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WEYL INVARIANT JACOBI FORMS: A NEW APPROACH

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ABSTRACT. For any irreducible root system not of type E_8, Wirthmüller proved in 1992 that the bigraded algebra of weak Jacobi forms invariant under the Weyl group is a polynomial algebra. In this paper we give a new automorphic proof of this result based on the general theory of Jacobi forms. We proved in a previous paper that the space of weak Jacobi forms for E_8 is not a polynomial algebra and every E_8 Jacobi form can be expressed uniquely as a polynomial in nine algebraically independent Jacobi forms introduced by Sakai with coefficients which are meromorphic $\mathrm{SL}_2(\mathbb{Z})$ modular forms. In this paper we further show that these coefficients are the quotients of some $\mathrm{SL}_2(\mathbb{Z})$ modular forms by a certain power of a fixed $\mathrm{SL}_2(\mathbb{Z})$ modular form which is completely determined by the nine generators.

1. Introduction

In 1992, Wirthmüller [17] studied the invariant theory of root systems with respect to the actions of the modular group and the Weyl group. More precisely, for any irreducible root system R of rank l, he defined the so-called Weyl invariant weak Jacobi forms for R. These forms are holomorphic functions in many variables which are modular in the complex upper half-plane $\mathbb{H}$ and are double periodic in the lattice variable $\mathfrak{z} \in R \otimes \mathbb{C}$, and are invariant with respect to the action of the Weyl group $W(R)$ on $\mathfrak{z}$. Weyl invariant Jacobi forms have many applications in Frobenius varieties, Gromov-Witten theory and string theory (see [4, 5, 11, 12, 13, 18]). All such Jacobi forms form a bigraded algebra over the ring $M_*(\mathrm{SL}_2(\mathbb{Z}))$ of usual elliptic modular forms, graded by the weight and index. Wirthmüller identified these forms with global sections of a selective reflexive sheaf on a certain abelian variety related to the root system. By means of the theory of algebraic geometry, he successfully proved that the bigraded algebra is freely generated by $l + 1$ basic Jacobi forms over $M_*(\mathrm{SL}_2(\mathbb{Z}))$ when R is not of type E_8. The weak Jacobi forms of even weight in the sense of Eichler-Zagier [8] are exactly the Weyl invariant Jacobi forms of type A_1. This is why we call these invariants introduced by Wirthmüller Weyl invariant Jacobi forms.

Unlike the classical case of A_1, Wirthmüller did not give a construction of generators and his proof is complicated and rather deep. Due to this defect and the importance of these Jacobi forms, some mathematicians were still studying this problem in the past 30 years. Now, all types of generators have been constructed explicitly in the literature. The generators of types A_n, B_n, G_2 and F_4 can be found in [4, 5]. The generators of type E_n were given in [14, 12, 13]. The generators of types D_n, C_n and F_4 were constructed in [2, 1]. Besides, Wirthmüller's theorem can also be reproved using the pull-back trick for all root systems except E_6 and E_7. We refer to [2] for a detailed proof of the tower of D_n. Since the root systems E_6 and E_7 are exceptional, the pull-back technique does not work.

In this paper, we introduce a new approach to recover Wirthmüller's theorem for all root systems. In [16], we proved a necessary and sufficient condition for the graded algebra of orthogonal modular forms being free. This condition is based on the differential operators introduced in [3] and the existence of a remarkable modular form which vanishes precisely on some mirrors of reflections with

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multiplicity one. We use a similar strategy to investigate Jacobi forms. We first define a differential operator on Jacobi forms, which can be regarded as the Jacobian of Jacobi forms (see Proposition 2.2). We then construct some weak Jacobi forms Φ_R which have special divisors associated to roots of root systems and are anti-invariant under the Weyl group. These forms are constructed as theta blocks, namely the product of Jacobi theta functions divided by a power of Dedekind η-function (see [10]). We observe that for each root system of not type E_8 the Jacobian of these generators is equal to this particular theta block. Inspired by this observation, we establish a sufficient condition for the bigraded algebra of weak Jacobi forms being free (see Proposition 2.3). This gives a simple proof of Wirthmüller's theorem.

This method also works for E_8. For the root system E_8, we proved in [15] that the space of weak Jacobi forms is not a polynomial algebra and each E_8 Jacobi form can be written uniquely as a polynomial in nine algebraically independent holomorphic Jacobi forms introduced by Sakai [12] with coefficients which are meromorphic $\mathrm{SL}_2(\mathbb{Z})$ modular forms. We still have the theta block Φ_{E_8} satisfying the properties mentioned above. But there are no algebraically independent weak Jacobi forms of type E_8 whose Jacobian has the same weight and index as Φ_{E_8}. Fortunately, there are indeed algebraically independent weak E_8 Jacobi forms whose Jacobian has the same index as Φ_{E_8}. Sakai's Jacobi forms satisfy such property. We conclude from this fact that the coefficients in the unique polynomial expression of a E_8 Jacobi form of index m can be represented as the quotients of some $\mathrm{SL}_2(\mathbb{Z})$ modular forms by g^{m-1}, where g is a $\mathrm{SL}_2(\mathbb{Z})$ modular form defined as the quotient of the Jacobian of Sakai's forms by Φ_{E_8} (see Theorem 5.1).

The layout of this paper is as follows. In §2 we define the Jacobian of Jacobi forms and establish the sufficient condition. We introduce Wirthmüller's theorem in §3 and present a new proof in §4. In §5 we prove the structure result of E_8 Jacobi forms.

2. The Jacobian of Jacobi forms

Let L be an even positive definite lattice of rank l with bilinear form $\langle\cdot,\cdot\rangle$ and dual lattice L^*. Let G be a subgroup of the integral orthogonal group $\mathrm{O}(L)$ of L. One defines Jacobi forms of lattice index L and invariant under the group G, which are a generalization of classical Jacobi forms introduced by Eichler-Zagier [8].

Definition 2.1. Let $k \in \mathbb{Z}$ and $t \in \mathbb{Z}_{\geq 0}$. If a holomorphic function $\varphi : \mathbb{H} \times (L \otimes \mathbb{C}) \to \mathbb{C}$ satisfies the following transformation laws

$$\varphi\left(\frac{a\tau+b}{c\tau+d}, \frac{\mathfrak{z}}{c\tau+d}\right) = (c\tau+d)^k \exp\left(t\pi i \frac{c\langle \mathfrak{z},\mathfrak{z}\rangle}{c\tau+d}\right)\varphi(\tau,\mathfrak{z}), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$
$$\varphi(\tau, \mathfrak{z}+x\tau+y) = \exp\left(-t\pi i[\langle x,x\rangle\tau + 2\langle x,\mathfrak{z}\rangle]\right)\varphi(\tau,\mathfrak{z}), \quad x,y \in L,$$

and if its Fourier expansion takes the form

$$\varphi(\tau,\mathfrak{z}) = \sum_{n=0}^{\infty}\sum_{\ell\in L^*} f(n,\ell) e^{2\pi i(n\tau + \langle \ell,\mathfrak{z}\rangle)},$$

and if it is invariant with respect to the action of G on the lattice variable

$$\varphi(\tau,\sigma(\mathfrak{z})) = \varphi(\tau,\mathfrak{z}), \quad \sigma \in G,$$

then φ is called a G-invariant weak Jacobi form of weight k and index t associated to L. If $f(n,\ell) = 0$ whenever $2nt - \langle \ell,\ell\rangle < 0$, then φ is called a G-invariant holomorphic Jacobi form. We denote by $J^{\mathrm{w},G}_{k,L,t}$ and $J^{G}_{k,L,t}$ the vector spaces of G-invariant weak and holomorphic Jacobi forms of weight k and index t, respectively.

We remark that Jacobi forms of index 0 do not depend on the lattice variable $\mathfrak{z}$ and their definition reduces to that of a classical modular form on $\mathrm{SL}_2(\mathbb{Z})$. Thus $J^{\mathrm{w},G}_{k,L,0} = M_k(\mathrm{SL}_2(\mathbb{Z}))$.

All G-invariant weak Jacobi forms associated to L form a bigraded algebra over $M_*(\mathrm{SL}_2(\mathbb{Z}))$

$$J^{\mathrm{w},G}_{*,L,*} := \bigoplus_{k\in\mathbb{Z},t\in\mathbb{Z}_{\geq 0}} J^{\mathrm{w},G}_{k,L,t}.$$

We will investigate the algebraic structure of this type of algebras.

We first define the Jacobian of Jacobi forms which is an analogue of the Jacobian of Siegel modular forms defined in [3]. This tool plays a vital role in this paper.

Proposition 2.2. *Let l be the rank of L. For $1 \leq j \leq l+1$, let ϕ_j be a G-invariant weak Jacobi form of weight k_j and index m_j associated to L. We fix a coordinate $\mathfrak{z} = (z_1, ..., z_l)$ of the space $L \otimes \mathbb{C}$. We define the Jacobian of the $l+1$ Jacobi forms as follows*

$$J := J(\phi_1, ..., \phi_{l+1}) = \begin{vmatrix} m_1\phi_1 & m_2\phi_2 & \cdots & m_{l+1}\phi_{l+1} \\ \frac{\partial \phi_1}{\partial z_1} & \frac{\partial \phi_2}{\partial z_1} & \cdots & \frac{\partial \phi_{l+1}}{\partial z_1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \phi_1}{\partial z_l} & \frac{\partial \phi_2}{\partial z_l} & \cdots & \frac{\partial \phi_{l+1}}{\partial z_l} \end{vmatrix}.$$

(1) *The function J is a weak Jacobi form of weight $l + \sum_{j=1}^{l+1} k_j$ and index $\sum_{j=1}^{l+1} m_j$ associated to L. Moreover, it is invariant under G up to the determinant character* det.

(2) *The function J is not identically zero if and only if the $l+1$ Jacobi forms ϕ_j are algebraically independent over $M_*(\mathrm{SL}_2(\mathbb{Z}))$.*

(3) *Let r be a primitive vector of L. If the reflection*

$$\sigma_r : \quad L \otimes \mathbb{C} \to L \otimes \mathbb{C}, \quad v \mapsto v - \frac{2\langle r, v\rangle}{\langle r, r\rangle} r$$

belongs to G, then J vanishes on the set

$$D_r(\tau) := \{(\tau, \mathfrak{z}) \in \mathbb{H} \times (L \otimes \mathbb{C}) : \langle r^\vee, \mathfrak{z}\rangle \in \mathbb{Z}\tau + \mathbb{Z}\},$$

where $r^\vee = 2r/(r, r)$ is the coroot of r.

Proof. (i) The proof is similar to that of [3, Proposition 2.1]. For $2 \leq j \leq l+1$, the function $\varphi_j = \phi_j^{m_1}/\phi_1^{m_j}$ is a meromorphic Jacobi form of weight $k_j m_1 - k_1 m_j$ and index 0 invariant under G associated to L. We define the usual Jacobian of the l forms φ_j with respect to $\mathfrak{z}$:

$$\Psi(\tau, \mathfrak{z}) = \frac{\partial(\varphi_2, ..., \varphi_{l+1})}{\partial(z_1, ..., z_l)}.$$

We find that Ψ is a meromorphic Jacobi form of weight $l + \sum_{j=2}^{l+1}(k_j m_1 - k_1 m_j)$ and index 0 associated to L by verifying the transformations under $\mathrm{SL}_2(\mathbb{Z})$ and the integral Heisenberg group of L. It is easy to check the following:

$$\frac{\partial}{\partial z_i}\left(\frac{\phi_j^{m_1}}{\phi_1^{m_j}}\right) = \frac{m_1 \phi_j^{m_1 - 1}}{\phi_1^{m_j}} \left[\frac{\partial \phi_j}{\partial z_i} - \frac{m_j \phi_j}{m_1 \phi_1} \times \frac{\partial \phi_1}{\partial z_i}\right], \quad 1 \leq i \leq l.$$

This implies that

$$J(\phi_1, ..., \phi_{l+1}) = \frac{\phi_1^{1+\sum_{j=2}^{l+1} m_j}}{m_1^{l-1}(\phi_2 \cdots \phi_{l+1})^{m_1 - 1}} \Psi.$$

It follows that J is a weak Jacobi form of weight $l + \sum_{j=1}^{l+1} k_j$ and index $\sum_{j=1}^{l+1} m_j$ associated to L. It is obvious that $\Psi(\tau, \sigma(\mathfrak{z})) = \det(\sigma)\Psi(\tau, \mathfrak{z})$ for any $\sigma \in G$. Therefore J is invariant under G up to the determinant character.

(ii) (⇐) If the $l+1$ Jacobi forms ϕ_j are algebraically independent over $M_*(\mathrm{SL}_2(\mathbb{Z}))$, then the l functions φ_j are local parameters of the l-dimensional variety $L\otimes\mathbb{C}/(L\cdot\tau+L)$. Thus the usual Jacobian Ψ does not vanish identically, which yields that J is not identically zero.

(⇒) Suppose that $J\neq 0$. If the $l+1$ modular forms ϕ_j are algebraically dependent over $M_*(\mathrm{SL}_2(\mathbb{Z}))$, then there exists a non-zero polynomial P over $M_*(\mathrm{SL}_2(\mathbb{Z}))$ in $l+1$ variables such that $P(\phi_1,...,\phi_{l+1})=0$. We write

$$P(X_1,...,X_{l+1})=\sum_{(i_1,...,i_{l+1})\in\mathbb{Z}_{\geq 0}^{l+1}} c(i_1,...,i_{l+1})X_1^{i_1}\cdots X_{l+1}^{i_{l+1}}.$$

We can assume that $\sum_{j=1}^{l+1}m_ji_j$ is a fixed constant c for any $(i_1,...,i_{l+1})\in\mathbb{Z}_{\geq 0}^{l+1}$ due to modularity with respect to the integral Heisenberg group of L. We note that c is the degree of P. By taking the differentials of $P(\phi_1,...,\phi_{l+1})$ with respect to z_1,...,z_n respectively, we obtain the following system of linear equations

$$\mathbf{J}\left(\frac{\partial P}{\partial\phi_1},\frac{\partial P}{\partial\phi_2},...,\frac{\partial P}{\partial\phi_{l+1}}\right)^t=\left(cP,\frac{\partial P}{\partial z_1},...,\frac{\partial P}{\partial z_l}\right)^t=0,$$

where $\mathbf{J}$ is the Jacobian matrix in the definition of J. This leads to a contradiction because we can assume that P has the minimal degree. Therefore these ϕ_j are algebraically independent over $M_*(\mathrm{SL}_2(\mathbb{Z}))$ if $J\neq 0$.

(iii) Let $\sigma_r\in G$ for a primitive $r\in L$. Then we have

$$J(\tau,\sigma_r(\mathfrak{z}))=-J(\tau,\mathfrak{z}). \tag{2.1}$$

The transformation with respect to the integral Heisenberg group gives

$$J(\tau,\mathfrak{z}+x\tau+y)=\exp(-\pi im(\langle x,x\rangle\tau+2\langle x,\mathfrak{z}\rangle))J(\tau,\mathfrak{z}),\quad x,y\in L, \tag{2.2}$$

where m is the index of J as a Jacobi form. Let a and b be two integers. For any vector $\mathfrak{z}\in L\otimes\mathbb{C}$ satisfying $\langle r^\vee,\mathfrak{z}\rangle=a\tau+b$, we can write $\mathfrak{z}=u\tau+v$ with $u,v\in L\otimes\mathbb{R}$. Then we have that $\langle r^\vee,u\rangle=a$ and $\langle r^\vee,v\rangle=b$. It is easy to check that $\sigma_r(u\tau+v)=u\tau+v-ar\tau-br$. By taking $\mathfrak{z}=u\tau+v$, $x=-ar$ and $y=-br$ in (2.1) and (2.2), we get

$$J(\tau,u\tau+v)=-J(\tau,u\tau+v),$$

which implies that J vanishes when $\mathfrak{z}=u\tau+v$. This finishes the proof of (3).

□

In [16, Theorem 5.1], we prove a sufficient condition for the graded algebra of orthogonal modular forms to be free. We here prove a similar criterion for Jacobi forms, which provides a simple method to determine the structure of a bigraded algebra of Jacobi forms.

Proposition 2.3. *Let L be an even positive definite lattice of rank l and G be a subgroup of $\mathrm{O}(L)$. Let ϕ_j be a G-invariant weak Jacobi form of weight k_j and index m_j associated to L for $1\leq j\leq l+1$. Assume that the Jacobian $J(\phi_1,...,\phi_{l+1})$ is not identically zero and we denote its weight and index by k_0 and m_0 respectively. We further assume that there is a weak Jacobi form $\hat{J}$ of weight $\hat{k}_0$ and index m_0 associated to L which vanishes precisely with multiplicity one on $D_r(\tau)$ for all primitive $r\in L$ satisfying $\sigma_r\in G$, and it has a non-zero Fourier coefficient of type $f(0,\ell)$.*

(1) *If $k_0=\hat{k}_0$, then the bigraded algebra $J_{*,L,*}^{\mathrm{w},G}$ is freely generated by the $l+1$ forms ϕ_j over $M_*(\mathrm{SL}_2(\mathbb{Z}))$.*

(2) *If $k_0>\hat{k}_0$, then for any $\varphi_m\in J_{k,L,m}^{\mathrm{w},G}$ there exists a unique polynomial P over $M_*(\mathrm{SL}_2(\mathbb{Z}))$ in $l+1$ variables such that*

$$g^{m-M+1}\varphi_m=P(\phi_1,...,\phi_{l+1}),$$

where

$$g = J(\phi_1, ..., \phi_j)/\hat{J} \in M_{k_0-\hat{k}_0}(\mathrm{SL}_2(\mathbb{Z})),$$

and M is the minimal index of G-invariant weak Jacobi forms not contained in the space $M_*(\mathrm{SL}_2(\mathbb{Z}))[\phi_j, 1 \leq j \leq l+1]$. We further define meromorphic Jacobi forms $\hat{\phi}_j = \phi_j/g^{m_j}$ for $1 \leq j \leq l+1$. Then we have

$$J^{\mathrm{w},G}_{*,L,*} \subsetneq M_*(\mathrm{SL}_2(\mathbb{Z}))[\hat{\phi}_j, 1 \leq j \leq l+1].$$

Proof. Let $\phi_{l+2} \in J^{\mathrm{w},G}_{k_{l+2},L,m_{l+2}}$. For $1 \leq t \leq l+2$ we define J_t as the Jacobian of the $l+1$ Jacobi forms ϕ_j except ϕ_t. It is clear that $J(\phi_1, ..., \phi_{l+1}) = J_{l+2}$. By Proposition 2.2, the quotient $g_t := J_t/\hat{J}$ is G-invariant and holomorphic on $\mathbb{H} \times (L \otimes \mathbb{C})$. Since the q^0-term of $\hat{J}$ is not zero, the Fourier expansion of g_t has no terms $c(n, \ell)$ with negative n. Therefore g_t is a G-invariant weak Jacobi form associated to L. It is easy to check that the following identity holds:

$$\sum_{t=1}^{l+2} (-1)^t m_t \phi_t J_t = 0.$$

By $J_t = \hat{J} g_t$, we have

$$\sum_{t=1}^{l+2} (-1)^t m_t \phi_t g_t = 0,$$

which yields

$$(-1)^{l+2} m_{l+2} \phi_{l+2} g = -\sum_{t=1}^{l+1} (-1)^t m_t \phi_t g_t \tag{2.3}$$

where $g := g_{l+2}$. Since J_{l+2} and $\hat{J}$ has the same index, the function g is a weak Jacobi form of index 0 associated to L. Thus g is independent of the variable $\mathfrak{z}$ and is a modular form of weight $k_0 - \hat{k}_0$ on $\mathrm{SL}_2(\mathbb{Z})$.

(i) When $k_0 = \hat{k}_0$, the modular form g has weight 0 and then it is a non-zero constant. We observe that the index of non-zero g_t is less than the index of ϕ_{l+2}. By induction on index, we prove the assertion (1) with the help of (2.3).

(ii) When $k_0 > \hat{k}_0$, we suppose that $J^{\mathrm{w},G}_{*,L,*}$ is not generated by these ϕ_j, otherwise there is nothing to prove. We assume that ϕ_{l+2} is a weak Jacobi form of index M not contained in $M_*(\mathrm{SL}_2(\mathbb{Z}))[\phi_j, 1 \leq j \leq l+1]$. Again, we note that the index of non-zero g_t is less than the index of ϕ_{l+2}. By (2.3) and the minimality of M, we assert that $g\phi_{l+1} \in M_*(\mathrm{SL}_2(\mathbb{Z}))[\phi_j, 1 \leq j \leq l+1]$. We then complete the proof of assertion (2) by induction on index.

□

3. Weyl invariant Jacobi forms

In this section we recall Wirthmüller's theorem. Let R be an irreducible root system of rank l. The classification of R is as follows (see [7])

$$A_l (l \geq 1), \quad B_l (l \geq 2), \quad C_l (l \geq 3), \quad D_l (l \geq 4), \quad E_6, \quad E_7, \quad E_8, \quad G_2, \quad F_4.$$

Let $W(R)$ be the Weyl group of R. The roots of R span an integral lattice with the standard bilinear form $(\cdot,\cdot)$ on $\mathbb{R}^l$. If this lattice is odd, we rescale its bilinear form by 2 such that it becomes an even lattice. We denote this even positive definite lattice by L_R and its normalized bilinear form by $\langle\cdot,\cdot\rangle$. Following Definition 2.1, a Weyl invariant weak Jacobi form of type R is just a $W(R)$-invariant weak Jacobi form associated to L_R.

We introduce some notations of root systems. The dual root system of R is defined as

$$R^\vee = \{r^\vee : r \in R\},$$

where $r^\vee = \frac{2}{(r,r)}r$ is the coroot of r. The weight lattice of R is defined as

$$\Lambda(R) = \{x \in R \otimes \mathbb{Q} : (x, r^\vee) \in \mathbb{Z}\}.$$

Let $\widetilde{\alpha}$ denote the highest root of $R^\vee$. The following significant theorem was proved by Wirthmüller in 1992.

Theorem 3.1 (Theorem 3.6 in [17])**.** *If R is an irreducible root system of rank l and not of type E_8, then $J^{\mathrm{w},W(R)}_{*,L_R,*}$ over $M_*(\mathrm{SL}_2(\mathbb{Z}))$ is freely generated by $l+1$ $W(R)$-invariant weak Jacobi forms of weight $-k_j$ and index m_j*

$$\phi_{-k_j,R,m_j}(\tau,\mathfrak{z}), \quad 1 \le j \le l+1.$$

Apart from $(k_1, m_1) = (0,1)$, the indices m_j are the coefficients of $\widetilde{\alpha}^\vee$ written as a linear combination of the simple roots of R. The integers k_j are the degrees of the generators of the ring of $W(R)$-invariant polynomials, namely the exponents of the Weyl group $W(R)$ increased by 1.

Table 1. Weights and indices of generators of Weyl invariant weak Jacobi forms ($B_l : l \ge 2$, $C_l : l \ge 3$, $D_l : l \ge 4$)

R	L_R	$W(R)$	(k_j, m_j)
A_l	A_l	$W(A_l)$	$(0,1), (s,1) : 2 \le s \le l+1$
B_l	lA_1	$\mathrm{O}(lA_1)$	$(2s,1) : 0 \le s \le l$
C_l	D_l	$W(C_l)$	$(0,1), (2,1), (4,1), (2s,2) : 3 \le s \le l$
D_l	D_l	$W(D_l)$	$(0,1), (2,1), (4,1), (n,1), (2s,2) : 3 \le s \le l-1$
E_6	E_6	$W(E_6)$	$(0,1), (2,1), (5,1), (6,2), (8,2), (9,2), (12,3)$
E_7	E_7	$W(E_7)$	$(0,1), (2,1), (6,2), (8,2), (10,2), (12,3), (14,3), (18,4)$
G_2	A_2	$\mathrm{O}(A_2)$	$(0,1), (2,1), (6,2)$
F_4	D_4	$\mathrm{O}(D_4)$	$(0,1), (2,1), (6,2), (8,2), (12,3)$

4. The proof of Wirthmüller's theorem

In this section we present a simple proof of Wirthmüller's theorem using Proposition 2.3 and some known constructions of generators.

Let R be an irreducible root system. We first construct the particular Jacobi form $\hat{J}$ appearing in Proposition 2.3 for R. Recall that the Jacobi theta function

$$\vartheta(\tau,z) = q^{\frac{1}{8}}(\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}) \prod_{n=1}^{\infty}(1-q^n\zeta)(1-q^n\zeta^{-1})(1-q^n), \quad q = e^{2\pi i\tau}, \zeta = e^{2\pi i z}$$

is a holomorphic Jacobi form of weight $1/2$ and index $1/2$ for A_1 with a multiplier system of order 8 (see e.g. [9, §1.5]). This function vanishes exactly on $\{(\tau,z) \in \mathbb{H}\times\mathbb{C} : z \in \mathbb{Z}\tau + \mathbb{Z}\}$ with multiplicity one. The Dedekind η-function is defined by

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n).$$

In terms of the two types of functions, we define a theta function related to R as follows

$$\Phi_R(\tau,\mathfrak{z}) = \prod_{\substack{r\in R^\vee \\ r>0}} \frac{\vartheta(\tau,(r,\mathfrak{z}))}{\eta^3(\tau)}, \quad \mathfrak{z}\in R\otimes\mathbb{C},$$

where the product takes over all positive roots of $R^\vee$. The Coxeter number $h^\vee$ of the dual root system $R^\vee$ is defined by the equality

$$h^\vee(\mathfrak{z},\mathfrak{z}) = \sum_{\substack{r\in R^\vee \\ r>0}} (r,\mathfrak{z})^2.$$

By the above identity, the lattice $\Lambda(R)(h^\vee)$, which is obtained by rescaling the weight lattice of R with $h^\vee$, is integral. The function Φ_R is a weak Jacobi form of weight $-|R|/2$ and index 1 for $\Lambda(R)(h^\vee)$, where $|R|$ is the number of roots of R. Let $\overline{R}$ stand for the lattice generated by roots of R. It is easy to check that $\overline{R}\subset\Lambda(R)$. We therefore conclude that

$$\Phi_R \in J^{\mathrm{w}}_{-|R|/2,L_R,h^\vee}, \quad \text{if } \overline{R} \text{ is even},$$
$$\Phi_R \in J^{\mathrm{w}}_{-|R|/2,L_R,h^\vee/2}, \quad \text{if } \overline{R} \text{ is odd}.$$

Remark 4.1. These functions Φ_R are the Kac-Weyl denominator functions of affine Lie algebras and also the infinite products occurring in Macdonald identities up to certain powers of Dedekind η-function. The automorphic properties of Φ_R are clear in the literature. We refer to [6, Theorem 6.5] for a direct proof given by Borcherds. In [9, Corollary 2.7 and formula (20)], Gritsenko gave another proof based on the automorphic properties of ϑ. We remark that Gritsenko, Skoruppa and Zagier discovered a new arithmetic proof of Macdonald identities in [10].

We formulate the data of root systems in Tables 2 and 3.

Table 2. even root systems

R	$\overline{R}$	L_R	$R^\vee$	$\Lambda(R)$	$h^\vee$	$\frac{1}{2}\lvert R\rvert$
A_l	A_l	A_l	A_l	A_l^*	$l+1$	$l(l+1)/2$
D_l	D_l	D_l	D_l	D_l^*	$2(l-1)$	$l(l-1)$
E_6	E_6	E_6	E_6	E_6^*	12	36
E_7	E_7	E_7	E_7	E_7^*	18	63
E_8	E_8	E_8	E_8	E_8	30	120
C_l	D_l	D_l	B_l	$\mathbb{Z}^l$	$2l-1$	l^2
G_2	A_2	A_2	$G_2(\frac{1}{3})$	A_2	4	6

Table 3. odd root systems

R	$\overline{R}$	L_R	$R^\vee$	$\Lambda(R)$	$h^\vee$	$\frac{1}{2}\lvert R\rvert$
B_l	$\mathbb{Z}^l$	lA_1	C_l	D_l^*	$2l+2$	l^2
F_4	D_4^*	D_4	$F_4(2)$	D_4^*	18	24

We have the following claims which can be verified directly.

(a) The function $\Phi_R(\tau,\mathfrak{z})$ vanishes precisely on $D_r(\tau)$ with multiplicity one for all roots r of R.
(b) The Jacobi form Φ_R has a non-zero Fourier coefficient of type $f(0,\ell)$.
(c) When $R\neq E_8$, the sum of indices of all generators of $W(R)$-invariant weak Jacobi forms in Theorem 3.1 equals the index of Φ_R.
(d) When $R\neq E_8$, the sum of weights of all generators of $W(R)$-invariant weak Jacobi forms in Theorem 3.1 equals $-\frac{1}{2}|R|-l$.

From the claims above, we see that for any irreducible root system R not of type E_8 the Jacobian of generators and Φ_R have the same weight and index. Thus, if we have some basic $W(R)$-invariant weak Jacobi forms with weights and indices as in Theorem 3.1 and if we can show that their Jacobian does not vanish identically or equivalently that they are algebraically independent over $M_*(\mathrm{SL}_2(\mathbb{Z}))$, then we deduce from Proposition 2.3 (1) that the algebra $J^{\mathrm{w},W(R)}_{*,L_R,*}$ is freely generated by these basic Jacobi forms over $M_*(\mathrm{SL}_2(\mathbb{Z}))$. This proves the Wirthmüller theorem.

We explain the existence of these expected algebraic independent $W(R)$-invariant weak Jacobi forms. They have been constructed in the literature as mentioned in the introduction. In the simplest case of A_1, there are two unique (up to scale) index one weak Jacobi forms of weights -2 and 0. A direct calculation shows that their Jacobian is not identically zero and thus equal to $\Phi_{A_1}=\vartheta(\tau,2z)/\eta^3(\tau)$ up to a constant.

We next discuss the two most complicated cases namely E_6 and E_7. For these two root systems, it is very hard to calculate the Fourier expansion of the Jacobian. Hence it is better to prove directly that the constructed Jacobi forms are algebraically independent over $M_*(\mathrm{SL}_2(\mathbb{Z}))$. The generators of types E_6 and E_7 were constructed by Sakai in [13]. In the case of E_7, Sakai first constructed eight holomorphic Jacobi forms

$$A_m^{E_7}\ (m=1,2,3),\quad B_m^{E_7}\ (m=2,3,4),\quad C_m^{E_7}\ (m=1,2)$$

of weights 4, 6, 6 and index m respectively. He then constructed eight weak Jacobi forms of weights and indices indicated in Theorem 3.1 as quotients of some polynomials over $M_*(\mathrm{SL}_2(\mathbb{Z}))$ in the above eight holomorphic Jacobi forms by certain powers of $\Delta=\eta^{24}$ and the Eisenstein series of weight 4 (see [13, pages 69–70]). Like the case of E_8 studied in [15, Lemma 3.3], for each n the q^n-term of the Fourier expansion of a $W(E_7)$-invariant Jacobi form can be expressed as a polynomial over $\mathbb{C}$ in seven Weyl orbits of fundamental weights of E_7 due to the Weyl invariance. These Weyl orbits appear in q^0-terms of Fourier expansions of Sakai's weak Jacobi forms. We can further construct eight weak Jacobi forms as polynomials over $M_*(\mathrm{SL}_2(\mathbb{Z}))$ in Sakai's weak Jacobi forms whose q^0-terms of Fourier expansions contain only the constant one and the seven Weyl orbits respectively. From the algebraic independence of the seven Weyl orbits over $\mathbb{C}$, we deduce the algebraic independence of the eight forms over $M_*(\mathrm{SL}_2(\mathbb{Z}))$, which forces that Sakai's weak Jacobi forms are also algebraically independent over $M_*(\mathrm{SL}_2(\mathbb{Z}))$. We then recover the Wirthmüller theorem for root system E_7. The case of E_6 is similar.

Remark 4.2. One can define the bigraded algebra $J^{\mathrm{w},W(R)}_{*,L_R,*}(\Gamma)$ of Weyl invariant weak Jacobi forms with respect to a congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$. In fact, Wirthmüller proved the structure result for any Γ in [17, Theorem 3.6]. More precisely, he showed that $J^{\mathrm{w},W(R)}_{*,L_R,*}(\Gamma)$ is freely generated by the $r+1$ basic forms ϕ_{-k_j,R,m_j} of Theorem 3.1 over the ring $M_*(\Gamma)$ of elliptic modular forms on Γ when $R\neq E_8$. We note that these ϕ_{-k_j,R,m_j} are independent of Γ. It is easy to see that our above proof holds for any $J^{\mathrm{w},W(R)}_{*,L_R,*}(\Gamma)$. Thus it also gives a proof of Wirthmüller's theorem for general Γ.

5. Weyl invariant E_8 Jacobi forms

In this section we study Weyl invariant E_8 Jacobi forms. Sakai [12, 13] constructed nine $W(E_8)$-invariant holomorphic Jacobi forms

$$A_m^{E_8}\ (m=1,2,3,4,5),\quad B_m^{E_8}\ (m=2,3,4,6)$$

of weights 4, 6 and index m respectively. We showed in the proof of [15, Theorem 4.1] that the nine Jacobi forms are algebraically independent over $M_*(\mathrm{SL}_2(\mathbb{Z}))$. The sum of the indices of the nine forms equals 30 which is also the index of Φ_{E_8}. Applying Proposition 2.3 to this case, we find $M=2$ and prove the following theorem which is stronger than [15, Theorem 4.1].

Theorem 5.1. *We define a modular form*

$$g := J(A_i^{E_8}, B_j^{E_8},\ i=1,2,3,4,5,\ j=2,3,4,6)/\Phi_{E_8} \in M_{172}(\mathrm{SL}_2(\mathbb{Z})).$$

For any $\varphi_m \in J^{\mathrm{w},W(E_8)}_{k,E_8,m}$, there exists a unique polynomial P over $M_*(\mathrm{SL}_2(\mathbb{Z}))$ in nine variables such that

$$g^{m-1}\varphi_m = P(A_i^{E_8}, B_j^{E_8},\ i=1,2,3,4,5,\ j=2,3,4,6).$$

Let us define $\hat{A}_i := A_i^{E_8}/g^i$ and $\hat{B}_j := B_j^{E_8}/g^j$. Then we have

$$J^{\mathrm{w},W(E_8)}_{*,E_8,*} \subsetneq M_*(\mathrm{SL}_2(\mathbb{Z}))[\hat{A}_i, \hat{B}_j,\ i=1,2,3,4,5,\ j=2,3,4,6].$$

Recall that $A_i^{E_8}$ and $B_j^{E_8}$ have Fourier expansions $1+O(q)$. By definition, $J(A_i^{E_8}, B_j^{E_8}) = O(q^8)$. Thus $g=O(q^8)$, which implies that $g \in \Delta^8 \cdot M_{76}(\mathrm{SL}_2(\mathbb{Z}))$. Therefore g is the product of $\Delta^8 E_4$ and a modular form of weight 72. It is hard to determine g explicitly because these forms have unwieldy Fourier expansions. We expect that g equals $\Delta^{14}E_4$ up to a constant.

It is possible to choose generators of lower weights instead of $A_i^{E_8}$ and $B_j^{E_8}$, in which case the corresponding modular form g has smaller weight. By the structure results of $W(E_8)$-invariant weak Jacobi forms of small index obtained in [15], the generators of the smallest possible weights should be chosen as weak Jacobi forms of weights k_j and indices m_j for $(k_j, m_j) = (4,1)$, $(-4,2)$, $(-2,2)$, $(-8,3)$, $(-6,3)$, $(-16,4)$, $(-14,4)$, $(-18,5)$, $(-26,6)$. If such generators exist, then the corresponding g has weight 38. Thus the smallest possible weight of g is 38. But we do not know if such forms of indices 5 and 6 exist. We only constructed a weak Jacobi form of weight -16 and index 5 and a weak Jacobi form of weight -24 and index 6 in [15]. Thus the modular form g that we can obtain at present has minimal weight 42. That g is the product of $E_4^3E_6$ and a modular form of weight 24 by the equality in the proof of [15, Theorem 5.8].

We do not know whether it can be inferred from Theorem 5.1 that $J^{\mathrm{w},W(E_8)}_{*,E_8,*}$ is finitely generated over $M_*(\mathrm{SL}_2(\mathbb{Z}))$. More generally, we would like to ask the following questions.

(1) Is $J^{\mathrm{w},G}_{*,L,*}$ always finitely generated over $M_*(\mathrm{SL}_2(\mathbb{Z}))$?
(2) Are there other lattices L and subgroups G of $\mathrm{O}(L)$ such that $J^{\mathrm{w},G}_{*,L,*}$ is freely generated over $M_*(\mathrm{SL}_2(\mathbb{Z}))$?

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