SEMICLASSICAL FOCUSING NLS

Spyridon Kamvissis

The present is an extended version of a Seminar given at the Max-Planck Institute on January 10, 2003.

I. SEMICLASSICAL NLS

The nonlinear Schrödinger equation arises by substituting a nonlinear term for the potential in the classical linear Schrödinger Equation. More precisely, let U be a real valued C^2 function defined on the positive real half-line. The NLS equation is

$$i\epsilon u_t + \frac{\epsilon^2}{2}\Delta u - U'(|u|^2)u = 0, \tag{1}$$

where in general $u(x,t)$ is a vector $(u_1, u_2, ..., u_N)^T$, and each $u_j(x,t)$ is a complex valued function defined for, say, $x \in \mathbb{R}^n, t \in \mathbb{R}^+$.

In the linear case, ϵ is of course the Planck constant. In the nonlinear case, there are several applications for (1) and it may not always be true that ϵ stands for the Planck constant. But we are always in the case where $x/\epsilon, t/\epsilon$ are large. So, mathematically, our problem is the limit of the solution of (1) as $\epsilon \to 0$. By analogy to the linear case, we still talk about a *semiclassical* limit.

One of the most important applications the semiclassical limit of (1) arises in nonlinear optics, in particular regarding the production and propagation of laser pulses in fiber optics. Here ϵ depends on the dimensions on the problem (amplitude of the initial pulse, dimensions of the fiber, etc.). Another application appears in

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

the study of the so-called Bose-Einstein condensate, where ϵ now *does* represent the Planck constant.

We will consider here initial value problems:

$$u(x,0)=u_0(x)=A(x)exp(\frac{iS(x)}{\epsilon}), \tag{2}$$

where the components of $A(x)$ are rapidly decreasing at infinity. $S(x)$ is a smooth function converging rapidly to limits $S_{\pm}$, as $x\to\pm\infty$, respectively.

II. THE COMPRESSIBLE EULER SYSTEM

Equation (1) is Hamiltonian. Indeed, it can be written as

$$i\epsilon\partial_t u=\frac{\delta H}{\delta\bar{u}},$$
$$H=\int_{\mathbb{R}^n}[\frac{\epsilon^2}{2}|\nabla u|^2+U(|u|^2)]d^N(x).$$

The associated Poisson bracket of any two functionals F,G is given by

$$\{F,G\}=\frac{1}{i\epsilon}\int_{\mathbb{R}^n}(\frac{\delta F}{\delta u}\frac{\delta G}{\delta\bar{u}}-\frac{\delta F}{\delta\bar{u}}\frac{\delta G}{\delta u})d^N(x).$$

The evolution of any functional under the NLS flow is then given by

$$\frac{dF}{dt}=\{F,H\}.$$

There are $N+2$ local conservation laws corresponding to conservation of mass, momentum and energy. Indeed, let the mass, momentum and energy densities be defined by

$$\rho=|u|^2,$$
$$\mu=-i\frac{\epsilon}{2}(\bar{u}\nabla u-u\nabla\bar{u}), \tag{3}$$
$$E=\frac{\epsilon^2}{2}|\nabla u|^2+U(|u|^2)=\frac{|\mu|^2}{2\rho}+\frac{\epsilon^2}{8}\frac{|\nabla\rho|^2}{\rho}+U(\rho),$$

respectively. The local conservation laws are

$$\partial_t\rho+\nabla\cdot\mu=0,$$
$$\partial_t\mu+\nabla\cdot(\frac{\mu\otimes\mu}{\rho})+\nabla P(\rho)=\frac{\epsilon^2}{4}\nabla\cdot(\rho\nabla^2 log\rho), \tag{4}$$
$$E_t+\nabla\cdot(\frac{\mu}{\rho}(E+P(\rho)))=\frac{\epsilon^2}{4}\nabla\cdot(\frac{\mu\Delta\rho}{\rho}-\frac{\nabla\cdot\mu\Delta\rho}{\rho}).$$

with

$$P(\rho) = \rho U'(\rho) - U(\rho). \tag{5}$$

The first two equations are a closed system for ρ, μ. In fact ρ, μ characterize u up to a constant phase.

The formal limit of the first two equations of (4) when ϵ goes to 0 is given by

$$\begin{aligned} &\partial_t \rho + \nabla \cdot \mu = 0, \\ &\partial_t \mu + \nabla \cdot (\frac{\mu \otimes \mu}{\rho}) + \nabla P(\rho) = 0. \end{aligned} \tag{4a}$$

System (4a) is a compressible Euler system where the pressure is given by (5) in terms of the mass density. If $P'(\rho) = \rho U''(\rho)$ is positive then the pressure is a strictly increasing function of ρ. In such a case it turns out that (4a) is strictly *hyperbolic*. Equation (1) is then called the *defocusing* NLS equation. In the case where $P'(\rho) = \rho U''(\rho)$ is negative, the hydrodynamic interpretation is awkward: pressure decreases with ρ. We call this the *focusing* NLS; system (4a) is now *elliptic*.

The hyperbolic system admits a classical solution for regular data until a "shock" appears. If the initial data are real analytic, the elliptic system is guaranteed to have a solution for small times by the Cauchy-Kovalevskaya theorem.

The semiclassical limit problem is equivalent to studying the behavior of (4), as $\epsilon \to 0$. In particular, a first question that arises is: does the solution of (4a) give the strong limit of the solution of (1)?

Numerical results suggest that the formal limit (4a) does govern the strong limit of (4), as long as (4a) has a classical solution. After the time where a classical solution for (4a) ceases to exist, the solution of (1) is highly oscillatory, with frequency of the order of $1/\epsilon$ and a limit can exist only in a weak (averaging) sense.

III. RESULTS

Before break time general WKB-type and microlocal analysis methods have been used by Gérard and Grenier to show that the formal limit of NLS (or more precisely the equivalent Euler system) governs the semiclassical (strong) limit of the solution to (1). After the "break" the only rigorous results known concern the two one-dimensional integrable cases:

$$i\epsilon u_t + \frac{\epsilon^2}{2}u_{xx} \pm |u|^2 u = 0, \tag{1a}$$

where in general $u(x,t)$ is a complex valued function defined for, say, $x \in \mathbb{R}, t \in \mathbb{R}^+$.

The initial data are

$$u(x,0) = u_0(x) = A(x)exp(\frac{iS(x)}{\epsilon}), \tag{2a}$$

where $A(x)$ is rapidly decreasing at infinity and $S(x)$ is a smooth function converging rapidly to limits $S_\pm$, as $x \to \pm\infty$, respectively.

The "+" sign in (1a) corresponds to the focusing case while the the "-" sign corresponds to the defocusing case.

The analysis of the semiclassical integrable defocusing NLS problem was done in the 90s by Jin, Levermore and D.McLaughlin (see [JLM]). In the focusing case, the integrable problem is analyzed in [KMM]. In all cases studied so far, weak limits of the densities ρ, μ, E are proved to exist even after the "break" time. The rigorous theory is provided for particular subclasses of initial data of the form (2a) and a connection is made with a variational problem for electrostatic Green potentials. Such a problem is stable and one-dimensional (over the real line) in the defocusing case, but unstable and two-dimensional (over the full complex plane) in the focusing case. Even though the formal limit (4a) does not yield the semiclassical limit of (1) for all time, there is a whole hierarchy of so-called Whitham systems of equations, which governs the weak semiclassical limit, and which is also hyperbolic or elliptic, depending on the sign of the nonlinearity in (1). It is vital to emphasize that the most important effect of the ellipticity in the formal limit of the (focusing) Euler-Whitham system is instability. Even though for fixed ϵ, equation (1) is well-posed, the formal limit is an ill-posed unstable initial value problem for an elliptic system (or more precisely, a hierarchy of systems).

IV. THE FOCUSING CASE AFTER THE BREAK. INTEGRABILITY

Integrability machinery is based on the scattering and inverse scattering theory for equation (1a). The Lax operator associated to (1a) is the Dirac operator

$$\begin{pmatrix} i\epsilon\partial_x & \mp iu(x) \\ -iu^*(x) & -i\epsilon\partial_x \end{pmatrix},$$

where sign "+" corresponds to the defocusing case, while sign "-" corresponds to the focusing case. * denotes complex conjugation. It is important to note that the Lax operator self-adjoint in the defocusing case and non-self-adjoint in the focusing case.

We now concentrate on the integrable focusing case, that is equation (1) with $N = n = 1, U' = -1$. We consider the initial value problem (2), in the special case $S(x) = 0$, with real analytic data $A(x)$ decaying rapidly as $|x| \to \infty$. This problem is solvable for arbitrary $\hbar$ because the focusing nonlinear Schrödinger equation can be represented as the compatibility condition for two systems of linear ordinary differential equations:

$$\epsilon\partial_x \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -i\lambda & u \\ -u^* & i\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{6}$$

and

$$i\epsilon\partial_t \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \lambda^2 - |u|^2/2 & i\lambda u - \epsilon\partial_x u/2 \\ -i\lambda u^* - \epsilon\partial_x u^*/2 & -\lambda^2 + |u|^2/2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{7}$$

where λ is an arbitrary complex parameter. The compatibility condition for (6) and (7) does not depend on the value of λ, and is equivalent to the nonlinear Schrödinger equation.

The N-soliton solutions of the nonlinear Schrödinger equation are those complex functions $u(x,t)$ for which there exist simultaneous column vector solutions of (6) and (7) of the particularly simple form:

$$v^+(x,t,\lambda) = \begin{pmatrix} \displaystyle\sum_{p=0}^{N-1} A_p(x,t)\lambda^p \\ \displaystyle\lambda^N + \sum_{p=0}^{N-1} B_p(x,t)\lambda^p \end{pmatrix} \exp(i(\lambda x + \lambda^2 t)/\epsilon)\,,$$

$$v^-(x,t,\lambda) = \begin{pmatrix} \displaystyle\lambda^N + \sum_{p=0}^{N-1} C_p(x,t)\lambda^p \\ \displaystyle\sum_{p=0}^{N-1} D_p(x,t)\lambda^p \end{pmatrix} \exp(-i(\lambda x + \lambda^2 t)/\epsilon)$$

satisfying the relations

$$v^+(x,t,\lambda_k) = \gamma_k v^-(x,t,\lambda_k),$$

$$-\gamma_k^* v^+(x,t,\lambda_k^*) = v^-(x,t,\lambda_k^*),\ \ k = 1,\dots,N,$$

for some distinct complex numbers $\lambda_0,\dots,\lambda_{N-1}$ in the upper half-plane and nonzero complex numbers (not necessarily distinct) $\gamma_0,\dots,\gamma_{N-1}$. It is easy to check that given the numbers $\{\lambda_k\}$ and $\{\gamma_k\}$, the relations above determine the coefficient functions $A_p(x,t)$, $B_p(x,t)$, $C_p(x,t)$ and $D_p(x,t)$ in terms of exponentials via the solution of a square inhomogeneous linear algebraic system. In Faddeev and Takhtajan [FT] it is shown that this linear system is always nonsingular assuming the $\{\lambda_k\}$ are distinct and nonreal and the $\{\gamma_k\}$ are nonzero. The solution of the nonlinear Schrödinger equation for which the column vectors $v^\pm(x,t,\lambda)$ are simultaneous solutions of (6) and (7) turns out to be

$$\psi(x,t) = 2iA_{N-1}(x,t)\,. \tag{8}$$

REMARKS.

1. This construction is equivalent to a classical problem of rational approximation, the construction of multipoint Padé interpolants for entire functions. Let $G(\lambda)$ be any polynomial satisfying $G(\lambda_k) = Log(\gamma_k)$ and $G(\lambda_k^*) = Log(-1/\gamma_k^*)$. Then, looking at the first row of (6), we see that we are seeking polynomials $P_{N-1}(\lambda)$ and $Q_{N-1}(\lambda)$ both of degree $N-1$ such that

(9)
$$\frac{P_{N-1}(\lambda)}{\lambda^N + Q_{N-1}(\lambda)} = \exp(G(\lambda))\exp(-2i(\lambda x + \lambda^2 t)/\epsilon), \;\; for\ \lambda = \lambda_0, \ldots.., \lambda_{N-1}, \lambda_0^*, \ldots, \lambda_{N-1}^*.$$

The coefficients of $P_{N-1}(\lambda)$ are the $\{A_p(x,t)\}$, and the coefficients of $Q_{N-1}(\lambda)$ are the $\{C_p(x,t)\}$.

Whereas the usual Padé approximants are constructed by demanding sufficiently high-order agreement in the asymptotic expansion of (9) for large or small λ, the multipoint approximants are constructed by demanding simple agreement of the function values on the left and right-hand sides of (9) at a sufficiently large number of distinct points. The problem in its most general form can fail to have a solution, i.e. given a set of nodes of interpolation, there exist isolated “unreachable” function values. In the context of the N-soliton solution problem, however, this undesirable situation does not occur due to the complex conjugation symmetry of the interpolation points and corresponding symmetry properties of the assigned values at those points.

2. A typical initial condition $u_0(x)$ for (1) will not correspond exactly to a multisoliton solution. As is well-known, the procedure for solving (1) generally begins with the study the solutions of (6) for real λ and for $u = u_0(x)$. One obtains from this analysis a complex-valued transmission coefficient $T(\lambda) = 1/a(\lambda)$, $\lambda \in \mathbb{R}$. Now, after the fact it turns out that the function $a(\lambda)$ has an analytic continuation into the whole upper half-plane, and its zeros occur at values of λ for which (6) has an $L^2(\mathbb{R})$ eigenfunction. In this sense, the study of the scattering problem for real

λ yields results for complex λ by unique analytic continuation. The function $a(\lambda)$ can be interpreted as a Wronskian between two particular solutions of (6) that have analytic continuations into the upper half-plane. Thus at each L^2 eigenvalue $\lambda = \lambda_k$, there is a complex number γ_k that is the ratio of these two analytic solutions. In addition to the transmission coefficient, one also finds a complex-valued function $b(\lambda)$ that gives rise to a reflection coefficient $r(\lambda) := b(\lambda)/a(\lambda)$, $\lambda \in \mathbb{R}$. The main results of Zakharov and Shabat are:

(i). When $u(x,t)$ is the solution of (1) with initial data $u_0(x)$, then for each $t > 0$ one has different coefficients in the linear problem (6), and therefore the eigenvalues $\{\lambda_k\}$, proportionality constants $\{\gamma_k\}$ and the function $b(\lambda)$, can be computed independently for each $t > 0$. However, it follows from (6) that the eigenvalues $\{\lambda_k\}$ (more generally the function $a(\lambda)$) and also $|b(\lambda)|$, $\lambda \in \mathbb{R}$, are independent of t, and the proportionality constants $\{\gamma_k\}$ and $\arg(b(\lambda))$, $\lambda \in \mathbb{R}$ evolve simply in time. Thus, $r(\lambda, t) = r(\lambda, 0)\exp(-2i\lambda^2 t/\epsilon)$ and $\gamma_k(t) = \gamma_k(0)\exp(-2i\lambda_k^2 t/\epsilon)$.

(ii). The function $u(x,t)$ can be reconstructed at later times $t > 0$ in terms of the discrete spectrum $\{\lambda_k\}$, $\{\gamma_k\}$, and the reflection coefficient $r(\lambda)$. If for the initial condition $u_0(x)$ we have $b(\lambda) \equiv 0$, then the step of reconstructing the solution of the initial value problem (1) is essentially what we have already described. Namely, one solves the linear equations (6) for the coefficient $A_{N-1}(x,t)$ and then the solution of (1) is given by (9). Note that N is the number of L^2 eigenvalues for $u_0(x)$ in the upper half-plane.

In general, the reconstruction of u from the scattering data can be recast in terms of the solution of a matrix-valued meromorphic Riemann-Hilbert problem. One seeks (for each x and t, which play the role of parameters) a matrix-valued function $m(\lambda)$ of λ that is jointly meromorphic in the upper and lower half-planes and for which

$$m(\lambda) \to I, \ as \ \lambda \to \infty,$$

in each half-plane.

The singularities of $m(\lambda)$ are completely specified. There are simple poles at the eigenvalues $\{\lambda_k\}$ and the complex conjugates with residues of a certain specified type (see below).

On the real axis, there is the jump relation

$$m_+(\lambda) = m_-(\lambda)w(\lambda), \;\; m_\pm(\lambda) := \lim_{\epsilon\to 0} m(\lambda \pm i\epsilon)$$

where $w(\lambda)$ is a certain jump matrix built out of $r(\lambda)$ and depending explicitly on x and t. The jump matrix becomes the identity matrix for $b(\lambda) \equiv 0$.

However, if the boundary values $m_\pm(\lambda)$ are continuous, and if $b(\lambda) \equiv 0$, then it is easy to see that the solution $m(\lambda)$ must be a rational function of λ. This is the case we will now develop in more detail. From the column vectors $v^\pm(x,t,\lambda)$, we build a matrix solution of (6)

$$\Psi(\lambda) := [v^-(x,t,\lambda), v^+(x,t,\lambda)]\cdot$$

$$diag\;\left(\;\; \Pi_{j=1}^N(\lambda-\lambda_j)^{-1}, \;\;\; \Pi_{j=1}^N(\lambda-\lambda_j^*)^{-1}\right)\exp(i\sigma_3\lambda^2 t/\epsilon).$$

This special matrix solution of (6) is called a Jost solution. Note that $\Psi(\lambda)$ would also satisfy (6) if it were not for the exponential factor in this formula. The reason for this exponential factor is that the Jost solution matrix has simple large x asymptotics that are, to leading order, independent of t. Indeed, if we define a matrix $m(\lambda)$ by

$$m(\lambda) := \Psi(\lambda)\exp(i\sigma_3\lambda x/\epsilon),$$

then we find using (6) that for all fixed complex λ different from the eigenvalues $\{\lambda_k\}$ and their complex conjugates, $m(\lambda)$ is a uniformly bounded function of x that satisfies $m(\lambda)\to I$ as $x\to+\infty$.

Continuing with the pure soliton case of $b(\lambda) \equiv 0$, we can deduce from the explicit form of the vectors $v^\pm(x,t,\lambda)$ and from the relations (6) that $m(\lambda)$ solves the following problem:

Given the discrete data $\{\lambda_k\}$ and $\{\gamma_k\}$, find a matrix $m(\lambda)$ with the following two properties:

1. Rationality. $m(\lambda)$ is a rational function of λ, with simple poles confined to the eigenvalues $\{\lambda_k\}$ and the complex conjugates. At the singularities:

$$Res_{\lambda=\lambda_k} m(\lambda) = \lim_{\lambda\to\lambda_k} m(\lambda) \begin{pmatrix} 0 & 0 \\ c_k(x,t) & 0 \end{pmatrix},$$
$$Res_{\lambda=\lambda_k^*} m(\lambda) = \lim_{\lambda\to\lambda_k^*} m(\lambda) \begin{pmatrix} 0 & -c_k(x,t)^* \\ 0 & 0 \end{pmatrix},$$

for $k = 0, \ldots, N-1$, with

$$c_k(x,t) = \frac{1}{\gamma_k} \frac{\Pi_{n=0}^{N-1}(\lambda_k - \lambda_n^*)}{\Pi_{n=0,n\neq k}^{N-1}(\lambda_k - \lambda_n)} \exp(2i(\lambda_k x + \lambda_k^2 t)/\epsilon).$$

2. Normalization.

$$m(\lambda) \to I, \ as\ \lambda \to \infty.$$

Whereas we deduced the two properties characterizing the Riemann-Hilbert Problem from the explicit construction of the vector solutions $v^{\pm}(x,t,\lambda)$, it is not difficult to see that these two properties actually characterize the matrix function $m(\lambda)$ uniquely.

PROPOSITION. The meromorphic Riemann-Hilbert Problem corresponding to the discrete data $\{\lambda_k\}$ and $\{\gamma_k\}$ has a unique solution whenever the λ_k are distinct in the upper half-plane and the γ_k are nonzero. The function defined from the solution by

$$u := 2i \lim_{\lambda\to\infty} \lambda m_{12}(\lambda)$$

(that this limit exists is part of the proposition) is a nontrivial N-soliton solution of the focusing nonlinear Schrödinger equation.

REMARK. In the general case when $b(\lambda) \not\equiv 0$, the meromorphic Riemann-Hilbert problem is altered. One only insists that $m(\lambda)$ be piecewise meromorphic in the upper and lower half-planes, and that the boundary values taken from above and below on the real λ-axis satisfy a jump relation with a matrix $w(\lambda)$ built out of

$r(\lambda)$ and going over into the identity matrix when $b(\lambda) \equiv 0$ (and thus $r(\lambda) \equiv 0$). When $r(\lambda) \not\equiv 0$, the corresponding meromorphic Riemann-Hilbert problem cannot be solved by algebraic operations alone, and in general the solution can be obtained by solving a system of integral equations. But even in this more general case, the function $u(x,t)$ defined as above when $r(\lambda)$ decays sufficiently rapidly satisfies the nonlinear Schrödinger equation.

Another very important property of the matrix $m(\lambda)$ is the following "reflection symmetry" in the real axis:

$$m(\lambda^*) = \sigma_2 m(\lambda)^* \sigma_2.$$

On the right-hand side and in other similar formulae, the star denotes component-wise complex conjugation; the matrix is not transposed.

We now show how to convert the meromorphic Riemann-Hilbert problem into a sectionally holomorphic Riemann-Hilbert problem, i.e. how to remove the poles from the problem. Let C be a simple closed contour that is the boundary of a simply-connected domain D in the upper half-plane that contains all of the eigenvalues $\{\lambda_k\}$. We assign to C an orientation ω ($\omega = +1$ means counterclockwise, and $\omega = -1$ means clockwise), and when this orientation is important (i.e. in contour integration and specifying Riemann-Hilbert jump relations), we will write the contour as C_ω. By C^* and D^* we mean the corresponding complex conjugate sets in the lower half-plane, and when we write $[C \cup C^*]_\omega$, we mean that both loops share the same orientation ω.

Next, we will need to interpolate the proportionality constants at the eigenvalues. Choose a constant Q and a function $X(\lambda)$ analytic in D so that

$$\gamma_k = Q \exp(X(\lambda_k)/\epsilon), \;\; k = 0, \ldots.., N-1.$$

In general, $X(\lambda)$ could be systematically constructed as an interpolating polynomial of degree $\sim N$. In other circumstances (see below) the phases γ_k can be highly correlated so that for very large N one can choose for $X(\lambda)$ a polynomial of low degree (or another simple expression). Note that the interpolant of the γ_k is not necessarily unique; for each K in some indexing set there is a distinct pair $(Q_K, X_K(\lambda))$ such that for all j, $\gamma_j = Q_K \exp(X_K(\lambda_j)/\epsilon)$. We will make use of this freedom later; for now we just carry the subscript K.

With the help of the interpolant of the proportionality constants, we define a new matrix $M(\lambda)$ for $\lambda \in \mathbb{C} \setminus (C \cup C^*)$ in the following way. First, for all $\lambda \in D$, set

$$M(\lambda) := m(\lambda) \begin{pmatrix} 1 & 0 \\ -\frac{1}{Q_K}\Pi_{n=0}^{N-1}\frac{\lambda-\lambda_n^*}{\lambda-\lambda_n} exp((2i\lambda x + 2i\lambda^2 t - X_K(\lambda))/\epsilon) & 1 \end{pmatrix}.$$

Next, for all $\lambda \in D^*$, set

$$M(\lambda) := \sigma_2 M(\lambda^*)^* \sigma_2.$$

Finally, for all $\lambda \in \mathbb{C} \setminus (\overline{D} \cup \overline{D}^*)$ i.e. in the rest of the complex plane minus $C \cup C^*$) simply set

$$M(\lambda) := m(\lambda).$$

It is straightforward to verify that by our choice of interpolants and the Blaschke factor that $M(\lambda)$ has no poles in D or D^* and hence is sectionally holomorphic in the complex λ plane. By definition, we have preserved the reflection symmetry of $m(\lambda)$ so that for all $\lambda \in \mathbb{C} \setminus (C \cup C^*)$ we have:

$$M(\lambda^*) = \sigma_2 M(\lambda)^* \sigma_2.$$

The matrix $M(\lambda)$ has continuous boundary values from either side on C and C^*. To describe these, let the left (respectively right) side of the oriented contour $[C \cup C^*]_\omega$ be denoted by "+" (respectively "−"). For $\lambda \in [C \cup C^*]_\omega$ define

$$M_\pm(\lambda) := \lim_{\mu \to \lambda} M(\mu),$$

where the nontangential limits from the left and right sides are to be taken respectively. Then, using the fact that $m(\lambda)$ is analytic on $C \cup C^*$ and the piecewise definition of $M(\lambda)$ we find

$$M_+(\lambda) = M_-(\lambda) w_M(\lambda), \ \lambda \in C_\omega, \tag{10}$$
$$M_+(\lambda) = M_-(\lambda)\sigma_2 v_M(\lambda^*)^* \sigma_2, \ \lambda \in [C^*]_\omega,$$

where for $\lambda \in C$,

$$w_M(\lambda) = \begin{pmatrix} 1 & 0 \\ -\frac{1}{Q_K}\Pi_{n=0}^{N-1}\frac{\lambda-\lambda_n^*}{\lambda-\lambda_n} exp((2i\lambda x + 2i\lambda^2 t - X_K(\lambda))/\epsilon) & 1 \end{pmatrix}. \tag{11}$$

Note that the orientation choice $\omega = \pm 1$ is arbitrary, leading to the same matrix $M(\lambda)$. We allow for both possibilities of orientation for later convenience. Also, observe that if one introduces a discrete measure in the complex plane by

$$d\mu = \sum_{k=0}^{N-1} [\epsilon\delta_{\lambda_k^*} - \epsilon\delta_{\lambda_k}],$$

then for any branch of the logarithm,

$$\Pi_{k=0}^{N-1}\frac{\lambda - \lambda_k^*}{\lambda - \lambda_k} = exp(\frac{1}{\epsilon}\int \log(\lambda - \eta) d\mu(\eta)).$$

The jump matrices are therefore conveniently written in terms of phases

$$\alpha = \int \log(\lambda - \eta)\, d\mu(\eta) + 2i\lambda x + 2i\lambda^2 t - X_K(\lambda), \tag{12}$$
$$mod(2\pi i\epsilon).$$

Suppose the eigenvalues $\{\lambda_k\}$ and proportionality constants $\{\gamma_k\}$ are given along with an appropriate interpolation $Q_K \exp(X_K(\lambda)/\hbar)$ of the γ_k and a smooth closed contour C enclosing the eigenvalues in the upper half-plane. We define a Riemann-Hilbert problem as follows:

HOLOMORPHIC PROBLEM

Given the eigenvalues $\{\lambda_k\}$, the interpolant $Q_K exp(X_K(\lambda)/\epsilon)$, and the oriented contour C_ω, find a matrix function $M(\lambda)$ that satisfies:

1. Analyticity: $M(\lambda)$ is analytic in each component of $\mathbb{C} \setminus (C \cup C^*)$.

2. Boundary behavior: $M(\lambda)$ assumes continuous boundary values on $C \cup C^*$.

3. Jump conditions: The boundary values taken on $[C \cup C^*]_\omega$ satisfy the relations (10) with $w_M(\lambda)$ given explicitly by (11).

4. Normalization: $M(\lambda)$ is normalized at infinity: $M(\lambda) \to I \ as \ \lambda \to \infty$.

PROPOSITION

The holomorphic Riemann-Hilbert Problem above has a unique solution $M(\lambda)$ whenever the λ_k are distinct and nonreal, and the γ_k are nonzero. The function defined by

$$u = 2i \lim_{\lambda \to \infty} \lambda M_{12}(\lambda), \tag{13}$$

is independent of the particular choice of loop contour C and interpolant index K, and is the N-soliton solution of the focusing nonlinear Schrödinger equation corresponding to the discrete data $\{\lambda_k\}$ and $\{\gamma_k\}$.

V. WKB APPROXIMATION. THE AMENDED HOLOMORPHIC RIEMANN-HILBERT PROBLEM

In the focusing case, rigorous results only exist in the case where $S = 0$, i.e. the initial data is real, independent of ϵ. Then WKB theory works and one has:

(i) The continuous spectrum consists of the real line. The reflection coefficient is exponentially small (at least away from $k = 0$, k being the spectral variable).

(ii) The eigenvalues accumulate near a subinterval of the imaginary axis. If, for simplicity, $A(x)$ is a bell-shaped function with a single maximum at $x = 0$, and say $A(0) = A$, then the asymptotic density of eigenvalues is

$$d\mu^{WKB}(\eta) = \rho^0(\eta)\chi_{[0,iA]}d\eta + (\rho^0(\eta))^*\chi_{[-iA,0]}d\eta,$$
$$for\ \eta \in [-iA, iA],$$

where χ_I is the characteristic function of the complex interval I, $d\eta$ is the Lebesgue measure and

$$\rho^0(\eta) = \frac{\eta}{\pi}\int_{x_-(\eta)}^{x_+(\eta)} \frac{dx}{(A(x)^2 + \eta^2)^{1/2}},$$

and $x_\pm(\eta)$ are the positive and negative solutions of $A(x)^2 + \eta^2 = 0$ respectively.

In fact, it has recently been shown [KS] that for any ϵ the eigenvalues are imaginary.

(iii) The norming constant $\gamma(\lambda)$ corresponding to the eigenvalue λ is

$$\gamma(\lambda) = i(-1)^K exp(\frac{i\pi(2K+1)}{\epsilon}\int_\lambda^{iA} \rho^0(\eta)d\eta).$$

In view of the WKB theory, the holomorphic Riemann-Hilbert problem given by (10)-(11) will be substituted by an amended problem.

THE AMENDED HOLOMORPHIC PROBLEM

Find a matrix function $\tilde{M}(\lambda)$ that satisfies:

1. Analyticity: $\tilde{M}(\lambda)$ is analytic in each component of $\mathbb{C} \setminus (C \cup C^*)$.

2. Boundary behavior: $\tilde{M}(\lambda)$ assumes continuous boundary values on $C \cup C^*$.

3. Jump conditions: The boundary values taken on $[C \cup C^*]_\omega$ satisfy the foll-lowing relations:

$$\begin{aligned} \tilde{M}_+(\lambda) &= \tilde{M}_-(\lambda) w_{\tilde{M}}(\lambda), \ \lambda \in C_\omega, \\ \tilde{M}_+(\lambda) &= \tilde{M}_-(\lambda) \sigma_2 v_{\tilde{M}}(\lambda^*)^* \sigma_2, \ \lambda \in [C^*]_\omega, \end{aligned} \tag{10a}$$

where for $\lambda \in C$,

$$w_{\tilde{M}}(\lambda) = \begin{pmatrix} 1 & 0 \\ i(-1)^K exp(\alpha/\epsilon) & 1 \end{pmatrix}, \tag{11a}$$

where α is given by equation (12) but with the measure $d\mu(\eta)$ replaced by $d\mu^{WKB}(\eta)$ as in the previous section.

4. Normalization: $\tilde{M}(\lambda)$ is normalized at infinity: $\tilde{M}(\lambda) \to I \ as \ \lambda \to \infty$.

It can be proved that the amended holomorphic Riemann-Hilbert Problem above has a unique solution $\tilde{M}(\lambda)$. Furthermore, the function defined by

$$u = 2i \lim_{\lambda \to \infty} \lambda \tilde{M}_{12}(\lambda), \tag{13a}$$

is independent of the particular choice of loop contour C and interpolant index K, and is the leading order approximation to the solution of the focusing nonlinear Schrödinger equation corresponding to the initial data $A(x)$.

VI. THE VARIATIONAL PROBLEM

In this section we introduce the variational problem related to semiclassical focusing NLS (see Chapter 8 of [KMM]). Its solution guarantees the possibility of deformation of the amended holomorphic Riemann-Hilbert problem defined in section V to an explicitly solvable Riemann-Hilbert problem. For simplicity we assume that the space variable x is non-negative. Otherwise the formula (14) below should have the opposite sign in front of $2i(zx + z^2t)$.

The space we have to work with is the upper half-plane minus a "spike": $\mathbb{H} = \{z : Imz > 0\} \setminus \{z : Rez = 0, 0 < Imz \leq A\}$. In the closure of this space, $\bar{\mathbb{H}}$, we consider the points ix_+ and ix_-, where $0 \leq x < A$ as different. In other words, we cut a slit in the upper half-plane along the segment $(0, iA)$ and distinguish between the two sides of the slit. The point infinity belongs to $\bar{\mathbb{H}}$, but not $\mathbb{H}$.

The set of distinguished points $\mathbb{A}$ contains the two points 0_+ and 0_-. We define $\mathbb{F}$ to be the set of all continua F in $\bar{\mathbb{H}}$ (i.e. connected compact sets containing more than one point) containing the distinguished points $0_+, 0_-$.

We need to introduce a topology on $\mathbb{F}$. We define the Hausdorff metric as follows. Let $F^\delta = \{z \in \mathbb{C} : \rho(z, F) < \delta\}$, where ρ is the usual distance between compact sets $\rho(E, F) = min_{z \in E, \zeta \in F} |z - \zeta|_C$ and $|z - \zeta|_C$ is the chordal distance (since we want $\bar{\mathbb{H}}$ to be compact). Then, define the Hausdorff distance

$$\rho_{\mathbb{H}}(E, F) = inf\{\delta : E \subset F^\delta, F \subset E^\delta\}.$$

It is known that, since $\bar{\mathbb{H}}$ is compact and complete, so is $\mathbb{F}$.

Our external field ϕ is the function defined as follows. Let $G(z; \eta)$ be the Green's function for the upper half-plane

$$G(z; \eta) = log \frac{|z - \eta^*|}{|z - \eta|}$$

and $d\mu^0(\eta)$ be the nonnegative measure $-\rho^0(\eta)d\eta$ on the segment $[0, iA]$ oriented from 0 to iA. The star denotes complex conjugation. Then

$$\phi(z) = -\int G(z;\eta)d\mu^0(\eta) - Re(i\pi\int_z^{iA}\rho^0(\eta)d\eta + 2i(zx + z^2t)). \tag{14}$$

Here we assume $t \geq 0$, so that $\sigma = 1$.

The WKB density ρ^0 is assumed to admit a holomorphic extension in the upper half-plane. Hence ϕ is a harmonic function in $\mathbb{H}$; it is also continuous in $\bar{\mathbb{H}}$ except at infinity.

Next, let $\mathbb{M}$ be the set of all positive measures on $\mathbb{H}$, such that both the (free) energy

$$E(\mu) = \int\int G(x,y)d\mu(x)d\mu(y), \;\; \mu \in \mathbb{M}$$

and $\int \phi d\mu$ are finite. Also, let

$$V^{\mu}(z) = \int G(z,x)d\mu(x), \;\; \mu \in \mathbb{M}.$$

The weighted energy of the field ϕ is

$$E_{\phi}(\mu) = E(\mu) + 2\int \phi d\mu,$$

for any $\mu \in \mathbb{M}$.

Now, given any $F \in \mathbb{F}$, the equilibrium measure λ^F supported in F is defined by

$$E_{\phi}(\lambda^F) = min_{\mu \in M(F)} E_{\phi}(\mu),$$

where $M(F)$ is the set of measures in $\mathbb{M}$ which are supported in F. It is not always true that such an equilibrium measure exists. The classical Gauss-Frostman theorem guarantees the existence of the equilibrium measure when F does not touch the boundary of the domain $\mathbb{H}$. This is not the case here. Still, one can show that, in the particular case of our special external field, for any given x, t and for any continuum F not containing infinity, the weighted energy is bounded below and

λ^F exists. So, in particular, we do know that the supremum of the equilibrium weighted energies over all continua is finite.

THEOREM 1. Let ϕ be given by (14). Let F be a continuum in $\bar{\mathbb{H}} \setminus \infty$ and suppose that F is contained in some sector $\pi < \alpha < arg(\lambda) < \beta < 0$ as $\lambda \to 0$. Let $M(F)$ be the set of measures $\in \mathbb{M}$ which are supported in F. (So, in particular their free energy is finite and $\phi \in L_1(\mu)$.) We have

$$inf_{\mu \in M(F)} E_\phi(\mu) > -\infty.$$

Furthermore the equilibrium measure on F exists, that is there is a non-negative measure λ^F with support contained in F such that $E_\phi[F] = E_\phi(\lambda^F) = inf_{\mu \in M(F)} E_\phi(\mu)$.

One can eventually prove the following existence theorem for a max-min variational problem.

THEOREM 2. For the external field given by (14), there exists a contour F such that the equilibrium measure λ^F exists and $E_\phi[F] = E_\phi(\lambda^F) = max_{F \in \mathbb{F}} min_{\mu \in M(F)} E_\phi(\mu)$.

The important fact is that the Euler-Lagrange conditions for the above max-min problem guarantee the possibility of deformation of the amended holomorphic Riemann-Hilbert problem described in section V to an explicitly solvable one!

VII. AN EXPLICITLY SOLVABLE RIEMANN-HILBERT PROBLEM

The amended Riemann-Hilbert problem can be asymptotically deformed to a Riemann-Hilbert problem that can be *explicitly* solved. The procedure by which this can be done is long and intricate. It involves the "trick" of conjugating by a "g-function" first introduced by Deift, Venakides and Zhou in their work on KdV (see e.g. [DVZ]). The g-function solves an auxiliary scalar Riemann-Hilbert problem with jump across the real line. Conjugation of the full matrix Riemann-Hilbert problem by the factor $exp(g\sigma_3/h)$ (where h is the appropriate small parameter) reduces it to an explicilty solvable one.

The semiclassical *focusing* NLS analysis extends this idea, in the sense that the scalar function g has cuts along a contour which itself is to be determined. We defer to the discussion in [KMM], mostly Chapter 4, for details. A very short exposition of the ideas is included in the Appendix. We want to underline once more the important fact that the Euler-Lagrange conditions for the above max-min problem of the previous section enable the construction of such a g-function with appropriate properties that will ensure the possibility of asymptotic deformation of the amended holomorphic Riemann-Hilbert problem described in section V to an explicitly solvable one.

The support of the jump matrix that we eventually end up with once all these deformations are pursued will consist of a finite union of analytic arcs, and the jump matrices will be off-diagonal, with determinant 1. Such a problem can be solved in terms of theta functions corresponding to a Riemann surface associated to the band structure defined by the support of the jump matrix.

The analytic arcs that consitute the support of the jump matrix are symmetrically placed with respect to the real axis. In fact, suppose F is the maxi-minimizing contour of Theorem 2 above and suppose that λ^F is the equilibrium measure. It can be shown that[1] the support of λ^F is a union of analytic arcs, say $I_j,\ j=0,...,G$. Then the Riemann-Hilbert problem given by (10a)-(11a) can be asymptotically deformed to an explicitly solvable Riemann-Hilbert problem with jump matrix supported on $\cup_j[I_j \cup I_j^*]$.

We also define the "gaps" $\Gamma_i,\ i=1,...,G/2+1$, as the connected components of $F \setminus \cup_j[I_j \cup I_j^*]$. The gap $\Gamma_i,\ i=1,...,G/2$ connects points λ_{2i-2} and λ_{2i-1} while the gap $\Gamma_{G/2+1}$ connects λ_G to zero.

The explicitly solvable Riemann-Hilbert problem which arises after the deforma-

[1]So far this has been proved under special assumptions on the real analytic data (2). See also section VIII. These assumptions can probably be removed.

tions suggested by the solution of the variational problem is as follows.

Find a matrix function $O(\lambda)$ satisfying:

1. Analyticity: $O(\lambda)$ is analytic for $\lambda \in \mathbb{C} \setminus ((C \setminus \Gamma^+_{G/2+1}) \cup (C^* \setminus \Gamma^-_{G/2+1}))$.

2. Boundary behavior: $O(\lambda)$ assumes boundary values that are continuous except at the endpoints of the bands and gaps, where at worst inverse fourth-root singularities are admitted.

3. Jump conditions: For $\lambda \in I_k^+ \cup I_k^-$, and $k = 0, \ldots, G/2$,

$$O_+(\lambda) = O_-(\lambda) \begin{pmatrix} 0 & iexp(-i\alpha_k/\epsilon) \\ iexp(i\alpha_k/\epsilon) & 0 \end{pmatrix} \tag{15}$$

For $\lambda \in \Gamma_k^+ \cup \Gamma_k^-$, and $k = 1, \ldots, G/2$,

$$O_+(\lambda) = O_-(\lambda) \begin{pmatrix} exp(i\theta_k/\epsilon) & 0 \\ 0 & exp(-i\theta_k/\epsilon) \end{pmatrix}. \tag{16}$$

4. Normalization: $O(\lambda)$ is normalized at infinity:

$$O(\lambda) \to I, \ as \ \lambda \to \infty. \tag{17}$$

Here α_k, θ_k are constants (with respect to λ) which depend explicilty on the initial data (2). They appear as the end product of the Riemann Hilbert problem deformations.

We next need to describe the underlying Riemann surface X. The arcs I_j can be written as $I_0 = [0, \lambda_0], I_j = [\lambda_{2j-1}, \lambda_{2j}],\ j = 1, ..., G/2$, for some even number G. The moduli of the Riemann surface X we will define consist of the points λ_j and their conjugates. The genus of X is G.

Let

$$\begin{aligned} \beta(\lambda)^4 &= \frac{\lambda - \lambda_0^*}{\lambda - \lambda_0} \Pi_{k=1}^{G/2} \frac{\lambda - \lambda_{2k-1}}{\lambda - \lambda_{2k-1}^*} \cdot \frac{\lambda - \lambda_{2k}^*}{\lambda - \lambda_{2k}}, \\ R(\lambda)^2 &:= \Pi_{k=0}^{G} (\lambda - \lambda_k)(\lambda - \lambda_k^*), \end{aligned} \tag{18}$$

choosing the particular branch that is cut along the bands I_k^+ and I_k^- and satisfies $\lim_{\lambda \to \infty} \frac{R(\lambda)}{\lambda^{G+1}} = -1$.

We define the homology cycles $a_j, b_j,\ j = 1, ..., G$ as follows. Cycle a_1 goes around the slit joining λ_0 to λ_0^*, remaining on the top sheet, oriented counterclockwise, a_2 goes through the slits I_{-1} and I_1 starting from the top sheet, also oriented counterclockwise, a_3 goes around the slits $I_{-1}, I_0 \cup I_0^*, I_1$ remaining on the top sheet, oriented counterclockwise, etc. Cycle b_1 goes through I_0 and I_1, cycle b_1 goes through I_{-1} and I_1, also oriented counterclockwise, cycle b_3 goes through I_{-1} and I_2, and around the slits $I_{-1}, I_0 \cup I_0^*, I_1$, oriented counterclockwise, etc.

Define function h by the conditions

$$h_+(\lambda) - h_-(\lambda) = -\theta_k,\ \lambda \in \Gamma_k^+ \cup \Gamma_k^-,\ k = 1, \dots, G/2,$$

$$h_+(\lambda) + h_-(\lambda) = -\alpha_k,\ \lambda \in I_k^+ \cup I_k^-,\ k = 0, \dots, G/2.$$

Let

$$h(\lambda) = k(\lambda) R(\lambda).$$

The jump conditions for k are

$$k_+(\lambda) - k_-(\lambda) = -\frac{\theta_k}{R(\lambda)},\ \lambda \in \Gamma_k^+ \cup \Gamma_k^-,\ k = 1, \dots, G/2,$$
$$k_+(\lambda) + k_-(\lambda) = -\frac{\alpha_k}{R(\lambda)},\ \lambda \in I_k^+ \cup I_k^-,\ k = 0, \dots, G/2.$$

By the Plemelj formula

$$k(\lambda) = \frac{1}{2\pi i} \Sigma_{n=1}^{G/2} \theta_n \int_{\Gamma_n^+ \cup \Gamma_n^-} \frac{d\eta}{(\lambda - \eta) R(\eta)} + \frac{1}{2\pi i} \Sigma_{n=0}^{G/2} \alpha_n \int_{I_n^+ \cup I_n^-} \frac{d\eta}{(\lambda - \eta) R_+(\eta)}.$$

One can see that this function has a singularity of order $(\lambda - \lambda_n)^{-1/2}$ near each endpoint, and vanishes like $1/\lambda$ for large λ. Hence

$$h(\lambda) = p(\lambda) + O(\lambda^{-1}),$$

near infinity, where $p(\lambda)$ is a polynomial of degree G.

On X there is a complex G-dimensional linear space of holomorphic differentials, with basis elements $\nu_k(P)$ for $k = 1, \dots, G$ that can be written in the form

$$\nu_k(P) = \frac{\sum_{j=0}^{G-1} c_{kj} \lambda(P)^j}{R_X(P)}\, d\lambda(P)\,,$$

where $R_X(P)$ is a "lifting" of the function $R(\lambda)$ from the cut plane to X: if P is on the first sheet of X then $R_X(P) = R(\lambda(P))$ and if P is on the second sheet of X then $R_X(P) = -R(\lambda(P))$. The coefficients c_{kj} are uniquely determined by the constraint that the differentials satisfy the normalization conditions:

$$\oint_{a_j} \nu_k(P) = 2\pi i \delta_{jk}.$$

From the normalized differentials, one defines a $G \times G$ matrix H (the period matrix) by the formula:

$$H_{jk} = \oint_{b_j} \nu_k(P).$$

It is a consequence of the standard theory of Riemann surfaces that H is a symmetric matrix whose real part is negative definite. In particular, we can define the theta function

$$\Theta(w) := \sum_{n \in \mathbb{Z}^G} \exp(\frac{1}{2} n^T H n + n^T w),$$

where H is the period matrix associated to X. Since the real part of H is negative definite, the series converges.

The Jacobian variety of X, $Jac(X)$, is simply the complex torus $\mathbb{C}^G/\Lambda$, where Λ is the lattice generated by linear combinations with integer coefficients of the vectors $2\pi i e_k$ and h_k for $k = 1, \ldots, G$. Here e_k denotes the standard unit vectors in $\mathbb{C}^G$ and let h_k denotes the kth column of the matrix H, i.e. $h_k := He_k$ for $k = 1, \ldots, G$.

We arbitrarily fix a base point P_0 on X. The Abel map $A : X \to Jac(X)$ is then defined componentwise as follows:

$$A_k(P; P_0) := \int_{P_0}^{P} \nu_k(P'), \ k = 1, \ldots, G,$$

where P' is an integration variable.

A particularly important element of the Jacobian is the Riemann constant vector K which is defined, modulo the lattice Λ, componentwise by

$$K_k := \pi i + \frac{H_{kk}}{2} - \frac{1}{2\pi i}\sum_{\substack{j=1\\ j\neq k}}^{G} \oint_{a_j} \left(\nu_j(P) \int_{P_0}^{P} \nu_k(P') \right),$$

where the index k varies between 1 and G.

Next, we will need to define a certain meromorphic differential on X. Let $\Omega(P)$ be holomorphic away from the points ∞_1 and ∞_2, where it has the behavior

$$\Omega(P) = dp(\lambda(P)) + \left(\frac{d\lambda(P)}{\lambda(P)^2}\right),\ P \to \infty_1,$$
$$\Omega(P) = -dp(\lambda(P)) + O\left(\frac{d\lambda(P)}{\lambda(P)^2}\right),\ P \to \infty_2,$$

and made unique by the normalization conditions

$$\oint_{a_j} \Omega(P) = 0, j = 1, \ldots, G.$$

Let the vector $U \in \mathbb{C}^G$ be defined componentwise by

$$U_j := \oint_{b_j} \Omega(P).$$

Note that $\Omega(P)$ has no residues.

Let the vectors V_1, V_2 be defined by

$$V_{1,k} = (A_k(\lambda_{1+}^*) + A_k(\lambda_{2+}) + A_k(\lambda_{3+}^*) + \cdots + A_k(\lambda_{G+})) + A_k(\infty) + \pi i + \frac{H_{kk}}{2},$$
$$V_{2,k} = (A_k(\lambda_{1+}^*) + A_k(\lambda_{2+}) + A_k(\lambda_{3+}^*) + \cdots + A_k(\lambda_{G+})) - A_k(\infty) + \pi i + \frac{H_{kk}}{2},$$

where $k = 1, ..., G$, and the + index means that the integral for A is to be taken on the first sheet of X, with base point λ_+^0.

Define $y_1(P)$ and $y_2(P)$ by setting

$$y_m(P) := \frac{\Theta(A(P;P_0) - V_m + iU/\hbar)}{\Theta(A(P;P_0) - V_m)} exp(\frac{i}{\epsilon}\int_{P_0}^{P} \Omega(P')),$$

where $m = 1$ or $m = 2$. The path of integration in the exponent is the same path as in the Abel-Jacobi mapping, but is otherwise unspecified. The fact that these formulae actually define functions that do not depend on that path follows from the transformation laws for the theta function:

$$\Theta(w+2\pi i e_k)=\Theta(w),\ \Theta(w+h_k)=exp(-H_{kk}/2-w_k)\Theta(w).$$

So, if the path of integration is augmented by adding one of the homology cycles a_k, then the exponent is invariant by the normalization condition for $\Omega(P)$. At the same time, we have $A(P;P_0)\to A(P;P_0)+2\pi i e_k$ and by the first transformation law we see that $y_m(P)$ is invariant. Similarly, if the homology cycle b_k is added to the path of integration, then the exponential transforms by producing a factor of $\exp(iU_k/\hbar)$. And at the same time, $A(P;P_0)\to A(P;P_0)+h_k$ and by the second transformation law we again deduce that $y_m(P)$ is invariant. This means that the functions $y_m(P)$ are well-defined given the choices of homology, base point P_0, and representatives V_m.

Now, the zeros of the denominator are exactly the points P_1^m, ... , P_G^m: the liftings of the points $\lambda_1^*,\lambda_2,...\lambda_{G-1}^*,\lambda_G$, to the top $(m=1)$ or bottom $(m=2)$ sheet of X. Since the theta function is entire and the differential $\Omega(P)$ is holomorphic away from the points ∞_1 and ∞_2, it follows that $y_m(P)$ is meromorphic on $X\setminus\{\infty_1,\infty_2\}$, with poles exactly at the points P_1^m, ... , P_G^m. Near the points ∞_1 and ∞_2, we have

$$y_m(P)=exp(ip(\lambda(P))/\epsilon)O(1), P\to\infty_1,$$

$$y_m(P)=exp(-ip(\lambda(P))/\epsilon)O(1), P\to\infty_2,$$

where the leading order term in each expansion depends on m. Let us appropriately normalize the functions $y_1(P)$ and $y_2(P)$.

$$N_1:=\lim_{P\to\infty_1}y_1(P)exp(-ip(\lambda(P))/\epsilon),$$

$$N_2:=\lim_{P\to\infty_2}y_2(P)exp(ip(\lambda(P))/\epsilon).$$

Then, we set

$$z_1(P):=\frac{y_1(P)}{N_1},\ z_2(P):=\frac{y_2(P)}{N_2}.$$

These so-called Baker-Akhiezer functions $z_1(P)$ and $z_2(P)$ do not depend on the choice of homology cycles, the choice of base point P_0, or the choice of representatives V_m. Note that

$$z_1(P) \sim exp(ip(\lambda)/\epsilon)(1+O(\lambda^{-1})),\ P \to \infty_1,$$

$$z_1(P) \sim exp(-ip(\lambda)/\epsilon))O(1),\ P \to \infty_2,$$

$$z_2(P) \sim exp(ip(\lambda)/\epsilon)O(1),\ P \to \infty_1,$$

$$z_2(P) \sim exp(-ip(\lambda)/\epsilon)(1+O(\lambda^{-1})), P \to \infty_2.$$

Now, let $p(\lambda)$ be the polynomial of degree G arising form the asymptotic behavior of h, as above. Construct a matrix function $Q(\lambda)$ by putting together the vectors $(z_1, z_2)^T$ defined for λ in the top and bottom sheet of X respectively. Then Q solves the following Riemann-Hilbert problem.

1. Analyticity. $Q(\lambda)$ is analytic for $\lambda \in \mathbb{C} \setminus \cup_k I_k^{\pm}$.

2. Boundary behavior. $Q(\lambda)$ takes continuous boundary values on $\cup_k I_k^{\pm}$ except at the alternating sequence of endpoints $\lambda_0, \lambda_1^*, \lambda_2, \lambda_3^*, \ldots, \lambda_G$, where inverse square root singularities in the matrix elements are admitted.

3. Jump conditions. On the interior of each oriented band $I_k^{\pm}$, the boundary values of $Q(\lambda)$ satisfy the jump conditions:

$$Q_+(\lambda) = Q_-(\lambda)\sigma_1,\ \lambda \in \cup_k(I_k^+ \cup I_k^-).$$

4. Normalization. $Q(\lambda)$ has an essential singularity at infinity, where it is normalized so that

$Q(\lambda)exp(-ip(\lambda)\sigma_3/\epsilon) = I + O(\lambda^{-1})$ and

$Q(\lambda)exp(-ip(\lambda)\sigma_3/\epsilon) = I + O(\lambda^{-1}),\ as\ \lambda \to \infty.$

Setting $Q = \beta P$, where β is defined by (18), we then see that P satisfies

$$P_+(\lambda) = iP_-(\lambda)\sigma_1,\ \lambda \in \cup_k(I_k^+ \cup I_k^-).$$

Finally, letting

$$P(\lambda) := \tilde{O}(\lambda) exp(ih(\lambda)\sigma_3/\epsilon),$$

with h as above, it is straightforward to check that we recover the (unique) solution of the Riemann-Hilbert problem (15)-(17).

VIII. THE EXPLICIT ASYMPTOTIC FORMULAE

From (13a), it is now easy to write down a formula for the asymptotic behavior of the solution to (1). The actual statement of so-far proved results is as follows.

THEOREM 3. Let $u = u^\epsilon(x,t)$ solve the initial value problem

$$i\epsilon u_t + \frac{\epsilon^2}{2} u_{xx} + |u|^2 u = 0, \tag{1a}$$

where $u^\epsilon(x,t)$ is a complex valued function defined for $x \in \mathbb{R}, t \in \mathbb{R}^+$. Let the "data" $A(x)$ be real analytic, rapidly decaying at infinity, even and "bell-shaped", and such that the function

$$\rho^0(\eta) = \frac{\eta}{\pi} \int_{x_-(\eta)}^{x_+(\eta)} \frac{dx}{(A(x)^2 + \eta^2)^{1/2}},$$

where $x_\pm(\eta)$ are the positive and negative solutions of $A(x)^2 + \eta^2 = 0$ respectively, extends to an analytic function in the upper half-plane.

Suppose ϵ takes values among the sequence

$$\epsilon_N = \frac{-1}{N} \int_0^{imaxA(x)} \rho^0(\eta) d\eta.$$

where N is any given integer.

Suppose $u^\epsilon(x,0)$ is reflectionless and hence the only spectral data consists of eigenvalues and their norming constants. Suppose also the eigenvalues λ_n^ϵ are defined by the Bohr-Sommerfeld condition

$$-\int_{\lambda_n^\epsilon}^{imaxA(x)} \rho^0(\eta) d\eta = \epsilon_N (n + 1/2), \;\; n = 0, 1, ..., N-1.$$

Suppose finally that the proportionality constants are

$$\gamma_n^\epsilon = -exp[\frac{i\pi n}{2}],\ \ n = 0, 1, ..., N-1.$$

Let $G = G(x,t)$ be the genus of the Riemann surface defined by the solution of the variational problem of section VI as the number of the analytic arcs of which the maximinizing contour consists minus one. (We have not proved that the maximinizing contour is uniquely defined, but one can simply choose the smallest such G for given x, t.) Under the above conditions on data (2a) (see also footnote 1), it is finite. Generically (in x, t) the condition that the genus G is applicable at the point x, t is open. Let x_0, t_0 be one of these generic points. The solution u to (1) is asymptotically described (locally) as a slowly modulated $G+1$ phase wavetrain. Setting $x = x_0 + \epsilon\hat{x}$ and $t = t_0 + \epsilon\hat{t}$, and recalling that all quantities depend parametrically on x and t:

$$u = a^0 e^{iU_0^0/\epsilon} e^{i(k_0^0\hat{x} - w_0^0\hat{t})} \frac{\Theta(Y^0 + iU^0/\epsilon + i(k^0\hat{x} - w^0\hat{t}))}{\Theta(Z^0 + iU^0/\epsilon + i(k^0\hat{x} - w^0\hat{t}))} \cdot (1 + O(\epsilon)), \tag{19}$$

where

$$\begin{gathered} a = \frac{\Theta(Z)}{\Theta(Y)} \sum_{k=0}^{G} (-1)^k \Im(\lambda_k), \\ k_n = \partial_x U_n,\ \ w_n = -\partial_t U_n,\ \ n = 0, \dots, G, \end{gathered} \tag{20}$$

and the superscript 0 indicates evaluation for $x = x_0$ and $t = t_0$. Here,

$$U_0 = -(\theta_1 + \alpha_0),\ \ Y = -A(\infty) - V_1,\ \ Z = A(\infty) - V_1,$$

where the $U, \theta_1, \alpha_0, V_1$ are as defined above.

PROOF: The proof that the variational problem of section VI admits a solution of finite genus can be found in [K]. Assuming the finite genus ansatz, the proof of the above Theorem is in [KMM].

In a sense, the Theorem above is unsatisfactory. Rather than dealing with the "real" problem of equation (1a) under data $A(x)$, it substitutes the data by a sequence of reflectionless data $u^\epsilon(x,0)$ as defined above. However, this substitution can be justfied. First, the reflection coefficient *can* be ignored because it is exponentially small everywhere except near 0. Second, the small perturbation of the eigenvalues has no effect on the semiclassical asumptotics to leading order.

THEOREM 4. Let $u = u^\epsilon(x,t)$ solve the initial value problem

$$i\epsilon u_t + \frac{\epsilon^2}{2}u_{xx} + |u|^2 u = 0, u^\epsilon(x,0) = A(x).$$

Suppose again that the function ρ defined above extends to an analytic function in the upper half-plane. Then the semiclassical asymptotics for $u = u^\epsilon(x,t)$ is still given by (19)-(20), up to leading order.

PROOF: There are two things one has to check. First one has to check that the step from the holomorphic Riemann-Hilbert problem defined by (10)-(11) to the amended holomorphic Riemann-Hilbert problem defined by (10a)-(11a) is justified. In other words, the validity of the WKB approximation for the eigenvalues (the Bohr-Sommerfeld rule) has to be justified. This problem has already been solved in the litterature by the introduction of the so-called exact WKB method of the French school (Ecalle, Voros, C.Gérard, Grigis, Ramond, etc.)

More precisely, we begin with the observation [KS] that all eigenvalues of the Zakharov-Shabat problem are imaginary. This has an important consequence.

It is straightforward to rewrite the system (6)-(7) of first-order equations as a semiclassically scaled Schrödinger equation with a slightly perturbed real potential [M]. To do this, make the change of variables

$$y_\pm(x,\lambda) = \frac{v_2(x,\lambda) \pm v_1(x,\lambda)}{(A(x) \pm i\lambda)^{1/2}}.$$

where an arbitrary smooth nonvanishing branch of the square root is selected in each case. One then finds that (6)-(7) is equivalent to the two independent eigenvalue equations

$$\epsilon^2 y''_{\pm}(x,\lambda) = [-(\lambda^2 + A(x)^2) + \epsilon^2 F_{\pm}(x,\lambda)]y_{\pm}(x,\lambda),$$

where

$$F_{\pm}(x,\lambda) = \frac{1}{2}[(log(A(x) \pm i\lambda))']^2 - \frac{[(A(x) \pm i\lambda)^{1/2}]''}{(A(x) \pm i\lambda)^{1/2}}.$$

This is a small $self-adjoint$ perturbation of the standard linear spectral problem for the $self - adjoint$ Schrödinger operator. Rigorous estimates of the eigenvalues with rigorous estimates of the error terms exist in the litterature (see e.g. [R]).

Second, one has to prove that the contribution of the reflection coefficient is indeed negligible. The fact that the reflection coefficient is exponentially small does not a priori guarantee its negligibility. To be precise, note that the inclusion of a non-zero reflection coefficient alters the (say amended) holomorphic Riemann-Hilbert problem by the addition of an extra nontrivial jump along the real line.

$$\tilde{M}_+(\lambda) = \tilde{M}_-(\lambda) \begin{pmatrix} 1 & r(\lambda)e^{\frac{-2i\lambda x - 2i\lambda^2 t}{h}} \\ r^*(\lambda)e^{\frac{2i\lambda x + 2i\lambda^2 x}{h}} & 1 + |r(\lambda)|^2 \end{pmatrix}. \tag{12a}$$

Even though r is indeed exponentially small (away from 0) this does not justify its immediate negligibility. For this to be true one would have to guarantee the uniform boundedness of the solution of the holomorphic Riemann-Hilbert problem defined by (10a)-(11a)-(12a). However, what is true is that $after\ the\ conjugation$ by the term $exp(g\sigma_3/h)$ involving the so-called $g - function$ (see [KMM] Chapter 4 for the detailed discussion of this procedure) the new Riemann-Hilbert problem *does* admit a solution that is uniformly bounded in L^1. The important observation now is that the jump on $\mathbb{R}$ becomes

$$\begin{pmatrix} 1 & r(\lambda)e^{\frac{-2i\lambda x - 2i\lambda^2 t - 2g}{h}} \\ r^*(\lambda)e^{\frac{2i\lambda x + 2i\lambda^2 x + 2g}{h}} & 1 + |r(\lambda)|^2 \end{pmatrix}. \tag{12b}$$

Now it just happens, beacuse of the symmetries involved in the geometry of the Zakharov-Shabat spectrum (see the actual formula for g in [KMM]), that $g \in i\mathbb{R}$ when $\lambda \in \mathbb{R}$. Hence the term $r(\lambda)e^{\frac{-2i\lambda x-2i\lambda^2 t-2g}{h}}$ is (still) exponentially small away from zero. Similarly $r^*(\lambda)e^{\frac{2i\lambda x+2i\lambda^2 x+2g}{h}}$ is exponentially small away from zero.

Now, the terms involving the reflection coefficient can finally be ignored. We omit the details but one may simply follow the discussion of [KMM] with the necessary changes.

IX. THE WHITHAM EQUATIONS

The Euler - Lagrange equations for the variational problem of section VI consist in part of a set of equations for the moduli $\lambda_j, \lambda_j^*, j = 0, ..., G$. In total one has $2G+2$ real-valued equations for the $2G+2$ real unknowns $Re(\lambda_j), Im(\lambda_j)$. A first set of $G+1$ equations are the real equations:

$$M_p := \int_{\cup_k I_k^{\pm}} \frac{2ix+4i\eta t}{R_+(\eta)}\eta^p \, d\eta + 2\Re\left(\int_{\Gamma_I\cap C_I} \frac{\pi i\rho^0(\eta)}{R(\eta)}\eta^p \, d\eta\right) = 0. \tag{21}$$

Let

$$F(\lambda) = \int_{C_I\cup C_I^*} \frac{d\lambda^F(\eta)}{\lambda-\eta}$$

where λ^F is the equilibrium measure of the variational problem in section VI.

The equations

(22)

$$V_k := \Re\left(\int_{\lambda_{2k}}^{\lambda_{2k+1}} \left[2ix+4i\lambda t+\frac{1}{2}(F_+(\lambda)+F_-(\lambda))\right] d\lambda\right) = 0, \;\; k = 0, \dots, G/2-1,$$

form a set of $G/2$ real-valued constraints. Here F_+, F_- denote the non-tangential limits of F along the "+" and "-" side of the contour $supp(d\lambda^F)$, oriented counter-clockwise (and with "+" to the left, "-" to the right).

In fact, the measure λ^F can be explicitly expressed:

$$\lambda^F(\lambda) = \rho^0(\lambda) - \frac{4t}{\pi}R_+(\lambda)\delta_{G,0} + \frac{R_+(\lambda)}{\pi i}\int_{\Gamma_I\cap C_I} \frac{\rho^0(\eta)\, d\eta}{(\lambda-\eta)R(\eta)} + \frac{R_+(\lambda)}{\pi i}\int_{\Gamma_I\cap C_I^*} \frac{\rho^0(\eta^*)^*\, d\eta}{(\lambda-\eta)R(\eta)}.$$

Finally, let

$$B_0(\lambda) = \int_0^\lambda d\lambda^F(\eta)d\eta,\ B_k(\lambda) = \int_{\lambda_{2k-1}}^\lambda d\lambda^F(\eta)d\eta,\ k = 1, \dots, G/2.$$

Then the "measure reality" condition

$$R_k := \Im(B_k(\lambda_{2k})) = 0,\ k = 0, \dots, G/2, \tag{23}$$

gives the remaining set of $G/2+1$ real-valued equations.

By differentiating the above equations (21)-(23) and a series of calculations we end up with a system of (elliptic) equations for the moduli $\lambda_j, \lambda_j^*,\ j = 0, ..., G$. Indeed let the matrix $\tilde{M}$ be defined by

$$\tilde{M}(\lambda) := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_0 & \lambda_0^* & \cdots & \lambda_G^* \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_0^{G+1} & \lambda_0^{*G+1} & \cdots & \lambda_G^{*G+1} \\ \int_{\Gamma_1^+\cup\Gamma_1^-} \frac{R(\eta)\ d\eta}{\eta-\lambda_0} & \int_{\Gamma_1^+\cup\Gamma_1^-} \frac{R(\eta)\ d\eta}{\eta-\lambda_0^*} & \cdots & \int_{\Gamma_1^+\cup\Gamma_1^-} \frac{R(\eta)\ d\eta}{\eta-\lambda_G^*} \\ \vdots & \vdots & \vdots & \vdots \\ \int_{\Gamma_{G/2}^+\cup\Gamma_{G/2}^-} \frac{R(\eta)\ d\eta}{\eta-\lambda_0} & \int_{\Gamma_{G/2}^+\cup\Gamma_{G/2}^-} \frac{R(\eta)\ d\eta}{\eta-\lambda_0^*} & \cdots & \int_{\Gamma_{G/2}^+\cup\Gamma_{G/2}^-} \frac{R(\eta)\ d\eta}{\eta-\lambda_G^*} \\ \int_{I_1^+\cup I_1^-} \frac{R_+(\eta)\ d\eta}{\eta-\lambda_0} & \int_{I_1^+\cup I_1^-} \frac{R_+(\eta)\ d\eta}{\eta-\lambda_0^*} & \cdots & \int_{I_1^+\cup I_1^-} \frac{R_+(\eta)\,d\eta}{\eta-\lambda_G^*} \\ \vdots & \vdots & \vdots & \vdots \\ \int_{I_{G/2}^+\cup I_{G/2}^-} \frac{R_+(\eta)\ d\eta}{\eta-\lambda_0} & \int_{I_{G/2}^+\cup I_{G/2}^-} \frac{R_+(\eta)\ d\eta}{\eta-\lambda_0^*} & \cdots & \int_{I_{G/2}^+\cup I_{G/2}^-} \frac{R_+(\eta)\ d\eta}{\eta-\lambda_G^*} \end{pmatrix}.$$

Then the Whitham equations are

$$\begin{gathered} \left(\frac{\partial\vec{\lambda}}{\partial t}\right)_k + c_k(\vec{\lambda})\left(\frac{\partial\vec{\lambda}}{\partial x}\right)_k = 0, \\ where\ c_k(\vec{\lambda}) := -\frac{\det \tilde{M}^{(k,t)}}{\det \tilde{M}^{(k,x)}} \end{gathered} \tag{24}$$

Here $\tilde{M}^{(k,x)}$ is the matrix obtained from $\tilde{M}$ by replacing the kth column with $\partial G/\partial x$ while $\tilde{M}^{(k,t)}$ is the matrix obtained from $\tilde{M}$ by replacing the kth column with $\partial G/\partial t$, where

$$G(\lambda_0, \lambda_0^*, ..., \lambda_G, \lambda_G^*)^T = [G_1, \dots, G_{2G+2}] := \left[M_0, \dots, M_{G+1}, V_0, \dots, V_{G/2-1}, R_1, \dots, R_{G/2}\right].$$

APPENDIX. THE g-FUNCTION

We begin with the amended holomorphic problem of section V. We seek a matrix function $\tilde{M}$ satisfying the jump conditions (10a)-(11a) and normalized to the identity at infinity.

Let $O = \tilde{M}e^{\frac{g\sigma_3}{\epsilon}}$ where g is a function to be eventually determined and $\sigma_3 = diag(1, -1)$. We want g to be holomorphic everywhere except across a finite union of oriented analytic arcs. On these arcs, the non-tangential limits of g from the +side and -side, say g_+, g_- respectively, are not equal but they will satisfy some prescribed relations. Along these arcs

$$O_+ = O_- w_O,$$

(A1) $$w_O = \begin{pmatrix} e^{\frac{(g_+ - g_-)}{\epsilon}} & 0 \\ i(-1)^K exp(\alpha/\epsilon) e^{\frac{(g_+ + g_-)}{\epsilon}} & e^{\frac{-(g_+ - g_-)}{\epsilon}} \end{pmatrix}, \; Im\lambda > 0,$$

while

(A2) $$w_O(\lambda) = \sigma_2 w_O(\lambda^*)^* \sigma_2, \; for \; Im\lambda < 0.$$

We also want g to vanish as $\lambda \to \infty$, so that the normalization does not change, and impose the symmetry $g(\lambda^*) + g(\lambda)^* = 0$, so that the Riemann-Hilbert problem for O is still symmetric.

It turns out to be convenient to write

(A3) $$g(\lambda) = \int_{C \cup C^*} log(\lambda - \eta)\rho(\eta)d\eta,$$

with a particular choice of the logarithm branch, where both the contour of integration $C \cup C^*$ and the measure ρ are to be determined. The decay and symmetry conditions become

$$\int_{C \cup C^*} \rho(\eta)d\eta = 0,$$

$$\rho(\lambda^*) = \rho(\lambda)^*, \; \lambda \in C \cup C^*.$$

One hopes to eventually be able to choose a "good" contour C such that

(i) $Re(g_+ + g_-) \leq 0$,

(ii) $Re(g_+ - g_-) = 0$,

for $\lambda \in C \cup C^*$.

This will enable us to control the growth of the deformed Riemann-Hilbert problem for O.

It turns out that

1. The conditions above are guaranteed once a solution of the variational problem of section VI whose support is a finite union of analytic arcs exists. In fact the Euler-Lagrange conditions for the variational problem essentially reduce to the equalities and inequalities needed. The equalities provide for the endpoints of the analytic arcs via some highly transcendental equations as well as for the "good" contour C via some ordinary differential equations. Also the solution of the variational problem consists of C and the measure ρ of (A3).

2. It is not hard to deform the new controlled Riemann-Hilbert problem for O to the problem of section VII given by conditions (15), (16), (17). This deformation involves the "openings of lenses" introduced in [DVZ]. All details are in [KMM].

X. REFERENCES

[DVZ] P.Deift, S.Venakides, X.Zhou, New Results in Small Dispersion KdV by an Extension of the Steepest Descent Method for Riemann-Hilbert Problems, IMRN, n.6, pp.285-299, 1997.

[FT] L.D. Fadeev, L.A. Takhtajan, Hamiltonian Methods in the Theory of Solitons, Springer-Verlag, 1987.

[JLM] S.Jin, C.D.Levermore, D.W.McLaughlin, The Semiclassical Limit of the Defocusing NLS Hierarchy, Comm.Pure Appl. Math. v.52, no. 5, 1999, pp.613–654.

[K] S.Kamvissis, A Variational Problem Related to Semiclassical Focusing NLS, Max-Planck-Institut für Mathematik, MPI-2002-93, 2002.

[KMM] S.Kamvissis, K.McLaughlin, P.Miller, Semiclassical Soliton Ensembles for the Focusing Nonlinear Schrödinger Equation, Annals of Mathematics Studies, v.155, Princeton University Press, 2003.

[KS] M.Klaus, J.K.Shaw, Purely Imaginary Eigenvalues of Zakharov-Shabat Systems, Phys. Rev. E (3), v.65, 2002, no. 3, 036607, 5 pp.

[M] P.D.Miller, Some Remarks on a WKB Method for the Nonselfadjoint Zakharov-Shabat Eigenvalue Problem with Analytic Potentials and Fast Phase, Physica D, v. 152/153, 2001, pp.145–162.

[R] T.Ramond, Semiclassical Study of Quantum Scattering on the Line, Comm. Math. Phys. v.177, n.1, 1996, pp.221–254.