{\mathbb{C}} + \Delta^-_{\mathbb{C}})$.

The index $ind(\partial_c(M; V, W))$ converges for $|q| < 1$ and defines a holomorphic function $ind(\partial_c(M; V, W))(\tau)$ for $\tau \in \mathcal{H}$ and $q = e^{2\pi i \tau}$.

Lemma 2: Let $\pm x_i, v_j$ and $\pm w_k$ be the roots of M, V and W. Then $ind(\partial_c(M; V, W))(\tau) =$

$$\langle e(M) \cdot \pi \, \phi(\tau, \tfrac{x_i}{2\pi i})^{-1} \cdot \pi \, \phi(\tau, \tfrac{v_j}{2\pi i}) \cdot \pi \frac{\phi(\tau, \frac{w_k}{2\pi i} + 1/2)}{\phi(\tau, 1/2)}, [M] \rangle .$$

Remark 3: 1) If $c=0$ $\operatorname{ind}(\partial_c(M;0,TM))(\tau) = \varphi_{ell}(M)$

2) If $c=0$ $\operatorname{ind}(\partial_c(M;0,0)) = \varphi_W(M)$

The Weierstraß' ϕ-function is a Jacobi function of "index 1/2" and weight -1 for $Sl_2(\mathbb{Z})$. This leads to

Proposition 4: If $p_1(V+W-TM)=0$ then

1) $\operatorname{ind}(\partial_c(M;V,W))(\tau)$ is a modular form for $\Gamma_0(2)$ of weight $m-s$

2) $\operatorname{ind}(\partial_c(M;V,W))(-1/\tau) = \tau^{m-s}\cdot$

$$\pi_!^M\Big(\bigotimes_{n\geq 1}^{\infty} S_{q^n}(\tilde{TM}_{\mathbb{C}})\otimes\Lambda_{-1}(V^*)\otimes\bigotimes\Lambda_{-q^n}(\tilde{V}_{\mathbb{C}})\otimes\bigotimes_{n\text{ odd}}\Lambda_{-q^{n/2}}(\tilde{W}_{\mathbb{C}})\Big)$$
$$\in \mathbb{Z}[[q]].$$

II) We now consider the equivariant situation. Let G be a compact connected Lie group. For a G-equivariant vectorbdl $E\to Y$ and a characteristic class x let $x(E)_G := x(E_G\to Y_G)$, where $(\,)_G$ is the Borel-construction, $Y_G := EG\times_G Y$.

We also assume that G acts on M and the action is lifted to P, V and W. Then the G-equivariant index $\partial_c(M;V,W)(\tau,g)$, $g\in G$, is defined.

The proof of the rigidity theorem of Liu for Spin-manifolds can also be applied in our situation and gives

<u>Theorem 5:</u> Assume for $S^1 \hookrightarrow G$

$p_1(V+W-TM)_{S^1} = \pi^*(\alpha x^2)$, where $\pi: M_{S^1} \to BS^1$ and $x \in H^2(BS^1;\mathbb{Z})$ is a generator

1) If $\alpha \geq 0$ $\quad ind(\partial_c(M;V,W))(\tau,\lambda)$ is constant in $\lambda \in S^1$
2) If $\alpha < 0$ $\quad ind(\partial_c(M;V,W)) \equiv 0$.

∎

We now assume that we only have an action of G on M. For G simply connected by Theorems of Stewart, Su and Petrie the action can be lifted to $P \to M$ and to complex line bdls.

<u>Theorem 6:</u> Let M be a Spin^c-mfd. with non-trivial S^3-action. Let V, W be sums of complex line-bdls over M, s.t. $c_1(V) \equiv w_2(M)\,(2)$ and W is spin. If $M^{S^3} \neq \emptyset$ and $p_1(V+W-TM) = 0$ then

$$ind(\partial_c(M;V,W)) \equiv 0.$$

<u>Idea of the proof:</u> First lift the action to P and to the line bdls. occuring in V and W. Since

$$H^4(BS^3;\mathbb{Z}) \to H^4(M_{S^3};\mathbb{Z}) \to H^4(M;\mathbb{Z}) \text{ is exact}$$

we get $p_1(V+W-TM)_{S^1} = \pi^*(\alpha x^2)$ for a choosen $S^1 \hookrightarrow S^3$.
Now $\alpha < 0$ since at the S^3-fixed pt. any S^3-equivariant line bdl. is just a trivial representation. Thus we can apply Theorem 5.

∎

III) We now give two applications

In 1972 Petrie gave the

Conjecture: If $M \simeq \mathbb{C}P^n$ and M has a non-trivial S^1-action then $p(M) = (1+x^2)^{n+1}$.

We prove the Conjecture to be true in the following situation

Theorem 7: Let $M \simeq \mathbb{C}P^n$ a S^3-mfd with fixed point. Define $\alpha \in \mathbb{Z}$ by $p_1(M) = \alpha x^2$. Then $\alpha \leq n+1$ and If $\alpha = n+1$ the total Pontrjagin class $p(M) = (1+x^2)^{n+1}$. ∎

Theorem 8: Let M be a 4-dim S^3-mfd with fixed point. Then $b_2(M) < 2$. ∎

Final Remarks: 1) There are also elliptic genera for Spin^c-bordism ring which generalize φ_{ell} and φ_W.

2) There are mod 8 divisibility theorems for $\mathrm{ind}(\partial_c(M; V, W))$.

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Titel: Generalized modular forms, Hecke algebras and operations in elliptic cohomology

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This is an informal introduction to "Operations and cooperations in elliptic cohomology", of which Part I appeared in the New York Journal of Mathematics, vol. 1 (1995), 39–74.

Let $E\ell\ell_{2n}$ be the group of weight n modular forms for $SL_2(\mathbb{Z})$ whose q-coefficients lie in $\mathbb{Z}[1/6]$. Then there is a genus

$$MU_* \longrightarrow E\ell\ell_* = \mathbb{Z}[1/6][Q, R, \Delta^{-1}]$$

for which

$$E\ell\ell^*(\ ) = MU^*(\ ) \underset{MU_*}{\otimes} E\ell\ell_*$$

is a cohomology theory on finite complexes. Here $\begin{cases} Q = E_4 \\ R = E_6 \end{cases}$ are Eisenstein functions and

$$\Delta = \frac{1}{1728}(Q^3 - R^2)$$

has weight 12.

We wish to understand the stable operation algebra $E\ell\ell^*(E\ell\ell)$. Dually we have the cooperation algebra $E\ell\ell_*(E\ell\ell)$ and a universal coefficient spectral sequence

$$E_2^{st} = \mathrm{Ext}^{st}_{Ell_*}(Ell_* Ell, Ell_*) \Longrightarrow Ell^*(Ell)$$

(this uses work of J.R. Hunton). In fact, $E_2^{s*} = 0$ if $s > 2$, and it is probably true that $E_2^{s*} = 0$ for $s > 0$. Hence we should be able to construct much of $Ell^*(Ell)$ from $\mathrm{Hom}_{Ell_*}(Ell_* Ell, Ell_*)$.

Thus the determination of $Ell_*(Ell)$ is a good first step in this programme.

Let $\mathcal{V} = \{(\omega_1, \omega_2) \in \mathbb{C}^2 : \omega_1/\omega_2 \in \mathcal{H}\}$

$$(\mathcal{H} = \{\tau \in \mathbb{C} : \operatorname{im} \tau > 0\})$$

$SL_2(\mathbb{Z})$ acts freely on $\mathcal{V}$ and we let

$\mathcal{L} = SL_2(\mathbb{Z}) \backslash \mathcal{V}$ which can be viewed as the space of oriented lattices in $\mathbb{R}$, where a frame $(\omega_1, \omega_2) \in \mathcal{V}$ spans a lattice $\langle \omega_1, \omega_2 \rangle$. $\mathcal{L}$ has a (non-free) action of $\mathbb{C}^\times$ by scaling:

$$\lambda \cdot \langle \omega_1, \omega_2 \rangle = \langle \lambda\omega_1, \lambda\omega_2 \rangle.$$

A modular form $f \in Ell_{2n}$ can be interpreted as a holomorphic function $F: \mathcal{L} \longrightarrow \mathbb{C}$ such that

$$\begin{cases} F(\lambda \langle w_1, w_2 \rangle) = \lambda^{-n} F(\langle w_1, w_2 \rangle) \\ F(\langle \tau, 1 \rangle) = \sum_{r_0 \le r_\phi} a_r q^r \quad , \quad q = e^{2\pi i \tau} \\ \qquad \text{and} \quad a_r \in \mathbb{Z}[1/6]. \end{cases}$$

Now let $\mathcal{L}(N) = \underset{SL_2(\mathbb{Z})}{\backslash} \left(\tfrac{1}{N}\right) M_2(N) \underset{SL_2(\mathbb{Z})}{\times} \mathcal{U}$

~~Then an element of $\mathcal{L}(N)$ can be viewed~~

where $\left(\tfrac{1}{N}\right) M_2(N) = \left\{ \left(\tfrac{1}{N}\right) A : \quad A \in M_2(\mathbb{Z}), \ \det A = N \right\}$

with $0 < N \in \mathbb{Z}$. Then an element of $\mathcal{L}(N)$ can be viewed as a lattice pair $L' \supseteq L$ where $[L', L] = N$. Associated to this is an isogeny of degree N of elliptic curves $\mathbb{C}/L \xrightarrow{[1]} \mathbb{C}/L'$ covered by the identity $\mathbb{C} \longrightarrow \mathbb{C}$. $\mathcal{L}(N)$ has two projections $\mathcal{L}(N) \rightrightarrows \mathcal{L}$ which pick off L, L' from the pair. ~~Also~~

We set $\mathcal{L}_\bullet = \coprod_{N>0} \mathcal{L}(N)$ and view this as a space over $\mathcal{L}$ in two ways. Also there is a partial product $\mathcal{L}_\bullet \times_{\mathcal{L}} \mathcal{L}_\bullet \longrightarrow \mathcal{L}_\bullet$ where we compose pairs $L'' \supseteq L'$ and $L' \supseteq L$ to give $L'' \supseteq L$. (i.e. the coproduct of pairings $\mathcal{L}(N') \times_{\mathcal{L}} \mathcal{L}(N) \longrightarrow \mathcal{L}(N'N)$).

We now define a generalised modular form of weight n to be a coproduct

$$F_\bullet = \left\{ F_N : \mathcal{L}(N) \longrightarrow \mathcal{L} \times \mathcal{L} \xrightarrow{F} \mathbb{C} \right\} : \coprod \mathcal{L}_\bullet \longrightarrow \mathbb{C}$$

where F is a bilinear combination of modular forms on $\mathcal{L} \times \mathcal{L}$ of total weight n,

i.e. $F = \sum_i G_i \otimes H_i$ where $\mathrm{wt}\, G_i + \mathrm{wt}\, H_i = n$.

We can define q-expansions for such $F_\bullet$ which are elements of $\mathbb{C}[[q, q']][q^{-1}, q'^{-1}]$. Also we may require that the coefficients of these q-expansions are arithmetical restricted. This is done by requiring that if $\underset{\in \mathcal{L}(N)}{(L' \supseteq \langle \tau, 1 \rangle)}$ we obtain q-coefficients in

$\mathbb{Z}[1/6N]$ (and actually Laurent polynomials in N).

The resulting ring of generalized modular forms M_*^{Gen} is a $\mathbb{Z}[1/6]$ algebra, and in fact a ~~bimodule~~ bialgebra over Ell_* using the two projections $\mathcal{L}_0 \longrightarrow \mathcal{L}$.

Lemma $\exists$ a ring homomorphism $Ell_* Ell \longrightarrow M_*^{Gen}$ extending the two units $Ell_* \longrightarrow M_*^{Gen}$.

Pf. This uses formal group laws for elliptic curves $\mathbb{C}/L$ and $\mathbb{C}/L'$ and existence of a local isomorphism $\mathbb{C}/L \longrightarrow \mathbb{C}/L'$ whenever $(L' \supseteq L) \in \mathcal{L}(N)$, This gives rise to an isomorphism of formal group laws over $\mathbb{Z}[1/6N][[q, q']]$ and in fact over M_*^{Gen}. By Quillen's work on MU_* and universality of formal group laws, we get a map $Ell_* Ell \longrightarrow M_*^{Gen}$ as required.

Cor $Ell_* Ell \otimes \mathbb{Q} \cong M_*^{Gen} \otimes \mathbb{Q}$.

Theorem $Ell_* Ell \cong M_*^{Gen}$

The proof uses ~~[illegible]~~ work of N. Katz who computed a certain "ring of divided congruences" amongst modular forms. In effect this amounts to a calculation of $K_0 Ell$!

Thus we have described $Ell_* Ell$ as a ring of functions on an analytic object $\mathcal{L}_*$. In fact, $\mathcal{L}_*$ is an analytic groupoid, at least after suitable localization which roughly amounts to inverting all isogenies $\mathbb{C}/L \xrightarrow{\cong} \mathbb{C}/L'$. This gives a cogroupoid structure on M_*^{Gen} agreeing with a known topological structure making these isomorphic Hopf algebroids.

It is possible to define Hecke-like operations as dual elements $T_N \in \mathrm{Hom}_{Ell_*}(Ell_*Ell, Ell_*[1/N])$

Hence we can view elliptic cohomology at least rationally as a module over the Hecke algebra $\mathbb{Z}[SL_2(\mathbb{Z}) \backslash GL_2^+(\mathbb{Q}) / SL_2(\mathbb{Z})]$.

This extends to an action of a "twisted" Hecke algebra $Ell_* \otimes \mathbb{Q}\{SL_2(\mathbb{Z}) \backslash GL_2^+(\mathbb{Q}) / SL_2(\mathbb{Z})\}$.

Each double coset $H \in SL_2(\mathbb{Z}) \backslash GL_2^+(\mathbb{Q}) / SL_2(\mathbb{Z})$ acts on $Ell^*(\ )[1/N]$ for some N depending on H.

Titel: CFT & LBG.

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§0. We view CFT (= conformal field theory) as the representation theory of VOAs (vertex operator algebras). From this point-of-view we discuss twisted sectors and their rôle in constructing certain bundles over the loop space LBG over ~~a the~~ classifying space of a (finite) group G. These are related to G. Segal's 'elliptic objects', and may well provide candidates for element of $Ell^*(BG)$

We consider the case of $M = G$ (Monster) in particular in view of the Preisfrage & the ideas of Mahowald/Hopkins (cf. talk of Charles Thomas)

§1. VOA's. Roughly, a VOA V is a $\mathbb{Z}$-graded linear space $/\mathbb{C}$, $V = \bigoplus_{n\in\mathbb{Z}} V_n$ s.t. $\dim V_n < \infty$ &

$V_n = 0$ for $n \ll 0$. There are, in addition, bilinear products $*_m : V \times V \longrightarrow V$ (for $m \in \mathbb{Z}$) denoted ~~given~~ by

$(u,v) \mapsto u *_m v$ for $u, v \in V$.

Define $u_m \in \operatorname{End} V$ via $u_m(v) = u *_m v$. Form the generating function (= vertex operator)

$$Y(u,z) = \sum_{m \in \mathbb{Z}} u_m z^{-m-1} \in (\operatorname{End} V)[[z, z^{-1}]]$$

for us, z is always a formal variable.

<u>Main Axiom for VOA</u>. Given $u, v \in V$, $\exists\, k = k(u,v)$ in $\mathbb{Z}$ with $k \geq 0$ and

$$(z_1 - z_2)^k [Y(u, z_1), Y(v, z_2)] = 0.$$

<u>Virasoro Axiom</u> $\exists$ distinguished $\omega \in V$ (conformal vector) with $Y(\omega, z) := \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ s.t. the L_n's close on the Virasoro algebra of central charge c. I.e

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{m^3 - m}{12} \delta_{m+n,0}\, c \cdot \mathrm{Id}_V$$

Also, $V_n = \{ v \in V \mid L_0 v = n v \}$
We omit other axioms!

<u>Definition</u>: $V = (V, Y)$ is a VOA. The automorphism gp. $\operatorname{Aut} V$ consists of invertible $g \in \operatorname{End} V$ s.t.

$$\begin{cases} g(u *_m v) = (gu) *_m gv & \forall u, v \in V \ \forall m \in \mathbb{Z} \\ g\omega = \omega \end{cases}$$

Note that $g\omega = \omega \Rightarrow g L_n = L_n g$ $\forall n$. In particular each V_n is a representation of $\operatorname{Aut} V$.

Example: $V = V^\natural$ (Moonshine Module). Here $c=24$, $\mathrm{Aut}\, V \cong \mathbb{M}$ (Monster). If we set

$ch_q V = \sum_n (\dim V_n) q^n$ then

$$q^{-1} ch_q V^\natural = J(q) = j(q) - 744 = q^{-1} + 196884 q + \dots$$

(Work of Borcherds, Frenkel, Lepowsky, Meurman)

§2 A. Baker, C. Thomas suggest that one consider the Borel constructions $EG \times_G V_n$ for VOA $V = \oplus V_n$ and gp. $G \leq \mathrm{Aut}\, V$, assembled into a VOA bundle over BG: $EG \times_G V$. Expected modular-invariance of $q^{-c/24} ch_q V$ in good cases suggests a connection with elliptic cohomology of BG.

G. Segal further suggests studying LBG, so one wants a "bigger" bundle over LBG which extends the above idea. Let's fix G finite from now on. Now

$$LBG \simeq \coprod_{*g} BC_G(g)$$

Here, $*$ denotes one choice of g in each conjugacy class of G. $C_G(g) = \{ h \in G \mid hg = gh \}$.

So we want bundles over $BC_G(g)$. CFT suggests what they should be.

§3 Twisted Sectors. A VOA V has a representation theory. For $g \in \mathrm{Aut}\, V$ (of finite order N) there is the notion of g-twisted module.

It is a linear space W equipped with twisted vertex operators $Y_g(u, z)$ $(u \in V)$ similar to before. But W is $\mathbb{Q}$-graded, and the bracket $[Y_g(u, z_1), Y_g(v, z_2)]$ for u, v in g-eigenspaces of V involves some Nth roots of unity.

Details suppressed!. For W a simple g-twisted V-module, W is graded in the form

$$W = \bigoplus_{n \geq 0} W_{p + \frac{n}{N}} \quad \text{for some fixed } p \in \mathbb{Q}, \; N = o(g), \; p = p(g).$$

It is not known if these things exist in general

Theorem (Dong, Mason) $\forall g \in \mathbb{M}$, $\exists$ unique g-twisted $V^\natural$-module. Call it $V^\natural(g)$.

~~[illegible]~~ Uniqueness forces a projective action of $C_M(g)$ on $V^{\natural}(g)$, so one can form the Borel construction $EC_M(g) \times_{C_M(g)} \mathbb{P}V^{\natural}(g)$. This gives the desired bundle over $BC_M(g)$: the fiber is a projective twisted $V^{\natural}$-module.

§4 Modular - Invariance

Let $V^{\natural}(g) = \bigoplus_{n \geq 0} V^{\natural}(g)_{p(g)+\frac{n}{N}}$, $N = o(g)$ as before. $V^{\natural}(g)_x$ is a projective $C_M(g)$-module, so for $h \in C_M(g)$ we can define (up to a scalar) the partition function

$$Z(g,h,\tau) = q^{-1}(\text{graded tr. } h \text{ on } V^{\natural}(g))$$
$$= q^{-1+p} \sum_{n \geq 0} (\operatorname{tr} V^{\natural}(g)_{p+\frac{n}{N}} | h) q^{n/N}$$

<u>Theorem (Dong, Mason)</u>. $Z(g,h,\tau)$ is holomorphic on upper half plane $\mathcal{H}$. (Here we have set $q = e^{2\pi i \tau}$, $\tau \in \mathcal{H}$)

The *conformal block* $\mathcal{B}$ is the $\mathbb{C}$-space with basis $Z(g,h,\tau)$ where $g \in \mathbb{M}$, $h \in C_{\mathbb{M}}(g)$. As $Z(g^x, h^x, \tau) = Z(g,h,\tau)$ where $g^x = xgx^{-1}$, take one pair (g,h) from each $\mathbb{M}$-conjugacy class. Hence

$$\dim \mathcal{B} = \dim \operatorname{Hom}_{\mathbb{M}}(\operatorname{Hom}(\mathbb{Z}^2, \mathbb{M}), \mathbb{C})$$

(cf. talk of N. Kuhn)

Theorem (Dong, Mason) Usual action of $\Gamma = SL(2,\mathbb{Z})$ on $\mathcal{H}$ induces a representation of Γ on $\mathcal{B}$. In fact

$$Z(g,h,\gamma\tau) = \epsilon(g,h,\gamma)\, Z((g,h)\gamma, \tau)$$

where $(g,h)\gamma = (g^a h^c, g^b h^d)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. $\epsilon(\ )$ is a scalar ind. of τ.

Rk: These theorems apply not just to VOA and $\mathbb{M}$ but to many other cases too.

§6 Examples. In our notation, for $g, h \in \mathbb{M}$,

$Z(1, g, \tau) = q^{-1}$(graded trace of g on $V^\natural$)

$Z(g, 1, \tau) = q^{-1}$(graded character of $V^\natural(g)$)

The last theorem says

$$(*) \quad Z(g, 1, \tau) = \epsilon(g)\, Z(1, g, -\tfrac{1}{\tau})$$

Thm (Borcherds = Monstrous conjecture)

Each $Z(1, g, \tau)$ is modular function on some $\Gamma_0(N)$ (N = small multiple of $o(g)$)

Now $(*) \Rightarrow$ the graded character of each $V^\natural(g)$ is also a modular function (namely the S-transform of $Z(1, g, \tau)$ up to a constant). It seems hard to get the constant $= 1$!

As an interesting example, take $g = 2A$ (a certain involution in $\mathbb{M}$)

Theorem (Dong, Li, Mason)

$$Z(2A, 1, \tau) = Z(1, 2A, -\tfrac{1}{\tau}) = Z(1, 2A, \tfrac{\tau}{2})$$

Thus

$$Z(2A, 1, \tau) = q^{-1/2} + 4372 q^{1/2} + \cdots\cdots$$

These are ~~module~~ dimensions of modules for $C_M(2A) = 2.(\text{Baby Monster})$ (cf. Lecture of G. Höhn)

We have, more generally:

<u>Theorem</u> (D, L, M) If $h \in C_M(2A)$ has odd order then

$$Z(2A, h, \tau) = Z(1, (2A)h, \tau/2)$$

(ie trace h on $V^\natural(2A)$ = trace $(2A).h$ on $V^\natural$ with $\tau \mapsto \tau/2$)

§7 Final Thought:

There is a special element $g_0 \in C_M(2A)$ also of type $2A$, characterized by the property that

$$Z(2A, g_0, \tau) = \sqrt{J(q) - 984}$$
$$= q^{-1/2} - 492 q^{1/2} + \cdots\cdots$$

More symmetrically, our relation is

$$Z(1A, 1A, \tau) = j(q) - 744$$

$$Z(2A, 2A, \tau) = \sqrt{J(q) - 984}$$

Ch. Thomas has discussed the numbers -744, -984 in the context of the 24 dim. manifold of Hopkins/Mahowald which has Witten genus $j(q) - 744$.

This suggests that there are non-trivial connections between this mnfld & $\mathbb{M}$......

To be continued elsewhere ...

Geoffrey Mason
June 14 1995.

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A Moonshine Module for the Baby Monster

In this talk we describe a construction of a vertex operator superalgebra (short SVOA) the Baby monster is acting on in a natural way.

VOA's were introduced in mathematics in connection with the "Moonshine of the Monster" — the strange connection between the Monster, the largest finite simple group ($|M| \sim 8 \cdot 10^{53}$), and modular functions (especially with $j - 744 = q^{-1} + 196884\,q + 21493760\,q^2 + \cdots$). In physics VOA's appeared as the so called chiral algebra of a CFT. VOA's are in some sense a "complexification" of the notation of Lie algebras, but they are related to many other structures in mathematics.

My original interest on VOA's comes from topology in connection with elliptic genera. The aim was to find a more geometrical description of elliptic cohomology theory. Let X^n be a Spin manifold. Then the "classical" Atyah-Singer index theorem states

$$\mathrm{Ind}(D \otimes E) = \hat{A}(X^n; E) \in \mathrm{KO}^{-n}(pt.) \cong \widetilde{\mathcal{M}_n}/i^*\widetilde{\mathcal{M}}_{n+1}.$$

Here $D \otimes E$ is the Dirac operator twisted by a vector bundle E which is associated to some G-principal bundle over X and $\widetilde{\mathcal{M}_n}/i^*\widetilde{\mathcal{M}}_{n+1}$ denotes the abelian group of isomorphism classes of graded Modules of the real Clifford algebra Cl_n in dimension n modulo the ones which come from Cl_{n+1}. For the loop space $\mathcal{L}X$ one has the following string theoretical wish

$$\mathrm{Ind}(\mathcal{L}D \otimes \widetilde{E}) = \widetilde{\varphi}_W(X, \widetilde{E}) \in \mathrm{Ell}^*(pt.).$$

Here $\mathcal{L}D \otimes \widetilde{E}$ should be the hypothetical Dirac operator on $\mathcal{L}X$ twisted by a bundle assoziiated to a positive energy representation of a loop group and $\widetilde{\varphi}_W$ is a refined version of the Witten genus with values in suitable defined elliptic cohomology group.

For one fibre one has the following picture of pairs:

(Cl_n, Δ)	$(\mathfrak{g}, V_\lambda)$
Spinor Representation of Cl_n	*a highest weight representation of a reductive Lie algebra*
$\downarrow$	$\downarrow$
$(\mathrm{Cliff}^n(\mathbf{Z}+\frac{1}{2}), V_{\mathrm{Fermi}}^{\otimes n})$	$(\hat{\mathfrak{g}}, V_{k,\lambda})$
infinite Spinor representation of infinite n-dim. komplex Clifford algebra	*a highest weight representation of a affine Kac-Moody Lie algebra of Level k*

The main remark is, that $V_{\mathrm{Fermi}}^{\otimes n}$ and $V_{k,\lambda}$ carries the much larger structure of a (S)VOA.

In this talk VOA's are studied more from number theoretic and combinatorial point of view. In fact there is a deep analogy between *Codes & Lattices & VOAs*. It was this analogy which motivated myself to compute $\chi_{1/2}^{47} - 47\,\chi_{1/2}^{23} = q^{-47/48}(1 + 4731q^{3/2} + 96256q^2 + \cdots)$, where $\chi_{1/2} = \sqrt{\frac{\theta_{\mathbf{Z}}}{\eta}}$. It is a modular function for $\Gamma_{\theta,24} := \mathrm{Ker}(\rho_{24} : \Gamma_\theta = \langle S, T^2\rangle \longrightarrow \mathbf{C}^*)$, $\rho_{24}(S) = 1$, $\rho(T^2) = e^{\frac{2\pi i}{24}}$ and it turns out, that the coefficients are simple sums of dimensions of irreducible representations of the Baby monster

I want to start by recalling the basic definitions for codes + lattices.

Codes: A code is a linear subspace $C \subset \mathbf{F}_2^n$. For $u, v \in \mathbf{F}_2^n$ let $(u,v) := \sum_i u_i v_i \pmod 2$ and $w(u) := \sum_i u_i \in \mathbf{Z}$ the weight. Definitions:

— *C even:* $2 \mid w(u)\ \forall u \in C$
— *C selfdual:* if $C = C^\perp \Rightarrow C$ even
— *C doubly even:* $4 \mid w(u)\ \forall u \in C$
— *weight enumerator:* $W_C(x) = \sum_{u\in C} x^{n-w(u)}$.

Lattices: A lattice L is an abelian discrete subgroups of maximal rank in $\mathbf{R}^n$. Definition:

— *L integral:* $\langle x, y\rangle \in Z\ \forall x, y \in L$
— *L selfdual:* $L^\perp := \{x \mid (x,y) \in \mathbf{Z}, \forall y \in L\} = L$
— *L even:* $\langle x, x\rangle \in 2\mathbf{Z}\ \forall x \in L$
— *Theta series:* $\Theta_L(q) = \sum_{x\in L} q^{\frac{1}{2}(x,x)}$

There is a construction from codes to lattices: $L_C := \frac{1}{\sqrt{2}}\rho^{-1}(C)$, where $\rho : \mathbf{Z}^n \longrightarrow \mathbf{F}_2^n$ is the reduction modulo 2. The lattice L_C is integral, even, resp. selfdual if C is even, doubly even or selfdual. Selfdual codes & lattices are classified for $n \leq 24$. Important examples are in the following table.

Rank	1/2	1	2	8	22	23	23 1/2	24
Codes	–	–	C_2	Hamming H_8	shorter Golay $\mathcal{G}_{22}$ $M_{22}\cdot 2$	–	–	Golay code $\mathcal{G}_{24}$ M_{24}
Lattices	–	$\mathbb{Z}$		Root lattice E_8 $W(E_8)$		shorter Leech O_{23} $Co_2\cdot 2$	–	Leech lattice Λ_{24} $2.Co_1$
(S)VOA's	V_{Fermi}			V_{E_8} E_8			Baby Monster SVOA V^{B} B	Monster VOA $V^\natural$ M
Type	odd			(doubly) even	odd			(doubly) even

Theorem: *The weight enumerator or theta series of an (doubly) even resp. odd (i.e. even or integral) code C or lattice L is an homogenous polynomial of weight n = dimension of C or L as shown in the following table*

	(doubly) even	*odd*
W_C	$\mathbf{C}[W_{H_8}, W_{G_{24}}]$	$\mathbf{C}[W_{C_2}, W_{H_8}]$
Θ_L	$\mathbf{C}[\Theta_{E_8}, \Theta_{\Lambda_{24}}]$	$\mathbf{C}[\Theta_{\mathbf{Z}}, \Theta_{E_8}]$

There is a third step after code + lattices: VOA's.

Remark on definition of SVOA's:

— *Data:* • Vector space $V = \bigoplus_{n\in\frac{1}{2}\mathbf{N}_0} V_n = V_{(0)} \oplus V_{(1)}$, V_n finite dim. $\mathbf{C}$-vector space.

• Vertex operator $Y(.,z) : V \longrightarrow \mathrm{End}(V)[[z,z^{-1}]]$ $Y(v,z) = \sum_{n\in\mathbf{Z}} v_n \cdot z^{-n-1}$ linear

• Elements $\mathbf{1} \in V_0 = \mathbf{C}\cdot\mathbf{1}$ (vacuum) and $\omega \in V_2$ (Virasoro element) with $2\,\omega_3\omega = c\cdot\mathbf{1}$, rank $V := c$.

— *Axioms:* • Jacobi identity $\Longleftrightarrow$ $\langle v', Y(a_1,z_1)Y(a_2,z_2)v\rangle = \pm\langle Y(a_2,z_2)Y(a_1,z_1)v\rangle$ as rational functions ("2-point correllation functions on the sphere") , where $v' \in V^*$, a_1, a_2, $v \in V$ ($\Longleftrightarrow$ causality in quantum field theory.)

• $\{\omega_n, \mathrm{id}_V\}$ form a Virasoro algebra,

• . . ..

— *Modul* M of a SVOA V: $Y_M(.,z) : V \longrightarrow \mathrm{End}(M)[[z,z^{-1}]]$.

— V *selfdual*: V has only one irreducible module: V itself.

— V *VOA*: $V = V_{(0)}$ (i.e. integral grading).

— *Character:* $\chi_V := q^{-c/24}\sum_{n\geq 0}\dim V_n \cdot q^n$, holomorphic function in $\tau \in \mathbf{H}$, where $q = e^{2\pi i\tau}$.

Examples:

— V_L, "String on Torus $\mathbf{R}^n/L$" for a integral lattice L, rank $V_L = \dim L$, $\chi_{V_L} = \frac{\theta_L}{\eta^c}$.

— $V_{\mathrm{Fermi}} = L_{1/2}(0) \oplus L_{1/2}(\frac{1}{2})$, $\chi_{V_{\mathrm{Fermi}}} = \sqrt{\frac{\theta_{\mathbf{Z}}}{\eta}} = \chi_{1/2}$.

— $V^\natural$ Monster VOA ("$\mathbf{Z}_2$-orbifold of $V_{\Lambda_{24}}$"), $\chi_{V^\natural} = j - 744$

Theorem: *V selfdual, "nice", rational VOA* $\Longrightarrow$ $\chi_V \in \mathbf{C}[\chi_{V_{E_8}}, \chi_{V^\natural}]$ *homogenous of* rank V $\Longrightarrow$ rank $V \in 8\mathbf{Z}$.

Theorem: **(H.)** *V selfdual, unitary, "very nice", rational SVOA* $\Longrightarrow$ $\chi_V \in \mathbf{C}[\chi_{V_{\mathrm{Fermi}}}, \chi_{V_{E_8}}]$ *homogenous of* rank V $\Longrightarrow$ rank $V \in \frac{1}{2}\mathbf{Z}$.

We denote the even part of $V_{\rm Fermi}^{\otimes k} = V_{(0)} \oplus V_{(1)}$ with $V_{\mathbf{SO}(k)}$. One has

$$V_{\mathbf{SO}(k)} = \begin{cases} V_{D_n,1} & \text{if } k = 2n, \\ V_{B_n,1} & \text{if } k = 2n+1 \text{ and } n \geq 0, \\ L_{1/2}(0) & \text{if } k = 1; \end{cases}$$

where $V_{D_n,1}$ and $V_{D_n,1}$ denote the VOA's associated to the level 1 highest weight representation of the corresponding affine Kac-Moody algebra.

In analogy to the situation for codes and lattices it is expected:

Let $d \in \frac{1}{2}\mathbf{Z}_+$, $d \in 8\mathbf{Z}$, $d > c$. Then there is a 1 : 1-correspondence between the set of unitary, "very nice", selfdual SVOA's of rank c *and* isomorphy classes of pairs $(V, V_{\mathbf{SO}(d-c)})$, where V selfdual unitary "very nice" VOA of rank d, $V_{\mathbf{SO}(d-c)}$ sub-VOA of V (and choice of isomorphy class of $V_{\mathbf{SO}(8)}$-module for $d-c=8$ because of triality).

In the most interesting case of $V = V^\natural$ the largest $V_{\mathbf{SO}(k)}$ inside $V^\natural$ is $V_{\mathbf{SO}(1)}$ because of $V_1^\natural = 0$.

Main Theorem (H.): *There exists a SVOA V^B of rank $23\frac{1}{2}$ on which the Baby monster B — the second largest sporadic simple groups — acts by automorphisms.* The SVOA V^B is in this sense the natural object which defines B.

Idea of proof: From a result of Meyer/Neutsch and indepentendly Dong/Mason/Zhu it follows that there is a sub-VOA $L_{1/2}(0)^{\otimes 48} \subset V^\natural$. Let $V_{(0)}^B := \mathrm{Com}_{V^\natural}(L_{1/2}(0))$ the commutante of one of this 48 Virasoro VOA's. Then there ist the decomposition

$$V^\natural = V_{(0)}^B \otimes L_{1/2}(0) \oplus V_{(1)}^B \otimes L_{1/2}(\frac{1}{2}) \oplus V_{(2)}^B \otimes L_{1/2}(\frac{1}{16}).$$

We define V^B as the sum $V_{(0)}^B \oplus V_{(1)}^B$. One can define on it a SVOA-structure in a natural way. The sub-VOA $L_{1/2}(0)$ is fixed by an $2A$-involution in M. So $2.B = \mathrm{Cent}_M(2A)$ acts on V^B. The central extension acts trivial and we have an action on V^B which respects in fact the full SVOA-structure. □

A selfdual unitary rational SVOA of rankc is called *extremal* if

$$\chi_V = \chi_{L_c(0)} + \mathrm{O}\left(q^{-\frac{c}{24}+\frac{1}{2}[\frac{c}{8}]+\frac{1}{2}}\right),$$

i.e. the first $\frac{1}{2}[\frac{c}{8}]$ coeffcients are as small as possible.

Theorem (H.): List of extremal, selfdual, unitary, rational SVOA's:

Rang	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	$\frac{9}{2}$	5	$\frac{11}{2}$	6	$\frac{13}{2}$	7	$\frac{15}{2}$
Codes				c_2				c_2^2				c_2^3			
				$\downarrow$				$\downarrow$				$\downarrow$			
Gitter		$\mathbf{Z}$		$\mathbf{Z}^2$		$\mathbf{Z}^3$		$\mathbf{Z}^4$		$\mathbf{Z}^5$		$\mathbf{Z}^6$		$\mathbf{Z}^7$	
		$\downarrow$		$\downarrow$		$\downarrow$		$\downarrow$		$\downarrow$		$\downarrow$		$\downarrow$	
SVOA's	V_F	V_F^2	V_F^3	V_F^4	V_F^5	V_F^6	V_F^7	V_F^8	V_F^9	V_F^{10}	V_F^{11}	V_F^{12}	V_F^{13}	V_F^{14}	V_F^{15}

Rang	8	12	14	15	$\frac{31}{2}$	22	23	$\frac{47}{2}$	24
Codes	e_8	d_{12}^+	$(e_7+e_7)^+$			g_{22}			g_{24}
	$\downarrow$	$\downarrow$	$\downarrow$						
Gitter	E_8	D_{12}^+	$(E_7+E_7)^+$	A_{15}^+			O_{23}		Λ_{24}
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$					
SVOA's	V_{E_8}	$V_{D_{12}^+}$	$V_{(E_7+E_7)^+}$	$V_{A_{15}^+}$	$V_{E_{8,2}^+}$			V^B	$V^\natural$

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Titel: *Elliptic Genera and Vertex Operator Algebras*

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On free loop spaces, various infinite dimensional objects, such as loop groups and their projective representations, come into play. In this talk, vertex operator algebra structure was introduced in the context of Witten genera and elliptic genera.

Vertex operator algebras (VOA) appear in geometric context through the normal bundle N to a manifold in its free loop space. Since $M = (LM)^{S^1} = (LM)^{\mathrm{Diff}^+S^1}$ the group Diff^+S^1 acts on the normal bundle preserving fibers. It is well known that a suitable completion of polynomial functions on $N_x \otimes \mathbb{C}$ for $x \in M$ splits into a tensor product of two irreducible projective representations of Diff^+S^1: $\hat{S}(N_x \otimes \mathbb{C}) = \hat{S}(A_x) \otimes \hat{S}(\overline{A}_x)$, where $A_x, \overline{A}_x$ consist of polynomials in vectors with positive Fourier coefficients (negative Fourier coefficients, respectively). Although Diff^+S^1 acts naturally on $\hat{S}(N_x \otimes \mathbb{C})$ by pulling back functions on $N_x \otimes \mathbb{C}$, it can only act projectively on $\hat{S}(A_x)$ and on $\hat{S}(\overline{A}_x)$ with central charges $c = 2N$, and $c = -2N$, respectively. These projective actions may be thought of as square root of the original geometric

action. There exists a canonical Diff^+S^1-invariant skew bilinear form on N_x associated with a given Riemannian metric. This skew form gives rise to a Heisenberg algebra $\widehat{N_x}$ and hence a bundle of Heisenberg algebras on M. associated to this bundle, there is a bundle of irreducible modules over $\widehat{N}$ whose fiber over $x \in M$ is $S(A_x)$. This graded vector space naturally has a structure of VOA and the bundle is of the form $S_q = \bigotimes_{m \geq 1} S_{q^m}(T_{\mathbb{C}})$. There is a canonical Virasoro section ω_S with central charge $c = 2N$.
Similarly, using spinors on S^1, one can construct a Clifford algebra and its representation for each $x \in M$, and these representations form a bundle of VOA of the form $\mathcal{V} = \bigotimes_{m \geq 1} \Lambda_{q^{m-\frac{1}{2}}}(T_{\mathbb{C}})$ with a canonical Virasoro section $\omega_{\mathcal{V}}$ of central charge $c = N$.

In general, when there is a bundle of VOAs with a connection, say $\underline{V}$, over a closed Riemannian Spin manifold M^{2N} one can construct a VOA and its module as follows. The VOA $\mathfrak{P}(\underline{V})$ is given by the graded vector space of parallel sections:

$$\mathfrak{P}(\underline{V}) = \{ s \in \Gamma(\underline{V}) \mid \nabla s = 0 \}.$$

Let $\Delta^{\pm}$ be the bundle of half Spin representations on M^{2N}. Let $\not{\partial}_{\underline{V}}$ be the graded Dirac operator twisted by $\underline{V}$. We have an exact sequence:

$$0 \to \operatorname{Ker} \not{\partial}_{\underline{V}} \to \Gamma(\Delta^{+} \otimes \underline{V}) \xrightarrow{\not{\partial}_{\underline{V}}} \Gamma(\Delta^{-} \otimes \underline{V}) \to \operatorname{Coker} \not{\partial}_{\underline{V}} \to 0$$

Here, $\operatorname{Ker} \not{\partial}_{\underline{V}}$ and $\operatorname{Coker} \not{\partial}_{\underline{V}}$ are graded vector spaces of modules over $\mathfrak{F}(\underline{V})$

The Witten genus and elliptic genus are usually considered as modular function (forms)-valued genera. Here we consider them as graded vector spaces:

$$\Phi_W(M^{2N}) = q^{-\frac{N}{12}} [\operatorname{Ker} \not{\partial}_{S_q} ; \operatorname{Coker} \not{\partial}_{S_q}], \text{ Witten genus}$$

$$\Phi_{ell}(M^{2N}) = q^{-\frac{N}{8}} [\operatorname{Ker} \not{\partial}_{S \otimes V} ; \operatorname{Coker} \not{\partial}_{S \otimes V}], \text{ elliptic genus}$$

If we take the graded dimension, one obtains the usual Witten genus and the elliptic genus for M^{2N}. It turns out that the Virasoro sections ω_V and ω_S are both parallel coming from the Riemannian metric. So they are in $\mathfrak{F}(S_q)$ or in $\mathfrak{F}(V_q)$ and the Witten genus $\Phi_W(M^{2N})$ is a Virasoro module with $c=2N$ and $\Phi_{ell}(M^{2N})$ is a Virasoro module with central charge $c=3N$.

The structure of the VOA $\mathcal{F}(S_q)$ or $\mathcal{F}(S_q \otimes V_q)$ depends on the Riemannian structure of a manifold. Since $\Lambda^*_{q^{\frac{1}{2}}} T^*M \subset V_q$ consists of only highest conformal weight vectors, it is natural to identify exterior algebra on $\Lambda^* T^*M$ with $\Lambda^*_{q^{\frac{1}{2}}} T^*M$. Thus, any parallel differential forms on M^{2N} give rise to elements in $\mathcal{F}(V_q)$, and then to vertex operators acting on geometric elliptic genera.

For example, on Kähler manifolds, Kähler forms are parallel and give rise to Heisenberg algebras. Squares of Kähler forms and Riemannian tensors give rise to commuting splitting of the Virasoro algebra. On hyperkähler manifolds, three prallel Kähler forms give rise to vertex operators forming an affine Lie algebra. On quaternion-Kähler manifolds, the canonical 4-form together with the Riemannian metric tensor splits the Virasoro algebras into two commuting sub-Virasoro algebras.

Bordism invariance of virtual characters was also mentioned.

Titel: *Applications of Modular Invariance in Topology*
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Elliptic genus is basically the index theory on loop space. The new features are <u>Modular invariance</u> and <u>Fusian rule</u> which allow us to prove many results in topology. I will discuss several results proved by using modular inv. and index theory, along the way I will raise Some questions concerning the applications of fusian rule in topology and of elliptic genus in repn theory of loop gps.

Let M^{2k} be a compact smooth spin mfd of dim $2k$ with an S^1-action. Let D be the Dirac operator on M.

<u>Thm</u> (Atiyah-Hirzebruch): $\operatorname{Ind} D = \hat{A}(M) = 0$.

The first application of modular inv. is the following loop space analogue of this thm.

<u>Thm 1</u>: If $P(M)_{S^1} = n \cdot \pi^* u^2$ for some integer n,

then $\operatorname{Ind} D \otimes \bigotimes_{n=1}^{\infty} S_{q^n} \widetilde{TM} = W_M(\tau) = 0$.

Here $p_1(M)_{S^1}$ is the first equiv. Pontryagin class and $\pi : ES^1 \times_{S^1} M \longrightarrow BS^1$, is the projection. u is the generator of $H^*(BS^1, \mathbb{Z})$. Also recall

$$\widetilde{TM} = TM - 2k, \quad S_t(E) = 1 + tE + tS^2E + \cdots$$

is the symmetric operation in $K(M)[[t]]$. We used $W_M(\tau)$ to denote the Witten genus which is the $\hat{A}$-genus ~~and~~ of LM.

Note $p_1(M)_{S^1} = n \cdot \pi^* u^2 \Rightarrow p_1(M) = 0 \Longrightarrow LM$ spin.

<u>Cor.</u> (1) If $S^1 \hookrightarrow K$, and K nonabelian acts on M, then $p_1(M) = 0 \Longrightarrow p_1(M)_{S^1} = n \cdot \pi^* u^2 \Longrightarrow$

$W_M(\tau) = 0$.

(2) If M is 2-connected, especially $MO\langle 8\rangle$, then $p_1(M) = 0 \Longrightarrow p_1(M)_{S^1} = n \cdot \pi^* u^2 \Longrightarrow$

$W_M(\tau) = 0$. (Used by S. Stolz to produce examples for his conj. with Höhn)

The first is due to A. Dessai. ~~The second~~

~~[illegible]~~

~~[illegible]~~.

<u>Def'n</u>: An elliptic oper. $D \otimes E$ on M is called rigid if the S^1-action commutes with it and its equivariant index, or Lefschetz number

$$L_E(g) = tr_g \operatorname{Ker} D\otimes E - tr_g \operatorname{CoKer} D\otimes E$$

is indep. of g.

<u>Examples</u>: (1). D; (2) the signature oper $d_s = D \otimes \Delta(M)$; (3) $D \otimes TM$.

(1) is the AH-thm.; (2) by definition of signature; (3) is proved only by elliptic genus method.

The <u>loop space analogues</u> are

$$(1)' \ \mathcal{D}_L = D \otimes \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{TM}) \ ; \quad (2)' \ d_s^L = \mathcal{D}_L \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n}(\widetilde{TM}).$$

(1)' is Thm 1; (2)' is the Witten Conj. proved by Taubes, Bott-Taubes. with partial work of Landweber-Stong, Ochanine. The complex analogue is due to Hirzebruch, Krichever.

More general rigid elliptic operators can be constructed through loop gp rep'ns. Kac-Weyl char. formula comes in naturally.

Consider $P \xrightarrow{G} M$, a principal bdle with G simply connected Lie gp. $\tilde{L}G$ central extension of LG.

E an irred. positive energy rep'n of level ℓ.

So E is a rep'n of $S^1 \ltimes \tilde{L}G$ with energy decom.

$$E = \bigoplus_{n \geq 0} E_n, \quad \text{each } E_n \text{ a finite dim'l rep'n}$$

of G.

Construct associated vector bdle to E_n and P, which we denote by $\tilde{E}_n$, and introduce a bdle coeff. power series

$$\psi(P, E) = q^{m_\lambda} \sum_{n \geq 0} \tilde{E}_n q^n \in K(M)[[q]].$$

Thm 2: If $p_1(M)_{S^1} = \ell \cdot p_1(P)_{S^1}$, then

$$\mathcal{D}_L \otimes \psi(P, E) \text{ is rigid}$$

Cor: 1). Take $\ell = 1$, $G = Spin(2k) \Longrightarrow$ Witten conjectures

That is, we have rigidity of

$$d_S^{\,L} = \mathcal{D}_L \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n}(\widetilde{TM}),\quad \mathcal{D}_L \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^{n-\frac{1}{2}}}(\widetilde{TM}),\ \mathcal{D}\otimes(\Delta^+ - \Delta^-)$$

2). Loop space analogue of the Mayer rigidity for quasi-symplectic mfd.

<u>Thm3</u>: If $p_1(M)_{S^1} = \ell\, p_1(P)_{S^1} + n\cdot\pi^* u^2$ with $n<0$, then $\mathrm{Ind}\,\mathcal{D}_L \otimes \Psi(P, E) = 0$.

As another corollary we have the Fusion ring structure of rigid elliptic operators. Let $K_\ell(G)$ be the level ℓ fusion ring of G. $(V_\lambda \hat{\otimes} V_\mu = \sum_\nu N_{\lambda\mu}^{\nu} V_\nu)$

$$K_\ell(G) = \{ V_\lambda,\ \lambda \text{ dominant root, and } (\ell, \theta) \le \ell \}$$

Where θ is the longest root. (here assume G simple and simply conn.).

<u>Cor.</u> (The fusion rule for rigid elliptic oper).

(1) If $p_1(M)_{S^1} = \ell\cdot p_1(P)_{S^1}$, then for any $V_\lambda \in K_\ell(G)$ $D\otimes \tilde{V}_\lambda$ is rigid

(2) If $p_1(M)_{S^1} = \ell\cdot p_1(P)_{S^1} + n\cdot \pi^* u^2$ with $n \neq 0$, then for any $V_\lambda \in K_\ell(G)$, $\mathrm{Ind}\, D\otimes \tilde{V}_\lambda = 0$.

This suggests a classification of all rigid elliptic operators.

Question: Can one prove the rigidity of $D\otimes \tilde{V}_\lambda$ for each $V_\lambda \in K_\ell(G)$, under condition $P_1(M)_{S^1} = \ell \cdot P_1(M)_{S^1}$ by using the fusian structure ?

Conjecture. Let K, a simply connected Lie gp act on M and P. E a pos. energy repn of $\tilde{L}G$ of level ℓ. If $P_1(M) = \ell \cdot P_1(P)$, then the equiv. index of $\mathcal{D}_L \otimes \Psi(P, E)$ is a virtual character (~~[illegible]~~) of $\tilde{L}K$.

We expect this K-equiv. index is the Kac-Weyl char. of a pos. energy repn. This conj., if holds, gives a way to construct repns of loop group by using elliptic genus, which is similar to the Borel-Weil-Bott model for compact Lie gps.

Example of $K_\ell(G)$: Take $\ell=1$. $G = \mathrm{Spin}(2k)$.

Then $K_\ell(G) = \{1, \Delta^+, \Delta^-, \text{vector rep}^n\}$ which induces rigid elliptic oper.

D. $D\otimes\Delta^\pm(M)$. $D\otimes TM$.

Exercise: 1). In the classical Hirzebruch genera. $\frac{x/2}{\sinh x/2}$, $x/\tanh x/2$, replace

$$\sinh\frac{x}{2} \longrightarrow \frac{\theta(x,\tau)}{\theta'(0,\tau)}$$

$$\cosh x/2 \longrightarrow \frac{\theta_1(x,\tau)}{\theta_1(0,\tau)}.$$

Then apply $SL_2(\mathbb{Z})$ transformation to get all elliptic genera of level 1.

Here $\theta(x,\tau)$, ~~…~~ $\theta_1(x,\tau)$ … are the classical Jacobi theta fns.

2. The $\hat{A}$-class can be viewed as

$$\frac{e(M)}{ch(\Delta^+(M)-\Delta^-(M))}$$ where $e(M)$ is the Euler

class of M, and $\Delta^\pm(M)$ are the half spinor bdles induced from the half spinor rep'ns $\Delta^\pm$ of Spin(2k). Replace $\Delta^\pm$ by the loop gp analogues $S^\pm$. Then $e(M)/ch(\psi(P,S^+)-\psi(P,S^-))$ gives the Witten class. Similarly L-class is $e(M)\cdot\dfrac{ch(\Delta^+(M)+\Delta^-(M))}{ch(\Delta^+(M)-\Delta^-(M))}$, same Replacement gives the elliptic genus class.

Titel: Kreck - Stolz type elliptic homology theories and applications

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Fix a bordism theory Ω_*, a manifold F and a group G of automorphisms of F. Define submodule $T_*(X) \subset \Omega_*(X)$ by $T_*(X) := \{[M,f] \in \Omega_*(X) \mid M$ is a fibre bundle with fibre F and structure group G s.t. the map f factors over the base space of the bundle$\}$. Furthermore let $\tilde{T}_*(X) \subset T_*(X)$ be the submodule defined by the additional condition, that the base space should be zero bordant in $\Omega_*(X)$.

Problem: In which cases are the following functors generalized homology theories (after suitable localization):

a) $el_*(X) = \Omega_*(X)/T_*(X)$

b) $ell_*(X) = \Omega_*(X)/\tilde{T}_*(X)$

c) $Ell^u_*(X) = ell_*(X)[u^{-1}]$ with $u \in ell_*$ of positive degree.

Such functors were first studied by Kreck and Stolz.

Theorem (Kreck-Stolz): Let $\Omega_* = \Omega^{Spin}_*$, $F = \mathbb{H}P^2$, $G = PSp(3)$;

a) $el_*(X) \xrightarrow{\cong} ko_*(X)$ at 2 where the isom. is induced by
$$\alpha : \Omega^{Spin}_*(X) \to ko_*(X)$$

b) $ell_*(X)$ at 2 is a generalized homology theory,
$$ell_*(pt) = \mathbb{Z}_{(2)}[s,k,h,b]/(2s, s^3, sk, k^2-4(h+64b))$$
where $s = [S^1]$, $k = [\text{Kummer surface}]$, $h = [\mathbb{H}P^2]$, $b = [B]$ with $\hat{A}(B) = 1$, $L(B) = 0$.

c) $Ell^u_*(X)$ is a generalized homology theory (over $\mathbb{Z}$) for all non-torsion $u \in Ell_*\mathbb{Z}$ of positive degree.

I studied analogous choices of Ω_*, F, G:

Theorem: $\Omega_* = \Omega^O_*$, F, G arbitrary. Then

$el_*(\cdot) = H_*(\cdot;\mathbb{Z}/2) \otimes el_*$

$ell_*(\cdot) = H_*(\cdot;\mathbb{Z}/2) \otimes ell_*$

$Ell^u_*(\cdot) = H_*(\cdot;\mathbb{Z}/2) \otimes Ell^u_*$, so

all are generalized homology theories.

As special cases we get for $F = \mathbb{R}P^2$, $G = (\mathbb{Z}/2\mathbb{Z})^2$:

$el_*(\cdot) = H_*(\cdot;\mathbb{Z}/2)$ given by fundamental class

$ell_*(\cdot) = H_*(\cdot;\mathbb{Z}/2[t])$ given by the square of the total Wu-clas

$Ell^u_*(\cdot) = H_*(\cdot;\mathbb{Z}/2[t,t^{-1}])$ for any u.

Problem: Which ideals in Ω^O_* can be obtained as I_* or $\tilde{I}_*$ for suitable pairs (F,G)?

Theorem: $\Omega_* = \Omega^{SO}_*$, $F = \mathbb{C}P^2$, $G = SU(3) \rtimes \mathbb{Z}/2$ where $\mathbb{Z}/2$ acts on $\mathbb{C}P^2$ by conjugation. Then

$el_*(\cdot) = H_*(\cdot;\mathbb{Z})$ at 2 given by fundamental class

$ell_*(\cdot) = H_*(\cdot;\mathbb{Z}[t])$ at 2 given by a 2 primary completion of the total L-class

$Ell_*(\cdot)$ is a generalized homology theory, more precisely

$Ell_*(\cdot) = H_*(\cdot;\mathbb{Z}[t,t^{-1}])$ at 2

$Ell_*(\cdot) = KO_*$ away from 2

Remark: The Theorem of the Ω^O_*-case with $F = \mathbb{R}P^2$ simplifies the proof in the Ω^{SO}_* case with $F = \mathbb{C}P^2$. Nevertheless there is no good geometric map between these two theories.

Problem: Will $\Omega_* = \Omega^{\langle 8\rangle}_*$ give a generalized homology theory $Ell^u_*(\cdot)$ for $F = CaP^2$ (Cayley projective plane) and $G =$ Isometries(F)?

The description of well-known homology theories by these new fibre bundle ideas leads to geometric application. The question of existence of a metric with positive scalar curvature on a manifold M is known to be a bordism question. Using the above Theorems and a similar construction in bordism with singularities it follows a

Theorem: Let M be a manifold with fundamental group $\pi = \pi_1(M)$. Let $f: M \to B\pi$ be the classifying map of the universal cover $\tilde{M}$ of M and let w_1, w_2 be the first two Stiefel-Whitney classes of M.

1) If $w_1 = 0, w_2 = 0$ then M admits a metric of positive scalar curvature if $\alpha(M,f) = \alpha(N,g) \in ko_*(B\pi)$ for some $g: N \to B\pi$ where N has such a metric. In particular if $\alpha(M,f) = 0$, M has such a metric.
2) If $w_1 = 0$, $w_2(\tilde{M}) \neq 0$ then M admits a metric of positive scalar curvature if $f_*([M]) = g_*([N]) \in H_*(B\pi; \mathbb{Z})$ for some $g: N \to B\pi$ where N has such a metric. In particular this holds if $f_*([M]) = 0$.
3) If $\pi = \pi' \times \mathbb{Z}/2$, $w_1(\pi') = 0$, $w_1 \neq 0$, $w_2(\tilde{M}) \neq 0$ the statement of 2) is true replacing $H_*(\cdot\,; \mathbb{Z})$ by $H_*(\cdot\,; \mathbb{Z}/2)$.

So e.g. for spin manifolds the vanishing of $\alpha(M,f) \in ko_*(B\pi)$ is sufficient for the existence of such a metric.
One can look at the stable question, e.g. does $M \times B^r$ carry such a metric for some r. If $p: ko_*(B\pi) \to KO_*(B\pi)$ is the natural map, the vanishing of $p\alpha(M,f)$ is sufficient for this. Now there is the assembly map $A: KO_*(B\pi) \to KO_*(C^*_{\mathbb{R}}(\pi))$ and the vanishing of $A \circ p\alpha(M,f)$ is necessary! So the obstruction in $KO_*(B\pi)$ is sharp if A is injective (the strong Novikov-conjecture, conjectured for torsion free groups). If A is not injective one can try to represent the kernel of A by manifolds with such metrics therefore showing that $KO_*(C^*_{\mathbb{R}}(\pi))$ is the right obstruction group. This was done by Stolz and Rosenberg for π finite. The hope is to show that for all groups, using construction methods for some finite or torsion "parts" of π and the strong Novikov conjecture for other parts.