A GEOMETRIC THEORY FOR SEMILINEAR ALMOST–PERIODIC PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS ON $\mathbb{R}^N$

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A.5 Zum Gradienten

Wir zeigen: Sei $U \subseteq \mathbf{R}^n$ offen und zu je zwei Punkten P und Q gebe es einen Weg $c : [a,b] \to U$ mit $c(a) = P$ und $c(b) = Q$. Weiter seien $f_1, f_2 : U \to \mathbf{R}$ zwei Funktionen mit $\operatorname{grad} f_1 = \operatorname{grad} f_2$. Dann gibt es ein $k \in \mathbf{R}$ mit $f_1 = f_2 + k$.

Beweis. Wir definieren eine Funktion $h : U \to \mathbf{R}$ durch $h := f_1 - f_2$. Dann gilt wegen der Voraussetzung $\operatorname{grad} h = 0$. Zu zeigen ist, das $h \equiv k$ für ein $k \in \mathbf{R}$ gilt. Dazu betrachten wir zwei beliebige Punkte $P, Q \in U$ und einen Weg $c : [a,b] \to U$ in U zwischen P und Q. Die Kettenregel liefert wieder

$$(h \circ c)'(t) = \langle \operatorname{grad} h(c(t)), c'(t) \rangle.$$

Wegen $\operatorname{grad} h = 0$ ist nun $\langle \operatorname{grad} h(c(t)), c'(t) \rangle = 0$, also $h \circ c$ konstant. Daraus folgt, daß $h(P) = h(c(a)) = h(c(b)) = h(Q)$ gilt. Also ist für beliebige Punkte $P, Q \in U$ gezeigt, daß $h(P) = h(Q)$ gilt, also ist h konstant.

A.6 Stammfunktionen zu Vektorfeldern

Es sei U eine Teilmenge des $\mathbf{R}^n$ und $F : U \to \mathbf{R}^n$ ein gegebenes Vektorfeld. Eine Funktion $\phi : U \to \mathbf{R}$ mit $\operatorname{grad} \phi = F$ heißt *Stammfunktion* zu F.

Uns interessiert nun, wann es solch eine Stammfunktion gibt. Dazu betrachten wir zunächst folgenden Spezialfall:

- Es sei $n = 2$ und F gegeben durch die Funktionen $f, g : U \to \mathbf{R}$. Nehmen wir an, es gebe eine Funktion ϕ mit $\operatorname{grad} \phi = F$, also $F = \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}\right)$. Dies bedeutet, daß gerade $f = \frac{\partial \phi}{\partial x_1}$ und $g = \frac{\partial \phi}{\partial x_2}$ ist. Dann ist aber $\frac{\partial f}{\partial x_2} = \frac{\partial^2 \phi}{\partial x_1 \partial x_2}$ und $\frac{\partial g}{\partial x_1} = \frac{\partial^2 \phi}{\partial x_2 \partial x_1}$. Wenn nun ϕ von der Klasse C^1 ist, dann ist $\frac{\partial^2 \phi}{\partial x_1 \partial x_2} = \frac{\partial^2 \phi}{\partial x_2 \partial x_1}$, also gilt dann $\frac{\partial f}{\partial x_2} = \frac{\partial g}{\partial x_1}$.

Analog zeigt man allgemein: Wenn es ein $\phi \in C^1$ mit $F = \operatorname{grad} \phi$ gibt, dann gilt $\frac{\partial f_j}{\partial x_i} = \frac{\partial f_i}{\partial x_j}$ für alle i, j.

Wir können uns nun fragen: Wenn umgekehrt $\frac{\partial f_j}{\partial x_i} = \frac{\partial f_i}{\partial x_j}$ gilt, gibt es dann eine Stammfunktion ?

Die Antwort liefert der folgende

Satz. *Es sei U ein Rechteck im $\mathbf{R}^n$, d.h ein kartesisches Produkt von offenen Intervallen in $\mathbf{R}$. Weiter sei* $F = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} : U \to \mathbf{R}^n$ *mit $f_i : U \to \mathbf{R}$ eine differenzierbare Funktion mit $\frac{\partial f_j}{\partial x_i} = \frac{\partial f_i}{\partial x_j}$. Dann gibt es eine Stammfunktion $\phi : U \to \mathbf{R}$ mit $F = \operatorname{grad} \phi$.*

Beweis. Wir führen den Beweis für den Fall $n = 2$ mit $f_1 = f$ und $f_2 = g$. Im allgemeinen Fall schließt man ganz analog.

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ABSTRACT

In this short expository article we review various applications of some geometric methods which have been recently devised to investigate the long time behaviour of classical solutions to certain semilinear almost–periodic reaction–diffusion equations on $\mathbb{R}^N$. As a consequence, we also show how to construct almost–periodic attractors for such equations and how to investigate their stability properties. The class of problems which we analyse here contains in particular well known equations of population genetics.

1. Introduction and Outline

In this expository article we discuss various applications of some geometric methods which have been recently devised to investigate the long time behaviour of classical solutions to semilinear parabolic Neumann boundary value problems of the form

$$\left\{\begin{array}{lll} u_t(x,t) = \Delta u(x,t) + s(t)g(u(x,t)) & , (x,t) \in \Omega\times\mathbb{R}^+ \\ \mathrm{Ran}(u) \subseteq (u_0,u_1) & \\ \frac{\partial u}{\partial \underline{n}}(x,t) = 0 & , (x,t) \in \partial\Omega\times\mathbb{R}^+ \end{array}\right\} \qquad (1.1)$$

In Eqs. (1.1) Ω denotes an open bounded connected subset of $\mathbb{R}^N$ with compact closure $\overline{\Omega}$, smooth boundary $\partial\Omega$ and $N \in [2,\infty) \cap \mathbb{N}^+$, while Δ stands for

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Laplace's operator in the x–variables. Furthermore $s : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is the restriction to $\mathbb{R}^+$ of a Bohr almost–periodic function on $\mathbb{R}$ which we shall also denote by s, while $g : \mathbb{R} \longrightarrow \mathbb{R}$ is sufficiently smooth and possesses two zeroes u_0 and u_1 such that $g(u) > 0$ for every $u \in (u_0,u_1)$ and $g'(u_0) > 0$, $g'(u_1) < 0$. Finally, $\mathrm{Ran}(u)$ denotes the range of u and $\mathbf{n}$ stands for the normalised outer normal vector to $\partial\Omega$.

Problems of the form (1.1) occur in various fields of sciences, such as the theory of nerve pulse propagation and population genetics [1, 2, 3, 4]. In the latter case, eqs. (1.1) model for instance the space–time evolution of the fraction u of one of two alleles in the population of a migrating diploid species located in Ω, when the so–called selection function s takes almost–periodic seasonal variations into account [5, 6]. It is then natural to ask whether there exist conditions on the function s, and perhaps additional restrictions concerning the nonlinearity g, such that every classical solution $(x,t) \longrightarrow u(x,t)$ to Problem (1.1) which exists globally in time stabilizes toward a stable almost–periodic attractor as $t \longrightarrow \infty$.

It is the purpose of this article to show that this is indeed possible, upon using the geometric theory developed in [7, 8] for the analysis of some hyperbolic problems, and further adapted and refined in [9, 10] within the realm ot nonautonomous parabolic equations. In Section 2 we assume that the primitive of s is not almost–periodic and moreover that the time average of s satisfies $\mu_B(s) < 0$ (resp. $\mu_B(s) > 0$) .Under further restrictions on s and g and upon combining our geometric arguments with the parabolic maximum principle, we can then prove that every classical solution to Problem (1.1) which exists globally in time converges to u_0 (resp. u_1) exponentially rapidly as $t \longrightarrow \infty$, with the rate of decay $r_{u_0} = g'(u_0)\mu_B(s) < 0$ (resp. $r_{u_1} = g'(u_1)\mu_B(s) < 0$). In this case, we can thus conclude that u_0 (resp. u_1) is an exponentially stable global attractor, and moreover that the stabilization phenomenon of the classical solutions of (1.1) is primarily governed by the reaction process in Eq. (1.1). This is in sharp contrast to the results of Section 3, in which we review the situation where the primitive of s is itself almost–periodic. In this case, we can prove that for every classical solution u to (1.1) which exists globally in time, there exists a non constant almost–periodic attractor $\hat{u}$, thereby neither equal to u_0 nor equal to u_1, which captures u in an appropriate topology as $t \longrightarrow \infty$. For certain particular solutions to Problem (1.1), we can moreover prove that the latter stabilization process also takes place exponentially rapidly, but with a rate of decay now determined by the largest negative eigenvalue of Laplace's operator. We can thus conclude that in those particular situations the stabilization phenomenon of the classical solutions to (1.1) is primarily governed by the diffusion process. Finally, Section 4 is devoted to the formulation of an open problem.

We should like to emphasize the fact that we have deliberately avoided too technical a presentation of our main results. The interested reader will therefore only find outlines or very brief sketches of their proof in these notes. More complete details were given to the participants in this school during the lectures. Those details can be found in references [9, 10, 11].

2. The Two Equilibria u_0 and u_1 as Global Exponential Attractors: The Role of the Reaction Process.

Consider Problem (1.1); the following hypotheses concerning g will be repeatedly used in the sequel:

(G_1) We have $g \in \mathscr{C}^{(5)}(\mathbb{R},\mathbb{R})$ and there exist u_0, $u_1 \in \mathbb{R}$ such that $g(u_0) = g(u_1) = 0$, $g'(u_0) > 0$, $g'(u_1) < 0$ and $g(u) > 0$ for every $u \in (u_0,u_1)$.

(G_2) If G denotes any primitive of $u \longrightarrow \frac{1}{g(u)}$ on the open interval (u_0,u_1), then $\lim_{u \to u_0} G(u) = -\infty$ and $\lim_{u \to u_1} G(u) = +\infty$.

Now consider the selection function s in (1.1) and assume that $s : \mathbb{R} \longrightarrow \mathbb{R}$ be almost-periodic [12, 13, 14, 15]. Write

$$\mu_B(s) = \lim_{\ell \to \infty} \ell^{-1} \int_0^\ell d\xi s(\xi)$$

for its time-average; in this section we shall assume that the following two hypotheses hold:

(S_1) We have $\mu_B(s) \neq 0$ and $t \longrightarrow \int_0^t d\xi \hat{s}(\xi) = 0(1)$ as $|t| \longrightarrow \infty$, where $\hat{s} = s - \mu_B(s)$.

(S_2) The restriction of s on $\mathbb{R}^+$ is Hölder continuous.

Remark. If s, $\hat{s}$ are periodic, then it follows immediately that $t \longrightarrow \int_0^t d\xi \hat{s}(\xi)$ remains bounded on $\mathbb{R}$ since $\mu_B(\hat{s}) = 0$. However, this is not any longer automatically true in the general almost-periodic case; for instance,

$\hat{s}(t) = \sum_{k=1}^{\infty} k^{-2}\exp[ik^{-2}t]$ has a zero time average but its primitive is unbounded. Finally, we proceed to give the definition of classical solution which we shall use throughout this article. Let $[N/2]$ be the integer part of $N/2$; in the remaining part of this paper we shall assume that Ω has a $\mathscr{C}^{5+[N/2]}$–boundary in the sense of [16], in such a way that Ω lies only on one side of its boundary, and that it satisfies the interior ball condition for every $x \in \partial\Omega$. We denote by $\mathscr{C}^{2,1}(\Omega \times \mathbb{R}^+,\mathbb{R})$ the set consisting of all functions $z \in \mathscr{C}(\Omega \times \mathbb{R}^+,\mathbb{R})$ such that $(x,t) \longrightarrow \partial_t^{\gamma} D^{\alpha} z(x,t) \in \mathscr{C}(\Omega \times \mathbb{R}^+,\mathbb{R})$ for all $\alpha = (\alpha_1, \dots ,\alpha_N) \in \mathbb{N}^N$, $\gamma \in \mathbb{N}$, satisfying $\sum_{j=1}^{N} \alpha_j + 2\gamma \leq 2$. In a similar way we define $\mathscr{C}^{1,0}(\overline{\Omega} \times \mathbb{R}^+,\mathbb{R})$ as the set consisting of all $z \in \mathscr{C}(\overline{\Omega} \times \mathbb{R}^+,\mathbb{R})$ with the property that $(x,t) \longrightarrow D^{\alpha} z(x,t) \in \mathscr{C}(\overline{\Omega} \times \mathbb{R}^+,\mathbb{R})$ for all $\alpha \in \mathbb{N}^N$ such that $\sum_{j=1}^{N} \alpha_j \leq 1$. Now fix $p \in (N,\infty)$; in the simplest case we then have the following

Definition 2.1. A function $u \in \mathscr{C}^{2,1}(\Omega\times\mathbb{R}^+,\mathbb{R}) \cap \mathscr{C}(\overline{\Omega}\times\mathbb{R}_0^+,\mathbb{R}) \cap \mathscr{C}^{1,0}(\overline{\Omega}\times\mathbb{R}^+,\mathbb{R})$ is said to be a <u>classical solution to Problem (1.1)</u> if the following conditions hold:

(C_1) There exists $c \in L^p(\Omega,\mathbb{R})$ such that $|u(x,t) - u(x,t')| \leq c(x)|t - t'|$ for every $x \in \Omega$ and every $t,t' \in \mathbb{R}^+$.

(C_2) $x \longrightarrow u(x,t) \in \mathscr{C}^{(2)}(\overline{\Omega},\mathbb{R})$ for every $t \in \mathbb{R}^+$.

(C_3) $(x,t) \longrightarrow u_t(x,t) \in \mathscr{C}(\overline{\Omega} \times \mathbb{R}^+,\mathbb{R})$ and $t \longrightarrow u_t(x,t) \in \mathscr{C}(\mathbb{R}^+,\mathbb{R})$ uniformly in x .

(C_4) u satisfies relations (1.1) identically.

The main result of this section is then the following

Theorem 2.1. Consider Problem (1.1) where g satisfies hypotheses (G_1) and (G_2); assume moreover that s satisfies hypotheses (S_1) and (S_2). Set $r_{u_0} = g'(u_0)\mu_B(s)$, $r_{u_1} = g'(u_1)\mu_B(s)$ and let u be <u>any</u> classical solution to Problem (1.1) in the sense of Definition 2.1. Then there exists $\varepsilon_0 \in (0,\infty)$,

$t_{\varepsilon_0} \in (0,\infty)$, a positive constant c depending only on N , p and the geometry of Ω , such that the following statements hold:

(1) If $\mu_B(s) < 0$, then the exponential decay estimates

$$\sup_{x\in\overline{\Omega}} |u(x,t) - u_0| \leq c\,\varepsilon_0 \exp[r_{u_0}(t - t_{\varepsilon_0})] \tag{2.1}$$

$$\sup_{x\in\overline{\Omega}} |\nabla u(x,t)| \leq c\,\varepsilon_0 \exp[r_{u_0}(t - t_{\varepsilon_0})] \tag{2.2}$$

$$\sup_{\substack{x,y\in\Omega \\ x\neq y}} |x-y|^{-\beta} |u(x,t) - u(y,t)| \leq c\varepsilon_0 \exp[r_{u_0}(t-t_{\varepsilon_0})] \tag{2.3}$$

$$\sup_{\substack{x,y\in\Omega \\ x\neq y}} |x-y|^{-\beta} |\nabla u(x,t)-\nabla u(y,t)| \leq c\varepsilon_0 \exp[r_{u_0}(t-t_{\varepsilon_0})] \tag{2.4}$$

hold for every $t \in [t_{\varepsilon_0},\infty)$ and every $\beta \in (0,1 - p^{-1}N]$.

(2) If $\mu_B(s) > 0$, a completely similar statement holds provided that we replace u_0 by u_1 everywhere in relations (2.1) – (2.4).

Remarks. (1) Upon invoking the more or less standard terminology of the theory of dynamical systems, we can say that under the hypotheses of Theorem 2.1, u_0 (resp. u_1) becomes an exponentially global attractor or simply a global exponential attractor for Problem (1.1). It is interesting to note that because of hypotheses (G_1), (G_2) and (S_1), the constants u_0 and u_1 are the only almost-periodic solutions to the problem

$$\begin{Bmatrix} \hat{u}'(t) = s(t)g(\hat{u}(t)) , & t \in \mathbb{R} \\ \mathrm{Ran}(u) \subseteq [u_0,u_1] & \end{Bmatrix} \tag{2.5}$$

(2) There are two contributions to the right–hand side of the first equation in (1.1), which model two very different physical phenomena: the diffusion term represented by Δu , and the nonlinear reaction term represented by $s(t)g(u)$. It is then natural to ask which one of the two is primarily responsible for the stabilisation properties of the classical solutions to (1.1). As we shall see the answer strongly depends on the properties of s ; for instance, in Theorem 2.1 where hypotheses (S_1) and (S_2) hold, the stabilization phenomenon is entirely governed by the reaction term. In fact, the spectral properties of Laplace's operator play no role in the proof of relations (2.1) – (2.4), and the rates of decay are uniquely

determined by $g'(u_0)\mu_B(s)$ (resp. $g'(u_1)\mu_B(s)$). To illustrate how irrelevant the diffusion process is in this case, for each $\hat{\nu} \in (u_0,u_1)$ we consider the function defined by

$$\hat{u}(t) = G^{-1}\left[\int_0^t d\xi s(\xi) + G(\hat{\nu})\right] \tag{2.6}$$

for every $t \in \mathbb{R}$, where G^{-1} denotes the monotone inverse of G. Then $\hat{u}$ is a classical solution to the problem

$$\left\{\begin{array}{l} \hat{u}'(t) = s(t)g(\hat{u}(t)),\ t \in \mathbb{R} \\ \mathrm{Ran}(u) \in [u_0,u_1] \end{array}\right\} \tag{2.7}$$

Since $\hat{u}$ is independent of x, it is then a fortiori a non diffusive classical solution to (1.1). However, according to Theorem 2.1, we still have $\hat{u}(t) \longrightarrow u_0$ or $\hat{u}(t) \longrightarrow u_1$ as $t \longrightarrow \infty$, depending on whether $\mu_B(s) < 0$ or $\mu_B(s) > 0$.

(3) Consider for instance the problem

$$\left\{\begin{array}{ll} u_t(x,t)=\Delta u(x,t)+(\cos(\omega_1 t)+\cos(\omega_2 t)\pm 1)u(x,t)(1-u(x,t)) & ,(x,t)\in\Omega\times\mathbb{R}^+ \\ \mathrm{Ran}(u) \subseteq (0,1) & \\ \frac{\partial u}{\partial \underline{n}}(x,t) = 0 & ,(x,t)\in\partial\Omega\times\mathbb{R}^+ \end{array}\right\} \tag{2.8}$$

where $\{\omega_1,\omega_2\} \subset \mathbb{R}/\{0\}$ is rationally independent. Problem (2.8) is of the form (1.1) with $s_\pm(t) = \cos(\omega_1 t) + \cos(\omega_2 t) \pm 1$ and it is easily verified that all of the hypotheses of Theorem 2.1 are satisfied. If $s = s_-$, we conclude that every classical solution to Problem (2.8) converges exponentially rapidly toward $u_0 = 0$ with $r_{u_0} = -1$. If $s = s_+$ the same conclusion holds true for $u_1 = 1$ and $r_{u_1} = -1$. In the context of population genetics, this result means that only one of the alleles will eventually survive in the population.

We devote the remaining part of this section to outlining the proof of Theorem 2.1, and we refer the reader to [9, 10, 11] for complete details. We write $H^{2,p}(\mathbb{C}) = H^{2,p}(\Omega,\mathbb{C})$ for the usual Sobolev space consisting of all complex L^p – functions z on Ω with L^p – distributional derivatives $D^\alpha z$ for

$|\alpha| \in [0,2]$, and equipped with the usual norm $\|\cdot\|_{2,p}$ [16]. $H^{2,p}(\mathbb{R})$ then denotes the real component in $H^{2,p}(\mathbb{C})$. We also write $\mathscr{C}^{1,\beta}(\mathbb{C}) = \mathscr{C}^{1,\beta}(\overline{\Omega},\mathbb{C})$ for the Banach space of all complex Hölder continuous functions on $\overline{\Omega}$ with Hölderian derivatives $D^{\alpha}z$ of exponent β for $|\alpha| \in [0,1]$, with respect to the usual pointwise operations and the usual norm [16]. If u denotes any classical solution to Problem (1.1) in the sense of Definition 2.1, and if we define $u(t)(x) = u(x,t)$ for every $(x,t) \in \overline{\Omega} \times \mathbb{R}_0^+$, it is then clear that $u(t) \in H^{2,p}(\mathbb{C})$ for every $t \in \mathbb{R}^+$ (compare with condition (C_2) of Definition 2.1). On the other hand, since there exists the continuous embedding $H^{2,p}(\mathbb{C}) \longrightarrow \mathscr{C}^{1,\beta}(\mathbb{C})$, it is sufficient to prove that

$$\|u(t) - u_0\|_{2,p} \leq \hat{c}\, \varepsilon_0 \exp[r_{u_0}(t - t_{\varepsilon_0})] \qquad (2.9)$$

for some positive constants $\hat{c}$, ε_0 and t_{ε_0}, in order to obtain estimates (2.1) – (2.4). We can then reduce the proof of inequality (2.9) to proving the following three statements:

Statement (A) : $u(t) \longrightarrow u_0$ strongly in $H^{2,p}(\mathbb{C})$ as $t \longrightarrow \infty$: this is already a statement which proves that u_0 is a global attractor, but which fails to provide the corresponding rate of stabilization.

Statement (B) : In $H^{2,p}(\mathbb{R})$, there exists a Banach manifold of classical solutions $t \longrightarrow \tilde{u}(t)$ of small norm to Problem (1.1), such that the estimate

$$\|\tilde{u}(t) - u_0\|_{2,p} \leq \hat{c}\, \varepsilon_0 \exp\, [r_{u_0} t] \qquad (2.10)$$

holds for some $\varepsilon_0 > 0$ and every $t \in \mathbb{R}_0^+$.

In contrast to Statement (A), Statement (B) does provide the appropriate exponential rate of decay, but the validity of estimate (2.10) is limited to certain classical solutions of small norm. However, we shall see below that we can in fact combine Statement (A) with Statement (B) to obtain

Statement (C) : Given $u(t)$ and the ε_0 of Statement (B), there exists $t_{\varepsilon_0} > 0$ and a small norm solution as in Statement (B), such that the equality

$$u(t) = \tilde{u}(t - t_{\varepsilon_0}) \qquad (2.11)$$

holds for every $t \in [t_{\varepsilon_0}, \infty)$.

It is then clear that the combination of (2.10) with (2.11) gives (2.9).

The proofs of the above statements are all very geometric; we begin with the following

Sketch of the Proof of Statement (A). Given $u(t)$, it is possible to define an auxiliary function $v(t) : \overline{\Omega} \longrightarrow \mathbb{R}$ such that $(x,t) \longrightarrow v(x,t)$ possesses the same regularity as $(x,t) \longrightarrow u(x,t)$, and which satisfies the linear parabolic differential inequality

$$\left\{ \begin{array}{ll} v_t(x,t) \leq \Delta v(x,t), & (x,t) \in \Omega \times \mathbb{R}^+ \\ \frac{\partial v}{\partial n}(x,t) = 0, & (x,t) \in \partial\Omega \times \mathbb{R}^+ \end{array} \right\} \tag{2.12}$$

This function may in fact be chosen as

$$v(t) = \varphi(t) \exp\, [\alpha\, G \circ u(t)] \tag{2.13}$$

where $\alpha \in \mathbb{R}^+ \cap (\max_{\xi \in [u_0, u_1]} g'(\xi), \infty)$, G is as in Hypothesis (G_2) and

$$\varphi(t) = \exp\left[-\alpha\left\{G(\hat{\mu}) + \int_0^t d\xi s(\xi)\right\}\right] \tag{2.14}$$

for some fixed $\hat{\mu} \in (u_0, u_1)$. Upon exploiting the properties of $u(t)$ along with the parabolic maximum principle applied to Problem (2.12), we then infer the existence of a constant $c_1 > 0$ such that the estimate

$$\|v(t)\|_{\infty,\overline{\Omega}} = \max_{x \in \overline{\Omega}} |v(x,t)| \leq c_1 \tag{2.15}$$

holds uniformly in $t \in \mathbb{R}_0^+$. From relations (2.13) and (2.15) it then follows that

$$\|\exp\, [\alpha\, G \circ u(t)]\,\|_{\infty,\overline{\Omega}} \leq \frac{c_1}{\varphi(t)} \longrightarrow 0 \tag{2.16}$$

as $t \longrightarrow \infty$, since $\mu_B(s) < 0$ implies that $\varphi(t) \longrightarrow \infty$ as $t \longrightarrow \infty$. This, combined with Hypothesis (G_2), shows that

$$\|u(t) - u_0\|_{\infty,\overline{\Omega}} \longrightarrow 0 \tag{2.17}$$

as $t \longrightarrow \infty$. Finally, if $\Delta_{p,\mathcal{N}}$ denotes the $L^p(\mathbb{C})$–realization of Laplace's operator on the domain $H^{2,p}_{\mathcal{N}} = \{ z \in H^{2,p}(\mathbb{C}) : \frac{\partial z}{\partial \underline{B}}(x) = 0 , x \in \partial\Omega \}$, we can prove that

$$\Delta_{p,\mathcal{N}} u(t) \longrightarrow 0 \tag{2.18}$$

strongly in $L^p(\mathbb{C})$ as $t \longrightarrow \infty$. This is exactly here that the geometry comes in: in order to obtain relation (2.18), we invoke the existence of exponential dichotomies for the diffusion semigroup generated by $\Delta_{p,\mathcal{N}}$. Relations (2.17) and (2.18) then imply that $u(t) \longrightarrow u_0$ strongly in $H^{2,p}(\mathbb{C})$, by standard elliptic theory. ■

Sketch of the Proof of Statement (B). Here we invoke the geometric techniques devised in [7, 8] for the analysis of some hyperbolic problems to construct a smooth local stable manifold of classical solutions to Problem (1.1) around u_0 , satisfying estimate (2.10). While it is not possible to give the details of this construction here, it is however important to point out that our method provides a manifold of solutions in $H^{2,p}(\mathbb{R})$ similar to an open ball, and not merely a lower dimensional manifold. This is because of the fact that the exponential dichotomies which we need to carry out our construction are valid on the whole of $H^{2,p}(\mathbb{R})$ when $\mu_B(s) < 0$, and not merely on a lower dimensional subspace. This point is essential if one wants to combine Statements (A) and (B) in order to obtain Statement (C). ■

Sketch of the Proof of Statement (C). Let $u(t)$ be as in Statement (A), and let $\hat{c}$ and ε_0 be as in Statement (B). Since we already know that $u(t) - u_0 \longrightarrow 0$ strongly in $H^{2,p}(\mathbb{R})$ as $t \longrightarrow \infty$, there exists $t_{\varepsilon_0} > 0$ such that $\|u(t) - u_0\|_{2,p} \le \hat{c}\varepsilon_0$ for every $t \ge t_{\varepsilon_0}$. The crucial point is that this implies that the function $u(t_{\varepsilon_0})$ lies on the stable manifold constructed under (B). What this means is that if we consider the initial boundary value problem

$$\left.\begin{cases} w(x,t) = \Delta w(x,t) + s(t)g(w(x,t)) \, , & (x,t) \in \Omega\times(t_{\varepsilon_0},\infty) \\ \mathrm{Ran}(w) \subsetneq (u_0,u_1) & \\ w(x,t_{\varepsilon_0}) = u(x,t_{\varepsilon_0}) & , x \in \overline{\Omega} \\ \frac{\partial w}{\partial B}(x,t) = 0 & , (x,t) \in \partial\Omega\times(t_{\varepsilon_0},\infty) \end{cases}\right\} \quad (2.19)$$

then there exists a classical solution $t \longrightarrow \tilde{u}(t)$ of small norm as in statement (B), such that $w(x,t) = \tilde{u}(x,t - t_{\varepsilon_0})$ provides a solution to Problem (2.19). But from the third condition in (2.19) and the fact that g is smooth and s is bounded, it follows from the parabolic maximum principle that $w(x,t) = u(x,t)$ for every $t \geq t_{\varepsilon_0}$. Hence $u(x,t) = \tilde{u}(x,t - t_{\varepsilon_0})$ for every $x \in \overline{\Omega}$, which implies relation (2.11).

■

Remarks. (1) The basic geometric idea in the above argument is the following: since we already know that $u(t) \longrightarrow u_0$ as $t \longrightarrow \infty$, we just have to wait long enough until $u(t)$ hits the stable manifold of statement (B) at time $t = t_{\varepsilon_0}$; we then proceed to identify $u(t)$ for $t \geq t_{\varepsilon_0}$ with a small norm solution of the type $\tilde{u}$, for which estimate (2.10) holds. However, the technical details of the proof of Statement (C) are not as elementary as it seems, since Problem (2.19) is non-autonomous. We refer the reader to [11] for complete details.

(2) The reason for which we have formulated Problem (1.1) as a dynamical system on $H^{2,p}(\mathbb{C})$ is that $H^{2,p}(\mathbb{C})$ becomes a commutative Banach algebra with respect to the usual pointwise operations and the norm $\|\cdot\|_{2,p}$, when $p > N$ [16]. This underlying Banach algebra structure is an essential ingredient of our theory, both in the hyperbolic and in the parabolic case [7, 8, 9, 10, 11].

(3) Within the scope of our method, the regularity hypothesis $g \in \mathscr{C}^{(5)}(\mathbb{R},\mathbb{R})$ is nearly optimal for the validity of the results of this paper. We do not know whether our results still hold if g satisfies only weaker differentiability properties.

(4) We leave it to the reader to prove Statement (2) of Theorem 2.1.

In the next section we review the situation where $\mu_B(s) = 0$ and $t \longrightarrow \int_0^t d\xi s(\xi) = 0(1)$ as $|t| \longrightarrow \infty$. As is perhaps expected, the nature of the stabilization phenomena then changes radically.

3. Asymptotic Almost–Periodicity for Classical Solutions to Problem (1.1): The Rôle of the Diffusion Process.

Throughout this section the hypotheses (G_1), (G_2) and (S_2) remain unaltered, whereas Hypothesis (S_1) is replaced by the following:

$(\hat{S}_1)$ We have $\mu_B(s) = 0$ and in fact $t \longrightarrow \int_0^t d\xi s(\xi) = 0(1)$ as $|t| \longrightarrow \infty$.

Finally, the notion of classical solution to Problem (1.1) is still the same as in Section 2. It is then clear that one cannot any longer expect the classical solutions of (1.1) to stabilize around u_0 or u_1. For instance, since it follows from a classic criterion of Bohr that the boundedness of $t \longrightarrow \int_0^t d\xi s(\xi)$ is equivalent to its almost–periodicity [12, 13, 14, 15], the non diffusive classical solutions to Problem (1.1) given by relation (2.6) are now all almost–periodic, and remain uniformly bounded away from u_0 and u_1 because of Hypotheses (G_2) and $(\hat{S}_1)$. In fact, the latter property remains true for every classical solution to Problem (1.1)[9]. It is then natural to wonder whether those classical solutions can stabilize at all. The answer is fortunately positive, and it turns out that it is precisely the almost–periodic functions (2.6) which now play the role of attractors. The precise result is the following

Theorem 3.1. Consider Problem (1.1) where g satisfies Hypotheses (G_1) and (G_2); assume moreover that s satisfies Hypotheses $(\hat{S}_1)$ and (S_2), and let u be any classical solution to (1.1) in the sense of Definition 2.1. Then there exists an almost–periodic solution $\hat{u}$ given by relation (2.6) such that

$$\lim_{t \to \infty} \sup_{x \in \Omega} |u(x,t) - \hat{u}(t)| = 0 \qquad (3.1)$$

Moreover, the given classical solution satisfies the relations

$$\lim_{t \to \infty} \sup_{x \in \Omega} |\nabla u(x,t)| = 0 \qquad (3.2)$$

and

$$\lim_{t\to\infty} \sup_{\substack{x,y\in\Omega \\ x\neq y}} |x-y|^{-\beta} |u(x,t)-u(y,t)| = 0 \tag{3.3}$$

$$\lim_{t\to\infty} \sup_{\substack{x,y\in\Omega \\ x\neq y}} |x-y|^{-\beta} |\nabla u(x,t)-\nabla u(y,t)| = 0 \tag{3.4}$$

for every $\beta \in (0,1-p^{-1}N]$. Finally, every Fourier exponent of $\hat{u}$ is a finite linear combination with integer coefficients of the Fourier exponents of s .

Remarks. (1) In contrast to the statements of Theorem 2.1, it is not possible to claim that $\hat{u}$ is a global attractor; in fact, $\hat{u}$ depends on u in general. Thus, every classical solution to (1.1) stabilizes around some $\hat{u}$, but Theorem 3.1 says absolutely nothing about the stability properties of such $\hat{u}$'s . It might even be a priori conceivable that for a given $\hat{u}$ of the form (2.6), there exists no classical solution u to Problem (1.1) satisfying relations (3.1)–(3.4). Fortunately, we shall see below that this is not the case.

(2) Theorem 3.1 provides no rate of decay for the stabilization of classical solutions. It is therefore impossible to infer from it whether the stabilization phenomenon described in Theorem 3.1 is governed by the diffusion process, or by the reaction process in (1.1). We shall see below that the solution to this problem is much more subtle than in Section 2.

(3) Consider for example the problem

$$\left\{\begin{array}{ll} u_t(x,t)= & \\ \quad =\Delta u(x,t)+(\cos(\omega_1 t)+\cos(\omega_2 t))u(x,t)((1-u(x,t))\exp[-u(x,t)] & \\ & ,\ (x,t)\in \partial\Omega\times\mathbb{R}^+ \\ \mathrm{Ran}(u) \subseteq (0,1) & \\ \dfrac{\partial u}{\partial \underset{\sim}{n}}(x,t) = 0 & ,\ (x,t)\in \partial\Omega\times\mathbb{R}^+ \end{array}\right\} \tag{3.5}$$

where $\{\omega_1,\omega_2\} \subset \mathbb{R}/\{0\}$ is rationally independent. Problem (3.5) is still of the form (1.1) with $s(t) = \cos(\omega_1 t) + \cos(\omega_2 t)$ and $g(u) = u(1-u)\exp[-u]$, and it is easily verified that all of the hypotheses of Theorem 3.1 are verified. We conclude that every classical solution to Problem (3.5) stabilizes around a quasi-periodic attractor of the form $\hat{u}$ which oscillates between the equilibria $u_0 = 0$ and $u_1 = 1$. Moreover, every Fourier exponent Λ of $\hat{u}$ is of the form

$\Lambda = k_1\omega_1 + k_2\omega_2$, where $k_{1,2} \in \mathbb{Z}$. In the context of population genetics, this result means that both alleles will persist in the population for all times, and that the fractions of the two alleles will eventually evolve quasiperiodically in time with oscillation properties entirely controlled by those of the seasonal variations.

Sketch of the Proof of Theorem 1.1. As in Section 2 let $u(t)$ be defined by $u(t)(x) = u(x,t)$ for every $(x,t) \in \overline{\Omega}\times\mathbb{R}^+$. Then $u(t) \in H^{2,p}(\mathbb{C})$, and it is sufficient to prove that there exists a $\hat{u}$ of the form (2.6) such that $\|u(t)-\hat{u}(t)\|_{2,p} \longrightarrow 0$ as $t \longrightarrow \infty$. But the proof of this is essentially the same as that of Statement (A) in Section 2. Finally, the very last statement of Theorem 3.1 follows from relation (2.6) and a classic criterion of Favard [15].

■

We shall now see that it is only for certain particular classical small norm solutions to Problem (1.1) that one can expect an exponential stabilization. More specifically, given any almost–periodic $\hat{u}$ of the form (2.6), we can prove that there always exists a smooth manifold of classical solutions to (1.1) which stabilize around $\hat{u}$ exponentially rapidly by diffusion. The precise result is the following

Theorem 3.2. Consider Problem (1.1) where g satisfies Hypotheses (G_1) and (G_2); assume moreover that s satisfies Hpotheses $(\hat{S}_1)$, (S_2) and let $\hat{u}$ be any non diffusive almost–periodic solution of the form (2.6). Finally, let λ_1 be the largest negative eigenvalue of Laplace's operator realized on the domain $\text{Dom}(\Delta_{\mathcal{N}}) = \{z \in H^{4,p}_{\mathcal{N}}(\mathbb{C}) : \Delta_{\mathcal{N}} z \in H^{2,p}_{\mathcal{N}}(\mathbb{C})\}$ of the Banach space $H^{2,p}_{\mathcal{N}}(\mathbb{C})$. Then there exists in $H^{2,p}_{\mathcal{N}}(\mathbb{R})$ a smooth codimension–one manifold of classical solutions $\tilde{u}$ to Problem (1.1) such that the following estimates hold for every $t \in \mathbb{R}^+_0$, every $\beta \in (0,1-p^{-1}N]$ and for some $\tilde{c}, \tilde{\varepsilon}_0 \in \mathbb{R}^+$:

$$\sup_{x\in\overline{\Omega}} |\tilde{u}(x,t)-\hat{u}(t)| \leq \tilde{c}\,\tilde{\varepsilon}_0 \exp[\lambda_1 t] \tag{3.6}$$

$$\sup_{x\in\overline{\Omega}} |\nabla\tilde{u}(x,t)| \leq \tilde{c}\,\tilde{\varepsilon}_0 \exp[\lambda_1 t] \tag{3.7}$$

$$\sup_{\substack{x,y\in\Omega \\ x\neq y}} |x-y|^{-\beta} |\tilde{u}(x,t)-\tilde{u}(y,t)| \leq \tilde{c}\,\tilde{\varepsilon}_0 \exp[\lambda_1 t] \tag{3.8}$$

$$\sup_{\substack{x,y\in\Omega\\ x\neq y}} |x-y|^{-\beta} |\nabla\tilde{u}(x,t)-\nabla\tilde{u}(y,t)| \leq \tilde{c}\,\tilde{\varepsilon}_0 \exp[\lambda_1 t] \tag{3.9}$$

Remarks. (1) Estimates (3.6) – (3.9) clearly play the role of inequalities (2.1) – (2.4) of Theorem 2.1, with the crucial difference that it is now the largest negative eigenvalue of Laplace's operator which determines the rate of exponential stabilization. However, this precise information is limited to those small norm solutions which lie on the codimension–one manifold of Theorem 3.2.

(2) In contrast to Theorem 2.1, it is not possible to hope that estimates (3.6)–(3.9) hold true for every classical solution to Problem (1.1). The reason for this is that the spectrum of Laplace's operator consists exclusively of discrete negative eigenvalues and $\lambda = 0$ (because of Neumann's boundary condition in (1.1)). It follows from this that besides the codimension–one stable manifolds of Theorem 3.2, there also exist one–dimensional center manifolds associated with Problem (1.1). The natural question is then to know how the coexistence of those two types of local manifolds influences the ultimate behaviour of the classical solutions to (1.1). First, given $\hat{u}$ it is possible to show that for t sufficiently large, $u(t)$ approaches the center manifold about $\hat{u}$ exponentially rapidly; moreover, it is the diffusion process in (1.1) that is primarily responsible for this phenomenon of exponential attractivity of the center manifold. With additional restrictions on the nonlinearity g, the classical solution $u(t)$ then ultimately stabilizes around some $\hat{u}$ according to the statement of Theorem 3.1, with a rate determined by the long time behaviour on the corresponding center manifold. Moreover, the latter behaviour is primarily governed by the reaction process in (1.1). These considerations eventually prove the asymptotic stability of each $\hat{u}$ given by (2.6), and provide a rate of decay which is in general slower than exponential. We refer the reader to [11] for complete details.

Sketch of the Proof of Theorem 3.2. Upon invoking the geometric theory of [7, 8], we can construct in $H^{2,p}(\mathbb{R})$ a one–codimensional local stable manifold of classical solutions to Problem (1.1) around each $\hat{u}$. Here again, the Banach algebra structure of $H^{2,p}(\mathbb{R})$ is essential. ■

Remark. Regarding Theorem 3.2, our observation concerning the regularity hypothesis $g \in \mathscr{C}^{(5)}(\mathbb{R},\mathbb{R})$ is the same as in Section 2. In particular, it is not known whether Theorem 3.2 still holds with only weaker differentiability properties concerning g. Theorem 3.1, however, actually holds when $g \in \mathscr{C}^{(1)}(\mathbb{R},\mathbb{R})$.

Finally, we devote the last section of this article to the discussion of an open problem.

4. On the Stabilization Properties of Solutions to Almost–Periodic Reaction–Diffusion Equations with Spatial Structure.

We noted that under the hypotheses of Theorem 3.1, the $\hat{u}$'s given by (2.6) cannot be global attractors. It is therefore natural to ask whether some kind of additional structure in Problem (1.1) might change this overall picture. Consider the problem

$$\left\{\begin{array}{ll} u_t(x,t) = \Delta u(x,t) + s(x,t)g(u(x,t)), & (x,t) \in \Omega\times\mathbb{R}^+ \\ \mathrm{Ran}(u) \subseteq (u_0,u_1) & \\ \frac{\partial u}{\partial \underset{\sim}{B}}(x,t) = 0 & , (x,t) \in \partial\Omega\times\mathbb{R}^+ \end{array}\right\} \tag{4.1}$$

where g is as before, and where the selection function now depends explicitly on $x \in \overline{\Omega}$ in such a way that $t \longrightarrow s(x,t)$ be almost–periodic for each $x \in \overline{\Omega}$. If s is sufficiently smooth on $\overline{\Omega} \times \mathbb{R}_0^+$, define $\bar{s}(t) = \max_{x\in\overline{\Omega}} s(x,t)$ and $\underline{s}(t) = \min_{x\in\overline{\Omega}} s(x,t)$. Then both $\bar{s}$ and $\underline{s}$ are Bohr almost–periodic, and it is natural to ask whether these two functions might play a role in the discussion of the stabilization properties of the classical solutions to Problem (4.1). A glance at the proofs outlined in the preceding sections shows that they do. For instance, if $\mu_B(\underline{s}) \leq \mu_B(\bar{s}) < 0$, it is easy to show that u_0 is the global attractor for Problem (4.1); the proof of this fact is entirely similar to that of Statement (A) in Section 2, with $\bar{s}$ replacing s (Note that we always have $\mu_B(\underline{s}) \leq \mu_B(\bar{s})$ by definition, so that the only non trivial condition in the above case is $\mu_B(\bar{s}) < 0$). Similarly, if $0 < \mu_B(\underline{s}) \leq \mu_B(\bar{s})$, then u_1 becomes the global attractor (here again, $\mu_B(\underline{s}) > 0$ is the only nontrivial condition, which plays the role of $\mu_B(s) > 0$ when s is independent of x). Thirdly, if $\mu_B(\bar{s}) = 0$ and if $t \longrightarrow \int_0^t d\xi\bar{s}(\xi) = 0(1)$ as $|t| \longrightarrow \infty$, then every classical solution to Problem (4.1) stabilizes around a spatially homogeneous, time almost–periodic solution to the equation

$$\hat{u}'(t) = \bar{s}(t)g(\hat{u}(t)), \quad t \in \mathbb{R} \tag{4.2}$$

a result which bears some analogy with Theorem 3.1. Of course, in light of the methods outlined in the preceding sections, none of the above statements is really surprising, and they all reduce to the corresponding statements of Sections 2 and

3 when s does not depend on x.
However, it is also worth mentioning that there exists in relation with Problem (4.1) the additional possibility of having $\mu_B(\underline{s}) < 0 < \mu_B(\overline{s})$. This, of course, does not occur for Problem (1.1) where $s(x,t) = s(t)$ for every $x \in \overline{\Omega}$. In this case, it is tempting to conjecture that there exists a unique time almost–periodic solution to Problem (4.1) which is neither identically equal to u_0 nor identically equal to u_1, and which is a global attractor for all classical solutions to (4.1). This was in fact proved in [17] when $t \longrightarrow s(x,t)$ is periodic, but remains an open problem in the general almost–periodic case. The source of this difficulty lies primarily in the fact that there is no natural substitute for the notion of Poincaré time–map in the almost–periodic case.

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