THE TENSOR PRODUCT IN THE THEORY OF FROBENIUS MANIFOLDS

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Abstract. We introduce the operation of forming the tensor product in the theory of analytic Frobenius manifolds. Building on the results for formal Frobenius manifolds which we extend to the additional structures of Euler fields and flat identities, we prove that the tensor product of pointed germs of Frobenius manifolds exists. Furthermore, we define the notion of a tensor product diagram of Frobenius manifolds with factorizable flat identity and prove the existence such a diagram and hence a tensor product Frobenius manifold. These diagrams and manifolds are unique up to equivalence. Finally, we derive the special initial conditions for a tensor product of semi–simple Frobenius manifolds in terms of the special initial conditions of the factors.

Introduction

This paper is devoted to the study of Frobenius manifolds and their tensor products.

The foundations of the theory of Frobenius manifolds were laid down by Dubrovin [D1]. Such manifolds play a central role in the study of quantum cohomology and mirror symmetry (cf. [Gi, KM, M3]). In the realm of mathematical physics they appear for instance as the canonical moduli spaces of Topological Field Theories (TFTs). Recently they have also emerged in the study of differential Gerstenhaber–Batalin–Vilkovisky algebras [BK]. Frobenius manifolds first arose in Saito's study of unfolding singularities (cf. [D2, M3]) where they were called flat structures [S]. Further examples of Frobenius manifolds built on extended affine Weyl groups were constructed in [DZh]. For an introduction to the subject the reader can consult [D2, H, M1].

In a way one can regard Frobenius manifolds as a non–linear structure on cohomology spaces. This non–linear structure is rigid and has weak functorial properties. However, it admits a remarkable tensor product operation.

In the formal setting this operation has been introduced in [KM]. In quantum cohomology it corresponds to the Künneth formula [B, K].

The main goal of this paper is to introduce and to study the tensor product of analytic Frobenius manifolds in the non–formal setting. This is done in two steps. First, we show that the formal tensor product of two convergent potentials is convergent, so that we can define the tensor product of two pointed germs of analytic Frobenius manifolds. The convergence proof is based upon the study of one–dimensional Frobenius manifolds, carried out in [KMZ]. Secondly, in the presence of flat identities, we show that inside the convergence domain the so defined tensor products corresponding to different base–points are canonically isomorphic. This observation is translated into the existence of a natural affine tensor product connection on the exterior product of the tangent bundles over the Cartesian product of two Frobenius manifolds. Using this connection we define the notion of tensor product diagrams for Frobenius manifolds with factorizable flat identities. To patch together the local pointed tensor products we need the technical assumption that the flat identities of the factors are factorizable which means that they can be split off as a

Cartesian factor $\mathbb{C}$. This is the case in all important examples. In this situation, we prove the existence of such diagrams and uniqueness up to equivalence. One of the pieces of data for these diagrams is a tensor product Frobenius manifold for two Frobenius manifolds, which contains a submanifold parameterizing all possible tensor products. The size of the manifold itself depends on the convergence domain of the tensor product potentials and cannot be controlled a priori. Therefore, we regard two tensor product manifolds as equivalent if they agree in an open neighborhood of this submanifold. We also extend the situation to a slighly more general setting and prove the analogous results.

In the examples stemming from the unfolding of singularities the tensor product corresponds to the direct sum of singularities [M3]. For TFTs it provides the canonical moduli space for the tensor product of two such theories. The theorem of the existence of a tensor product then implies that the moduli space obtained by tensoring all possible natural perturbations of two given TFTs is included as a subspace in the natural moduli space of the tensor theory.

Frobenius manifolds often carry the additional structures of an Euler field and a flat identity which are sometimes included in the definition [D2]. Our tensor product can also be extended into this category.

In the special case of semi–simplicity the structure of Frobenius manifolds becomes particularly transparent. Roughly speaking, the Frobenius structure is determined by the Schlesinger special initial conditions [D2, M1, MM] at a given tame base–point. In this setting we calculate the special initial conditions of the tensor product.

Since we need to review the formalism of formal Frobenius manifolds and their tensor product, this paper gives a complete analysis of the tensor product in the theory of Frobenius manifolds in all of its presently known facets.

The paper is organized as follows: We begin by recalling the necessary definitions and facts of the theory of formal Frobenius manifolds and Frobenius manifolds, including the tensor product in the formal setting in section 1. We also introduce the notion of pointed germs of analytic Frobenius manifolds and give a one–to–one correspondence with convergent formal Frobenius manifolds. In section 2 we define the tensor product in the category of formal Frobenius manifolds with flat identity and Euler field and prove that the tensor product of two convergent potentials is again convergent yielding a tensor product for pointed germs of Frobenius manifolds. Section 3 contains the definition of a global version of the tensor product in terms of an affine tensor product connection on the exterior product of the tangent bundles over the Cartesian product of two Frobenius manifolds. In the framework of tensor product diagrams for Frobenius manifolds with factorizable flat identities, we prove the existence of such a diagram and show uniqueness up to equivalence. By generalizing the setting to general tensor product diagrams and introducing new natural restrictions, we are again able to show existence and uniqueness up to equivalence. The last section is an application of the previous results to semi–simple Frobenius manifolds. In particular, we calculate the special initial conditions at a tame semi–simple point of the tensor product Frobenius manifold in terms of those of the pre–images.

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1. Frobenius manifolds

We begin by reviewing the necessary material from the theory of Frobenius manifolds:

1.1. **Formal Frobenius manifolds.** We will follow the definition from [M1]. Let k be a supercommutative $\mathbb{Q}$–algebra, $H = \oplus_{a\in A} k\partial_a$ a free ($\mathbb{Z}_2$–graded) k–module of finite rank, $g : H \otimes H \to k$ an even symmetric pairing which is non–degenerate in the sense that it induces an isomorphism $g' : H \to H^t$ where H^t is the dual module.

Denote by $K = k[[H^t]]$ the completed symmetric algebra of H^t. This means that if $\sum_a x^a \partial_a$ is a generic element of H, then K is the algebra of formal series $k[[x^a]]$. We will also regard elements of K as derivations on $H_K := K \otimes_k H$ with H acting via contractions. We will call the elements of H *flat*.

1.1.1. **Definition.** The structure of *a formal Frobenius manifold on* (H, g) is given by a potential $\Phi \in K$ defined up to quadratic terms which satisfies the associativity of WDVV–equations:

$$\forall a, b, c, d : \sum_{ef} \Phi_{abe} g^{ef} \Phi_{fcd} = (-1)^{\tilde{a}(\tilde{b}+\tilde{c})} \sum_{ef} \Phi_{bce} g^{ef} \Phi_{fad} \tag{1.1}$$

where $\Phi_{abc} = \partial_a\partial_b\partial_c\Phi$, g^{ij} is the inverse metric and $\tilde{a} := \widetilde{x^a} = \widetilde{\partial_a}$ is the $\mathbb{Z}_2$–degree.

From the equations (1.1) it follows that the multiplication law given by $\partial_a \circ \partial_b = \sum_c \Phi^c_{ab}\partial_c$ turns $H_K = K \otimes_k H$ into a supercommutative K–algebra.

There are two other equivalent descriptions of formal Frobenius manifolds using abstract correlation functions and $Comm_\infty$–algebras (cf. [M1]).

1.1.2. **Definition.** *An abstract tree level system of correlation functions (ACFs) on* (H, g) is a family of S_n–symmetric even polynomials

$$Y_n : H^{\otimes n} \to k,\ n \geq 3 \tag{1.2}$$

satisfying the Coherence axiom (1.3) below.

Set $\Delta = \sum \partial_a g^{ab} \partial_b$. Choose any pairwise distinct $1 \leq i, j, k, l \geq n$ and denote by $ijSkl$ any partition $S = \{S_1, S_2\}$ of $\{1, \ldots, n\}$ which separates i, j and k, l, i.e. $i, j \in S_1$ and $k, l \in S_2$. The axiom now reads:

Coherence: For any choice of i, j, k, l

$$\sum_{ijSkl} \sum_{a,b} Y_{|S_1|+1}(\bigotimes_{r\in S_1} \gamma_r \otimes \partial_a) g^{ab} Y_{|S_2|+1}(\partial_b \otimes \bigotimes_{r\in S_2} \gamma_r)$$
$$= \sum_{ikTjl} \sum_{a,b} Y_{|T_1|+1}(\bigotimes_{r\in T_1} \gamma_r \otimes \partial_a) g^{ab} Y_{|T_2|+1}(\partial_b \otimes \bigotimes_{r\in T_2} \gamma_r). \tag{1.3}$$

1.1.3. **Correspondence between formal series and families of polynomials.** Given a formal series $\Phi \in K$, we can expand it up to terms of order two as

$$\Phi = \sum_{n\geq 3}^{\infty} \frac{1}{n!} Y_n \tag{1.4}$$

where the $Y_n \in (H^t)^{\otimes n}$. We will consider the Y_n as even symmetric maps $H^{\otimes n} \to k$. One can check that the WDVV–equations (1.1) and the Coherence axiom (1.3) are equivalent under this identification, see e.g. [M1].

1.1.4. **Remark.** Using Y_n one can define multiplications $\circ_n$ by dualizing with g

$$g(\circ_n(\gamma_1,\dots,\gamma_n),\gamma_{n+1}) := Y_{n+1} : H^{\otimes(n+1)} \to k,\ Y_{n+1}(\gamma_1\otimes\cdots\otimes\gamma_n) \tag{1.5}$$

which define a so–called $Comm_\infty$–algebras.

1.1.5. **Theorem (III.1.5 of [M1]).** *The correspondence of 1.1.3 establishes a bijection between the following structures on (H, g).*

(i) *Formal Frobenius manifolds.*
(ii) *Cyclic $Comm_\infty$–algebras.*
(iii) *Abstract correlation functions.*

1.1.6. **Definition.** An even element e in H_K is called an *identity*, if it is an identity for the multiplication $\circ$. It is called flat, if $e \in H$. In this case, we will denote e by ∂_0 and include it as a basis element.

1.1.7. **Euler Operator.** An even element $E \in K$ is called *conformal*, if $\mathrm{Lie}_E(g) = Dg$ for some $D \in k$. Here, we take the Lie derivative of the tensor g bilinearly extended to K w.r.t. the derivation E. In other words:

$$\forall X, Y \in K : \quad \mathrm{Lie}_E(g) := Eg(X,Y) - g([E,X],Y) - g(X,[E,Y]) = D\,g(X,Y). \tag{1.6}$$

It follows that E is the sum of infinitesimal rotation, dilation and constant shift, hence, we can write E as:

$$E = \sum_{a,b\in A} d_{ab}x^a\partial_b + \sum_{a\in A} r^a\partial_a := E_1 + E_0, \tag{1.7}$$

for some $d_{ab} \in k$. Specializing $X = \partial_a, Y = \partial_b$ we can rewrite (1.6)

$$\forall a, b : \quad \sum_c d_{ac}g_{cb} + \sum_c d_{bc}g_{ac} = Dg_{ab}. \tag{1.8}$$

In particular, we see that $[E, H] \subset H$ and that the operator

$$\mathcal{V} : H \to H : \quad \mathcal{V}(X) := [X, E] - \frac{D}{2}X \tag{1.9}$$

is skew–symmetric.

A conformal operator E is called *Euler*, if it additionally satisfies $\mathrm{Lie}_E(\circ) = d_0\circ$ for some constant d_0.

1.1.8. **Quasi–homogeneity.** The last condition is equivalent to the quasi–homogeneity condition (Proposition 2.2.2. of [M1])

$$E\Phi = (d_0 + D)\Phi + \text{a quadratic polynomial in flat coordinates.} \tag{1.10}$$

1.2. **The tensor product of formal Frobenius manifolds.** The tensor product of formal Frobenius manifolds is naturally defined via the respective Cohomological Field Theories (Cf. [KM, KMK, K]) which in terms of correlation functions manifests itself in the appearance of operadic correlation functions and the diagonal class $\Delta_{\overline{M}_{0n}} \in A^{n-3}(\overline{M}_{0n} \times \overline{M}_{0n})$.

1.2.1. **Trees and the cohomology of the spaces $\overline{M}_{0S}$.** We will consider a tree τ as quadruple $(F_\tau, V_\tau, \partial_\tau, j_\tau)$ of a (finite) set of (of flags) F_τ, a (finite) set (of vertices) V_τ, the boundary map $\partial_\tau : F_\tau \to V_\tau$, and an involution $j_\tau F_\tau \to F_\tau, j_\tau^2 = j_\tau$.

We call a a tree S–labeled or an S–tree if there is a fixed isomorphism of the tails (one element orbits of j) and S. We will only consider trees at least three tails. A tree is called stable if the set of flags at each vertex is at least of cardinality 3: $\forall v \in V(\tau) |F_\tau(v)| \geq 3$.

If a tree τ is unstable we define the *stabilization* of τ to be the tree obtained from τ by contracting one edge at each unstable vertex. There are just three possible configurations at a given unstable vertex and it is easily seen that the result of the stabilization is indeed stable and that the stabilization does not depend on the chosen edge.

1.2.2. **Keel's presentation.** As was shown in [Ke], the cohomology ring of $\overline{M}_{0S}$ can be presented in terms of classes of boundary divisors as generators and quadratic relations as introduced by [Ke]. Thus we have a map

$$[\,] : \{\text{stable S–trees}\} \to H^*(M_{0S}) \tag{1.11}$$

The additive structure of this ring and the respective relations can then be naturally described in terms of stable trees (see [KM] and [KMK]).

1.2.3. **Operadic Correlation Functions.** By identifying the index set $\overline{n} = \{1, \ldots, n\}$ of ACFs or more generally any finite set S with a set of markings of a S–tree, one can extend the notion of ACFs to operadic correlation functions. These are maps from $H^{\otimes S}$ which also depend on a choice of a stable S–tree τ (cf. [KM]).

$$Y(\tau) : H^{\otimes T_\tau} \to k \tag{1.12}$$

In fact under certain natural restrictions there exists a unique extension to trees for any system of ACFs $\{Y_n\}$ (cf. Lemma 8.4.1 of [KM]).

1.2.4. **Remark.** Given a set of ACFs $\{Y_n\}$ the correlation function of the above cited Lemma for a stable n–tree τ is given by the formula

$$Y(\tau)(\partial_{a_1} \otimes \cdots \otimes \partial_{a_n}) = (\bigotimes_{v \in V_\tau} Y_{F_v})(\partial_{a_1} \otimes \cdots \otimes \partial_{a_n} \otimes \Delta^{\otimes |E_\tau|}). \tag{1.13}$$

If fact due to the Coherence axiom (1.3) the operadic correlation functions only depend on the class of the tree $[\tau] \in H^*(\overline{M}_{0n})$: $Y(\tau) = Y([\tau])$.

1.2.5. **Remark.** To shorten the formulas, by abuse of notation, we will also denote the following function from $H^{\otimes F_\tau}$ to k by $Y(\tau)$:

$$\bigotimes_{v \in V_\tau} Y_{F_v} =: Y(\tau). \tag{1.14}$$

Which function is meant will be clear from the index set of the arguments.

1.2.6. **The diagonal of $\overline{M}_{0n} \times \overline{M}_{0n}$.** Denote the class of the diagonal in $H^{n-3}(\overline{M}_{0n} \times \overline{M}_{0n})$ by $\Delta_{\overline{M}_{0n}}$ and write it in terms of a tree basis:

$$\Delta_{\overline{M}_{0n}} = \sum_{[\sigma],[\tau]\in\mathcal{B}_n} [\sigma] g^{[\sigma][\tau]} \otimes [\tau] \tag{1.15}$$

where $g_{[\sigma][\tau]} = \int_{\overline{M}_{0n}} [\sigma] \cup [\tau]$ and $\mathcal{B}_n$ is some basis of $H^*(\overline{M}_{0n})$. Notice that

$$g^{[\sigma][\tau]} = 0 \;\text{ unless }\; |E_\sigma| + |E_\tau| = n-3. \tag{1.16}$$

For explicit computations one can use the basis $\mathcal{B}_n$ presented in [K].

1.2.7. **Tensor product for ACFs.** The tensor product of two systems of ACFs ($H^{(1)}$, $\Delta^{(1)}$, $\{Y_n^{(1)}\}$) and ($H^{(2)}$, $\Delta^{(2)}$, $\{Y_n^{(2)}\}$) is the system of ACFs ($H^{(1)}\otimes H^{(2)}$, $\Delta^{(1)}\otimes\Delta^{(2)}$, $\{Y_n\}$) defined by

$$\begin{aligned} &Y_n((\gamma_1^{(1)} \otimes \gamma_1^{(2)}) \otimes \cdots \otimes (\gamma_n^{(1)} \otimes \gamma_n^{(2)})) := \\ &\qquad \epsilon(\gamma^{(1)}, \gamma^{(2)})(Y^{(1)} \otimes Y^{(2)})(\Delta_{\overline{M}_{0n}})((\gamma_1^{(1)} \otimes \gamma_1^{(2)}) \otimes \cdots \otimes (\gamma_n^{(1)} \otimes \gamma_n^{(2)})). \end{aligned} \tag{1.17}$$

where for each summand

$$\begin{aligned} &(Y^{(1)} \otimes Y^{(2)})([\tau] \otimes [\sigma])((\gamma_1^{(1)} \otimes \gamma_1^{(2)}) \otimes \cdots \otimes (\gamma_n^{(1)} \otimes \gamma_n^{(2)})) = \\ &\qquad \epsilon(\gamma^{(1)}, \gamma^{(2)}) Y^{(1)}(\tau)(\gamma_1^{(1)} \otimes \cdots \otimes \gamma_n^{(1)}) Y^{(2)}(\sigma)(\gamma_1^{(2)} \otimes \cdots \otimes \gamma_n^{(2)}). \end{aligned} \tag{1.18}$$

1.2.8. **Definition.** Given two formal Frobenius manifolds ($H^{(1)}$, $g^{(1)}$, $\Phi^{(1)}$) and ($H^{(2)}$, $g^{(2)}$, $\Phi^{(2)}$), let $\{Y_n^{(1)}\}$ and $\{Y_n^{(2)}\}$ be the corresponding ACFs. *The tensor product* (H, g, Φ) *of* $(H^{(1)}, g^{(1)}, \Phi^{(1)})$ *and* $(H^{(2)}, g^{(2)}, \Phi^{(2)})$ is defined to be $(H^{(1)} \otimes H^{(2)}, g^{(1)} \otimes g^{(2)}, \Phi)$ where the potential Φ is given by:

$$\Phi(\gamma) = \sum_{n\geq 3} \frac{1}{n!}(Y^{(1)} \otimes Y^{(2)})(\Delta_{\overline{M}_{0n}})(\gamma^{\otimes n}). \tag{1.19}$$

As in 1.1.3, to make sense of (1.19) one should expand $\gamma = \sum x^{a',a''}\partial_{a'a''}$ in terms of the tensor product basis ($\partial_{a'a''} := \partial_{a'}^{(1)} \otimes \partial_{a''}^{(2)}$) of the two basis $\{\partial_{a'}^{(1)}\}$ and $\{\partial_{a''}^{(2)}\}$ and the dual coordinates $x^{a'a''}$ for this basis.

Inserting the explicit basis $\mathcal{B}_n$ with its intersection form given in [K] allows to make (1.19) explicit (cf. [K]). W In the case of quantum cohomology this provides the explicit Künneth formula. Here the tensor product of the potentials Φ^V and Φ^W belonging to some smooth projective varieties V and W is the Gromov–Witten potential of $\Phi^{V\times W}$ (cf. [KM, KMK, K, B]).

1.3. **Frobenius manifolds.**

1.3.1. **Definition.** A *Frobenius manifold* $\mathbf{M}$ is a quadruple $(M, \mathcal{T}_M^f, g, \Phi)$ of a (super)-manifold M, an affine flat structure $\mathcal{T}_M^f$, a compatible metric g and a potential function whose tensor of third derivatives defines an associative commutative multiplication $\circ$ on each fiber of $\mathcal{T}_M$.

For the notion of supermanifolds and supergeometry in general we refer to the book [M2]. Below we will consider only manifolds in the analytic category.

1.3.2. **Definition.** A *pointed Frobenius manifold* is a pair $(\mathbf{M}, m_0)$ of a Frobenius manifold $\mathbf{M}$ and a point $m_0 \in M$ called the base–point.

When considering *flat coordinates* in a neighborhood of the base–point m_0 of a pointed Frobenius manifold, we require that the coordinates of m_0 are all zero. In other words, the base–point corresponds to a choice of a zero in flat coordinates.

1.3.3. **Euler field and Identity.** Just as in the formal case, a Frobenius manifold may carry two additional structures; an Euler field and an identity. They are defined analogously.

1.3.4. **Definition.** An even vector field E on a Frobenius manifold with a flat metric g is called *conformal* of conformal weight D, for some constant D, if it satisfies $\mathrm{Lie}_E(g) = Dg$. A conformal field E is called *Euler*, if it additionally satisfies $\mathrm{Lie}_E(\circ) = d_0 \circ$ for some constant d_0.

1.3.5. **From germs of pointed Frobenius manifolds to convergent formal Frobenius manifolds.** Regarding a germ of a pointed Frobenius manifold $(\mathbf{M}, m_0)$ over a field k of characteristic zero, choose a flat basis of vector fields (∂_a) and set $H = \oplus_a k\partial_a$ and keep the metric g. Choose corresponding unique local flat coordinates x^a s.t. $\forall a : x^a(m_0) = 0$ as we demanded in 1.3.2. A structure of a formal Frobenius manifold on (H, g) is then given by the expansion of the potential into a power series in local flat coordinates (x^a) at m_0. Up to quadratic terms we obtain:

$$\Phi_{m_0}(\mathbf{x}) = \sum_{n\geq 3} \frac{1}{n!} \sum_{a_1,\dots,a_n\in\{1,\dots,n\}} x^{a_n} \cdots x^{a_1} Y_n^{m_0}(\partial_{a_1} \otimes \cdots \otimes \partial_{a_n}) \tag{1.20}$$

where the functions Y_n are defined via

$$Y_n^{m_0}(\partial_{a_1} \otimes \cdots \otimes \partial_{a_n}) := \partial_{a_1} \cdots \partial_{a_n} \Phi|_{m_0} = (\partial_{a_1} \otimes \cdots \otimes \partial_{a_n})\Phi_{m_0}(\mathbf{x})|_0 \tag{1.21}$$

Φ_{m_0} obviously obeys the WDVV–equations.

Furthermore, in the presence of an Euler field or a flat identity writing E and $e = \partial_0$ in flat coordinates defines the same structures in the formal situation.

We stress again that we are dealing with pointed Frobenius manifolds. Due to this a zero in flat coordinates has been fixed and $\{Y_n^{m_0}\}$, E and e are uniquely defined.

On the other hand, the functions in (1.21) and E are dependent on the choice of the base–point. Choosing a different base–point $\widehat{m}_0$ with x–coordinates $x^a(\widehat{m}_0) = x_0^a$ in the domain of convergence of Φ_{m_0} yields the new standard flat coordinates $\widehat{x}^a = x^a - x_0^a$. The corresponding functions Y transform via:

$$\begin{aligned}\widehat{Y}_n^{\widehat{m}_0}(\partial_{a_1} \otimes \cdots \otimes \partial_{a_n}) &:= \partial_{a_1} \dots \partial_{a_n} \Phi|_{\widehat{m}_0} = \partial_{a_1} \dots \partial_{a_n} \Phi_{m_0}|_{x_0^a} \\ &= \sum_{N\geq 0} \frac{1}{N!} \sum_{(b_1,\dots,b_N):b_i\in A} x_0^{b_N} \cdots x_0^{b_1} Y_{n+N}^{m_0}(\partial_{b_1} \otimes \cdots \otimes \partial_{b_N} \otimes \partial_{a_1} \otimes \cdots \otimes \partial_{a_n}) \\ &= \sum_{N\geq 0} \frac{1}{N!} \sum_{(b_1,\dots,b_N):b_i\in A} \epsilon(b|a)\, x_0^{b_N} \cdots x_0^{b_1} Y_{n+N}^{m_0}(\partial_{a_1} \otimes \cdots \otimes \partial_{a_n} \otimes \partial_{b_1} \otimes \cdots \otimes \partial_{b_N})\end{aligned} \tag{1.22}$$

where $\epsilon(b|a)$ is a shorthand notation for $\epsilon(\partial_{b_1} \cdots \partial_{b_N} | \partial_{a_1} \dots \partial_{a_n})$ which we define as the superalgebra sign acquired by permuting $\partial_{b_1}, \dots, \partial_{b_N}$ past the $\partial_{a_1}, \dots, \partial_{a_n}$:

$$\partial_{b_1} \cdots \partial_{b_N} \partial_{a_1} \cdots \partial_{a_n} = \epsilon(b|a)\, \partial_{a_1} \cdots \partial_{a_n} \partial_{b_1} \cdots \partial_{b_N}. \tag{1.23}$$

1.3.6. **Notation.** We denote the convergent formal Frobenius structure obtained from a pointed Frobenius manifold $((M, T^f_M, g_M, \Phi^M), p)$ with a choice of a basis (∂_a) of T^f_M by

$$(\bigoplus k\ \partial_a, g_M, \Phi^M_p) \tag{1.24}$$

1.3.7. **From convergent formal Frobenius manifolds to germs of pointed Frobenius manifolds.** Starting with any formal Frobenius manifold (H, g) with a potential Φ, we can produce a germ of a manifold with a flat structure by identifying the x^a as coordinate functions around some point m_0, choosing H as the space of flat fields and considering g as the metric. To get a Frobenius manifold, however, we need that the formal potential Φ has some nonempty domain of convergence. If

$$\Phi(\gamma) = \sum_{n\geq 3} \frac{1}{n!} Y_n(\gamma^{\otimes n}) \tag{1.25}$$

with $\gamma = \sum x^a \Delta_a$ is convergent, we can pass to a germ of a pointed Frobenius manifold. If necessary, we can, in this situation, even move the base–point as indicated above.

2. The Tensor product for Euler fields, flat identities and germs of Frobenius manifolds

2.1. **The tensor product for Euler fields and flat identities.** In this section, we extend the operation of forming the tensor product to the additional structures of an Euler field and an identity. In order to achieve this, we first rewrite the quasi–homogeneity condition and the defining relation for an identity in terms of operadic correlation functions. To this end we introduce the morphisms π^*, π_* on trees.

2.1.1. **Forgetful morphisms and trees.** The flat and proper morphisms $\pi_s : \overline{M}_{0S} \to \overline{M}_{0,S\setminus\{s\}}$ which forget the point marked by s and stabilize if necessary induce the maps π_* and π^* on the Chow rings where we omitted the subscript s which we will always do, if there is no risk of confusion.

We will now define the maps π_*, π^* on trees corresponding under [ ] to the respective maps in the Chow rings of $\overline{M}_{0S}$.

Define π_* via

$$\pi_{s*}(\tau) = \begin{cases} \text{forget the tail number } s \text{ and stabilize, if the stabilization is necessary} \\ 0 \text{ otherwise} \end{cases} \tag{2.1}$$

For any S–tree τ and any $s \notin S$ set

$$\tau^s_v = \text{ the } (S \cup \{s\})\text{–tree obtained from } \tau \text{ by adding an additional tail marked by } s \text{ at the vertex } v. \tag{2.2}$$

For the notion of stabilization of a tree cf. 1.2.1.

Now we define:

$$\pi^{s*}(\tau) = \sum_{v\in V_\tau} \tau^s_v. \tag{2.3}$$

Taking the definition of [ ] from [KM] it is a straightforward calculation using e.g. [Ke] to check that indeed $[\pi^*(\tau)] = \pi^*([\tau])$ and $[\pi_*(\tau)] = \pi_*([\tau])$.

2.1.2. **Quasi–homogeneity condition in terms of correlation functions.**

2.1.3. **Lemma.** *In terms of the abstract correlation functions Y_n the quasi–homogeneity condition (1.10) is given by*

$$\sum_{a\in A}(\sum_{i=1}^{n} d_{a_ia}Y_n(\partial_{a_1}\otimes\cdots\otimes\widehat{\partial_{a_i}}\otimes\cdots\otimes\partial_{a_n}\otimes\partial_a)+r^aY_{n+1}(\partial_{a_1}\otimes\cdots\otimes\partial_{a_n}\otimes\partial_a)) = (d_0+D)Y_n(\partial_{a_1}\otimes\cdots\otimes\partial_{a_n}). \quad (2.4)$$

Proof. Applying the vector field E in the form (1.7) to (1.4) and making a coefficient check yields (2.4).

2.1.4. **Lemma.** *The correlation functions (1.14) obey the following relation. For a given n–tree τ:*

$$\sum_{a\in A}(\sum_{f\in F_\tau} d_{fa}Y(\tau)((\bigotimes_{f'\in F_\tau\setminus\{f\}}\partial_{f'})\otimes\partial_a)+r^aY(\pi^*(\tau))((\bigotimes_{f\in F_\tau}\partial_f)\otimes\partial_a)) = |V_\tau|(D+d_0)Y(\tau)(\bigotimes_{f\in F_\tau}\partial_f). \quad (2.5)$$

Proof. Recall that by definition $Y(\pi^*(\tau))=\sum_{v\in V_\tau}Y(\tau_v^{n+1})$. By applying (2.4) at every vertex v of τ, we obtain

$$\begin{aligned}
&\sum_{f\in F_\tau}\sum_{a} d_{fa}\,Y(\tau)((\bigotimes_{f'\in F_\tau\setminus\{f\}}\partial_f)\otimes\partial_a)\\
&=\sum_{v\in V_\tau}[\sum_{f\in F_\tau(v)} d_{fa}\bigotimes_{v'\in V_\tau}(Y_{|F_\tau(v')|})((\bigotimes_{f'\in F_\tau\setminus\{f\}}\partial_{f'})\otimes\partial_a)]\\
&=\sum_{v\in V_\tau}[(D+d_0)Y(\tau)(\bigotimes_{f\in F_\tau}\partial_f)-\sum_{a\in A}r^aY(\tau_v^{n+1})((\bigotimes_{f\in F_\tau}\partial_f)\otimes\partial_a)]\\
&=|V_\tau|(D+d_0)Y(\tau)(\bigotimes_{f\in F_\tau}\partial_f)-\sum_{a\in A}r^aY(\pi^*(\tau))((\bigotimes_{f\in F_\tau}\partial_f)\otimes\partial_a).
\end{aligned} \quad (2.6)$$

2.1.5. **Proposition.** *For the operadic correlation functions $\{Y(\tau)\}$ the quasi–homogeneity condition is equivalent to*

$$\sum_{i=1}^{n}\sum_{a\in A} d_{a_ia}Y(\tau)(\partial_{a_1}\otimes\cdots\otimes\widehat{\partial_{a_i}}\otimes\cdots\otimes\partial_{a_n}\otimes\partial_a)-|E_\tau|\,d_0Y(\tau)(\partial_{a_1}\otimes\cdots\otimes\partial_{a_n}) + \sum_{a\in A}r^aY(\pi^*(\tau))(\partial_{a_1}\otimes\cdots\otimes\partial_{a_n}\otimes\partial_a) = (d_0+D)Y(\tau)(\partial_{a_1}\otimes\cdots\otimes\partial_{a_n}). \quad (2.7)$$

Proof. Writing out the Casimir elements $\Delta = \sum \partial_p g^{pq} \partial_q$ and applying Lemma 2.1.4 yields:

$$\sum_{i=1}^{n} \sum_{a\in A} d_{a_i a} Y(\tau)(\partial_{a_1} \otimes \cdots \otimes \widehat{\partial_{a_i}} \otimes \cdots \otimes \partial_{a_n} \otimes \partial_a) + |E_\tau|\, D\, Y(\tau)(\partial_{a_1} \otimes \cdots \otimes \partial_{a_n})$$
$$= \sum_{i=1}^{n} \sum_{a\in A} d_{a_i a} Y(\tau)(\partial_{a_1} \otimes \cdots \otimes \widehat{\partial_{a_i}} \otimes \cdots \otimes \partial_{a_n} \otimes \partial_a)$$
$$+ |E_\tau|\, D \sum_{\substack{(p_1,\dots p_{|E_\tau|}) p_i \in A \\ (q_1,\dots q_{|E_\tau|}) q_i \in A}} (\bigotimes_{v\in V_\tau} Y_{F_v})(\partial_{a_1} \otimes \cdots \otimes \partial_{a_n} \otimes \bigotimes_{j=1}^{|E_\tau|} (\partial_{p_j} g^{p_j q_j} \otimes \partial_{q_i}))$$
$$\stackrel{(*)}{=} \sum_{\substack{(p_1,\dots p_{|E_\tau|}) p_i \in A \\ (q_1,\dots q_{|E_\tau|}) q_i \in A}} [\sum_{i=1}^{n} \sum_{a\in A} d_{a_i a} Y(\tau)(\partial_{a_1} \otimes \cdots \otimes \widehat{\partial_{a_i}} \otimes \cdots \otimes \partial_{a_n} \otimes \partial_a \otimes \bigotimes_{j=1}^{|E_\tau|} (\partial_{p_j} g^{p_j q_j} \otimes \partial_{q_i}))$$
$$+ \sum_{i=1}^{|E_\tau|} \sum_{a\in A} d_{p_i a} (\bigotimes_{v\in V_\tau} Y_{F_v})(\partial_{a_1} \otimes \cdots \otimes \partial_{a_n} \otimes \bigotimes_{j=1, j\neq i}^{|E_\tau|} (\partial_{p_j} g^{p_j q_j} \otimes \partial_{q_i}) \otimes \widehat{\partial_{p_i}} g^{p_i a} \partial_{q_i} \otimes \partial_a)$$
$$+ \sum_{i=1}^{|E_\tau|} \sum_{a\in A} d_{p_j a} (\bigotimes_{v\in V_\tau} Y_{F_v})(\partial_{a_1} \otimes \cdots \otimes \partial_{a_n} \otimes \bigotimes_{i=1, i\neq j}^{|E_\tau|} (\partial_{p_i} g^{p_i q_i} \otimes \partial_{q_i}) \otimes \partial_{p_i} g^{p_i a} \widehat{\partial_{q_i}} \otimes \partial_a)]$$
$$= (|E|_\tau + 1)(D + d_0) Y(\tau)(\partial_{a_1} \otimes \cdots \otimes \partial_{a_n})$$
$$- \sum_{a\in A} r^a Y(\pi^*(\tau))(\partial_{a_1} \otimes \cdots \otimes \partial_{a_n} \otimes \partial_a). \quad (2.8)$$

The equality $(*)$ holds due to (1.8). Rewriting (2.8), we obtain (2.7). Vice versa postulating (2.7), we see that it reduces to (2.4) for the one–vertex tree (ρ_n).

2.1.6. **The identity in terms of correlation functions.** As previously remarked, we will assume that the identity is a flat vector field $e = \partial_0$. As the semi–simplicity of E this restriction is satisfied in the case of quantum cohomology.

2.1.7. **Remark.** From Corollary 2.1.1 of [M1], we have that

$$Y_3(\partial_a, \partial_b, \partial_0) = g_{ab} \text{ and } Y_n(\partial_{a_1} \otimes \cdots \otimes \partial_{a_{n-1}} \otimes \partial_0) = 0 \quad \forall n > 3 \quad (2.9)$$

are equivalent to the fact that ∂_0 is a flat identity.
In terms of operadic ACFs one obtains:

2.1.8. **Proposition.** *For a flat identity $e = \partial_0$ and for any stable n–tree τ with $n > 3$*

$$Y(\tau)(\partial_{a_1} \otimes \cdots \otimes \partial_{a_{n-1}} \otimes \partial_0) = Y(\pi_*(\tau))(\partial_{a_1} \otimes \cdots \otimes \partial_{a_{n-1}}). \quad (2.10)$$

Proof. From (2.9) we know that $Y(\tau)(\partial_{a_1} \otimes \cdots \otimes \partial_{a_{n-1}} \otimes \partial_0) = 0$, if the valence of the vertex v_0 with the tail marked with n is greater than three or, in other words, if the vertex remains stable after forgetting the tail n. Assume now that the vertex has valence three. Noticing that for a flat identity $Y_3(\partial_a, \partial_b, \partial_0) = g_{ab}$ the result follows by direct calculation. There are two cases: either v_0 has two tails marked n and i for some i and is joined to one other vertex v' by the edge e or v_0 just has one tail and is joined to two other vertices by the edges e_1 and e_2. In the first case we get

$$\begin{aligned}
&Y(\tau)(\partial_{a_1}\otimes\cdots\otimes\partial_{a_{n-1}}\otimes\partial_0\otimes\Delta^{\otimes|E_\tau|})\\
&\quad=(\bigotimes_{v\in V_\tau\setminus\{v_0\}}(\bigotimes_{f\in F_\tau(v)}Y_{F_\tau(v)})\otimes Y_{F_\tau(v_0)})\\
&\quad\quad(\partial_{a_1}\otimes\cdots\otimes\widehat{\partial_{a_i}}\otimes\cdots\otimes\partial_{a_{n-1}}\otimes\Delta^{\otimes|E_\tau|-1}\otimes\Delta_e\otimes\partial_{a_i}\otimes\partial_0)\\
&\quad=\sum_{pq}(\bigotimes_{v\in V_\tau\setminus\{v_0\}}(\bigotimes_{f\in F_\tau(v)}Y_{F_\tau(v)}))\\
&\quad\quad(\partial_{a_1}\otimes\cdots\otimes\widehat{\partial_{a_i}}\otimes\cdots\otimes\partial_{a_{n-1}}\otimes\Delta^{\otimes|E_\tau|-1}\otimes\partial_p g^{pq}g_{qa_i})\\
&\quad=(\bigotimes_{v\in V_\tau\setminus\{v_0\}}(\bigotimes_{f\in F_\tau(v)}Y_{F_\tau(v)}))(\partial_{a_1}\otimes\cdots\otimes\partial_{a_{n-1}}\otimes\Delta^{\otimes|E_\tau|-1})\\
&\quad=Y(\pi_*(\tau))(\partial_{a_1}\otimes\cdots\otimes\partial_{a_{n-1}})
\end{aligned}$$

likewise in the second case

$$\begin{aligned}
&Y(\tau)(\partial_{a_1}\otimes\cdots\otimes\partial_{a_{n-1}}\otimes\partial_0\otimes\Delta^{\otimes|E_\tau|})\\
&\quad=(\bigotimes_{v\in V_\tau\setminus\{v_0\}}(\bigotimes_{f\in F_\tau(v)}Y_{F_\tau(v)})\otimes Y_{F_\tau(v_0)})\\
&\quad\quad(\partial_{a_1}\otimes\cdots\otimes\partial_{a_{n-1}}\otimes\Delta^{\otimes|E_\tau|-2}\otimes\Delta_{e_1}\otimes\Delta_{e_2}\otimes\partial_0)\\
&\quad=\sum_{pq,rs}(\bigotimes_{v\in V_\tau\setminus\{v_0\}}(\bigotimes_{f\in F_\tau(v)}Y_{F_\tau(v)}))\\
&\quad\quad(\partial_{a_1}\otimes\cdots\otimes\partial_{a_{n-1}}\otimes\Delta^{\otimes|E_\tau|-2}\otimes\partial_p g^{pq}g_{qr}g^{rs}\otimes\partial_s)\\
&\quad=(\bigotimes_{v\in V_\tau\setminus\{v_0\}}(\bigotimes_{f\in F_\tau(v)}Y_{F_\tau(v)}))(\partial_{a_1}\otimes\cdots\otimes\partial_{a_{n-1}}\otimes\Delta^{\otimes|E_\tau|-1})\\
&\quad=Y(\pi_*(\tau))(\partial_{a_1}\otimes\cdots\otimes\partial_{a_{n-1}}).
\end{aligned}$$

2.1.9. **Remark.** In the setting of operads and higher order multiplications ([G, GK]), the formulas (2.9) for a flat identity $e=\partial_0$ correspond to the statements that e is an identity for $\circ_2$ and acts as a zero for all higher multiplications $\circ_n, n\geq 3$. The contents of Proposition 2.1.8 is the extension of these properties to any concatenation of these multiplications.

After these preparations, we come to the main result of this section:

2.1.10. **Theorem.** *Given two formal Frobenius manifolds* $(H^{(1)},g^{(1)},\Phi^{(1)})$ *and* $(H^{(2)},g^{(2)},\Phi^{(2)})$ *with Euler fields*

$$E^{(1)}=\sum_{a'b'\in A^{(1)}}d^{(1)}_{a'b'}x^{(1)a'}\partial^{(1)}_{b'}+\sum_{a'\in A^{(1)}}r^{(1)a'}\partial^{(1)}_{a'}\qquad\textit{of weight } D^{(1)} \textit{ and} \tag{2.11}$$

$$E^{(2)}=\sum_{a''b''\in A^{(2)}}d^{(2)}_{a''b''}x^{(2)a''}\partial^{(2)}_{b''}+\sum_{a''\in A^{(2)}}r^{(2)a''}\partial^{(2)}_{a''}\qquad\textit{of weight } D^{(2)} \tag{2.12}$$

and with flat identities $e^{(1)},e^{(2)}$ *of the same weight* $d^{(1)}_0=d^{(2)}_0=d$, *then*

$$e=e^{(1)}\otimes e^{(2)}=\partial^{(1)}_0\otimes\partial^{(2)}_0=\partial_{00} \tag{2.13}$$

and

$$E = \sum_{(b',b'')\in A^{(1)}\times A^{(2)}} [\sum_{a'\in A^{(1)}} (d^{(1)}_{a'b'} x^{a'b''}) + \sum_{a''\in A^{(2)}} (d^{(2)}_{a''b''} x^{b'a''}) - dx^{b'b''}]\,\partial_{b'b''}$$
$$+ \sum_{a'\in A^{(1)}} r^{(1)a'}\partial_{a'0} + \sum_{a''\in A^{(2)}} r^{(2)a''}\partial_{0a''} \quad (2.14)$$

define a flat identity of weight d and an Euler field of weight $D^{(1)}+D^{(2)}-2d$ on the tensor product (H,g,Φ) of $(H^{(1)},g^{(1)},\Phi^{(1)})$ and $(H^{(2)},g^{(2)},\Phi^{(2)})$.

Before we can prove the above theorem, we need one more Lemma about the properties of the diagonal class $\Delta_{\overline{M}_{0n}}$.

2.1.11. **Lemma.**

$$(id,\pi_*)(\Delta_{\overline{M}_{0n}}) = (\pi^*,id)(\Delta_{\overline{M}_{0n-1}}) \quad (2.15)$$

and

$$(\pi_*,\pi_*)(\Delta_{\overline{M}_{0n}}) = 0. \quad (2.16)$$

Proof. Consider two any strata classes $D_\tau \in A^*(\overline{M}_{0n}), D_\sigma \in A^*(\overline{M}_{0n-1})$. Using the projection formula twice, we obtain

$$\int_{\overline{M}_{0n}} D_\tau \cup \pi^*(D_\sigma) = \int_{\overline{M}_{0n-1}} \pi_*(D_\tau)\cup D_\sigma$$
$$\Leftrightarrow \int_{\overline{M}_{0n}\times\overline{M}_{0n}} (D_\tau\boxtimes\pi^*(D_\sigma))\cup\Delta_{\overline{M}_{0n}} = \int_{\overline{M}_{0n-1}\times\overline{M}_{0n-1}} (\pi_*(D_\tau)\boxtimes D_\sigma)\cup\Delta_{\overline{M}_{0n-1}}$$
$$\Leftrightarrow \int_{\overline{M}_{0n}\times\overline{M}_{0n-1}} (D_\tau\boxtimes D_\sigma)\cup(id,\pi_*)\Delta_{\overline{M}_{0n}} = \int_{\overline{M}_{0n}\times\overline{M}_{0n-1}} (D_\tau\boxtimes D_\sigma)\cup(\pi^*,id)\Delta_{\overline{M}_{0n-1}}.$$

Since the intersection pairing is non–degenerate and the classes $D_\tau\boxtimes D_\sigma$ generate $A^*(\overline{M}_{0n-1}\times\overline{M}_{0n})$, the formula (2.15) follows. Using the same type of argument for

$$\int_{\overline{M}_{0n-1}\times\overline{M}_{0n-1}} (D_\tau\boxtimes D_\sigma)\cup(\pi_*,\pi_*)\Delta_{\overline{M}_{0n}}$$
$$= \int_{\overline{M}_{0n-1}\times\overline{M}_{0n-1}} (\pi^*,\pi^*)(D_\tau\boxtimes D_\sigma)\cup\Delta_{\overline{M}_{0n}}$$
$$= \int_{\overline{M}_{0n}} \pi^*(D_\tau)\cup\pi^*(D_\sigma) = \int_{\overline{M}_{0n}} \pi^*(D_\tau\cup D_\sigma) = 0$$

where the last zero is due to dimensional reasons, we obtain the second claim (2.16).

Proof of the Theorem.

As in 1.2.8, we choose the coordinates $x^{a'a''}$ corresponding to the basis $\partial_{a'}\otimes\partial_{a''}$. The metric for the tensor product is given by

$$g_{a'b',a''b''} := g(\partial_{a'}\otimes\partial_{a''},\partial_{b'}\otimes\partial_{b''}) = g^{(1)}(\partial_{a'},\partial_{b'})g^{(2)}(\partial_{a''},\partial_{b''}) = g^{(1)}_{a',b'}g^{(2)}_{a'',b''}. \quad (2.17)$$

Euler field.

First we check that E is conformal of weight $D^{(1)} + D^{(2)} - 2d$. On the basis of flat vector fields we calculate:

$$\begin{aligned} & g([\partial_{a'a''}, E], \partial_{b'b''}) + g(\partial_{a'a''}, [\partial_{b'b''}, E]) \\ & = \sum_{c'} d^{(1)}_{a'c'} g^{(1)}_{c'b'} g^{(2)}_{a''b''} + \sum_{c''} d^{(2)}_{a''c''} g^{(1)}_{c'b'} g^{(2)}_{c''b''} + \sum_{c'} d^{(1)}_{b'c'} g^{(1)}_{a'c'} g^{(2)}_{a''b''} + \sum_{c''} d^{(2)}_{b''c''} g^{(1)}_{a'b'} g^{(2)}_{a''c''} \\ & \quad - 2d g^{(1)}_{a'b'} g^{(2)}_{a''b''} \\ & = (D^{(1)} + D^{(2)} - 2d) g_{a'a'',b'b''}. \end{aligned} \tag{2.18}$$

We will prove the fact that E is indeed an Euler field by verifying the quasi–homogeneity condition (1.10).

Set $D = D^{(1)} + D^{(2)} - 2d$ and $\gamma = \sum x^{a'a''} \partial^{(1)}_{a'} \otimes \partial^{(2)}_{a''}$:

$$\begin{aligned} & E_1 \Phi(\gamma) = E_1 \sum_{n\geq 3} \frac{1}{n!} Y_n(\gamma^{\otimes n}) = E_1 \sum_{n \geq 3} \frac{1}{n!} (Y^{(1)} \otimes Y^{(2)})(\Delta_{\overline{M}_{0n}})(\gamma^{\otimes n}) \\ & = \sum_{n\geq 3} \frac{1}{n!} x^{a'_n a''_n} \cdots x^{a'_1 a''_1} \Big(\sum_{a' \in A^{(1)}} \sum_{i=1}^{n} d^{(1)}_{a'_i a'} (Y^{(1)} \otimes Y^{(2)})(\Delta_{\overline{M}_{0n}}) \\ & \qquad ((\partial^{(1)}_{a'_1} \otimes \partial^{(2)}_{a''_1}) \otimes \cdots \otimes \widehat{(\partial^{(1)}_{a'_i} \otimes \partial^{(2)}_{a''_i})} \otimes \cdots \otimes (\partial^{(1)}_{a'_n} \otimes \partial^{(2)}_{a''_n}) \otimes (\partial^{(1)}_{a'} \otimes \partial^{(2)}_{a''_i})) \\ & \quad + \sum_{a'' \in A^{(2)}} \sum_{i=1}^{n} d^{(2)}_{a''_i a''} (Y^{(1)} \otimes Y^{(2)})(\Delta_{\overline{M}_{0n}}) \\ & \qquad ((\partial^{(1)}_{a'_1} \otimes \partial^{(2)}_{a''_1}) \otimes \cdots \otimes \widehat{(\partial^{(1)}_{a'_i} \otimes \partial^{(2)}_{a''_i})} \otimes \cdots \otimes (\partial^{(1)}_{a'_n} \otimes \partial^{(2)}_{a''_n}) \otimes (\partial^{(1)}_{a'_i} \otimes \partial^{(2)}_{a''})) \\ & \quad - n\, d\, (Y^{(1)} \otimes Y^{(2)})(\Delta_{\overline{M}_{0n}})((\partial^{(1)}_{a'_1} \otimes \partial^{(2)}_{a''_1}) \otimes \cdots \otimes (\partial^{(1)}_{a'_n} \otimes \partial^{(2)}_{a''_n}))\Big) \\ & \overset{(*)}{=} \sum_{n\geq 3} \frac{1}{n!} x^{a'_n a''_n} \cdots x^{a'_1 a''_1} \Big((D^{(1)} + D^{(2)} - d)(Y^{(1)} \otimes Y^{(2)})(\Delta_{\overline{M}_{0n}}) \\ & \qquad ((\partial^{(1)}_{a'_1} \otimes \partial^{(2)}_{a''_1}) \otimes \cdots \otimes (\partial^{(1)}_{a'_n} \otimes \partial^{(2)}_{a''_n})) \\ & \quad - \sum_{a' \in A^{(1)}} r^{(1)a'} (Y^{(1)} \otimes Y^{(2)})((\pi^*, id)(\Delta_{\overline{M}_{0n}}))((\partial^{(1)}_{a'_1} \otimes \partial^{(2)}_{a''_1}) \otimes \cdots \otimes (\partial^{(1)}_{a'_n} \otimes \partial^{(2)}_{a''_n}) \otimes \partial^{(1)}_{a'}) \\ & \quad - \sum_{a'' \in A^{(2)}} r^{(2)a''} (Y^{(1)} \otimes Y^{(2)})((id, \pi^*)(\Delta_{\overline{M}_{0n}}))((\partial^{(1)}_{a'_1} \otimes \partial^{(2)}_{a''_1}) \otimes \cdots \otimes (\partial^{(1)}_{a'_n} \otimes \partial^{(2)}_{a''_n}) \otimes \partial^{(2)}_{a''})\Big) \\ & = \sum_{n\geq 3} \frac{1}{n!} \Big((D + d)(Y^{(1)} \otimes Y^{(2)})(\Delta_{\overline{M}_{0n}})(\gamma^{\otimes n}) \\ & \quad - \sum_{a' \in A^{(1)}} r^{(1)a'} (Y^{(1)} \otimes Y^{(2)})((\pi^*, id)(\Delta_{\overline{M}_{0n}}))(\gamma^{\otimes n} \otimes \partial^{(1)}_{a'}) \\ & \quad - \sum_{a'' \in A^{(2)}} r^{(2)a''} (Y^{(1)} \otimes Y^{(2)})((id, \pi^*)(\Delta_{\overline{M}_{0n}}))(\gamma^{\otimes n} \otimes \partial^{(2)}_{a''})\Big). \end{aligned} \tag{2.19}$$

To obtain $(*)$ write $\Delta_{\overline{M}_{0n}} = \sum [\tau] g^{[\tau][\sigma]} \otimes [\sigma]$ as in 1.2.6 and apply Proposition 2.1.5 to both tensor factors of each summand. Furthermore, notice that the $[\tau], [\sigma]$ are homogeneous and $g^{[\sigma][\tau]} = 0$ unless $|E_\sigma| + |E_\tau| = n - 3$ (1.16).

On the other hand, applying Proposition 2.1.8, we obtain up to quadratic terms

$$
\begin{aligned}
E_0\Phi(\gamma) = & \sum_{n\geq 3}\frac{1}{(n-1)!}\Big(\sum_{a'\in A^{(1)}} r^{(1)a'}(Y^{(1)}\otimes Y^{(2)})(\Delta_{\overline{M}_{0n}})(\gamma^{\otimes n-1}\otimes \partial^{(1)}_{a'}\otimes\partial^{(2)}_0) \\
& + \sum_{a''\in A^{(2)}} r^{(2)a''}(Y^{(1)}\otimes Y^{(2)})(\Delta_{\overline{M}_{0n}})(\gamma^{\otimes n-1}\otimes \partial^{(1)}_{0}\otimes\partial^{(2)}_{a''})\Big) \\
= & \sum_{n\geq 3}\frac{1}{n!}\Big(\sum_{a'\in A^{(1)}} r^{(1)a'}(Y^{(1)}\otimes Y^{(2)})(id,\pi_*)(\Delta_{\overline{M}_{n+1}})(\gamma^{\otimes n}\otimes \partial^{(1)}_{a'}) \\
& + \sum_{a''\in A^{(2)}} (Y^{(1)}\otimes Y^{(2)})(\pi_*,id)(\Delta_{0\overline{M}_{n+1}})(\gamma^{\otimes n}\otimes \partial^{(2)}_{a''})\Big)
\end{aligned} \tag{2.20}
$$

Applying the formula (2.15), we see that the sum of (2.19) and (2.20) is just the the quasi–homogeneity condition for E and therefore E is an Euler field.

Identity.

The proposed identity $\partial^{(1)}_0\otimes\partial^{(2)}_0$ is a flat field by definition. Furthermore,

$$
\begin{aligned}
& Y_3(\partial^{(1)}_{a'}\otimes\partial^{(2)}_{a''}\otimes\partial^{(1)}_{b'}\otimes\partial^{(2)}_{b''}\otimes\partial^{(1)}_{0}\otimes\partial^{(2)}_{0}) = \\
& \qquad Y^{(1)}_3(\partial^{(1)}_{a'}\otimes\partial^{(1)}_{b'}\otimes\partial^{(1)}_{0})Y^{(2)}_3(\partial^{(2)}_{a''}\otimes\partial^{(2)}_{b''}\otimes\partial^{(2)}_{0}) = g_{a'b',a''b''}
\end{aligned} \tag{2.21}
$$

and for $n\geq 3$ by Proposition 2.1.8 and (2.16)

$$
\begin{aligned}
& Y_n((\partial^{(1)}_{a'_1}\otimes\partial^{(2)}_{a''_1})\otimes\cdots\otimes(\partial^{(1)}_{a'_{n-1}}\otimes\partial^{(2)}_{a''_{n-1}})\otimes(\partial^{(1)}_{0}\otimes\partial^{(2)}_{0})) \\
& = (Y^{(1)}\otimes Y^{(2)})((\pi_*,\pi_*)\Delta_{\overline{M}_{0n}})((\partial^{(1)}_{a'_1}\otimes\partial^{(2)}_{a''_1})\otimes\cdots\otimes(\partial^{(1)}_{a'_{n-1}}\otimes\partial^{(2)}_{a''_{n-1}})) \\
& = 0
\end{aligned} \tag{2.22}
$$

which proves that $\partial^{(1)}_0\otimes\partial^{(2)}_0$ is indeed an identity by Remark 2.1.7. The weight of this identity can be read off the Euler field as $d+d-d=d$, proving the theorem.

2.1.12. **Remarks.** The condition that the weights of the identities are equal can be met by a rescaling of the Euler fields as long as not only one of the weights is 0. In the following, we will always assume this when considering the tensor product.

Since, given a metric and the multiplication on the fibers of a Frobenius manifold, the identity is uniquely determined —cf .[M1]—, the above identity is the only identity compatible with the choice of the tensor metric (2.17).

The theorem, however, contains no such uniqueness property for the Euler field, but there are several reasons for the choice of this particular type of Euler field. If the E_1–part is regarded as providing the operator $\mathcal{V}$ of (1.9), then our choice of E_1 for the tensor product is equivalent up to the shift by d which is necessary to accommodate the dependence of the tensor product on the diagonal in $H^*(\overline{M}_{0n}\times\overline{M}_{0n})$ to the natural definition:

$$
\mathcal{V} := \mathcal{V}^{(1)}\otimes \mathrm{id} + \mathrm{id}\otimes\mathcal{V}^{(2)}. \tag{2.23}
$$

As remarked in [M1], if the action of $\mathrm{ad}(E)$ is semi–simple on H, there is a natural grading of H induced by the action of $\mathrm{ad}(E)$, shifted by d_0. This grading basically fixes the E_1 component. In the setting of quantum cohomology, this grading is just (half) the usual grading for the cohomology groups. The additivity is just the fact that under the Künneth formula the total degree of a class is the sum of the degrees of the two components. The natural grading on the space of $H^{(1)}\otimes H^{(2)}$ is consequently given by the grading operator $\mathrm{ad}(E^{(1)} \otimes \mathrm{id} + \mathrm{id} \otimes E^{(2)})$ shifted by d, so that the tensor product of $\partial_a^{(1)}$ and $\partial_b^{(2)}$ of degrees $\delta_{a'}^{(1)}$ and $\delta_{a''}^{(2)}$ is of degree $\delta_{a'}^{(1)} + d + \delta_{a''}^{(2)} + d - d$. Recalling that d_a was the eigenvalue of $-\mathrm{ad}(E)$, we obtain $d_{a'a''} = d_{a'}^{(1)} + d_{a''}^{(2)} - d$.

In the physical realm of topological field theories [DVV], the above argument for the choice of E_1 just reflects the additivity of a $U(1)$ charge.

The choice for E_0 is motivated by quantum cohomology where the E_0–part corresponds to the canonical class. Thus, the definition of $E_0 = E_0^{(1)} \otimes \partial_0^{(2)} + \partial_0^{(1)} \otimes E_0^{(2)}$ corresponds to the formula $K_{X\times Y} = K_X \otimes 1 + 1 \otimes K_Y$. More generally, it corresponds to the map $H^*(V) \times H^*(W) \to H^*(V \times W) : (v, w) \to pr_1^*(v) + pr_2^*(w)$ which generally reflects the structure of the tensor product of Frobenius manifolds in the presence of flat identities, see Section 3.4 below.

Furthermore, in view of (2.19) and Lemma 2.1.11, E_0 seems to be the only possible choice, if one postulates (2.23).

2.2. **The tensor product for two germs of pointed Frobenius manifolds.** Due to the following main Theorem of this section, we can define the tensor product in the category of pointed germs of Frobenius manifolds.

2.2.1. **Theorem.** *The tensor product potential of two convergent potentials is convergent.*

2.2.2. **Definition.** Given two germs of pointed Frobenius manifolds $(\mathbf{M}^{(1)}, m_0^{(1)})$ and $(\mathbf{M}^{(2)}, m_0^{(2)})$, let $(H^{(1)}, g^{(1)}, \Phi^{(1)})$ and $(H^{(2)}, g^{(2)}, \Phi^{(2)})$ be the associated formal Frobenius manifolds. We define the tensor product $(\mathbf{M}, m_0)$ of $(\mathbf{M}^{(1)}, m_0^{(1)})$ and $(\mathbf{M}^{(2)}, m_0^{(2)})$ to be the associated germ of a pointed Frobenius manifold.

We will now prove the main Theorem of this section in several steps starting with invertible 1-dimensional CohFTs and proceeding to full generality. In the course of the proof, we will utilize a Theorem on complex series cited below for convenience (cf. e.g. Grauert, Einführung in die Funktionentheorie mehrerer Veränderlicher, Satz 1.1).

2.2.3. **Theorem.** *Let* $\mathbf{z}_1 \in \overset{\circ}{\mathbb{C}^n} := \{\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n \mid z_k \neq 0; 1 \leq k \leq n\}$. *If the power series* $\sum_{\boldsymbol{\nu}=0}^{\infty} a_{\boldsymbol{\nu}} \mathbf{z}^{\boldsymbol{\nu}}$ *converges at* $\mathbf{z_1}$*, then the series is uniformly convergent inside the polycylinder* $P_{\mathbf{z_1}} := \{\mathbf{z} \in \mathbb{C}^n \mid |z_k| < |z_k^{(1)}|\}$.

2.2.4. **Invertible rank one CohFTs.** For invertible rank one CohFTs (i.e. $C_3 \neq 0$) we have the following property:

2.2.5. **Theorem (3.4.2 of [KMZ]).** *Define the bijections*

$$\mathbf{CohFT_1}(k) \;\longleftrightarrow\; \frac{x^3}{6} + x^4\, k[[x]] \;\longleftrightarrow\; 1 + \eta\, k[[\eta]]\,, \tag{2.24}$$

where the first map assigns to a theory $\mathcal{A}$ *its potential* $\Phi_{\mathcal{A}}(x)$ *and the second map is defined by*

$$\Phi(x) \;\leftrightarrow\; U(\eta) = \int_0^\infty e^{-\Phi''(\eta x)/\eta}\, dx \tag{2.25}$$

or alternatively by assigning to $\Phi(x) = \frac{1}{6}x^3 + \dots$ *the power series* $U(\eta) = \sum_{n=0}^{\infty} B_n \eta^n$ *where* $x = \sum B_n \frac{y^{n+1}}{(n+1)!} = y + \cdots$ *is the inverse power series of* $y = \Phi^{(2)}(x) = x + \cdots$.

Then the tensor product of 1–dimensional CohFTs corresponds to multiplication in $1 + \eta k[[\eta]]$ *:* $U_{\mathcal{A}^{(1)}\otimes\mathcal{A}^{(2)}}(\eta) = U_{\mathcal{A}^{(1)}}(\eta)\, U_{\mathcal{A}^{(2)}}(\eta)$*. The coefficients of* $-\log U_{\mathcal{A}}(\eta)$ *are the canonical coordinates of* $\mathcal{A}$ *.*

2.2.6. **Explicit formulas.** The above theorem can be used to give explicit formulas for the coefficients of $U(\eta)$ in terms of the coefficients of $\Phi(x)$; see [KMZ] section 3.5. The explicit law for the tensor product of two normalized invertible CohFTs in terms of the coefficients of their potential functions can be derived by combining these formulas with the identity $U_{\mathcal{A}^{(1)}\otimes\mathcal{A}^{(2)}}(\eta) = U_{\mathcal{A}^{(1)}}(\eta)\, U_{\mathcal{A}^{(2)}}(\eta)$:

$$\begin{aligned}
C_4 &= C_4^{(1)} + C_4^{(2)}\,,\\
C_5 &= C_5^{(1)} + 5C_4^{(1)}C_4^{(2)} + C_5^{(2)}\,,\\
C_6 &= C_6^{(1)} + (8\,{C_4^{(1)}}^2 + C_5^{(1)})\,C_4^{(2)} + C_4^{(1)}\,(8\,{C_4^{(2)}}^2 + C_5^{(2)}) + C_6^{(2)}\,,\\
C_7 &= C_7^{(1)} + (35\,C_4^{(1)}\,C_5^{(1)} + 14\,C_6^{(1)})\,C_4^{(2)} + (61\,{C_4^{(1)}}^2\,{C_4^{(2)}}^2 + 33\,{C_4^{(1)}}^2 C_5^{(2)} +\\
&\qquad 33\,C_5^{(1)}\,{C_4^{(2)}}^2 + 19\,C_5^{(1)}\,C_5^{(2)}) + C_4^{(1)}\,(35\,C_4^{(2)}\,C_5^{(2)} + 14\,C_6^{(2)}) + C_7^{(2)}\,, \qquad \ldots
\end{aligned}$$

2.2.7. **Proposition.** *The tensor product potential of two convergent invertible rank one CohFTs is a convergent invertible rank one CohFT.*

Proof. First assume that $C_3^{(1)} = C_3^{(2)} = 1$.

As in Theorem 2.2.5 write the inverse power series of the second derivatives of the potentials

$$\Psi^{(1)} := \sum \frac{B_n^{(1)}}{(n+1)!}y^{n+1} \qquad \Psi^{(2)} := \sum \frac{B_n^{(2)}}{(n+1)!}y^{n+1}.$$

These series are convergent, if the respective potentials are and so is the product of their positive counterparts $|\Psi^{(1)}| := \sum \frac{|B_n^{(1)}|}{(n+1)!}y^{n+1}$ and $|\Psi^{(2)}| := \sum \frac{|B_n^{(2)}|}{(n+1)!}y^{n+1}$ as well as its derivative:

$$\widehat{\Psi} := \frac{\partial}{\partial y}(|\Psi^{(1)}|\,|\Psi^{(2)}|) = \sum_{n\geq 0}\sum_{i=0}^{n} \frac{n+1}{i+1}\frac{|B_i^{(1)}B_{n-i}^{(2)}|}{i!(n+1-i)!}y^{n+1}.$$

By [KMZ] (see Theorem 2.2.5 above) we have the following expansion for the inverse of the second derivative of the tensor potential Ψ:

$$\Psi = \sum_{n\geq 0}\sum_{i=0}^{n} \frac{B_i^{(1)}B_{n-i}^{(2)}}{(n+1)!}y^{n+1}. \tag{2.26}$$

This series is dominated by $\widehat{\Psi}$ at any point inside the domain of convergence of $\widehat{\Psi}$, since

$$\frac{n+2}{i+1}\binom{n+1}{i} > 1, \text{ for } 0 \leq i \leq n,$$

and thus

$$|\sum_{i=0}^{n} \frac{B_i^{(1)}B_{n-i}^{(2)}}{(n+1)!}| \leq \sum_{i=0}^{n} \frac{|B_i^{(1)}B_{n-i}^{(2)}|}{(n+1)!} < \sum_{i=0}^{n} \frac{n+1}{i+1}\frac{|B_i^{(1)}B_{n-i}^{(2)}|}{i!(n+1-i)!}$$

proving the convergence of the tensor product potential if $C_3^{(1)} = C_3^{(2)} = 1$.

In case that $C_3^{(1)}$ and/or $C_3^{(2)}$ are not equal to one, but not equal to zero, we can scale $\partial^{(1)}$ and $\partial^{(2)}$ in such a way that they are one. Notice the following scaling behavior for potentials of one–dimensional theories

$$\Phi_{\lambda\partial}(x) = \Phi_{\partial}(\lambda x) = \Phi_{\partial}^{\lambda}(x),$$

where the subscript ∂ refers to the chosen basis vector of the theory and $\Phi_{\partial}^{\lambda}(x)$ is the potential for the scaled theory $C_i^{\lambda} = \lambda^i C_i, i \geq 3$ which is convergent, if Φ is.

Furthermore, for the tensor product potential of two such theories:

$$\begin{aligned}\Phi^{(1)}_{\partial^{(1)}} \otimes \Phi^{(2)}_{\partial^{(2)}}(\lambda\mu x) &= \Phi_{\partial^{(1)}\otimes\partial^{(2)}}(\lambda\mu x) = \Phi_{\lambda\partial^{(1)}\otimes\mu\partial^{(2)}}(x)\\ &= \Phi^{(1)}_{\lambda\partial^{(1)}} \otimes \Phi^{(2)}_{\mu\partial^{(2)}}(x) = \Phi^{(1)\lambda}_{\partial^{(1)}} \otimes \Phi^{(2)\mu}_{\partial^{(2)}}(x).\end{aligned}$$

Choosing the appropriate scalings the right hand side is the tensor product of two convergent potentials with $C_3^{(1)}$=$C_3^{(2)}$=1 which converges by the first argument. It follows that $\Phi^{(1)}_{\partial^{(1)}} \otimes \Phi^{(2)}_{\partial^{(2)}}$ also converges, proving the proposition.

2.2.8. **The case of general rank one CohFTs.** In the previous section, we used the fact that we have a good handle on the tensor product potential in the case that the two rank one theories are invertible. We will show below that one can basically use the same formula even if the theories in question are not–necessarily invertible.

Denote by C_n the coefficients of the tensor product potential of two rank one CohFTs.

For a monomial $p =$ const. $\times\, C_{i_1} \dots C_{i_n}$ define the degree $deg(p) := i_1 + \cdots + i_n$ and the length $length(p) := n$

2.2.9. **Lemma.** *In the notation of the previous section*

$$C_n = Y^{(1)} \otimes Y^{(2)}(\Delta_{0n})(\partial^{\otimes n}) = P_n(C_3^{(1)}, \dots, C_n^{(1)}, C_3^{(2)}, \dots, C_n^{(2)}) \tag{2.27}$$

where $P_n(C_3^{(1)}, \dots, C_n^{(1)}, C_3^{(2)}, \dots, C_n^{(2)})$ is a universal polynomial. Furthermore,

$$P_n = \sum \text{monomials } p^{(i)}_{n,(k^{(1)},k^{(2)}),(l^{(1)},l^{(2)})} \text{ in the } C_i^{(1)}, C_j^{(2)} \tag{2.28}$$

with $bideg(p^{(i)}_{n,(k^{(1)},k^{(2)}),(l^{(1)},l^{(2)})}) = (k^{(1)}, k^{(2)})$, $bilength(p^{(i)}_{n,(k^{(1)},k^{(2)}),(l^{(1)},l^{(2)})}) = (l^{(1)}, l^{(2)})$ and the bi–degrees and bi–lengths satisfy:

$$k^{(1)} - 2l^{(1)} = k^{(2)} - 2l^{(2)} = n - 2 \quad and \quad l^{(1)} + l^{(2)} = n - 1. \tag{2.29}$$

Proof. Just express $Y^{(1)} \otimes Y^{(2)}(\Delta_{0n})(\partial^{\otimes n})$ as a sum over trees $(\tau^{(1)}, \tau^{(2)})$. The restriction then follows from the observation that $bideg(p)$ is $(|F_{\tau^{(1)}}|, |F_{\tau^{(2)}}|)$ and $bilength(p)$ is $(|V_{\tau^{(1)}}|, |V_{\tau^{(2)}}|)$. Finally, notice that $|E_{\tau^{(1)}}| + |E_{\tau^{(2)}}| = n - 3$.

2.2.10. **Lemma.** *For a fixed n there is a unique way of extending a monomial p^n_{red} in $C_i^{(1)}, C_j^{(2)}; 4 \leq i, j \leq n$ of given bi–degree $(k^{(1)}, k^{(2)})$ and bi–length $(l^{(1)}, l^{(2)})$ into a monomial p^n in the $C_i^{(1)}, C_j^{(2)}; 3 \leq i, j \leq n$, s.t. the bi–degree and bi–length of p^n satisfy the equations (2.29) and p^n coincides with p^n_{red} for $C_3^{(1)} = C_3^{(2)} = 1$.*

Proof. The monomials must be of the form $p^n = p^n_{red} C_3^{(1)i} C_3^{(2)j}$ and using the restrictions (2.29) we find:

$$i = n - 2 - k^{(1)} + 2l^{(1)} \text{ and } j = n - 2 - k^{(2)} + 2l^{(2)}. \tag{2.30}$$

2.2.11. **Corollary.** *The universal polynomials P_n are given by the unique polynomials extending the $C_n =: p^n_{red}$ given in [KMZ].*

2.2.12. **Proposition.** *The tensor product of two convergent rank one CohFTs is again convergent.*

Proof. Given the potentials $\Phi^{(1)}$ and $\Phi^{(2)}$, we assume after scaling that $C_3^{(1)}, C_3^{(2)} \in \{0,1\}$. Denote by $\widehat{\Phi}^{(1)/(2)}$ the potential with $\widehat{C_3}^{(1)/(2)} = 1$ and $\widehat{C}_i^{(1)/(2)} = C_i^{(1)/(2)}, i \geq 4$. These potentials are both convergent and invertible. Using the proposition for convergent and invertible potentials, we obtain that their tensor potential $\widehat{\Phi}$ is convergent. Now, due to the Corollary 2.2.11 there is a unique power series $\widetilde{\Phi} \in \mathbb{C}[[C_3^{(1)}, C_3^{(2)}, x]]$ extending $\widehat{\Phi}$, s.t. $\widetilde{\Phi}|_{C_3^{(1)}=C_3^{(2)}=1} = \widehat{\Phi}$ and the conditions (2.29) are satisfied.

First, assume that only one of the potentials is not invertible say $C_3^{(1)} = 1, C_3^{(2)} = 0$. Regarding the power series $\widetilde{\Phi}|_{C_3^{(1)}=1} =: \widetilde{\Phi}_1 \in \mathbb{C}[[C_3^{(2)}, x]]$, notice that $\widetilde{\Phi}_1$ converges at all points $(1, x_0)$ with x_0 inside the domain of convergence of $\widehat{\Phi}$ and is therefore —again by Theorem 2.2.3— convergent at points $(0, x_0)$. However, $\widetilde{\Phi_1}|_{C_3^{(2)}=0} = \Phi$ and thus Φ is also convergent.

In case that both $C_3^{(1)} = 0$ and $C_3^{(2)} = 0$, we see that $\widetilde{\Phi}|_{C_3^{(1)}=1, C_3^{(2)}=1} = \widehat{\Phi}$ and $\widetilde{\Phi}$ converges at all points $(1, 1, x_0)$ with x_0 inside the domain of convergence of $\widehat{\Phi}$. Therefore —again by Theorem 2.2.3— it is also convergent at points $(0, 0, x_0)$. Now, $\widetilde{\Phi}|_{C_3^{(1)}=C_3^{(2)}=0} = \Phi$ and we again obtain that Φ is convergent.

2.2.13. **Proposition.** *In case that both potentials are non-invertible, i.e. $C_3^{(1)} = C_3^{(2)} = 0$, we even have that $\widetilde{\Phi}|_{C_3^{(1)}=C_3^{(2)}=0} = \Phi \equiv 0$.*

Proof. By Lemma 2.2.10, all summands of Φ are of the form $p^n = p^n_{red} C_3^{(1)i} C_3^{(2)j}$ with p_{red} of given bi–length $(l^{(1)}, l^{(2)})$ and bi–degree $(k^{(1)}, k^{(2)})$ and i, j given by (2.30). Furthermore from the last equation in (2.29) we obtain $l^{(1)} + l^{(2)} = n - 1 - (i + j)$. Thus, using the inequalities

$$k^{(1)} \geq 4l^{(1)} \quad k^{(2)} \geq 4l^{(2)}$$

we find:

$$0 = 2n - 4 - (k^{(1)} + k^{(2)}) + 2(l^{(1)} + l^{(2)}) - (i + j) \leq -2 + (i + j).$$

So that $(i + j) \geq 2$ and all p^n vanish for $C_3^{(1)} = C_3^{(2)} = 0$.

2.2.14. **The higher dimensional case.** Given two formal Frobenius manifolds ($V^{(1)}$, $g^{(1)}$, $\Phi^{(1)}$) and ($V^{(2)}$, $g^{(2)}$, $\Phi^{(2)}$) with convergent potentials $\Phi^{(1)}$ and $\Phi^{(2)}$, denote the corresponding ACFs by $Y^{(1)}$ and $Y^{(2)}$.

Let (V, g, Φ) be the tensor product formal Frobenius manifold. Choosing a basis $(\partial_a^{(1)}), a \in A$ resp. $(\partial_b^{(2)}), b \in B$ for $V^{(1)}$ resp. $V^{(2)}$, the tensor potential in the tensor basis $\partial_{ab} := \partial_a^{(1)} \otimes \partial_b^{(2)}$ takes the form

$$\Phi = \sum_{n=3}^{\infty} \frac{1}{n!} \sum_{\substack{(a_1,\dots,a_n) \\ (b_1,\dots,b_n)}} x_{a_n b_n} \dots x_{a_1 b_1} Y^{(1)} \otimes Y^{(2)}(\Delta_{0n})(\partial_{a_1 b_1} \otimes \dots \otimes \partial_{a_n b_n}) \tag{2.31}$$

2.2.15. **Pure even case.** In the pure even case, we can consider the points y_{diag} whose coordinates are given by $x_{ab} \equiv y$, $y \in \mathbb{C}$ constant $\forall a \in A$ and $b \in B$. The potential at these points reads

$$\Phi = \sum_{n=3}^{\infty} \frac{1}{n!} y^n Y^{(1)} \otimes Y^{(2)}(\Delta_{0n})((\partial^{(1)} \otimes \partial^{(2)})^{\otimes n}) \tag{2.32}$$

with

$$\partial^{(1)} := \sum_{a \in A} \partial_a^{(1)} \quad \text{and} \quad \partial^{(2)} := \sum_{b \in B} \partial_b^{(2)}. \tag{2.33}$$

2.2.16. **Proposition.** *The potential of two convergent pure even CohFTs is convergent.*

Proof. First scale the chosen basis in such a way that $|g^{ab}| \leq 1$. Now, consider the series

$$\sum_{n=3}^{\infty} \frac{1}{n!} y^n |Y^{(1)} \otimes Y^{(2)}(\Delta_{0n})((\partial^{(1)} \otimes \partial^{(2)})^{\otimes n})|.$$

Due to the condition $|g^{ab}| \leq 1$, this series is dominated by the tensor potential for two rank one CohFTs given by the coordinates:

$$C_n^{(1)} := |Y_n^{(1)}(\partial^{(1)\otimes n})|, \qquad C_n^{(2)} := |Y_n^{(2)}(\partial^{(2)\otimes n})|. \tag{2.34}$$

Since the two given potentials $\Phi^{(1)}$ and $\Phi^{(2)}$ are convergent, so are their restrictions to the line $x_a^{(1)} \equiv y$ resp. $x_b^{(2)} \equiv y$ as power series in $\mathbb{C}[[y]]$. Thus they are also absolutely convergent and the positive counterparts of these restrictions are just the rank one CohFT given by the coordinates (2.34). The potential of the tensor product of two convergent CohFTs of rank one is convergent by Proposition 2.2.12. Therefore, we have convergence of the tensor potential Φ of $\Phi^{(1)}$ and $\Phi^{(2)}$ at some points y_{diag}, by the remarks above. Using the theorem on complex series 2.2.3, we find that the potential Φ is indeed convergent.

2.2.17. **The general case.** Consider the underlying $\mathbb{Z}_2$ graded space of the theory $V = V_0 \oplus V_1$ with a basis $(\partial_{a_i} \mid i \in I_0)$ of V_0 and $(\partial_{a_i} \mid i \in I_1)$ for some subsets I_0, I_1 of a set $I = I_0 \amalg I_1$ with an order $<$. Again chose the basis in such a way that $|g^{ab}| \leq 1$. Denote the dual coordinates of V_0 by $(x_i \mid i \in I_0)$ and those of V_1 by $(y_i \mid i \in I_1)$.

The potential can now be written as

$$\Phi = \sum_n \sum_{a_n > \cdots > a_1 | a_i \in I_1} y_{a_n} \cdots y_{a_1} \Phi_{a_1,\dots,a_n} \tag{2.35}$$

$$\Phi_{a_1,\dots,a_n} := \sum_m \frac{1}{m!} \sum_{(b_m,\cdots b_1) \in I_0^{\times n}} x_{b_m} \cdots x_{b_1} Y_{n+m}(\partial_{a_1} \otimes \cdots \otimes \partial_{a_n} \otimes \partial_{b_1} \otimes \cdots \otimes \partial_{b_m}) \tag{2.36}$$

The potential Φ is by definition convergent, if all the $\Phi_{a_1,\dots,a_n}$ are.

2.2.18. **Proposition.** *If Φ is the tensor product of two convergent series Φ and $\Phi^{(2)}$, then all the $\Phi_{a_1b_1,\dots,a_nb_n}$ with $(a_i,b_i)\in I_1$ are convergent with the usual notation for the variables and index sets for tensor products.*

Proof. Consider the following auxiliary pure even series in the even variables $x_{ij} \mid i \in I^{(1)}, j\in I^{(2)}$

$$\Psi = \sum_n \sum_{\substack{(a_1,\dots,a_n)\in I^{(1)\times n}\\ (b_1,\dots,b_n)\in I^{(2)\times n}}} \frac{1}{n!} x_{a_nb_n}\dots x_{a_1b_1}|Y_n(\partial_{a_1b_1}\otimes\dots\otimes\partial_{a_nb_n})| \tag{2.37}$$

To prove convergence we again look at points $x_i\equiv y$ with $y\neq 0$. In the tensor case, i.e. $Y_n = Y^{(1)}\otimes Y^{(2)}(\Delta_{0n})$ this series is dominated by the tensor product potential of the following two one-dimensional series:

$$\Psi^{(1)} = \sum_n \frac{1}{n!} y^n C_n^{(1)} \qquad C_n^{(1)} = |Y_n(\textstyle\sum_{i\in I^{(1)}} \partial_{a_i'}^{(1)\otimes n})| \tag{2.38}$$

$$\Psi^{(2)} = \sum_n \frac{1}{n!} y^n C_n^{(2)} \qquad C_n^{(2)} = |Y_n(\textstyle\sum_{i\in I^{(2)}} \partial_{a_i''}^{(2)\otimes n})| \tag{2.39}$$

where $I^{(1)}$ and $I^{(2)}$ are the index sets of the two original theories $\Phi^{(1)}$ and $\Phi^{(2)}$. Notice that the $\mathbb{Z}_2$–grading and the order of the ∂_{a_i} is irrelevant, since we take the absolute values of the correlators. Because the series $\Psi^{(1)}$ and $\Psi^{(2)}$ are convergent, by the assumption of convergence of $\Phi^{(1)}$ and $\Phi^{(2)}$, their tensor product potential converges by Proposition 2.2.12. Therefore, Ψ converges as well.

Finally notice that

$$\frac{\partial}{\partial x_{a_1b_1}}\cdots\frac{\partial}{\partial x_{a_nb_n}}\Psi \,\Big|_{x_{a_ib_i}=0:(a_i,b_i)\in I_1} =$$
$$\sum_m \frac{1}{m!}\sum_{((c_1,d_1),\cdots,(c_m,d_m))\in I_0^{\times n}} x_{c_md_m}\cdots x_{c_1d_1}|Y_{n+m}(\partial_{a_1b_1}\otimes\dots\otimes\partial_{a_nb_n}\otimes\partial_{c_1d_1}\otimes\dots\otimes\partial_{c_md_m})| \tag{2.40}$$

and these functions again converge for some points y_{diag} with $x_{ij}(y_{diag})\equiv y \mid (i,j)\in I_0$. This shows that $\Phi_{a_1b_1,\dots,a_nb_n}$ is absolutely convergent at some points on y_{diag} of the above type and thus the proposition follows by Theorem 2.2.3.

Collecting all results, we arrive at the general Theorem 2.2.1.

3. The tensor product for Frobenius manifolds

In this section we will only deal with analytic Frobenius manifolds.

3.1. **The exterior product of two Frobenius manifolds.** Given two Frobenius manifolds $\mathbf{M}^{(1)},\mathbf{M}^{(2)}$ we can consider the vector bundle $T_{M^{(1)}}\boxtimes T_{M^{(2)}}$ on $M^{(1)}\times M^{(2)}$ which we call the exterior product bundle. Since we have an affine flat structure on both M and N, we also have such a structure on $M^{(1)}\times M^{(2)}$ given by $T^f_{M^{(1)}}\boxplus T^f_{M^{(2)}}$ on $T_{M^{(1)}\times M^{(2)}}$. In addition we have an affine flat structure on $T_M\boxtimes T_N$ in the sense that

$$T^f_{M^{(1)}}\otimes T^f_{M^{(2)}}\otimes \mathcal{O}_{M^{(1)}\times M^{(2)}} \cong T_{M^{(1)}}\boxtimes T_{M^{(2)}} \tag{3.1}$$

3.2. The tensor product relative to a pair of base–points.

3.2.1. **Reminder.** As explained in section 1.3.5 given any pointed Frobenius manifold $(\mathbf{M}, p)$ there is an associated convergent formal Frobenius manifold (T_p, g, Φ_p) given by the expansion of the potential Φ of M at p in terms of the coordinates (x_i) of a chosen basis of flat vector fields (X_i).

3.2.2. **The tensor product of Frobenius manifolds relative to a pair of base–points.** Using the notion tensor product for formal Frobenius manifolds in the context of pointed Frobenius manifolds, we arrive at the following construction:

Given a pair of points (p, q) in the product $M^{(1)} \times M^{(2)}$ there is an associated convergent series $\Phi_{pq} := \Phi_p^{M^{(1)}} \otimes \Phi_q^{M^{(2)}}$ in the dual coordinates of $T_{p,M^{(1)}} \otimes T_{q,M^{(2)}}$, which defines a Frobenius manifold structure on the domain of convergence of Φ_{pq} (cf. section 1.3.7). We will denote the resulting pointed Frobenius manifold by:

$$(T_{p,M^{(1)}} \otimes T_{q,M^{(2)}}, 0). \tag{3.2}$$

Its germ corresponds to the convergent formal Frobenius structure

$$(\bigoplus \mathbb{C}\partial_a \otimes \partial_b, g_{M^{(1)}} \otimes g_{M^{(2)}}, \Phi_p^{M^{(1)}} \otimes \Phi_q^{M^{(2)}}). \tag{3.3}$$

3.3. Frobenius structures on the affine exterior bundle.

In the construction of the previous section we have defined over each point $(p, q) \in M^{(1)} \times M^{(2)}$ a pointed Frobenius manifold on a neighborhood V_{pq} the zero section s of the fiber of the exterior product bundle. Let V be the union of all the V_{pq}.

$$\begin{tikzcd} V \arrow[r, hook] & T_{M^{(1)}} \boxtimes T_{M^{(2)}} \arrow[d] \\ & M^{(1)} \times M^{(2)} \arrow[ul, "s"] \end{tikzcd}$$

3.3.1. **Definition.** A bundle will be called *a bundle of pointed Frobenius manifolds* if there exists a neighborhood V of the zero section s.t. the intersection of each fiber with this neighborhood is a Frobenius manifold. Let V_p is the intersection of V with the fiber at p then it is naturally a pointed Frobenius manifold with base–point zero.

3.3.2. **Examples.**

1) By the previous remarks, the exterior product bundle is actually a bundle of pointed Frobenius manifolds.
2) Every tangent bundle of a Frobenius manifold is naturally a bundle of pointed Frobenius manifolds. This can be shown in two equivalent ways. Either one uses the pointed Frobenius manifold (Φ, p) to define a potential near zero on the fiber over p or one uses the affine connection associated to the flat structure connection of the Frobenius manifold to define on each fiber $T_{p,N}$ the potential $\Psi_p(\xi) := \Phi(p')$ where p is the development of a path joining p' and $\xi \in T_{p,N}$ is the point of the development into the fiber at p of the point p'. The latter construction is defined locally since the local holonomy groups vanish, due to the flatness of the connection.

3.3.3. **Definition.** A flat affine connection on an exterior product bundle over the Cartesian product of two Frobenius manifolds $\mathbf{M}^{(1)}$ and $\mathbf{M}^{(2)}$ which is an extension of the linear connection defined by the canonical flat structure is called *tensor product connection* if it respects the flat and the Frobenius structures, i.e. it satisfies the following two conditions:

i) Let $\theta_\tau : T_{M^{(1)}} \boxplus T_{M^{(2)}} \to T_{M^{(1)}} \boxtimes T_{M^{(2)}}$ be the map corresponding to the tensorial $(GL_n \otimes GL_m, \mathbb{C}^{nm})$ 1–form defined by the affine connection (cf. e.g. [KN]) then θ_τ induces a linear map
$$\theta^f_\tau : T^f_{M^{(1)}} \boxplus T^f_{M^{(2)}} \to T^f_{M^{(1)}} \boxtimes T^f_{M^{(2)}} \tag{3.4}$$
ii) The parallel displacement w.r.t. the affine tensor projection preserves the germs of Frobenius manifolds. I.e. for all local horizontal lifts $\tilde{x}_t$ a of curve x_t in $M^{(1)} \times M^{(2)}$ with $x_0 = (p, q)$ and $\tilde{x}_0 = 0 \in T_{M^{(1)}} \boxtimes T_{M^{(2)}}|_{(p,q)}$ into $V \subset T_{M^{(1)}} \boxtimes T_{M^{(2)}}$, where 0 is the zero section of the corresponding vector bundle:
$$(T_{M^{(1)}} \boxtimes T_{M^{(2)}}|_{(p,q)}, \tilde{x}_0) \overset{\theta_\tau}{\cong} (T_{M^{(1)}} \boxtimes T_{M^{(2)}}|_{x_t}, \tilde{x}_t) \quad \forall \tilde{x}_t \in V. \tag{3.5}$$

3.3.4. **Remark.** Since the linear connection is flat and torsion free, such a tensor product connection locally identifies the germs of Frobenius manifolds in different fibers of the bundle of pointed Frobenius manifolds via affine parallel displacement along arbitrary path connecting the base–points.

3.3.5. **Proposition.** *If the Frobenius manifolds $\mathbf{M}^{(1)}$ and $\mathbf{M}^{(2)}$ both carry flat identities then the affine connection defined by the 1–forms:*
$$\theta^U_\tau : T_{M^{(1)}} \boxplus T_{M^{(2)}}|_U \to T_{M^{(1)}} \boxtimes T_{M^{(2)}}|_U$$
$$\theta_\tau(\partial^{(1)}_a) = \delta_{a0}\partial_{a0}, \quad \theta_\tau(\partial^{(2)}_b) = \delta_{0b}\partial_{0b}$$
is a tensor product connection. Here again $(\partial^{(1)}_a, \partial^{(2)}_b)$ is the restriction of a chosen basis of $T^f_{M^{(1)}} \boxplus T^f_{M^{(2)}}$ and $(\partial_{ab} = \partial^{(1)}_a, \otimes\partial^{(2)}_b)$ the tensor basis of $T^f_{M^{(1)}} \boxtimes T^f_{M^{(2)}}$

Proof. It is clear that the locally defined forms glue together and that the condition i) is met. The proof of the condition ii) is given below by calculating the respective ACFs. We give the proof including odd coordinates.

3.3.6. **Lemma.** *Let $\{Y^p_n\}$ be the ACFs corresponding to Φ_p and $\{Y^{p'}_n\}$ be the ACFs corresponding to Φ'_p where p' is some point which lies inside the domain of convergence of the potential Φ_p. Let $x^a(p') =: x^a_0$ be the x–coordinates of this point. The new operadic ACFs are then given by, see (1.22):*

For any stable n–tree τ:
$$Y^{p'}(\tau)(\partial_{a_1} \otimes \cdots \otimes \partial_{a_n}) = \sum_{N \geq 0} \frac{1}{N!} \sum_{(b_1,\dots,b_N): b_i \in A} \epsilon(b|a)\, x^{b_N}_0 \cdots x^{b_1}_0$$
$$Y^p(\pi^*_{\{n+1,\dots,n+N\}}(\tau))(\partial_{a_1} \otimes \cdots \otimes \partial_{a_n} \otimes \partial_{b_1} \otimes \cdots \otimes \partial_{b_N}). \tag{3.6}$$

Proof. Inserting (1.22) into the definition of $Y(\tau)$ (1.13), we see that the correlation functions having a pre–factor $x^{b_N}_0 \cdots x^{b_1}_0$ are those belonging to trees with $N - n$ tails added in an arbitrary fashion to τ. The sum over all of these trees is just $\pi^*_{\{n+1,\dots,n+N\}}(\tau)$, whence the Lemma follows.

In order to prove the Proposition, we also need a Lemma about the diagonal $\Delta_{\overline{M}_{0S}}$ which extends Lemma 2.1.11.

3.3.7. **Lemma.** *For any two disjoint subsets $S, T \subset \{1, \dots, n\}$*

$$(\pi_S^*, \pi_T^*)(\Delta_{\overline{M}_{0\{1,\dots,n\}\setminus(S\cup T)}}) = (\pi_{T*}, \pi_{S*})(\Delta_{\overline{M}_{0n}}). \tag{3.7}$$

Proof. Writing (π_S^*, π_T^*) as $(\pi_S^*, id) \circ (id, \pi_T^*)$, we obtain, after repeated application of Lemma 2.1.11 in an appropriate version, that

$$(\pi_S^*, \pi_T^*)(\Delta_{\overline{M}_{0\{1,\dots,n\}\setminus(S\cup T)}}) = (\pi_S^* \circ \pi_{T*}, id)(\Delta_{\overline{M}_{0\{1,\dots,n\}\setminus S}}).$$

Since π_S and π_T commute if $T \cap S = \emptyset$, we can prove the equality (3.7) again by Lemma 2.1.11.

Proof of the Proposition 3.3.5.

For any point $(p, q) \in M^{(1)} \times M^{(2)}$ denote the domain of convergence of $\Phi_p^{(1)}$ by U_p and the domain of convergence of $\Phi_q^{(2)}$ by U_q and set $U_{(p,q)} = U_p \times U_q$.

Choose a point $(p', q') \in U_{(p,q)}$. Let p' have the coordinates $x_{a'}^{(1)}(p') = x_0^{(1)a'}$ and q' the coordinates $x_{b''}^{(2)}(q') = x_0^{(2)b''}$.

Denote the ACFs corresponding to $\Phi_p^{M^{(1)}}$ by $\{Y_n^p\}$ and denote the ACFs corresponding to $\Phi_{p'}^{M^{(1)}}$ by $\{Y_n^{p'}\}$. Likewise denote the ACFs corresponding to $\Phi_q^{M^{(2)}}$ by $\{Y_n^q\}$ and denote the ACFs for the expansion $\Phi_{q'}^{M^{(2)}}$ by $\{Y_n^{q'}\}$.

The correlation functions of the formal tensor product potential Φ_{pq} are be given by $\{Y_n^{pq} = Y^p \otimes Y^q(\Delta_{\overline{M}_{0n}})\}$. The ACFs of the potential $\Phi_{p'q'}$ will be denoted $\{Y_n^{p'q'}\}$. Finally let $\theta_\tau((p', q'))$ be the image of the affine development w.r.t. θ_τ of $0 \in T_{(p',q')}$ into $T_{M^{(1)},p} \otimes T_{M^{(2)},q}$ and suppose that $\theta_\tau((p', q')) \in V_{pq}$.

Denote the ACFs corresponding to the expansion of the holomorphic function given by Φ_{pq} at $\theta_\tau((p', q'))$ by $\{Y_n^{\theta_\tau((p',q'))}\}$.

Since θ_τ is linear, the point $\theta_\tau((p', q')) \in T_{M^{(1)},p} \otimes T_{M^{(2)},q}$ has the coordinates

$$x_{a'b''}(\theta_\tau((p', q'))) = \delta_{a'0} x_0^{(1)a'} + \delta_{0b''} x_0^{(2)b''}.$$

The equation (3.5) in terms of these correlation functions reads:

$$Y_n^{\theta_\tau((p',q'))} = Y_n^{p'q'} \qquad \forall n \geq 3. \tag{3.8}$$

Now:

$$Y_n^{\theta_\tau((p',q'))}(\partial_{a_1'a_1''} \otimes \dots \otimes \partial_{a_n'a_n''})$$
$$= \sum_{N\geq 0} \frac{1}{N!} \sum_{l=0}^{N} \sum_{\substack{(b_1',\dots,b_l')|b'\in A' \\ (b_1'',\dots,b_{N-l}'')|b''\in A''}} \binom{N}{l} \epsilon(b'0|a'a'')\epsilon(0b''|a'a'')\, x_0^{(2)b_{N-l}''} \cdots x_0^{(2)b_1''} x_0^{(1)b_l'} \cdots x_0^{(1)b_1'}$$
$$(Y^p \otimes Y^q)(\Delta_{\overline{M}_{0,N+n}})(\partial_{a_1'a_1''} \otimes \dots \otimes \partial_{a_n'a_n''} \otimes \partial_{b_1'0} \otimes \dots \partial_{b_l'0} \otimes \partial_{0b_1''} \otimes \dots \otimes \partial_{0b_{N-l}''})$$

$$= \sum_{N\geq 0}\frac{1}{N!}\sum_{l=0}^{N}\sum_{\substack{(b'_1,\dots,b'_l)|b'\in A'\\(b''_1,\dots,b''_{N-l})|b''\in A''}}\binom{N}{l}\epsilon(b'|a')\epsilon(b''|a'')\epsilon(b'|a'')\epsilon(b''|a')$$

$$x_0^{(2)b''_{N-l}}\cdots x_0^{(2)b''_1}x_0^{(1)b'_l}\cdots x_0^{(1)b'_1}(Y^p\otimes Y^q)((\pi_{\{n+l+1,\dots,n+N\}*},\pi_{\{n+1,\dots,n+l\}*})(\Delta_{\overline{M}_{0,N+n}}))$$
$$(\partial_{a'_1a''_1}\otimes\cdots\otimes\partial_{a'_na''_n}\otimes\partial_{b'_1}\otimes\dots\partial_{b'_l}\otimes\partial_{b''_1}\otimes\cdots\otimes\partial_{b''_{N-l}}) \quad (3.10)$$

due to Proposition 2.1.8.

On the other hand, tensoring the ACFs $\{Y_n^{p'}\}$ and $\{Y_n^{q'}\}$ and then utilizing Lemma 3.3.6 yields:

$$Y_n^{p'q'}(\partial_{a'_1a''_1}\otimes\cdots\otimes\partial_{a'_na''_n}) := (Y^{p'}\otimes Y^{q'})(\Delta_{\overline{M}_{0n}})(\partial_{a'_1a''_1}\otimes\cdots\otimes\partial_{a'_na''_n})$$
$$=\sum_{N\geq 0}\sum_{l=0}^{N}\sum_{\substack{(b'_1,\dots,b'_l)|b'\in A'\\(b''_1,\dots,b''_{N-l})|b''\in A''}}\frac{1}{N!(N-l)!}\epsilon(b'|a')\epsilon(b''|a'')\epsilon(b'|a'')\epsilon(b''|a')$$
$$\times\, x_0^{(2)b''_{N-l}}\cdots x_0^{(2)b''_1}x_0^{(1)b'_l}\cdots x_0^{(1)b'_1}(Y^p\otimes Y^q)((\pi^*_{\{n+1,\dots,n+l\}},\pi^*_{\{n+l+1,\dots,n+N\}})(\Delta_{\overline{M}_{0,N+n}}))$$
$$(\partial_{a'_1a''_1}\otimes\cdots\otimes\partial_{a'_na''_n}\otimes\partial_{b'_1}\otimes\dots\partial_{b'_l}\otimes\partial_{b''_1}\otimes\cdots\otimes\partial_{b''_{N-l}}). \quad (3.11)$$

Applying Lemma 3.3.7 with $S=\{n=1,\dots,n+l\}, T=\{n+l+1,\dots,n+N\}$, we see that (3.10) and (3.11) and thus the multiplications, respectively the potentials modulo quadratic terms, coincide.

3.3.8. **Remark.** One should view the existence of a tensor product connection as an expression of the independence of the choice of base–points for the operation of forming the tensor product of pointed germs of Frobenius manifolds. More precisely, consider two germs of pointed Frobenius manifolds realized as two small neighborhoods of zero on $\mathbb{C}^{n_i}$ where n_i are the appropriate dimensions. In this case, the pointed tensor product is just the germ on the fiber of the exterior product bundle over zero which we can again realize as some small open neighborhood of zero. This neighborhood then contains all nearby germs (e.g. the germs of tensor product with base–points near zero) via the affine connection. Hence all continuations of the initial germ over zero will likewise be continuations of these germs.

3.4. **The tensor product for Frobenius manifolds.** In this section we will give a way to patch together all germs on the exterior product bundle. More precisely, we will construct a Frobenius manifold which contains a submanifold parameterizing all these germs. In order to give a general construction we will need the following technical assumption.

3.4.1. **Definition.** A flat identity on Frobenius manifold **M** is called *factorizable* if $M=\bar{M}\times\mathbb{C}$, where the factor $\mathbb{C}$ is coordinatized by the identity.

3.4.2. **Definition.** Consider a commutative diagram of the type

$$\begin{tikzcd}
T_{M^{(1)}} \boxtimes T_{M^{(2)}} \arrow[r, "\Theta"] \arrow[d] & \tau^*(T_N) \arrow[r, "\hat{\tau}"] \arrow[dl] & T_N \arrow[d] \\
M^{(1)} \times M^{(2)} \arrow[rr, "\tau"'] \arrow[d, "p"] & & N \\
\bar{M}^{(1)} \times \bar{M}^{(2)} \times \mathbb{C} \arrow[urr, hook, "i"] & &
\end{tikzcd} \tag{3.12}$$

where $(M^{(1)}, T^f_{M^{(1)}}, g_{M^{(1)}}, \Phi^{M^{(1)}})$, $(M^{(2)}, T^f_{M^{(2)}}, g_{M^{(2)}}, \Phi^{M^{(2)}})$ and (N, T^f_N, g_N, Φ^N) are Frobenius manifolds with factorizable flat identities, τ is an affine map, which factors through $p: M^{(1)} \times M^{(2)} = \bar{M}^{(1)} \times \mathbb{C} \times \bar{M}^{(2)} \times \mathbb{C} \to \bar{M}^{(1)} \times \bar{M}^{(2)} \times \mathbb{C}$ which is given in some fixed choice of coordinates on the factors $\mathbb{C}$ by $p(m_1, x, m_2, y) = (m_1, m_2, x + y)$, i is an embedding of affine flat manifolds and Θ is an isomorphism of metric bundles with affine flat structure between the pulled back tangent bundle of N and the exterior product bundle over $M^{(1)} \times M^{(2)}$. Where the statement that Θ is an isomorphism of metric bundles with affine flat structure means that it is an isomorphism of metric bundles and $\tau^* T^f_N = \Theta(T^f_{M^{(1)}} \otimes T^f_{M^{(2)}})$.

We will call such a diagram a *tensor product diagram* and $\mathbf{N}$ a tensor product manifold for $\mathbf{M}^{(1)}$ and $\mathbf{M}^{(2)}$ if it additionally preserves the structure of bundles of Frobenius manifolds i.e. it satisfies the following conditions:

i) for all points $(p, q) \in M^{(1)} \times M^{(2)}$:

$$(\bigoplus \mathbb{C}\, \partial_a \otimes \partial_b, g_{M^{(1)}} \otimes g_{M^{(2)}}, \Phi^{M^{(1)}}_p \otimes \Phi^{M^{(2)}}_q) \overset{\hat{\tau}\circ\Theta}{\cong} \\ (\bigoplus \mathbb{C}\, (\hat{\tau} \circ \Theta(\partial_a \otimes \partial_b)), g_N, \Phi^N_{\tau((p,q))}), \tag{3.13}$$

where $\hat{\tau} \circ \Theta(\partial_a \otimes \partial_b) \in T_{\tau((p,q)),N}$

In other words, at all points in the image of τ, $\hat{\tau} \circ \Theta$ gives an isomorphism of pointed germs of Frobenius manifolds defined in Example 3.3.2:

$$(T_{p,M^{(1)}} \otimes T_{q,M^{(2)}}, 0) \overset{\hat{\tau}\circ\Theta}{\cong} (T_{\tau((p,q)),N}, 0) = (\mathbf{N}, \tau((p,q))) \tag{3.14}$$

ii) The affine connection defined on $T_{M^{(1)}} \boxtimes T_{M^{(2)}}$ by the pullback of the canonical affine connection on T_N — defined by the flat structure on T_N and the canonical 1–form θ_{can}— is the tensor product connection θ_τ of the Proposition 3.3.5:

$$\tau^*(\theta_{can}) = \Theta \circ \theta_\tau. \tag{3.15}$$

as maps.

3.4.3. **Remark.** Due to the condition i) for a tensor product diagram the condition ii) for a tensor product connection is already satisfied by the pulled back affine connection cf. Example 3.3.2. The condition i) for a tensor product connection then forces that τ is affine in the affine coordinates of the source and target spaces.

3.4.4. **Definition.** Two tensor product diagrams

$$
\begin{tikzcd}
T_{M^{(1)}} \boxtimes T_{M^{(2)}} \arrow[r, "\Theta"] \arrow[d] & \tau^*(T_N) \arrow[r, "\hat{\tau}"] \arrow[dl] & T_N \arrow[d] \\
M^{(1)} \times M^{(2)} \arrow[rr, "\tau"'] \arrow[d, "p"] & & N \\
\bar{M}^{(1)} \times \bar{M}^{(2)} \times \mathbb{C} \arrow[urr, hook, "i"] & &
\end{tikzcd}
\tag{3.16}
$$

$$
\begin{tikzcd}
T_{M^{(1)}} \boxtimes T_{M^{(2)}} \arrow[r, "\Theta'"] \arrow[d] & \tau'^*(T_N) \arrow[r, "\hat{\tau}'"] \arrow[dl] & T_{N'} \arrow[d] \\
M^{(1)} \times M^{(2)} \arrow[rr, "\tau'"'] \arrow[d, "p'"] & & N' \\
\bar{M}^{(1)} \times \bar{M}^{(2)} \times \mathbb{C} \arrow[urr, hook, "i'"] & &
\end{tikzcd}
\tag{3.17}
$$

are called *equivalent* if there exist open neighborhoods $U_N, U_{N'}$ of the images of τ, τ' and an isomorphism ϕ of of Frobenius manifolds $U_N \to U_{N'}$ s.t. the induced diagram satisfying the conditions of a tensor product diagram is commutative.

$$
\begin{tikzcd}
T_N|_{U_N} \arrow[d] \arrow[rrrr, dotted, bend left, "\phi_*"] & \tau^* T_N \arrow[l, "\hat{\tau}"'] \arrow[dr] & T_{M^{(1)}} \boxtimes T_{M^{(2)}} \arrow[l, "\Theta"'] \arrow[r, "\Theta'"] \arrow[d] & \tau'^* T_{N'} \arrow[r, "\hat{\tau}'"] \arrow[dl] & T_{N'}|_{U_{N'}} \arrow[d] \\
 & & M^{(1)} \times M^{(2)} \arrow[dll, "\tau"'] \arrow[drr, "\tau'"] \arrow[d, shift right, "p"'] \arrow[d, shift left, "p'"] & & \\
U_N \arrow[rrrr, dotted, bend right, "\phi"'] & & \bar{M}^{(1)} \times \bar{M}^{(2)} \times \mathbb{C} \arrow[ll, hook', "i"'] \arrow[rr, hook, "i'"] & & U_{N'}
\end{tikzcd}
\tag{3.18}
$$

Note that we take the notion of isomorphism of Frobenius manifold in the strict sense that all data should be compatible and we do not allow for instance a conformal change in the metric as in [D2].

3.4.5. **Theorem.** *For any two Frobenius manifolds with factorizable flat identities there exists a tensor product diagram and hence a tensor product manifold of these two manifolds. Furthermore, any two tensor product diagrams for two given Frobenius manifolds* $\mathbf{M}^{(1)}$ *and* $\mathbf{M}^{(2)}$ *are equivalent.*

Proof.

Construction. We construct a tensor product manifold and the structure isomorphism for Frobenius manifolds with factorizable flat identities (for a generalization see below). We start from a pointed cover which is a refinement of the cover given by the domains of convergence of the various $\Phi_p^{(1)}$ and $\Phi_q^{(2)}$. Another choice of refinement would lead to an equivalent diagram. We define $\mathcal{W} = \{U_{(p,q)} | (p,q) \in M^{(1)} \times M^{(2)}\}$ where the $U_{(p,q)}$ are defined as above. The affine structure provides the affine transition functions for the respective coordinate maps φ_U:

$$\varphi_U^V = \varphi_V \varphi_U^{-1} : \varphi_U(U \cap V) \to \varphi_V(V \cap U) \quad \forall\, U, V \in \mathcal{W},\ U \cap V \neq \emptyset$$

which can be written in matrix form relative to the chosen basis

$$\varphi_U^V = \begin{pmatrix} A_U^V & \xi_U^V \\ 0 & 1 \end{pmatrix}$$

They satisfy the conditions

$$\begin{pmatrix} A_U^V & \xi_U^V \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A_V^U & -A_V^U \xi_V^U \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} A_U^W & \xi_U^W \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A_U^V A_V^W & A_V^W \xi_U^V + \xi_V^W \\ 0 & 1 \end{pmatrix} \tag{3.19}$$

where φ_U^V are product transition functions, i.e. $A_U^V = (A^{(1)} \oplus A^{(2)})_U^V$ and $\xi_U^V = x_U^V + y_U^V$.

Notice that the $U_{p,q}$ can be decomposed as $U_{(p,q)} = \check{U}_{(p,q)} \times \mathbb{C}$ where we choose the direction $\mathbb{C}$ to be the "anti–diagonal" in $\mathbb{C}^2$ given by $e^{(1)} - e^{(2)}$. Furthermore, the map p has a section s defined by $s(m_1, m_2, x) := (m_1, \frac{1}{2}x, m_2, \frac{1}{2}x)$.

Using the tensor product connection we can map each U into the fiber of the exterior product bundle over its base–point, say (p, q). This yields an embedding of $\check{U}$ into V_{pq}, where $V_{p_i q_i}$ is, as defined above, the domain of convergence of $\Phi_{p_i}^{(1)} \otimes \Phi_{q_i}^{(2)}$. Let θ_τ be the tensor connection of 3.3.5. Denote the affine parallel displacement w.r.t. θ_τ of $T_{M^{(1)}} \boxtimes T_{M^{(2)}}|_{(p,q)}$ into $T_{M^{(1)}} \boxtimes T_{M^{(2)}}|_{(p_0,q_0)}$ by $(\theta_\tau)_{(p,q)}^{(p_0,q_0)}$. We define a pointed coordinate neighborhood, to be a pair $(U, (p_0, q_0))$ s.t. U is a connected simply–connected coordinate neighborhood of a fixed point $(p_0, q_0) \in U$. For such a pointed coordinate neighborhood $(U, (p_0, q_0))$ with $U \subset U_{(p_0,q_0)}$ we define $\tau^U : U \to T_{M^{(1)}} \boxtimes T_{M^{(2)}}|_{(p_0,q_0)}$ by

$$\tau^U(p, q) = (\theta_\tau)_{(p,q)}^{(p_0,q_0)}(0). \tag{3.20}$$

From now on we will always use pointed neighborhoods and sometimes drop the explicit mention of the base–point.

Denote the matrix in the chosen basis defined by θ_τ^f on an open pointed coordinate neighborhood $U \in \mathcal{W}$ by θ_τ^U. These matrices satisfy

$$(A^{(1)} \otimes A^{(2)})_U^V \theta_\tau^U = \theta_\tau^V (A^{(1)} \oplus A^{(2)})_V^U. \tag{3.21}$$

In this notation:

$$\tau^U(p, q) = \theta_\tau^U(p, q) \tag{3.22}$$

where we identified the point (p, q) with its coordinate vector in $T_{(p_0,q_0)}$

Now choose an open pointed cover of $\mathcal{U}$ the image of s inside $M^{(1)} \times M^{(2)}$ subordinate to $\mathcal{W}$ along which is small enough for our purposes. I.e. all open sets of the cover are connected simply–connected coordinate neighborhoods, their union is a tubular neighborhood of $s(\bar{M})$ and all intersections of these opens are connected simply–connected as well. Furthermore, $\forall (U, (p, q)) \in \mathcal{U} : (p, q) \in \mathrm{Im}(s), U \subset U_{(p,q)}$, and $\tau^U(U) \in V_{p_i q_i}$.

For each $U \in \mathcal{U}$ we choose $\bar{U}$ to be a tubular neighborhood of the image of U inside V_{pq}. Again, different choices lead to equivalent diagrams.

We obtain the desired tensor product manifold by gluing the open sets $\bar{U}$ together using the tensor product of the transition functions. More precisely: For a given pair $U, V \in \mathcal{U}, U \cap V \neq \emptyset$ we define the affine transformation

$$\bar{\varphi}_{\bar{U}}^{\bar{V}} := \begin{pmatrix} \bar{A}_{\bar{U}}^{\bar{V}} & \bar{\xi}_{\bar{U}}^{\bar{V}} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A_U^{(1)V} \otimes A_U^{(2)V} & \theta_\tau^V(\xi_U^V) \\ 0 & 1 \end{pmatrix} \tag{3.23}$$

where $A_U^V = A_{(U,(p,q))}^{(V,(p',q'))}$. It is straightforward to check using (3.19) and (3.21) that

$$\bar{\varphi}_{\bar{U}}^{\bar{V}} = \bar{\varphi}_{\bar{V}}^{\bar{U}-1}, \quad \bar{\varphi}_{\bar{V}}^{\bar{W}} \bar{\varphi}_{\bar{U}}^{\bar{V}} = \bar{\varphi}_{\bar{U}}^{\bar{W}} \tag{3.24}$$

and therefore we get a manifold

$$N := (\amalg_{U \in \mathcal{U}} \bar{U})/\mathcal{R} \tag{3.25}$$

where $\mathcal{R}$ is the equivalence relation induced by the $\bar{\varphi}$.

The cover $\bar{\mathcal{U}} := \{\bar{U} | U \in \mathcal{U}\}$ together with the inclusion maps $i^{\bar{U}} : \bar{U} \to \mathbb{C}^{n_1 n_2}$ and the transition functions $\varphi_{\bar{U}}^{\bar{V}}$ (3.23) define an affine atlas of N. The canonical flat structure $\bigoplus_{a,b} \mathbb{C}\ (\partial_a \otimes \partial_b)$ on each $\bar{U}$ together with the tensor metric $g^{\bar{U}} = g_p \otimes g_q$ glue together under the affine transformations to form an affine flat structure on N.

Due to the condition (3.21) the maps $\tau^U : U \to \bar{U} \subset V_{pq}$ of (3.22) satisfy:

$$\bar{\varphi}_{\bar{U}}^{\bar{V}} \tau^U(u) = \tau^V(u) \tag{3.26}$$

for $u \in U \cap V$ and thus glue together to a map $t : \bigcup_{U \in \mathcal{U}} U \to N$. We can now define the map τ as

$$\tau := t \circ s \circ p \tag{3.27}$$

Note that since $T_{M^{(1)}}$ and $T_{M^{(2)}}$ are trivial along the factors $\mathbb{C}$ of the decomposition $M^{(i)} = \bar{M}^{(i)} \times \mathbb{C}$, $s \circ p$ induces the following identity between bundles

$$(s \circ p)^*(T_{M^{(1)}} \boxtimes T_{M^{(2)}}) \cong T_{M^{(1)}} \boxtimes T_{M^{(2)}} \tag{3.28}$$

Due to the fact that all higher (i.e. higher than the third) derivatives of the potentials $\Phi^{(i)}$ are independent of the coordinates of the identities, the above identity of bundles induces an isomorphism of germs of pointed Frobenius manifolds

$$(T_{M^{(1)}} \boxtimes T_{M^{(2)}}|_m, 0) \cong (T_{M^{(1)}} \boxtimes T_{M^{(2)}}|_{s \circ p(m)}, 0). \tag{3.29}$$

So far we have constructed an affine flat manifold N a map $\tau : M^{(1)} \times M^{(2)} \to N$ and it is easy to confirm that $\tau^*(\theta_{can}) = \Theta \circ \theta_\tau$ and that $\tau^*(T_N)$ and $T_{M^{(1)}} \boxtimes T_{M^{(2)}}$ are naturally isomorphic under the composition Θ of the identification of the tangent space of a vector space at a point with the vector space itself and the isomorphism of (3.28).

Furthermore, the by construction the map τ factors through p and since t is an injection on $\mathrm{Im}(s)$, $i := t \circ s$ is an embedding.

To endow N with the desired Frobenius manifold structure, we have to check that the $\bar{U}$ glue together as Frobenius manifolds.

This follows from the properties of the tensor product connection. Let $u \in U \cap V$, U have the base–point (p, q) and (p', q') be the base–point of V. The Frobenius structure of $\bar{U}$ is given by the potential Φ_{pq} and the Frobenius structure of $\bar{V}$ by the potential $\Phi_{p'q'}$. Using the tensor connection we see that the germ of Φ_{pq} at $\tau^U(u)$ and the germ of Φ_u at 0 in V_u can be given by the same power series. This equality between the germs holds as well for the germ of Φ_u at 0 in V_u and the germ of $\Phi_{p'q'}$ at $\tau^V(u)$ and since the whole germs coincide so do the functions:

$$\Phi_{pq}(x) = \Phi_{p'q'}(y) \quad \forall y = \bar{\varphi}_{\bar{U}}^{\bar{V}}(x). \tag{3.30}$$

Since θ_τ is a tensor connection the definition of t and the isomorphism (3.28) show that the condition i) for a tensor product diagram also holds.

Uniqueness

Let two diagrams as in 3.4.4 be given. To construct the open subsets U_N, $U_{N'}$ and the isomorphism of Frobenius manifolds ϕ we choose a pointed cover $\mathcal{U}_N$ of a tubular neighborhood of the image of τ which additionally has the following properties:

i) $\forall (U,n) \in \mathcal{U}_N : n \in \mathrm{Im}(\tau), U \subset U_n$ where U_n is again the notation for the domain of convergence of Φ_n^N.
ii) The opens and their intersections should be connected and simply–connected and all open sets should be coordinate neighborhoods.
iii) Using the isomorphisms Θ and Θ' we can identify a small neighborhood of 0 on the fiber of $T_{M^{(1)}} \boxtimes T_{M^{(2)}}$ at (p,q) with a neighborhood of $\tau((p,q))$ on N and a neighborhood of $\tau'((p,q))$ on N'. The condition for our cover is that all open neighborhoods of the cover are so small that the above identifications exist.

We define the map ϕ as the concatenation of these identifications, i.e.

$$\phi^{(U,n)} = \exp|_{\tau'(s \circ i^{-1}(n))} \circ \hat{\tau}' \circ \Theta' \circ \Theta^{-1} \circ \hat{\tau}|^{-1}_{s \circ i^{-1}(n)} \circ \exp^{-1}|_n \tag{3.31}$$

where $\hat{\tau}|^{-1}_{s \circ i^{-1}(n)}$ is the inverse of $\hat{\tau}$ restricted to the fiber of $T_{M^{(1)} \times M^{(2)}}$ at $s \circ i^{-1}(n)$. We set $U_N = \bigcup_{(U,n) \in \mathcal{U}_N} U$ and $U_{N'} = \bigcup_{(U,n) \in \mathcal{U}_N} \phi^{(U,n)}(U)$. Since $\tau^* \theta_{can} = \Theta \circ \theta_\tau$ and $\Theta' \circ \theta_\tau = \tau'^* \theta'_{can}$ and all maps preserve the relevant germs of Frobenius manifolds, it is clear that the maps $\phi^{(U,n)}$ patch together as a morphism of Frobenius manifolds. On the image of i we have $\phi|_{\mathrm{Im}(i)} = \tau' \circ s \circ i^{-1}$. A short calculation shows that this map yields a bijection between $\mathrm{Im}(i)$ and $\mathrm{Im}(i')$ with inverse $\tau \circ s' \circ i'^{-1}$. After making the original cover smaller if necessary, we can assume that the induced cover $\mathcal{U}_{N'} := \{(\phi(U), \phi(n)) | (U,n) \in \mathcal{U}_N\}$ also satisfies the conditions i)–iii) and the union of the opens of this cover is again a tubular neighborhood of $\mathrm{Im}(\tau')$. Hence, we can perform the analogous construction starting from $\mathcal{U}_{N'}$ yielding an inverse morphism. Thus ϕ is an isomorphism of Frobenius manifolds which satisfies the condition $\phi \circ \tau = \tau'$ and the commutativity of the upper part of the diagram (3.18) follows directly from the construction.

3.4.6. **Proposition.** *Given two Frobenius manifolds with factorizable flat identities and Euler fields (with $d = 1$) then any tensor product manifold carries the natural tensor product identity and can be endowed with an Euler field locally defined by 2.1.10.*

Proof.

We will define the identity and the Euler field on a tensor on the product manifold N constructed above. The results can be pushed to any equivalent manifold. Let

$$e_N^{\bar{U}} := \partial_0^{(1)} \otimes \partial_0^{(2)} \quad E^{\bar{U}} := E_{pq} \tag{3.32}$$

where E_{pq} is the Euler field constructed for the formal tensor product in 2.1.10. The gluing condition for the identities is clear from the compatibility of the flat structures. What still remains to be shown is that the locally defined Euler fields glue together $J_{\bar{\varphi}_{\bar{U}}^{\bar{V}}} E^{\bar{U}}(x) = E^{\bar{V}}(\bar{\varphi}_{\bar{U}}^{\bar{V}}(x))$, where J is the Jacobian.

Let $\bar{\varphi}_{\bar{U}}^{\bar{V}}(x) = A^{(1)} \otimes A^{(2)} x + \theta_\tau^V(\xi_U^V)$. Furthermore, we will write all Euler fields in matrix form; for $U = U^{(1)} \times U^{(2)}$, $V = V^{(1)} \times V^{(2)}$

$$E^{U^{(i)}}(x) = D^{U^{(i)}} x + r^{U^{(i)}}, \quad i = 1,2$$

The gluing conditions for $E^{(1)}$ and $E^{(2)}$ read

$$A^{(i)} D^{U^{(i)}} = D^{V^{(i)}} A^{(i)}, \quad A^{(i)} r^{U^{(i)}} = D^{V^{(i)}} \xi_{U^{(i)}}^{V^{(i)}} + r^{V^{(i)}}, \quad i = 1,2.$$

In this notation for all (p, q):

$$E_{pq} = (D^{U^{(1)}} \otimes Id + Id \otimes D^{U^{(2)}} - Id \otimes Id)(x) + \theta_\tau^U (r^{U^{(1)}} + r^{U^{(2)}})$$

And hence

$$\begin{aligned} A^{(1)} \otimes A^{(2)} E_{pq}(x) =& (D^{V^{(1)}} \otimes Id + Id \otimes D^{V^{(2)}} - Id \otimes Id)(\bar{\varphi}_{\bar{U}}^{\bar{V}}(x)) \\ &+ \theta_\tau^V \big(- (D^{U^{(1)}} \oplus Id + Id \oplus D^{U^{(2)}} - Id \oplus Id)(\xi_V^U) + A^{(1)} r^{U^{(1)}} + A^{(2)} r^{U^{(2)}} \big) \\ =& E_{p'q'}(\bar{\varphi}_{\bar{U}}^{\bar{V}}(x)) \end{aligned}$$

3.4.7. **Remarks.**

i) The restriction of factorizable identity is not too severe. In all presently known examples this is the case. This includes all semi–simple Frobenius manifolds considered on $\mathbb{C}^n$ without the diagonals, as well as the split semi–simple Frobenius manifolds on the universal cover of the previous space given by special initial conditions.

ii) Locally one can always complete the direction of the identity by using an appropriate embedding into $\mathbb{C}^n$ (see below).

3.5. **Embedded Frobenius manifolds.** Notice that the universal cover of every affine manifold of dimension n has an immersion into $\mathbb{C}^n$ (cf. e.g. [KW]). This immersion can be quite non–trivial however, see e.g. [ST]. We will call a Frobenius manifold an *embedded Frobenius manifold* if the manifold itself has an embedding into $\mathbb{C}^n$. In this case we will identify the Frobenius manifold with its image under the embedding. It then has global coordinates given by a choice of basis for the affine flat tangent bundle.

Actually, most constructions of Frobenius manifolds use global coordinates e.g. the ones coming from quantum cohomology, unfolding of singularities or Landau–Ginzburg models.

3.5.1. **Lemma.** *An embedded Frobenius manifold* $\mathbf{M}$*,* $M \subset \mathbb{C}^n$*, with flat identity can be completed in the direction of the identity. I.e. there exists a Frobenius manifold* $\tilde{\mathbf{M}}$ *with factorizable flat identity which contains* $\mathbf{M}$ *as a Frobenius manifolds* $\tilde{M} = \bar{M} \times \mathbb{C} \supset M$ *and the other structures are given by restriction.*

Proof. Since the potential a polynomial of order less or equal three in the coordinate of the identity for a Frobenius manifold with flat identity, the respective three–tensor defining the Frobenius structure is independent of the coordinate of the flat identity. Hence, since the tangent bundle is trivial, we can enlarge the domain of definition of this three–tensor so that it contains all lines in the direction of the identity. Likewise, we can extend the metric to these points too.

The structure of a tensor product manifold for two embedded Frobenius manifolds with factorizable flat identity allows a more explicit description.

Let $M^{(1)}$ and $M^{(2)}$ be realized in $\mathbb{C}^{n_1}$ respectively $\mathbb{C}^{n_2}$ and consider the map $\tau : \mathbb{C}^{n_1+n_2} \to \mathbb{C}^{n_1 n_2} = \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$ given by the matrix $\tau_{ab} = \partial_{0a} + \partial_{b0}$. Looking at the construction we arrive immediately at the following:

3.5.2. **Proposition.** *The tensor product manifold for embedded Frobenius manifolds is equivalent to an embedded Frobenius manifold given by a neighborhood of the image* $\tau(M^{(1)} \times M^{(2)}) \subset \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$ *and the image is isomorphic to* $\bar{M}^{(1)} \times \bar{M}^{(2)} \times \mathbb{C}$ *in the notation of Lemma 3.5.1.*

3.5.3. **Remark.** The tensor product in the particular cases considered in this subsection thus contains a subset isomorphic to the image of p parameterized by the coordinates x_{a0}, x_{0b} ($x_{ab} = 0$ for $ab \neq 0$) The third derivatives of the potential Φ of the tensor product satisfies:

$$\Phi_{aa'bb'cc'}|_{\mathrm{Im}(\tau)}(x_{i0}, x_{0j}) = \Phi^{(1)}_{abc}(x^{(1)}_i \equiv x_{i0})\Phi^{(2)}_{a'b'c'}(x^{(2)}_j \equiv x_{0j}), \quad i, j \neq 0 \tag{3.33}$$

but moreover along the image of τ the *whole germs* of Φ are given by the tensor product of the associated pointed germs. Vice versa, the condition (3.33) does not suffice to identify a tensor product manifold, since it does not determine the higher derivatives in the ∂_{ab} directions for $ab \neq 0$.

3.6. **General tensor products.** The construction and the universality statement easily generalize to the following setting.

3.6.1. **Definition.** Let θ_τ be a tensor product connection on the exterior product bundle over a Cartesian product of two Frobenius manifolds $\mathbf{M}^{(1)}$ and $\mathbf{M}^{(2)}$. We call the Cartesian product $\mathbf{M}^{(1)} \times \mathbf{M}^{(2)}$ *θ_τ–reducible* if there exists a flat affine manifold $\bar{M}$, an affine projection $p : M^{(1)} \times M^{(2)} \to \bar{M}$ together with an affine section s of this projection which is an embedding of affine flat manifolds satisfying the following conditions

i) The projection condition of (3.28)

$$(s \circ p)^*(T_{M^{(1)}} \boxtimes T_{M^{(2)}}) \cong T_{M^{(1)}} \boxtimes T_{M^{(2)}} \tag{3.34}$$

ii) The condition on the respective germs of Frobenius manifolds (3.29)

$$(T_{M^{(1)}} \boxtimes T_{M^{(2)}}|_{s\circ p(m)}, 0) \overset{(s\circ p)^*}{\cong} (T_{M^{(1)}} \boxtimes T_{M^{(2)}}|_m, 0) \tag{3.35}$$

iii) The compatibility with the tensor product connection; under the identification of i):

$$(s \circ p)^*\theta_\tau = \theta_\tau \tag{3.36}$$

vi) And the embedding condition

$$T_{M^{(1)}} \boxplus T_{M^{(2)}}|_{\mathrm{Im}(s)} = T_{\mathrm{Im}(s)} \oplus ker(\theta^f_\tau)|_{\mathrm{Im}(s)} \tag{3.37}$$

The triple $(\bar{M}, p, s)$ is then called *a θ_τ–reduction*

3.6.2. **Lemma.** *If in a Cartesian product $M^{(1)} \times M^{(2)}$ each of the factors $M^{(i)}$ can be decomposed as $\bar{M}^{(i)} \times \mathbb{C}^{n_i}$ as flat affine manifolds and the whole Cartesian product can be decomposed as $\bar{M} \times \mathbb{C}^n$ where the third derivatives of the potential are constant in the $\mathbb{C}$–directions and the kernel of θ_τ gives the coordinates the $\mathbb{C}$–directions and furthermore the factor $\mathbb{C}^n$ is an affine flat factor of $\mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$, then $\mathbf{M}^{(1)} \times \mathbf{M}^{(2)}$ is θ_τ–reducible.*

Proof. We can decompose $M^{(1)} \times M^{(2)}$ as $\bar{M}^{(1)} \times \bar{M}^{(2)} \times \mathbb{C}^{n_1+n_2-n} \times \mathbb{C}^n$ where $\mathbb{C}^{n_1+n_2-n} \times \mathbb{C}^n$ is the postulated decomposition of $\mathbb{C}^{n_1+n_2}$. Now consider the projection $p : M^{(1)} \times M^{(2)} \to \bar{M} = \bar{M}^{(1)} \times \bar{M}^{(2)} \times \mathbb{C}^{n_1+n_2-n}$ and the zero section s. They satisfy all the conditions of a θ_τ–reduction.

3.6.3. **Lemma.** For two Frobenius manifolds with not necessarily factorizable flat identities a general tensor product diagram for the canonical tensor product connection θ_τ (3.3.5) exists.

Proof. Locally we can achieve the following situation: let U in $M^{(1)} \times M^{(2)}$ be an open set which satisfies: $U = U^{(1)} \times U^{(2)} = \bar{U}^{(1)} \times D_r \times \bar{U}^{(2)} \times D_r$ where D_r is a disc of radius r in $\mathbb{C}$ centered at 0 —$D_r = \{z \mid |z| < r\}$— which is coordinatized by the identity. Now take $\bar{M} := \bar{U}^{(1)} \times \bar{U}^{(2)} \times D_{2r}$ where $D_{2r} = \{z \mid |z| < 2r\}$ and define p and s as in the construction of 3.4.5. Since again the potential is constant on the factors D all necessary properties directly follow.

3.6.4. **Definition.** For three Frobenius manifolds $\mathbf{M}^{(1)}, \mathbf{M}^{(2)}$ and $\mathbf{N}$ consider a diagram of the type

$$\begin{tikzcd}
T_{M^{(1)}} \boxtimes T_{M^{(2)}} \arrow[r, "\Theta"] \arrow[d] & \tau^* T_N \arrow[r, "\hat{\tau}"] \arrow[dr] & T_N \arrow[d] \\
M^{(1)} \times M^{(2)} \arrow[rr, "\tau"] \arrow[d, "p"] & & N \\
\bar{M} \arrow[u, bend left, "s"] \arrow[urr, hook, "i"] & &
\end{tikzcd} \tag{3.38}$$

together with a tensor product connection θ_τ, such that $\mathbf{M}^{(1)} \times \mathbf{M}^{(2)}$ is θ_τ–reducible, $(\bar{M}, p, s)$ is a θ_τ–reduction, τ is an affine map and Θ is an isomorphism of metric bundles with affine flat structure.

We will call such a diagram a θ_τ*–tensor product diagram* if it satisfies the condition i) and ii) of a tensor product diagram where now θ_τ is given in the data.

3.6.5. **Definition.** Two general tensor product diagrams are called *equivalent* if there exist open neighborhoods $U_N, U_{N'}$ of the images of τ, τ' and an isomorphism ϕ of of Frobenius manifolds $U_N \to U_{N'}$ s.t. the induced diagram of the form (3.18) satisfying the conditions of a general tensor product diagram is commutative.

3.6.6. **Theorem.** *Given a tensor product connection θ_τ on an exterior product bundle over the Cartesian product of two Frobenius manifolds $\mathbf{M}^{(1)}$ and $\mathbf{M}^{(2)}$ with θ_τ–reducible Cartesian product, there exists a θ_τ–tensor product diagram and thus a tensor product manifold. Furthermore, fixing a θ_τ–reduction all diagrams involving this reduction are equivalent.*

Proof. In the case that and θ_τ is not identically zero we can retrace the proof of 3.4.5, since we only used that θ_τ is a non–zero tensor product connection and the existence of a θ_τ–reduction. In case $\theta_\tau \equiv 0$ the image of p is just a point and the tensor product already exists by the construction for convergent germs of Frobenius manifolds. The uniqueness then follows directly from the condition i) for a general tensor product diagram.

3.6.7. **Remark.** The local situation for Frobenius manifolds with flat identities can be described in three different ways:

i) Via the *tensor product connection.*
   Given any point $(p, q) \in M^{(1)} \times M^{(2)}$ we have the corresponding pointed germ of the tensor product relative to the pair of base–points (p, q) on the fiber of $T_{M^{(1)}} \boxtimes T_{M^{(2)}}$ over (p, q). This germ can be seen as the germ describing the situation locally since

by virtue of the existence of the tensor product connection of 3.3.5 all continuations of this germ will contain all neighboring germs.

ii) Via an *embedding and completion.*
Locally we can embed any complex affine manifold of dimension n into $\mathbb{C}^n$. Using the analysis of embedded Frobenius manifolds with flat identity we can complete our embedded manifold and use the Theorem for tensor product diagrams to find a local tensor product manifold.

iii) Via the *local general tensor product.*
See Lemma 3.6.3.

Of course all these descriptions are compatible. The compatibility of i) with ii) and iii) is manifest in the condition i) of the definition of (general) tensor product diagrams. For a suitable neighborhood, we can pass from the ii) to iii) by restricting everything to the image of this neighborhood in the completion of its embedding and its image under τ.

3.6.8. **Remark.** One necessary condition for the existence of a general tensor product diagram is the existence of a tensor product connection. In some cases there are many such connections in others there may be none. If there is none one this shows that the tensor product can only defined w.r.t. a fixed base–point and there is no way of naturally parameterizing the tensor products by a submanifold in a Frobenius manifold.

3.6.9. **Examples.**

i) Choose two vector spaces V, V' with a constant product and consider the constant tensor product multiplication in the sense of algebras on $V \otimes V'$. Now any linear map $\theta_\tau : V \times V' \to V \otimes V'$ will provide a tensor product connection.

ii) Consider two one–dimensional Frobenius manifolds whose potentials both have a zero; say at the points p and q, but are not constantly zero near these points. Using the Proposition 2.2.13 we see that the Frobenius structure on the tangent space at (p, q) would be given by a vanishing potential. On the other hand, near the point (p, q) the potentials are non–vanishing by assumption and so is their product which is the value of Ψ on the zero section of the exterior product bundle. Thus there is no tensor product connection on a neighborhood of the point (p, q). (For a general statement about the one–dimensional situation see below.)

iii) A tensor product connection always exists for the product of any Frobenius manifold $\mathbf{M}$ with $\mathbb{C}$ carrying a constant multiplication, i.e. the third derivative potential Φ is constant $\Phi_{zzz} \equiv \alpha$ — where z is a fixed coordinate on $\mathbb{C}$. Notice that after scaling $\frac{\partial}{\partial z}$ we can assume that the multiplication is either constantly zero or $\frac{\partial}{\partial z}$ is a flat identity for $\mathbb{C}$. In this case, $\theta_\tau(\partial_a) = \partial_{a0}, \theta_\tau(\partial_0') = 0$ provides a tensor product connection. The tensor product manifold is M itself and the map $\tau = \pi_1$ the first projection of $M \times \mathbb{C}$. Here $p = \tau$ and s is the zero section.

3.6.10. **The tensor product of one–dimensional Frobenius manifolds.** In this subsection we give a complete answer to the existence question of a tensor product connection in the case of two one–dimensional Frobenius manifolds.

3.6.11. **Proposition.** *A tensor product connection for two one–dimensional Frobenius manifolds exists if and only if one of the factors has a locally constant multiplication i.e. locally $\Phi_{zzz} = \alpha$ where z is the coordinate function of the flat vector field $\frac{d}{dz}$ or equivalently it carries a flat identity or a zero multiplication.*

In other words, the tensor product of one–dimensional theories is essentially pointed and does not contain perturbations of the base–points.

Proof. Using the notation of the previous subsection, suppose a tensor product connection exists. Choose any point $(p,q) \in M^{(1)} \times M^{(2)}$ and a coordinate neighborhood U of (p,q) with the local normalized product coordinate (z_1, z_2), i.e. $(z_1,z_2)(p,q) = (0,0)$. Write $\varphi_1 := \Phi^{(1)}_{z_1 z_1 z_1}$ and $\varphi_2 := \Phi^{(2)}_{z_2 z_2 z_2}$ and $\varphi_{z_1,z_2} := \frac{\partial^3}{\partial^3 z}\Phi_{z_1} \otimes \Phi_{z_2}$

Set

$$\psi(z_1, z_2, z) := \varphi_{z_1,z_2}(z - \tau(z_1, z_2)).$$

The function ψ is independent of z_1, z_2 since by definition of a tensor product connection

$$\psi(z_1, z_2, z) = \varphi_{0,0}(z)$$

Therefore:

$$\frac{\partial}{\partial z_i}\psi \equiv 0$$

and thus

$$\begin{aligned} \frac{\partial}{\partial z_i}\psi(z_1, z_2, \tau(z_1, z_2)) &= \frac{\partial \varphi_{z_1,z_2}}{\partial z_i}(0) - \frac{\partial \tau}{\partial z_i}|_{(z_1,z_2)}\varphi'_{z1,z2}(0) = 0 \\ \Leftrightarrow \frac{\partial}{\partial z_i}(\varphi_1\varphi_2) &- \frac{\partial \tau}{\partial z_i}|_{(z_1,z_2)}(\varphi'_1\varphi_2^2 + \varphi_1^2\varphi'_2) = 0 \end{aligned}$$

where $'$ denotes the derivative and we used the short–hand notation φ_1, φ_2 for $\varphi_1(z_1), \varphi_2(z_2)$. Furthermore,

$$\begin{aligned} \frac{\partial}{\partial z_i}\psi'(z_1, z_2, \tau(z_1, z_2)) &= \frac{\partial \varphi'_{z_1,z_2}}{\partial z_i}(0) - \frac{\partial \tau}{\partial z_i}|_{(z_1,z_2)}\varphi''_{z1,z2}(0) = 0 \\ \Leftrightarrow \frac{\partial}{\partial z_i}(\varphi'_1\varphi_2^2 + \varphi_1^2\varphi'_2) &- \frac{\partial \tau}{\partial z_i}|_{(z_1,z_2)}(\varphi''_1\varphi_2^3 + 5\varphi_1\varphi_2\varphi'_1\varphi'_2 + \varphi_1^3\varphi''_2) = 0. \end{aligned} \tag{3.39}$$

Therefore we have

$$((\varphi_2\frac{\partial}{\partial z_1} + \varphi_1\frac{\partial}{\partial z_2})\psi'(z_1, z_2, \tau(z_1, z_2)))\psi'(\tau(z_1, z_2)) = \varphi_1\varphi_2\varphi'_1\varphi'_2(\varphi'_1\varphi_2^2 + \varphi_1^2\varphi'_2) = 0.$$

Therefore either φ_1 or φ_2 constantly vanish, or we may assume that on some open set $\varphi_1(z_1) \neq 0$ and $\varphi_2(z_2) \neq 0$ and therefore if also neither φ'_1 nor φ'_2 constantly vanish we must have

$$\frac{\varphi'_1(z_1)}{\varphi_1(z_1)} = -\frac{\varphi'_2(z_2)}{\varphi_2(z_2)} = c \tag{3.40}$$

where c is a constant.

The solution to these simple differential equations is $\varphi_1(z_1) = \frac{1}{cx+d_1}$ and $\varphi_2(z_2) = \frac{2}{cx+d_2}$. Inserting (3.40) into the equation (3.39) we find

$$c^3\varphi_1^5\varphi_2^4 \equiv 0$$

which yields that $c = 0$ a contradiction to the last assumption.

Therefore either $\varphi'_1 \equiv 0$ or $\varphi'_2 \equiv 0$ and the proposition follows.

4. Semi–simple Frobenius manifolds

4.1. **Semi–simple Frobenius manifolds.** We will briefly recall the main notions of semi–simple Frobenius manifolds as explained in [M1]. For other versions see [D2] or [H]. A Frobenius manifold of dimension n is called semi–simple (respectively split semi–simple), if an isomorphism of the sheaves of $\mathcal{O}_M$–algebras

$$(\mathcal{T}_M, \circ) \simeq (\mathcal{O}_M^n, \text{ componentwise multiplication}) \tag{4.1}$$

exists everywhere locally (respectively globally).

If a Frobenius manifold M is semi–simple, one can find so–called canonical coordinates u_i —unique up to constant shifts and renumbering— s. t. the metric and the three–tensor A defining the multiplication become particularly simple. Let $e_i = \frac{\partial}{\partial u_i}, \nu_i = du_i$, then

$$g = \sum_i \eta_i(\nu_i^2), \quad A = \sum_i \eta_i(\nu_i^3). \tag{4.2}$$

If in addition an Euler field exists, then it has the form $E = \sum(u^i + c^i)e_i$. In this situation, we will normalize the coordinates in such a way that

$$E = \sum u^i e_i. \tag{4.3}$$

This normalization fixes the ambiguity in the coordinates u^i and renders them unique up to the $\mathbb{S}_n$–action.

4.1.1. **Definition.** In the above situation, we will call a point $m \in M$ *tame*, if it satisfies $u_i(m) \neq u_j(m)$ for all $i \neq j$. In other words, the point m is tame, if the spectrum of the operator $E\circ$ on $\mathcal{T}_M$ is simple.

In the theory of semi–simple Frobenius manifolds one then defines certain natural structure connections which give rise to isomonodromic deformations. These deformations are governed by the Schlesinger differential equations [Sch, Mal], thus providing a link between Frobenius manifolds and solutions of the Schlesinger equations; the details can be found in [D2, M1, MM].

4.1.2. **Theorem (2.6.1 of [MM]).** *Let $(M, (u^i), T, (A_i))$ be a strictly special solution and e an identity of weight D, then these data come from a unique structure of semi–simple split Frobenius manifold M with an identity $(d_0 = 1)$ and an Euler field via*

$$T = \Gamma(M, \mathcal{T}_M^f), \quad (u^i)\text{: the canonical coordinates}$$

$$A_j(e_i) = 0 \quad \text{for } i \neq j, \quad A_i(e_i) = -\frac{1}{2}e_i + \sum_{j:j\neq i}(u^j - u^i)\frac{\eta_{ij}}{\eta_i}e_j \tag{4.4}$$

$$\text{The operator } \mathcal{V} \text{ is given by: } \mathcal{V}(X) = \nabla_{0,X}(E) - \frac{D}{2}X.$$

Here, the manifold M only has tame points which means that by definition $u^i(m) \neq u^j(m), \forall i \neq j, m \in M$. M should be regarded as a splitting cover of the subspace of tame points of a given Frobenius manifold.

For the notion of strictly special solutions consult [MM].

4.1.3. **Special initial conditions.** Fixing a base–point in a solution to Schlesinger's equations and taking the coordinates e_i for T call a family of matrices $A^0_1, \dots A^0_m \in \mathrm{End}(T)$ *special initial conditions*, if there exists a diagonal metric g and a skew–symmetric operator $\mathcal{V}$ s. t. $A^0_j = -(\mathcal{V} + \frac{1}{2}\mathrm{Id})P_j$, where P_j is the projector onto $\mathbb{C}e_j$.

In the case of semi–simple Frobenius manifolds with an Euler field and a flat identity, the special initial conditions are given by the value of the structures listed in 4.1.2 at a fixed tame point $m_0 \in M$ with coordinates (u^i_0); more precisely, the metric is given by the $\eta_i(m_0)$ and the operator $\mathcal{V}$ by the matrix $(v_{ij})_{ij}$ defined by $(\nabla_{0,e_i}(E) - \frac{D}{2}(e_i))(m_0) = (\sum_j v_{ij} e_j)(m_0)$. The matrix coefficients v_{ij} can be calculated as follows:

$$v_{ij} = (u^i - u^j)\frac{\eta_{ij}}{\eta_i}(m_0). \tag{4.5}$$

4.2. **The tensor product for split semi–simple Frobenius manifolds with Euler field and flat identity.** In the previous section we constructed a tensor product of Frobenius manifolds with factorizable flat identity. Moreover any split semi–simple Frobenius manifold is already determined by the special initial conditions at a tame semi–simple point.

If there is a pair of tame semi–simple points $(p, q) \in M' \times M''$, then the image $\tau((p, q))$ is again a semi–simple point, since the algebra in the tangent space over the base–point of the tensor product is just the tensor product of two semi–simple algebras and thus it is itself semi–simple.

Thus, the tensor product manifold is given locally near $\tau((p, q))$ by the special initial conditions at $\tau((p, q))$ if this new base–point is again tame. The tensor product is globally given by these conditions for the tensor product of two split semi–simple Frobenius manifolds.

This condition, however, is not very restrictive and if there is a pair of tame semi–simple points in some open U one can always find a pair of tame semi–simple points $(p', q') \in U$ whose image is also tame semi–simple as we will show later.

4.2.1. **Canonical coordinates.** Since the proof of existence of the tensor product makes extensive use of the flat coordinates, a natural question to ask in the setting of semi–simple Frobenius manifolds is: Is there also a nice formulation in terms of canonical coordinates? Generally, one can not expect simple formulas, since the algebra in the tangent space over a given point in the tensor product manifold is generally not a tensor product of algebras —this locus is described by the image of the Cartesian product— and the "coupling" of algebras results in a destruction of the pure tensor form of the idempotents.

Using the definitions of the tensor product for formal Frobenius manifolds, we can, however, calculate the idempotents of the tensor product in terms of flat coordinates in the formal situation. They are given by the following Proposition up to terms of order two in flat coordinates which is the precision needed to calculate the special initial conditions.

4.2.2. **Proposition.** *Given two semi–simple Frobenius manifolds* $\mathbf{M}', \mathbf{M}''$ *let the idempotents near the base–points* m'_0, m''_0 *have the expansions* $e'_i = e'^0_i + \sum x'^{a'} e'^{a'}_i + O(x'^2)$, *and* $e''_i = e''^0_i + \sum x''^{a''} e''^{a''}_i + O(x''^2)$ *in the flat coordinates* x' *and* x'', *then the idempotents* e_{ij} *of* *the tensor product* $(\mathbf{M}' \otimes_{(m_0', m_0'')} \mathbf{M}'', 0)$ *have the following expansion in the flat coordinates* x *around* 0*:*

$$e_{ij}(x) = e_i'^0 \otimes e_j''^0 + \sum_{a',a''} x^{a'a''} (\lambda_j^{a''} e_i'^{a'} \otimes e_j''^0 + \lambda_i^{a'} e_i'^0 \otimes e_j''^{a''}) + O(x^2) \tag{4.6}$$

where $\partial'_{a'} = \sum \lambda_i^{a'} e_i'^0$ *and* $\partial''_{a''} = \sum \lambda_j^{a''} e_j''^0$.

Furthermore, the respective coordinate functions for the tensor metric $\eta_{ij} := \eta(e_{ij}, e_{ij})$ *have the expansions:*

$$\eta_{ij}(x) = \eta_i'(m_0')\eta_j''(m_0'') + \sum_{a',a''} x^{a'a''} (\lambda_j^{a''} (\partial'_{a'}\eta_i')(m_0')\eta_j''(m_0'') + \lambda_i^{a'} \eta_i'(m_0')(\partial''_{a''}\eta_j''(m_0''))) \tag{4.7}$$

and their derivatives $\eta_{ij,kl} := e_{kl}\eta_{ij}$ *have the following values at the base–point* m_0*:*

$$\eta_{ij,kl}(0) = \delta_{j,l}\eta'_{ik}(m_0')\eta_j''(m_0'') + \delta_{i,k}\eta_i'(m_0')\partial''_{a''}\eta''_{jl}(m_0'') \tag{4.8}$$

where $\delta_{i,k}$ *is the Kronecker delta symbol.*

If the factors carry flat identities and Euler fields, then the normalized canonical coordinates of m_0 *are:*

$$u^{ij}(0) = u'^i(m_0') + u''^j(m_0''). \tag{4.9}$$

Proof. To check the formula (4.6), expand the potential Φ up to order four in the flat coordinates and verify the idempotency by direct calculation.

The equations for the idempotents $e_i = e_i^0 + \sum x^a e_i^a + \mathrm{O}(x^2)$ in flat coordinates are in zeroth order:

$$e_i^0 = (e_i^0, e_i^0) \tag{4.10}$$

and in first order

$$e_i^a = (e_i^0, e_i^0, \partial_a) + 2(e_i^a, e_i^0). \tag{4.11}$$

Here we used the notation $(\partial_i, \dots, \partial_j)$ for the higher order multiplications.

We can now check both conditions for (4.6). For the zeroth order we obtain:

$$(e_{ij}^0, e_{ij}^0) = (e_i'^0, e_i'^0) \otimes (e_j''^0, e_j''^0) = e_i'^0 \otimes e_j''^0 = e_{ij}^0$$

And for the first order terms:

$$\begin{aligned}
&(e_{ij}^0, e_{ij}^0, \partial_{a'a''}) + 2(e_{ij}^{a'a''} e_{ij}^0) \\
&\quad = (e_i'^0, e_i'^0, \partial_{a'}) \otimes ((e_j''^0, e_j''^0)\partial_{a''}) + ((e_i'^0, e_i'^0)\partial_{a'}) \otimes (e_j''^0, e_j''^0, \partial_{a''}) \\
&\qquad + 2\lambda_j^{a''}(e_i'^{a'}, e_i'^0) \otimes (e_j''^0, e_j''^0) + 2\lambda_i^{a'}(e_i'^0, e_i'^0) \otimes (e_j''^{a''}, e_j''^0) \\
&\quad = \lambda_j^{a''}[(e_i'^0, e_i'^0, \partial_{a'}) + 2(e_i'^{a'}, e_i'^0)] \otimes e_j''^0 + \lambda_i^{a'} e_i'^0 \otimes [(e_j''^0, e_j''^0, \partial_{a''}) + 2(e_j''^{a''}, e_j''^0)] \\
&\quad = \lambda_j^{a''} e_i'^{a'} \otimes e_j''^0 + \lambda_i^{a'} e_i'^0 \otimes e_j''^{a''} \\
&\quad = e_{ij}^0
\end{aligned}$$

since $((e_j''^0, e_j''^0)\partial_{a''}) = (e_j''^0, \partial_{a''}) = \lambda_j^{a''} e_j''^0$ and $((e_i'^0, e_i'^0)\partial_{a'}) = (e_i'^0, \partial_{a'}) = \lambda_i^{a'} e_i'^0$.

The expansion for the metrics of the factors reads:

$$\eta_i'(x') = \eta_i'(m_0') + \sum x'^{a'} 2g(e_i'^{a'}, e'^0) + O(x'^2)$$

$$\eta_j''(x'') = \eta_j''(m_0'') + \sum x''^{a''} 2g(e_i''^{a''}, e''^0) + O(x''^2)$$

Inserting (4.6) into the tensor metric we obtain (4.7):

$$\begin{aligned}
\eta_{ij}(x) &= g(e_{ij}, e_{ij}) \\
&= g(e_i'^0 \otimes e_j''^0, e_i'^0 \otimes e_j''^0) \\
&\quad + 2\sum_{a',a''} x^{a'a''} g(\lambda_j^{a''} e_i'^{a'} \otimes e_j''^0 + \lambda_i^{a'} e_i'^0 \otimes e_j''^{a''}, e_i'^0 \otimes e_j''^0) + O(x^2) \\
&= g'(e_i'^0 \otimes e_i'^0) g''(e_j''^0, e_j''^0) \\
&\quad + 2\sum_{a',a''} x^{a'a''} [\lambda_j^{a''} g'(e_i'^{a'}, e_i'^0) g''(e_j''^0, e_j''^0) + \lambda_i^{a'} g'(e_i'^0, e_i'^0) g''(e_j''^{a''}, e_j''^0)] + O(x^2) \\
&= \eta_i'(m_0')\eta_j''(m_0'') + \sum_{a',a''} x^{a'a''} (\lambda_j^{a''} (\partial_{a'}'\eta_i')(m_0')\eta_j''(m_0'') + \lambda_i^{a'} \eta_i'(m_0')(\partial_{a''}''\eta_j'')(m_0''))
\end{aligned}$$

The formula (4.8) then follows by derivating (4.7) w.r.t. $e_{kl} = \sum_{a'a''} \lambda_{a'}^i \lambda_{a''}^j$ where $(\lambda_{a'}^i)$ and $(\lambda_{a''}^j)$ are the inverse matrices of $(\lambda_i^{a'})$ and $(\lambda_j^{a''})$ defined above.

$$\begin{aligned}
(e_{kl}\eta_{ij})(0) &= (\sum_{a'a''} \lambda_{a'}^k \lambda_{a''}^l \partial_{a'a''}\eta_{ij})(0) \\
&= \sum_{a'a''} \lambda_{a'}^k \lambda_{a''}^l (\lambda_j^{a''} (\partial_{a'}'\eta_i')(m_0')\eta_j''(m_0'') + \lambda_i^{a'} \eta_i'(m_0')(\partial_{a''}''\eta_j'')(m_0'')) \\
&= \delta_{jl}(\sum_{a'} \lambda_{a'}^k \partial_{a'}'\eta_i')(m_0')\eta_j''(m_0'') + \delta_{ik}\eta_i'(m_0')(\sum_{a''} \lambda_{a''}^l \partial_{a''}''\eta_j'')(m_0'') \\
&= \delta_{j,l}\eta_{ik}'(m_0')\eta_j''(m_0'') + \delta_{i,k}\eta_i'(m_0')\partial_{a''}''\eta_{jl}''(m_0'')
\end{aligned}$$

Finally, (4.9) can be derived from the expansion of the equation $E = \sum u^{ij} e_{ij}$ with the Euler field E given by Theorem 2.2.2.

$$\begin{aligned}
\sum_{ij} u^{ij}(0) e_{ij}^0 = E(0) &= E'(m_0') \otimes \partial_0'' + \partial_0'' \otimes E''(m_0'') \\
&= (\sum_i u'^i(m_0') e_i'^0) \otimes (\sum_j e_j''^0) + (\sum_i e_i'^0) \otimes (\sum_j u''^j(m_0'') e_j''^0) \\
&= \sum_{ij} (u'^i(m_0') + u''^j(m_0'')) e_{ij}^0
\end{aligned}$$

4.2.3. **Remark.** Notice that since $\partial_0 = \sum_i e_i$ we always have that $\lambda_i^0 = \lambda_j^0 = 1$, so that we retrieve the previous result for the tensor product constructed in the last section

$$e_{ij}|_{\mathrm{Im}\tau} = e_i \otimes e_j$$

up to order two as it should be.

4.3. **Tensor product of special initial conditions.** In the Lemma 4.2.2, we have calculated all of the structures (4.4) necessary to determine the special initial conditions. We find:

$$\begin{aligned} v_{ij,kl} &= (u^{ij} - u^{kl})\frac{\eta_{ij,kl}}{\eta_{kl}} \\ &= (u^i + u^j - u^k - u^l)\frac{\delta_{jl}\eta'_{ik}\eta''_j + \delta_{ik}\eta'_i\eta''_{jl}}{\eta'_k\eta'_l} \\ &= \delta_{jl}(u^i - u^k)\frac{\eta'_{ik}}{\eta'_k} + \delta_{ik}(u^j - u^l)\frac{\eta''_{jl}}{\eta'_l} \\ &= \delta_{jl}v_{ik} + \delta_{ik}v_{jl} \end{aligned} \tag{4.12}$$

Another approach using the Euler field is given by the following observation:

4.3.1. **Remark.** To give the special initial conditions for a tensor product with the choice of tensor metric and the Euler field (2.14), it suffices to determine the operator

$$\mathcal{V} : \mathcal{V}(X) = \nabla_{0,X}(E) - \frac{D}{2}X \tag{4.13}$$

in the tangent space to the base–point $\mathcal{T}_{M,m_0}$. Since $\mathcal{V}$ is an $\mathcal{O}_M$–linear tensor, its value on a vector field X is already determined by $X\mid_{m_0} \in \mathcal{T}_{M,m_0}$, so that, if we are only interested in the operator $\mathcal{V}$ restricted to $\mathcal{T}_{M,m_0}$, we can use any extension of the vector $X\mid_{m_0}$ to a vector field in a neighborhood of m_0. Choosing a flat extension X^f, the formula (4.13) simplifies to

$$\mathcal{V}(X)\mid_{m_0} = ([X^f, E] - \frac{D}{2}X^f)\mid_{m_0}. \tag{4.14}$$

In particular, in the situation of Theorem 2.1.10, we can extend the idempotents $e_{ij}\mid_{m_0}$ to flat vector fields $e^f_{i,j}$ and use the formula (4.14) to calculate the special initial conditions via the operator $\mathcal{V}$ for the semi–simple tensor Euler field. Now it is clear that $(e_{ij})\mid_{m_0} = e'_i\mid_{m'_0} \otimes e''_j\mid_{m''_0}$, since the algebra over m_0 is just the tensor of the algebras at the chosen zeros m'_0 and m''_0. Recalling the form of E given by (2.14), we find for flat X, Y

$$[X \otimes Y, E] = [X, E'] \otimes Y + X \otimes [Y, E''] - dX \otimes Y. \tag{4.15}$$

Thus,

$$\mathcal{V}(e^f_{ij}) = [e^f_{i,j}, E] - \frac{D}{2}e^f_{i,j} = \\ ([e'^f_i, E'] - \frac{D'}{2}e'^f_i) \otimes e''^f_j + e'^f_i \otimes ([e''^f_j, E''] - \frac{D''}{2}e''^f_j). \tag{4.16}$$

Using the explicit formulas of Lemma 4.2.2 or the Remark 4.3.1, we obtain:

4.3.2. **Theorem.** *Let (N', p) and (N'', q) be two germs of semi–simple Frobenius manifolds with tame base–points, Euler fields and flat identities which satisfy $u'^i(p) + u''^j(q) \neq u'^k(p) + u''^l(q)$ for $i \neq k$ and $j \neq l$ and let the corresponding special initial conditions be given by $(\mathcal{V}', \eta')$ and $(\mathcal{V}'', \eta'')$, then the special initial conditions for the Schlesinger equations corresponding to the tensor product with the flat identity and the Euler field of the product chosen as in Theorem 2.1.10 are given by:*

$$\begin{aligned} \eta_{ij} &= \eta'_i\eta''_j \\ v_{ij,kl} &= \delta_{jl}v'_{ik} + \delta_{ik}v''_{jl}. \quad \square \end{aligned} \tag{4.17}$$

4.3.3. **Corollary.** *In the neighborhood of a pair of tame semi–simple base points the tensor product can be locally given in terms of special initial conditions.*

Proof. Since the condition $u'^i(p) + u''^j(q) \neq u'^k(p) + u''^l(q)$ for $i \neq k$ and $j \neq l$ is an open condition we can always find a pair of tame semi–simple points satisfying the equation. Since the tensor product is locally unique up to isomorphism the Corollary follows.

4.3.4. **Remark.** The virtue of the Corollary above (together with the existence theorem of the last section and the Theorem on the existence of an Euler field) is that it is thus possible to consider special initial conditions to identify a tensor product which was originally defined by the tensor product of two germs with nilpotent base–point.

In many examples this is exactly the case. For instance in quantum cohomology as well as in Saito's unfolding spaces [S, M3] and the constructions in [D2, DZh].

4.4. **Example: Special initial conditions for $\mathbb{P}^n \times \mathbb{P}^m$.** Using the Theorem we can calculate the special initial conditions for $\mathbb{P}^n \times \mathbb{P}^m$ using the results of [MM]. Set $\zeta_n = \exp(\frac{2\pi i}{n+1})$.

4.4.1. **Proposition.** *The point $(x^{00}, x^{10}, x^{01}, 0, \dots)$ has canonical coordinates $u_{ij} = x^{00} + \zeta_n^i (n+1) e^{\frac{x^{10}}{n+1}} + \zeta_m^i (m+1) e^{\frac{x^{01}}{m+1}}$*

The special initial conditions at this point corresponding to $H_{quant}(\mathbb{P}^n \times \mathbb{P}^m)$ are given by

$$v_{ij,kl} = -\left(\frac{\zeta_n^{i-k}}{1-\zeta_n^{i-k}}\delta_{jl} + \frac{\zeta_m^{j-l}}{1-\zeta_m^{j-l}}\delta_{ik}\right) \tag{4.18}$$

and

$$\eta_{ij} = \frac{\zeta_n^i \zeta_m^j}{(n+1)(m+1)} e^{-x^{10}\frac{n}{n+1} - x^{01}\frac{m}{m+1}} \tag{4.19}$$

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