# *A Geometric Method in the Study of Subgroups of the Modular Group*

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§1. Introduction. Let $\Gamma \cong PSL_2(\mathbb{Z})$ be the modular group acting on the upper-half plane

$$\mathcal{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$$

in the standard way and let $\mathcal{T}$ be the corresponding well-known tesselation of $\mathcal{H}$ into black and white triangles with angles $\frac{\pi}{2}$, $\frac{\pi}{3}$ and 0. There is a good deal of number-theoretic interest in the study of the subgroups of finite index in $\Gamma$. A useful method of obtaining information about a subgroup is to construct a "good" fundamental domain for it. In principle such a fundamental domain may be constructed out of the tiles of $\mathcal{T}$. See e.g. Gunning [4] ch. 1, Schoneberg, [12] ch. 4 for some pictures of such fundamental polygons. A general reference on fundamental polygons for Fuchsian groups is Lehner [8], ch. IV and VII. L. Keen [5] has justified and made more precise an older method due to Fricke for constructing fundamental polygons for arbitrary finitely generated Fuchsian groups. However if one applies her method to a subgroup of $\Gamma$ the associated fundamental polygon usually would not be a union of the tiles in $\mathcal{T}$. Moreover to apply this method one already needs

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a canonical system of generators for the subgroup. Our present concern is closer to that of Rankin [11] pp. 47 - 69. We consider a special type of polygons, here called "admissible", cf. (2.2) below, and construct an admissible fundamental polygon for each subgroup of finite index in $\Gamma$. Moreover for schematic purposes, to admissible polygons we associate certain s-diagrams which are certain types of finite trees all of whose internal vertices have valence 3. From its s-diagram, one can quickly obtain the geometric invariants of a subgroup, and in fact the classes of s-diagrams with respect to a certain equivalence relation actually classify the conjugacy classes of subgroups.

Let p be an odd prime. If one replaces 3 by p in the definition of an s-diagram then the equivalence classes of these modified s-diagrams would classify the conjugacy classes of subgroups of finite index in the Hecke group $\approx \mathbb{Z}_2 * \mathbb{Z}_p$. In fact, the method generalizes to all finitely generated fuchsian groups of genus zero and one cusp. But since it is most transparent in the case of the modular group, and possibly of wider interest we have restricted to this special case in the paper.

In an earlier paper [7] we had introduced certain "thickened diagrams" for the subgroups of a non-cocompact fuchsian group which also classify their conjugacy classes. The s-diagrams of this paper are obtained essentially by cutting these thickened diagrams for subgroups of $\Gamma$, thinning them, and inserting the relevant information at the cuts. In fact the notion of an s-diagram suggested itself while trying to correlate the thickened diagrams to tesselations.

For other and earlier methods of studying subgroups of the modular group we refer to Atkin and Swinnerton-Dyer [1], Rankin [11], Millington [9], Singerman [13], Stothers [14], Brenner and Lyndon [3] and the references there. (For a more extensive list of references see section F05 of the Reviews in Number Theory, ed. by R.K. Guy, Amer. Math. Soc.). These methods are various combinations of algebraic and geometric techniques and have different advantages. An advantage in the present method appears to be in its possible intuitive appeal and facility in obtaining a minimal system of generators for a subgroup in terms of Möbius transformations.

## §2. Admissible Polygons and s-diagrams

(2.1) First some terminology. All geometric notions in $\mathcal{H}$ refer to hyperbolic geometry. Let $i = \sqrt{-1}$ and $\rho = \exp(\pi i/3)$. There are two $\Gamma$-orbits of the vertices in $\mathcal{J}$. The vertices in $\Gamma i$ (resp. $\Gamma\rho$) will be referred to as 2-vertices (resp. 3-vertices). The set of edges in $\mathcal{J}$ is denoted by $\mathcal{E}$. The subset of edges of finite length is denoted by $\mathcal{E}_f$. It forms a single $\Gamma$-orbit. Among the edges of infinite length there are two $\Gamma$-orbits, namely $\mathcal{E}_2$ and $\mathcal{E}_3$ consisting of those incident with 2-vertices and 3-vertices respectively. The elements of $\mathcal{E}_2$, $\mathcal{E}_3$, $\mathcal{E}_f$ will also be referred to as 2-edges, 3-edges and f-edges respectively. Throughout this paper, unless otherwise stated, a polygon will always mean a connected, simply connected closed subset of $\mathcal{H} \cup \{\mathbb{R} \cup \{\infty\}\}$ with non-empty interior, finite hyperbolic area and bounded by finitely many geodesic arcs. These arcs are called the sides of the polygon.

(2.2) An admissible polygon is a polygon $P$ with the following structure. First

(2.2.1) $$P = P_0 \cup \tau_1 \cup \tau_2 \cup \dots \cup \tau_b, \quad b \geq 0$$

where i) $P_0$ is a polygon with all vertices in $\mathbb{R} \cup \{\infty\}$ and each side of $P_0$ is a union of a pair of 2-edges, ii) Each $\tau_j$ is a triangle. Two of its vertices are in $\mathbb{R} \cup \{\infty\}$ and the side joining them called the diagonal of $\tau_j$ is a union of a pair of 2-edges. The third vertex of $\tau_j$ is a 3-vertex joined to the other two vertices by 3-edges subtending an angle $2\pi/3$, iii) Each $\tau_j$ is attached externally along its diagonal to a side of $P_0$.

Secondly we need to specify the <u>sides</u> of $P$. The sides of $P_0$ not attached to any $\tau_j$ are called the <u>free sides</u> of $P_0$. A side of $P$ is either a 3-edge of some $\tau_j$, or a free side of $P_0$, (in which case it will be called a <u>free side</u> of $P$) or a 2-edge contained in a free side of $P_0$.

Thirdly $P$ comes equipped with a <u>side-pairing</u> satisfying the following rules: i) The 3-edges of each $\tau_j$ are paired, ii) If a free side of $P_0$ is divided into a pair of 2-edges which are considered as sides of $P$ then they are paired, iii) A free side of $P$ is paired to another such free side.

(2.3) Going to the disk-model for $\mathcal{H} \cup \mathbb{R} \cup \{\infty\}$ an admissible polygon will have the shape as shown in figure 1.

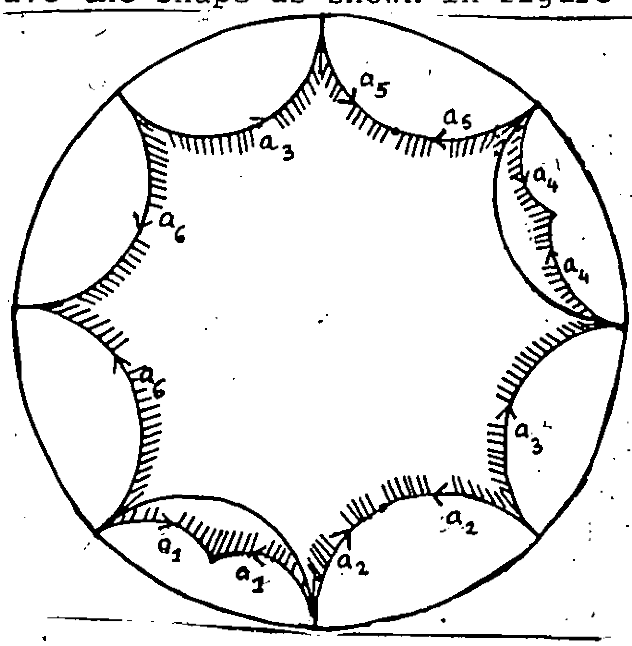


Figure 1

Notice that an admissible polygon is automatically convex. Also if two sides are paired then geometrically they are to be considered as identified by an element of $\Gamma$ (necessarily) in an orientation-reversing way.

(2.4) We now introduce two notions of equivalence among admissible polygons. These equivalence relations are denoted by $\approx$ (strong equivalence) and $\sim$ (weak equivalence). Let $P_1, P_2$ be two admissible polygons. The $\approx$ is easy to describe.

(2.4.1) $P_1 \approx P_2$ iff there exists $\alpha \in \Gamma$ s.t. $\alpha P_1 = P_2$.

Next suppose that two non-adjacent vertices of $P_1$ can be joined by a geodesic $\gamma$ which is a union of a pair of 2-edges, and $\gamma$ separates two paired free sides $\alpha, \beta$ of $P_1$. We can then cut $P_1$ along $\gamma$ and attach the component containing $\alpha$ externally to the second component attaching $\alpha$ along $\beta$, thus obtaining a new polygon $P_3$. In this process the sides $\alpha, \beta$ of $P_1$ have become a single geodesic in the interior of $P_3$, and there has arisen a new pair $\gamma_1, \gamma_2$ of free sides corresponding to the cut along $\gamma$. We now pair $\gamma_1$ with $\gamma_2$ and other side-pairs remain the same as in $P_1$ - thus turning $P_3$ into an admissible polygon. We shall say that $P_3$ is <u>obtained from</u> $P_1$ by an <u>elementary move</u>. Now we are in a position to define the weak equivalence:

(2.4.2) $P_1 \sim P_2$ iff there exists an admissible polygon $P$ such that $P_2 \approx P$ and $P$ is obtained from $P_1$ by a finite sequence of elementary moves.

(2.5) We shall now describe another class of objects which are convenient for schematic purposes. Recall that a vertex in a graph is called <u>terminal</u> if it has valence 1; otherwise it is called <u>internal</u>.

A _subgroup-diagram_, or an s-_diagram_ for short, is a finite tree all of whose internal vertices have valence 3, and the terminal vertices are divided into two subsets the elements of which will be called 2-_vertices_ and 3-_vertices_ respectively together with i) a cyclic order of the edges incident with each internal vertex and ii) an invòlution on the set of 2-vertices. Moreover to avoid degenerate cases we also assume iii) there are at least two vertices, and if there are exactly two then at least one of them is a 3-vertex.

(2.6) We shall picture an s-diagram by a tree considered as embedded in the plane of the paper so that the cyclic order of the edges at an internal vertex agrees with the anticlockwise orientation of the plane. Moreover a 2-vertex (resp. a 3-vertex) will be denoted by a o (resp. ● ), and a pair of distinct 2-vertices paired by the involution will be labelled by the same integer. Of course different pairs will have different labels. See the appendix (A.2) for some pictures together with their relevance to the subgroups of $\Gamma$.

(2.7) We shall now again introduce two notions of equivalence among the s-diagrams, denoted by $\approx$ (strong equivalence) and $\sim$ (weak equivalence). Let $\Sigma_1, \Sigma_2$ be two s-diagrams. First

(2.7.1) $\Sigma_1 \approx \Sigma_2$ iff there exists an isomorphism of underlying trees preserving the extra structure imposed in the definition of an s-diagram.

Next suppose that $v_1, v_2$ are the 2-vertices of $\Sigma_1$ which are paired by the involution and the shortest path $\pi$ joining $v_1$

to $v_2$ has length[⊕] $\geq 3$ so that $\pi$ contains two internal vertices say $w_1$ and $w_2$. Now insert a new vertex $u$ on the edge $(w_1 w_2)$, cut $\Sigma_1$ at $u$ and join the component containing $v_1$ to the other identifying $v_1$ with $v_2$, thus obtaining a new tree, say $\Sigma_3$. We shall delete the vertex (of valence 2) in $\Sigma_3$ obtained by the identification of $v_1$ with $v_2$. So in the process we have lost a pair of 2-vertices, but we have created two new terminal vertices say $u_1$ and $u_2$ corresponding to the cut at $u$. We consider $u_1$, $u_2$ as 2-vertices of $\Sigma_3$ and pair them thus defining an involution on all terminal 2-vertices of $\Sigma_3$ (which agrees with that on all the terminal 2-vertices $\neq \{v_1, v_2\}$ of $\Sigma_1$). This makes $\Sigma_3$ an s-diagram. We say that $\Sigma_3$ is obtained from $\Sigma_1$ by an elementary move, and set

(2.7.2) $\Sigma_1 \sim \Sigma_2$ iff there exists an s-diagram $\Sigma$ such that $\Sigma_2 \cong \Sigma$ and $\Sigma$ is obtained from $\Sigma_1$ by a finite sequence of elementary moves.

(2.8) Proposition. There is a natural 1-1 correspondence between the strong equivalence classes of admissible polygons and those of s-diagrams.

Proof. Let $P$ be an admissible polygon. Let $\Sigma$ be the union of all the f-edges contained in $P$. Let int $P$ and $\partial P$ denote the interior of $P$ and the boundary of $P$ respectively. We make $\Sigma$ into a graph as follows. Its vertices are all the vertices of $\mathcal{T}$ lying in $\partial P$ and all the 3-vertices of $\mathcal{T}$ lying in int $P$. An

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⊕ Each edge in the graph is supposed to be of unit length.

edge of $\Sigma$ is either a union of a pair of f-edges joining two 3-vertices of $\mathcal{T}$ lying in $P$ or an f-edge joining a 3-vertex of $\mathcal{T}$ lying in int $P$ and a 2-vertex of $\mathcal{T}$ lying on $\partial P$. Since $P$ is connected and every point of $P$ can be joined to an f-edge by an arc within a tile it follows that $\Sigma$ is connected. Now it is well-known that the f-edges of $\mathcal{T}$, not counting the vertices of valence 2 as vertices, has a structure of the so-called universal 3-regular tree. It follows that $\Sigma$ is a finite tree all of whose internal vertices have valence 3. The terminal vertices of $\Sigma$ are those lying in $\partial P$. We call a terminal vertex in $\Sigma$ a 2-vertex resp. a 3-vertex if it is such as a vertex of $\mathcal{T}$. The involution on the 2-vertices of $\Sigma$ is induced by the side-pairing of $P$ in an obvious way. This makes $\Sigma$ an s-diagram.

Conversely given an s-diagram $\Sigma$ we embed$^{\oplus}$ it in the f-edges of $\mathcal{T}$ so that the internal vertices and the 3-vertices of $\Sigma$ are mapped into 3-vertices of $\mathcal{T}$ and so that the cyclic order among the edges incident with an internal vertex of $\Sigma$ agrees with that among the edges incident with the corresponding 3-vertex induced by the standard orientation of $\mathcal{H}$; moreover the 2-vertices of $\Sigma$ are mapped into those of $\mathcal{T}$. Let $\bar{\Sigma}$ be the image of $\Sigma$ under the embedding. The involution on the 2-vertices of $\Sigma$ induces one on the terminal 2-vertices of $\bar{\Sigma}$. Let $P$ be the polygon bounded by the 2-edges which are incident to the terminal 2-vertices of $\bar{\Sigma}$ and also the 3-edges which are incident to the terminal 3-vertices of $\bar{\Sigma}$ and make an angle $\pi/3$ with the corresponding f-edges.

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$^{\oplus}$We have used here the non-degeneracy condition iii) in the definition of an s-diagram.

(Since $\bar{\Sigma}$ is finite, P is indeed a polygon ! ). The sides of P are taken to be the 3-edges incident with a terminal 3-vertex of $\bar{\Sigma}$, or the 2-edges incident with a terminal 2-vertex of $\bar{\Sigma}$ fixed by the involution, or the unions of 2-edges incident with a terminal 2-vertex not fixed by the involution. The side-pairing for P is now defined in an obvious way and it is easy to make P into an admissible polygon.

The map $P \to \Sigma$ defined in the first paragraph clearly induces the map on their strong equivalence classes. The second paragraph shows that this induced map is onto. Finally observe that two embeddings of $\Sigma$ into $\mathfrak{I}$ as described above, which agree on one edge, must agree everywhere. This shows that the induced map on the strong equivalence classes is also injective. This finishes the proof.

q.e.d.

(2.9) Proposition. There is a natural 1-1 correspondence between the weak equivalence classes of admissible polygons and those of s-diagrams.

Proof. Consider the map $P \to \Sigma$ defined in (2.8). It is easy to see that corresponding to an elementary move on P there is a canonical elementary move which can be performed on $\Sigma$, and conversely. Further details may be left to the reader.

q.e.d.

## §3. Main Theorem.

(3.1) Theorem. Each subgroup of finite index in $\Gamma$ admits an admissible fundamental polygon. Moreover the conjugacy classes of subgroups of finite index in $\Gamma$ are in a natural 1-1 correspondence with the weak equivalence classes of admissible polygons (and so also in a natural 1-1 correspondence with the weak equivalence classes of s-diagrams by (2.9).)

The proof will be divided into several steps.

(3.2) Let $\Phi$ be a subgroup of finite index in $\Gamma$. First we describe the nature of all the fundamental polygons for $\Phi$ which are made up of the tiles of $\mathfrak{J}$. Let $p: \mathcal{H} \to S_\Phi \overset{\text{def}}{=\!=} \Phi\backslash\mathcal{H}$ be the canonical projection. Since $\Phi$ preserves $\mathfrak{J}$, we have an induced tesselation $\mathfrak{J}_\Phi$ of $S_\Phi$. Using the notation in (2.1) we also write $p(\mathcal{E}) = \mathcal{E}_\Phi$, $p(\mathcal{E}_2) = \mathcal{E}_{2,\Phi}$, $p(\mathcal{E}_3) = \mathcal{E}_{3,\Phi}$ etc. An image by $p$ of a 2-vertex, 2-edge etc. will also be called a 2-vertex, 2-edge etc. in $\mathfrak{J}_\Phi$. Notice however that there are two types of 2-vertices (resp. 3 vertices) in $\mathfrak{J}_\Phi$, namely type 1: those which are incident to a single f-edge and type 2: those which are incident to 2 f-edges (resp. 3 f-edges). Now if $P$ is a fundamental polygon for $\Phi$ made up of the tiles of $\mathfrak{J}$ then $S_\Phi = p(P)$, $p$ is 1-1 on int $P$ and $p$ identifies the sides of $P$ in pairs. Conversely let $A$ be a subset of $\mathcal{E}_\Phi$ and let $P_A = \{S_\Phi$ cut along $A\}$. If $P_A$ is connected and simply connected then developing $P_A$ along the tiles of $\mathfrak{J}$ we obtain a polygon $P$, and some translate of $P$ by an element of $\Gamma$ serves as a fundamental polygon for $\Phi$.

(3.3) Starting with $S_\Phi$ we now construct an <u>admissible</u> polygon. Let $T$ be a maximal tree in $\mathcal{E}_{f,\Phi}$. Since $\mathcal{E}_f$ is connected so is $\mathcal{E}_{f,\Phi}$. Hence $T$ contains all vertices of $\mathcal{T}_\Phi$. Let $A \subseteq \mathcal{E}_\Phi$ be the subset of 2- resp. 3-edges incident with the 2- resp. 3-vertices of type 1 which are the terminal vertices in $T$, together with the pairs of 2-edges incident with the 2-vertices of type 2 which are also terminal in $T$. Let $P_A$ be $S$ cut along $A$.

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We claim that $P_A$ is connected and simply connected. Indeed since each tile in $\mathcal{T}$ can be continuously retracted onto its f-edge we easily see that $S_\Phi$ can be retracted onto $\mathcal{E}_{f,\Phi}$. To see the effect of cutting, it is convenient to consider $S_\Phi - A = P_A - \partial P_A \stackrel{def}{=} \text{int } P_A$. It suffices to show that int $P_A$ is connected and simply connected. Since int $P_A$ = $S_\Phi - A$ continuously retracts into $\mathcal{E}_\Phi$ - (the terminal vertices in T) $\stackrel{\text{def}}{=\!=}$ $\eta_\Phi$ it suffices to show that $\eta_\Phi$ is simply connected. Now an edge of $\mathcal{E}_{f,\Phi}$ not in T joins a 2-vertex (necessarily) of type 2 which is terminal in T to some 3-vertex. This 2-vertex is removed while passing to $\eta_\Phi$ — so one circuit is broken. Since T is chosen to be a maximal tree in $\mathcal{E}_{f,\Phi}$ we see that T is a deformation retract of $\eta_\Phi$ so $\eta_\Phi$ is connected and simply connected. So $P_A$ also is connected and simply connected.

Since $P_A$ is simply connected we may develop it into $\mathcal{H}$. In fact once one tile in $P_A$ is developed onto a tile of $\mathcal{T}$, $P_A$ develops uniquely into a polygon P.

(3.4) We now show that P can be turned into an admissible polygon in the sense of (2.2). One has to consider various cases depending on the nature of the terminal vertices in T. In the following v denotes a terminal vertex in T.

<u>case 1</u>. <u>v is a 2-vertex of type 1</u>: There is a unique 2-edge incident to v in $S_\Phi$. Corresponding to it we obtain a pair of 2-edges in P which form a single complete geodesic. These form two sides of P which are paired.

<u>case 2</u>. <u>v is a 2-vertex of type 2</u>: There is a pair of 2-edges incident to v in $S_\Phi$. These form a single complete geodesic. Corresponding to it there are two free sides of P which in turn are paired.

<u>case 3</u>. <u>v is a 3-vertex of type 1</u>: There is a unique 3-edge incident to v in $S_\Phi$. Correspondingly we obtain a pair of 3-edges

in P making an angle $\frac{2\pi}{3}$ at a 3-vertex. These 3-edges are two sides of P which are paired.

Now let w be the unique 2-vertex joined to v by an edge in T.

<u>subcase 1</u>. w is to type 1: Then v,w are the only vertices in T, hence also in $\mathfrak{J}_\Phi$ and so $T = \mathcal{E}_{\mathfrak{f},\Phi}$. As in case 1 corresponding to the 2-edge incident to w there is a side of P which is a complete geodesic. So P in fact is a triangle with angles 0,0 and $\frac{2\pi}{3}$ . Of course this happens precisely when $\Phi = \Gamma$.

<u>subcase 2</u>. w is to type 2: Then the pair of 2-edges incident to w in $S_\Phi$ form a complete geodesic say $\gamma_v$. Clearly $S_\Phi - \gamma_v$ consists of two components for the vertex v cannot be connected to any other vertex without intersecting $\gamma_v$. So corresponding to the component of $S_\Phi$ cut along $\gamma_v$ which contains v, we again have a triangle $\tau_v$ with angles 0,0, $\frac{2\pi}{3}$ in P which is attached to $P-\tau_v$ along its diagonal.

If we cut $S_\Phi$ along all $\gamma_v$'s for each terminal 3-vertex of type 1 in T, correspondingly we obtain a decomposition of P as

$$P = P_0 \cup \tau_1 \cup \tau_2 \cup \dots \cup \tau_b$$

(b = the number of such v's) which has all the properties mentioned in ( 2.2).

From the remarks in (3.2) we have now shown that every subgroup of finite index admits an admissible fundamental polygon.

(3.5) Conversely let $P$ be an admissible polygon. If the sides are paired there exists a unique element in $\Gamma$ carrying one side onto the other. Identifying the paired sides via such maps we obtain a space $S$ which has a canonical tesselation, and so has a canonical projection $p: S \to \Gamma\backslash\mathcal{H}$ which is an isometry on each tile. It follows that there exists a subgroup $\Phi$ of $\Gamma$ necessarily of finite index and determined upto conjugacy such that there is a tile-preserving isometry of $\Phi\backslash\mathcal{H}$ onto $S$.

(3.6) If $P$ is an admissible polygon, and $P_1$ is obtained from $P$ by an elementary move, and $S, S_1$ the spaces obtained as in (3.5) by identifying the paired sides, clearly there is a tile-preserving isometry of $S$ onto $S_1$. So $S, S_1$ define the same conjugacy class of subgroups. In particular there is a surjective map of the weak equivalence classes of admissible polygons onto the conjugacy classes of subgroups of finite index in $\Gamma$.

(3.7) Finally we have to show that the map described at the end of (3.6) is 1-1. For this purpose it is perhaps best to interpret it in terms of the s-diagrams. Given an s-diagram $\Sigma$ let $\sigma$ be the graph obtained by identifying each pair of 2-vertices paired by the involution to a single vertex. Then $\sigma$ has the following structure: i) all internal vertices of $\sigma$ have valence 3 and there is a cyclic order prescribed among the edges incident with each internal vertex, ii) the terminal vertices of $\sigma$ are divided into two subsets, one of "2-vertices", and the other of "3-vertices".

We call such $\sigma$ a reduced s-diagram. Conversely $\Sigma$ is obtained from $\sigma$ by cutting a minimum number of edges so that $\Sigma$ is a (connected) tree, and pairing the two new vertices obtained from each cut. Equivalence of the graphs satisfying i) and ii) noted above is defined in the obvious way. The injectivity of the map described at the end of (3.6) now amounts to an easily verified fact that two s-diagrams are weakly equivalent iff the associated reduced s-diagrams are equivalent. This finishes the proof of the theorem.

## §4. Two properties of the admissible polygons

(4.1) Let $\Phi$ be a subgroup of finite index in $\Gamma$ and P an admissible polygon associated to it as in §2. Suppose that the boundary of P contains b 3-vertices and 2r+a+2b sides so that the pairing on precisely a sides is made by dividing them into two halves. It readily follows from surface topology that $\Phi$ is $\approx$ a free product of a free group of rank r, and a resp. b copies of $\mathbb{Z}_2$ resp. $\mathbb{Z}_3$. In particular by Grushko's theorem the minimum number of generators for $\Phi$ is r+a+b. Now if Q is any fundamental polygon for $\Phi$ such that $\{\eta Q\}$, $\eta \in \Phi$ forms a locally finite tesselation of $\mathcal{H}$, and Q has 2s sides then $\Phi$ is generated by s side-pairing transformations, cf. [2] theorem 9.2.7. It follows that $s \geq r+a+b$. In other words we have shown that

(4.2) Proposition. An admissible polygon associated to a conjugacy class of a subgroup $\Phi$ of finite index has least number of sides among all fundamental polygons for $\Phi$ whose $\Phi$-translates give a locally finite tesselation of $\mathcal{H}$. Also $\Phi$ is isomorphic to the free product of the cyclic groups generated by the side-pairing transformations.

(4.3) The second property of admissible polygons which we wish to point out is a curious connection among the vertices of admissible polygons and partial Farey sequences. By a partial Farey sequence we mean a finite sequence

$$(4.3.1) \quad \frac{0}{1} = \frac{a_0}{b_0} < \frac{a_1}{b_1} < \frac{a_2}{b_2} < \dots < \frac{a_n}{b_n} = \frac{1}{1}$$

where $\frac{a_i}{b_i}$ are reduced fractions and $a_0 = 0$, $b_0 = a_n = b_n = 1$, such that

(4.3.2) $|a_i b_{i+1} - a_{i+1} b_i| = 1$, $i = 0,1,\ldots,n-1$.

(4.4) Proposition. An admissible polygon with at least three vertices lying in $\mathbb{R} \cup \{\infty\}$ is strongly equivalent to one with $\infty$ as a vertex so that the remaining vertices which lie in $\mathbb{R} \cup \{\infty\}$ form a partial Farey sequence.

Proof. First note that if $\gamma$ is any hyperbolic geodesic which is a union of a pair of 2-edges of $\mathfrak{J}$ then either one of its end-points is $\infty$ (and the other end-point an integer) or else its end-points are of the form $\{\frac{a}{c}, \frac{b}{d}\}$ with $|ad-bc| = 1$.

Let $P$ be an admissible polygon. Decompose $P$ in the form

$$P = P_0 \cup \tau_1 \cup \tau_2 \cup \ldots \cup \tau_b, \quad b \geq 0$$

as in (2.2). The vertices of $P$ which lie in $\mathbb{R} \cup \{\infty\}$ are exactly the vertices of $P_0$. Let $\Sigma_0$ be the union of the f-edges lying in $P_0$. We consider $\Sigma_0$ as an s-diagram as explained in (2.8). All terminal vertices of $\Sigma_0$ are 2-vertices. Notice that at least two terminal vertices of $\Sigma_0$, say $u$ and $v$, are connected to the same internal vertex each by a single edge. (If it were not so, remove all the terminal vertices and the corresponding incident edges from $\Sigma_0$. In the remaining graph each vertex would have valence $\geq 2$. But then $\Sigma_0$ would not be a tree - a contradiction.) Now we may choose $\sigma$ in $\Gamma$ such that $\sigma(u) = i$, $\sigma(v) = i+1$. Evidently $\sigma P_0$ lies within the strip $0 \leq x \leq 1$ and its boundary contains the lines $x = 0$ and $x = 1$. Moreover it is easy to see from the earlier remarks that $\sigma P \cong P$ has the stated properties.

q.e.d.

## §5. Reading the geometric invariants from an s-diagram

(5.1) Let $\Sigma$ be an s-diagram and $\Phi$ a representative of the associated conjugacy class. The geometric invariants of $\Phi$ include

(5.1.1) $d$ = the degree of the branched covering $\Phi\backslash\mathcal{H} \to \Gamma\backslash\mathcal{H}$

(5.1.2) $e_2 = \#$ {branch points of $\Phi\backslash\mathcal{H}$ with branching index 2}

(5.1.3) $e_3 = \#$ {branch points of $\Phi\backslash\mathcal{H}$ with branching index 3}

(5.1.4) $g$ = the genus of $\Phi\backslash\mathcal{H}$

(5.1.5) $t = \#$ {cusps of $\Phi\backslash\mathcal{H}$ }.

(5.1.6) $w_k$ = cusp-width at the k-th cusp, $k = 1,2,\dots,t$.

We now briefly show how to read these invariants from $\Sigma$. We set also

(5.1.7) $r = 2g+t-1$.

The fundamental group of $\Phi\backslash\mathcal{H}$ is a free group, and its rank is $r$.

(5.2) Let $a$ (resp. $2\alpha$) be the number of 2-vertices of $\Sigma$ which are fixed by the involution, (resp. not fixed by the involution). Let $\beta$ be the number of internal vertices of $\Sigma$, and $b$ the number of 3-vertices of $\Sigma$. If $P$ is the associated admissible polygon it has $6\beta+2b$ tiles. It follows that

(5.2.1) $d = 3\beta+b$.

Also

(5.2.2) $e_2 = a$.

(5.2.3) $e_3 = b$.

Let $S$ be the space obtained from $P$ by the identification of paired sides. It is clear from surface topology that $\pi_1(S)$ is a free group of rank $\alpha$. So

(5.2.4) $r = \alpha$.

So by (5.2.1) - (5.2.4) and (5.1.7)

(5.5.2) $d = 3\beta+b = 3e_2+4e_3+12g+6t-6$

which is a form of the Riemann-Hurwitz formula.

It is more subtle to compute $t$ (or $g$) and $w_k$, but one can easily formulate the rules from the associated $P$.

(5.3) The terminal vertices of $\Sigma$ have a canonical cyclic order. (Imagine a traveller walking very close to $\Sigma$ but always keeping it to the left). Let $v_0, v_1, \ldots, v_{s-1}$, $(s = a+2\alpha+b)$ be the terminal vertices of $\Sigma$ in a cyclic order. Let $\pi_i$ be the shortest path of edges from $v_i$ to $v_{i+1}$ ($i$ counted mod $s$). Consider the following equivalence relation on the set $\{\pi_i\}$. If $v_i$ is either a 3-vertex or a 2-vertex fixed by the involution we let $\pi_{i-1} \sim \pi_i$. If $v_i$ is paired to $v_j$ by the involution and $i \neq j$ then we let $\pi_{i-1} \sim \pi_{j+1}$, $\pi_{i+1} \sim \pi_{j-1}$. Then

(5.3.1) $t = \#\{\text{equivalence classes in } \{\pi_i\}\}$.

Then $g$ can be read from (5.3.1) and (5.2.4), thanks to (5.1.7), namely $g = \frac{1}{2}(\alpha+1-t)$.

(5.4) We attach the weight $\frac{1}{2}$ (resp. 1) to an edge of $\Sigma$ if it is incident with a 2-vertex (resp. otherwise). To each $\pi_i$ we attach the weight $w(\pi_i)$ = the sum of the weights of the edges in $\pi_i$. Finally the width of a cusp is the sum of $w(\pi_i)$ where $\pi_i$ runs over the equivalence class defining the cusp.

(5.5) The sum of vertex-valences in any graph equals twice the number of edges. For $\Sigma$ this sum is $3\beta+a+2\alpha+b$. Also $\Sigma$ has $\beta+a+2\alpha+b$ vertices. Since $\Sigma$ is a tree we have $\#$ vertices - $\#$ edges = 1, which allows one to solve for $\beta$ namely,

(5.5.1) $\beta = a+b+2\alpha-2$.

§ Appendix

(A.1) In this appendix we reprove some known results by the methods of this paper. We first give a new proof of Millington's theorem cf. [9] based on a direct construction of an admissible polygon. Millington's proof used permutations. In [7] we gave another proof based on thickened diagrams, which also extends to a situation not covered by this paper. The following proof however is perhaps the simplest.

Theorem. (Millington). Let $a \geq 0$, $b \geq 0$, $g \geq 0$, $t \geq 1$ be integers s.t. $d \stackrel{\text{def}}{=\!=} 3a+4b+12g+6t-12 \geq 1$. Then there exists a subgroup $\Phi$ in $\Gamma$ of index d, genus g, # cusps = t, # the conjugacy classes of elliptics of order 2 (resp. 3) = a (resp. b).

Proof. The case $d = 1$ is trivial so assume $d \geq 2$.

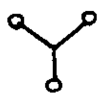
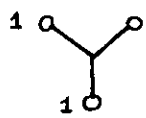
It is easy to construct a polygon $P_0$ with $s \stackrel{\text{def}}{=\!=} 2(2g+t-1)+a+b$ sides, say $\lambda_1,\ldots,\lambda_s$ in cyclic order, so that all angles are zero and each side is a union of a pair of 2-edges. The condition $d \geq 2$ implies $s \geq 3$. Divide each of $\lambda_1,\ldots,\lambda_a$ at the 2-vertex it contains and pair the two halves. Attach externally a triangle $\tau_j$, with angles $0, 0, \frac{2\pi}{3}$ along its diagonal to $\lambda_j$, $a+1 \leq j \leq a+b$ so that the other two sides of $\tau_j$ are 3-edges meeting at a 3-vertex and making an angle $2\pi/3$ and these two sides are paired off. Now pair off the sides in the next consecutive t-1 pairs. There still remain 4g sides, which are to be paired off in the well-known $a_1 b_1 a_1^{-1} b_1^{-1} \ldots$ fashion. If either of a,b,g,t-1 are zero the corresponding identifications are absent. The polygon so obtained is admissible and it is easy to see that the corresponding subgroup has the required invariants.

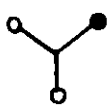
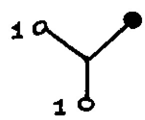
q.e.d.

(A.2) <u>Conjugacy classes of subgroups of index</u> $\leq 6$ : In view of (5.2.1) we have only to list the s-diagrams with $\beta$ internal vertices and $b$ black vertices with $3\beta+b \leq 6$, and check for the possible weak equivalences. The list is easily compiled by inspection:

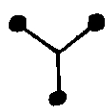
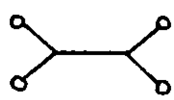
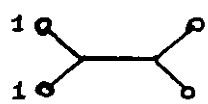
(A.2.1) $d = 1$

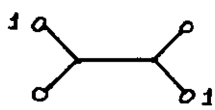
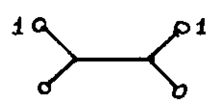
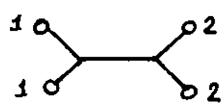
(A.2.2) $d = 2$

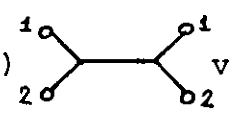
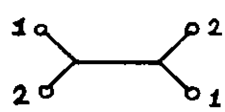
(A.2.3) $d = 3$ i)  ii) 

(A.2.4) $d = 4$ i)  ii) 

(A.2.5) $d = 5$

(A.2.6) $d = 6$ i)  ii)  iii) 

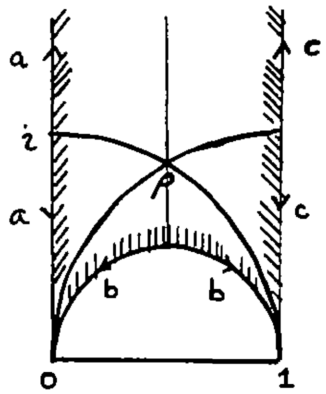
iv)  v)  vi) 

vii)  viii) 

It is easy to check that there are no further weak equivalences among these s-diagrams.

(A.3) Perhaps an advantage of the method developed in this paper over the previously available methods is the facility with which one can construct subgroups of finite index minimally generated by an explicit set of Möbius transformations. We illustrate this point on subgroups of index 3.

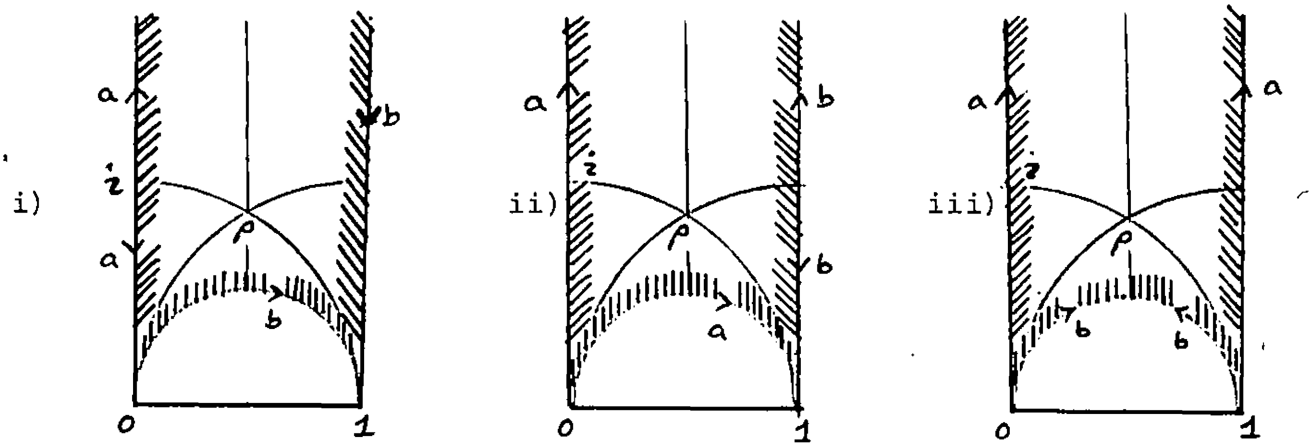
An admissible polygon corresponding to (A.2.3) i) is




The $\mathbb{Z}_3$-symmetry of the s-diagram, and of the polygon shows that there is a unique normal subgroup corresponding to this s-diagram. (cf. also the comments below). This subgroup is

$$\langle z \to -\frac{1}{z}, \quad z \to \frac{z-1}{2z-1}, \quad z \to \frac{z-2}{z-1} \rangle$$

Now consider the case (A.2.3) ii). The s-diagram clearly shows that the corresponding subgroup is not normal. Moreover fix an f-edge $e_0$ in $\mathcal{T}$. If $\Sigma$ is any s-diagram then for each edge $e$ in $\Sigma$ there are at most two embeddings of $\Sigma$ in $\mathcal{E}_f$ so that the image of $e$ contains $e_0$. Clearly these embeddings (as $e$ varies over all edges of $\Sigma$) determine <u>all</u> the subgroups of $\Gamma$ in the corresponding conjugacy class. Applying these considerations to the case in hand we see that there are three subgroups in the conjugacy class corresponding to this s-diagram, and their admissible fundamental polygons are




The corresponding subgroups are

i) $\langle z \to -\frac{1}{z}\, , \quad z \to \frac{2z-1}{z} \rangle = \Gamma_\theta$

ii) $\langle z \to \frac{-z}{z-1}\, , \quad z \to \frac{z-2}{z-1} \rangle = \Gamma^0(2)$

iii) $\langle z \to z+1\, , \quad z \to \frac{z-1}{2z-1} \rangle = \Gamma_0(2)$.

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