PU(2) MONOPOLES. II: TOP-LEVEL SEIBERG-WITTEN MODULI SPACES AND WITTEN'S CONJECTURE IN LOW DEGREES

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Abstract. In this article, a continuation of [10], we complete the proof—for a broad class of four-manifolds—of Witten's conjecture that the Donaldson and Seiberg-Witten series coincide, at least through terms of degree less than or equal to $c-2$, where $c=-\frac{1}{4}(7\chi+11\sigma)$ and χ and σ are the Euler characteristic and signature of the four-manifold. We use our computations of Chern classes for the virtual normal bundles for the Seiberg-Witten strata from the companion article [10], a comparison of all the orientations, and the PU(2) monopole cobordism to compute pairings with the links of level-zero Seiberg-Witten moduli subspaces of the moduli space of PU(2) monopoles. These calculations then allow us to compute low-degree Donaldson invariants in terms of Seiberg-Witten invariants and provide a partial verification of Witten's conjecture.

1. Introduction

1.1. **Main results.** The purpose of the present article, a continuation of [10], is to complete the proof that Witten's conjecture [58] relating the Donaldson and Seiberg-Witten invariants holds in "low degrees" for a broad class of four-manifolds, using the PU(2)-monopole cobordism of Pidstrigatch and Tyurin [51]. We assume throughout that X is a closed, connected, smooth four-manifold with an orientation for which $b_2^+(X)>0$. The Seiberg-Witten (SW) invariants (see §4.1) comprise a function, $SW_X:\mathrm{Spin}^c(X)\to\mathbb{Z}$, where $\mathrm{Spin}^c(X)$ is the set of isomorphism classes of spin^c structures on X. For $w\in H^2(X;\mathbb{Z})$, define

$$\mathbf{SW}^w_X(h)=\sum_{\mathfrak{s}\in\mathrm{Spin}^c(X)}(-1)^{\frac{1}{2}(w^2+c_1(\mathfrak{s})\cdot w)}SW_X(\mathfrak{s})e^{\langle c_1(\mathfrak{s}),h\rangle},\quad h\in H_2(X;\mathbb{R}), \tag{1.1}$$

by analogy with the structure of the Donaldson series $\mathbf{D}^w_X(h)$ [35, Theorem 1.7]. There is a map $c_1:\mathrm{Spin}^c(X)\to H^2(X;\mathbb{Z})$ and the image of the support of SW_X is the set B of SW-basic classes [58]. A four-manifold X has SW-simple type when $b_1(X)=0$ if $c_1(\mathfrak{s})^2=2\chi+3\sigma$ for all $c_1(\mathfrak{s})\in B$, where χ and σ are the Euler characteristic and signature of X. Let $B^\perp\subset H^2(X;\mathbb{Z})$ denote the orthogonal complement of B with respect to the intersection form Q_X on $H^2(X;\mathbb{Z})$. Let $c(X)=-\frac{1}{4}(7\chi+11\sigma)$. As stated in [10], we have:

Theorem 1.1. *Let X be four-manifold with $b_1(X)=0$ and odd $b_2^+(X)\ge 3$. Assume X is abundant, SW-simple type, and effective. For any $\Lambda\in B^\perp$ and $w\in H^2(X;\mathbb{Z})$ for which $\Lambda^2=2-(\chi+\sigma)$ and $w-\Lambda\equiv w_2(X) \pmod 2$, and any $h\in H_2(X;\mathbb{R})$, one has*

$$\begin{aligned}\mathbf{D}^w_X(h)&\equiv 0\equiv \mathbf{SW}^w_X(h)\pmod{h^{c(X)-2}},\\ \mathbf{D}^w_X(h)&\equiv 2^{2-c(X)}e^{\frac{1}{2}Q_X(h,h)}\mathbf{SW}^w_X(h)\pmod{h^{c(X)}}.\end{aligned}\tag{1.2}$$

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The order-of-vanishing assertion for the series $\mathbf{D}^w_X(h)$ and $\mathbf{SW}^w_X(h)$ in equation (1.2) was proved in joint work with Kronheimer and Mrowka [8], based on the results in an earlier version [11] of this article and its companion [10].

The background material underlying the statement of Theorem 1.1—including the definition and significance of "abundant" and "effective" four-manifolds—was explained in [10, §1], so we refer the reader to [10] for details and just briefly mention here some aspects of the statement which may be less familiar.

As customary, $b_2^+(X)$ denotes the dimension of a maximal positive-definite linear subspace $H^{2,+}(X;\mathbb{R})$ for the intersection pairing Q_X on $H^2(X;\mathbb{R})$. It is implicit in the statement of Theorem 1.1 that we have selected an orientation for $H^1(X;\mathbb{R})\oplus H^{2,+}(X;\mathbb{R})$, and the Donaldson and Seiberg-Witten invariants are computed with respect to this choice.

A four-manifold is *abundant* if the restriction of Q_X to $B^\perp$ contains a hyperbolic sublattice [10, Definition 1.2]. This condition ensures that there exist classes $\Lambda\in B^\perp$ with prescribed even square, such as $\Lambda^2 = 2-(\chi+\sigma)$. All compact, complex algebraic, simply connected surfaces with $b_2^+\geq 3$ are abundant. There exist simply connected four-manifolds with $b_2^+\geq 3$ which are not abundant, but which nonetheless admit classes $\Lambda\in B^\perp$ with prescribed even squares [8, p. 175].

As described in [10, Definition 1.3], a four-manifold is *effective* if it satisfies Conjecture 3.1 in [8], restated as Conjecture 3.34 in this article. This conjecture asserts that the pairings of Donaldson-type cohomology classes with the link of a Seiberg-Witten moduli subspace of the (compactified) moduli space of PU(2) monopoles are multiples of its Seiberg-Witten invariants, so these pairings are zero when the Seiberg-Witten invariants for that Seiberg-Witten moduli space are trivial.

For any $w\in H^2(X;\mathbb{Z})$, one can define a Donaldson invariant (see §3.4.2 for a detailed description) as a real-linear function [35, p. 595]

$$D^w_X : \mathbb{A}(X)\to\mathbb{R},$$

where [35]

$$\mathbb{A}(X) = \operatorname{Sym}\left(H_{\text{even}}(X;\mathbb{R})\right)\otimes\Lambda^\bullet(H_{\text{odd}}(X;\mathbb{R})) \tag{1.3}$$

is the graded algebra. If $z\in\mathbb{A}(X)$ is a monomial then $D^w_X(z)=0$ unless

$$\deg(z)\equiv -2w^2-\tfrac{3}{2}(\chi+\sigma)\pmod 8. \tag{1.4}$$

Recall from [35, Equation (1.5)] that the Donaldson series is defined by

$$\mathbf{D}^w_X(h) = D^w_X((1+\tfrac{1}{2}x)e^h) = \sum_{d\geq 0}\tfrac{1}{d!}D^w_X(h^d)+\tfrac{1}{2d!}D^w_X(xh^d),\quad h\in H_2(X;\mathbb{R}). \tag{1.5}$$

A four-manifold with $b_1(X)=0$ and odd $b_2^+(X)\geq 3$ has *Kronheimer-Mrowka (KM) simple type* [35] if for some w and all $z\in\mathbb{A}(X)$,

$$D^w_X(x^2z) = 4D^w_X(z).$$

If as in Theorem 1.1, we do not assume that X has KM-simple type, then equation (1.5) only defines $\mathbf{D}^w_X(h)$ as a formal power series and one cannot necessarily recover all invariants of the form $D^w_X(x^mh^{d-2m})$ from the series (1.5). According to [35, Theorem 1.7], when X has KM-simple type the series $\mathbf{D}^w_X(h)$ is an analytic function of h and there are finitely many characteristic cohomology classes $K_1,\dots,K_m$ (the KM-basic classes) and constants $a_1,\dots,a_m$ (independent of w) so that

$$\mathbf{D}^w_X(h) = e^{\frac{1}{2}Q_X(h,h)}\sum_{r=1}^s(-1)^{\frac{1}{2}(w^2+K_r\cdot w)}a_re^{\langle K_r,h\rangle},\quad h\in H_2(X;\mathbb{R}).$$

More generally [33], a four-manifold X has *finite type* or *type* τ if

$$D^w_X((x^2-4)^\tau z) = 0,$$

for some $\tau\in\mathbb{N}$ and all $z\in\mathbb{A}(X)$. Kronheimer and Mrowka conjectured that all four-manifolds X with $b_2^+(X)>1$ have finite type and state an analogous formula for the series $\mathbf{D}^w_X(h)$; proofs of different parts of their conjecture have been reported Frøyshov [23, Corollary 1], Muñoz [49, Corollary 7.2 & Proposition 7.6], and Wieczorek [57, Theorem 1.3].

For a four-manifold X with $b_1(X)=0$ and odd $b_2^+(X)\geq 3$, Witten's conjecture [58] asserts that X has KM-simple type if and only if it has SW-simple type; if X has simple type, then

$$\mathbf{D}^w_X(h) = 2^{2-c(X)}e^{\frac{1}{2}Q_X(h,h)}\mathbf{SW}^w_X(h),\quad h\in H_2(X;\mathbb{R}). \tag{1.6}$$

Equation (1.2) therefore tells us that Witten's formula holds, modulo terms of degree greater than or equal to $c(X)$, at least for four-manifolds satisfying the hypotheses of Theorem 1.1. Equation (1.2) is proved by considering Seiberg-Witten moduli spaces in the top level, $\ell=0$, of the compactified PU(2) monopole moduli space; in order to prove that equation (1.6) holds modulo h^d for arbitrary $d\geq c(X)$ (and the same w, Λ), one needs to compute the contributions of Seiberg-Witten moduli spaces in arbitrary levels $\ell\geq 0$. In [14] we use the case $\ell=1$ to show that equation (1.6) holds mod $h^{c(X)+2}$.

Equation (1.2) is a special case of a more general formula for Donaldson invariants which we now describe; the hypotheses still include an important restriction which guarantees that the only Seiberg-Witten moduli spaces with non-trivial invariants lie in the top level of the PU(2)-monopole moduli space. When $b_1(X)\geq 0$, the Seiberg-Witten invariants for $(X,\mathfrak{s})$, with $\mathfrak{s}\in\mathrm{Spin}^c(X)$, are defined collectively as a real-linear function (see §4.1 for a detailed description),

$$SW_{X,\mathfrak{s}}:\mathbb{B}(X)\to\mathbb{R}, \tag{1.7}$$

where the graded algebra is given by

$$\mathbb{B}(X) = \mathbb{R}[x]\otimes\Lambda^\bullet(H_1(X;\mathbb{R})). \tag{1.8}$$

Here, $\Lambda^\bullet(H_1(X;\mathbb{R}))$ is the exterior algebra on $H_1(X;\mathbb{R})$, with $\gamma\in H_1(X;\mathbb{R})$ having degree one, and $\mathbb{R}[x]$ is the polynomial algebra with generator x of degree two. If $z\in\mathbb{B}(X)$ then $SW_{X,\mathfrak{s}}(z)=0$ unless

$$\deg(z) = d_s(\mathfrak{s}),$$

where $d_s(\mathfrak{s})$ is the dimension of the Seiberg-Witten moduli space $M_\mathfrak{s}$,

$$d_s(\mathfrak{s}) = \tfrac{1}{4}(c_1(\mathfrak{s})^2-2\chi-3\sigma). \tag{1.9}$$

If $z=x^m\in\mathbb{B}(X)$ and $2m=d_s(\mathfrak{s})$, then as customary [34], [46], [58] one has

$$SW_X(\mathfrak{s}) = SW_{X,\mathfrak{s}}(z). \tag{1.10}$$

As in [50, §1], when $b_1(X)\geq 0$ we call $c_1(\mathfrak{s})\in H^2(X;\mathbb{Z})$ an *SW-basic class* if the Seiberg-Witten function (1.7) is non-trivial. If $b_2^+(X)=1$ or $b_1(X)>0$, there are examples of four-manifolds whose basic classes have positive-dimensional Seiberg-Witten moduli spaces: $\mathbb{CP}^2$ and its blow-ups give examples with $b_2^+(X)=1$ [52], and connected sums of $S^1\times S^3$ and a four-manifold with non-trivial Seiberg-Witten invariants provide examples with $b_1(X)>0$ [50, §2].

For $\Lambda\in H^2(X;\mathbb{Z})$, define

$$i(\Lambda) = \Lambda^2+c(X)+\chi+\sigma. \tag{1.11}$$

If $S(X) \subset \mathrm{Spin}^c(X)$ is the subset yielding non-trivial Seiberg-Witten functions (1.7), let

$$r(\Lambda, c_1(\mathfrak{s})) = -(c_1(\mathfrak{s}) - \Lambda)^2 - \tfrac{3}{4}(\chi + \sigma) \quad \text{and} \quad r(\Lambda) = \min_{\mathfrak{s} \in S(X)} r(\Lambda, c_1(\mathfrak{s})). \tag{1.12}$$

See Remark 3.36 for a discussion of the significance of $r(\Lambda, c_1(\mathfrak{s}))$ and $r(\Lambda)$. We then have:

Theorem 1.2. *Let X be a four-manifold with $b_2^+(X) \geq 1$. Assume $\alpha \smile \alpha' = 0$ for every $\alpha, \alpha' \in H^1(X;\mathbb{Z})$ and that X is effective. Suppose $\Lambda, w \in H^2(X;\mathbb{Z})$ are classes such that $w - \Lambda \equiv w_2(X) \pmod 2$ and, if $b_2^+(X) = 1$, the class $w \pmod 2$ admits no torsion integral lifts. Let $z = x^{\delta_0}\vartheta h^{\delta_2}$, where $h \in H_2(X;\mathbb{R})$, $\vartheta \in \Lambda^{\delta_1}(H_1(X;\mathbb{R}))$, and $x \in H_0(X;\mathbb{Z})$ is the positive generator, and write $\deg(z) = 2\delta$, for $\delta \in \frac{1}{2}\mathbb{Z}$.*

(a) *If $\delta < i(\Lambda)$ and $\delta < r(\Lambda)$, then*

$$D_X^w(z) = 0. \tag{1.13}$$

(b) *If $\delta < i(\Lambda)$ and $\delta = r(\Lambda)$, then*

$$\begin{aligned} D_X^w(z) &= 2^{1-\frac{1}{4}(i(\Lambda)-\delta)-\delta_2-2\delta_0}(-1)^{\delta_0+\delta_1+\frac{1}{2}(\sigma-w^2)+1} \\ &\quad \times \sum_{\{\mathfrak{s} \in S(X):\ r(\Lambda, c_1(\mathfrak{s})) = r(\Lambda)\}} (-1)^{\frac{1}{2}(w^2 + c_1(\mathfrak{s})\cdot(w-\Lambda))} \\ &\quad \times H_{\chi,\sigma}(\Lambda^2, \deg(z), d_s(\mathfrak{s}), \delta_1) SW_{X,\mathfrak{s}}(\vartheta x^{\frac{1}{2}(d_s(\mathfrak{s})-\delta_1)}) \langle c_1(\mathfrak{s}) - \Lambda, h\rangle^{\delta_2}, \end{aligned} \tag{1.14}$$

with $H_{\chi,\sigma}$ defined in equation (1.15). If $\frac{1}{2}(d_s(\mathfrak{s}) - \delta_1) = 0$, then $H_{\chi,\sigma}(\Lambda^2, \deg(z), d_s(\mathfrak{s}), \delta_1) = 1$. If $b_2^+(X) = 1$, then all invariants in equation (1.14) are evaluated with respect to the chambers determined by the same period point in the positive cone of $H^2(X;\mathbb{R})$.

(c) *If Λ^2 and δ satisfy*

$$2r(\Lambda) < 2\delta \leq r(\Lambda) + \tfrac{1}{2}(r(\Lambda) + i(\Lambda)) - 2,$$

then equation (1.14) holds with $D_X^w(z) = 0$.

Remark 1.3.
1. When $\delta_1 > d_s(\mathfrak{s})$, the Seiberg-Witten invariant $SW_{X,\mathfrak{s}}(\vartheta x^{\frac{1}{2}(d_s(\mathfrak{s})-\delta_1)})$ in equation (1.14) is zero by definition.
2. If $z = Yz'$ where $Y \in H_3(X;\mathbb{Z})$ and z' cannot be written as $z' = xz''$ for $x \in H_0(X;\mathbb{Z})$, then equations (1.13) and (1.14) hold but the right-hand-side of (1.14) vanishes.

The hypothesis in Theorems 1.1, 1.2, 1.4, and Corollary 1.5 that X is effective can be eliminated if, in the definition (1.12) of $r(\Lambda)$, we replace $S(X)$ with the (possibly larger) set of all $\mathfrak{s} \in \mathrm{Spin}^c(X)$ for which the Seiberg-Witten moduli space $M_{\mathfrak{s}}$ (as defined in [10, §2.3], with perturbations depending on Λ) is non-empty. This additional generality does not seem to be useful in practice, however.

When $b_2^+(X) = 1$ and $w \pmod 2$ admits no torsion integral lifts, Lemma 4.1 implies that the walls defining the chamber structure for the Donaldson invariant D_X^w are given by the walls for the Seiberg-Witten invariants appearing in equation (1.14). Thus, both sides of the equation will change when the period point crosses one of these walls.

The function $H_{\chi,\sigma}$ in equation (1.14) is defined as

$$H_{\chi,\sigma}(\Lambda^2, \deg(z), d_s(\mathfrak{s}), \delta_1) = 2^d P_d^{a,b}(0), \tag{1.15}$$

where d is a natural number and a, b are integers given by

$$\begin{aligned} d &= \tfrac{1}{2}(d_s(\mathfrak{s}) - \delta_1), \\ a &= \tfrac{1}{4}\left(3r(\Lambda) + i(\Lambda)\right) - \tfrac{1}{2}\deg(z) - d - 2, \\ b &= \tfrac{1}{2}\left(\deg(z) - 2r(\Lambda) - d_s(\mathfrak{s})\right) - \tfrac{1}{4}(\chi + \sigma), \end{aligned}$$

and $P_d^{a,b}(\xi)$ is a *Jacobi polynomial* [29, §8.96],

$$P_d^{a,b}(\xi) = \sum_{u=0}^{d} (-1)^u \binom{a+d}{d-u} \binom{b+d}{u} (\xi - 1)^u (\xi + 1)^{d-u}, \quad \xi \in \mathbb{C}. \tag{1.16}$$

The polynomials $P_d^{a,b}(\xi)$ may in turn be expressed in terms of *hypergeometric functions* [29, §9.10], [38], as we explain in §4.4. Since $c_1(\mathfrak{s})^2 \equiv \sigma \pmod 8$, equation (1.9) for d_s implies that the expression $-\frac{1}{2}d_s - \frac{1}{4}(\chi + \sigma)$ in the definition of b is an integer.

When $\Lambda \in B^\perp \subset H^2(X;\mathbb{Z})$ and X has SW-simple type, the expression (1.12) for $r(\Lambda)$ becomes

$$r(\Lambda) = -\Lambda^2 + c(X) - (\chi + \sigma), \tag{1.17}$$

and $i(\Lambda) + r(\Lambda) = 2c(X)$, by equation (1.11) for $i(\Lambda)$. Theorem 1.2 then simplifies to:

Theorem 1.4. *Let X be a four-manifold with odd $b_2^+(X) \geq 3$ and $b_1(X) = 0$. Assume that X is effective and has SW-simple type. Suppose that $\Lambda \in B^\perp$ and that $w \in H^2(X;\mathbb{Z})$ is a class with $w - \Lambda \equiv w_2(X) \pmod 2$. Let $\delta \geq 0$ and $0 \leq m \leq [\delta/2]$ be integers.*

(a) *If $\delta < i(\Lambda)$ and $\delta < r(\Lambda)$, then for all $h \in H_2(X;\mathbb{R})$ we have*

$$D_X^w(h^{\delta-2m}x^m) = 0. \tag{1.18}$$

(b) *If $\delta < i(\Lambda)$ and $\delta = r(\Lambda)$ we have*

$$\begin{aligned} D_X^w(h^{\delta-2m}x^m) &= 2^{1-\frac{1}{2}(c(X)+\delta)}(-1)^{m+1+\frac{1}{2}(\sigma-w^2)} \\ &\quad \times \sum_{\mathfrak{s}\in\operatorname{Spin}^c(X)} (-1)^{\frac{1}{2}(w^2+c_1(\mathfrak{s})\cdot w)} SW_X(\mathfrak{s}) \langle c_1(\mathfrak{s}) - \Lambda, h\rangle^{\delta-2m}. \end{aligned} \tag{1.19}$$

(c) *If $r(\Lambda) < \delta \leq \frac{1}{2}(r(\Lambda) + c(X) - 2)$, then equation (1.19) holds with $D_X^w(h^{\delta-2m}x^m) = 0$.*

We recall from [22, §6.1.1] that the symmetric algebra $\operatorname{Sym}(H_2(X;\mathbb{R}))$ is canonically isomorphic (as a graded algebra) to the algebra $P(H_2(X;\mathbb{R}))$ of polynomial functions on $H_2(X;\mathbb{R})$. Thus, given the Donaldson invariants $D_X^w(h^{\delta-2m}x^m)$, we can recover all invariants of the form $D_X^w(h_1 \cdots h_{\delta-2m}x^m)$.

Corollary 1.5. [8, Theorem 1.1] *Let X be a four-manifold with odd $b_2^+(X) \geq 3$ and $b_1(X) = 0$. Assume that X is effective, abundant, and SW-simple type, with $c(X) \geq 3$. Then for any $w \in H^2(X;\mathbb{Z})$ with $w \equiv w_2(X) \pmod 2$ we have*

$$\mathbf{SW}_X^w(h) \equiv 0 \pmod{h^{c(X)-2}}.$$

In §4.6 we give a slightly different and more geometric proof of Corollary 1.5 than provided in [8, §2], using the final case of Theorem 1.4. This result proves a conjecture of Mariño, Moore, and Peradze [40], [41] for four-manifolds of Seiberg-Witten simple type, albeit with the additional hypotheses that the four-manifolds are abundant and effective. The vanishing result for Donaldson invariants in Theorem 1.1 can be sharpened: see Theorem 1.2 in [8].

1.2. **Remarks on the hypotheses of Theorems 1.2, 1.4, and 1.1.** To prove Theorem 1.2 (and thus Theorems 1.1, 1.4, and Corollary 1.5), we employ the moduli space of PU(2) monopoles, $\mathcal{M}_{\mathfrak{t}}/S^1$, as a cobordism between a link of the moduli space of anti-self-dual connections, M_κ^w, and the links of moduli spaces of Seiberg-Witten monopoles, $M_{\mathfrak{s}}$. Our application of the cobordism method in this article requires that

1. The codimension of M_κ^w in $\mathcal{M}_{\mathfrak{t}}$, given by twice the complex index of a Dirac operator, is positive (used in Proposition 3.29), and
2. Only the top level of the Uhlenbeck compactification $\bar{\mathcal{M}}_{\mathfrak{t}}$ of the moduli space of PU(2) monopoles contains Seiberg-Witten moduli spaces $M_{\mathfrak{s}}$ with non-trivial invariants (used in Corollary 3.35).

In the proof of Assertions (a) and (b) of Theorem 1.2, one has $2\delta = \deg(z) = \dim M_\kappa^w$ and the hypotheses $\delta < i(\Lambda)$ and $\delta \leq r(\Lambda)$ ensure that Conditions (1) and (2) hold, respectively. In the proof of Assertion (c), one has $\dim M_\kappa^w = 2r(\Lambda)$, which implies that Condition (2) holds while the inequality in the hypothesis of (c) implies that Condition (1) is satisfied.

Assertion (a) follows because the hypothesis $\delta < r(\Lambda)$ implies that there are no Seiberg-Witten moduli spaces contained in $\bar{\mathcal{M}}_{\mathfrak{t}}$ with non-trivial invariants, and so $\mathcal{M}_{\mathfrak{t}}/S^1$ is essentially a null-cobordism of the link of M_κ^w. Assertion (c) follows because the hypothesis on $\deg(z)$ implies that $\deg(z) > M_\kappa^w$, and only the pairings of Donaldson-type cohomology classes with links of Seiberg-Witten moduli spaces can be non-trivial. In the remaining Assertion (b), the hypotheses imply that the cobordism yields an equality between pairings with the link of M_κ^w and a sum of pairings with the links of $M_{\mathfrak{s}}$. The same remarks apply to the hypotheses in Assertions (a), (b), and (c) in Theorem 1.4.

The assumption that $\alpha \smile \alpha' = 0$ for all $\alpha, \alpha' \in H^1(X;\mathbb{Z})$ greatly simplifies the calculation of the Chern classes of the virtual normal bundle of $M_{\mathfrak{s}}$ in $\mathcal{M}_{\mathfrak{t}}$ (see Corollary 3.30 in [10]) and hence its Segre classes (see Lemma 4.11). It should be possible to remove this condition with more work, but this would take us a little beyond the scope of this article and [10]. We plan to address this point elsewhere.

When $b_2^+(X) = 1$, we assume that $w \pmod 2$ does not admit a torsion integral lift in order to avoid complications in defining the chamber in the positive cone of $H^2(X;\mathbb{R})$ with respect to which the Donaldson and Seiberg-Witten invariants are computed. See the comments at the end of §3.4.2 and before Lemma 4.1 for further discussion.

The proof of Theorem 1.1 requires one to choose classes $\Lambda \in B^\perp$ with optimally prescribed even square in order to obtain the indicated vanishing results for the Donaldson and Seiberg-Witten series, as well as compute the first non-vanishing terms. The hypothesis that X is abundant guarantees that one can find such classes, though such choices are also possible for some non-abundant four-manifolds [8].

1.3. **Remarks and conjectures for formulas for Donaldson invariants involving Seiberg-Witten strata in arbitrary levels.** The following remarks are intended to convey an outline of the remainder of our work on a proof of Witten's conjecture in [12], [13], [15], [14]. While some details still remain to be checked, we are confident that the conclusions stated below are correct based on our work thus far, despite their conservatively-stated current status as conjectures rather than firm assertions.

As envisaged in [16], the PU(2)-monopole program proposed by Pidstrigatch and Tyurin [51] for proving Witten's conjecture [44], [58] uses the oriented cobordism $\mathcal{M}_{\mathfrak{t}}/S^1$ between

- The links $\mathbf{L}_{\mathfrak{t},\kappa}^w$ in $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ of the anti-self-dual moduli subspace M_κ^w of $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$, and
- The links $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ in $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ of the Seiberg-Witten moduli subspaces, $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$, of the space of ideal PU(2) monopoles, $\cup_{\ell=0}^\infty(\mathcal{M}_{\mathfrak{t}_\ell} \times \mathrm{Sym}^\ell(X))$, containing $\bar{\mathcal{M}}_{\mathfrak{t}}$.

The program therefore has two principal steps, which we now outline. The first step is to define the links $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ of $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$ for arbitrary $\ell \geq 0$ using the gluing construction of [12], [13], extending the construction in [10] which just treats the case $\ell = 0$. The oriented cobordism $\mathcal{M}_{\mathfrak{t}}/S^1$ then yields a formula (with $\deg(z) = 2\delta$),

$$D^w_X(z) = -2^{-\delta_c} \sum_{\mathfrak{s} \in \mathrm{Spin}^c(X)} (-1)^{\frac{1}{4}(w-\Lambda+c_1(\mathfrak{s}))^2} \langle \mu_p(z) \smile \mu_c^{\delta_c}, [\mathbf{L}_{\mathfrak{t},\mathfrak{s}}] \rangle, \tag{1.20}$$

where $\mu_p(z)$ and μ_c are Donaldson-type cohomology classes, and $\delta_c = \frac{1}{4}(i(\Lambda) - \delta) - 1$. Work in progress [15] then strongly indicates that the pairings on the right-hand side of equation (1.20) have the following general form, when $b_1(X) = 0$ and $z = x^m h^{\delta-2m}$,

$$\langle \mu_p(z) \smile \mu_c^{\delta_c}, [\mathbf{L}_{\mathfrak{t},\mathfrak{s}}] \rangle = SW_X(\mathfrak{s}) \sum_{i=0}^{r} \boldsymbol{\delta}_{d,i} \left(\langle c_1(\mathfrak{s}), h \rangle, \langle \Lambda, h \rangle \right) Q_X^{\ell-i}(h,h), \tag{1.21}$$

where $r = \min(\ell, [\delta/2] - m)$, $\ell = \frac{1}{4}(\delta - r(\Lambda, c_1(\mathfrak{s})))$, $d = \delta - 2(m+\ell-i)$, and $\boldsymbol{\delta}_{d,i}$ is a degree-d, homogeneous polynomial (aside from stray powers of (-1) and 2) in two variables with coefficients which are degree-i polynomials in $2\chi \pm 3\sigma$, $(c_1(\mathfrak{s}) - \Lambda)^2$, Λ^2, and $(c_1(\mathfrak{s} - \Lambda) \cdot c_1(\mathfrak{s})$.

An interesting feature of the formula (1.21) is that one sees a factorization of the pairings $\langle \mu_p(z) \smile \mu_c^{\delta_c}, [\mathbf{L}_{\mathfrak{t},\mathfrak{s}}] \rangle$ into a product of $SW_X(\mathfrak{s})$ and the term $\boldsymbol{\delta}_d = \sum_{i=1}^r \boldsymbol{\delta}_{d,i} Q_X^{\ell-i}$. In particular, the pairing (1.21) vanishes when $SW_X(\mathfrak{s}) = 0$, implying that X is "effective" in the sense described in §1.1. The factors $\boldsymbol{\delta}_{d,i}$ are similar to those appearing in the Kotschick-Morgan conjecture [9], [32] for the form of the wall-crossing formula for the Donaldson invariants of a four-manifold X with $b_2^+(X) = 1$. However, $\boldsymbol{\delta}_{d,i}$ is a polynomial in two variables while the corresponding term in the conjectured wall-crossing formula for Donaldson invariants is a polynomial in only one variable.

Explicit, direct computations of pairings with the links $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ of ideal Seiberg-Witten moduli spaces, $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$, are possible when $\ell = 0, 1$ or 2: indeed, Theorem 1.1 is proved using the case $\ell = 0$ and we prove an $\ell = 1$ analogue of Theorem 1.1 in [14], while the case $\ell = 2$ would follow by adapting work of Leness in [36]. However, direct computations appear intractable when ℓ is large.

The idea underlying the second step of the program is to use the existence of formulas (1.20) and (1.21) in conjunction with auxiliary arguments to prove Witten's conjecture, since more direct calculations of the link pairings appear difficult. The work of Göttsche [27] suggests that such indirect strategies should succeed, as he was able to compute the wall-crossing formula for the Donaldson invariants of four-manifolds with $b_2^+(X) = 1$ and $b_1(X) = 0$, under the assumption that the Kotschick-Morgan conjecture [32] holds for such four-manifolds. The facts that $\boldsymbol{\delta}_{d,i}$ is a function of two variables and both χ and σ may vary independently indicate that this second step in the PU(2)-monopole program is potentially more complicated than that of [27], where the assumption that $b_2^+(X) = 1$ implies that $\sigma = 1 - b_2^-(X)$ and $\chi = 3 + b_2^-(X)$ (when $b_1(X) = 0$). However, in the case of the PU(2)-monopole program there are more sources of "recursion relations" of the type used by Göttsche, in addition to those arising from the blow-up formulas of Fintushel-Stern [20], [19] (for Donaldson and Seiberg-Witten invariants). Moreover, there is a rich supply of examples of four-manifolds where Witten's conjecture is known to hold.

1.4. **Remarks on abelian localization.** One of the first observations concerning the moduli space of PU(2) monopoles is that the Donaldson stratum, $\iota(M^w_\kappa)$, and the Seiberg-Witten strata, $\iota(M_{\mathfrak{s}})$, are fixed-point sets under the circle action given by scalar multiplication on the spinor components of PU(2)-monopole pairs; see [10, §3.1] for a detailed account. This raises the question of whether the technique of abelian localization, as applied to circle actions on compact manifolds [1] or its generalizations to singular algebraic varieties (for example, see [28]), can be usefully applied here to prove Witten's conjecture. As we indicate below, if the moduli space of PU(2) monopoles were a compact manifold and the Donaldson and Seiberg-Witten strata were smooth submanifolds, then an application of the localization formula would be equivalent to our construction of links and application of the PU(2)-monopole cobordism. There is no saving of labor and the essential point remains, with either equivalent view, to compute the Chern classes of the normal bundles of the fixed-point sets. As the moduli space of PU(2) monopoles is non-compact, equipped with the highly singular Uhlenbeck compactification or somewhat less singular but more complicated bubble-tree compactification, our application of the PU(2)-monopole cobordism can be thought of as an extension of the localization method to those differential-geometric, singular settings.

The technique of abelian localization does not reduce the information about neighborhoods of singularities needed to compute intersection pairings because the localization formula requires a computation of an equivariant Thom or Euler class of the normal bundle of the fixed point set. For example, if $F \subset M$ is the fixed point set and $N \to F$ is its normal bundle, the equivariant Euler class of N is the Euler class $e(N_{S^1})$ of the bundle

$$N_{S^1} = \mathrm{ES}^1 \times_{S^1} N \to \mathrm{BS}^1 \times F,$$

where $\mathrm{ES}^1 \to \mathrm{BS}^1$ is the universal S^1 bundle over the classifying space BS^1. Let

$$\pi_{S^1} : \mathrm{ES}^1 \times_{S^1} M \to \mathrm{BS}^1 \quad \text{and} \quad \iota : \mathrm{ES}^1 \times_{S^1} N \to \mathrm{ES}^1 \times_{S^1} M,$$

be the projection and embedding maps, respectively. If $\dim M = m$, then because

$$H^m_{S^1}(M \setminus F; \mathbb{R}) \cong H^m((M \setminus F)/S^1; \mathbb{R}),$$

any class $x \in H^m_{S^1}(M;\mathbb{R})$ has compact support in $\mathrm{ES}^1 \times_{S^1} N$ by dimension counting. The abelian localization formula [1, Equation (3.8)] states that

$$(\pi_{S^1})_* x = \frac{\iota^* x}{e(N_{S^1})}/[F]. \tag{1.22}$$

If h is the pullback of the universal first Chern class from BS^1 to $\mathrm{BS}^1 \times F$ and $\pi_F : \mathrm{BS}^1 \times F \to F$ is the projection, then the splitting principle shows that $e(N_{S^1}) = \sum_{i=0}^r h^i \pi_F^* c_{r-i}(N)$, where $r = \operatorname{rank} N$. A simple algebraic computation (see [14]) shows that computing $(\pi_{S^1})_* x$ using formula (1.22) and finding the inverse of $e(N_{S^1})$ is equivalent to computing the Segre classes of N.

Thus, if $\mathcal{M}_{\mathfrak{t}}$ were a compact manifold and the Seiberg-Witten moduli spaces $M_{\mathfrak{s}}$ were smooth submanifolds of $\mathcal{M}_{\mathfrak{t}}$, the abelian localization method would be equivalent to the one used in this article.

If X is a complex algebraic surface, it should be possible to construct the Gieseker compactifiication for the moduli space of PU(2) monopoles over X, by analogy with the construction of Morgan [45] and Li [37] for the moduli space of PU(2) monopoles, and then apply the results of [28] to this compactification. However, one would still need to compute the equivariant Euler classes of the normal sheaves of strata of ideal, reducible PU(2) monopoles, in order to apply [28, Equation (1)]. If X is not algebraic, one would need to solve the non-trivial problem of defining the normal sheaves of these strata in gauge-theoretic compactifications.

1.5. **A guide to the article and outline of the proofs of the main results.** The present article is a direct continuation of [10] and rather than repeat many of the definitions here, we shall refer to [10]. The reader may wish to consult the notational index in [10], as well as the notational index for the present article which appears just before §2. The first version of this article was distributed in December 1997 as sections 4–7 of the preprint [11].

As in [10], we let $\mathfrak{s} = (\rho, W)$ denote a spin^c structure on X, where W is a Hermitian, rank-four bundle over X and use $\mathfrak{t} = (\rho, V)$ to denote a "spin^u structure" on X, where V is a Hermitian, rank-eight bundle over X [10, Definition 2.2].

One of the main results (Theorem 3.31) of [10] is a calculation the Chern characters of vector bundles defining tubular neighborhoods of links of Seiberg-Witten moduli spaces in local, "thickened" moduli spaces of PU(2) monopoles $\mathcal{M}_{\mathfrak{t}}$. In §2 of this article, we compare the orientations of the moduli spaces and anti-self-dual connections, Seiberg-Witten monopoles, and their links in the moduli space of PU(2) monopoles. In §3 we define cohomology classes and dual geometric representatives on the moduli space of PU(2) monopoles and in §4 we prove Theorem 1.2 by counting the intersection of these representatives with links of the moduli spaces of anti-self-dual connections and top-level Seiberg-Witten monopoles.

Compatible choices of orientations for all the moduli spaces appearing in the stratification [10, Equation 3.17] of $\mathcal{M}_{\mathfrak{t}}$ and of the links $\mathbf{L}^w_{\mathfrak{t},\kappa}$ and $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ are a basic requirement of the cobordism method and a discussion of our orientation conventions is taken up in §2. As in [10, Equation (2.40)], we let $\mathcal{M}^{*,0}_{\mathfrak{t}} \subset \mathcal{M}_{\mathfrak{t}}$ denote subspace represented by PU(2) monopoles which are neither zero-section pairs (corresponding to anti-self-dual connections) or reducible pairs (corresponding to Seiberg-Witten monopoles). In §2.1 we show that $\mathcal{M}^{*,0}_{\mathfrak{t}}/S^1$ (the smooth locus or top stratum of $\mathcal{M}_{\mathfrak{t}}/S^1$) is orientable, with an orientation determined by a choice of an orientation for the moduli space of anti-self-dual connections, M^w_κ, as explained further in §2.2: this allows us to compute the oriented intersection of one-manifolds with the link $\mathbf{L}^w_{\mathfrak{t},\kappa}$, where the one-manifolds arise as the intersection of geometric representatives of the cohomology classes on $\mathcal{M}^{*,0}_{\mathfrak{t}}/S^1$. We also define an orientation for $\mathcal{M}^{*,0}_{\mathfrak{t}}/S^1$ induced by an orientation of $M_{\mathfrak{s}}$, as in §2.3, and this allows us to compute the oriented intersection of one-manifolds with the link $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$. In §2.4 we compare the two orientations of $\mathcal{M}^{*,0}_{\mathfrak{t}}/S^1$ naturally induced by those of M^w_κ and $M_{\mathfrak{s}}$. Finally, in §2.5 we compare the natural orientations of these links with the orientations obtained by considering them as boundaries of the complement in $\mathcal{M}_{\mathfrak{t}}/S^1$ of small open neighborhoods of the anti-self-dual and Seiberg-Witten strata.

In §3.1 and 3.2 we describe the cohomology classes $\mu_p(\beta)$ on the moduli space $M^*_{\mathfrak{t}}/S^1$ and μ_c on $\mathcal{M}^{*,0}_{\mathfrak{t}}/S^1$ and their dual geometric representatives $\mathcal{V}(\beta)$ and $\mathcal{W}$, following the methods of [4], [6], [7], [35] for the classes $\mu_p(\beta)$. A technical complication not present when dealing solely with M^w_κ is that the lower strata of $\bar{\mathcal{M}}_{\mathfrak{t}}$ have smaller codimension than those of $\bar{M}^w_\kappa$. The unique continuation property for reducible PU(2) monopoles [17, Theorem 5.11] plays a role here analogous to that of the unique continuation property for reducible anti-self-dual SO(3) connections in [7], [35]. In §3.3, we show that the closures $\bar{\mathcal{V}}(\beta)$ and $\bar{\mathcal{W}}$ of these geometric representatives in $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ meet the lower strata of $\bar{\mathcal{M}}^{*,0}_{\mathfrak{t}}/S^1$ transversely, that is, in a subspace of the expected codimension away from the reducible and zero section solutions appearing in lower levels. Thus, the closure of the one-manifolds will have boundaries in $\bar{M}^w_\kappa$ or in some stratum $M_{\mathfrak{s}} \times \Sigma$ of reducible PU(2) monopoles, where $\Sigma \subset \mathrm{Sym}^\ell(X)$. The hypotheses of Theorem 1.2 exclude consideration of the more difficult case $\ell > 0$. In §3.4 we show that the number of points, counted with sign, in the boundaries of the one-manifolds defined by an appropriate choice of geometric representatives in the link $\mathbf{L}^w_{\mathfrak{t},\kappa}$ of the stratum M^w_κ is given by a multiple of the Donaldson invariant, thus completing the proof of Theorem 3.33. In the course of proving this result we also show that $\mathcal{M}_{\mathfrak{t}}^{*,0}$ is nonempty for sufficiently negative $p_1(\mathfrak{t})$ — see Proposition 3.30 in §3.4.

In §4 we calculate the intersection of these geometric representatives with the link $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ of the stratum $M_{\mathfrak{s}}$ and show that it is given by a multiple of the Seiberg-Witten invariant associated to the spin^c structure $\mathfrak{s}$ (see Theorem 4.13). The geometric representatives, in general, intersect the strata $M_{\mathfrak{s}}$ in sets of higher than expected dimension, so our calculation of the link pairings here may be viewed as a differential-geometric analogue of the "excess intersection theory" calculations discussed in [24]. Combining the link pairing calculations of §3 and §4 and applying the cobordism $\mathcal{M}_{\mathfrak{t}}^{*,0}/S^1$ then yields the formulas for Donaldson invariants in terms of Seiberg-Witten invariants in Theorem 1.2, from which Theorems 1.1 and 1.4 and Corollary 1.5 are derived in §4.6.

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Index of Notation

$\mathbb{A}(X)$	Equation (1.3)
$\mathbb{B}(X)$	Equation (1.8)
$\mathcal{B}^w_\kappa$	[10, §2.1.6]
$C_{\chi,\sigma}(\cdot)$	Theorem 4.13
$\mathcal{C}_\mathfrak{t}$, $\tilde{\mathcal{C}}_\mathfrak{t}$, $\mathcal{C}^{*,0}_\mathfrak{t}$, $\mathcal{C}^*_\mathfrak{t}$, $\mathcal{C}^0_\mathfrak{t}$	[10, §2.1.5]
D_A, $D_{A,\vartheta}$, D_B, $D_{B,\vartheta}$	Equation (2.4)
$\boldsymbol{\mathcal{D}}_\mathfrak{t}$	§2.1
$\mathcal{D}^n$, $\mathcal{D}^t$	[10, Equation (3.35)]
D^w_X	Equation (3.31)
$\mathbf{D}^w_X$	Equation (1.5)
$\mathbb{F}^w_\kappa$	Equation (3.4)
$\mathbb{F}_\mathfrak{t}$	Equation (3.2)
$\mathcal{G}_\mathfrak{s}$	[10, §2.3]
$\mathcal{G}_\mathfrak{t}$	[10, Definition 2.6]
$H_{\chi,\sigma}(\cdot)$	Equation (1.15)
$\mathbf{K}_{A,\delta}$	Equation (3.46)
$\mathbb{L}_\mathfrak{s}$	Equation (4.1)
$\mathbb{L}_\mathfrak{t}$	Equation (3.12)
$\mathbf{L}^w_{\mathfrak{t},\kappa}$, $\mathbf{L}^{w,\varepsilon}_{\mathfrak{t},\kappa}$	Equation (3.43)
$\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$	[10, Definition 3.22]
M^w_κ, $\bar{M}^w_\kappa$	[10, §2.1.6 & 2.2]
$M_\mathfrak{s}$, $\tilde{M}_\mathfrak{s}$	[10, §2.3]
$\mathcal{M}_\mathfrak{t}$, $\mathcal{M}^{*,0}_\mathfrak{t}$	[10, Equations (2.33) & (2.40)]
$\bar{\mathcal{M}}_\mathfrak{t}$	[10, Equation (2.46)]
$\bar{\mathcal{M}}^{\text{asd}}_\mathfrak{t}$	Equation (3.28)
$\bar{\mathcal{M}}^*_\mathfrak{t}$, $\bar{\mathcal{M}}^0_\mathfrak{t}$, $\bar{\mathcal{M}}^{*,0}_\mathfrak{t}$	Equation (3.18)
$\bar{\mathcal{M}}^{\geq\varepsilon}_\mathfrak{t}$, $\bar{\mathcal{M}}^{*,\geq\varepsilon}_\mathfrak{t}$	Equation (3.19)
$\mathcal{M}_\mathfrak{t}(\Xi,\mathfrak{s})$	[10, Definition 3.20]
$N_\mathfrak{t}(\Xi,\mathfrak{s})$, $\tilde{N}_\mathfrak{t}(\Xi,\mathfrak{s})$,	[10, §3.5]
$N^\varepsilon_\mathfrak{t}(\Xi,\mathfrak{s})$, $N^{\leq\varepsilon}_\mathfrak{t}(\Xi,\mathfrak{s})$	§2.5
$O^{\text{asd}}(\Omega,w)$	Definition 2.3
$O^{\text{red}}(\Omega,\mathfrak{t},\mathfrak{s})$	Definition 2.5
$\mathcal{O}_A$, $\mathcal{O}'_A$	Lemma 3.22
$P^{a,b}_d$	Equation (1.16)
SW_X	Equation (1.10)
$SW_{X,\mathfrak{s}}$	Equation (4.5)
$\mathbf{SW}^w_X$	Equation (1.1)
$\mathcal{V}(z)$, $\mathcal{W}$	§3.2.3 & §3.2.4
$\bar{\mathcal{V}}(z)$, $\bar{\mathcal{W}}$	Definition 3.14
$\tilde{X}$	§3.4.2
$\mathcal{Z}_A$	Equation (3.40)
$(a)_n$	Equation (4.38)
$c(X)$	§1.1
$c_1(\mathfrak{t})$, $p_1(\mathfrak{t})$, $w_2(\mathfrak{t})$	[10, Equation (2.19)]
$d_a(\mathfrak{t})$	Equation (3.21)
$d_s(\mathfrak{s})$	Equation (1.9)
$\deg(z)$	Equation (3.15)
$\det\mathcal{D}_{A,\Phi}$, $\det\boldsymbol{\mathcal{D}}_\mathfrak{t}$	Equation (2.1)
$\det\delta_{\hat{A}}$, $\det\boldsymbol{\delta}_E$	Equation (2.2)
f_A	Lemma 3.22
g_A	Equation (3.53)
$i(\Lambda)$	Equation (1.11)
$\boldsymbol{\ell}$	Equation (3.42)
$n_a(\mathfrak{t})$	Equation (3.21)
$n'_s(\mathfrak{t},\mathfrak{s})$, $n''_s(\mathfrak{t},\mathfrak{s})$	Equation (4.25)
$o(\Omega,w)$	§2.2
$o_\mathfrak{t}(w,\mathfrak{s})$	Equation (3.66)
$r(\Lambda)$, $r(\Lambda,\mathfrak{s})$	Equations (1.12) & (1.17)
$\mathfrak{s}$	[10, Definition 2.1]
$\mathfrak{s}^\pm$	§4.1
s_i, $s_i(N)$	Lemma 4.10
$\mathfrak{t}$	[10, Definition 2.2]
$\tilde{\mathfrak{t}}$	Lemma 4.19
$\mathfrak{t}_\ell$	[10, Equation (2.44)]
Ξ	[10, Definition 3.20]
$\boldsymbol{\gamma}_A$	Equation (3.36)
$\boldsymbol{\delta}_E$	§2.1
δ_c	Equation (3.26)
δ_i	Equation (3.15)
δ_p	Equation (4.33)
ι	[10, Equations (3.4) & (3.9)]
κ	§3.1
μ_c	Equation (3.12)
μ_p	Equation (3.3)
$\mu_\mathfrak{s}$	Equations (4.3) & (4.4)
ν	Definition 4.3
$\pi_\mathcal{B}$	Equation (2.6)
ϱ	Equation (4.10)
ϱ_L	Equation (4.11)
σ, χ,	§1.1
$\boldsymbol{\varphi}_A$	Equation (3.37)
$\omega(\mathbf{K},o)$	Equation (3.48)
$\omega(\mathbf{L},\partial O)$	Equation (3.47)
$\omega(\mathcal{Z}\cap\mathbf{K},\partial O)$	Equation (3.50)
$\omega(Z)$, $\omega(Z)$	Convention 3.23

2. Orientations of moduli spaces

Following the pattern in [5] and [7, §5.4 & §7.1.6], we first show that $\mathcal{M}_{\mathfrak{t}}^{*,0}$ is an orientable manifold and then show that its orientation is canonically determined by a choice of homology orientation of our four-manifold X and an integral lift w of $w_2(\mathfrak{t})$. The orientation for $\mathcal{M}_{\mathfrak{t}}^{*,0}$ will be invariant under the circle action and thus give an orientation for $\mathcal{M}_{\mathfrak{t}}^{*,0}/S^1$. We also obtain relations between the orientations of the smooth, top stratum $\mathcal{M}_{\mathfrak{t}}^{*,0}/S^1$, the stratum $M_\kappa^w \hookrightarrow \mathcal{M}_{\mathfrak{t}}/S^1$ defined by the anti-self-dual moduli space, and the strata $M_{\mathfrak{s}} \hookrightarrow \mathcal{M}_{\mathfrak{t}}/S^1$ defined by the Seiberg-Witten moduli spaces.

2.1. Orientability of moduli spaces of PU(2) monopoles.

In this section we show that $\mathcal{M}_{\mathfrak{t}}^{*,0}$ is orientable.

As in [10, §2.1.5], we let $\tilde{\mathcal{C}}_{\mathfrak{t}}$ denote the pre-configuration space of pairs (A,Φ), where A is a spin connection on $V = V^+ \oplus V^-$ with fixed determinant connection $A^{\det} = 2A_\Lambda$ on $\det(V^+)$ and Φ is a section of V^+. We defined $\mathcal{C}_{\mathfrak{t}} = \tilde{\mathcal{C}}_{\mathfrak{t}}/\mathcal{G}_{\mathfrak{t}}$, where $\mathcal{G}_{\mathfrak{t}}$ is the group spin^u automorphisms of V [10, Definition 2.6].

Recall that $\mathcal{D}_{A,\Phi} = d_{A,\Phi}^{*,0} + d_{A,\Phi}^1$ is the "deformation operator" corresponding to the elliptic deformation complex [10, Equation (2.47] for the moduli spaces $\mathcal{M}_{\mathfrak{t}}$. Let $\det \boldsymbol{\mathcal{D}}_{\mathfrak{t}}$ be the real determinant line bundle over the pre-configuration space $\tilde{\mathcal{C}}_{\mathfrak{t}}$, with fiber over $(A,\Phi) \in \tilde{\mathcal{C}}_{\mathfrak{t}}$ given by

$$\det \mathcal{D}_{A,\Phi} = \Lambda^{\max}(\operatorname{Ker} \mathcal{D}_{A,\Phi}) \otimes \Lambda^{\max}(\operatorname{Coker} \mathcal{D}_{A,\Phi})^*. \tag{2.1}$$

(See [7, §5.2.1] for the construction of determinant line bundles for families of elliptic operators.) The kernel and cokernel of $\mathcal{D}_{A,\Phi}$ are equivariant with respect to the actions of group $\mathcal{G}_{\mathfrak{t}} \times_{\{\pm 1\}} S^1$. The stabilizer, in $\mathcal{G}_{\mathfrak{t}} \times_{\{\pm 1\}} S^1$, of the pair (A,Φ) acts trivially on the fibers of $\det \boldsymbol{\mathcal{D}}_{\mathfrak{t}}$ because this stabilizer is connected and the structure group of the fiber of $\det \boldsymbol{\mathcal{D}}_{\mathfrak{t}}$ (which is a real line) is $\{\pm 1\}$ and thus disconnected. Hence, the bundle $\det \boldsymbol{\mathcal{D}}_{\mathfrak{t}}$ descends to $\mathcal{C}_{\mathfrak{t}}/S^1$ and so to $\mathcal{C}_{\mathfrak{t}}$ as well. We will show that the bundle $\det \boldsymbol{\mathcal{D}}_{\mathfrak{t}} \to \mathcal{C}_{\mathfrak{t}}/S^1$ is trivial.

Motivated by the remarks of [22, p. 330], we say that an orientation for $\mathcal{M}_{\mathfrak{t}}$ is a choice of orientation for the real line bundle $\det \boldsymbol{\mathcal{D}}_{\mathfrak{t}}$ (restricted to $\mathcal{M}_{\mathfrak{t}}$): an orientation of the fibers of $\det \boldsymbol{\mathcal{D}}_{\mathfrak{t}}$ gives orientations for the real lines

$$\det \mathcal{D}_{A,\Phi} = \Lambda^{\max}(H^1_{A,\Phi}) \otimes \Lambda^{\max}(H^0_{A,\Phi} \oplus H^2_{A,\Phi})^*, \qquad [A,\Phi] \in \mathcal{M}_{\mathfrak{t}}.$$

If $[A,\Phi]$ is a smooth point of $\mathcal{M}_{\mathfrak{t}}$, so $\operatorname{Coker} \mathcal{D}_{A,\Phi} \cong H^0_{A,\Phi} \oplus H^2_{A,\Phi} = 0$, then $\operatorname{Ker} \mathcal{D}_{A,\Phi} = H^1_{A,\Phi} = T_{A,\Phi}\mathcal{M}_{\mathfrak{t}}$ and

$$\det \mathcal{D}_{A,\Phi} = \Lambda^{\max}(H^1_{A,\Phi}) = \Lambda^{\max}(T_{[A,\Phi]}\mathcal{M}_{\mathfrak{t}}),$$

so an orientation for $\det \boldsymbol{\mathcal{D}}_{\mathfrak{t}}$ defines an orientation for the open manifold $\mathcal{M}_{\mathfrak{t}}^{*,0}$ of smooth points of $\mathcal{M}_{\mathfrak{t}}$. Therefore, $\det \boldsymbol{\mathcal{D}}_{\mathfrak{t}}$ is an orientation bundle for $\mathcal{M}_{\mathfrak{t}}$ and $\mathcal{M}_{\mathfrak{t}}$ is orientable if $\det \boldsymbol{\mathcal{D}}_{\mathfrak{t}}$ is trivial. As in [5], we show that $\mathcal{M}_{\mathfrak{t}}$ is orientable because the bundle $\det \boldsymbol{\mathcal{D}}_{\mathfrak{t}} \to \mathcal{C}_{\mathfrak{t}}/S^1$ has a nowhere vanishing section, that is, its first Stiefel-Whitney class vanishes.

Suppose $E \to X$ is a rank-two, complex Hermitian bundle with $c_1(E) = w$ and $p_1(\mathfrak{su}(E)) = -4\kappa$. Denote the group of determinant-one, unitary automorphisms of E by $\mathcal{G}_\kappa^w$ and the space of SO(3) connections on $\mathfrak{su}(E)$ by $\mathcal{A}_\kappa^w$. Over the quotient space of connections $\mathcal{B}_\kappa^w = \mathcal{A}_\kappa^w/\mathcal{G}_\kappa^w$ there is an orientation bundle $\det \boldsymbol{\delta}_E$ (see [7, Equation (5.4.2)]) with fiber over $[\hat{A}] \in \mathcal{B}_\kappa^w$ given by

$$\det \delta_{\hat{A}} = \Lambda^{\max}(\operatorname{Ker} \delta_{\hat{A}}) \otimes \Lambda^{\max}(\operatorname{Coker} \delta_{\hat{A}}), \tag{2.2}$$

coming from the rolled-up deformation complex for the anti-self-dual moduli space M^w_κ,

$$\delta_{\hat{A}} = d^*_{\hat{A}} + d^+_{\hat{A}} : C^\infty(\Lambda^1 \otimes \mathfrak{su}(E)) \to C^\infty((\Lambda^0 \oplus \Lambda^+) \otimes \mathfrak{su}(E)). \tag{2.3}$$

Thus, an orientation for $\det \boldsymbol{\delta}_E$ defines an orientation for manifold M^w_κ, since

$$\det(d^*_{\hat{A}} + d^+_{\hat{A}}) = \Lambda^{\max}(H^1_{\hat{A}}) \otimes \Lambda^{\max}(H^0_{\hat{A}} \oplus H^2_{\hat{A}})^*,$$

We recall the following result of Donaldson:

Proposition 2.1. [5, Corollary 3.27] *The bundle* $\det \boldsymbol{\delta}_E \to \mathcal{B}^w_\kappa$ *is topologically trivial.*

We now show that $\det \boldsymbol{\mathcal{D}}_{\mathfrak{t}}$ is trivial using the fact that $\det \boldsymbol{\delta}_E$ is trivial, where $\mathfrak{t} = (\rho, V)$, $V = W \otimes E$, we identify $\mathfrak{g}_{\mathfrak{t}} \cong \mathfrak{su}(E)$ and so $w_2(\mathfrak{t}) \equiv w \pmod 2$ and $p_1(\mathfrak{t}) = -4\kappa$. We shall denote

$$D_{A,\vartheta} = D_A + \rho(\vartheta) \quad \text{and} \quad D_{B,\vartheta} = D_B + \rho(\vartheta), \tag{2.4}$$

where $D_A : C^\infty(V^+) \to C^\infty(V^-)$ and $D_B : C^\infty(W^+) \to C^\infty(W^-)$ are the Dirac operators defined by spin connections A on V and B on W, respectively [10, §2.2 & §2.3].

Lemma 2.2. *The bundle* $\det \boldsymbol{\mathcal{D}}_{\mathfrak{t}} \to \mathcal{C}_{\mathfrak{t}}/S^1$ *is topologically trivial.*

Proof. We recall that the K-theory isomorphism class of an index bundle over a compact topological space depends only on the homotopy class of its defining family of Fredholm operators (see, for example, [2, p. 69]). Moreover, the isomorphism class of the determinant line bundle (over a possibly non-compact topological space) depends only on the homotopy class of the family of Fredholm operators [46, Lemma 6.6.1]. In particular, the defining family of Fredholm operators $\mathcal{D}_{A,\Phi}$, parameterized by $[A,\Phi] \in \mathcal{C}_{\mathfrak{t}}/S^1$, is homotopic through $\mathcal{D}_{A,t\Phi}$, $t \in [0,1]$, to the family of Fredholm operators $\mathcal{D}_{A,0} = (d^*_{\hat{A}} + d^+_{\hat{A}}) \oplus D_{A,\vartheta}$ parameterized by $[A,\Phi] \in \mathcal{C}_{\mathfrak{t}}/S^1$. Thus,

$$\det \mathcal{D}_{A,\Phi} \cong \det(d^*_{\hat{A}} + d^+_{\hat{A}}) \otimes \det D_{A,\vartheta}. \tag{2.5}$$

Let $\det \mathbf{D}_V$ be the real determinant line bundle over $\mathcal{C}_{\mathfrak{t}}/S^1$ associated to the family of perturbed Dirac operators, $D_{A,\vartheta}$, where $[A,\Phi] \in \mathcal{C}_{\mathfrak{t}}$. Let

$$\pi_{\mathcal{B}} : \mathcal{C}_{\mathfrak{t}}/S^1 \to \mathcal{B}^w_\kappa, \quad [A,\Phi] \mapsto [\hat{A}] \tag{2.6}$$

be the projection. Equation (2.5) implies there is an isomorphism $\det \boldsymbol{\mathcal{D}}_{\mathfrak{t}} \cong \pi^*_{\mathcal{B}} \det \boldsymbol{\delta}_E \otimes \det \mathbf{D}_V$ of real determinant line bundles, so

$$w_1(\det \boldsymbol{\mathcal{D}}_{\mathfrak{t}}) = \pi^*_{\mathcal{B}} w_1(\det \boldsymbol{\delta}_E) + w_1(\det \mathbf{D}_V).$$

Because the Dirac operators $D_{A,\vartheta}$ have complex kernels and cokernels, the real line bundle $\det \mathbf{D}_V \to \mathcal{C}_{\mathfrak{t}}/S^1$ is topologically trivial and hence $w_1(\det \mathbf{D}_V) = 0$. By Proposition 2.1 we have $w_1(\det \boldsymbol{\delta}_E) = 0$. Combining these observations yields $w_1(\det \boldsymbol{\mathcal{D}}_{\mathfrak{t}}) = 0$. □

2.2. **Orientations of moduli spaces of PU(2) monopoles and anti-self-dual connections.** We introduce an orientation for the PU(2)-monopole moduli space $\mathcal{M}_{\mathfrak{t}}$ determined by an orientation for the moduli space $M^w_\kappa \hookrightarrow M_{\mathfrak{t}}$ of anti-self-dual connections.

An orientation for the line bundle $\det \boldsymbol{\mathcal{D}}_{\mathfrak{t}}$ determines an orientation for $\mathcal{M}_{\mathfrak{t}}$. The space $\tilde{\mathcal{C}}_{\mathfrak{t}}$ is connected, so the quotients $\mathcal{C}_{\mathfrak{t}}$ and $\mathcal{C}_{\mathfrak{t}}/S^1$ are connected and a choice of orientation for $\det \boldsymbol{\mathcal{D}}_{\mathfrak{t}}$ is equivalent to a choice of orientation for a fiber $\det \mathcal{D}_{A,\Phi}$ over a point $[A,\Phi]$. The proof of Lemma 2.2 provides a method of orienting $\det \boldsymbol{\mathcal{D}}_{\mathfrak{t}}$ from an orientation for $\det \boldsymbol{\delta}_E$, and thus from a homology orientation and integral lift w of $w_2(\mathfrak{t})$, using the isomorphism (2.5) of real determinant lines. Indeed, it suffices to choose an orientation for the line $\det(d^*_{\hat{A}} + d^+_{\hat{A}})$

and thus an orientation for $\det \boldsymbol{\delta}_E$ and choose the orientation of $\det \mathbf{D}_V$ induced from the complex orientations of the complex kernel and cokernel of $D_{A,\vartheta}$.

To fix our conventions and notation, we outline Donaldson's method for orienting $\det(d^*_{\hat{A}}+d^+_{\hat{A}})$, and thus $\det \boldsymbol{\delta}_E$, given a homology orientation Ω and an integral lift w of $w_2(\mathfrak{t})$: the detailed construction is described in [5, §3]. Suppose $E \cong \underline{\mathbb{C}} \oplus L$ is a Hermitian, rank-two vector bundle over X, where $\underline{\mathbb{C}} = X \times \mathbb{C}$ and L is a complex line bundle with $c_1(L) = w$. Then $\mathfrak{su}(E) \cong i\underline{\mathbb{R}} \oplus L$ has $w_2(\mathfrak{su}(E)) \equiv c_1(L) \pmod 2$, where $\underline{\mathbb{R}} = X \times \mathbb{R}$. If $d_{\mathbb{C}} \oplus A_L$ is a reducible connection with respect to the splitting of E, where $d_{\mathbb{C}}$ is the trivial connection on $\underline{\mathbb{C}}$, and $\hat{A} = d_{\mathbb{R}} \oplus A_L$ is the corresponding reducible connection on $\mathfrak{su}(E)$, where $d_{\mathbb{R}}$ the trivial connection on $\underline{\mathbb{R}}$. Then the induced rolled-up deformation complex for the anti-self-dual equation (2.3) splits as

$$d^*_{\hat{A}} + d^+_{\hat{A}} = (d^* + d^+) \oplus (d^*_{A_L} + d^+_{A_L}), \tag{2.7}$$

where,

$$d^* + d^+ : C^\infty(i\Lambda^1) \to C^\infty(i\Lambda^0 \oplus i\Lambda^+), \tag{2.8}$$

$$d^*_{A_L} + d^+_{A_L} : C^\infty(\Lambda^1 \otimes_{\mathbb{R}} L) \to C^\infty((\Lambda^0 \oplus \Lambda^+) \otimes_{\mathbb{R}} L). \tag{2.9}$$

The real determinant line,

$$\det(d^* + d^+) \cong \Lambda^{\max}(H^1(X;\mathbb{R})) \otimes \Lambda^{\max}(H^0(X;\mathbb{R}) \oplus H^+(X;\mathbb{R}))^*,$$

is oriented by a choice of "homology orientation" Ω [5, §3], that is, an orientation for $H^1(X;\mathbb{R}) \oplus H^{2,+}(X;\mathbb{R})$, while $H^0(X;\mathbb{R})$ is oriented by the choice of orientation for X [46, §6.6]. The operator $d^*_{A_L} + d^+_{A_L}$ is complex linear, and hence the complex orientations of its complex kernel and cokernel determine an orientation for the real line $\det(d^*_{A_L} + d^+_{A_L})$. Thus, an orientation for $\det(d^*_{\hat{A}} + d^+_{\hat{A}})$ is defined by the class w and homology orientation Ω.

An isomorphism between any two pairs of Hermitian, rank-two complex vector bundles E, E' over X with first Chern class w can be constructed by splicing in $|c_2(E) - c_2(E')|$ copies of SU(2) bundles over S^4 with second Chern class one. Given a U(2) connection on the bundle over X with smaller second Chern class, we obtain a U(2) connection on the other U(2) bundle by splicing in copies of the one-instanton on S^4. The excision principle [5, §3], [7, §7.1] implies that an orientation for one of the pair $\det \boldsymbol{\delta}_E$, $\det \boldsymbol{\delta}_{E'}$ determines an orientation for the other.

For the moduli space M^w_κ of anti-self-dual SO(3) connections, we let $o(\Omega, w)$ denote the orientation determined by the class $w \in H^2(X;\mathbb{Z})$ and corresponding split U(2) bundle, $\underline{\mathbb{C}} \oplus L$, with $c_1(L) = w$, together with a homology orientation Ω.

Definition 2.3. Let $w \in H^2(X;\mathbb{Z})$ be an integral lift of $w_2(\mathfrak{t})$. The orientation $O^{\text{asd}}(\Omega, w)$ for the line bundle $\det \boldsymbol{\mathcal{D}}_{\mathfrak{t}}$ over $\mathcal{C}_{\mathfrak{t}}/S^1$, and so for the moduli space $\mathcal{M}_{\mathfrak{t}}$, is defined by:

- The orientation of a fiber $\det \mathcal{D}_{A,\Phi}$ over a point $[A,\Phi] \in \mathcal{C}_{\mathfrak{t}}$, via isomorphism (2.5),
- The complex orientation for $\det D_{A,\vartheta}$, and
- The orientation $o(\Omega, w)$ for $\det(d^*_{\hat{A}} + d^+_{\hat{A}})$.

For the moduli space of anti-self-dual connections on an SO(3) bundle, we shall need to compare orientations defined by different integral lifts of its second Stiefel-Whitney class:

Lemma 2.4. [7, p. 283] *Let X be a closed, oriented, Riemannian four-manifold and let Ω be a homology orientation. If $w, w' \in H^2(X;\mathbb{Z})$ obey $w \equiv w' \pmod 2$, then*

$$o(\Omega, w') = (-1)^{\frac{1}{4}(w-w')^2} o(\Omega, w).$$

2.3. **Orientations of moduli spaces of PU(2) and Seiberg-Witten monopoles.** We introduce an orientation for the PU(2)-monopole moduli space $\mathcal{M}_{\mathfrak{t}}$ determined by an orientation for a Seiberg-Witten moduli space $M_{\mathfrak{s}} \hookrightarrow \mathcal{M}_{\mathfrak{t}}$.

Let $(A,\Phi) = \iota(B,\Psi) = (B\oplus B\otimes A_L, \Psi)$ be a reducible pair in $\tilde{\mathcal{C}}_{\mathfrak{t}}$, with respect to a splitting $V = W\oplus W\otimes L$, where $\mathfrak{s} = (\rho, W)$ and $\mathfrak{t} = (\rho, V)$ and $\iota : \tilde{\mathcal{C}}_{\mathfrak{s}} \hookrightarrow \tilde{\mathcal{C}}_{\mathfrak{t}}$ denotes the embedding (see Lemma 3.11 in [10]). Recall from [10, §3.4] that the deformation operator $\mathcal{D}_{\iota(B,\Psi)}$ admits a splitting $\mathcal{D}_{\iota(B,\Psi)} = \mathcal{D}^t_{\iota(B,\Psi)} \oplus \mathcal{D}^n_{\iota(B,\Psi)}$ into tangential and normal components given by [10, Equations (3.21) & (3.22)]; the splitting is $\mathcal{G}_{\mathfrak{s}}$-equivariant with respect to the inclusion $\mathcal{G}_{\mathfrak{s}} \hookrightarrow \mathcal{G}_{\mathfrak{t}}$ of automorphism groups in [10, Equation (3.10)]. Hence, we have an isomorphism of real determinant lines,

$$\det \mathcal{D}_{\iota(B,\Psi)} \cong \det \mathcal{D}^t_{\iota(B,\Psi)} \otimes \det \mathcal{D}^n_{\iota(B,\Psi)}. \tag{2.10}$$

Furthermore, by comparing equations (2.12), (2.17), and (2.19) with [10, Equations (3.26) & (3.32], we see that the rolled-up Seiberg-Witten elliptic deformation complex is identified with the rolled-up tangential deformation complex (3.34) in [10]. This identifies an orientation for the line $\det \mathcal{D}_{B,\Psi}$ with an orientation for $\det \mathcal{D}^t_{\iota(B,\Psi)}$. Combined with the isomorphism (2.10), this yields

$$\det \mathcal{D}_{\iota(B,\Psi)} \cong \det \mathcal{D}_{B,\Psi} \otimes \det \mathcal{D}^n_{\iota(B,\Psi)}. \tag{2.11}$$

The pair $\iota(B,\Psi) \in \tilde{\mathcal{C}}_{\mathfrak{t}}$ is a fixed point of the S^1 action on $\tilde{\mathcal{C}}_{\mathfrak{t}}$ induced by the S^1 action on $V = W \oplus W \otimes L$ given by the trivial action on the factor W and the action by scalar multiplication on L (see [10, Equation (3.2)]). The operator

$$\mathcal{D}^n_{\iota(B,\Psi)} : C^\infty(\Lambda^1 \otimes L) \oplus C^\infty(W^+ \otimes L) \to C^\infty(L) \oplus C^\infty(\Lambda^+ \otimes L) \oplus C^\infty(W^- \otimes L)$$

is gauge equivariant and thus, because $\iota(B,\Psi)$ is a fixed point of this S^1 action, is complex linear. Hence, $\mathcal{D}^n_{\iota(B,\Psi)}$ is complex linear and and the complex orientations on its complex kernel and cokernel induce an orientation for $\det \mathcal{D}^n_{\iota(B,\Psi)}$.

We recall that a homology orientation Ω defines an orientation for $M_{\mathfrak{s}}$ [46, §6.6]. As in [10, §2.3], we let $\tilde{\mathcal{C}}_{\mathfrak{s}}$ denote the pre-configuration space of pairs (B,Ψ), where $\mathfrak{s} = (\rho, W)$, B is a spin connection on W, and Φ is a section of W^+; then $\mathcal{C}_{\mathfrak{s}} = \tilde{\mathcal{C}}_{\mathfrak{s}}/\mathcal{G}_{\mathfrak{s}}$ is the configuration space, where $\mathcal{G}_{\mathfrak{s}} \cong \mathrm{Map}(X, S^1)$ is the group of spin^c automorphisms of W. If $(B,0)$ is a point in $\tilde{\mathcal{C}}_{\mathfrak{s}}$ then from [10, Equations (2.61) & (2.62)], the rolled-up Seiberg-Witten elliptic deformation complex is given by

$$\mathcal{D}_{B,0} : C^\infty(i\Lambda^1) \oplus C^\infty(W^+) \to C^\infty(i\Lambda^0 \oplus i\Lambda^+) \oplus C^\infty(W^-).$$

According to [10, Equations (2.59), (2.60), & (2.62)], we have

$$\mathcal{D}_{B,0} = (d^* + d^+) \oplus D_{B,\vartheta},$$

where $d^* + d^+$ is the operator in (2.8) and $D_{B,\vartheta}$ is the Dirac operator in (2.4). Thus,

$$\det \mathcal{D}_{B,0} \cong \det(d^* + d^+) \otimes \det D_{B,\vartheta}. \tag{2.12}$$

The determinant line bundle $\det \boldsymbol{\mathcal{D}}_{\mathfrak{s}}$ with fibers $\det \mathcal{D}_{B,\Psi}$ is topologically trivial over $\mathcal{C}_{\mathfrak{s}}$, so $M_{\mathfrak{s}}$ is orientable and, as $\mathcal{C}_{\mathfrak{s}}$ is connected, an orientation for the real line $\det \mathcal{D}_{B,0}$ defines an orientation for $\det \boldsymbol{\mathcal{D}}_{\mathfrak{s}}$. A homology orientation Ω determines an orientation for $\det(d^*+d^+)$. Since the Dirac operator $D_{B,\vartheta}$ is complex linear, the complex orientation for its complex kernel and cokernel defines an orientation for the real line $\det D_{B,\vartheta}$. The product of these orientations then defines an orientation for $\det \mathcal{D}_{B,0}$ and hence for $\det \boldsymbol{\mathcal{D}}_{\mathfrak{s}}$ and $M_{\mathfrak{s}}$.

Definition 2.5. The orientation $O^{\mathrm{red}}(\Omega,\mathfrak{t},\mathfrak{s})$ for the real line $\det\mathcal{D}_{A,\Phi}$, and so for the line bundle $\det\boldsymbol{\mathcal{D}}_{\mathfrak{t}}$ and the moduli space $\mathcal{M}_{\mathfrak{t}}$, is defined, through the isomorphism (2.11), by:

- The orientation for $\det\mathcal{D}_{B,\Psi}$, and thus $\det\boldsymbol{\mathcal{D}}_{\mathfrak{s}}$, given by the homology orientation Ω,
- The complex orientation for $\det\mathcal{D}^n_{\iota(B,\Psi)}$.

2.4. **Comparison of orientations of moduli spaces of PU(2) monopoles.** We now compare the different possible orientations for $\mathcal{M}_{\mathfrak{t}}$ which we have defined in the preceding sections.

Lemma 2.6. *Let $\mathfrak{t}$ be a spin^u structure on an oriented four-manifold X and let Ω be a homology orientation. Suppose that w is an integral lift of $w_2(\mathfrak{t})$ and that $\mathfrak{t}$ admits a splitting $\mathfrak{t}=\mathfrak{s}\oplus\mathfrak{s}\otimes L$, for some complex line bundle L. Then,*

$$O^{\mathrm{asd}}(\Omega,w)=(-1)^{\frac{1}{4}(w-c_1(L))^2}O^{\mathrm{asd}}(\Omega,c_1(L)),$$
$$O^{\mathrm{asd}}(\Omega,c_1(L))=O^{\mathrm{red}}(\Omega,\mathfrak{t},\mathfrak{s}).$$

Proof. By Definition 2.3, the difference between $O^{\mathrm{asd}}(\Omega,w)$ and $O^{\mathrm{asd}}(\Omega,c_1(L))$ is equal to the difference between the orientations $o(\Omega,w)$ and $o(\Omega,c_1(L))$ for the moduli spaces of anti-self-dual connections on SO(3) bundles with second Stiefel-Whitney classes $w \pmod 2$ and $c_1(L) \pmod 2$, respectively. Since $\mathfrak{g}_{\mathfrak{t}}\cong i\underline{\mathbb{R}}\oplus L$ and $w_2(\mathfrak{t})\equiv w \pmod 2$ by hypothesis, we have $c_1(L)\equiv w \pmod 2$ and so Lemma 2.4 applies to compute the difference in orientations.

To see the second equality, write $\mathfrak{t}=(\rho,V)$ and $\mathfrak{s}=(\rho,W)$ and let

$$(A,\Phi)=\iota(B,0)=(B\oplus B\otimes A_L,0)$$

be a pair in $\tilde{\mathcal{C}}_{\mathfrak{t}}$ which is reducible with respect to the splitting $V=W\oplus W\otimes L$ and which has a vanishing spinor component, with $A_L=A_\Lambda\otimes(B^{\det})^*$. Recall from [10, Lemma 2.9] that $\hat{A}$ is then reducible with respect to the splitting $\mathfrak{g}_{\mathfrak{t}}=i\underline{\mathbb{R}}\oplus L$ and can be written as $\hat{A}=d_{\mathbb{R}}\oplus A_L$. The Dirac operator $D_{A,\vartheta}$ also splits,

$$D_{A,\vartheta}=D_{B,\vartheta}\oplus D_{B\otimes A_L,\vartheta}, \tag{2.13}$$

where

$$D_{B,\vartheta}:C^\infty(W^+)\to C^\infty(W^-)\quad\text{and}\quad D_{B\otimes A_L,\vartheta}:C^\infty(W^+\otimes L)\to C^\infty(W^-\otimes L).$$

The isomorphism (2.5) of determinant lines giving the orientation $O^{\mathrm{asd}}(\Omega,c_1(L))$ to the line $\det\mathcal{D}_{A,0}$ and the decompositions (2.7) of $d^*_{\hat{A}}+d^+_{\hat{A}}$ and (2.13) of $D_{A,\vartheta}$ at a reducible connection A yield the isomorphisms

$$\begin{aligned}\det\mathcal{D}_{A,0}&\cong\det(d^*_{\hat{A}}+d^+_{\hat{A}})\otimes\det D_{A,\vartheta}\\&\cong\det(d^*+d^+)\otimes\det(d^*_{A_L}+d^+_{A_L})\otimes\det D_{B,\vartheta}\otimes\det D_{B\otimes A_L,\vartheta}.\end{aligned}\tag{2.14}$$

The operators $d^*_{A_L}+d^+_{A_L}$, $D_{B,\vartheta}$, and $D_{B\otimes A_L,\vartheta}$ are complex linear and thus have complex kernels and cokernels. By Definition 2.3, the orientation $O^{\mathrm{asd}}(\Omega,c_1(L))$ is defined by choosing a homology orientation Ω for $\det(d^*+d^+)$, and the complex orientation on the remaining factors on the right-hand-side of (2.14).

On the other hand, the isomorphisms (2.11) and (2.12) of determinant lines giving the orientation $O^{\mathrm{red}}(\Omega,\mathfrak{t},\mathfrak{s})$ to the line $\det\mathcal{D}_{A,0}$ yield the isomorphisms

$$\begin{aligned}\det\mathcal{D}_{A,0}&\cong\det\mathcal{D}_{B,0}\otimes\det\mathcal{D}^n_{\iota(B,0)}\\&\cong\det(d^*+d^+)\otimes\det D_{B,\vartheta}\otimes\det(d^*_{A_L}+d^+_{A_L})\otimes\det D_{B\otimes A_L,\vartheta}.\end{aligned}\tag{2.15}$$

By Definition 2.5, the orientation $O^{\mathrm{red}}(\Omega,\mathfrak{t},\mathfrak{s})$ for $\det \mathcal{D}_{A,0}$ is induced by the isomorphism (2.15), a choice of homology orientation Ω for $\det(d^*+d^+)$, and the complex orientation on the remaining factors on the right-hand side of (2.15).

The isomorphisms (2.14) and (2.15) thus yield the same orientation of $\det \mathcal{D}_{A,0}$, and therefore $O^{\mathrm{asd}}(\Omega,c_1(L)) = O^{\mathrm{red}}(\Omega,\mathfrak{t},\mathfrak{s})$. □

2.5. **Orientations of links of strata of reducible PU(2) monopoles.** We shall need to compute the oriented intersections of codimension-one submanifolds of $\mathcal{M}_{\mathfrak{t}}^{*,0}/S^1$ with links $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ in $\mathcal{M}_{\mathfrak{t}}^{*,0}/S^1$ of the strata $\iota(M_{\mathfrak{s}})$. These computations (see §4) are performed most naturally with a "complex orientation" of the link $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ induced from the complex structure on the fibers of the "virtual normal bundle" of $M_{\mathfrak{s}}$. We then compare this orientation with the "boundary orientation" of $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ induced from an orientation of $\mathcal{M}_{\mathfrak{t}}^{*,0}/S^1$ when the link is oriented as a boundary of an open subspace of $\mathcal{M}_{\mathfrak{t}}^{*,0}/S^1$. Our orientation conventions for the link $\mathbf{L}_{\mathfrak{t},\kappa}^w$ of the stratum $\iota(M_\kappa^w)$ are explained in §3.4.3. We assume throughout this subsection that there are no zero-section pairs in $M_{\mathfrak{s}}$.

Suppose that Y is a connected, finite-dimensional, orientable manifold with a free circle action. We give $S^1\subset\mathbb{C}$ its usual orientation. If λ_{S^1} is a vector in T_yY which is tangent to an S^1 orbit through $y\in Y$, then an orientation λ_Y for $\det(T_yY)$ and an orientation λ_{Y/S^1} for $\det(T_y(Y/S^1))$ determine one another through the convention

$$\lambda_Y = \lambda_{S^1}\wedge\tilde{\lambda}_{Y/S^1}, \tag{2.16}$$

where $\tilde{\lambda}_{Y/S^1}\in\Lambda^{\dim Y-1}(T_yY)$ satisfies $\pi_*(\tilde{\lambda}_{Y/S^1})=\lambda_{Y/S^1}$ and $\pi: Y\to Y/S^1$ is the projection. In particular, orientations for $\mathcal{M}_{\mathfrak{t}}$ and $\mathcal{M}_{\mathfrak{t}}/S^1$ determine one another via convention (2.16).

Recall from [10, §3.5.3] that the "thickened moduli space" $\mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s})\subset\mathcal{C}_{\mathfrak{t}}^{*,0}$ is a finite-dimensional S^1-invariant manifold, defined by a choice of finite-rank, S^1-equivariant, trivial "stabilizing" or "obstruction" bundle Ξ over an open neighborhood of $\iota(M_{\mathfrak{s}})$ in $\mathcal{C}_{\mathfrak{t}}^0$. Then $M_{\mathfrak{s}}\hookrightarrow\mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s})$ is a smooth submanifold with S^1-equivariant normal bundle $N_{\mathfrak{t}}(\Xi,\mathfrak{s})\to M_{\mathfrak{s}}$ and tubular neighborhood defined by the image of the S^1-equivariant smooth embedding,

$$\boldsymbol{\gamma}: N_{\mathfrak{t}}(\Xi,\mathfrak{s})\hookrightarrow\mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s}).$$

An open neighborhood of $\iota(M_{\mathfrak{s}})$ in the moduli space $\mathcal{M}_{\mathfrak{t}}$ is recovered as the zero locus of an S^1-equivariant section $\boldsymbol{\varphi}$ of the S^1-equivariant vector bundle $\boldsymbol{\gamma}^*\Xi\to N_{\mathfrak{t}}(\Xi,\mathfrak{s})$:

$$\boldsymbol{\gamma}\left(\boldsymbol{\varphi}^{-1}(0)\cap N_{\mathfrak{t}}(\Xi,\mathfrak{s})\right)\subset\mathcal{M}_{\mathfrak{t}}.$$

The section $\boldsymbol{\varphi}$ vanishes transversely on $N_{\mathfrak{t}}(\Xi,\mathfrak{s})-M_{\mathfrak{s}}$. As in [10, Definition 3.22] we define the *link* of the stratum $\iota(M_{\mathfrak{s}})$ to be

$$\mathbf{L}_{\mathfrak{t},\mathfrak{s}}=\boldsymbol{\gamma}\left(\boldsymbol{\varphi}^{-1}(0)\cap\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})\right)\subset\mathcal{M}_{\mathfrak{t}}/S^1,$$

where $\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})$ is the projectivization of the complex vector bundle $N_{\mathfrak{t}}(\Xi,\mathfrak{s})$. Via the diffeomorphism,

$$\mathbf{L}_{\mathfrak{t},\mathfrak{s}}\cong\boldsymbol{\varphi}^{-1}(0)\cap\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s}),$$

we can take the right-hand side as our model for the link, where $\boldsymbol{\varphi}$ is a section of the complex vector bundle $\boldsymbol{\gamma}^*\Xi\to\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})$.

We now define the complex orientation for the link $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$. The tangent space of $\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})$ is oriented by an orientation on $M_{\mathfrak{s}}$ and the complex structure on the fibers. To be precise, at a point $[B,\Psi,\eta]$ in the fiber of $\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})$ over $[B,\Psi]\in M_{\mathfrak{s}}$, the inclusion of the fiber gives an exact sequence of tangent spaces,

$$0\to T_{[\eta]}(\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})|_{[B,\Psi]})\to T_{[B,\Psi,\eta]}\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})\to T_{[B,\Psi]}M_{\mathfrak{s}}\to 0,$$

and thus an isomorphism of determinant lines,

$$\Lambda^{\max}(T_{[\eta]}(\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})|_{[B,\Psi]}))\otimes\Lambda^{\max}(T_{[B,\Psi]}M_{\mathfrak{s}})\cong\Lambda^{\max}(T_{[B,\Psi,\eta]}\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})). \tag{2.17}$$

According to [10, Lemma 3.23], the section $\boldsymbol{\varphi}$ vanishes transversely at any point $[B,\Psi,\eta]$ in an open neighborhood of the zero section $M_{\mathfrak{s}}$ of $N_{\mathfrak{t}}(\Xi,\mathfrak{s})$, provided $[B,\Psi,\eta]\notin M_{\mathfrak{s}}$. Thus, at $[B,\Psi,\eta]\in\boldsymbol{\gamma}^{-1}(\mathbf{L}_{\mathfrak{t},\mathfrak{s}})=\boldsymbol{\varphi}^{-1}(0)\cap\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})$, for $\eta\neq 0$, the differential of $\boldsymbol{\varphi}$ and the diffeomorphism $\boldsymbol{\gamma}$ induce an exact sequence,

$$0\to T_{\boldsymbol{\gamma}[B,\Psi,\eta]}\mathbf{L}_{\mathfrak{t},\mathfrak{s}}\to T_{[B,\Psi,\eta]}\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})\to(\boldsymbol{\gamma}^*\Xi)_{[B,\Psi,\eta]}\to 0,$$

since $T_{\boldsymbol{\gamma}[B,\Psi,\eta]}\mathbf{L}_{\mathfrak{t},\mathfrak{s}}\cong\operatorname{Ker}(D\boldsymbol{\varphi})_{[B,\Psi,\eta]}$ and $\operatorname{Ran}(D\boldsymbol{\varphi})_{[B,\Psi,\eta]}=(\boldsymbol{\gamma}^*\Xi)_{[B,\Psi,\eta]}$. This exact sequence and the isomorphism (2.17) induce an isomorphism

$$\begin{aligned}\Lambda^{\max}(T_{\boldsymbol{\gamma}[B,\Psi,\eta]}\mathbf{L}_{\mathfrak{t},\mathfrak{s}})&\cong\Lambda^{\max}(T_{[B,\Psi,\eta]}\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s}))\otimes(\Lambda^{\max}(\Xi_{\boldsymbol{\gamma}[B,\Psi,\eta]}))^*\\&\cong\Lambda^{\max}(T_{[B,\Psi]}M_{\mathfrak{s}})\otimes\Lambda^{\max}(T_{[\eta]}(\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})|_{[B,\Psi,\eta]}))\\&\quad\otimes(\Lambda^{\max}(\Xi_{\boldsymbol{\gamma}[B,\Psi,\eta]}))^*.\end{aligned} \tag{2.18}$$

The fibers of the bundle $\Xi\to\mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s})$ are preserved under the S^1 action. The complex structure defined by this S^1 action gives an orientation for $\Lambda^{\max}(\Xi_{\boldsymbol{\gamma}[B,\Psi,\eta]})$.

Definition 2.7. The *complex orientation* of the link $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ is defined through the isomorphism (2.18) with the orientations of the terms on the right-hand-side of (2.18) given by:

- The orientation of $M_{\mathfrak{s}}$ defined by a choice of homology orientation Ω,
- The complex orientation of the bundle $\boldsymbol{\gamma}^*\Xi\to\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})$,
- The complex orientation of the tangent space of a fiber of $\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})$.

Although the complex orientation given by Definition 2.7 is the natural orientation to use when computing intersection numbers with $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$, we shall need to orient $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ as a boundary when using $\mathcal{M}_{\mathfrak{t}}^{*,0}/S^1$ as a cobordism. We describe this procedure next.

Suppose $Z\subset Y$ is a compact submanifold of an oriented, Riemannian manifold Y, with normal bundle $N\to Z$. If $\vec r$ is the outward-pointing radial vector at a point y on the boundary ∂N of the tubular neighborhood, also denoted N, then an orientation λ_Y for $\det(T_yY)$ and an orientation $\lambda_{\partial N}$ for $\det(T_y(\partial N))$ determine one another through the convention

$$\lambda_Y=-\vec r\wedge\lambda_{\partial N}, \tag{2.19}$$

choosing the sign in equation (2.19) so that the link ∂N is the boundary of $Y-N$.

For $[A,\Phi]\in\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$, choose an outward-pointing radial vector with respect to the thickened tubular neighborhood $N_{\mathfrak{t}}^{<\varepsilon}(\Xi,\mathfrak{s})/S^1$,

$$\vec r\in T_{[A,\Phi]}(\mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s})/S^1)\cong T_{[A,\Phi]}(N_{\mathfrak{t}}(\Xi,\mathfrak{s})/S^1) \tag{2.20}$$

$$\cong\mathbb{R}\cdot\vec r\oplus T_{[A,\Phi]}(N_{\mathfrak{t}}^{\varepsilon}(\Xi,\mathfrak{s})/S^1). \tag{2.21}$$

Because the section $\boldsymbol{\varphi}$ of $\boldsymbol{\gamma}^*\Xi$ vanishes transversely on both $N_{\mathfrak{t}}(\Xi,\mathfrak{s})/S^1$ and its ε-sphere bundle, for generic ε, we have isomorphisms

$$T_{[A,\Phi]}(N_{\mathfrak{t}}(\Xi,\mathfrak{s})/S^1)\cong T_{[A,\Phi]}(\mathcal{M}_{\mathfrak{t}}/S^1)\oplus\Xi_{[A,\Phi]}, \tag{2.22}$$

$$T_{[A,\Phi]}(N_{\mathfrak{t}}^{\varepsilon}(\Xi,\mathfrak{s})/S^1)\cong T_{[A,\Phi]}\mathbf{L}_{\mathfrak{t},\mathfrak{s}}\oplus\Xi_{[A,\Phi]}. \tag{2.23}$$

Through the isomorphism (2.20), let $\pi_{\mathcal{M}/S^1}\vec{r}$ be the orthogonal projection of $\vec{r}$ onto the subspace $T_{[A,\Phi]}(\mathcal{M}_{\mathfrak{t}}/S^1)$ in equation (2.22). If $\pi_{\mathcal{M}/S^1}\vec{r}=0$, we would have $\vec{r}\in\Xi_{[A,\Phi]}$ and thus tangent to $N^{\varepsilon}_{\mathfrak{t}}(\Xi,\mathfrak{s})/S^1$ at $[A,\Phi]$ by equation (2.23), contradicting our choice of $\vec{r}$. Since $\pi_{\mathcal{M}/S^1}\vec{r}\neq 0$, a comparison of the isomorphisms (2.21), (2.22), and (2.23) yields

$$T_{[A,\Phi]}(\mathcal{M}_{\mathfrak{t}}/S^1)\cong(\pi_{\mathcal{M}/S^1}\vec{r})\cdot\mathbb{R}\oplus T_{[A,\Phi]}\mathbf{L}_{\mathfrak{t},\mathfrak{s}}. \tag{2.24}$$

Hence, we make the

Definition 2.8. Given an orientation $\lambda_{\mathcal{M}/S^1}$ of $\mathcal{M}_{\mathfrak{t}}/S^1$ and an outward-pointing radial vector $\vec{r}$ with respect to the tubular neighborhood $N^{<\varepsilon}_{\mathfrak{t}}(\Xi,\mathfrak{s})/S^1$, we define the *boundary orientation* $\lambda_{\partial\mathcal{M}/S^1}$ of $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ by

$$\lambda_{\mathcal{M}/S^1}=-\pi_{\mathcal{M}/S^1}\vec{r}\wedge\lambda_{\partial\mathcal{M}/S^1}. \tag{2.25}$$

Lemma 2.9. *The complex orientation (Definition 2.7) of the link $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ agrees with the boundary orientation (Definition 2.8) of $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ determined by the orientation $O^{\mathrm{red}}(\Omega,\mathfrak{t},\mathfrak{s})$ for $\mathcal{M}_{\mathfrak{t}}/S^1$.*

Proof. The orientation $O^{\mathrm{red}}(\Omega,\mathfrak{t},\mathfrak{s})$ of $\mathcal{M}_{\mathfrak{t}}$ is defined through the isomorphism (2.10), using the orientation for $\det\mathcal{D}^t_{\iota(B,\Psi)}$ (and thus the tangent space for $M_{\mathfrak{s}}$) given by the homology orientation Ω, and the complex orientation for $\det\mathcal{D}^n_{\iota(B,\Psi)}$. From [10, Equation (3.55)] we have an isomorphism $\det(\boldsymbol{\mathcal{D}}^n)\cong\det([N_{\mathfrak{t}}(\Xi,\mathfrak{s})]-[\Xi])$ and thus, for any $[B,\Psi]\in M_{\mathfrak{s}}$ an isomorphism,

$$\det\mathcal{D}^n_{\iota(B,\Psi)}\cong\Lambda^{\max}(N_{\mathfrak{t}}(\Xi,\mathfrak{s})|_{[B,\Psi]})\otimes(\Lambda^{\max}(\Xi_{[B,\Psi]}))^*, \tag{2.26}$$

which preserves the orientations defined by the complex structures of both sides. The orientation $O^{\mathrm{red}}(\Omega,\mathfrak{t},\mathfrak{s})$ of $\mathcal{M}_{\mathfrak{t}}$ determines one for $\mathcal{M}_{\mathfrak{t}}/S^1$ through convention (2.16) and a boundary orientation for $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ through convention (2.25).

On the other hand, the complex orientation for $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ uses, through equation (2.18), the complex orientation for the complex projective space given by the fiber of $\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})$. Comparing equation (2.26) with equation (2.18) shows that the difference between the two orientations lies in how the fibers of the projections $\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})\to M_{\mathfrak{s}}$ and $N_{\mathfrak{t}}(\Xi,\mathfrak{s})\to M_{\mathfrak{s}}$ are oriented. The boundary orientation $\lambda_{\partial\mathcal{M}/S^1}$ for $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ induced by $O^{\mathrm{red}}(\Omega,\mathfrak{t},\mathfrak{s})$ on $\mathcal{M}_{\mathfrak{t}}$ begins with the complex orientation for the fiber of the projection $N_{\mathfrak{s}}(\Xi,\mathfrak{s})\to M_{\mathfrak{s}}$, uses convention (2.16) to define an orientation for the fiber of $N_{\mathfrak{s}}(\Xi,\mathfrak{s})/S^1\to M_{\mathfrak{s}}$, and then uses convention (2.19) to define an orientation for the boundary $N^{\varepsilon}_{\mathfrak{t}}(\Xi,\mathfrak{s})/S^1=\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})$ of the bundle $N^{\leq\varepsilon}_{\mathfrak{t}}(\Xi,\mathfrak{s})/S^1$. Hence, it is enough to compare these two methods of orienting the fibers of $\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})$.

We denote the fibers of $N_{\mathfrak{t}}(\Xi,\mathfrak{s})$, $N^{\varepsilon}_{\mathfrak{t}}(\Xi,\mathfrak{s})$, and $\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})$ by $\mathbb{C}^k$, $(\mathbb{C}^k)^{\varepsilon}$, and $\mathbb{P}^{k-1}=(\mathbb{C}^k)^{\varepsilon}/S^1$, respectively. If $\mathbb{C}^k$ has a complex basis $\{\vec{r},v_1,\dots,v_{k-1}\}$, then the complex orientation of $\mathbb{C}^k$ is defined by

$$\begin{aligned}\lambda_{\mathbb{C}^k}&=\vec{r}\wedge i\vec{r}\wedge v_1\wedge iv_1\wedge\cdots\wedge v_{k-1}\wedge iv_{k-1}\\&=-i\vec{r}\wedge\vec{r}\wedge v_1\wedge iv_1\wedge\cdots\wedge v_{k-1}\wedge iv_{k-1}.\end{aligned} \tag{2.27}$$

If $\lambda_{\mathbb{P}^{k-1}}$ is the complex orientation for $\mathbb{P}^{k-1}$ and $\pi:\mathbb{C}^k\setminus\{0\}\to\mathbb{P}^{k-1}$ is the projection, then

$$\pi_*(v_1\wedge iv_1\wedge\cdots\wedge v_{k-1}\wedge iv_{k-1})=\lambda_{\mathbb{P}^{k-1}},$$

because $\{v_1,\dots,v_{k-1}\}$ is a complex basis for the tangent space to $\mathbb{P}^{k-1}$ at $\pi(\vec{r})$, so equation (2.27) yields the following relation between the complex orientations of $\mathbb{C}^k$ and $\mathbb{P}^{k-1}$:

$$\lambda_{\mathbb{C}^k}=-i\vec{r}\wedge\vec{r}\wedge\lambda_{\mathbb{P}^{k-1}}. \tag{2.28}$$

On the other hand, if $\mathbb{C}^k/S^1$ is oriented through convention (2.19) by $-\vec{r}\wedge\lambda_{\mathbb{P}^{k-1}}$, the boundary orientation of the link $\mathbb{P}^{k-1}$ is equal to its complex orientation, $\lambda_{\mathbb{P}^{k-1}}$. By convention (2.16), the orientation $-\vec{r}\wedge\lambda_{\mathbb{P}^{k-1}}$ for $\mathbb{C}^k/S^1$ is induced by the orientation $-i\vec{r}\wedge\vec{r}\wedge\pi^*\lambda_{\mathbb{P}^{k-1}}$ for $\mathbb{C}^k$, which is equal to the complex orientation $\lambda_{\mathbb{C}^k}$ by equation (2.28). Hence, the complex and boundary orientations of $\mathbb{P}^{k-1}$ agree.

Therefore, the complex orientation agrees with the boundary orientation for $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$, induced by the orientation $O^{\mathrm{red}}(\Omega,\mathfrak{t},\mathfrak{s})$ through the conventions (2.25) and (2.16). □

3. Cohomology classes on moduli spaces

In this section we introduce cohomology classes on the moduli space $\mathcal{M}_{\mathfrak{t}}^{*,0}$ (see §3.1) and construct geometric representatives for these cohomology classes (see §3.2). The PU(2) monopole program uses the moduli space $\mathcal{M}_{\mathfrak{t}}^{*,0}/S^1$ as a cobordism between the link $\mathbf{L}_{\mathfrak{t},\kappa}^{w}$ of the anti-self-dual moduli space stratum, $\iota(M_\kappa^w)\subset\mathcal{M}_{\mathfrak{t}}$, and the links $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ of the Seiberg-Witten strata, $\iota(M_{\mathfrak{s}})\subset\mathcal{M}_{\mathfrak{t}}$, giving an equality between the pairings of the cohomology classes with these links. The following geometric description should help motivate the constructions of this section.

The intersection of the geometric representatives with $\mathcal{M}_{\mathfrak{t}}^{*,0}$ is a family of oriented one-manifolds. The links $\mathbf{L}_{\mathfrak{t},\kappa}^{w}$ and $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ of the strata of zero-section and reducible monopoles described in [10, Definitions 3.7 & 3.22] are oriented hypersurfaces in $\mathcal{M}_{\mathfrak{t}}^{*,0}/S^1$. The intersection of these hypersurfaces with the one-dimensional manifolds given by the intersection of the geometric representatives is thus an oriented collection of points. We would like to use the family of oriented one-manifolds to show that the total signed count of the points in the intersection of the geometric representatives with the links is zero (being an oriented boundary). This would give an equality between the oriented count of points in the link of the stratum of zero-section monopoles with the oriented count of points in the links of the strata of the reducible monopoles. In §3.4 we show that the intersection of the geometric representatives with the link $\mathbf{L}_{\mathfrak{t},\kappa}^{w}$ is a multiple of the Donaldson invariant. In §4 we show that the intersection of the geometric representatives with the links $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ can be expressed in terms of Seiberg-Witten invariants. Hence, the cobordism gives a relation between these two invariants.

In practice, the above argument does not work in the simple manner just described because $\mathcal{M}_{\mathfrak{t}}^{*,0}/S^1$ is non-compact: the non-compactness phenomenon responsible for the difficulty is due to Uhlenbeck bubbling. Geometrically, this means there can be one-manifolds in the intersection of the geometric representatives with one boundary on a link and the other end approaching a reducible in a lower level of $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$. Let

$$\bar{\mathcal{M}}_{\mathfrak{t}}^{*,0}\subset\bar{\mathcal{M}}_{\mathfrak{t}} \tag{3.1}$$

be the subspace consisting of points $[A,\Phi,\mathbf{x}]$ where (A,Φ) is neither a zero-section nor a reducible pair. In §3.3 we describe the intersection of the closure of the geometric representatives with the lower Uhlenbeck levels of $\bar{\mathcal{M}}_{\mathfrak{t}}^{*,0}/S^1$ and show that for appropriate choices of geometric representatives these intersections are empty. Therefore, the ends of the one-manifolds in $\mathcal{M}_{\mathfrak{t}}^{*,0}/S^1$ do not approach the lower levels of $\bar{\mathcal{M}}_{\mathfrak{t}}^{*,0}/S^1$.

However, there may still be one-manifolds in the intersection of the geometric representatives with ends approaching reducible monopoles in lower Uhlenbeck levels of $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$. Theorem 3.33 gives a relationship between the Donaldson and Seiberg-Witten invariants when there are no reducible monopoles in the lower levels of $\bar{\mathcal{M}}_{\mathfrak{t}}$ and thus the ends of the one-manifolds do not approach the lower levels of $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$. To extend this argument to the case where there are reducible monopoles in the lower levels of the compactification requires a description of neighborhoods of the lower strata precise enough to allow the definition of links of these strata of lower-level reducibles. As we show in §4, the geometric representatives intersect the strata of reducible monopoles in sets of larger than expected codimension. Thus, in the case of the reducibles, we cannot cut down by geometric representatives as we do with the zero-section monopoles and restrict our attention to a generic point. Rather, we are forced to describe the entire link. When the reducible monopoles lie in a lower level, these links can be extremely complicated. A description of neighborhoods the strata of lower-level reducibles, sufficient to define links, will be given in [12], [13].

We work with geometric representatives rather than cohomology classes for two reasons. First, describing the closure of a geometric representative in a compactification appears to be simpler than calculating the extension of a cohomology class. Second, the topology near points in the lower levels of $\bar{\mathcal{M}}_{\mathfrak{t}}$ need not be locally finite (for example, there may be infinitely many path-connected components). Hence, it is not known if $\mathbf{L}^w_{\mathfrak{t},\kappa}$ is triangulable and thus it may not have a fundamental class to pair with the cohomology classes described in §3.1. This problem also leads us to work in the category of smoothly stratified spaces rather than that of piecewise-linear spaces.

3.1. **The cohomology classes.** In this subsection, we define the cohomology classes on the moduli spaces, following the prescriptions of [4], [6], [7], [35]. These classes arise from a universal SO(3) bundle, just as in Donaldson theory, and a universal line bundle.

Recall that $\tilde{\mathcal{C}}^*_{\mathfrak{t}}$ denotes the subspace of pairs which are not reducible, $\tilde{\mathcal{C}}^0_{\mathfrak{t}}$ denotes the subspace of pairs which are not zero-section pairs, and $\tilde{\mathcal{C}}^{*,0}_{\mathfrak{t}} = \tilde{\mathcal{C}}^*_{\mathfrak{t}} \cap \tilde{\mathcal{C}}^0_{\mathfrak{t}}$ (see [10, §2.1.5]). We define a universal SO(3) bundle:

$$\mathbb{F}_{\mathfrak{t}} = \tilde{\mathcal{C}}^*_{\mathfrak{t}}/S^1 \times_{\mathcal{G}_{\mathfrak{t}}} \mathfrak{g}_{\mathfrak{t}} \to \mathcal{C}^*_{\mathfrak{t}}/S^1 \times X. \tag{3.2}$$

The action of $\mathcal{G}_{\mathfrak{t}}$ in (3.2) is diagonal so, for $u \in \mathcal{G}_{\mathfrak{t}}$ and $(A,\Phi,\xi) \in \tilde{\mathcal{C}}^{*,0}_{\mathfrak{t}} \times \mathfrak{g}_{\mathfrak{t}}$, one has

$$((A,\Phi),\xi) \mapsto (u(A,\Phi),u\xi).$$

We now define

$$\mu_p : H_\bullet(X;\mathbb{R}) \to H^{4-\bullet}(\mathcal{C}^*_{\mathfrak{t}}/S^1;\mathbb{R}), \quad \beta \mapsto -\tfrac{1}{4}p_1(\mathbb{F}_{\mathfrak{t}})/\beta. \tag{3.3}$$

Following [7, Definition 5.1.11] we can also define a universal SO(3) bundle over the quotient space of SO(3) connections,

$$\mathbb{F}^w_\kappa = \mathcal{A}^{w,*}_\kappa \times_{\mathcal{G}^w_\kappa} F \to \mathcal{B}^{w,*}_\kappa \times X, \tag{3.4}$$

where F is an SO(3) bundle over X with $\kappa = -\frac{1}{4}p_1(F)$ and w is an integral lift of $w_2(F)$, and $\mathcal{G}^w_\kappa$ is the group of special unitary automorphisms of the U(2) bundle E with $\mathfrak{su}(E) = F$, so $p_1(F) = p_1(\mathfrak{su}(E))$, and $c_1(E) = w$. As in [7, Definition 5.1.11], we define cohomology classes on $\mathcal{B}^{w,*}_\kappa$ via

$$\nu_p : H_\bullet(X;\mathbb{R}) \to H^{4-\bullet}(\mathcal{B}^{w,*}_\kappa;\mathbb{R}), \quad \beta \mapsto -\tfrac{1}{4}p_1(\mathbb{F}^w_\kappa)/\beta. \tag{3.5}$$

Comparing (3.3) and (3.5), we see that there must be a simple relation, which we now describe, between the cohomology classes defined by these two SO(3) bundles.

Recall from [10, §2.1.3] that if $F = \mathfrak{su}(E)$ and $V = W \otimes E$, then we have an identification of automorphism groups, $\mathcal{G}^w_\kappa \cong \mathcal{G}_{\mathfrak{t}}$, and isomorphisms

$$\mathcal{A}^w_\kappa(X) \cong \mathcal{A}_{\mathfrak{t}}, \quad \hat{A} \mapsto A, \quad \text{and} \quad \mathcal{B}^w_\kappa(X) \cong \mathcal{B}_{\mathfrak{t}}, \quad [\hat{A}] \mapsto [A]. \tag{3.6}$$

Hence, denoting $\mathfrak{g}_{\mathfrak{t}} = \mathfrak{su}(E)$, we have an isomorphism of SO(3) bundles

$$\mathbb{F}^w_\kappa \cong \mathcal{A}^*_{\mathfrak{t}} \times_{\mathcal{G}_{\mathfrak{t}}} \mathfrak{g}_{\mathfrak{t}} \to \mathcal{B}^*_{\mathfrak{t}} \times X. \tag{3.7}$$

Furthermore, there are natural embeddings

$$\iota : \mathcal{A}_{\mathfrak{t}} \hookrightarrow \tilde{\mathcal{C}}_{\mathfrak{t}}, \quad A \mapsto (A, 0), \quad \text{and} \quad \iota : \mathcal{B}_{\mathfrak{t}} \hookrightarrow \mathcal{C}_{\mathfrak{t}}, \quad [A] \mapsto [A, 0]. \tag{3.8}$$

Using (3.8) together with the isomorphism (3.7) and the definition (3.2) of $\mathbb{F}_{\mathfrak{t}}$, we see that

$$(\pi_{\mathcal{B}} \times \mathrm{id}_X)^* \mathbb{F}^w_\kappa = \mathbb{F}_{\mathfrak{t}} \quad \text{and} \quad (\iota \times \mathrm{id}_X)^* \mathbb{F}_{\mathfrak{t}} = \mathbb{F}^w_\kappa, \tag{3.9}$$

where $\pi_{\mathcal{B}} : \mathcal{C}^*_{\mathfrak{t}}/S^1 \to \mathcal{B}^{w,*}_\kappa$ is the restriction of the map (2.6) to $\mathcal{C}^*_{\mathfrak{t}}/S^1$. Since $\pi_{\mathcal{B}}$ is a deformation retract, we obtain the following relation between the cohomology classes on $\mathcal{C}^*_{\mathfrak{t}}/S^1$ and $\mathcal{B}^{w,*}_\kappa$.

Lemma 3.1. *If $\beta \in H_\bullet(X;\mathbb{R})$, then $\pi^*_{\mathcal{B}}\nu_p(\beta) = \mu_p(\beta)$ or, equivalently, $\iota^*\mu_p(\beta) = \nu_p(\beta)$.*

Because $(\iota \times \mathrm{id}_X)^*\mu_p = \nu_p$, we shall henceforth write μ_p for both μ_p and ν_p.
Lastly, we define a universal complex line bundle,

$$\mathbb{L}_{\mathfrak{t}} = \mathcal{C}^{*,0}_{\mathfrak{t}} \times_{(S^1, \times -2)} \mathbb{C} \to \mathcal{C}^{*,0}_{\mathfrak{t}}/S^1, \tag{3.10}$$

where the S^1 action defining $\mathbb{L}_{\mathfrak{t}}$ is given, for $[A,\Phi] \in \mathcal{C}^{*,0}_{\mathfrak{t}}$, $e^{i\theta} \in S^1$, and $z \in \mathbb{C}$, by

$$([A,\Phi], z) \mapsto \left([A, e^{i\theta}\Phi], e^{2i\theta} z\right). \tag{3.11}$$

The factor of 2 is necessary in the action (3.11) because $-1 \in S^1$ acts on $\tilde{\mathcal{C}}_{\mathfrak{t}}$ as $-\mathrm{id}_V \in \mathcal{G}_t$ and thus $-1 \in S^1$ acts trivially on $\mathcal{C}^{*,0}_{\mathfrak{t}}$. The negative sign in the quotient (3.10) indicates that the S^1 action is diagonal, and is chosen to give a more convenient sign in Lemma 3.28. We then define an additional cohomology class in $H^2(\mathcal{C}^{*,0}_{\mathfrak{t}}/S^1;\mathbb{R})$,

$$\mu_c = c_1(\mathbb{L}_{\mathfrak{t}}). \tag{3.12}$$

In Lemma 3.28 we will show that the class μ_c is non-trivial on the link of the subspace $\iota(M^w_\kappa) \subset \mathcal{M}_{\mathfrak{t}}/S^1$. Thus, μ_c does not extend over $\iota(\mathcal{B}^*_{\mathfrak{t}}) \subset \mathcal{C}^*_{\mathfrak{t}}/S^1$ as the restriction of an extension to a contractible neighborhood of point $[A,0] \in \iota(\mathcal{B}^*_{\mathfrak{t}})$ would have to be trivial, contradicting Lemma 3.28.

3.2. **Geometric representatives.** To avoid having to define the link of M^w_κ in $\mathcal{M}_{\mathfrak{t}}$ as a homology class, we work with geometric representatives of these cohomology classes. We define geometric representatives $\mathcal{V}(\beta)$ and $\mathcal{W}$ to represent $\mu_p(\beta)$ and μ_c respectively. To facilitate the description of the intersection of the closures of $\mathcal{V}(\beta)$ and $\mathcal{W}$ in $\bar{\mathcal{M}}_{\mathfrak{t}}$ with the lower strata, we construct geometric representatives with certain localization properties—they are pulled back from configuration spaces over proper open subsets $U \subset X$. We let $\mathcal{B}^w_\kappa(U)$ and $\mathcal{C}_{\mathfrak{t}}(U)$ denote the quotient spaces of connections and pairs respectively on $U \subseteq X$, where $\kappa = -\frac{1}{4}p_1(\mathfrak{t})$ and w is an integral lift of $w_2(\mathfrak{t})$.

3.2.1. *Stratified spaces.* We begin by recalling a definition of a stratified space (see [48, Definition 11.0.1]) that will be sufficient for the purposes of defining our intersection pairings.

Definition 3.2. [26], [42], [48] A *smoothly stratified space* Z is a topological space with a *smooth stratification* given by a disjoint union, $Z = Z_0 \cup Z_1 \cup \cdots \cup Z_n$, where the *strata* Z_i are smooth manifolds. There is a partial ordering among the strata, given by $Z_i < Z_j$ if $Z_i \subset \bar{Z}_j$. There is a unique stratum of highest dimension, Z_0, such that $\bar{Z}_0 = Z$, called the *top stratum*. If Y, Z are smoothly stratified spaces, a map $f : Y \to Z$ is *smoothly stratified* if f is a continuous map, there are smooth stratifications of Z and Y such that f preserves strata, and restricted to each stratum f is a smooth map. A subspace $Y \subset Z$ is *smoothly stratified* if the inclusion is a smoothly stratified map.

Remark 3.3. If Z is a smoothly stratified space and $f : Z \to \mathbb{R}$ is a smoothly stratified map, that is, f is a continuous map which is smooth on each stratum, then for generic values of ε, the preimage $f^{-1}(\varepsilon)$ is a smoothly stratified subspace of Z.

We shall use the following definition of a geometric representative:

Definition 3.4. [35, p 588]. Let Z be a smoothly stratified space. A *geometric representative* for a rational cohomology class μ of dimension c on Z is a closed, smoothly stratified subspace $\mathcal{V}$ of Z together with a rational coefficient q, the *multiplicity*, satisfying

1. The intersection $Z_0 \cap \mathcal{V}$ of $\mathcal{V}$ with the top stratum Z_0 of Z has codimension c in Z_0 and has an oriented normal bundle.
2. The intersection of $\mathcal{V}$ with all strata of Z other than the top stratum has codimension 2 or more in $\mathcal{V}$.
3. The pairing of μ with a homology class h of dimension c is obtained by choosing a smooth singular cycle representing h whose intersection with all strata of $\mathcal{V}$ has the codimension $\dim Z_0 - c$ in that stratum of $\mathcal{V}$, and then taking q times the count (with signs) of the intersection points between the cycle and the top stratum of $\mathcal{V}$.

Definition 3.5. Let $\mathcal{V}_1, \dots, \mathcal{V}_n$ be geometric representatives on a compact, smoothly stratified space Z with multiplicities $q_1, \dots, q_n$. Assume

1. The sum of the codimensions of the $\mathcal{V}_i$ is equal to the dimension of the top stratum Z_0 of Z.
2. For every smooth stratum Z_s of Z, the smooth submanifolds $\mathcal{V}_i \cap Z_s$ intersect transversely.

Then dimension-counting and the definition of a geometric representative imply that the intersection $\cap_i \mathcal{V}_i$ is a finite collection of points in the top stratum Z_0:

$$\mathcal{V}_1 \cap \cdots \cap \mathcal{V}_n = \{v_1, \dots, v_N\} \subset Z_0.$$

Let $\varepsilon_j = \pm 1$ be the sign of this intersection at v_j. Then we define the *intersection number of the $\mathcal{V}_i$ in Z* by setting

$$\#\left(\mathcal{V}_1 \cap \cdots \cap \mathcal{V}_n \cap Z\right) = \left(\prod_{i=1}^{n} q_i\right) \sum_{j=1}^{N} \varepsilon_j.$$

A cobordism between two geometric representatives $\mathcal{V}$ and $\mathcal{V}'$ in Z with the same multiplicity is a geometric representative $\mathcal{W} \subset Z \times [0,1]$ which is transverse to the boundary and with $\mathcal{W} \cap Z \times \{0\} = \mathcal{V}$ and $\mathcal{W} \cap Z \times \{1\} = \mathcal{V}'$, with the obvious orientations of normal bundles.

The definition of intersection number does not change if $\mathcal{V}_i$ is replaced by $\mathcal{V}_i'$ and there is a cobordism between $\mathcal{V}_i$ and $\mathcal{V}_i'$ whose intersection with the other geometric representatives is transverse in each stratum. One can see this by observing that the intersection of the cobordism $\mathcal{W}$ with the other geometric representatives will be a collection of one-manifolds contained in Z_0 because the lower strata of Z have codimension two. The boundaries of these one manifolds are the points in the two intersections

$$\mathcal{V}_1 \cap \cdots \cap \mathcal{V}_n \quad \text{and} \quad \mathcal{V}_1 \cap \dots \mathcal{V}_{i-1} \cap \mathcal{V}_i' \cap \cdots \cap \mathcal{V}_n,$$

giving the equality of oriented intersection numbers.

Remark 3.6. In Definition 3.5, it is necessary to assume that the geometric representatives have transverse intersection in each stratum because we cannot assume there are perturbations of the geometric representatives which do intersect transversely. The definition of a smoothly-stratified space in Definition 3.2 does not control the topology of one strata near another. If there is "control data" on a neighborhood of one strata in another (see [26, p. 42], as is true for Whitney-stratified spaces, then such perturbations are constructed in [26, §1.3]. Instead, we will construct our geometric representatives pulling them back from a smooth manifold where one can assume that generic choices of the geometric representatives intersect transversely.

3.2.2. *Preliminaries for localization.* By construction, our geometric representatives will be "determined by restriction to submanifolds" of X, in the sense that they have the following localization property:

Definition 3.7. Let $U \subset X$ be a submanifold. A geometric representative $\mathcal{V}$ in $\mathcal{B}_\kappa^{w,*}$ or $\mathcal{C}_\mathfrak{t}^{*,0}/S^1$ is *determined by restriction to* $U \subset X$ if there is a geometric representative $\mathcal{V}_U$ in $\mathcal{B}_\kappa^{w,*}(U)$ or $\mathcal{C}_\mathfrak{t}^{*,0}(U)/S^1$ such that $\mathcal{V} = r_U^{-1}(\mathcal{V}_U)$, where r_U is the map given by restricting connections or pairs to the submanifold U.

This localization property will allow a partial description of the intersection of the closures of the geometric representatives in the subspace $\bar{\mathcal{M}}_\mathfrak{t}^{*,0}/S^1$ with the lower strata in this compactification. The technical issue which has complicated this localization technique since its introduction in [6], [7] (see [7, p. 192]) is that there can be pairs (connections) which are irreducible on X but are reducible when restricted to a submanifold $Y \subset X$. The bundles over $\mathcal{B}_\kappa^{w,*}(Y)$, whose sections define the geometric representative, do not extend over $\mathcal{B}_\kappa^{w}(Y)$. Therefore, the pullback of these sections do not have good properties (transversality, for example) over the subspace of connections in $\mathcal{B}_\kappa^{w,*}(X)$ which are reducible when restricted to Y. When working with the moduli space M_κ^w of anti-self-dual SO(3) connections, this problem can be overcome, if X is simply connected, by working with a tubular neighborhood $\nu(Y)$ of $Y \subset X$. The local-to-global reducibility result of [7, Lemma 4.3.21] implies that any anti-self-dual connection which is irreducible on X must be irreducible on $\nu(Y)$ if X is simply connected. If X is not simply connected, there can be "twisted reducible" connections, (see [10, Lemma 3.5] or [35, Lemma 2.4]) which are irreducible on X but reducible when restricted to a tubular neighborhood. The notion of a "suitable open neighborhood" of Y (see Definition 3.8) was introduced in [35] to deal with the problem of twisted reducibles. Any anti-self-dual connection which is irreducible on X must be irreducible on a suitable open neighborhood. We can then define geometric representatives by pulling back sections of bundles over $\mathcal{B}_\kappa^{w,*}(U(Y))$, where $U(Y)$ is a suitable open neighborhood of Y. When points $[\hat{A}_\alpha]$ in this geometric representative approach a point $[\hat{A}_0, \mathbf{x}]$ in a lower Uhlenbeck level where the support of $\mathbf{x}$ is disjoint from $U(Y)$, the point $[\hat{A}_0]$ will also be in this geometric representative. Thus, either $[\hat{A}_0]$ is in the geometric representative or the support of $\mathbf{x}$ meets $U(Y)$. Both of these conditions have high enough codimension in the lower Uhlenbeck levels of the compactification to ensure, via dimension-counting arguments (see [35, pp. 592–593]), that the intersection of the geometric representatives is compactly supported in the top level M_κ^w of the Uhlenbeck compactification $\bar{M}_\kappa^w$.

We begin by recalling the following definition of Kronheimer and Mrowka:

Definition 3.8. [35, p. 589] A smooth submanifold-with-boundary or open set $U \subsetneq X$ is *suitable* if the induced map $H_1(U;\mathbb{Z}/2\mathbb{Z}) \to H_1(X;\mathbb{Z}/2\mathbb{Z})$ is surjective.

Let $Y \subset X$ be a submanifold with tubular neighborhood $\nu(Y)$. If D is a set of embedded loops generating $H_1(X;\mathbb{Z}/2\mathbb{Z})$, which are mutually disjoint and transversal to Y, then a tubular neighborhood $\nu(Y \cup D)$ of $Y \cup D \subset X$ is a suitable neighborhood Y. By tubular neighborhood of the possibly singular space $Y \cup D$, we mean a smoothing of the union of the tubular neighborhoods of Y and of D.

Remark 3.9. If $[\hat{A}] \in M_\kappa^{w,*}$ then the restriction of $\hat{A}$ to any suitable open neighborhood is irreducible by unique continuation [7, Lemma 4.3.21]. Hence, the only irreducible anti-self-dual connections which could be reducible when restricted to an open set are the twisted reducibles (see [10, §3.2] or [35, p. 586]). However, the homology condition in Definition 3.8 excludes this possibility.

The corresponding local-to-global reducibility result for PU(2) monopoles which are not zero-section pairs [17, Theorem 5.11] is stronger than that for anti-self-dual connections:

Theorem 3.10. [17, Theorem 5.10] *Suppose* (A,Φ) *is a solution to the perturbed* PU(2) *monopole equations* [10, Equation (2.32)] *over a connected, oriented, smooth four-manifold* X *with smooth Riemannian metric such that* (A,Φ) *is reducible on a non-empty open subset* $U \subset X$. *Then* (A,Φ) *is reducible on* X *if*

- $\Phi \not\equiv 0$ *on* X, *or*
- $\Phi \equiv 0$, *and* M_κ^w *contains no twisted reducibles or* U *is suitable.*

Both suitable and tubular neighborhoods of submanifolds are open subsets of X and thus have codimension zero. However, a tubular neighborhood admits a retraction onto the submanifold while a suitable neighborhood admits a retraction onto the union of the submanifold and a collection of loops. Hence, these neighborhoods may, for the purposes of counting intersection points, be thought of as having codimension equal to that of the submanifold or to that of the union of the submanifold and some loops in X. In this sense, the suitable neighborhood of a point has smaller codimension than a tubular neighborhood of a point. Because the lower strata of $\bar{\mathcal{M}}_\mathfrak{t}$ do not have codimension as large as those of $\bar{M}_\kappa^w$ and because the suitable neighborhoods do not have codimension as large as tubular neighborhoods, we will require an additional technical condition on the elements of $H_\bullet(X;\mathbb{R})$ to ensure the intersection of the geometric representatives does not intersect the lower levels away from the reducible pairs.

Let $Y \subset X$ be a smooth submanifold. If Y is a manifold with boundary, the manifold structure of the configuration space $\mathcal{B}_\kappa^w(Y)$ is described in, say, [6, p. 262], [7, p. 192], [56, §2(a)]; the corresponding slice result for $\mathcal{C}_\mathfrak{t}(Y)/S^1$ can be obtained from the slice result for manifolds without boundary [17, Proposition 2.12] by taking into account the Neumann boundary conditions as in [56]. Let $r_Y : \mathcal{C}_\mathfrak{t}/S^1 \to \mathcal{C}_\mathfrak{t}(Y)/S^1$ and $r_Y : \mathcal{B}_\kappa^w \to \mathcal{B}_\kappa^w(Y)$ denote the restriction maps defined by $[A,\Phi] \mapsto [A|_Y, \Phi|_Y]$ and $[\hat{A}] \mapsto [\hat{A}|_Y]$, respectively. We will use the same notation for the restriction map on any domain.

We define $\mathcal{C}_\mathfrak{t}^*(X,U)$ to be the quotient space of pairs on X which are irreducible when restricted to U, let $\mathcal{C}_\mathfrak{t}^0(X,U)$ denote the quotient space of pairs on X which are not zero-section pairs when restricted to U, and let $\mathcal{C}_\mathfrak{t}^{*,0}(X,U) = \mathcal{C}_\mathfrak{t}^*(X,U) \cap \mathcal{C}_\mathfrak{t}^0(X,U)$. The space $\mathcal{B}_\kappa^{w,*}(X,U)$ is similarly defined.

If $[A,\Phi] \in \mathcal{M}_\mathfrak{t}^*$, then Theorem 3.10 implies that the restriction of the SO(3) connection $\hat{A}$ to the suitable neighborhood $\nu(Y \cup D)$ cannot be reducible; if $[A,\Phi] \in \mathcal{M}_\mathfrak{t}^{*,0}$, so we further assume $\Phi \not\equiv 0$, then Theorem 3.10 implies that the restriction of the connection $\hat{A}$ to $\nu(Y)$ cannot be reducible. There is a disjoint decomposition

$$\mathcal{M}_\mathfrak{t}^* = \mathcal{M}_\mathfrak{t}^{*,0} \cup \iota(M_\kappa^{w,*}).$$

The unique continuation result for reducible anti-self-dual SO(3) connections [7, Lemma 4.3.21] and PU(2) monopoles (Theorem 3.10), and the preceding decomposition and remarks yield

Lemma 3.11. *Let $U \subset X$ be an open subset and let $Y \subset X$ be a submanifold. Then the following inclusions hold:*

$$\pi_{\mathcal{B}}(\mathcal{M}_{\mathfrak{t}}^{*,0}) \subset \mathcal{B}_{\kappa}^{w,*}(X,U) \quad \text{and} \quad M_{\kappa}^{w,*} \subset \mathcal{B}_{\kappa}^{w,*}(X,\nu(Y \cup D)),$$

where, as in equation (2.6), $\pi_{\mathcal{B}} : \mathcal{C}_{\mathfrak{t}} \to \mathcal{B}_{\kappa}^{w}$ *is the projection* $[A,\Phi] \mapsto [\hat{A}]$.

We can now proceed to construct geometric representatives for the classes $\mu_p(\beta) \in H^{\bullet}(\mathcal{M}_{\mathfrak{t}}^{*}/S^1;\mathbb{R})$ and $\mu_c \in H^{\bullet}(\mathcal{M}_{\mathfrak{t}}^{*,0}/S^1;\mathbb{R})$.

3.2.3. *The geometric representatives for μ_p.* Let $Y \subset X$ be a smooth submanifold and let $\beta = [Y] \in H_{\bullet}(X;\mathbb{R})$. Let $\nu(Y \cup D)$ be a suitable open neighborhood of Y. In [35, p. 588–595], geometric representatives for the classes $\mu_p(\beta) \in H^{\bullet}(M_{\kappa}^{w,*};\mathbb{R})$,

$$r_{\nu(Y\cup D)}^{-1}(\mathcal{V}(\beta)) \subset \mathcal{B}_{\kappa}^{w,*}(X,\nu(Y \cup D)),$$

are defined which have the property that they are determined by

$$\mathcal{V}(\beta) \subset \mathcal{B}_{\kappa}^{w,*}(\nu(Y \cup D)).$$

Let $(r_{\nu(Y\cup D)}\pi_{\mathcal{B}})^{-1}(\mathcal{V}(\beta))$ be the preimage of this geometric representative with respect to the map

$$\pi_{\mathcal{B}} : \mathcal{C}_{\mathfrak{t}}^{*}(X,\nu(Y \cup D))/S^1 \to \mathcal{B}_{\kappa}^{w,*}(X,\nu(Y \cup D)).$$

The following result is a clear consequence of the definitions and Lemma 3.11.

Lemma 3.12. *If $Y \subset X$ is a smooth submanifold, representing a class $\beta \in H_{\bullet}(X;\mathbb{R})$, then $(r_{\nu(Y\cup D)}\pi_{\mathcal{B}})^{-1}(\mathcal{V}(\beta)) \subset \mathcal{M}_{\mathfrak{t}}^{*}/S^1$ is a geometric representative for $\mu_p(\beta) \in H^{4-\bullet}(\mathcal{M}_{\mathfrak{t}}^{*}/S^1;\mathbb{R})$ and is determined by restriction to $\nu(Y \cup D) \subset X$.*

Henceforth, we shall abuse notation slightly and write $\mathcal{V}(\beta)$ for $\mathcal{V}(\beta) \subset \mathcal{B}_{\kappa}^{w,*}(\nu(Y \cup D))$, for its preimage $r_{\nu(Y\cup D)}^{-1}(\mathcal{V}(\beta)) \subset \mathcal{B}_{\kappa}^{w,*}(X,\nu(Y \cup D))$, and for $(r_{\nu(Y\cup D)}\pi_{\mathcal{B}})^{-1}(\mathcal{V}(\beta)) \subset \mathcal{C}_{\mathfrak{t}}^{*}(X,\nu(Y \cup D))/S^1$.

3.2.4. *A representative for the determinant class.* Recall that $\mathcal{C}_{\mathfrak{t}}^{*,0}$ is an S^1 bundle over $\mathcal{C}_{\mathfrak{t}}^{*,0}/S^1$. Let $\nu(x)$ be a tubular neighborhood of x and let s be a generic, smooth, C^0 bounded section of the line bundle

$$\mathbb{L}_{\mathfrak{t}}(\nu(x)) = \mathcal{C}_{\mathfrak{t}}^{*,0}(\nu(x)) \times_{(S^1,\times -2)} \mathbb{C} \to \mathcal{C}_{\mathfrak{t}}^{*,0}(\nu(x)). \tag{3.13}$$

The action of S^1 in the above is the same as that in (3.11), for the definition (3.10) of the universal line bundle $\mathbb{L}_{\mathfrak{t}} \to \mathcal{C}_{\mathfrak{t}}^{*,0}$. The pullback $r_{\nu(x)}^{*}\mathbb{L}_{\mathfrak{t}}(\nu(x))$ is thus isomorphic to the restriction of $\mathbb{L}_{\mathfrak{t}}$ to $\mathcal{C}_{\mathfrak{t}}^{*,0}(X,\nu(x))$. We define a geometric representative by

$$\mathcal{W} = (r_{\nu(x)}^{*}s)^{-1}(0) \subset \mathcal{C}_{\mathfrak{t}}^{*,0}(X,\nu(x)). \tag{3.14}$$

Since $\mathcal{M}_{\mathfrak{t}}^{*,0} \subset \mathcal{C}_{\mathfrak{t}}^{*,0}(X,\nu(x))$ by Theorem 3.10 and the unique continuation theorem for the Dirac operator, [17, Lemma 5.12], the proof of the next lemma is then clear.

Lemma 3.13. *For a generic choice of section s, the zero locus $\mathcal{W}$ of the section $r_{\nu(x)}^{*}s$ is a geometric representative for $\mu_c \in H^2(\mathcal{M}_{\mathfrak{t}}^{*,0}/S^1;\mathbb{R})$ and is determined by restriction to $\nu(x)$.*

Let $1 \in \mathbb{A}(X)$ be the element of degree zero. If $z = \beta_1 \cdots \beta_r \in \mathbb{A}(X)$, we write

$$\delta_i = \sum_{\{p \mid \beta_p \in H_i(X;\mathbb{R})\}} 1 \quad \text{and} \quad \deg(z) = \sum_{i=0}^{4}(4-i)\deg_i(z). \tag{3.15}$$

For monomials $z = \beta_1 \cdots \beta_r$, we set

$$\begin{aligned}\mu_p(z) &= \mu_p(\beta_1) \smile \cdots \smile \mu_p(\beta_r), \\ \mathcal{V}(z) &= \mathcal{V}(\beta_1) \cap \cdots \cap \mathcal{V}(\beta_r),\end{aligned} \tag{3.16}$$

and define $\mu_p(z)$ and $\mathcal{V}(z)$ for arbitrary elements $z \in \mathbb{A}(X)$ by $\mathbb{R}$-linearity [35, p. 595]. We write

$$\mu_c^m = \underbrace{\mu_c \smile \cdots \smile \mu_c}_{m \text{ times}} \quad \text{and} \quad \mathcal{W}^m = \underbrace{\mathcal{W} \cap \cdots \cap \mathcal{W}}_{m \text{ times}}, \tag{3.17}$$

for products of the class μ_c and its dual $\mathcal{W}$ (with the understanding that the copies of $\mathcal{W}$ in the above representative are defined with different points x and different transversely intersecting sections s in Lemma 3.13).

3.3. **Extension of the geometric representatives.** The Uhlenbeck closure $\bar{\mathcal{M}}_{\mathfrak{t}}$ of the PU(2) monopole moduli space $\mathcal{M}_{\mathfrak{t}}$ is described in [10, §2.2]. The space $\bar{\mathcal{M}}_{\mathfrak{t}}$ is compact [10, Theorem 2.12], [17, Theorem 1.2]. We shall need to consider the following subspaces of $\bar{\mathcal{M}}_{\mathfrak{t}}$:

$$\begin{aligned}\bar{\mathcal{M}}_{\mathfrak{t}}^* &= \{[A, \Phi, \mathbf{x}] \in \bar{\mathcal{M}}_{\mathfrak{t}} : A \text{ is irreducible}\}, \\ \bar{\mathcal{M}}_{\mathfrak{t}}^0 &= \{[A, \Phi, \mathbf{x}] \in \bar{\mathcal{M}}_{\mathfrak{t}} : \Phi \not\equiv 0\},\end{aligned} \tag{3.18}$$

and so, as defined in (3.1), $\bar{\mathcal{M}}_{\mathfrak{t}}^{*,0} = \bar{\mathcal{M}}_{\mathfrak{t}}^* \cap \bar{\mathcal{M}}_{\mathfrak{t}}^0$. We also define

$$\mathcal{M}_{\mathfrak{t}}^{\geq\varepsilon} = \{[A, \Phi] \in \mathcal{M}_{\mathfrak{t}} : \|\Phi\|_{L^2}^2 \geq \varepsilon\} \subset \mathcal{M}_{\mathfrak{t}}^0, \tag{3.19}$$

and the subspace $\mathcal{M}_{\mathfrak{t}}^{*,\geq\varepsilon} \subset \mathcal{M}_{\mathfrak{t}}^{*,0}$ is defined analogously. By [10, Theorem 2.13], the dimension of the highest stratum $\mathcal{M}_{\mathfrak{t}}^{*,0}$ of $\bar{\mathcal{M}}_{\mathfrak{t}}$ is given by

$$\dim \mathcal{M}_{\mathfrak{t}}^{*,0} = d_a(\mathfrak{t}) + 2n_a(\mathfrak{t}), \tag{3.20}$$

where

$$\begin{aligned}d_a(\mathfrak{t}) &= -2p_1(\mathfrak{t}) - \tfrac{3}{2}(\chi + 2\sigma), \\ n_a(\mathfrak{t}) &= \tfrac{1}{4}(p_1(\mathfrak{t}) + c_1(\mathfrak{t})^2 - \sigma).\end{aligned} \tag{3.21}$$

Recall that the spin^u structure $\mathfrak{t}_\ell$ defined in [10, Equation (2.44)] has $p_1(\mathfrak{t}_\ell) = p_1(\mathfrak{t}) + 4\ell$ and so equation (3.20) implies $\dim \mathcal{M}_{\mathfrak{t}_\ell} = \dim \mathcal{M}_{\mathfrak{t}} - 6\ell$. The strata of $\mathcal{M}_{\mathfrak{t}_\ell}/S^1 \times \mathrm{Sym}^\ell(X)$ then have codimension at least 2ℓ relative to the top stratum. Thus we can calculate the intersection of geometric representatives whose intersections with the lower strata have the expected codimensions because this ensures that (by the usual dimension-counting argument) the intersection of the geometric representatives will be in the top stratum.

Definition 3.14. The closures of the geometric representatives, $\mathcal{V}(\beta)$, $\mathcal{W}$, in $\bar{\mathcal{M}}_{\mathfrak{t}}$ are denoted by $\bar{\mathcal{V}}(\beta)$, $\bar{\mathcal{W}}$, respectively. For $z = \beta_1 \cdots \beta_r \in \mathbb{A}(X)$, the positive generator $x \in H_0(X; \mathbb{Z})$, and an integer $m \geq 0$, we denote

$$\bar{\mathcal{V}}(z) = \bar{\mathcal{V}}(\beta_1) \cap \cdots \cap \bar{\mathcal{V}}(\beta_r) \quad \text{and} \quad \bar{\mathcal{W}}^m = \underbrace{\bar{\mathcal{W}} \cap \cdots \cap \bar{\mathcal{W}}}_{m \text{ times}}. \tag{3.22}$$

We shall see in Lemma 3.15 that these closures intersect the lower strata of $\bar{\mathcal{M}}_{\mathfrak{t}}^*/S^1$ in sets of the appropriate codimension, except for $\bar{\mathcal{V}}(x)$ where $x \in H_0(X; \mathbb{Z})$ (see the remarks following the proof of Lemma 3.15), and thus are geometric representatives on the compactification, away from the zero-section and reducible monopoles. The description of the intersection of $\bar{\mathcal{V}}(\beta)$ and $\bar{\mathcal{W}}$ with the lower strata given in this section is incomplete, as it does not give the multiplicities of all components of these intersections. A more complete description will be given in [15] using information about neighborhoods of the lower strata in $\bar{\mathcal{M}}_{\mathfrak{t}}^*$ obtained from gluing maps.

Recall that $\operatorname{Sym}^\ell(X)$ is a smoothly stratified space, the strata being enumerated by partitions of $\ell \in \mathbb{N}$. For $i = 1, \dots, \ell$, let $\pi_i : X \times \cdots \times X \to X$ be projection onto the ith factor. For any subset $Y \subset X$, let $S^\ell(Y)$ be the image of $\pi_1^{-1}(Y) \cup \cdots \cup \pi_\ell^{-1}(Y)$ in $\operatorname{Sym}^\ell(X)$ under the projection $X^\ell \to \operatorname{Sym}^\ell(Y)$. If $\Sigma \subset \operatorname{Sym}^\ell(X)$ is a smooth stratum, we define $S_\Sigma(Y) = S^\ell(Y) \cap \Sigma$. Let $\pi_\Sigma : \mathcal{M}_{\mathfrak{t}_\ell} \times \Sigma \to \Sigma$ be the projection.

On each space $\mathcal{M}_{\mathfrak{t}_\ell}^*/S^1$ and $\mathcal{M}_{\mathfrak{t}_\ell}^{*,0}/S^1$ there are geometric representatives $\mathcal{V}_\ell(\beta)$ and $\mathcal{W}_\ell$ defined by the same construction as $\mathcal{V}(\beta)$ and $\mathcal{W}$, except using the bundles $\mathfrak{g}_{\mathfrak{t}_\ell}$ instead of $\mathfrak{g}_{\mathfrak{t}}$. We write $\mathcal{V}_\ell(\beta)$ and $\mathcal{W}_\ell$ for the pullbacks of these geometric representatives to $\mathcal{M}_{\mathfrak{t}_\ell} \times \operatorname{Sym}^\ell(X)$.

Lemma 3.15. *Let $\ell \geq 0$ be an integer, let $\Sigma \subset \operatorname{Sym}^\ell(X)$ be a smooth stratum, and let $\beta \in H_\bullet(X;\mathbb{R})$.*

1. *If β has a smoothly embedded representative $Y \subset X$ with a suitable neighborhood $\nu(D \cup Y)$ and $x \in X$ has a tubular neighborhood $\nu(x)$, then*
   (a) $\bar{\mathcal{V}}(\beta) \cap (M_{\mathfrak{t}_\ell}^*/S^1 \times \Sigma) \subset \mathcal{V}_\ell(\beta) \cup \pi_\Sigma^{-1}(S_\Sigma(\nu(Y \cup D)))$,
   (b) $\bar{\mathcal{W}} \cap (M_{\mathfrak{t}_\ell}^{*,0}/S^1 \times \Sigma) \subset \mathcal{W}_\ell \cup \pi_\Sigma^{-1}(S_\Sigma(\nu(x)))$.
2. *If $\iota(M_{\mathfrak{s}}) \subset \bar{\mathcal{M}}_{\mathfrak{t}}$ and $\beta \in H_2(X;\mathbb{R})$ is a two-dimensional class with $\langle c_1(\mathfrak{t}) - c_1(\mathfrak{s}), \beta\rangle \neq 0$, and $\gamma \in H_1(X;\mathbb{R})$, then we have the following reverse inclusions:*
   (a) $\iota(M_{\mathfrak{s}}) \subset \bar{\mathcal{V}}(\beta)$,
   (b) $\iota(M_{\mathfrak{s}}) \subset \bar{\mathcal{V}}(\gamma)$,
   (c) $\iota(M_{\mathfrak{s}}) \subset \bar{\mathcal{V}}(x)$,
   (d) $\iota(\bar{M}_\kappa^w) \cup \iota(M_{\mathfrak{s}}) \subset \bar{\mathcal{W}}$.

Remark 3.16. From the expression for $\mu_p(\beta)$ in Corollary 4.7, when $\beta \in H_3(X;\mathbb{R})$, one can see that $\mu_p(\beta)$ extends across $\iota(M_{\mathfrak{s}})$. Thus, $\bar{\mathcal{V}}(\beta)$ should be transverse to $\iota(M_{\mathfrak{s}})$ and so $\bar{\mathcal{V}}(\beta) \cap \iota(M_{\mathfrak{s}})$ would be a codimension-one subset of $\iota(M_{\mathfrak{s}})$ in this case.

Proof. Here we only prove Assertion (1) . Assertions (2a), (2b), and (2c) will be shown in Corollary 4.7, while Assertion (2d) will follow from Lemma 3.28.

We prove Assertion (1a) about $\bar{\mathcal{V}}(\beta)$ by restricting pairs to the complement of the set $\pi_\Sigma^{-1}(S_\Sigma(\nu(Y \cup D)))$. We assume that $[A_\alpha, \Phi_\alpha] \in \mathcal{V}(\beta)$ is a sequence of points in $\mathcal{M}_{\mathfrak{t}}^*/S^1$ converging to the point

$$[A_\infty, \Phi_\infty, \mathbf{y}] \in (M_{\mathfrak{t}_\ell}^*/S^1 \times \Sigma) - \pi_\Sigma^{-1}(S_\Sigma(\nu(Y \cup D))).$$

Given a suitable neighborhood $U = \nu(Y \cup D)$ of $Y \subset X - \mathbf{y}$, we may choose a positive constant r such that

$$U \subset X - \cup_{y \in \mathbf{y}} B(y, r).$$

By the definition of Uhlenbeck convergence, (A_α, Φ_α) converges in the C^∞ topology to (A_∞, Φ_∞) on $X - \cup_{y\in\mathbf{y}} B(y,r)$, modulo gauge transformations, and thus

$$\lim_{\alpha\to\infty} [A_\alpha|_U, \Phi_\alpha|_U] = [A_\infty|_U, \Phi_\infty|_U]. \tag{3.23}$$

Let $\mathcal{V}_Y(\beta) \subset \mathcal{B}_\kappa^{w,*}(\nu(Y \cup D))$ be the geometric representative whose pullback defines $\mathcal{V}(\beta)$. By Lemma 3.12, if $[A_\alpha, \Phi_\alpha] \in \mathcal{V}(\beta)$, then $[(A_\alpha, \Phi_\alpha)|_U] \in \mathcal{V}_Y(\beta)$. Because $\mathcal{V}_Y(\beta)$ is closed (see the definition in [35, pp. 588–592]), equation (3.23) implies that $[(A_\infty, \Phi_\infty)|_U] \in \mathcal{V}_Y(\beta)$ and thus $[A_\infty, \Phi_\infty] \in \mathcal{V}_\ell(\beta)$.

The same argument proves Assertion (1b) concerning $\bar{\mathcal{W}}$, except one observes that one can replace a suitable neighborhood U of x with a tubular neighborhood $\nu(x)$. $\square$

Lemma 3.15 shows that the intersection of $\bar{\mathcal{V}}(\beta)$ with the lower levels of $\bar{\mathcal{M}}_{\mathfrak{t}}^*/S^1$ has the same codimension as that of $\mathcal{V}(\beta)$ in $\mathcal{M}_{\mathfrak{t}}^*/S^1$, unless $\beta \in H_0(X;\mathbb{Z})$. This is only significant if $z = \beta_1 \cdots \beta_r$ contains both a three-dimensional and a four-dimensional homology class. Then the loops in the suitable neighborhood $\nu(D \cup x)$ may intersect the three-manifold Y. In general, if U_i is a suitable neighborhood of a smooth representative of β_i , then for any $x \in X$, the inequality

$$\sum_{\{i:\ x \in U_i\}} (4 - \dim \beta_i) \leq 5, \tag{3.24}$$

holds [35, Equation (2.7)]. If there are either no three-dimensional classes among the β_i or no four-dimensional classes among the β_i, the inequality (3.24) can be improved to

$$\sum_{\{i:\ x \in U_i\}} (4 - \dim \beta_i) \leq 4. \tag{3.25}$$

If $z = \beta_1 \cdots \beta_r \in \mathbb{A}(X)$ and there is a collection of suitable neighborhoods U_i of smooth representatives of β_i satisfying (3.25), then we call z *intersection-suitable*.

Lemma 3.17. *If $z = \beta_1 \cdots \beta_r \in \mathbb{A}(X)$ and either $\beta_i \notin H_0(X;\mathbb{Z})$ for $i = 1, \ldots, r$ or $\beta_i \notin H_3(X;\mathbb{R})$ for $i = 1, \ldots, r$ then z is intersection-suitable.*

Let $z \in \mathbb{A}(X)$ and δ_c be a non-negative integer which satisfy

$$\deg(z) + 2\delta_c = d_a + 2n_a - 2, \tag{3.26}$$

so for generic choices of the geometric representatives, $\mathcal{V}(z) \cap \mathcal{W}^{\delta_c} \cap \mathcal{M}_{\mathfrak{t}}^{*,0}/S^1$ is a collection of one-manifolds. If

$$[A_0, \Phi_0, \mathbf{y}] \in \bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^{\delta_c} \cap \bar{\mathcal{M}}_{\mathfrak{t}}^{*,0}/S^1,$$

where $\mathbf{y} \in \mathrm{Sym}^\ell(X)$, then by equation (3.24) and Lemma 3.15 we have

$$[A_0, \Phi_0] \in \mathcal{V}_\ell(z') \cap \mathcal{W}_\ell^{\delta_c - j},$$

for some $z' = \beta_{i_1} \cdots \beta_{i_q}$ and $1 \leq i_1 < \cdots < i_q \leq r$, where $0 \leq j \leq \delta_c$. The preceding intersection has codimension greater than or equal to

$$\deg(z) + 2\delta_c - 5\ell = d_a + 2n_a - 2 - 5\ell.$$

Then, because $\dim(\mathcal{M}_{\mathfrak{t}_\ell}/S^1) = d_a + 2n_a - 1 - 6\ell$, the intersection

$$\mathcal{V}_\ell(z') \cap \mathcal{W}_\ell^{\delta_c - j} \cap \mathcal{M}_{\mathfrak{t}_\ell}/S^1 \tag{3.27}$$

has dimension less than or equal to

$$\dim(\mathcal{M}_{\mathfrak{t}_\ell}/S^1) - (\deg(z) + 2\delta_c - 5\ell) = 1 - \ell.$$

Hence, there could be a point in $\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^{\delta_c} \cap \bar{\mathcal{M}}_{\mathfrak{t}}^{*,0}/S^1$ contained in the level $X \times \mathcal{M}_{\mathfrak{t}_1}^{*,0}/S^1$, where $\ell = 1$. However, if z is intersection-suitable, the dimension of the intersection (3.27) is $1 - 2\ell$, using equation (3.25), so intersection will be empty if $\ell > 0$. Thus, we have the following corollary to Lemma 3.15.

Corollary 3.18. *Let $z \in \mathbb{A}(X)$ be intersection-suitable and let δ_c be a non-negative integer satisfying*

$$\deg(z) + 2\delta_c = \dim(\mathcal{M}_{\mathfrak{t}}^{*,0}/S^1) - 1 = d_a + 2n_a - 2.$$

Then for generic choices of geometric representatives, the intersection

$$\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^{\delta_c} \cap \bar{\mathcal{M}}_{\mathfrak{t}}^{*,0}/S^1,$$

is a collection of one-dimensional manifolds, disjoint from the lower strata of $\bar{\mathcal{M}}_{\mathfrak{t}}^*/S^1$.

Remark 3.19. The restriction that $z \in \mathbb{A}(X)$ be intersection-suitable is a technical one and it should be possible to remove it; we plan to address this point in a subsequent paper.

3.4. **Geometric representatives and zero-section monopoles.** Our goal in this subsection is to show that the signed count of the points in the intersection of the geometric representatives with the link of Donaldson moduli space, $\#(\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^{n_a-1} \cap \mathbf{L}^w_{\mathfrak{t},\kappa})$, can be expressed in terms of the intersection number $\#(\bar{\mathcal{V}}(z) \cap \bar{M}^w_\kappa)$, which defines a Donaldson invariant (at least when w is chosen so that no SO(3) bundle with second Stiefel-Whitney class $w \pmod 2$ admits a flat connection); the conclusion is stated as Proposition 3.29.

3.4.1. *Geometric representatives on the stratum of zero-section monopoles.* From the construction of the Uhlenbeck compactifications for M^w_κ and $\mathcal{M}_\mathfrak{t}$, the smoothly stratified embedding [10, Equation (3.5)],

$$\iota : M^w_\kappa \hookrightarrow \mathcal{M}_\mathfrak{t}, \quad [\hat{A}] \mapsto [A, 0],$$

extends (see [10, §2.2], [17, §4], [7, §4.4]) to smoothly stratified embedding

$$\iota : \bar{M}^w_\kappa \hookrightarrow \bar{\mathcal{M}}_\mathfrak{t}, \quad [\hat{A}, \mathbf{x}] \mapsto [A, 0, \mathbf{x}],$$

where $\kappa = -\frac{1}{4}p_1(\mathfrak{t})$ and w is any integral lift of $w_2(\mathfrak{t})$. Define

$$\bar{M}^{\mathrm{asd}}_\mathfrak{t} = \{[A, \Phi, \mathbf{x}] \in \bar{\mathcal{M}}_\mathfrak{t} : \Phi = 0\} \subset \bar{\mathcal{M}}_\mathfrak{t}. \tag{3.28}$$

The space $\mathbf{L}^w_{\mathfrak{t},\kappa}$ defined in [10, Definition 3.7] serves as a link for $\bar{M}^{\mathrm{asd}}_\mathfrak{t}$. Although the definitions imply that

$$\iota(\bar{M}^w_\kappa) \subset \bar{M}^{\mathrm{asd}}_\mathfrak{t},$$

the reverse inclusion might not be true. For example, suppose $[\hat{A}_0]$ is the gauge-equivalence class of a flat connection on an SO(3) bundle F over X with $w_2(F) = w \pmod 2$ and $-\frac{1}{4}p_1(F) = 0$. The Uhlenbeck compactification $\bar{M}^w_\kappa$ might not contain all points $[\hat{A}_0, \mathbf{x}] \in M^w_0 \times \mathrm{Sym}^\ell(X)$ because there can be obstructions to gluing [54] onto flat connections, as the Freed-Uhlenbeck generic metrics theorem does not guarantee they are smooth points of their moduli spaces [7], [21]. However, there could be a sequence of points $[A_\alpha, \Phi_\alpha] \in \mathcal{M}_\mathfrak{t}$ converging to $[A_0, 0, \mathbf{x}]$ in the Uhlenbeck topology but no sequence $[\hat{A}'_\alpha] \in M^w_\kappa$ also converging to $[\hat{A}_0, \mathbf{x}]$.

Definition 3.20. A class $v \in H^2(X; \mathbb{Z}/2\mathbb{Z})$ is *good* if no integral lift of v is torsion.

Observe that a class $v \in H^2(X; \mathbb{Z}/2\mathbb{Z})$ is good if and only if no line bundle L over X with first Chern class $c_1(L) \equiv v \pmod 2$ admits a flat connection or if and only if no SO(3) bundle over X with second Stiefel-Whitney class v admits a flat connection. Thus, Lemma 3.2 in [10] gives a criterion for v to be good.

Hence, if $w_2(\mathfrak{t})$ is good, there are no flat SO(3) connections in $\bar{\mathcal{M}}_\mathfrak{t}$ and, when there no obstructions to gluing (for example, when the metric on X is generic in the sense of [7, 21]), it follows from Taubes' gluing theorem for anti-self-dual SO(3) connections [53], [55] that

$$\bar{M}^{\mathrm{asd}}_\mathfrak{t} \subset \iota(\bar{M}^w_\kappa).$$

The preceding discussion yields

Lemma 3.21. *Let $\mathfrak{t}$ be a spin^u structure on a closed, oriented four-manifold X with generic metric, $b_2^+(X) > 0$ and $w_2(\mathfrak{t}) \equiv w \pmod 2$, for $w \in H^2(X; \mathbb{Z})$. If $w \pmod 2$ is good, then*

$$\bar{M}^{\mathrm{asd}}_\mathfrak{t} = \iota(\bar{M}^w_\kappa). \tag{3.29}$$

The constraint that $w \pmod 2$ is good is also used to separate the strata of zero-section PU(2) monopoles from the strata of reducible monopoles, so that the moduli space of PU(2) monopoles gives a smooth cobordism between their links. Therefore, when $w \pmod 2$ is good, equation (3.29) holds, we have a disjoint union

$$\bar{\mathcal{M}}_{\mathfrak{t}} = \bar{\mathcal{M}}_{\mathfrak{t}}^{0} \cup \iota(\bar{M}_{\kappa}^{w}),$$

and $\mathbf{L}_{\mathfrak{t},\kappa}^{w}$ is a link of $\iota(\bar{M}_{\kappa}^{w}) \subset \bar{\mathcal{M}}_{\mathfrak{t}}/S^1$.

3.4.2. *A definition of the Donaldson invariants.* Fix $w \in H^2(X;\mathbb{Z})$ and let $z \in \mathbb{A}(X)$ be a monomial whose degree $\deg(z)$ satisfies equation (1.4). Then let $\kappa \in \frac{1}{4}\mathbb{Z}$ be defined by

$$\deg(z) = 8\kappa - \tfrac{3}{2}(\chi + \sigma)\,. \tag{3.30}$$

Let $\tilde{X} = X\#\overline{\mathbb{CP}}^2$ denote the blow-up of X and let $e \in H_2(\tilde{X};\mathbb{Z})$ be the exceptional class and let $\mathrm{PD}[e]$ be its Poincaré dual. Since $-4\kappa = w^2 \pmod 4$ by equations (1.4) and (3.30) and thus $-4(\kappa + 1/4) = (w + \mathrm{PD}[e])^2 \pmod 4$, we can find an SO(3) bundle $\tilde{F} \to \tilde{X}$ with $p_1(\tilde{F}) = -4(\kappa + 1/4)$ and $w_2(\tilde{F}) = (w + \mathrm{PD}[e]) \pmod 2$ [25, Theorem 1.4.20]. We can therefore define $M_{\kappa+1/4}^{w+\mathrm{PD}[e]}(\tilde{X})$, the moduli space of anti-self-dual SO(3) connections on $\tilde{F}$. Then the *Donaldson invariant* is defined by [35, p. 594]

$$D_X^w(z) = \#\left(\bar{\mathcal{V}}(ze) \cap \bar{M}_{\kappa+1/4}^{w+\mathrm{PD}[e]}(\tilde{X})\right), \tag{3.31}$$

where the moduli space $M_{\kappa+1/4}^{w+\mathrm{PD}[e]}(\tilde{X})$ is given the orientation $o(\Omega, w + \mathrm{PD}[e])$ and on the right-hand side, we consider z to be a monomial in $\mathbb{A}(\tilde{X})$ via the inclusion

$$H_2(X;\mathbb{R}) \subset H_2(\tilde{X};\mathbb{R}) = H_2(X;\mathbb{R}) \oplus \mathbb{R}[e] \quad \text{and} \quad \mathbb{A}(X) \subset \mathbb{A}(\tilde{X}).$$

The Donaldson invariant is independent of the choice of generic geometric representatives and, when $b_2^+(X) > 1$, independent of the metric.

If $M_\kappa^w(X)$ is given the orientation $o(\Omega, w)$, a well-known special case of the blow-up formula [18, Lemma 3.13], [31, Theorem 6.9] implies that

$$\#\left(\bar{\mathcal{V}}(z) \cap \bar{M}_\kappa^w(X)\right) = \#\left(\bar{\mathcal{V}}(ze) \cap \bar{M}_{\kappa+1/4}^{w+\mathrm{PD}[e]}(\tilde{X})\right), \tag{3.32}$$

when the intersection number on the left is well-defined, for example, when w is good in the sense of Definition 3.20. However, the blow-up trick [47] ensures that the intersection number on the right is well-defined for arbitrary $w \in H^2(X;\mathbb{Z})$, since $w + \mathrm{PD}[e] \in H^2(\tilde{X};\mathbb{Z})$ is good.

In [35, p. 585], Kronheimer and Mrowka require that $b_2^+(X) - b_1(X)$ be odd. However, for the purposes of defining the Donaldson invariants for a closed four-manifold, with $b_1(X)$ possibly non-zero, one can have non-trivial Donaldson invariants when $b_2^+(X) - b_1(X)$ is even as they point out in [35, p. 595]. The reason for the constraint is that their structure theorem is only stated for the case $b_1(X) = 0$; the invariants become more difficult to compute when $b_1(X) > 0$. If $b_1(X) = 0$, then the Donaldson invariants are necessarily trivial unless $b_2^+(X)$ is odd (and the moduli spaces of anti-self-dual connections are even dimensional).

When $b_2^+(X) = 1$, the Donaldson invariant (3.31) depends on the "chamber" in $H^2(\tilde{X};\mathbb{R})$ defined by metric $\tilde{g}$ on $\tilde{X}$. Specifically, if $\omega(\tilde{g})$ is the unique unit-length harmonic two-form which is self-dual with respect to $\tilde{g}$—the *period point* for $\tilde{g}$—and lies in the positive cone (determined by the homology orientation) of $H^2(\tilde{X};\mathbb{R})$, then the intersection number on the right-hand-side of equation (3.31) changes whenever the sign of $\omega(\tilde{g}) \smile \alpha$ changes for some $\alpha \in H^2(\tilde{X};\mathbb{Z})$ satisfying

$$\alpha \equiv w + \mathrm{PD}[e] \pmod 2 \quad \text{and} \quad \alpha^2 = -4(\kappa + 1/4) + 4\ell, \quad \text{for some } \ell \in \mathbb{N}. \tag{3.33}$$

(The classes α correspond to split SO(3) bundles over $\tilde{X}$, namely $\underline{\mathbb{R}} \oplus L$ with $c_1(L) = \alpha$, so they have first Pontrjagin number α^2 and second Stiefel-Whitney class $\alpha \pmod 2$; see [31], [32] for further explanation.) For any $\alpha \in H^2(\tilde{X};\mathbb{R})$ which is *not torsion*, the subset

$$\{h \in H^2(\tilde{X};\mathbb{R}) : h \smile h > 0 \text{ and } h \smile \alpha = 0\}$$

of the positive cone of $H^2(\tilde{X};\mathbb{R})$ is an *α-wall*. If α obeys condition (3.33), then α is non-torsion since $w+\mathrm{PD}[e] \pmod 2$ is good and the resulting α-wall is called a *$(w+\mathrm{PD}[e], -4\kappa-1)$-wall*. The connected components of the complement in the positive cone of $H^2(\tilde{X};\mathbb{R})$ of the union of $(w+\mathrm{PD}[e], -4\kappa-1)$-walls are called *$(w+\mathrm{PD}[e], -4\kappa-1)$-chambers*. Hence, the intersection pairing in definition (3.31) changes if the period point $\omega(\tilde{g})$ moves from one $(w+\mathrm{PD}[e], -4\kappa-1)$-chamber to another.

We now discuss how a choice of a metric g on X determines a chamber in the positive cone of $H^2(\tilde{X};\mathbb{R}) \cong \mathbb{R}[e] \oplus H^2(X;\mathbb{R})$. Assume first that $w \pmod 2$ is good in the sense of Definition 3.20. Therefore, if $\beta \in H^2(X;\mathbb{Z})$ satisfies

$$\beta \equiv w \pmod 2 \quad \text{and} \quad \beta^2 = -4\kappa + 4\ell, \quad \text{for some } \ell \in \mathbb{N}, \tag{3.34}$$

then β is non-torsion and thus defines a $(w, -4\kappa)$-wall in $H^2(X;\mathbb{R})$. Moreover, $\alpha = \beta + \mathrm{PD}[e]$ satisfies condition (3.33) and defines a $(w+\mathrm{PD}[e], -4\kappa-1)$-wall in $H^2(\tilde{X};\mathbb{R})$. This establishes an inclusion of $(w, -4\kappa)$-chambers in $H^2(X;\mathbb{R})$ into *related $(w+\mathrm{PD}[e], -4\kappa-1)$-chambers* in $H^2(\tilde{X};\mathbb{R})$.

If g is a generic metric on X, then the period point $\omega(g) \in H^2(X;\mathbb{R})$ does not lie on any wall and there is a unique chamber in the positive cone of $H^2(X;\mathbb{R})$ which contains $\omega(g)$ [7, Corollary 4.3.15].

If the metric $\tilde{g}$ on $\tilde{X}$ is constructed by splicing together a generic metric g on X and the Fubini-Study metric on $\overline{\mathbb{CP}}^2$ along a "long neck", then $\omega(\tilde{g})$ converges to $\omega(g)$ in L^2 as the length of the neck tends to infinity, viewing both $\omega(\tilde{g})$ and $\omega(g)$ as elements of $H^2(\tilde{X};\mathbb{R})$. Thus, $\omega(\tilde{g})$ will lie in the chamber in the positive cone of $H^2(\tilde{X};\mathbb{R})$ related to the chamber in the positive cone of $H^2(X;\mathbb{R})$ containing $\omega(g)$. Thus, when $w \pmod 2$ is good, the intersection pairing (3.31) defining the invariant $D^w_X(z)$ is defined with respect to the chamber in $H^2(\tilde{X};\mathbb{R})$ related to the chamber in $H^2(X;\mathbb{R})$ determined by the period point $\omega(g)$.

If $w \pmod 2$ is not good, one can have torsion classes $\beta \in H^2(X;\mathbb{Z})$ satisfying condition (3.34) and thus $\alpha = \beta + \mathrm{PD}[e] \in H^2(\tilde{X};\mathbb{Z})$ satisfying (3.33). The corresponding $(w+\mathrm{PD}[e], -4\kappa-1)$-wall is given by

$$\{h \in H^2(\tilde{X};\mathbb{R}) : h \smile h > 0 \text{ and } h \smile \mathrm{PD}[e] = 0\}.$$

Hence, the period point $\omega(g) \in H^2(X;\mathbb{R})$ for any metric g on X lies in this $\mathrm{PD}[e]$-wall, since $H^2(\tilde{X};\mathbb{R}) \cong \mathbb{R}[e] \oplus H^2(X;\mathbb{R})$, and the fact that $\omega(\tilde{g})$ converges to $\omega(g)$ as the length of the neck tends to infinity does not determine the chamber of $\omega(\tilde{g})$ without a delicate analysis of the sign of $\omega(\tilde{g}) \smile \mathrm{PD}[e]$ (see [59]). We plan to address the case $b_2^+(X) = 1$ elsewhere and so in the present article, if $b_2^+(X) = 1$, we only consider the dependence of the invariants $D^w_X(z)$ on the chamber in $H^2(X;\mathbb{R})$ when w is good.

3.4.3. *Geometric representatives on the link of the stratum of zero-section monopoles.* We now turn to the arguments leading to a proof of Proposition 3.29, which expresses the intersection number

$$\#\left(\bar{\mathcal{V}}(z)\cap\bar{\mathcal{W}}^{n_a-1}\cap\mathbf{L}^w_{\mathfrak{t},\kappa}\right) \tag{3.35}$$

in terms of

$$\#\left(\bar{\mathcal{V}}(z)\cap\iota(\bar{M}^w_\kappa)\right)=\#(\bar{\mathcal{V}}(z)\cap\bar{M}^w_\kappa),$$

which is equal to the Donaldson invariant $D^w_X(z)$ when w is good.

Note that by the construction of the geometric representatives and definition of the Donaldson invariants [7], [35, §2], one has

$$\bar{\mathcal{V}}(z)\cap\bar{M}^w_\kappa=\mathcal{V}(z)\cap M^w_\kappa.$$

That is, the intersection is contained in the top stratum M^w_κ of the compactification $\bar{M}^w_\kappa$. Therefore, to calculate the intersection number (3.35) it will be enough to examine small neighborhoods of the points in the intersection $\mathcal{V}(z)\cap\iota(M^w_\kappa)$. Such neighborhoods are described by Kuranishi models, which we now describe.

Suppose $[A,0]\in\mathcal{V}(z)\cap\iota(M^w_\kappa)$. Applying the Kuranishi method to describe the zero locus of the PU(2) monopole equations using [10, Corollary 3.6], we obtain a smooth S^1-equivariant embedding

$$\boldsymbol{\gamma}_A:\mathcal{O}_A\subset T_{[\hat{A}]}M^w_\kappa\oplus\operatorname{Ker}D_{A,\vartheta}\to(A,0)+\operatorname{Ker}d^*_{A,0}\subset\tilde{\mathcal{C}}_{\mathfrak{t}}, \tag{3.36}$$

of a precompact, open S^1-invariant neighborhood $\mathcal{O}_A$ of the origin with image $\boldsymbol{\gamma}_A(\mathcal{O}_A/S^1)\subset\mathcal{C}_{\mathfrak{t}}/S^1$, where S^1 acts on the domain by scalar multiplication on $\operatorname{Ker}D_{A,\vartheta}$, and a smooth S^1-equivariant map

$$\boldsymbol{\varphi}_A:\mathcal{O}_A\subset T_{[\hat{A}]}M^w_\kappa\oplus\operatorname{Ker}D_{A,\vartheta}\to\operatorname{Coker}D_{A,\vartheta}, \tag{3.37}$$

such that

$$\boldsymbol{\gamma}_A\left((\boldsymbol{\varphi}_A^{-1}(0)\cap\mathcal{O}_A)/S^1\right)=\mathcal{M}_{\mathfrak{t}}/S^1\cap\boldsymbol{\gamma}_A(\mathcal{O}_A/S^1), \tag{3.38}$$

$$\boldsymbol{\gamma}_A\left((T_{[\hat{A}]}M^w_\kappa\oplus\{0\})\cap\mathcal{O}_A\right)=\iota(M^w_\kappa)\cap\boldsymbol{\gamma}_A(\mathcal{O}_A), \tag{3.39}$$

are open neighborhoods of the point $[A,0]$ in $\mathcal{M}_{\mathfrak{t}}/S^1$ and $\iota(M^w_\kappa)$, respectively. Because points in $\mathcal{M}^{*,0}_{\mathfrak{t}}$ are regular, the map $\boldsymbol{\varphi}_A$ vanishes transversely on

$$\mathcal{O}_A-(T_{[\hat{A}]}M^w_\kappa\oplus\{0\}).$$

Compare the proof of Assertion (4) in [10, Theorem 3.21]. For convenience, we set

$$\mathcal{Z}_A=\boldsymbol{\varphi}_A^{-1}(0)\cap\mathcal{O}_A. \tag{3.40}$$

Equation (3.38) implies that the S^1-equivariant embedding $\boldsymbol{\gamma}_A$ descends to a homeomorphism from a neighborhood of the origin onto a neighborhood of $[A,0]$,

$$\boldsymbol{\gamma}_A:(\boldsymbol{\varphi}_A^{-1}(0)\cap\mathcal{O}_A)/S^1\cong\mathcal{M}_{\mathfrak{t}}/S^1\cap\boldsymbol{\gamma}_A(\mathcal{O}_A/S^1), \tag{3.41}$$

which restricts to a diffeomorphism on a smooth stratum.

In [10, §3.2], we constructed the link $\mathbf{L}^w_{\mathfrak{t},\kappa}$ using the S^1-invariant "distance function",

$$\boldsymbol{\ell}:\mathcal{C}_{\mathfrak{t}}\to[0,\infty),\quad[A,\Phi]\mapsto\|\Phi\|^2_{L^2}. \tag{3.42}$$

The function $\boldsymbol{\ell}$ extends continuously over $\bar{\mathcal{M}}_{\mathfrak{t}}$ if we set $\boldsymbol{\ell}([A,\Phi,\mathbf{x}])=\|\Phi\|^2_{L^2}$. For generic, positive, small ε we have [10, Definition 3.7]

$$\mathbf{L}^{w,\varepsilon}_{\mathfrak{t},\kappa}=\boldsymbol{\ell}^{-1}(\varepsilon)\cap\bar{\mathcal{M}}_{\mathfrak{t}}/S^1, \tag{3.43}$$

and denote $\mathbf{L}^{w,\varepsilon}_{\mathfrak{t},\kappa}$ by $\mathbf{L}^{w}_{\mathfrak{t},\kappa}$ when the value of ε is not relevant. To compute the pairing (3.35) with $\mathbf{L}^{w,\varepsilon}_{\mathfrak{t},\kappa}$, we must first describe $\bar{\mathcal{V}}(z)$ in a neighborhood of $\iota(M^w_\kappa)$ in $\mathcal{M}_\mathfrak{t}$.

Lemma 3.22. *Let $\mathfrak{t}$ be a spin^u structure over a four-manifold X with $w_2(\mathfrak{t}) \equiv w \pmod 2$, where $w \in H^2(X;\mathbb{Z})$ and $w \pmod 2$ is good. Suppose that $\deg(z) \geq \dim M^w_\kappa$ and z is intersection-suitable, as defined before Lemma 3.17. Denote $\bar{\mathcal{V}}(z) \cap \bar{M}^w_\kappa = \{[\hat{A}_i]_{i=1}^N\}$. Then for each $[\hat{A}] \in \bar{\mathcal{V}}(z) \cap \bar{M}^w_\kappa$, there is an open neighborhood $\mathcal{O}'_A \subset \mathcal{O}_A$ of the origin in $T_{[\hat{A}]}M^w_\kappa \oplus \operatorname{Ker} D_{A,\vartheta}$, where $\mathcal{O}_A$ is the open neighborhood defining the Kuranishi model (3.36), such that:*

1. *There is a smooth, S^1-invariant map,*
$$f_A : \mathcal{O}'_A \cap (\{0\} \oplus \operatorname{Ker} D_{A,\vartheta}) \to T_{[\hat{A}]}M^w_\kappa,$$
with $f_A(0) = 0$ and $(Df_A)_0 = 0$ such that
$$\boldsymbol{\gamma}_A^{-1}(\mathcal{V}(z)) \cap \mathcal{O}'_A = \{(f_A(\phi), \phi) : \phi \in \mathcal{O}'_A \cap (\{0\} \oplus \operatorname{Ker} D_{A,\vartheta})\}.$$
2. *There is a positive constant ε_0 such that for all $\varepsilon < \varepsilon_0$,*
$$\bar{\mathcal{V}}(z) \cap \mathbf{L}^{w,\varepsilon}_{\mathfrak{t},\kappa} \subset \bigcup_{i=1}^N \boldsymbol{\gamma}_{A_i}\left(\mathcal{O}'_{A_i}\right)/S^1,$$
where the union on the right above is disjoint.
3. *For each $\varepsilon \in (0, \varepsilon_0)$ with ε_0 as in (2), there is a positive constant δ such that all $(a, \phi) \in \boldsymbol{\gamma}_A^{-1}(\mathcal{V}(z)) \cap \mathcal{O}'_A$ satisfy $\|\phi\|^2_{L^2} > \delta$.*

Proof. Consider $\mathcal{V}(z)$ as a smooth submanifold of $\mathcal{B}^{w,*}_\kappa$. If $\pi_\mathcal{B} : \mathcal{C}^*_\mathfrak{t} \to \mathcal{B}^{w,*}_\kappa$ is the projection, then the composition $\pi_\mathcal{B} \circ \boldsymbol{\gamma}_A$ is a smooth map from $\mathcal{O}_A$ to $\mathcal{B}^{w,*}_\kappa$. The manifolds M^w_κ and $\mathcal{V}(z)$ intersect transversely in $\mathcal{B}^{w,*}_\kappa$ at $[\hat{A}] = \pi_\mathcal{B} \circ \boldsymbol{\gamma}_A(0,0)$. The restriction of $\boldsymbol{\gamma}_A$ to
$$\mathcal{O}_A \cap \left(T_{[\hat{A}]}M^w_\kappa \oplus \{0\}\right)$$
is an embedding onto an open neighborhood of $[\hat{A}]$ in M^w_κ, so the composition $\pi_\mathcal{B} \circ \boldsymbol{\gamma}_A$ is transverse to $\mathcal{V}(z)$ at the origin. Thus, restricted to a sufficiently small open neighborhood $\mathcal{O}''_A \subset \mathcal{O}_A$ of the origin $(0,0)$ in $T_{[\hat{A}]}M^w_\kappa \oplus \operatorname{Ker} D_{A,\vartheta}$. the map $\pi_\mathcal{B} \circ \boldsymbol{\gamma}_A$ is transverse to $\mathcal{V}(z)$ in $\mathcal{B}^{w,*}_\kappa$. Hence $\boldsymbol{\gamma}_A^{-1}(\mathcal{V}(z)) \cap \mathcal{O}''_A$ is a smooth manifold. By shrinking the neighborhood $\mathcal{O}''_A$ we can assume that $\boldsymbol{\gamma}_A^{-1}(\mathcal{V}(z)) \cap \mathcal{O}''_A$ and $T_{[\hat{A}]}M^w_\kappa \oplus \{0\}$ intersect only at the origin, since $\mathcal{V}(z) \cap M^w_\kappa = [\hat{A}] = \pi_\mathcal{B} \circ \boldsymbol{\gamma}_A(0,0)$. We now prove that

$$T_{(0,0)}\left(\boldsymbol{\gamma}_A^{-1}(\mathcal{V}(z)) \cap \mathcal{O}''_A\right) = \{0\} \oplus \operatorname{Ker} D_{A,\vartheta}. \tag{3.44}$$

First, note that because the derivative of $\boldsymbol{\gamma}_A$ at the origin is the inclusion of $T_{[\hat{A}]}M^w_\kappa \oplus \operatorname{Ker} D_{A,\vartheta}$ into $\operatorname{Ker} d^{0,*}_{A,0}$ by construction of the Kuranishi model, we have the inclusion:
$$\{0\} \oplus \operatorname{Ker} D_{A,\vartheta} \subset \operatorname{Ker}(D(\pi_\mathcal{B} \circ \boldsymbol{\gamma}_A))_{(0,0)} \subset T_{(0,0)}\left(\boldsymbol{\gamma}_A^{-1}(\mathcal{V}(z)) \cap \mathcal{O}''_A\right).$$
Because $(D(\pi_\mathcal{B} \circ \boldsymbol{\gamma}_A))_{(0,0)}$ maps $T_{[\hat{A}]}M^{w,*}_\kappa \oplus \{0\}$ onto the normal bundle of $\mathcal{V}(z)$ in $\mathcal{B}^{w,*}_\kappa$, the above inclusion is an equality. Equation (3.44) implies that if
$$\pi_{K,A} : T_{[\hat{A}]}M^w_\kappa \oplus \operatorname{Ker} D_{A,\vartheta} \to \operatorname{Ker} D_{A,\vartheta}$$
is the projection onto $\operatorname{Ker} D_{A,\vartheta}$ then the derivative of the restriction of $\pi_{K,A}$,

$$\pi_{K,A} : \boldsymbol{\gamma}_A^{-1}(\mathcal{V}(z)) \cap \mathcal{O}''_A \to \operatorname{Ker} D_{A,\vartheta} \tag{3.45}$$

at the origin is an isomorphism. Therefore, for small enough $\mathcal{O}''_A$ the map (3.45) is a diffeomorphism onto a neighborhood $\mathcal{O}_{K,A}$ of the origin in $\operatorname{Ker} D_{A,\vartheta}$. If $\mathcal{O}'_A = \mathcal{O}''_A \cap \pi_{K,A}^{-1}(\mathcal{O}_{K,A})$, then $\mathcal{O}'_A \cap \{0\} \oplus \operatorname{Ker} D_{A,\vartheta} \subset \mathcal{O}_{K,A}$ and $\pi_{K,A}$ restricted to $\boldsymbol{\gamma}_A^{-1}(\mathcal{V}(z)) \cap \mathcal{O}'_A$ is still a diffeomorphism onto $\mathcal{O}_{K,A}$ with inverse as described.

By shrinking the sets $\mathcal{O}'_A$, we can assume the images $\boldsymbol{\gamma}_A(\mathcal{O}'_A)$ are disjoint, proving the final statement in Assertion (2). Suppose ε_α is a sequence of positive numbers converging to zero. If Assertion (2) were not true, there would be a sequence

$$\{[A_\alpha, \Phi_\alpha, \mathbf{x}_\alpha]\}_{\alpha=1}^\infty \subset \bar{\mathcal{V}}(z) \cap \bar{\mathcal{M}}_{\mathfrak{t}},$$

satisfying $\boldsymbol{\ell}([A_\alpha, \Phi_\alpha, \mathbf{x}_\alpha]) = \varepsilon_\alpha$ and $[A_\alpha, \Phi_\alpha, \mathbf{x}_\alpha]$ not in any $\boldsymbol{\gamma}_{A_i}(\mathcal{O}'_{A_i})$. Since $\bar{\mathcal{V}}(z) \cap \bar{\mathcal{M}}_{\mathfrak{t}}$ is compact, there would be a convergent subsequence, also denoted $\{[A_\alpha, \Phi_\alpha, \mathbf{x}_\alpha]\}_{\alpha=1}^\infty$, converging to $[A_\infty, \Phi_\infty, \mathbf{x}_\infty]$. Because $\boldsymbol{\ell}$ is continuous on $\bar{\mathcal{M}}_{\mathfrak{t}}$, we would have $\boldsymbol{\ell}([A_\infty, \Phi_\infty, \mathbf{x}_\infty]) = 0$, so $[A_\infty, \Phi_\infty, \mathbf{x}_\infty] \in \iota(\bar{M}_\kappa^w)$ by Lemma 3.21, with $\Phi_\infty = 0$. Then,

$$[A_\infty, 0, \mathbf{x}_\infty] \in \bar{\mathcal{V}}(z) \cap \iota(\bar{M}_\kappa^w) = \mathcal{V}(z) \cap \iota(M_\kappa^w),$$

and thus $\mathbf{x} = \varnothing$ and $[\hat{A}_\infty] \in \{[\hat{A}_1, \dots, \hat{A}_N]\}$. The images $\boldsymbol{\gamma}_{A_i}(\mathcal{O}'_{A_i})$ contain open neighborhoods of the points $\{[\hat{A}_1], \dots, [\hat{A}_N]\}$ in $\bar{\mathcal{M}}_{\mathfrak{t}}$, so for large enough α the sequence must lie in the union of these images, contradicting the assumption that $[A_\alpha, \Phi_\alpha, \mathbf{x}_\alpha]$ is not in any $\boldsymbol{\gamma}_{A_i}(\mathcal{O}'_{A_i})$. This proves Assertion (2).

We use contradiction to prove Assertion (3): if it were not true, then there would be a sequence $\{(a_\alpha, \phi_\alpha)\}_{\alpha=1}^\infty$ in $\boldsymbol{\gamma}_A^{-1}(\mathcal{V}(z)) \cap \mathcal{O}'_A$ with $\boldsymbol{\ell}(\boldsymbol{\gamma}_A(a_\alpha, \phi_\alpha)) = \varepsilon$ and $\phi_\alpha \in \mathcal{O}_{K,A}$ with $\lim_{\alpha\to\infty} \|\phi_\alpha\|_{L^2}^2 = 0$. By Assertion (1) this sequence could be written $(a_\alpha, \phi_\alpha) = (f_A(\phi_\alpha), \phi_\alpha)$. Because the sequence $\{\phi_\alpha\} \subset \operatorname{Ker} D_{A,\vartheta}$ converged to zero in L^2 and $D_{A,\vartheta}$ is elliptic, it would converge to zero in L^2_ℓ (for $\ell \geq 2$). Since f_A is continuous on $\mathcal{O}_{K,A}$, we would have $\lim_{\alpha\to\infty} a_\alpha = \lim_{\alpha\to\infty} f_A(\phi_\alpha) = 0$ and, as $\boldsymbol{\ell} \circ \boldsymbol{\gamma}_A$ is continuous on $\mathcal{O}_{K,A}$,

$$\lim_{\alpha\to\infty} \boldsymbol{\ell}(\boldsymbol{\gamma}_A(a_\alpha, \phi_\alpha)) = \boldsymbol{\ell}(\boldsymbol{\gamma}_A(0,0)) = \boldsymbol{\ell}(0) = 0,$$

contradicting our assumption that for all α we have $\boldsymbol{\ell}(\boldsymbol{\gamma}_A(a_\alpha, \phi_\alpha)) = \varepsilon$. This proves Assertion (3) and completes the proof of the lemma. □

Because the spaces $\boldsymbol{\gamma}_A(\mathcal{O}'_A)$ suffice to cover the intersection $\bar{\mathcal{V}}(z) \cap \mathbf{L}_{\mathfrak{t},\kappa}^{w,\varepsilon}$ for ε sufficiently small by Lemma 3.22, we shall henceforth restrict the domain of $\boldsymbol{\gamma}_A$ to $\mathcal{O}'_A$.

We define a link of the submanifold $T_{[\hat{A}]}M_\kappa^w \oplus \{0\} \subset T_{[\hat{A}]}M_\kappa^w \oplus \operatorname{Ker} D_{A,\vartheta}$ by setting

$$\mathbf{K}_{A,\delta} = \{(a, \phi) \in T_{[\hat{A}]}M_\kappa^w \oplus \operatorname{Ker} D_{A,\vartheta} : \|\phi\|_{L^2}^2 = \delta\}. \tag{3.46}$$

The link $\mathbf{K}_{A,\delta}/S^1$ is more convenient to work with than the level sets of $\boldsymbol{\ell} \circ \boldsymbol{\gamma}_A$ defining $\boldsymbol{\gamma}_A^{-1}\left(\mathbf{L}_{\mathfrak{t},\kappa}^{w,\varepsilon}\right)$. We will in Lemma 3.26 that the two links are related by an oriented cobordism. Therefore, prior to showing this equivalence, we first discuss the orientation of the spaces.

An orientation O for $\mathcal{M}_{\mathfrak{t}}^{*,0}$ determines an orientation

$$\omega(\mathbf{L}, \partial O) \tag{3.47}$$

for $\mathbf{L}_{\mathfrak{t},\kappa}^{w,\varepsilon}$ by considering $\mathbf{L}_{\mathfrak{t},\kappa}^{w,\varepsilon}$ as the boundary of the subspace $\ell^{-1}([\varepsilon, \infty)) \subset \mathcal{M}_{\mathfrak{t}}/S^1$ and using the convention (2.19).

An orientation o of $T_{[\hat{A}]}M_\kappa^w$ determines an orientation of $\mathbf{K}_{A,\delta}/S^1$ by identifying $\mathbf{K}_{A,\delta}/S^1$ with $T_{[\hat{A}]}M_\kappa^w \times \mathbb{CP}^{k-1}$, where $k = \dim_{\mathbb{C}} \operatorname{Ker} D_{A,\vartheta}$ and taking the complex orientation of $\mathbb{CP}^{k-1}$, denoted by

$$\omega(\mathbf{K}, o). \tag{3.48}$$

We now describe a convention for orienting smooth submanifolds.

Convention 3.23. Suppose Z and M are manifolds which intersect transversely. Then an orientation O for M and an orientation $\omega(Z)$ for the normal bundle of Z determine an orientation for $M \cap Z$, which we write as $O/\omega(Z)$.

From equation (3.37) and the fact that φ_A vanishes transversely on $(\mathcal{O}'_A - (T_{[\hat{A}]}M^w_\kappa \oplus \{0\})/S^1$, the fibers of the normal bundle of $\mathcal{Z}_A/S^1$ in $(\mathcal{O}'_A - (T_{[\hat{A}]}M^w_\kappa \oplus \{0\})/S^1$ are naturally identified with $\operatorname{Coker} D_{A,\vartheta}$. Let $\omega(\mathcal{Z})$ be the orientation of this normal bundle of $\mathcal{Z}_A/S^1$ obtained by giving $\operatorname{Coker} D_{A,\vartheta}$ the complex orientation.

By Convention 3.23, the orientation $\omega(\mathbf{K}, o)$ of $\mathbf{K}_{A,\delta}/S^1$ and orientation $\omega(\mathcal{Z})$ of the normal bundle of $\mathcal{Z}_A/S^1$ determine a "complex orientation" of $\mathcal{Z}_A \cap \mathbf{K}_{A,\delta}/S^1$,

$$\omega(\mathbf{K}, o)/\omega(\mathcal{Z}), \tag{3.49}$$

where o is the orientation for $T_{[\hat{A}]}M^w_\kappa$, by analogy with Definition 2.7 which this construction matches.

However, to compare the orientations of $\mathbf{L}^{w,\varepsilon}_{\mathfrak{t},\kappa}$ and $\mathcal{Z}_A \cap \mathbf{K}_{A,\delta}/S^1$ with those of other links in a cobordism formula such as equation (3.70), it is natural to orient $\mathcal{Z}_A \cap \mathbf{K}_{A,\delta}/S^1$ as a boundary of the cobordism. If O is an orientation of $\mathcal{M}^{*,0}_{\mathfrak{t}}/S^1$, we obtain an orientation

$$\omega(\mathcal{Z} \cap \mathbf{K}, \partial O) \tag{3.50}$$

for $\mathcal{Z}_A \cap \mathbf{K}_{A,\delta}/S^1$ by identifying this manifold, via the map $\boldsymbol{\gamma}_A$, with the boundary of

$$\mathcal{M}_{\mathfrak{t}} \setminus \boldsymbol{\gamma}_A\left(T_{[\hat{A}]}M^w_\kappa \times B_A(0,\delta) \cap \mathcal{O}'_A\right),$$

where $B_A(0,\delta) \subset \operatorname{Ker} D_{A,\vartheta}$ is the ball of radius δ, and using convention (2.19). The proof of the following lemma is the same as that of Lemma 2.9.

Lemma 3.24. *Let w be an integral lift of $w_2(\mathfrak{t})$. Fix an orientation $o = o(\Omega, w)$ of M^w_κ. Let $O = O^{\mathrm{asd}}(\Omega, w)$ be the orientation for $\mathcal{M}^{*,0}/S^1$ in Definition 2.3. Then, the orientations* (3.49) *and* (3.50) *for $\mathcal{Z}_A \cap \mathbf{K}_{A,\delta}/S^1$ agree, that is*

$$\omega(\mathbf{K}, o)/\omega(\mathcal{Z}) = \omega(\mathcal{Z} \cap \mathbf{K}, \partial O).$$

By the definition of a geometric representative (Definition 3.4), the normal bundle of $\mathcal{V}(z)$ in $\mathcal{B}^{w,*}_\kappa$ has an orientation, which we denote by $\omega(\mathcal{V})$. Because $\mathcal{V}(z)$ intersects

$$\pi_{\mathcal{B}} \circ \boldsymbol{\gamma}_A(\mathbf{K}_{A,\delta}/S^1), \quad \pi_{\mathcal{B}} \circ \boldsymbol{\gamma}_A(\mathcal{Z}_A \cap \mathbf{K}_{A,\delta}/S^1), \quad \text{and} \quad \pi_{\mathcal{B}}(\mathbf{L}^{w,\varepsilon}_{\mathfrak{t},\kappa})$$

transversely, the orientations $\omega(\mathcal{V})$, $\omega(\mathbf{K}, o)$, $\omega(\mathcal{Z})$, $\omega(\mathcal{Z} \cap \mathbf{K}, \partial O)$ and $\omega(\mathbf{L}, \partial O)$ determine orientations

$$\begin{array}{rcl} \omega(\mathbf{K}, o)/\omega(\mathcal{V}) & \text{for} & \boldsymbol{\gamma}_A^{-1}(\mathcal{V}(z)) \cap \mathbf{K}_{A,\delta}/S^1, \\ \left.\begin{array}{r} \left(\omega(\mathbf{K}, o)/\omega(\mathcal{Z})\right)/\omega(\mathcal{V}) \\ \omega(\mathcal{Z} \cap \mathbf{K}, \partial O)/\omega(\mathcal{V}) \end{array}\right\} & \text{for} & \boldsymbol{\gamma}_A^{-1}(\mathcal{V}(z)) \cap \mathcal{Z}_A \cap \mathbf{K}_{A,\delta}/S^1, \\ \omega(\mathbf{L}, \partial O)/\omega(\mathcal{V}) & \text{for} & \mathcal{V}(z) \cap \mathbf{L}^{w,\varepsilon}_{\mathfrak{t},\varepsilon}. \end{array} \tag{3.51}$$

Observe that Lemma 3.24 implies that

$$\left(\omega(\mathbf{K}, o)/\omega(\mathcal{Z})\right)/\omega(\mathcal{V}) = \omega(\mathcal{Z} \cap \mathbf{K}, \partial O)/\omega(\mathcal{V}), \tag{3.52}$$

if $o = o(\Omega, w)$ and $O = \Omega^{\mathrm{asd}}(\Omega, w)$.

Lemma 3.25. *Let w be an integral lift of $w_2(\mathfrak{t})$. Let $\varepsilon(A)=\pm 1$ be the signed intersection number of $\mathcal{V}(z)$ and M^w_κ at $[\hat{A}]$, where M^w_κ is given the orientation $o(\Omega,w)$. Then the following map is a diffeomorphism,*

$$g_A = f_A \times \mathrm{id}_{\mathrm{Ker}\, D_{A,\vartheta}} : \mathbb{CP}^{k-1} \to \boldsymbol{\gamma}_A^{-1}(\mathcal{V}(z)) \cap \mathbf{K}_{A,\delta}/S^1, \tag{3.53}$$

where $\mathbb{CP}^{k-1}=\mathbb{P}(\mathrm{Ker}\, D_{A,\vartheta})$ and f_A is defined in Assertion (1) of Lemma 3.22. If $\mathbb{CP}^{k-1}$ has the complex orientation and $\boldsymbol{\gamma}_A^{-1}(\mathcal{V}(z)) \cap \mathbf{K}_{A,\delta}/S^1$ has the orientation $\omega(\mathbf{K},o)/\omega(\mathcal{V})$ of (3.51) for $o=o(\Omega,w)$, then g_A preserves orientation if and only if $\varepsilon(A)=1$.

Proof. By Assertion (1) of Lemma 3.22, the intersection $\boldsymbol{\gamma}_A^{-1}(\mathcal{V}(z)) \cap \mathbf{K}_{A,\delta}/S^1$ is given by the S^1 quotient of the graph of f_A restricted to $S^{2k-1} \subset \mathrm{Ker}\, D_{A,\vartheta}$. Thus, g_A gives the desired diffeomorphism. Because $\mathcal{V}(z)$ and M^w_κ intersect transversely in $\mathcal{B}^{w,*}_\kappa$ at $[\hat{A}]$, the normal bundle of $\mathcal{V}(z)$ in $\mathcal{B}^{w,*}_\kappa$ is identified with $T_{[\hat{A}]}M^w_\kappa$ but their orientations agree if and only if $\varepsilon(A)=1$. The result then follows from the definition (3.48) of the orientation $\omega(\mathbf{K},o)$ of $\mathbf{K}_{A,\delta}/S^1$ determined by the orientation $o(\Omega,w)$ of $T_{[\hat{A}]}M^w_\kappa$. □

Lemma 3.26. *Continue the assumptions and notation of Lemma 3.22. For ε sufficiently small and δ as in Assertion (3) of Lemma 3.22 and generic, there is a smooth, compact, and oriented cobordism between*

$$\boldsymbol{\gamma}_A^{-1}\left(\mathcal{V}(z) \cap \mathbf{L}^{w,\varepsilon}_{\mathfrak{t},\kappa}\right), \tag{3.54}$$

with the orientation $\omega(\mathbf{L},\partial O)/\omega(\mathcal{V})$ of (3.51) for $O=O^{\mathrm{asd}}(\Omega,w)$, and the manifold

$$\mathcal{Z}_A \cap \boldsymbol{\gamma}_A^{-1}(\mathcal{V}(z)) \cap \mathbf{K}_{A,\delta}/S^1, \tag{3.55}$$

with the orientation $(\omega(\mathbf{K},o)/\omega(\mathcal{Z}))/\omega(\mathcal{V})$ of (3.51), where $o=o(\Omega,w)$.

Proof. As before, we let $B_A(0,\delta) \subset \mathrm{Ker}\, D_{A,\vartheta}$ be the open ball of radius δ. Assertion (3) of Lemma 3.22 yields the inclusion

$$\boldsymbol{\gamma}_A\left(\left(T_{[\hat{A}]}M^w_\kappa \times B_A(0,\delta)\right) \cap \mathcal{O}'_A/S^1\right) \cap \bar{\mathcal{V}}(z) \cap \bar{\mathcal{M}}_{\mathfrak{t}}/S^1 \ \subset\ \boldsymbol{\ell}^{-1}([0,\varepsilon)).$$

Then for generic values of ε and δ,

$$\bar{\mathcal{M}}_{\mathfrak{t}}/S^1 \cap \bar{\mathcal{V}}(z) \cap \boldsymbol{\ell}^{-1}([0,\varepsilon]) \cap \boldsymbol{\gamma}_A(\mathcal{O}'_A) - \boldsymbol{\gamma}_A\left(T_{[\hat{A}]}M^w_\kappa \times B_A(0,\delta)/S^1\right) \tag{3.56}$$

is a smooth manifold with boundaries given by the manifold (3.54) and by

$$\bar{\mathcal{M}}^0_{\mathfrak{t}} \cap \bar{\mathcal{V}}(z) \cap \boldsymbol{\gamma}_A\left(\left(T_{[\hat{A}]}M^w_\kappa \times \partial\bar{B}_A(0,\delta)\right) \cap \mathcal{O}'_A/S^1\right), \tag{3.57}$$

which is diffeomorphic, via $\boldsymbol{\gamma}_A$, to the manifold (3.55).

By Assertion (2) of Lemma 3.22, the compact set $\bar{\mathcal{V}}(z) \cap \bar{\mathcal{M}}_{\mathfrak{t}} \cap \boldsymbol{\ell}^{-1}([0,\varepsilon])$ is contained in a finite, disjoint union $\cup_{i=1}^N \boldsymbol{\gamma}_{A_i}\left(\mathcal{O}'_{A_i}/S^1\right)$, where $\bar{\mathcal{V}}(z) \cap \bar{M}^w_\kappa = \{[\hat{A}_i]\}_{i=1}^N$. Thus, each component

$$\bar{\mathcal{V}}(z) \cap \bar{\mathcal{M}}_{\mathfrak{t}} \cap \boldsymbol{\ell}^{-1}([0,\varepsilon]) \cap \boldsymbol{\gamma}_A\left(\mathcal{O}'_A/S^1\right),$$

of this disjoint union is compact. Therefore, the space (3.56) is a compact and smooth cobordism between the manifold (3.54) and the manifold (3.57) which, as previously noted, is diffeomorphic to the manifold (3.55).

Let $\mathcal{M}^{*,0}_{\mathfrak{t}}/S^1 \cap \mathcal{V}(z)$ have the orientation determined by the orientation $O^{\mathrm{asd}}(\Omega,w)$ of $\mathcal{M}_{\mathfrak{t}}/S^1$ and the orientation $\omega(\mathcal{V})$ of the normal bundle of $\mathcal{V}(z)$. The manifold (3.56) has codimension zero in $\mathcal{M}^{*,0}_{\mathfrak{t}}/S^1 \cap \mathcal{V}(z)$ and thus inherits an orientation from $\mathcal{M}^{*,0}_{\mathfrak{t}}/S^1 \cap \mathcal{V}(z)$. Hence, the manifold (3.56) defines an oriented cobordism. The orientation of the manifold (3.54) given by viewing it as a component of the boundary of the oriented manifold (3.56) is then equal to $-\omega(\mathbf{L}, \partial O)/\omega(\mathcal{V})$ (as defined in (3.51)) for $O = O^{\mathrm{asd}}(\Omega, w)$. The negative sign arises because the orientation $\omega(\mathbf{L}, \partial O)$ of (3.47) is defined by viewing $\mathbf{L}^{w,\varepsilon}_{\mathfrak{t},\kappa}$ as the boundary of $\ell^{-1}([\varepsilon, \infty))$.

The orientation of the manifold (3.55) given by considering it as a component of the boundary of the oriented manifold (3.56) is then equal to $\omega(\mathcal{Z} \cap \mathbf{K}, \partial O)/\omega(\mathcal{V})$ (as defined in (3.51)) for $O = O^{\mathrm{asd}}(\Omega, w)$ which, by equation (3.52) is equal to $(\omega(\mathbf{K}, o)/\omega(\mathcal{Z}))/\omega(\mathcal{V})$ for $o = o(\Omega, w)$.

Recall that two oriented manifolds (M_i, ω_i) for $i = 0, 1$ are cobordant if there is an oriented manifold W whose oriented boundary is $(M_0, -\omega_0) \cup (M_1, \omega_1)$ [30, p. 170]. Therefore, the manifold (3.56) gives the desired cobordism. □

We now give a cohomological description of the zero locus $\mathcal{Z}_A$ of the obstruction map:

Lemma 3.27. *Continue the hypotheses and notation of Lemmas 3.22 and 3.25. Assume that $n_a(\mathfrak{t}) = \mathrm{Index}_{\mathbb{C}} D_{A,\vartheta}$ is positive and let $c = \dim_{\mathbb{C}} \mathrm{Coker}\, D_{A,\vartheta}$. If $\mathbb{CP}^{k-1} \cong \mathbb{P}(\mathrm{Ker}\, D_{A,\vartheta})$ has the complex orientation and $h \in H^2(\mathbb{CP}^{k-1}; \mathbb{Z})$ is the positive generator, then for generic $\delta > 0$, there is a smooth submanifold T of $\mathbb{CP}^{k-1}$ which is Poincaré dual to h^c such that the restriction of the map g_A of definition (3.53) to T gives a diffeomorphism,*

$$g_A : T \simeq \mathcal{Z}_A \cap \boldsymbol{\gamma}_A^{-1}(\mathcal{V}(z)) \cap \mathbf{K}_{A,\delta}/S^1.$$

If T is oriented as the Poincaré dual of h^c and $\mathcal{Z}_A \cap \boldsymbol{\gamma}_A^{-1}(\mathcal{V}(z)) \cap \mathbf{K}_{A,\delta}/S^1$ is oriented by $\omega(\mathcal{Z} \cap \mathbf{K}, o)/\omega(\mathcal{V})$ from (3.51) where $o = o(\Omega, w)$, then the restriction of g_A to T is orientation preserving if and only if $\varepsilon(A) = 1$, where $\varepsilon(A)$ is defined in Lemma 3.25.

Proof. As noted before Lemma 3.22, the Kuranishi map $\boldsymbol{\varphi}_A$ in (3.37) vanishes transversely on $\mathcal{O}'_A - (T_{[\hat{A}]} M^w_\kappa \oplus \{0\})$. For generic δ, the map $\boldsymbol{\varphi}_A$ vanishes transversely on $\boldsymbol{\gamma}_A^{-1}(\mathcal{V}(z)) \cap \mathbf{K}_{A,\delta}/S^1$ because $\mathcal{V}(z)$ is transverse to $\mathcal{M}^{*,0}_{\mathfrak{t}}/S^1$ by construction. This implies that the zero locus of $\boldsymbol{\varphi}_A$ is Poincaré dual to the Euler class of the vector bundle (3.58) of which $\boldsymbol{\varphi}_A$ is a section. We define a smooth submanifold,

$$T = g_A^{-1}(\boldsymbol{\varphi}_A^{-1}(0)) \subset \mathbb{CP}^{k-1},$$

and observe that the diffeomorphism g_A in equation (3.53) restricts to the desired diffeomorphism of T.

From the definition of $\boldsymbol{\varphi}_A$ in equation (3.37) and of g_A in equation (3.53), we see that the composition $\boldsymbol{\varphi}_A \circ g_A$ can be viewed as an S^1-equivariant map

$$\boldsymbol{\varphi}_A \circ g_A : S^{2k-1} \to \mathbb{C}^c,$$

where $\mathbb{C}^c \cong \mathrm{Coker}\, D_{A,\vartheta}$, and thus a section of the vector bundle

$$S^{2k-1} \times_{S^1} \mathbb{C}^c \to \mathbb{CP}^{k-1}, \tag{3.58}$$

where the S^1 action is diagonal since $(\boldsymbol{\varphi}_A \circ g_A)(e^{i\theta} z) = e^{i\theta}(\boldsymbol{\varphi}_A \circ g_A)(z)$, for $z \in S^{2k-1}$ and $e^{i\theta} \in S^1$. Because the action is diagonal, the Euler class of this bundle is h^c (see [10, Lemma 3.27] for a further explanation of the sign).

By Lemma 3.25, the diffeomorphism g_A defines an orientation-preserving diffeomorphism from $\mathbb{CP}^{k-1}$ to $\boldsymbol{\gamma}_A^{-1}(\mathcal{V}(z)) \cap \mathbf{K}_{A,\delta}/S^1$ (with the orientation $\omega(\mathbf{K}, o)/\omega(\mathcal{V})$ of (3.51)) if and only if $\varepsilon(A) = 1$. Recall that the orientation $(\omega(\mathbf{K}, o)/\omega(\mathcal{Z}))/\omega(\mathcal{V})$ from (3.51) of $\mathcal{Z}_A \cap \boldsymbol{\gamma}_A^{-1}(\mathcal{V}(z)) \cap \mathbf{K}_{A,\delta}/S^1$ is given by the orientation $\omega(\mathbf{K}, o)/\omega(\mathcal{V})$ of $\boldsymbol{\gamma}_A^{-1}(\mathcal{V}(z)) \cap \mathbf{K}_{A,\delta}/S^1$ and the orientation $\omega(\mathcal{Z})$ of the normal bundle of $\mathcal{Z}_A/S^1$. Thus, if T is oriented as the Poincaré dual of h^c and thus has the orientation determined by the complex orientation of $\mathbb{CP}^{k-1}$

and the complex orientation of the normal bundle of T, then the restriction of g_A to T is orientation preserving if and only if $\varepsilon(A) = 1$. $\square$

The final tool needed to compute intersection numbers with $\mathbf{L}^w_{\mathfrak{t},\kappa}$ in equation (3.60) is the following description of $\boldsymbol{\gamma}_A^{-1}(\mathcal{W})$:

Lemma 3.28. *Continue the notation and assumptions of Lemmas 3.22, 3.25, and 3.27. Then* $(\boldsymbol{\gamma}_A \circ g_A)^*\mu_c = 2h$, *where* μ_c *is the cohomology class* (3.12).

Proof. Recall that μ_c is the first Chern class of the line bundle $\mathbb{L}_{\mathfrak{t}}$ in definition (3.10). The embedding $\boldsymbol{\gamma}_A$ and the map g_A are S^1-equivariant so, noting that $\mathbb{L}_{\mathfrak{t}}$ is defined by the S^1 action in equation (3.11), we have

$$(\boldsymbol{\gamma}_A \circ g_A)^*\mathbb{L}_{\mathfrak{t}} \cong S^{2k-1} \times_{(S^1,\times -2)} \mathbb{C} \to S^{2k-1}/S^1, \tag{3.59}$$

The bundle (3.59) has first Chern class $2h$, the sign being positive because the S^1 action is diagonal (see [10, Lemma 3.27]). $\square$

Using Lemma 3.28 we can prove the assertion of Lemma 3.15 that $\iota(\bar{M}^w_\kappa) \subset \bar{\mathcal{W}}$:

Proof of Assertion (2d) in Lemma 3.15. Lemma 3.28 shows that $\mathcal{W}$ will have non-trivial intersection with the normal cone of any point in $\iota(M^w_\kappa) \subset \mathcal{M}_{\mathfrak{t}}$, where by "normal cone" we mean $\boldsymbol{\gamma}_A(\mathcal{Z}_A \cap \{0\} \oplus \operatorname{Ker} D_{A,\vartheta})/S^1$. Therefore, the closure of $\mathcal{W}$ will contain all points in $\iota(M^w_\kappa)$ and thus $\iota(\bar{M}^w_\kappa) \subset \bar{\mathcal{W}}$. $\square$

We can now compute the intersection with the link.

Proposition 3.29. *Let* $\mathfrak{t}$ *be a spin*^{u} *structure on a four-manifold* X, *with* w *an integral lift of* $w_2(\mathfrak{t})$ *and* w (mod 2) *is good. We further assume that* $d_a(\mathfrak{t}) = \dim M^w_\kappa \geq 0$ *and that* $n_a(\mathfrak{t}) = \operatorname{Index}_{\mathbb{C}} D_A > 0$. *Let* δ_c *be a non-negative integer such that*

$$\deg(z) + 2\delta_c = d_a + 2n_a - 2 = \dim(\mathcal{M}^{*,0}_{\mathfrak{t}}/S^1) - 1.$$

Suppose $z \in \mathbb{A}(X)$ *has degree* $\deg(z) \geq d_a$ *and is intersection-suitable. If* $\mathbf{L}^{w,\varepsilon}_{\mathfrak{t},\kappa} \cap \iota(M^w_\kappa)$ *is oriented as the boundary of* $\mathcal{M}^{*,\geq\varepsilon}_{\mathfrak{t}}/S^1$, *where* $\mathcal{M}^{*,0}_{\mathfrak{t}}/S^1$ *is given the orientation* $O^{\mathrm{asd}}(\Omega, w)$, *then there is a positive constant* ε_0 *such that for generic* $\varepsilon \in (0, \varepsilon_0)$,

$$\#\left(\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^{\delta_c} \cap \mathbf{L}^{w,\varepsilon}_{\mathfrak{t},\kappa}\right) = \begin{cases} 2^{n_a-1}\#(\bar{\mathcal{V}}(z) \cap \bar{M}^w_\kappa), & \text{if } \deg(z) = d_a, \\ 0, & \text{if } \deg(z) > d_a. \end{cases} \tag{3.60}$$

Moreover, these intersection numbers are independent of the choice of generic $\varepsilon < \varepsilon_0$.

Proof. By Assertion (2) of Lemma 3.22, the pairing (3.60) is a sum of local terms,

$$\begin{aligned} &\#\left(\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^{\delta_c} \cap \mathbf{L}^{w,\varepsilon}_{\mathfrak{t},\kappa}\right) \\ &= \sum_{[A] \in \bar{\mathcal{V}}(z) \cap \bar{M}^w_\kappa} \#\left(\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^{\delta_c} \cap \mathbf{L}^{w,\varepsilon}_{\mathfrak{t},\kappa} \cap \boldsymbol{\gamma}_A\left(\mathcal{O}'_A\right)\right), \end{aligned} \tag{3.61}$$

where $\mathcal{O}'_A$ is the neighborhood defined in Lemma 3.22. If $\deg(z) > \dim M^w_\kappa$, the intersection $\bar{\mathcal{V}}(z) \cap \bar{M}^w_\kappa$ is empty and so the sum is trivial. Hence, we can assume $\deg(z) = \dim M^w_\kappa$, so $\delta_c = n_a(\mathfrak{t}) - 1$. Let $c = \dim_{\mathbb{C}} \operatorname{Coker} D_{A,\vartheta}$, so $k = \dim \operatorname{Ker} D_{A,\vartheta} = n_a + c$. If $\bar{\mathcal{V}}(z)$ has multiplicity q (in the sense of Definition 3.4), then we can evaluate the terms in the sum in equation (3.61) as

$$\begin{aligned}
&\#\left(\bar{\mathcal{V}}(z)\cap\bar{\mathcal{W}}^{n_a-1}\cap\mathbf{L}^{w,\varepsilon}_{\mathfrak{t},\kappa}\cap\boldsymbol{\gamma}_A\left(\mathcal{O}'_A\right)\right)\\
&=\#\left(\boldsymbol{\gamma}_A^{-1}\left(\mathcal{V}(z)\cap\bar{\mathcal{W}}^{n_a-1}\cap\boldsymbol{\ell}^{-1}(\varepsilon)\right)\cap\mathcal{Z}_A/S^1\right)\quad\text{(Equations (3.41) \& (3.43))}\\
&=\#\left(\boldsymbol{\gamma}_A^{-1}\left(\bar{\mathcal{W}}^{n_a-1}\right)\cap\boldsymbol{\gamma}_A^{-1}\left(sV(z)\right)\cap\mathbf{K}_{A,\delta}\cap\mathcal{Z}_A/S^1\right)\quad\text{(Lemma 3.26)}\\
&=q\#\left((\boldsymbol{\gamma}_A\circ g_A)^{-1}\left(\bar{\mathcal{W}}^{n_a-1}\right)\cap g_A^{-1}(\mathcal{Z}_A/S^1)\cap\mathbb{CP}^{n_a+c-1}\right)\quad\text{(Lemma 3.25)}\\
&=q\varepsilon(A)\left\langle(2h)^{n_a-1}\smile h^c,[\mathbb{CP}^{n_a+c-1}]\right\rangle\quad\text{(Lemmas 3.27 and 3.28)}\\
&=q\varepsilon(A)2^{n_a-1}.
\end{aligned}$$

Hence, equation (3.61) simplifies to give

$$\#\left(\bar{\mathcal{V}}(z)\cap\bar{\mathcal{W}}^{\delta_c}\cap\mathbf{L}^{w,\varepsilon}_{\mathfrak{t},\kappa}\right)=q2^{n_a-1}\sum_{[A]\in\bar{\mathcal{V}}(z)\cap\bar{M}^w_\kappa}\varepsilon(A)=2^{n_a-1}\#\left(\bar{\mathcal{V}}(z)\cap\bar{M}^w_\kappa\right),$$

completing the proof of the proposition. □

As an application of Lemma 3.27, we explain why the moduli space $\mathcal{M}_\mathfrak{t}$ contains solutions to the PU(2) monopole equations [10, Equation (2.32)] which are distinct from the anti-self-dual or reducible solutions. Lemma 3.27 yields the following analogue of Taubes' existence theorem for solutions to the anti-self-dual equation for SO(3) connections:

Proposition 3.30. *Let $\mathfrak{t}$ be a spin^u structure on a four-manifold X, where we allow $b_2^+(X)\geq 0$, and suppose $w_2(\mathfrak{t})\equiv w \pmod 2$, for $w\in H^2(X;\mathbb{Z})$. Assume that $w \pmod 2$ is good. If $n_a(\mathfrak{t})>0$, then for a generic, C^∞ pair (g,ρ), consisting of a Riemannian metric and Clifford map, and generic, C^∞ parameters (τ,ϑ), the moduli space $\mathcal{M}^{*,0}_\mathfrak{t}(g,\rho,\tau,\vartheta)$ of irreducible, non-zero-section* PU(2) *monopoles is non-empty if the moduli space $M^{w,*}_\kappa$ of irreducible, anti-self-dual* SO(3) *connections on $\mathfrak{g}_\mathfrak{t}$ is non-empty.*

Proof. This follows from the fact that the Euler class of the obstruction bundle in the Kuranishi model of $[A,0]\in\mathcal{M}_\mathfrak{t}$ is non-trivial by Lemma 3.27. □

3.5. **The cobordism formula.** If $z\in\mathbb{A}(X)$ is intersection-suitable and $\deg(z)+2\delta_c=\dim(\mathcal{M}^{*,0}_\mathfrak{t}/S^1)-1=d_a+2n_a-2$, then Corollary 3.18 tells us that the intersection

$$\bar{\mathcal{V}}(z)\cap\bar{\mathcal{W}}^{\delta_c}\cap\bar{\mathcal{M}}^{*,\geq\varepsilon}_\mathfrak{t}/S^1 \tag{3.62}$$

is a union of smooth one-dimensional manifolds in $M^{*,\geq\varepsilon}_\mathfrak{t}/S^1$. The boundaries of these one-manifolds will lie either on $\mathbf{L}^w_{\mathfrak{t},\kappa}$ or in a neighborhood of some reducible monopole, possibly in a lower level. Proposition 3.29 describes the intersection of this family of one-manifolds with the component $\mathbf{L}^w_{\mathfrak{t},\kappa}$ of the boundary of $\bar{\mathcal{M}}^{*,\geq\varepsilon}_\mathfrak{t}/S^1$. In particular, we see there are finitely many points in this boundary.

If $w_2(\mathfrak{t})$ is good then (noting that we always assume $b_2^+(X)>0$), for any splitting $\mathfrak{t}=\mathfrak{s}\oplus\mathfrak{s}\otimes L$, the space $M_\mathfrak{s}$ contains no zero-section monopoles, [10, Corollary 3.3]. In [10, Definition 3.22] we constructed a homology class $[\mathbf{L}_{\mathfrak{t},\mathfrak{s}}]$ of the link of the family of reducibles $M_\mathfrak{s}$ contained in the top level $\mathcal{M}^0_\mathfrak{t}/S^1$. By Lemma 2.6, the orientations $O^{\text{asd}}(\Omega,w)$ and $O^{\text{red}}(\Omega,\mathfrak{t},\mathfrak{s})$ differ by $\frac{1}{4}(w-c_1(L))^2$. From the definition of the geometric representatives and Lemma 2.9, we have:

Lemma 3.31. *Let $\mathfrak{t}$ be a spin^u structure over a four-manifold X, with w an integral lift of $w_2(\mathfrak{t})$. Assume that $w \pmod 2$ is good. If $z \in \mathbb{A}(X)$ is intersection-suitable and $\deg(z) + 2\delta_c = d_a + 2n_a - 2$, then*

$$\#\left(\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^{\delta_c} \cap \mathbf{L}_{\mathfrak{t},\mathfrak{s}}\right) = (-1)^{\frac{1}{4}(w - c_1(L))^2} \langle \mu_p(z) \smile \mu_c^{\delta_c}, [\mathbf{L}_{\mathfrak{t},\mathfrak{s}}]\rangle,$$

where $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ is given the boundary orientation determined by $O^{\mathrm{asd}}(\Omega, w)$ on the left hand side and the complex orientation of Definition 2.7 on the right hand side of the above identity.

We now characterize the spin^c structures, $\mathfrak{s}$, for which $\mathfrak{t} = \mathfrak{s} \oplus \mathfrak{s}'$.

Lemma 3.32. *A spin^u structure $\mathfrak{t}$ on X admits a splitting, $\mathfrak{t} = \mathfrak{s} \oplus \mathfrak{s}'$, if and only if*

$$(c_1(\mathfrak{t}) - c_1(\mathfrak{s}))^2 = p_1(\mathfrak{t}). \tag{3.63}$$

Proof. Assume $\mathfrak{t} = \mathfrak{s}\oplus\mathfrak{s}'$. We may write $\mathfrak{s}' = \mathfrak{s}\otimes L$ for some line bundle L, with $\mathfrak{t} = (\rho, V)$ and $\mathfrak{s} = (\rho, W)$. Then $V = W\otimes E$, where $E = \underline{\mathbb{C}}\oplus L$, and $\mathfrak{g}_{\mathfrak{t}} \cong i\underline{\mathbb{R}}\oplus L$. Thus, $c_1(L) = c_1(\mathfrak{t}) - c_1(\mathfrak{s})$ obeys $c_1(L)^2 = p_1(\mathfrak{g}_{\mathfrak{t}}) = p_1(\mathfrak{t})$, as desired.

Conversely, suppose $c_1(\mathfrak{t}) - c_1(\mathfrak{s})$ obeys condition (3.63). Let L be a complex line bundle with $c_1(L) = c_1(\mathfrak{t}) - c_1(\mathfrak{s})$. From Lemma 2.3 in [10] we know that $V \cong W \otimes E$ for a complex rank-two bundle E determined up to isomorphism by $\mathfrak{s}$ and $\mathfrak{t}$. Then

$$c_1(E) = \tfrac{1}{2}c_1(V^+) - c_1(W^+) = c_1(\mathfrak{t}) - c_1(\mathfrak{s}) = c_1(L),$$

while

$$c_2(E) = -\tfrac{1}{4}(p_1(\mathfrak{su}(E)) - c_1(E)^2) = -\tfrac{1}{4}(p_1(\mathfrak{t}) - c_1(L)^2) = 0,$$

where the final equality follows from the fact that $p_1(\mathfrak{t}) = c_1(L)^2$ by hypothesis. Hence, $E \cong \underline{\mathbb{C}} \oplus L$ and so $V \cong W \oplus W \otimes L$, as desired. □

If $\mathfrak{s}$ is a spin^c structure with $c_1(\mathfrak{s})$ obeying condition (3.63), there is a topological embedding $M_{\mathfrak{s}} \hookrightarrow \mathcal{M}_{\mathfrak{t}}$ of $M_{\mathfrak{s}}$ into the top level of $\bar{\mathcal{M}}_{\mathfrak{t}}$ [10, Lemma 3.13]. More generally, if $c_1(\mathfrak{s})$ obeys

$$(c_1(\mathfrak{t}) - c_1(\mathfrak{s}))^2 = p_1(\mathfrak{t}) + 4\ell, \tag{3.64}$$

for some non-negative integer ℓ, then there is a topological embedding of $M_{\mathfrak{s}}$ into the lower-level PU(2)-monopole moduli space $\mathcal{M}_{\mathfrak{t}_\ell} \times \mathrm{Sym}^\ell(X)$, where $\mathfrak{t}_\ell$ is a spin^u structure with $p_1(\mathfrak{t}_\ell) = p_1(\mathfrak{t}) + 4\ell$ [10, Equation (2.44)].

If the reducibles in $\bar{\mathcal{M}}_{\mathfrak{t}}$ appear only in the top Uhlenbeck level $\mathcal{M}_{\mathfrak{t}}$, then $\bar{\mathcal{M}}_{\mathfrak{t}}^{*,\geq\varepsilon}/S^1$ is a cobordism between the link $\mathbf{L}_{\mathfrak{t},\kappa}^{w,\varepsilon}$ of the stratum defined by the anti-self-dual moduli space, $\iota(M_\kappa^w)$, and the links $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ of the strata of reducibles, $\iota(M_{\mathfrak{s}})$. Counting the points in the boundary of the oriented, one-dimensional manifold (3.62) then gives the identity:

$$\#\left(\bar{\mathcal{V}}(z) \cap \mathcal{W}^{\delta_c} \cap \mathbf{L}_{\mathfrak{t},\kappa}^{w}\right) = -\sum_{\{\mathfrak{s}:\ \mathfrak{s}\oplus\mathfrak{s}'=\mathfrak{t}\}} \#\left(\bar{\mathcal{V}}(z) \cap \mathcal{W}^{\delta_c} \cap \mathbf{L}_{\mathfrak{t},\mathfrak{s}}\right). \tag{3.65}$$

Let w be an integral lift of $w_2(\mathfrak{t})$ defining the orientation $O^{\mathrm{asd}}(\Omega, w)$ of $\mathcal{M}_{\mathfrak{t}}^{*,0}/S^1$. Lemmas 2.6 and 2.9 imply that the orientation $O^{\mathrm{asd}}(\Omega, w)$ and the complex orientation for the link $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ differ by

$$(-1)^{o_{\mathfrak{t}}(w,\mathfrak{s})}, \quad \text{where we define } o_{\mathfrak{t}}(w,\mathfrak{s}) = \tfrac{1}{4}(w - c_1(L))^2 = \tfrac{1}{4}(w - c_1(\mathfrak{t}) + c_1(\mathfrak{s}))^2. \tag{3.66}$$

Equation (3.65), Proposition 3.29 and Lemma 3.31 then yield the following result.

Theorem 3.33. *Let $\mathfrak{t}$ be a spin^u structure on an oriented, smooth four-manifold X with $b_2^+(X) > 0$ and $w_2(\mathfrak{t}) \equiv w \pmod 2$, for $w \in H^2(X;\mathbb{Z})$. Assume that $w \pmod 2$ is good. Suppose $z \in \mathbb{A}(X)$ has degree*

$$d_a(\mathfrak{t}) \leq \deg(z) \leq d_a(\mathfrak{t}) + 2n_a(\mathfrak{t}) - 2, \tag{3.67}$$

and is intersection-suitable. Assume that the set of isomorphism classes of spin^c structures, $\mathfrak{s} \in Spin^c(X)$, defining reducible $\mathrm{PU}(2)$ monopoles in $\bar{\mathcal{M}}_{\mathfrak{t}}$ all obey condition (3.63), and so non-empty Seiberg-Witten moduli strata $\iota(M_{\mathfrak{s}})$ appear only in the top level, $\mathcal{M}_{\mathfrak{t}}$.

(a) *If for all $\mathfrak{s} \in Spin^c(X)$ with $M_{\mathfrak{s}}$ non-empty we have*

$$(c_1(\mathfrak{t}) - c_1(\mathfrak{s}))^2 < p_1(\mathfrak{t}), \tag{3.68}$$

so $\bar{\mathcal{M}}_{\mathfrak{t}}$ contains no reducible monopoles, then

$$\#\left(\bar{\mathcal{V}}(z) \cap \bar{M}^w_\kappa(X)\right) = 0. \tag{3.69}$$

(b) *If $\deg(z) = d_a$ and $o_{\mathfrak{t}}(w,\mathfrak{s})$ is as defined in equation (3.66), then*

$$\#\left(\bar{\mathcal{V}}(z) \cap \bar{M}^w_\kappa(X)\right) = -2^{1-n_a} \sum_{\{\mathfrak{s}:\ \mathfrak{s}\oplus\mathfrak{s}'=\mathfrak{t}\}} (-1)^{o_{\mathfrak{t}}(w,\mathfrak{s})} \langle \mu_p(z) \smile \mu_c^{n_a-1}, [\mathbf{L}_{\mathfrak{t},\mathfrak{s}}]\rangle, \tag{3.70}$$

where the class $[\mathbf{L}_{\mathfrak{t},\mathfrak{s}}]$ is defined by the complex orientation of Definition 2.7 on $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$.
(c) *If $d_a < \deg(z) \leq d_a + 2n_a - 2$ and $\delta_c \in \mathbb{N}$ is defined by $\deg(z) + \delta_c = d_a + 2n_a - 2$, then*

$$\sum_{\{\mathfrak{s}:\ \mathfrak{s}\oplus\mathfrak{s}'=\mathfrak{t}\}} (-1)^{o_{\mathfrak{t}}(w,\mathfrak{s})} \langle \mu_p(z) \smile \mu_c^{\delta_c}, [\mathbf{L}_{\mathfrak{t},\mathfrak{s}}]\rangle = 0. \tag{3.71}$$

The sums in equations (3.70) and (3.71) are necessarily finite, because there are only finitely many spin^c structures $\mathfrak{s}$ with $M_{\mathfrak{s}}$ non-empty and thus $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ non-empty. Theorem 3.33 can be strengthened to a more useful form if we assume

Conjecture 3.34. [8, Conjecture 3.1] Continue the notation of Theorem 3.33. Suppose that $\mathfrak{t}_\ell = \mathfrak{s}\oplus\mathfrak{s}'$, where $p_1(\mathfrak{t}_\ell) = p_1(\mathfrak{t}) + 4\ell$ and $\iota(M_{\mathfrak{s}})$ is contained in the level $\mathcal{M}_{\mathfrak{t}_\ell} \times \mathrm{Sym}^\ell(X)$, for some natural number $\ell \geq 0$. Then the pairing $\#(\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^{\delta_c} \cap \mathbf{L}_{\mathfrak{t},\mathfrak{s}})$ is a multiple of $SW_X(\mathfrak{s})$ and thus vanishes if the Seiberg-Witten function $SW_{X,\mathfrak{s}}$ is trivial.

See §4.1 for a definition of the Seiberg-Witten invariants. The motivation for this conjecture is discussed in [16] and almost certainly does hold, one of our current goals being to provide a complete proof in the near future. The difficulty lies in the construction of the link $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ of a Seiberg-Witten moduli space when $\ell = \ell(\mathfrak{t},\mathfrak{s}) > 0$. We show that Conjecture 3.34 holds when $\ell = 0$ in Theorem 4.13 and when $\ell = 1$ in [14]. By adapting Leness's proof of the wall-crossing formula in [36], we can also see that the conjecture holds when $\ell = 2$.

Corollary 3.35. *Given Conjecture 3.34, we can relax the hypothesis of Theorem 3.33 — that* non-empty *Seiberg-Witten moduli spaces $M_{\mathfrak{s}}$ appear only in the top level $\mathcal{M}_{\mathfrak{t}}$—to the weaker requirement that Seiberg-Witten moduli spaces $M_{\mathfrak{s}}$ with* non-trivial Seiberg-Witten functions *$SW_{X,\mathfrak{s}}$ appear only in the top level $\mathcal{M}_{\mathfrak{t}}$. Then the conclusions of Theorem 3.33 hold without change.*

Remark 3.36. The Seiberg-Witten stratum $\iota(M_{\mathfrak{s}})$ corresponding to a splitting $\mathfrak{t}_\ell = \mathfrak{s}\oplus\mathfrak{s}'$ lies in level

$$\ell(\mathfrak{t},\mathfrak{s}) = \tfrac{1}{8}(d_a - 2r(\Lambda,\mathfrak{s}))$$

of the space of ideal $\mathrm{PU}(2)$ monopoles containing $\bar{\mathcal{M}}_{\mathfrak{t}}$, from the definition (1.12) of $r(\Lambda,\mathfrak{s})$, where d_a is the dimension of the anti-self-dual moduli space $M^w_\kappa \hookrightarrow \mathcal{M}_{\mathfrak{t}}$. Then, by definition

(1.12) of $r(\Lambda)$, the Seiberg-Witten strata with non-trivial invariants are contained only in levels ℓ of this space of ideal PU(2) monopoles, where

$$0 \leq \ell \leq \tfrac{1}{8}(d_a - 2r(\Lambda)).$$

See the proof of Theorem 1.2 in §4.6.

4. Intersection with the link of a stratum of top-level reducibles

In this section we calculate the pairings appearing on the right-hand-side of equation (3.70) in Theorem 3.33 under the additional assumption that for all $\alpha, \alpha' \in H^1(X;\mathbb{Z})$ one has $\alpha \cup \alpha' = 0$. We begin by giving a definition of the Seiberg-Witten invariants which is appropriate for the perturbations we use in our version of the Seiberg-Witten equations [10, §2.3]. Recall that the link $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ constructed in [10, §3.5] is diffeomorphic via a map $\boldsymbol{\gamma}$ to the zero-locus of a section of an obstruction bundle $\boldsymbol{\gamma}^*\Xi\Xi/S^1$ over the complex projectivization of a bundle $N_{\mathfrak{t}}(\Xi,\mathfrak{s}) \to M_{\mathfrak{s}}$. Thus, in §4.2, we calculate pullbacks of the classes μ_p and μ_c by $\boldsymbol{\gamma}$ to $\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})$. In §4.3 we compute the total Segre class—the formal inverse of the total Chern class as defined in Lemma 4.10—of the virtual normal bundle $N_{\mathfrak{t}}(\Xi,\mathfrak{s})$ of $M_{\mathfrak{s}}$ and the Euler class of the obstruction bundle $\boldsymbol{\gamma}^*\Xi/S^1$. Finally, in §4.4 we perform the actual computation and complete the proofs of Theorems 1.1, 1.2, 1.4, and Corollary 1.5.

4.1. **A definition of the Seiberg-Witten invariants.** In this subsection we give a definition of the Seiberg-Witten invariants for a closed, smooth four-manifold X; we allow $b_1(X) \geq 0$ and $b_2^+(X) \geq 1$.

Recall that X is equipped with an orientation for which $b_2^+(X) > 0$ and that we have fixed an orientation for $H^1(X;\mathbb{R}) \oplus H^+(X;\mathbb{R})$: the Seiberg-Witten moduli spaces are then oriented according to the conventions of [46, §6.6] (or see our §2.3). In [10, §2.4.2] we defined a universal complex line bundle,

$$\mathbb{L}_{\mathfrak{s}} = \tilde{\mathcal{C}}^0_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} \underline{\mathbb{C}} \to \mathcal{C}^0_{\mathfrak{s}} \times X, \tag{4.1}$$

where $\underline{\mathbb{C}} = X \times \mathbb{C}$ and the action of $\mathcal{G}_{\mathfrak{s}}$ is given for $s \in \mathcal{G}_{\mathfrak{s}}$, $x \in X$ and $z \in \mathbb{C}$ by

$$(s, (B,\Psi), (x,z)) \mapsto \left(s(B,\Psi), (x, s(x)^{-1}z)\right). \tag{4.2}$$

We then defined cohomology classes on $\mathcal{C}^0_{\mathfrak{s}}$ by

$$\mu_{\mathfrak{s}} : H_\bullet(X;\mathbb{R}) \to H^{2-\bullet}(\mathcal{C}^0_{\mathfrak{s}};\mathbb{R}), \quad \mu_{\mathfrak{s}}(\alpha) = c_1(\mathbb{L}_{\mathfrak{s}})/\alpha, \tag{4.3}$$

where α is either the positive generator $x \in H_0(X;\mathbb{Z})$ or a class $\gamma \in H_1(X;\mathbb{R})$.

If $z \in \mathbb{B}(X)$ is a monomial $\alpha_1 \cdots \alpha_p$ with $\alpha_i \in H_0(X;\mathbb{Z})$ or $H_1(X;\mathbb{R})$, then it has total degree $\deg(z) = \sum_{i=1}^p \deg(\alpha_i)$. If $z = x^m\gamma_1\cdots\gamma_n \in \mathbb{B}(X)$, we set

$$\mu_{\mathfrak{s}}(z) = \underbrace{\mu_{\mathfrak{s}}(x) \smile \cdots \smile \mu_{\mathfrak{s}}(x)}_{m \text{ times}} \smile \mu_{\mathfrak{s}}(\gamma_1) \smile \cdots \smile \mu_{\mathfrak{s}}(\gamma_n), \tag{4.4}$$

and define $\mu_{\mathfrak{s}}(z)$ for arbitrary elements $z \in \mathbb{B}(X)$ by $\mathbb{R}$-linearity.

Let $\tilde{X} = X\#\overline{\mathbb{CP}}^2$ be the blow-up of X with exceptional class $e \in H_2(\tilde{X};\mathbb{Z})$ and denote its Poincaré dual by $\mathrm{PD}[e] \in H^2(\tilde{X};\mathbb{Z})$. Let $\mathfrak{s}^\pm = (\tilde{\rho}, \tilde{W})$ denote the spin^c structure on $\tilde{X}$ with $c_1(\mathfrak{s}^\pm) = c_1(\mathfrak{s}) \pm \mathrm{PD}[e]$ obtained by splicing the spin^c structure $\mathfrak{s} = (\rho, W)$ on X with the spin^c structure on $\overline{\mathbb{CP}}^2$ with first Chern class $\pm\mathrm{PD}[e]$. (See §4.5 for a more detailed explanation of the relation between spin^c and spin^u structures on X and $\tilde{X}$, as well as the blow-up formula for Seiberg-Witten invariants, which we shall invoke below.) One easily checks that $\dim M_{\mathfrak{s}^\pm}(\tilde{X}) = \dim M_{\mathfrak{s}}(X)$, where the Seiberg-Witten moduli spaces $M_{\mathfrak{s}}(X)$ and $M_{\mathfrak{s}^\pm}(\tilde{X})$ are defined in [10, Equation (2.55) & Lemma 3.12] with perturbation

$\eta = F^+(A_\Lambda)$. Here, $2A_\Lambda$ is the fixed connection on $\det(V^+) \cong \det(\tilde V^+)$ and $\tilde{\mathfrak{t}} = (\tilde\rho, \tilde V)$ is the spin^u structure on $\tilde X$ defined in Lemma 4.19, with $c_1(\tilde{\mathfrak{t}}) = c_1(\mathfrak{t})$, $p_1(\tilde{\mathfrak{t}}) = p_1(\mathfrak{t}) - 1$, and $w_2(\tilde{\mathfrak{t}}) = w_2(\mathfrak{t}) + \mathrm{PD}[e]$. Now

$$c_1(\mathfrak{s}^\pm) - c_1(\tilde{\mathfrak{t}}) = c_1(\mathfrak{s}) \pm \mathrm{PD}[e] - \Lambda \in H^2(\tilde X; \mathbb{Z})$$

is not a torsion class and so—for $b_2^+(X) > 0$, generic Riemannian metrics g on X and related metrics on the connect sum $\tilde X$—the moduli spaces $M_{\mathfrak{s}^\pm}(\tilde X)$ contain no zero-section pairs [46, Proposition 6.3.1]. Thus, for our choice of generic perturbations, the moduli spaces $M_{\mathfrak{s}^\pm}(\tilde X)$ are compact, oriented, smooth manifolds.

Noting that $\mathbb{B}(X) \cong \mathbb{B}(\tilde X)$, we define the *Seiberg-Witten invariants for* $(X, \mathfrak{s})$ as an $\mathbb{R}$-linear function (1.7) by setting

$$SW_{X,\mathfrak{s}}(z) = \langle \mu_{\mathfrak{s}^+}(z), [M_{\mathfrak{s}^+}(\tilde X)] \rangle, \tag{4.5}$$

with $SW_{X,\mathfrak{s}}(z) = 0$ when $\deg(z) \neq \dim M_{\mathfrak{s}}(X)$. The blow-up formula for Seiberg-Witten invariants (Theorem 4.20) implies that

$$\langle \mu_{\mathfrak{s}}(z), [M_{\mathfrak{s}}(X)] \rangle = \langle \mu_{\mathfrak{s}^\pm}(z), [M_{\mathfrak{s}^\pm}(\tilde X)] \rangle, \tag{4.6}$$

when the pairing on the left is well-defined, that is, when $M_{\mathfrak{s}}(X)$ contains no zero-section monopoles. For example, with our version of the Seiberg-Witten equations [10, Equation (2.55)], this situation arises when $c_1(\mathfrak{s}) - \Lambda \in H^2(X; \mathbb{Z})$ is not a torsion class and thus $M_{\mathfrak{s}}(X)$ contains no zero-section pairs if the metric g is generic and $b_2^+(X) > 0$ [46, Proposition 6.3.1]. Therefore, our definition of the Seiberg-Witten invariants coincides with the usual one [46] in this case, but has the advantage that it is valid even when $c_1(\mathfrak{s}) - \Lambda$ is torsion and one cannot perturb the Seiberg-Witten equations by a generic two-form η (see Remark 2.14 in [10]). When $b_2^+(X) > 1$, the pairing on the right-hand-side of equation (4.5) is independent of the metric, [46, Lemma 6.7.1].

When $b_2^+(X) = 1$, however, the pairing on the right-hand-side of definition (4.5) depends on the period point $\omega(\tilde g)$ of the metric $\tilde g$ on $\tilde X$, as in the case of the Donaldson invariants (see §3.4.2). To explain this dependence when the Seiberg-Witten moduli spaces are defined as in [10, §2.3] with the perturbation parameters $\eta = F^+(A_\Lambda)$ described above, we note that the moduli space $M_{\mathfrak{s}^+}(\tilde g)$ contains zero-section pairs if and only if the period point $\omega(\tilde g)$ lies on the $(c_1(\mathfrak{s}^+) - \Lambda)$-wall in the positive cone of $H^2(\tilde X; \mathbb{R})$, that is

$$\omega(\tilde g) \smile (c_1(\mathfrak{s}^+) - \Lambda) = 0. \tag{4.7}$$

When $\omega(\tilde g)$ does not lie on the wall, the pairing in definition (4.5) may depend on the sign of $\omega(\tilde g) \smile (c_1(\mathfrak{s}^+) - \Lambda)$. The chambers for the Seiberg-Witten invariants of $M_{\mathfrak{s}^+}$ are thus connected components of the complement of the $(c_1(\mathfrak{s}^+) - \Lambda)$-wall in the positive cone of $H^2(\tilde X; \mathbb{R})$, which we call $(c_1(\mathfrak{s}^+) - \Lambda)$-chambers.

By an argument which is the same as the one we gave for the Donaldson invariants in §3.4.2, if $c_1(\mathfrak{s}) - \Lambda$ is not torsion then each $(c_1(\mathfrak{s}) - \Lambda)$-chamber in the positive cone of $H^2(X; \mathbb{R})$ is contained in a unique $(c_1(\mathfrak{s}^+) - \Lambda)$-chamber in the positive cone of $H^2(\tilde X; \mathbb{R}) \cong \mathbb{R}[e] \oplus H^2(X; \mathbb{R})$, the *related chamber*. The Seiberg-Witten invariant associated to a $(c_1(\mathfrak{s}) - \Lambda)$-chamber is then defined by evaluating the pairing in equation (4.5) with a metric whose period point lies in the related $(c_1(\mathfrak{s}^+) - \Lambda)$-chamber.

Suppose $w_2(X) - \Lambda \pmod 2$ is good, in the sense of Definition 3.20. For any spin^c structure $\mathfrak{s}$ over X, we have $c_1(\mathfrak{s}) \equiv w_2(X) \pmod 2$ and so $c_1(\mathfrak{s}) - \Lambda \pmod 2$ is good. Then $c_1(\mathfrak{s}) - \Lambda$ is not torsion and the Seiberg-Witten invariants for $M_{\mathfrak{s}}$ depend only on the metric g through the $(c_1(\mathfrak{s}) - \Lambda)$-chamber for $\omega(g)$.

If $w_2(X) - \Lambda \pmod 2$ is not good, then $c_1(\mathfrak{s}) - \Lambda$ may be torsion and in this situation

$$\omega(\tilde{g}) \smile (c_1(\mathfrak{s}^+) - \Lambda) = \omega(\tilde{g}) \smile \mathrm{PD}[e],$$

so the sign of the cup-product would depend on the sign of $\omega(\tilde{g}) \smile \mathrm{PD}[e]$, which converges to zero as the neck is stretched. Hence, the definition (4.5) of the Seiberg-Witten invariant in this case requires a more delicate analysis of the sign of $\omega(\tilde{g}) \smile \mathrm{PD}[e]$ as the length of the neck converges to infinity [59], which we shall not consider here.

Thus, when $b_2^+(X) = 1$, we shall assume that $w_2(X) - \Lambda \pmod 2$ is good. Since

$$w \equiv w_2(X) - \Lambda \pmod 2,$$

this coincides with the constraint we used to define the Donaldson invariants in §3.4.2.

We now compare the Donaldson and Seiberg-Witten chamber structures:

Lemma 4.1. *Let $\mathfrak{t}$ be a spin^u structure on a four-manifold X with $b_2^+(X) = 1$, where w is an integral lift of $w_2(\mathfrak{t})$ and $w \pmod 2$ is good, and $c_1(\mathfrak{t}) = \Lambda$. Then there is a one-to-one correspondence between the set of $(w, p_1(\mathfrak{t}))$-walls and the set of $(c_1(\mathfrak{s}) - \Lambda)$-walls, where $M_{\mathfrak{s}}$ is contained in the space of ideal* $\mathrm{PU}(2)$ *monopoles, $\cup_{\ell=0}^{\infty} \left(\mathcal{M}_{\mathfrak{t}_\ell} \times \mathrm{Sym}^\ell(X)\right)$.*

Proof. A $(w, p_1(\mathfrak{t}))$-wall is defined by class $\alpha \in H^2(X;\mathbb{Z})$ with $\alpha \equiv w \pmod 2$ and $\alpha^2 = p_1(\mathfrak{t}) + 4\ell$ for $\ell \geq 0$ (see equation (3.33)). Because α is an integral lift of $w_2(\mathfrak{t})$, the class $\Lambda - \alpha$ is characteristic. Hence, there is a spin^c structure $\mathfrak{s}$ with $c_1(\mathfrak{s}) = \Lambda - \alpha$. By Lemma 3.32 and the identity $\alpha^2 = p_1(\mathfrak{t}) + 4\ell$, a spin^u structure $\mathfrak{t}_\ell$ with $c_1(\mathfrak{t}_\ell) = \Lambda$ and $p_1(\mathfrak{t}_\ell) = p_1(\mathfrak{t}) + 4\ell$ admits a splitting $\mathfrak{t}_\ell = \mathfrak{s} \oplus \mathfrak{s}'$ for any such spin^c structures. Conversely, given a spin^c structure $\mathfrak{s}$ with $M_{\mathfrak{s}}$ contained in the space of ideal monopoles, Lemma 3.32 implies that $(c_1(\mathfrak{s}) - \Lambda)^2 = p_1(\mathfrak{t}) + 4\ell$ and $c_1(\mathfrak{s}) - \Lambda \equiv w_2(\mathfrak{t}) \pmod 2$, so the $(c_1(\mathfrak{s}) - \Lambda)$-wall is a $(w, p_1(\mathfrak{t}))$-wall. □

Remark 4.2. Lemma 4.1 implies that in formulas such as (1.20) and (1.21) derived using the cobordism $\mathcal{M}_{\mathfrak{t}}^{*,0}/S^1$ to compare Donaldson and Seiberg-Witten invariants, if a period point $\omega(g)$ crosses a wall for the Donaldson invariant, it will also cross a wall for one of the Seiberg-Witten invariants in the formula; thus both sides of the identity will change.

4.2. **Pullbacks of cohomology classes to the link of a stratum of reducibles.** We now compute the pullbacks of the cohomology classes $\mu_p(z)$ and μ_c by $\boldsymbol{\gamma} : N_{\mathfrak{t}}^{\varepsilon}(\Xi, \mathfrak{s})/S^1 \to \mathcal{C}_{\mathfrak{t}}/S^1$, the restriction of the S^1-equivariant embedding defined in [10, Equation (3.44)] to the ε-sphere of the bundle $N_{\mathfrak{t}}(\Xi, \mathfrak{s})$. The result will be expressed in terms of the Seiberg-Witten $\mu_{\mathfrak{s}}$-classes and one additional cohomology class:

Definition 4.3. Let $\nu \in H^2(\mathbb{P}N_{\mathfrak{t}}(\Xi, \mathfrak{s}); \mathbb{Z})$ be the negative of the first Chern class of the S^1 bundle $N_{\mathfrak{t}}^{\varepsilon}(\Xi, \mathfrak{s}) \to \mathbb{P}N_{\mathfrak{t}}(\Xi, \mathfrak{s})$. Restricted to each fiber of $\mathbb{P}N_{\mathfrak{t}}(\Xi, \mathfrak{s})$, the class ν is the positive generator of the cohomology. With the conventions of [24, §3.1], the class ν is the first Chern class of the line bundle $\mathcal{O}_{\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})}(1)$, the dual of the tautological bundle.

Let $\tilde{N}_{\mathfrak{t}}(\Xi, \mathfrak{s}) \to \tilde{M}_{\mathfrak{s}}$ be the pullback of $N_{\mathfrak{t}}(\Xi, \mathfrak{s})$ by the projection $\tilde{M}_{\mathfrak{s}} \to M_{\mathfrak{s}}$. To compute the pullbacks by $\boldsymbol{\gamma}$ of the cohomology classes $\mu_p(\beta)$ to $\mathbb{P}N_{\mathfrak{t}}(\Xi, \mathfrak{s})$, we first compute the pullback of the universal bundle $\mathbb{F}_{\mathfrak{t}}$ defined in equation (3.2).

Lemma 4.4. *Let $\mathfrak{t}$ be a spin^u structure which admits a splitting $\mathfrak{t} = \mathfrak{s} \oplus \mathfrak{s} \otimes L$. Assume $M_{\mathfrak{s}}$ contains no zero-section pairs. Let $\boldsymbol{\gamma} : N_{\mathfrak{t}}^{\varepsilon}(\Xi, \mathfrak{s}) \hookrightarrow \mathcal{C}_{\mathfrak{t}}$ be the embedding constructed in* [10, §3.5.3]*, and let $\tilde{N}_{\mathfrak{t}}^{\varepsilon}(\Xi, \mathfrak{s})$ denote the ε-sphere bundle of $\tilde{N}_{\mathfrak{t}}(\Xi, \mathfrak{s})$. Then, we have an isomorphism of* $\mathrm{SO}(3)$ *bundles over $\mathbb{P}N_{\mathfrak{t}}(\Xi, \mathfrak{s}) \times X$,*

$$(\boldsymbol{\gamma} \times \mathrm{id}_X)^* \mathbb{F}_{\mathfrak{t}} \cong \tilde{N}_{\mathfrak{t}}^{\varepsilon}(\Xi, \mathfrak{s}) \times_{\mathcal{G}_{\mathfrak{s}} \times S^1} (i\underline{\mathbb{R}} \oplus L), \tag{4.8}$$

where $s \in \mathcal{G}_\mathfrak{s}$ and $e^{i\theta} \in S^1$ act on $\tilde{N}^\varepsilon_\mathfrak{t}(\Xi,\mathfrak{s}) \times (i\underline{\mathbb{R}} \oplus L)$ by

$$((B,\Psi,\eta), f \oplus z) \mapsto \left(s(B,\Psi,e^{i\theta}\eta), f \oplus s^{-2}e^{i\theta}z\right), \tag{4.9}$$

where $(B,\Psi,\eta) \in \tilde{N}_\mathfrak{t}(\Xi,\mathfrak{s})$, $(B,\Psi) \in \tilde{M}_\mathfrak{s}$, and $f \oplus z \in i\underline{\mathbb{R}} \oplus L$.

Proof. Since $V = W \oplus W \otimes L$, where $\mathfrak{s} = (\rho, W)$ and $\mathfrak{t} = (\rho, V)$, we have an isomorphism of SO(3) bundles $\mathfrak{g}_\mathfrak{t} \cong i\underline{\mathbb{R}} \oplus L$ and the definition (3.2) of $\mathbb{F}_\mathfrak{t}$ yields isomorphism of SO(3) bundles over $\mathcal{C}^*_\mathfrak{t} \times X$,

$$\mathbb{F}_\mathfrak{t} \cong \tilde{\mathcal{C}}^*_\mathfrak{t} \times_{\mathcal{G}_\mathfrak{t}\times S^1} (i\underline{\mathbb{R}} \oplus L).$$

From [10, §3.5.4] we recall that the embedding $\boldsymbol{\gamma} : N_\mathfrak{t}(\Xi,\mathfrak{s}) \to \mathcal{C}_\mathfrak{t}$ lifts to a map

$$\tilde{\boldsymbol{\gamma}} : \tilde{N}_\mathfrak{t}(\Xi,\mathfrak{s}) \to \tilde{\mathcal{C}}_\mathfrak{t},$$

which is $\mathcal{G}_\mathfrak{s}$ equivariant when $s \in \mathcal{G}_\mathfrak{s}$ acts on $\tilde{\mathcal{C}}_\mathfrak{t}$ via the embedding

$$\varrho : \mathcal{G}_\mathfrak{s} \hookrightarrow \mathcal{G}_\mathfrak{t}, \quad s \mapsto \varrho(s) = s\,\mathrm{id}_W \oplus s^{-1}\mathrm{id}_{W\otimes L}, \tag{4.10}$$

while s acts on the base $\tilde{M}_\mathfrak{s}$ by the usual action of $\mathcal{G}_\mathfrak{s}$ and on the fibers of $\tilde{N}_\mathfrak{t}(\Xi,\mathfrak{s}) \to \tilde{M}_\mathfrak{s}$ by the action on $L^2_k(\Lambda^1 \otimes L) \oplus L^2_k(W^+ \otimes L)$ induced by the isomorphisms $\mathfrak{g}_\mathfrak{t} \cong i\underline{\mathbb{R}} \oplus L$ and $V = W \oplus W \otimes L$ and the action of $\varrho(s)$ on V.

We also recall from [10, §3.5.4] that the map $\tilde{\boldsymbol{\gamma}}$ is S^1 equivariant with respect to the action on the complex fibers of $\tilde{N}_\mathfrak{t}(\Xi,\mathfrak{s})$ by scalar multiplication and the trivial action on the base $\tilde{M}_\mathfrak{s}$, while S^1 acts on $\tilde{\mathcal{C}}_\mathfrak{t}$ through

$$\varrho_L : S^1 \times V \to V, \quad \text{where} \quad \varrho_L(e^{i\theta}) = \mathrm{id}_W \oplus e^{i\theta}\,\mathrm{id}_{W\otimes L}. \tag{4.11}$$

Therefore, we have an isomorphism of SO(3) bundles:

$$(\tilde{\boldsymbol{\gamma}} \times \mathrm{id}_X)^*\left(\tilde{\mathcal{C}}^*_\mathfrak{t} \times_{\mathcal{G}_\mathfrak{t}} (i\underline{\mathbb{R}} \oplus L)\right) \cong \tilde{N}_\mathfrak{t}(\Xi,\mathfrak{s}) \times_{\mathcal{G}_\mathfrak{s}} (i\underline{\mathbb{R}} \oplus L).$$

We obtain $(\tilde{\boldsymbol{\gamma}} \times \mathrm{id}_X)^*\mathbb{F}_\mathfrak{t}$ on the left above after we take the S^1 quotient, with S^1 acting on $\tilde{\mathcal{C}}_\mathfrak{t}$ through complex multiplication on V and trivially on $\mathfrak{g}_\mathfrak{t}$. Given

$$[A,\Phi,f,z] = [\tilde{\boldsymbol{\gamma}}(B,\Psi,\eta), f, z] \in \tilde{\mathcal{C}}^*_\mathfrak{t} \times_{\mathcal{G}_\mathfrak{t}} (i\underline{\mathbb{R}} \oplus L),$$

and noting that

$$e^{i\theta}\,\mathrm{id}_V = \varrho(e^{i\theta})\varrho_L(e^{2i\theta}), \quad e^{i\theta} \in S^1, \tag{4.12}$$

then we can identify the pull-back of the S^1 action:

$$\begin{aligned}
[e^{i\theta}(A,\Phi), f, z] &= [e^{i\theta}\tilde{\boldsymbol{\gamma}}(B,\Psi,\eta), f, z] \\
&= [\varrho(e^{i\theta})\varrho_L(e^{2i\theta})\tilde{\boldsymbol{\gamma}}(B,\Psi,\eta), f, z] \quad \text{(Equation (4.12))} \\
&= [\varrho(e^{i\theta})\tilde{\boldsymbol{\gamma}}(B,\Psi,e^{2i\theta}\eta), f, z] \quad \text{(see [10, §3.5.4])} \\
&= [\tilde{\boldsymbol{\gamma}}(B,\Psi,e^{2i\theta}\eta), \varrho(e^{-i\theta})(f,z)] \\
&= [\tilde{\boldsymbol{\gamma}}(B,\Psi,e^{i2\theta}\eta), f, e^{2i\theta}z] \quad \text{(see [10, §3.4.2])}.
\end{aligned}$$

The final equality follows from the observation that the action of $s \in \mathcal{G}_\mathfrak{s}$ induced by the embedding $\varrho : \mathcal{G}_\mathfrak{s} \to \mathcal{G}_\mathfrak{t}$, the homomorphism $\mathrm{Ad} : \mathrm{Aut}(V) \to \mathfrak{su}(V)$, the projection $\mathfrak{su}(V) \to \mathfrak{g}_\mathfrak{t}$, and the isomorphism $\mathfrak{g}_\mathfrak{t} \cong i\underline{\mathbb{R}} \oplus L$, is given by $(f,z) \mapsto (f, s^{-2}z)$; see [10, §3.4.2] for details. □

Lemma 4.4 shows that we can compute $\boldsymbol{\gamma}^* p_1(\mathbb{F}_\mathfrak{t}) \in H^4(\mathbb{P}N_\mathfrak{t}(\Xi,\mathfrak{s}) \times X;\mathbb{R})$ once we know the Chern class of the line-bundle component of the SO(3) bundle (4.8):

Lemma 4.5. *Continue the hypotheses of Lemma 4.4 and let ν be the cohomology class in Definition 4.3. Then the complex line bundle,*

$$\tilde{N}_{\mathfrak{t}}^{\varepsilon}(\Xi,\mathfrak{s}) \times_{\mathcal{G}_{\mathfrak{s}}\times S^1} L \to \mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s}) \times X, \tag{4.13}$$

has first Chern class

$$\pi_{\mathbb{P}N}^*\nu + 2(\pi_{\mathfrak{s}} \times \pi_X)^* c_1(\mathbb{L}_{\mathfrak{s}}) + \pi_X^* c_1(L) \in H^2(\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s}) \times X; \mathbb{Z}),$$

where $\pi_{\mathbb{P}n}$, $\pi_{\mathfrak{s}}$ and π_X are the projections from $\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s}) \times X$ to $\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})$, $M_{\mathfrak{s}}$, and X, respectively, and $\mathbb{L}_{\mathfrak{s}}$ is the universal Seiberg-Witten line bundle (4.1).

Proof. The projection $\tilde{N}_{\mathfrak{s}}^{\varepsilon}(\Xi,\mathfrak{s}) \to \mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})$ is a principal $\mathcal{G}_{\mathfrak{s}} \times S^1$ bundle, where S^1 acts by scalar multiplication on the fibers of $\tilde{N}_{\mathfrak{t}}(\Xi,\mathfrak{s})$. One has an isomorphism of $\mathcal{G}_{\mathfrak{s}} \times S^1$ bundles

$$\tilde{N}_{\mathfrak{t}}^{\varepsilon}(\Xi,\mathfrak{s}) \cong N_{\mathfrak{t}}^{\varepsilon}(\Xi,\mathfrak{s}) \times_{M_{\mathfrak{s}}} \tilde{M}_{\mathfrak{s}}, \tag{4.14}$$

defined, for $(B,\Psi) \in \tilde{M}_{\mathfrak{s}}$ and $(B,\Psi,\eta) \in \tilde{N}_{\mathfrak{t}}^{\varepsilon}(\Xi,\mathfrak{s})$, by the map

$$(B,\Psi,\eta) \mapsto ([B,\Psi,\eta],(B,\Psi)),$$

where the square brackets indicate equivalence modulo $\mathcal{G}_{\mathfrak{s}}$. Applying the isomorphism (4.14) to the $\mathcal{G}_{\mathfrak{s}} \times S^1$ bundle (4.13) yields an isomorphism of complex line bundles,

$$\tilde{N}_{\mathfrak{t}}^{\varepsilon}(\Xi,\mathfrak{s}) \times_{\mathcal{G}_{\mathfrak{s}}\times S^1} L \cong (N_{\mathfrak{t}}^{\varepsilon}(\Xi,\mathfrak{s}) \times_{M_{\mathfrak{s}}} (\tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} L))/S^1 \tag{4.15}$$

where, as in definition (4.9), an element $s \in \mathcal{G}_{\mathfrak{s}}$ acts on $\tilde{M}_{\mathfrak{s}} \times L$ by $(B,\Psi,z) \mapsto (s(B,\Psi), s^{-2}z)$ and S^1 acts diagonally on $N_{\mathfrak{t}}^{\varepsilon}(\Xi,\mathfrak{s}) \times L$.

The isomorphism in [10, Equation (6.68)] gives

$$\tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} L \cong \mathbb{L}_{\mathfrak{s}}^{\otimes 2} \otimes \pi_X^* L. \tag{4.16}$$

Substituting the isomorphism (4.16) into equation (4.15) yields an isomorphism of complex line bundles,

$$\tilde{N}_{\mathfrak{t}}^{\varepsilon}(\Xi,\mathfrak{s}) \times_{\mathcal{G}_{\mathfrak{s}}\times S^1} L \cong (N_{\mathfrak{t}}^{\varepsilon}(\Xi,\mathfrak{s}) \times_{M_{\mathfrak{s}}} \mathbb{L}_{\mathfrak{s}}^{\otimes 2} \otimes \pi_X^* L)/S^1. \tag{4.17}$$

The proof is completed by applying Lemma 3.27 in [10] to compute the first Chern class of a fiber product with an S^1 action, the observation that the S^1 action in (4.17) is diagonal, and the fact that $\nu = -c_1(N_{\mathfrak{t}}^{\varepsilon}(\Xi,\mathfrak{s}))$. □

The reduction of $(\boldsymbol{\gamma} \times \mathrm{id}_X)^*\mathbb{F}_{\mathfrak{t}}$ in Lemma 4.4, the computation in Lemma 4.5, and the fact that $c_1(L) = c_1(\mathfrak{t}) - c_1(\mathfrak{s})$ give the following expression for $(\boldsymbol{\gamma} \times \mathrm{id}_X)^* p_1(\mathbb{F}_{\mathfrak{t}})$.

Corollary 4.6. *Continue the hypotheses and notation of Lemmas 4.4 and 4.5. Then,*

$$\begin{aligned}(\boldsymbol{\gamma} \times \mathrm{id}_X)^* p_1(\mathbb{F}_{\mathfrak{t}}) &= (\pi_{\mathbb{P}N}^*\nu + 2(\pi_{\mathfrak{s}} \times \pi_X)^* c_1(\mathbb{L}_{\mathfrak{s}}) + \pi_X^*(c_1(\mathfrak{t}) - c_1(\mathfrak{s})))^2 \\ &\in H^2(\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s}) \times X; \mathbb{Z}).\end{aligned} \tag{4.18}$$

We compute the pullbacks of the cohomology classes $\mu_p(\beta)$ in $\mathcal{C}_{\mathfrak{t}}^{*,0}/S^1$ to $\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})$:

Corollary 4.7. *Continue the hypotheses of Lemma 4.4. Let $\{\gamma_i\}$ be a basis for $H_1(X;\mathbb{Z})/\operatorname{Tor}$ and let $\{\gamma_i^*\}$ be the dual basis for $H^1(X;\mathbb{Z})$. Then if $x \in H_0(X;\mathbb{Z})$ is the positive generator, $\gamma \in H_1(X;\mathbb{R})$, $h \in H_2(X;\mathbb{R})$, and $[Y] \in H_3(X;\mathbb{R})$, the pullbacks of the cohomology classes*

$\mu_p(\beta)$ in $H^\bullet(\mathcal{C}_{\mathfrak{t}}^{*,0}/S^1;\mathbb{R})$ by the embedding $\boldsymbol{\gamma}:\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})\hookrightarrow\mathcal{C}_{\mathfrak{t}}^{*,0}/S^1$ to cohomology classes $\boldsymbol{\gamma}^*\mu_p(\beta)$ in $H^\bullet(\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s});\mathbb{R})$ are given by

$$
\begin{aligned}
\boldsymbol{\gamma}^*\mu_p([Y]) &= \sum_{i=1}^{b_1(X)} \langle (c_1(\mathfrak{s})-c_1(\mathfrak{t}))\smile\gamma_i^*,[Y]\rangle\,\mu_{\mathfrak{s}}(\gamma_i),\\
\boldsymbol{\gamma}^*\mu_p(h) &= \tfrac{1}{2}\langle c_1(\mathfrak{s})-c_1(\mathfrak{t}),h\rangle\,(2\mu_{\mathfrak{s}}(x)+\nu) - 2\sum_{i<j}\langle\gamma_i^*\smile\gamma_j^*,h\rangle\mu_{\mathfrak{s}}(\gamma_i\gamma_j),\\
\boldsymbol{\gamma}^*\mu_p(\gamma) &= -\sum_{i=1}^{b_1(X)}\langle\gamma_i^*,\gamma\rangle\,(2\mu_{\mathfrak{s}}(x)+\nu))\smile\mu_{\mathfrak{s}}(\gamma_i),\\
\boldsymbol{\gamma}^*\mu_p(x) &= -\tfrac{1}{4}\left(2\mu_{\mathfrak{s}}(x)+\nu\right)^2,
\end{aligned}
\tag{4.19}
$$

where we have written $\mu_{\mathfrak{s}}(\beta)$ for the pullback of this class to $\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})$.

Proof. Recall from [10, Lemma 2.24] that

$$
c_1(\mathbb{L}_{\mathfrak{s}}) = \mu_{\mathfrak{s}}(x)\times 1 + \sum_{i=1}^{b_1(X)}\mu_{\mathfrak{s}}(\gamma_i)\times\gamma_i^* \in H^2(\mathcal{C}_{\mathfrak{s}}^0\times X;\mathbb{R}).
$$

The identities (4.19) then follow from equation (4.18), the definition (3.3) of the cohomology classes $\mu_p(\beta) = -\frac{1}{4}p_1(\mathbb{F}_{\mathfrak{t}})/\beta$, and standard computations (compare the proof of [7, Proposition 5.1.21]). □

Finally, we compute the pullback of the class μ_c in $\mathcal{C}_{\mathfrak{t}}^{*,0}/S^1$ to $\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})$:

Lemma 4.8. *Continue the hypotheses of Lemma 4.4 and let ν be the cohomology class in Definition 4.3. Then*

$$
\boldsymbol{\gamma}^*\mu_c = \nu \in H^\bullet(\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s});\mathbb{R}).
\tag{4.20}
$$

Proof. We compute $c_1(\boldsymbol{\gamma}^*\mathbb{L}_{\mathfrak{t}})$, where $\mathbb{L}_{\mathfrak{t}}$ is the line bundle (3.10) with $c_1(\mathbb{L}_{\mathfrak{t}}) = \mu_c$, so

$$
\mathbb{L}_{\mathfrak{t}} = (\mathcal{C}_{\mathfrak{t}}^{*,0}\times\mathbb{C})/S^1,
$$

with circle action given by

$$
\left(e^{i\theta},([A,\Phi],z)\right)\to\left(e^{i\theta}[A,\Phi],e^{2i\theta}z\right) = \left(\varrho_L(e^{2i\theta})[A,\Phi],e^{2i\theta}z\right),
\tag{4.21}
$$

where ϱ_L is given by definition (4.11) and the preceding equality follows from the relation (4.12) between the circle actions. The embedding $\boldsymbol{\gamma}: N_{\mathfrak{t}}^{\varepsilon}(\Xi,\mathfrak{s})\to\mathcal{C}_{\mathfrak{t}}^{*,0}$ is S^1 equivariant with respect to scalar multiplication on the fibers of $N_{\mathfrak{t}}(\Xi,\mathfrak{s})$ and the action induced by $\varrho_L: S^1\times V\to V$ on $\mathcal{C}_{\mathfrak{t}}^{*,0}$. Hence, by definition of $\mathbb{L}_{\mathfrak{t}}$ and the S^1 equivariance of $\boldsymbol{\gamma}$, we obtain an isomorphism of complex line bundles

$$
\boldsymbol{\gamma}^*\mathbb{L}_{\mathfrak{t}} \cong (N_{\mathfrak{t}}^{\varepsilon}(\Xi,\mathfrak{s})\times\mathbb{C})/S^1,
$$

where the circle acts by

$$
(e^{i\theta},([B,\Psi,\eta],z))\mapsto([B,\Psi,e^{i\theta}\eta),e^{i\theta}z).
$$

Therefore, $\boldsymbol{\gamma}^*\mu_c = \boldsymbol{\gamma}^*c_1(\mathbb{L}_{\mathfrak{t}}) = c_1(\boldsymbol{\gamma}^*\mathbb{L}_{\mathfrak{t}}) = \nu$, as desired. □

Corollary 4.7 completes the proof of Lemma 3.15:

Proof of Assertions 2 (a), (b) and (c) in Lemma 3.15. Because the class ν is non-trivial on the fiber of the projection $\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s}) \to M_{\mathfrak{s}}$, the closures of the geometric representatives $\mathcal{V}(\beta)$, $\mathcal{V}(\gamma)$, and $\mathcal{V}(x)$ dual to the cohomology classes $\mu_p(\beta)$, $\mu_p(\gamma)$, and $\mu_p(x)$ must contain each point in $\iota(M_{\mathfrak{s}})$. Hence, the closures $\bar{\mathcal{V}}(\beta)$, $\bar{\mathcal{V}}(\gamma)$, and $\bar{\mathcal{V}}(x)$ contain $\iota(M_{\mathfrak{s}})$. □

4.3. **Euler and Segre classes.** In [10, Equation (3.48)], the homology class of the link is given by

$$[\mathbf{L}_{\mathfrak{t},\mathfrak{s}}] = e(\boldsymbol{\gamma}^*(\Xi/S^1)) \cap [\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})],$$

where Ξ/S^1 is an obstruction bundle over an open neighborhood of $\iota(M_{\mathfrak{s}}) \subset \mathcal{C}_{\mathfrak{t}}$, as defined in [10, Theorem 3.19]. In this section, we compute the Euler class of this obstruction bundle and then compute the Segre classes of $N_{\mathfrak{t}}(\Xi,\mathfrak{s})$ in order to relate intersection pairings on $\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})$ with pairings on $M_{\mathfrak{s}}$.

The obstruction bundle Ξ is given by

$$\Xi \cong \mathcal{U} \times \mathbb{C}^{r_\Xi} \to \mathcal{U},$$

where (see [10, Theorem 3.19]) $\mathcal{U}$ is a neighborhood of $M_{\mathfrak{s}}$ in $\mathcal{C}_{\mathfrak{t}}$ and the S^1 action is given by, for $[A,\Phi] \in \mathcal{U}$, $z \in \mathbb{C}^{r_\Xi}$, and ϱ_L is the map (4.11),

$$([A,\Phi],z) \to \left(\varrho_L(e^{i\theta})[A,\Phi], e^{i\theta}z\right). \tag{4.22}$$

Because the embedding $\boldsymbol{\gamma} : N_{\mathfrak{t}}^{\varepsilon}(\Xi,\mathfrak{s}) \to \mathcal{U} \subset \mathcal{C}_{\mathfrak{t}}$ is S^1 equivariant with respect to scalar multiplication on the fibers of $N_{\mathfrak{t}}(\Xi,\mathfrak{s})$ and the action induced by the map ϱ_L on $\mathcal{C}_{\mathfrak{t}}$, we have an isomorphism:

$$\boldsymbol{\gamma}^*\left(\Xi/S^1\right) \cong N_{\mathfrak{t}}^{\varepsilon}(\Xi,\mathfrak{s}) \times_{(S^1,\times -1)} \mathbb{C}^{r_\Xi},$$

where the factor -1 indicates that the S^1 action in equation (4.22) is diagonal. Thus, we can calculate the Euler class of $\boldsymbol{\gamma}^*\Xi/S^1$:

Lemma 4.9. *The vector bundle $\boldsymbol{\gamma}^*\Xi/S^1$ has Euler class $e(\boldsymbol{\gamma}^*\Xi/S^1) = \nu^{r_\Xi}$, where ν is the cohomology class in Definition 4.3, $r_\Xi = \operatorname{rank}_{\mathbb{C}}\Xi$, and*

$$[\mathbf{L}_{\mathfrak{t},\mathfrak{s}}] = \nu^{r_\Xi} \cap [\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})],$$

where $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ is given the complex orientation of Definition 2.7 and $[\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})]$ is given the orientation defined by the orientation of $TM_{\mathfrak{s}}$ and the complex orientation of the fibers.

Proof. The obstruction section vanishes transversely, so its zero locus, $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ is Poincaré dual to $e(\boldsymbol{\gamma}^*\Xi/S^1) = \nu^{r_\Xi}$ by [3, Proposition 12.8]. Note that the top Chern class is the Euler class associated to the complex orientation of the fibers of a complex vector bundle [43, Lemma 14.1 & Definition, p. 158]. □

Although the definition of Segre classes is well-known [24, p. 69], we include a definition here via the following lemma in order to make our conventions clear.

Lemma 4.10. *Let N be a complex rank-r vector bundle with Chern classes $c_i = c_i(N)$ over an oriented, real m-dimensional manifold M. Let $N^\varepsilon \subset N$ be the associated ε-sphere bundle. Define Segre classes $s_i = s_i(N) \in H^{2i}(X;\mathbb{Z})$ by the relation*

$$(1 + c_1 + c_2 + \cdots + c_r)(s_0 + s_1 + \cdots) = 1. \tag{4.23}$$

Let $\pi : \mathbb{P}(N) \to M$ be the projectivization of N and h the negative of the first Chern class of the bundle $N^\varepsilon \to \mathbb{P}(N)$. Then, for any $\alpha \in H^{m-2i}(M;\mathbb{Z})$,

$$\langle h^{r+i-1} \smile \pi^*\alpha, [\mathbb{P}(N)]\rangle = \langle s_i \smile \alpha, [M]\rangle, \tag{4.24}$$

where $\mathbb{P}(N)$ is given the orientation arising from that of M and the complex orientation of the fibers of π.

Proof. The cohomology ring of $\mathbb{P}(N)$ is given by [3, Equation (20.7)],

$$H^\bullet(\mathbb{P}(N);\mathbb{Z}) = \pi^* H^\bullet(M;\mathbb{Z})[h]/(h^r + \pi^* c_1 h^{r-1} + \cdots + \pi^* c_r).$$

We then have $h^r = -\sum_{i=1}^r \pi^* c_i h^{r-i}$. Suppose $\alpha \in H^m(M;\mathbb{Z})$, so $i=0$ in the assertion of the lemma, and α is dual to $\langle \alpha, [M]\rangle p \in H_0(M;\mathbb{Z})$, where p is a point. Then

$$\begin{aligned}\langle h^{r-1} \smile \pi^*\alpha, [\mathbb{P}(N)]\rangle &= \langle h^{r-1}, [\pi^{-1}(p)]\rangle \langle \alpha, [M]\rangle \\ &= \langle h^{r-1}, \mathbb{CP}^{r-1}]\rangle\langle \alpha,[M]\rangle = \langle \alpha, [M]\rangle.\end{aligned}$$

Because $s_0(N) = 1$ by equation (4.23), equation (4.24) holds for $i=0$. We now use induction on i and consider $\alpha \in H^{m-2i}(M,\mathbb{Z})$:

$$\begin{aligned}\langle h^{r+i-1} \smile \pi^*\alpha, [\mathbb{P}(N)]\rangle &= -\Big\langle \Big(\sum_{j=1}^i \pi^* c_j h^{r+i-1-j}\Big) \smile \pi^*\alpha, [\mathbb{P}(N)]\Big\rangle \\ &= -\sum_{j=1}^i \langle c_j s_{i-j} \smile \alpha, [M]\rangle = \langle s_i \smile \alpha, [M]\rangle.\end{aligned}$$

The last equality follows from the identity $(\sum_i c_i)(\sum_j s_j) = 1$, by equating degrees. This gives the desired relation. □

Next we compute the Segre classes of $N_{\mathfrak{t}}(\Xi,\mathfrak{s})$. Recall from [10, Equation (3.72)] that

$$\begin{aligned} n_s'(\mathfrak{t},\mathfrak{s}) &= -(c_1(\mathfrak{t}) - c_1(\mathfrak{s}))^2 - \tfrac{1}{2}(\chi+\sigma), \\ n_s''(\mathfrak{t},\mathfrak{s}) &= \tfrac{1}{8}\left((2c_1(\mathfrak{t}) - c_1(\mathfrak{s}))^2 - \sigma\right), \end{aligned} \tag{4.25}$$

where $n_s = n_s' + n_s''$ is the index of the elliptic "normal deformation operator" for the Seiberg-Witten stratum $\iota(M_{\mathfrak{s}}) \subset \mathcal{M}_{\mathfrak{t}}$.

Lemma 4.11. *Suppose that for all $\alpha, \alpha' \in H^1(X;\mathbb{Z})$ one has $\alpha \smile \alpha' = 0$. Let $\mathfrak{t}$ be a spin^u structure with $\mathfrak{t} = \mathfrak{s} \oplus \mathfrak{s}\otimes L$ and assume $M_{\mathfrak{s}}$ contains no zero-section pairs. Then the bundle $N_{\mathfrak{t}}(\Xi,\mathfrak{s}) \to M_{\mathfrak{s}}$ has Segre classes*

$$s_i(N_{\mathfrak{t}}(\Xi,\mathfrak{s})) = \mu_{\mathfrak{s}}(x)^i \sum_{j=1}^i 2^j \binom{-n_s'}{j}\binom{-n_s''}{i-j}, \qquad i = 0,1,2,\ldots. \tag{4.26}$$

Proof. With the hypothesis on $H^1(X;\mathbb{Z})$, Corollary 3.30 in [10] asserts that $N_{\mathfrak{t}}(\Xi,\mathfrak{s})$ has total Chern class

$$c(N_{\mathfrak{t}}(\Xi,\mathfrak{s})) = (1 + 2\mu_{\mathfrak{s}}(x))^{n_s'}(1+\mu_{\mathfrak{s}}(x))^{n_s''}.$$

As described in Lemma 4.10, the total Segre class $s = s_0 + s_1 + s_2 + \cdots$ is the formal inverse of the total Chern class $c = 1 + c_1 + c_2 + \cdots$, so

$$s(N_{\mathfrak{t}}(\Xi,\mathfrak{s})) = (1 + 2\mu_{\mathfrak{s}}(x))^{-n_s'}(1+\mu_{\mathfrak{s}}(x))^{-n_s''}.$$

The lemma follows by computing the formal power series expansions for the above expression, using equation (2) in Lemma 4.16 to simplify before multiplying the two series. □

Remark 4.12. The assumption $\alpha \smile \alpha' = 0$ in Lemma 4.11 is used to simplify the expression for the Chern character of $N_{\mathfrak{t}}(\Xi,\mathfrak{s})$ computed in [10, Theorem 3.29]. Without this assumption, the universal expression of the Segre classes in terms of the Chern character given in [39, Equation (2.11)] and Theorem 3.29 of [10] still show that the Segre classes of $N_{\mathfrak{t}}(\Xi,\mathfrak{s})$ are expressible in terms of the $\mu_{\mathfrak{s}}$-classes, though not as explicitly.

4.4. **Computation of the intersection pairing.** We now compute intersection pairings with $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ of the type encountered in the cobordism formula (3.70). Some combinatorial factors appearing in this computation can be expressed in terms of the *Jacobi polynomials* [29, §8.96], which are defined by

$$P_n^{a,b}(\xi) = \frac{1}{2^n}\sum_k \binom{a+n}{n-k}\binom{n+b}{k}(\xi-1)^k(\xi+1)^{n-k}, \quad \xi \in \mathbb{C}. \tag{4.27}$$

Functional relations, relations with other special functions, and the generating function for the Jacobi polynomials can be found in [29, pp. 1034–1035].

Theorem 4.13. *Let X be a four-manifold with $\alpha \smile \alpha' = 0$ for every $\alpha, \alpha' \in H^1(X;\mathbb{Z})$, and let Ω be a homology orientation. Let $\mathfrak{s}$ and $\mathfrak{t}$ be a spin^c and spin^u structure on X for which $\mathfrak{t} = \mathfrak{s} \oplus \mathfrak{s} \otimes L$, and assume $M_{\mathfrak{s}}$ contains no zero-section pairs. Give $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ the complex orientation, determined by the orientation for $M_{\mathfrak{s}}$ fixed by the homology orientation Ω as in Definition 2.5. Let $z \in \mathbb{A}(X)$ and let δ_c be a non-negative integer satisfying*

$$\deg(z) + 2\delta_c = \dim(\mathcal{M}_{\mathfrak{t}}^{*,0}/S^1) - 1.$$

If $z = z'Y$ for some $Y \in H_3(X;\mathbb{Z})$ and $z' \in \mathbb{A}(X)$, then

$$\langle \mu_p(z) \smile \mu_c^{\delta_c}, [\mathbf{L}_{\mathfrak{t},\mathfrak{s}}]\rangle = 0. \tag{4.28}$$

If $z = x^{\delta_0}\vartheta h^{\delta_2}$, where $h \in H_2(X;\mathbb{R})$, $\vartheta \in \Lambda^{\delta_1}(H_1(X;\mathbb{R}))$, and $x \in H_0(X;\mathbb{Z})$ is the positive generator, then $d_s(\mathfrak{s}) \equiv \delta_1 \pmod 2$ and if we set

$$2d = d_s(\mathfrak{s}) - \delta_1, \tag{4.29}$$

then

$$\begin{aligned}\Big\langle \mu_p(z) \smile \mu_c^{\delta_c}, [\mathbf{L}_{\mathfrak{t},\mathfrak{s}}]\Big\rangle &= (-1)^{\delta_0+\delta_1}2^{-\delta_2-2\delta_0}C_{\chi,\sigma}\left(\deg(z), \delta_c, d_a(\mathfrak{t}), d_s(\mathfrak{s}), \delta_1\right) \\ &\quad \times \Big\langle \mu_{\mathfrak{s}}(x^d\vartheta), [M_{\mathfrak{s}}]\Big\rangle \langle c_1(\mathfrak{s}) - c_1(\mathfrak{t}), h\rangle^{\delta_2},\end{aligned} \tag{4.30}$$

where

$$C_{\chi,\sigma}\left(\deg(z), \delta_c, d_a(\mathfrak{t}), d_s(\mathfrak{s}), \delta_1\right) = (-2)^d P_d^{a,b}(0),$$

for

$$a = \delta_c - d - 1 \quad \text{and} \quad b = \tfrac{1}{2}\left(\deg(z) - d_a(\mathfrak{t}) - d_s(\mathfrak{s})\right) - \tfrac{1}{4}\left(\chi + \sigma\right).$$

If $d = 0$, then $C_{\chi,\sigma} = 1$.

Remark 4.14. If $\delta_1 > d_s(\mathfrak{s})$, then the pairing $\langle \mu_{\mathfrak{s}}(x^d\vartheta), [M_{\mathfrak{s}}]\rangle$ vanishes and so the pairing (4.30) also vanishes.

Proof of Theorem 4.13. By the multilinearity of the pairing, we can assume that ϑ is a monomial, $\vartheta = \gamma_1 \cdots \gamma_{\delta_1}$, where $\{\gamma_1, \dots, \gamma_{\delta_1}\}$ is a subset of a basis for $H_1(X;\mathbb{Z})/\operatorname{Tor}$. Extend it to a basis and let $\{\gamma_i^*\}$ be a dual basis for $H^1(X;\mathbb{R})$, so $\langle \gamma_i^*, \gamma_j\rangle = \delta_{ij}$.

Suppose $z = z'Y$ for $Y \in H_3(X;\mathbb{Z})$ and $z' \in \mathbb{A}(X)$. The expression for $\mu_p([Y])$ in equation (4.19) is a sum of terms of the form

$$\langle (c_1(\mathfrak{s}) - c_1(\mathfrak{t})) \smile \gamma_i^*, [Y]\rangle \mu_{\mathfrak{s}}(\gamma_i) = \langle (c_1(\mathfrak{s}) - c_1(\mathfrak{t})) \smile \gamma_i^* \smile \mathrm{PD}[Y], [X]\rangle \mu_{\mathfrak{s}}(\gamma_i),$$

which vanish by our hypothesis on $H^1(X;\mathbb{Z})$. This yields identity (4.28).

The integers δ_i and δ_c satisfy

$$\begin{aligned}2\delta_2 + 3\delta_1 + 4\delta_0 + 2\delta_c &= \deg(z) + 2\delta_c = \dim(\mathcal{M}_{\mathfrak{t}}^{*,0}/S^1) - 1 \\ &= d_s(\mathfrak{s}) + 2n_s' + 2n_s'' - 2.\end{aligned} \tag{4.31}$$

Thus $\delta_1 \equiv d_s(\mathfrak{s}) \pmod 2$ and $d = \frac{1}{2}(d_s(\mathfrak{s}) - \delta_1)$ is an integer. We use Corollary 4.7, the comments following Lemma 3.23 in [10], and Lemma 4.9 to write the pairing in equation (4.30) as

$$
\begin{aligned}
&\left\langle \mu_p(h^{\delta_2}\vartheta x^{\delta_0})\mu_c^{\delta_c}, [\mathbf{L}_{\mathfrak{t},\mathfrak{s}}]\right\rangle \\
&= 2^{-\delta_2}\langle c_1(\mathfrak{s}) - c_1(\mathfrak{t}), h\rangle^{\delta_2} \\
&\quad \times \left\langle (2\mu_{\mathfrak{s}}(x) + \nu)^{\delta_2}\left(\prod_{k=1}^{\delta_1}(-2\mu_{\mathfrak{s}}(x) - \nu)\mu_{\mathfrak{s}}(\gamma_k)\right)\left(-\tfrac{1}{4}(2\mu_{\mathfrak{s}}(x)+\nu)^2\right)^{\delta_0}\nu^{\delta_c}, \right. \\
&\qquad \left. \nu^{r_\Xi} \cap [\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})]\right\rangle .
\end{aligned} \tag{4.32}
$$

Write $C_1 = (-1)^{\delta_1+\delta_0}2^{-\delta_2-2\delta_0}\langle c_1(\mathfrak{s}) - c_1(\mathfrak{t}), h\rangle^{\delta_2}$ and define

$$
\delta_p = \tfrac{1}{2}(\deg(z) - \delta_1) = \delta_2 + \delta_1 + 2\delta_0. \tag{4.33}
$$

The pairing (4.32) is then equal to

$$
\begin{aligned}
&C_1\left\langle (2\mu_{\mathfrak{s}}(x)+\nu)^{\delta_p}\mu_{\mathfrak{s}}(\vartheta)\nu^{\delta_c + r_\Xi}, [\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})]\right\rangle \\
&\qquad = C_1\left\langle \sum_{i=0}^{\delta_p} 2^i\binom{\delta_p}{i}\mu_{\mathfrak{s}}(x^i\vartheta)\nu^{\delta_p+\delta_c+r_\Xi - i}, [\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})]\right\rangle .
\end{aligned} \tag{4.34}
$$

Then $\mu_{\mathfrak{s}}(x^i\vartheta) \in H^{d_s(\mathfrak{s})-2(d-i)}(M_{\mathfrak{s}};\mathbb{Z})$ and

$$
\begin{aligned}
\delta_p + \delta_c + r_\Xi - i &= \delta_2 + \delta_1 + 2\delta_0 + \delta_c + r_\Xi - i \quad (\text{as } \delta_p = \delta_2 + \delta_1 + 2\delta_0) \\
&= n_s' + n_s'' + r_\Xi + \tfrac{1}{2}(d_s(\mathfrak{s}) - \delta_1) - 1 - i \quad (\text{Equation (4.31)}) \\
&= n_s' + n_s'' + r_\Xi + (d-i) - 1 \quad (\text{as } d = \tfrac{1}{2}(d_s(\mathfrak{s}) - \delta_1)).
\end{aligned}
$$

We use the preceding equation, the Segre class relation (4.24), and the formulas (4.26) for the Segre classes $s_i(N_{\mathfrak{t}}(\Xi,\mathfrak{s}))$ to calculate

$$
\begin{aligned}
\left\langle \mu_{\mathfrak{s}}(x^i\vartheta)\nu^{\delta_p+\delta_c+r_\Xi-i}, [\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})]\right\rangle &= \left\langle \mu_{\mathfrak{s}}(x^i\vartheta)s_{d-i}, [M_{\mathfrak{s}}]\right\rangle \\
&= \sum_{j=0}^{d-i} 2^j\binom{-n_s'}{j}\binom{-n_s''}{d-i-j}\left\langle \mu_{\mathfrak{s}}(x^d\vartheta), [M_{\mathfrak{s}}]\right\rangle .
\end{aligned} \tag{4.35}
$$

Writing $C_2 = C_1\langle \mu_{\mathfrak{s}}(x^d\vartheta), [M_{\mathfrak{s}}]\rangle$ and substituting the formula (4.35) into equation (4.34) yields a simplified expression for that pairing:

$$
\begin{aligned}
&C_2\sum_{i=0}^{d} 2^i\binom{\delta_p}{i}\sum_{j=0}^{d-i} 2^j\binom{-n_s'}{j}\binom{-n_s''}{d-i-j} \\
&\qquad = C_2\sum_{i=0}^{d}\sum_{j=0}^{d-i} 2^{i+j}\binom{\delta_p}{i}\binom{-n_s'}{j}\binom{-n_s''}{d-i-j}.
\end{aligned} \tag{4.36}
$$

If we write $u = i + j$, then the pairing (4.36) becomes

$$\begin{aligned} C_2 \sum_{u=0}^{d} \sum_{i=0}^{u} 2^u \binom{\delta_p}{i} \binom{-n'_s}{u-i} \binom{-n''_s}{d-u} &= C_2 \sum_{u=0}^{d} 2^u \binom{-n''_s}{d-u} \sum_{i=0}^{u} \binom{\delta_p}{i} \binom{-n'_s}{u-i} \\ &= C_2 \sum_{u=0}^{d} 2^u \binom{-n''_s}{d-u} \binom{\delta_p - n'_s}{u}, \end{aligned} \tag{4.37}$$

where the second equality follows from the Vandermonde convolution identity (see equation (5) in Lemma 4.16). Equation (4.30) will follow from equation (4.37) and Lemma 4.18 which expresses the last sum in equation (4.37) in terms of the Jacobi polynomial given by $C_{\chi,\sigma}(\deg(z), \delta_c, d_a(\mathfrak{t}), d_s(\mathfrak{s}), \delta_1)$. This completes the proof of the theorem. □

Remark 4.15. The proof of Theorem 4.13 implies that Conjecture 3.1 of [8] holds for level-zero reducibles, even without the assumption that $\alpha \smile \alpha' = 0$ for every $\alpha, \alpha' \in H^1(X;\mathbb{Z})$. We only used this condition on $H^1(X;\mathbb{Z})$ in equation (4.35) in order to apply the Segre class computations of Lemma 4.11. If the condition on $H^1(X;\mathbb{Z})$ is is omitted, then—as noted in Remark 4.12—the Segre classes can still be computed in terms of $\mu_{\mathfrak{s}}$-classes, though less explicitly. In the general situation, the pairing (4.30) could still be expressed in terms of a pairing of $\mu_{\mathfrak{s}}$-classes with $M_{\mathfrak{s}}$ and, when $c_1(\mathfrak{s})$ is not a basic class, the pairing (4.30) would be zero.

Before proving the relation between the combinatorial expression in equation (4.37) and the Jacobi polynomial used at the end of the proof of Theorem 4.13, it is convenient to collect the combinatorial identities we shall need here. For $a \in \mathbb{R}$ and $n \in \mathbb{N}$, define

$$(a)_n = a(a+1)\cdots(a+n-1), \quad \text{and} \quad (a)_0 = 1. \tag{4.38}$$

We then have:

Lemma 4.16. [38, p. 9] *Let $a, b \in \mathbb{R}$ and let $k, n \in \mathbb{N}$ be non-negative integers. Then:*

$$(a)_k = (-1)^k (1-a-k)_k, \tag{1}$$

$$\binom{a}{k} = \frac{(-1)^k(-a)_k}{k!}, \tag{2}$$

$$(a)_{n-k} = \frac{(-1)^k (a)_n}{(1-a-n)_k}, \quad n \geq k, \tag{3}$$

$$(n-k)! = \frac{n!}{(-1)^k(-n)_k}, \tag{4}$$

$$\sum_{i=0}^{u} \binom{a}{i}\binom{b}{u-i} = \binom{a+b}{u}. \tag{5}$$

Identity (5) in Lemma 4.16 (the Vandermonde convolution identity) follows by comparing coefficients in binomial expansions of the two sides of the identity $(x+y)^a(x+y)^b = (x+y)^{a+b}$.

Remark 4.17. Note that Identity (2) in Lemma 4.16 allows one to extend the definition of a binomial coefficient $\binom{n}{r}$ to the case when $n \leq 0$. If $r = 0$, the identity $\binom{n}{r} = 1$ still holds.

We now prove the relation between the combinatorial expression in equation (4.37) and the Jacobi polynomial.

Lemma 4.18. *Continue the notation of Theorem 4.13. Then*

$$\sum_{u=0}^{d} 2^u \binom{-n_s''}{d-u}\binom{\delta_p - n_s'}{u} = (-2)^d P_d^{a,b}(0),$$

where δ_p *is defined in equation* (4.33) *and*

$$a = \delta_c - d - 1 \quad \text{and} \quad b = \tfrac{1}{2}\left(\deg(z) - d_a(\mathfrak{t}) - d_s(\mathfrak{s})\right) - \tfrac{1}{4}\left(\chi + \sigma\right).$$

Proof. We first recall that the *hypergeometric functions* [29, §9.10] are defined by

$$ {}_2F_1(a,b;c;\xi) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!}\xi^k, \quad \xi \in \mathbb{C}. \tag{4.39}$$

We shall use the following identities [38, Equation (23), p. 40 & Equation (2), p. 274]:

$$\begin{aligned} {}_2F_1(-m,b;c;\xi) &= \frac{(b)_m(-1)^m \xi^m}{(c)_m} {}_2F_1(-m, 1-m-c; 1-m-b; \xi^{-1}), \\ P_n^{a,b}(\xi) &= \frac{(-1)^n (b+1)_n}{n!} {}_2F_1(-n, n+a+b+1; b+1; \tfrac{1}{2}(1+\xi)). \end{aligned} \tag{4.40}$$

By equation (2) of Lemma 4.16, the combinatorial expression in equation (4.37) can be written as

$$\begin{aligned} C(n_s'', n_s', \delta_p, d) &:= \sum_{u=0}^{d} 2^u \binom{-n_s''}{d-u}\binom{\delta_p - n_s'}{u} \\ &= \sum_{u=0}^{d} 2^u (-1)^{d-u} \frac{(n_s'')_{d-u}}{(d-u)!}(-1)^u \frac{(n_s' - \delta_p)_u}{u!}. \end{aligned} \tag{4.41}$$

Applying equation (3) of Lemma 4.16 to $(n_s'')_{d-u}$ and equation (4) to $(d-u)!$ yields

$$\begin{aligned} C(n_s'', n_s', \delta_p, d) &= (-1)^d \sum_{u=0}^{d} 2^u \frac{(-1)^u (n_s'')_d (-1)^u (-d)_u (n_s' - \delta_p)_u}{(1 - n_s'' - d)_u (d)! u!} \\ &= \frac{(-1)^d (n_s'')_d}{(d)!} \sum_{u=0}^{d} \frac{(-d)_u (n_s' - \delta_p)_u}{(1 - n_s'' - d)_u u!} 2^u \\ &= \frac{(-1)^d (n_s'')_d}{(d)!} {}_2F_1(-d, n_s' - \delta_p; 1 - n_s'' - d; 2). \end{aligned} \tag{4.42}$$

Substituting the first identity in equation (4.40) into equation (4.42) gives

$$C(n_s'', n_s', \delta_p, d) = \frac{(n_s'')_d 2^d (n_s' - \delta_p)_d}{(d)!(1 - n_s'' - d)_d} {}_2F_1(-d, n_s''; \delta_p - n_s' - d + 1; \tfrac{1}{2}). \tag{4.43}$$

By substituting equation (1) from Lemma 4.16 into equation (4.43) we obtain

$$C(n_s'', n_s', \delta_p, d) = \frac{2^d (-1)^d (n_s' - \delta_p)_d}{(d)!} {}_2F_1(-d, n_s''; \delta_p - n_s' - d + 1; \tfrac{1}{2}). \tag{4.44}$$

Applying the second identity in (4.40) to equation (4.44) yields

$$C(n_s'', n_s', \delta_p, d) = \frac{2^d (n_s' - \delta_p)_d (d)!}{(d)!(\delta_p - n_s' - d + 1)_d} P_d^{n_s'' + n_s' - \delta_p - 1, \delta_p - n_s' - d}(0). \tag{4.45}$$

By applying equation (1) from Lemma 4.16, we can then simplify the right-hand side of equation (4.45) to give

$$C(n''_s, n'_s, \delta_p, d) = (-2)^d P_d^{n''_s+n'_s-\delta_p-1,\delta_p-n'_s-d}(0). \tag{4.46}$$

Then, the equality

$$d_s(\mathfrak{s}) + 2n'_s + 2n''_s - 2 = \dim(\mathcal{M}_{\mathfrak{t}}^{*,0}/S^1) - 1 = \deg(z) + 2\delta_c = 2\delta_p + \delta_1 + 2\delta_c$$

and $d_s(\mathfrak{s}) = 2d + \delta_1$ imply

$$n''_s + n'_s - \delta_p - 1 = \delta_c - d - 1. \tag{4.47}$$

The definition (4.25) of n'_s implies that

$$n'_s = \tfrac{1}{2}d_a(\mathfrak{t}) + \tfrac{1}{4}(\chi+\sigma),$$

which, together with the identities $d = \frac{1}{2}(d_s(\mathfrak{s}) - \delta_1)$ and $\delta_p = \frac{1}{2}(\deg(z) - \delta_1)$, yields

$$\delta_p - n'_s - d = \tfrac{1}{2}\left(\deg(z) - d_a(\mathfrak{t}) - d_s(\mathfrak{s})\right) - \tfrac{1}{4}\left(\chi+\sigma\right). \tag{4.48}$$

Substituting equations (4.47) and (4.48) into equation (4.46) then completes the proof. □

4.5. **A blow-up formula for Seiberg-Witten link pairings.** Our formula (4.30) in Theorem 4.13 for pairings with Seiberg-Witten links $[\mathbf{L}_{\mathfrak{t},\mathfrak{s}}]$ only applies when $M_{\mathfrak{s}}$ contains no zero-section pairs. In the same vein, our formula (3.70) in Theorem 3.33 for $\#(\bar{\mathcal{V}}(z)\cap \bar{M}_\kappa^w)$ in terms of pairings with $[\mathbf{L}_{\mathfrak{t},\mathfrak{s}}]$ only applies when $w_2(\mathfrak{t}) \equiv w \pmod 2$ and $w \pmod 2$ is good; when $\mathfrak{t} = \mathfrak{s}\oplus\mathfrak{s}'$ and $w_2(\mathfrak{t})$ is good, then $M_{\mathfrak{s}}$ contains no zero-section pairs. The definition (3.31) of the Donaldson invariants $D_X^w(z)$ incorporates the blow-up formula in order to remove any such constraint on w. Therefore, we derive a "blow-up" formula for the Seiberg-Witten link pairings which, together with equations (3.70) and (4.30), will allow us to compute $D_X^w(z)$ for arbitrary $w\in H^2(X;\mathbb{Z})$.

As before, we let $\tilde{X} = X\#\overline{\mathbb{CP}}^2$ be the blow-up of X with exceptional class $e\in H_2(\tilde{X};\mathbb{Z})$ and denote its Poincaré dual by $\mathrm{PD}[e]\in H^2(\tilde{X};\mathbb{Z})$. We first need to relate spin^c structures on X with those on $\tilde{X}$. Because $\overline{\mathbb{CP}}^2$ is simply-connected, the following map is a bijection:

$$\mathrm{Spin}^c(\overline{\mathbb{CP}}^2) \to \{(2k-1)\,\mathrm{PD}[e] : k\in\mathbb{Z}\} \subset H^2(\overline{\mathbb{CP}}^2;\mathbb{Z}), \quad \mathfrak{s}\mapsto c_1(\mathfrak{s}).$$

Let $\mathfrak{s}_{2k-1}$ denote the spinc structure on $\overline{\mathbb{CP}}^2$ with $c_1(\mathfrak{s}_{2k-1}) = (2k-1)\,\mathrm{PD}[e]$. By the discussion in [52, §12.4], a spin^c structure $\mathfrak{s}$ on X and $\mathfrak{s}_{2k-1}$ on $\overline{\mathbb{CP}}^2$ can be spliced together to yield a spin^c structure $\mathfrak{s}\#\mathfrak{s}_{2k-1}$ on $\tilde{X}$ with

$$c_1(\mathfrak{s}\#\mathfrak{s}_{2k-1}) = c_1(\mathfrak{s}) + (2k-1)\,\mathrm{PD}[e]. \tag{4.49}$$

Moreover, every spin^c structure on $\tilde{X}$ can be realized in this way. The dimensions of the Seiberg-Witten moduli spaces are related by

$$\begin{aligned} d_s(\mathfrak{s}\#\mathfrak{s}_{2k-1}) &= \tfrac{1}{4}\left(c_1(\mathfrak{s}\#\mathfrak{s}_{2k-1})^2 - 2\tilde{\chi} - 3\tilde{\sigma}\right) \\ &= d_s(\mathfrak{s}) - k(k-1), \end{aligned} \tag{4.50}$$

where $\tilde{\chi} = \chi+1$ denotes the Euler characteristic of $\tilde{X}$ and $\tilde{\sigma} = \sigma - 1$ is the signature of $\tilde{X}$. We now define a spin^u structure $\tilde{\mathfrak{t}}$ on $\tilde{X}$ related to a spin^u structure $\mathfrak{t}$ on X, and relate reducible PU(2) monopoles in $\bar{\mathcal{M}}_{\tilde{\mathfrak{t}}}$ to those in $\bar{\mathcal{M}}_{\mathfrak{t}}$.

Lemma 4.19. *Let $\mathfrak{t}$ be a spin^u structure on X with the property that reducible* PU(2) *monopoles in $\bar{\mathcal{M}}_{\mathfrak{t}}$ appear only in the top level $\mathcal{M}_{\mathfrak{t}}$. For $\tilde{X} = X\#\overline{\mathbb{CP}}^2$, let $e\in H_2(\tilde{X};\mathbb{Z})$ be the exceptional class, and let $\mathrm{PD}[e]\in H^2(\tilde{X};\mathbb{Z})$ be its Poincaré dual. Then we have:*

1. *There is a spin^u structure $\tilde{\mathfrak{t}}$ on $\tilde{X}$ satisfying*
$$c_1(\tilde{\mathfrak{t}}) = c_1(\mathfrak{t}), \quad p_1(\tilde{\mathfrak{t}}) = p_1(\mathfrak{t}) - 1, \quad \textit{and} \quad w_2(\tilde{\mathfrak{t}}) \equiv w_2(\mathfrak{t}) + \mathrm{PD}[e] \pmod 2.$$
2. *The reducible* $\mathrm{PU}(2)$ *monopoles in $\bar{\mathcal{M}}_{\tilde{\mathfrak{t}}}$ appear only in the top level, $\mathcal{M}_{\tilde{\mathfrak{t}}}$, and are defined by spin^c structures $\mathfrak{s}^\pm$ on $\tilde{X}$ with $c_1(\mathfrak{s}^\pm) = c_1(\mathfrak{s}) \pm \mathrm{PD}[e]$, where $\iota(M_\mathfrak{s}) \subset \mathcal{M}_\mathfrak{t}$.*
3. *Suppose we relax the assumption that reducible* $\mathrm{PU}(2)$ *monopoles in $\bar{\mathcal{M}}_\mathfrak{t}$ appear only in the top level $\mathcal{M}_\mathfrak{t}$, to the assumption that reducible* $\mathrm{PU}(2)$ *monopoles in $\bar{\mathcal{M}}_\mathfrak{t}$ with non-trivial Seiberg-Witten functions appear only in the top level $\mathcal{M}_\mathfrak{t}$. Then reducible* $\mathrm{PU}(2)$ *monopoles in $\bar{\mathcal{M}}_{\tilde{\mathfrak{t}}}$ with non-trivial Seiberg-Witten functions appear only in the top-level $\mathcal{M}_{\tilde{\mathfrak{t}}}$.*

Proof. Suppose $\mathfrak{t} = (\rho, V)$. By Lemma 2.3 in [10], we may assume $V = W \otimes E$, where $\mathfrak{s} = (\rho, W)$ is a spin^c structure on X and $E \to X$ is a complex rank-two vector bundle. Let $\tilde{E} \to \tilde{X}$ be the complex, rank-two bundle with $c_1(\tilde{E}) = c_1(E) + \mathrm{PD}[e]$ and $c_2(\tilde{E}) = c_2(E)$, and let $\tilde{\mathfrak{s}} = (\tilde{\rho}, \tilde{W})$ be the spin^c structure on $\tilde{X}$ with $c_1(\tilde{\mathfrak{s}}) = c_1(\mathfrak{s}) - \mathrm{PD}[e]$. Then set $\tilde{V} = \tilde{W} \otimes \tilde{E}$ and $\tilde{\mathfrak{t}} = (\tilde{\rho}, \tilde{V})$, and observe that $\tilde{\mathfrak{t}}$ has the desired characteristic classes.

By the neck-stretching argument described in [19], the only non-empty Seiberg-Witten moduli spaces on $\tilde{X}$ are defined by spin^c structures $\mathfrak{s}\#\mathfrak{s}_{2k-1}$, where $M_\mathfrak{s}$ is non-empty. Since $\iota(M_\mathfrak{s}) \subset \mathcal{M}_\mathfrak{t}$ by hypothesis, equation (3.63) implies that $c_1(\mathfrak{s})$ obeys
$$(c_1(\mathfrak{s}) - c_1(\mathfrak{t}))^2 = p_1(\mathfrak{t}).$$
To see which Seiberg-Witten moduli spaces $\iota(M_{\mathfrak{s}\#\mathfrak{s}_{2k-1}})$ can be contained in $\bar{\mathcal{M}}_\mathfrak{t}$ and in which level, we need to check the corresponding equation for $c_1(\mathfrak{s}\#\mathfrak{s}_{2k-1})$. Equation (4.49) and the relations between the characteristic classes of $\mathfrak{t}$ and $\tilde{\mathfrak{t}}$ yields
$$\begin{aligned}\left(c_1(\mathfrak{s}\#\mathfrak{s}_{2k-1}) - c_1(\tilde{\mathfrak{t}})\right)^2 &= (c_1(\mathfrak{s}) + (2k-1)\,\mathrm{PD}[e] - c_1(\mathfrak{t}))^2 \\ &= p_1(\mathfrak{t}) - (2k-1)^2 \\ &= p_1(\tilde{\mathfrak{t}}) - 4k(k-1).\end{aligned}$$
Restricted to integers, the function $-4k(k-1)$ takes its maximum value at $k=0$ and $k=1$. Hence, only the spaces $\iota(M_{\mathfrak{s}\#\mathfrak{s}_{2k-1}})$ with $k = 0, 1$, appear in $\bar{\mathcal{M}}_{\tilde{\mathfrak{t}}}$, as all other spin^c structures $\mathfrak{s}\#\mathfrak{s}_{2k-1}$ would require an $\mathrm{SO}(3)$ bundle with Pontrjagin class smaller than $p_1(\tilde{\mathfrak{t}})$. ☐

Theorem 4.20. [50, Theorem 3.2], [19, Theorem 1.4] *Let X be a four-manifold, and let $\tilde{X} = X\#\overline{\mathbb{CP}}^2$ denote its blow-up, with exceptional class $e \in H_2(\tilde{X};\mathbb{Z})$. If $b_2^+(X) > 1$, then for each spin^c structure $\tilde{\mathfrak{s}}$ on $\tilde{X}$ with $d_s(\tilde{\mathfrak{s}}) \geq 0$ and each $z \in \mathbb{B}(X) \cong \mathbb{B}(\tilde{X})$, we have*
$$SW_{\tilde{X},\tilde{\mathfrak{s}}}(z) = SW_{X,\mathfrak{s}}(x^m z), \tag{4.51}$$
where $\mathfrak{s}$ is the spin^c structure induced on X by restriction, and $2m = d_s(\mathfrak{s}) - d_s(\tilde{\mathfrak{s}})$. If $b_2^+(X) = 1$ and $c_1(\mathfrak{s}) - \Lambda$ is not torsion, there is a one-to-one correspondence between $(c_1(\mathfrak{s}) - \Lambda)$-chambers in the positive cone of $H^2(X;\mathbb{R})$ and $(c_1(\tilde{\mathfrak{s}}) - \Lambda)$-chambers in the positive cone of $H^2(\tilde{X};\mathbb{R})$, and the above relation holds provided both invariants are calculated in related chambers.

Remark 4.21. The presence of the class Λ in the hypotheses of Theorem 4.20 when describing the chambers arises because of the nature of the fixed perturbation used in our definition of the Seiberg-Witten moduli spaces; see the discussion in §4.1.

Lemma 4.19, the blow-up formula for Seiberg-Witten invariants (Theorem 4.20), and Theorem 3.33 then yield the following "blow-up" formula for Seiberg-Witten link pairings:

Proposition 4.22. *Continue the hypotheses and notation of Theorem 4.13 leading to equation* (4.30)*, except we omit the requirement that $M_{\mathfrak{s}}$ contains no zero-section pairs and define z as given below. Let $\tilde X = X\#\overline{\mathbb{CP}}^2$ be the blow-up, let $e \in H_2(\tilde X;\mathbb{Z})$ be the exceptional class, and let $\mathrm{PD}[e] \in H^2(\tilde X;\mathbb{Z})$ be its Poincaré dual. Let $z = x^{\delta_0}\vartheta h^{\delta_2-k} \in \mathbb{A}(X) \subset \mathbb{A}(\tilde X)$. Let $\mathfrak{t}$ and $\tilde{\mathfrak{t}}$ be related spin^u structures on X and $\tilde X$, respectively, as in Lemma 4.19. Then, for k even,*

$$
\begin{aligned}
&(-1)^{o_{\tilde{\mathfrak{t}}}(w+\mathrm{PD}[e],\mathfrak{s}^+)}\left\langle \mu_p(e^{k+1}z)\smile \mu_c^{\delta_c},[\mathbf{L}_{\tilde{\mathfrak{t}},\mathfrak{s}^+}]\right\rangle\\
&+(-1)^{o_{\tilde{\mathfrak{t}}}(w+\mathrm{PD}[e],\mathfrak{s}^-)}\left\langle \mu_p(e^{k+1}z)\smile \mu_c^{\delta_c},[\mathbf{L}_{\tilde{\mathfrak{t}},\mathfrak{s}^-}]\right\rangle\\
&= (-1)^{o_{\mathfrak{t}}(w,\mathfrak{s})+\delta_0+\delta_1}2^{-\delta_2-2\delta_0}C_{\chi,\sigma}\left(\deg(z)+2k,\delta_c,d_a(\mathfrak{t}),d_s(\mathfrak{s}),\delta_1\right)\\
&\qquad\times SW_{X,\mathfrak{s}}(x^d\vartheta)\langle c_1(\mathfrak{s})-c_1(\mathfrak{t}),h\rangle^{\delta_2-k},
\end{aligned}
\tag{4.52}
$$

while the left-hand side is zero if k is odd or z is replaced by $z'Y$ and $Y \in H_3(X;\mathbb{Z})$.

Proof. The result follows by applying equation (4.30) in Theorem 4.13 to the links $\mathbf{L}_{\tilde{\mathfrak{t}},\mathfrak{s}^\pm}$ of $M_{\mathfrak{s}^\pm}(\tilde X)$ in $\mathcal{M}_{\tilde{\mathfrak{t}}}(\tilde X)$, together with the following observations.

The vanishing result in the case $z = z'Y$ follows immediately from equation (4.28).

Because $d_s(\mathfrak{s}^\pm) = d_s(\mathfrak{s})$ by equation (4.50), and $d_a(\tilde{\mathfrak{t}}) = d_a(\mathfrak{t}) + 2$ by equation (3.21) (noting that $p_1(\mathfrak{t}) = p_1(\tilde{\mathfrak{t}}) - 1$ from Lemma 4.19), and $\tilde\chi + \tilde\sigma = \chi + \sigma$, we have:

$$
C_{\tilde\chi,\tilde\sigma}(\deg(ze^{2k+1}),\delta_c,d_a(\tilde{\mathfrak{t}}),d_s(\mathfrak{s}^\pm),\delta_1) = C_{\chi,\sigma}\left(\deg(z)+2k,\delta_c,d_a(\mathfrak{t}),d_s(\mathfrak{s}),\delta_1\right). \tag{4.53}
$$

The proof (see [19, §4]) of the blow-up formula, Theorem 4.20, gives an identity

$$
\left\langle \mu_{\mathfrak{s}^+}(x^d\vartheta),[M_{\mathfrak{s}^+}(\tilde X)]\right\rangle = \left\langle \mu_{\mathfrak{s}^-}(x^d\vartheta),[M_{\mathfrak{s}^-}(\tilde X)]\right\rangle,
$$

and thus our definition (4.5) of the Seiberg-Witten invariants yields

$$
\left\langle \mu_{\mathfrak{s}^\pm}(x^d\vartheta),[M_{\mathfrak{s}^\pm}(\tilde X)]\right\rangle = SW_{X,\mathfrak{s}}(x^d\vartheta).
$$

Noting that $c_1(\tilde{\mathfrak{t}}) = c_1(\mathfrak{t})$ by Lemma 4.19, the product

$$
\langle c_1(\mathfrak{s}^\pm)-c_1(\tilde{\mathfrak{t}}),e\rangle^{k+1}\langle c_1(\mathfrak{s}^\pm)-c_1(\mathfrak{t}),h\rangle^{\delta_2-k}
$$

appearing in equation (4.30) can be simplified to

$$
\langle c_1(\mathfrak{s}^\pm)-c_1(\tilde{\mathfrak{t}}),e\rangle^{k+1}\langle c_1(\mathfrak{s}^\pm)-c_1(\tilde{\mathfrak{t}}),h\rangle^{\delta_2-k} = (\mp 1)^{k+1}\langle c_1(\mathfrak{s})-c_1(\mathfrak{t}),h\rangle^{\delta_2-k}. \tag{4.54}
$$

From its definition (3.66), the orientation term is given by

$$
\begin{aligned}
o_{\tilde{\mathfrak{t}}}(w+\mathrm{PD}[e],\mathfrak{s}^\pm) &= \tfrac{1}{4}\left(w+\mathrm{PD}[e]-c_1(\tilde{\mathfrak{t}})+c_1(\mathfrak{s}^\pm)\right)^2\\
&= \tfrac{1}{4}\left(w-c_1(\mathfrak{t})+c_1(\mathfrak{s})\right)^2-\tfrac{1}{4}(1\pm 1)^2\\
&= o_{\mathfrak{t}}(w,\mathfrak{s})-\tfrac{1}{4}(1\pm 1)^2.
\end{aligned}
\tag{4.55}
$$

Hence, applying equation (4.30) to the pair $\mathbf{L}_{\tilde{\mathfrak{t}},\mathfrak{s}^\pm}$, using equations (4.54) and (4.55) to compute the sign differences between the pairings with $\mathbf{L}_{\tilde{\mathfrak{t}},\mathfrak{s}^+}$ and $\mathbf{L}_{\tilde{\mathfrak{t}},\mathfrak{s}^-}$, and using equation

(4.53) to relate the constants yields

$$\begin{aligned}&(-1)^{o_{\tilde{\mathfrak{t}}}(w+\mathrm{PD}[e],\mathfrak{s}^+)}\left\langle \mu_p(e^{k+1}z)\smile\mu_c^{\delta_c},[\mathbf{L}_{\tilde{\mathfrak{t}},\mathfrak{s}^+}]\right\rangle\\ &+(-1)^{o_{\tilde{\mathfrak{t}}}(w+\mathrm{PD}[e],\mathfrak{s}^-)}\left\langle \mu_p(e^{k+1}z)\smile\mu_c^{\delta_c},[\mathbf{L}_{\tilde{\mathfrak{t}},\mathfrak{s}^-}]\right\rangle\\ &=\Big((-1)(-1)^{k+1}+1\Big)(-1)^{o_{\mathfrak{t}}(w,\mathfrak{s})+\delta_0+\delta_1}2^{-\delta_2-1-2\delta_0}C_{\chi,\sigma}\left(\deg(z)+2k,\delta_c,d_a(\mathfrak{t}),d_s(\mathfrak{s}),\delta_1\right)\\ &\quad\times SW_{X,\mathfrak{s}}(x^d\vartheta)\langle c_1(\mathfrak{s})-c_1(\mathfrak{t}),h\rangle^{\delta_2-k}.\end{aligned}$$

Since $(-1)(-1)^{k+1}+1=2$ if k is even and is zero if k is odd, the preceding equation reduces to the desired formula (4.52). □

4.6. **Proofs of main theorems.** Combining Theorems 3.33 and 4.13 and a brief discussion will complete the proofs of Theorems 1.1, 1.2, 1.4, and Corollary 1.5:

Proof of Theorem 1.2. By hypothesis we have $w-\Lambda\equiv w_2(X)\pmod 2$ and the invariants $D_X^w(z)$ are zero unless $\deg(z)$ obeys the constraint (1.4). Let $p\in\mathbb{Z}$ satisfy $p\equiv w^2\pmod 4$ and recall that—see the paragraph following equation (2.20) in [10]—we may choose a spin^u structure $\mathfrak{t}$ on X for which

$$c_1(\mathfrak{t})=\Lambda,\quad p_1(\mathfrak{t})=p,\quad\text{and}\quad w_2(\mathfrak{t})\equiv w\pmod 2.$$

Then $d_a(\mathfrak{t})=-2p-\frac{3}{2}(\chi+\sigma)$ and $n_a(\mathfrak{t})=\frac{1}{4}(p+\Lambda^2-\sigma)$, by equation (3.21).

From equation (3.64), a reducible PU(2) monopole in $\bar{\mathcal{M}}_{\mathfrak{t}}$ defined by a spin^c structure $\mathfrak{s}$ lies in the level $\mathcal{M}_{\mathfrak{t}_\ell}\times\mathrm{Sym}^\ell(X)$, where $\ell=\ell(\mathfrak{t},\mathfrak{s})$ and

$$4\ell(\mathfrak{t},\mathfrak{s})=(c_1(\mathfrak{t})-c_1(\mathfrak{s}))^2-p_1(\mathfrak{t}).$$

But $c_1(\mathfrak{t})=\Lambda$ and $p_1(\mathfrak{t})=-\frac{1}{2}d_a(\mathfrak{t})-\frac{3}{4}(\chi+\sigma)$ by equation (3.21), so the definition (1.12) of $r(\Lambda)$ and $r(\Lambda,c_1(\mathfrak{s}))$ implies that

$$\begin{aligned}4\ell(\mathfrak{t},\mathfrak{s})&=\tfrac{1}{2}d_a(\mathfrak{t})+(\Lambda-c_1(\mathfrak{s}))^2+\tfrac{3}{4}(\chi+\sigma)\\ &=\tfrac{1}{2}d_a(\mathfrak{t})-r(\Lambda,c_1(\mathfrak{s}))\\ &\le\tfrac{1}{2}d_a(\mathfrak{t})-r(\Lambda).\end{aligned}\tag{4.56}$$

Hence, when $d_a\le 2r(\Lambda)$, the strata $\iota(M_{\mathfrak{s}})$ with non-trivial Seiberg-Witten functions $SW_{X,\mathfrak{s}}$ can only appear in the top level $\mathcal{M}_{\mathfrak{t}}$ of $\bar{\mathcal{M}}_{\mathfrak{t}}$ (if at all), where they correspond to splittings $\mathfrak{t}=\mathfrak{s}\oplus\mathfrak{s}'$.

From equation (3.21), the stratum $\iota(M_\kappa^w)$ has real codimension $2n_a(\mathfrak{t})$ in $\mathcal{M}_{\mathfrak{t}}$, with

$$\begin{aligned}4n_a(\mathfrak{t})&=p_1(\mathfrak{t})+c_1(\mathfrak{t})^2-\sigma\\ &=-\tfrac{1}{2}d_a(\mathfrak{t})-\tfrac{3}{4}(\chi+\sigma)+\Lambda^2-\sigma\\ &=-\tfrac{1}{2}d_a(\mathfrak{t})-\tfrac{1}{4}(7\chi+11\sigma)+\Lambda^2+\chi+\sigma\\ &=-\tfrac{1}{2}d_a(\mathfrak{t})+i(\Lambda),\end{aligned}\tag{4.57}$$

where the second equality follows from equation (3.21) and the final one by definition (1.11) of $i(\Lambda)$. Thus, our hypotheses on $\deg(z)$ and Λ imply that $n_a(\mathfrak{t})>0$ in Cases (a) and (b) below (where $d_a=\deg(z)$), and also in Case (c) (where $d_a<\deg(z)$), recalling that $\deg(z)$ is denoted by 2δ, for $\delta\in\frac{1}{2}\mathbb{Z}$, in the hypotheses of Theorem 1.2.

Therefore, provided $w\pmod 2$ is good, we can apply Theorem 3.33 to the cobordism $\mathcal{M}_{\mathfrak{t}}^{*,0}$. To eliminate this last constraint on w when $b_2^+(X)>1$, we shall instead apply Theorem 3.33 to the cobordism $\mathcal{M}_{\tilde{\mathfrak{t}}}^{*,0}$, where $\tilde{\mathfrak{t}}$ is the related spin^u structure on the blow-up

$\tilde{X} = X\#\overline{\mathbb{CP}}^2$ produced by Lemma 4.19. When $b_2^+(X) = 1$, we assume that $w \equiv w_2(X) - \Lambda$ (mod 2) is good so that the Donaldson and Seiberg-Witten invariants are well-defined in this case (see §3.4.2 and §4.1) and $\mathcal{M}_{\mathfrak{t}}$ contains no zero-section pairs.

From Lemma 4.19, Seiberg-Witten strata $\iota(M_{\mathfrak{s}^\pm})$ with non-trivial invariants appear only in the top level $\mathcal{M}_{\tilde{\mathfrak{t}}}$ of $\bar{\mathcal{M}}_{\tilde{\mathfrak{t}}}$ if and only if Seiberg-Witten strata $\iota(M_{\mathfrak{s}}) \subset \mathcal{M}_{\mathfrak{t}}$ with non-trivial invariants appear only in the top level $\mathcal{M}_{\mathfrak{t}}$ of $\bar{\mathcal{M}}_{\mathfrak{t}}$. Since X is "effective" by hypothesis, we may assume Conjecture 3.34 holds. Also,

$$n_a(\tilde{\mathfrak{t}}) = n_a(\mathfrak{t}) > 0,$$

using equation (4.57) and the facts that $p_1(\tilde{\mathfrak{t}}) = p_1(\mathfrak{t}) - 1$ and $c_1(\tilde{\mathfrak{t}}) = c_1(\mathfrak{t})$ by Lemma 4.19 and $\sigma(\tilde{X}) = \sigma(X) - 1$. Hence, Corollary 3.35 applies to $\mathcal{M}_{\tilde{\mathfrak{t}}}^{*,0}$.

Theorem 1.2 now follows by applying Proposition 4.22 in conjunction with the relation (3.70) for the cobordism $\bar{\mathcal{M}}_{\tilde{\mathfrak{t}}}^{*,0}$. Equation (3.70) (with the additional hypothesis of Corollary 3.35) gives

$$\begin{aligned}
&\#\left(\bar{\mathcal{V}}(ez) \cap \bar{M}^{w+\mathrm{PD}[e]}_{\kappa+1/4}(\tilde{X})\right) \\
&= -2^{1-n_a} \sum_{\{\mathfrak{s}:\ \mathfrak{s}\oplus\mathfrak{s}'=\mathfrak{t}\}} \Big((-1)^{o_{\tilde{\mathfrak{t}}}(w+\mathrm{PD}[e],\mathfrak{s}^+)} \langle \mu_p(ez) \smile \mu_c^{n_a-1}, [\mathbf{L}_{\tilde{\mathfrak{t}},\mathfrak{s}^+}]\rangle \\
&\quad + (-1)^{o_{\tilde{\mathfrak{t}}}(w+\mathrm{PD}[e],\mathfrak{s}^-)} \langle \mu_p(ez) \smile \mu_c^{n_a-1}, [\mathbf{L}_{\tilde{\mathfrak{t}},\mathfrak{s}^-}]\rangle \Big).
\end{aligned} \tag{4.58}$$

From Theorem 3.33 we see that we need to to consider the following cases, when $n_a > 0$:

(a) $\deg(z) = d_a < 2r(\Lambda)$,
(b) $d_a = 2r(\Lambda)$ and $\deg(z) = d_a$,
(c) $d_a = 2r(\Lambda)$ and $d_a < \deg(z) \le d_a + 2n_a - 2$.

Case (a). The condition $n_a > 0$ is equivalent to $\delta < i(\Lambda)$, since $n_a = \frac{1}{8}(2i(\Lambda) - d_a)$ by equation (4.57) and $\deg(z) = d_a = 2\delta$ in this case.

Using the definition (3.31) of the Donaldson invariants and using $c_1(\mathfrak{t}) = \Lambda$ in Theorem 3.33 and Corollary 3.35 yields

$$D_X^w(z) = 0, \quad \text{for} \quad \deg(z) < 2r(\Lambda). \tag{4.59}$$

This proves Case (a).

Case (b). The condition $n_a > 0$ is again equivalent to $\delta < i(\Lambda)$, since $\deg(z) = d_a = 2\delta$ in this case. We also have $\deg(z) = 2r(\Lambda)$.

Using the definition (3.31) of the Donaldson invariants, applying our blow-up formula (4.52) (with $k = 0$) for link pairings to equation (4.58), and using $c_1(\mathfrak{t}) = \Lambda$ yields

$$\begin{aligned}
D_X^w(z) &= 2^{1-n_a} 2^{-\delta_2 - 2\delta_0} (-1)^{\delta_0+\delta_1+1} \\
&\quad \times \sum_{\{\mathfrak{s}:\ \mathfrak{s}\oplus\mathfrak{s}'=\mathfrak{t}\}} (-1)^{o_{\mathfrak{t}}(w,\mathfrak{s})} C_{\chi,\sigma}\left(\deg(z), \delta_c, 2r(\Lambda), d_s(\mathfrak{s}), \delta_1\right) \\
&\quad \times SW_{X,\mathfrak{s}}(x^d \vartheta) \langle c_1(\mathfrak{s}) - \Lambda, h\rangle^{\delta_2}.
\end{aligned} \tag{4.60}$$

Note that although we assume $\delta_c = n_a - 1$ in Case (b), we allow $\delta_c \ge n_a - 1$ in the above sum, as $\delta_c > n_a - 1$ in Case (c). The inequality (4.56) implies that the subset of $\mathfrak{s} \in \mathrm{Spin}^c(X)$ giving a splitting $\mathfrak{t} = \mathfrak{s} \oplus \mathfrak{s}'$ coincides with the subset for which $r(\Lambda, c_1(\mathfrak{s})) = r(\Lambda)$. Hence, the sum in (4.60) is over the same subset of $\mathrm{Spin}^c(X)$ as that in equation (1.14).

We simplify the sign factor $(-1)^{o_{\mathfrak{t}}(w,\mathfrak{s})}$ in equation (4.60) by writing

$$\begin{aligned} o_{\mathfrak{t}}(w,\mathfrak{s}) &= \tfrac{1}{4}\left(w-\Lambda+c_1(\mathfrak{s})\right)^2 \quad \text{(from definition (3.66))} \\ &= \tfrac{1}{2}c_1(\mathfrak{s})\cdot(w-\Lambda)+\tfrac{1}{4}\left((w-\Lambda)^2+c_1(\mathfrak{s})^2\right). \end{aligned} \tag{4.61}$$

Because $c_1(\mathfrak{s})$ and $\Lambda - w$ are characteristic, we have $c_1(\mathfrak{s})^2 \equiv (\Lambda-w)^2 \equiv \sigma \pmod 8$. Thus, equation (4.61) yields

$$\begin{aligned} o_{\mathfrak{t}}(w,\mathfrak{s}) &\equiv \tfrac{1}{2}c_1(\mathfrak{s})\cdot(w-\Lambda)+\tfrac{1}{2}\sigma \pmod 2 \\ &\equiv \tfrac{1}{2}\left(w^2+c_1(\mathfrak{s})\cdot(w-\Lambda)\right)+\tfrac{1}{2}\left(\sigma-w^2\right) \pmod 2. \end{aligned} \tag{4.62}$$

Substituting equation (4.62) for $o_{\mathfrak{t}}(w,\mathfrak{s}) \pmod 2$ into equation (4.60) implies that the power of (-1) in that formula for $D^w_X(z)$ becomes

$$(-1)^{\delta_0+\delta_1+1}(-1)^{\frac{1}{2}(\sigma-w^2)}(-1)^{\frac{1}{2}(w^2+c_1(\mathfrak{s})\cdot(w-\Lambda))},$$

matching the power of (-1) appearing in equation (1.14).

Equation (4.57) gives $n_a(\mathfrak{t}) = \frac{1}{4}(i(\Lambda)-\delta)$ and so the power of 2 in equation (4.60) for $D^w_X(z)$ then becomes

$$2^{1-\frac{1}{4}(i(\Lambda)-\delta)}2^{-\delta_2-2\delta_0},$$

matching the power of 2 appearing in equation (1.14).

Finally we simplify the expression for the constant $C_{\chi,\sigma}(\deg(z),\delta_c,2r(\Lambda),d_s(\mathfrak{s}),\delta_1)$. Equation (4.57), the equality $\deg(z)+2\delta_c = d_a+2n_a-2$ and the assumption that $d_a = 2r(\Lambda)$ gives

$$\delta_c = \tfrac{1}{2}\left(d_a+2n_a-2-\deg(z)\right) = \tfrac{1}{4}\left(3r(\Lambda)+i(\Lambda)\right)-\tfrac{1}{2}\deg(z)-1.$$

(Note that this holds without the assumption $\delta_c = n_a - 1$.) Then, by the expression for $C_{\chi,\sigma}$ in Lemma 4.18,

$$C_{\chi,\sigma}(\deg(z),\delta_c,2r(\Lambda),d_s(\mathfrak{s}),\delta_1) = H_{\chi,\sigma}(\Lambda^2,\deg(z),d_s(\mathfrak{s}),\delta_1),$$

where the function H is defined in equation (1.15). This completes the proof of the formula (1.14) in Case (b).

The result mentioned in Remark 1.3 for $z = z'Y$ can be proved by the same argument, noting that z as described there is intersection-suitable in the sense of Lemma 3.17 and that the pairings with $\mathbf{L}_{\tilde{\mathfrak{t}},\mathfrak{s}^\pm}$ all vanish by Proposition 4.22.

Case (c). Continue to assume $d_a = 2r(\Lambda)$, so the reducibles (with non-trivial Seiberg-Witten functions) can lie in the top level (but not in any lower level). This case follows in exactly the same way as Case (b), except that we now use equation (3.71) in place of equation (3.70) when $\deg(z)$ lies in the range

$$d_a < \deg(z) \le d_a + 2n_a - 2, \tag{4.63}$$

so we obtain non-trivial relations among the Seiberg-Witten invariants from the cobordism. Using $n_a = \frac{1}{8}(2i(\Lambda)-d_a)$, the upper bound in equation (4.63) becomes

$$\begin{aligned} d_a+2n_a-2 &= d_a+\tfrac{1}{4}(2i(\Lambda)-d_a)-2 \\ &= \tfrac{3}{4}d_a+\tfrac{1}{2}i(\Lambda)-2 \\ &= \tfrac{3}{2}r(\Lambda)+\tfrac{1}{2}i(\Lambda)-2 \\ &= \tfrac{1}{2}(3r(\Lambda)+i(\Lambda))-2. \end{aligned}$$

Thus, our pair of inequalities reduces to

$$2r(\Lambda) < \deg(z) \le r(\Lambda)+\tfrac{1}{2}(r(\Lambda)+i(\Lambda))-2. \tag{4.64}$$

Therefore we obtain a non-trivial relation amongst the Seiberg-Witten invariants and a vanishing result for Donaldson invariants when the constraint (4.64) on Λ^2 and $\deg(z)$ holds. □

Proof of Theorem 1.4. By hypothesis, $\Lambda \cdot c_1(\mathfrak{s}) = 0$ for all $\mathfrak{s} \in \mathrm{Spin}^c(X)$ with $SW_X(\mathfrak{s}) \neq 0$, so from equation (1.12) for $r(\Lambda, c_1(\mathfrak{s}))$ we have

$$\begin{aligned} r(\Lambda, c_1(\mathfrak{s})) &= -c_1(\mathfrak{s})^2 - \Lambda^2 - \tfrac{3}{4}(\chi + \sigma) \\ &= -(2\chi + 3\sigma) - \Lambda^2 - \tfrac{3}{4}(\chi + \sigma) \\ &= -(\chi + \sigma) - \Lambda^2 + c(X) \\ &= r(\Lambda), \end{aligned}$$

using the definition of $c(X)$ (see §1.1) and the definition (1.12) of $r(\Lambda)$, and the formula (§1.1) for $c_1(\mathfrak{s})^2$ when X has SW-simple type. A reducible PU(2) monopole in $\bar{\mathcal{M}}_{\mathfrak{t}}$ defined by a splitting $\mathfrak{t}_\ell = \mathfrak{s} \oplus \mathfrak{s}'$ appears in level

$$\ell(\mathfrak{t}, \mathfrak{s}) = \tfrac{1}{8}(d_a(\mathfrak{t}) - 2r(\Lambda, c_1(\mathfrak{s}))) = \tfrac{1}{8}(d_a(\mathfrak{t}) - 2r(\Lambda)),$$

and thus all reducibles appear in the same level of $\bar{\mathcal{M}}_{\mathfrak{t}}$. Hence, the sum over $\mathfrak{s} \in \mathrm{Spin}^c(X)$ with $r(\Lambda, c_1(\mathfrak{s})) = r(\Lambda)$ can be replaced by a sum over $\mathfrak{s} \in \mathrm{Spin}^c(X)$ when $\Lambda \in B^\perp$. We write $\deg(z) = 2\delta$, as in the hypothesis of the theorem.

Case (a). In this situation, $\delta < r(\Lambda)$, $\delta < i(\Lambda)$, and

$$D^w_X(z) = 0, \quad \text{when } 0 \leq \delta < r(\Lambda),$$

by Theorem 1.2.

Case (b). In this situation, $\delta = r(\Lambda)$ and $\delta < i(\Lambda)$. We can further simplify the formula (1.14). First, using $i(\Lambda) = 2c(X) - r(\Lambda) = 2c(X) - \delta$, $\delta_2 = \delta - 2m$, and $\delta_0 = m$, the power of 2 in equation (1.14) becomes

$$2^{1-\frac{1}{4}(2c(X)-2\delta)-\delta} = 2^{1-\frac{1}{2}(c(X)+\delta)},$$

matching the power of 2 in equation (1.19). The power of (-1) in equation (1.14) is

$$(-1)^{m+1}(-1)^{\frac{1}{2}(\sigma - w^2)}(-1)^{\frac{1}{2}(w^2 + c_1(\mathfrak{s})\cdot w)},$$

since $\delta_0 = m$ and $c_1(\mathfrak{s}) \cdot \Lambda = 0$, and also matches that in (1.19). Finally, $d_s(\mathfrak{s}) = 0$ because we assume X is SW-simple type, so the constant $H_{\chi,\sigma}(\cdot)$ is equal to one and thus equation (1.19) follows from equation (1.14).

Case (c). Using $\deg(z) = 2\delta$, $r(\Lambda) + i(\Lambda) = c(X)$, and equation (1.17) for $r(\Lambda)$, the constraint (4.64) simplifies to

$$2r(\Lambda) < 2\delta \leq r(\Lambda) + c(X) - 2. \tag{4.65}$$

The vanishing relation follows from Case (c) in Theorem 1.2. This completes the proof. □

Proof of Corollary 1.5. We consider the last case of Theorem 1.4, where δ and Λ^2 obey the constraints (4.65) and so

$$r(\Lambda) < c(X) - 2.$$

Therefore, the choice of $r(\Lambda) < c(X) - 2$ giving the largest possible integer δ (admitting a non-trivial vanishing relation) is $r(\Lambda) = c(X) - 4$, achieved when $\Lambda = \Lambda_0$ with $\Lambda_0^2 = 4 - (\chi + \sigma)$. By hypothesis, such a class $\Lambda_0 \in B^\perp$ exists. Therefore, the pair (4.65) of inequalities constrains

$$\delta = c(X) - 3.$$

Thus, using $z = x^m h^{\delta-2m}$ with $0 \leq m \leq [\delta/2]$, we obtain for $w_0 \in H^2(X;\mathbb{Z})$ with $w_0 - \Lambda_0 \equiv w_2(X) \pmod 2$

$$\sum_{\mathfrak{s}\in\operatorname{Spin}^c(X)} (-1)^{\frac{1}{2}(w_0^2+c_1(\mathfrak{s})\cdot w_0)} SW_X(\mathfrak{s})\langle c_1(\mathfrak{s}) - \Lambda_0, h\rangle^d = 0, \quad 0 \leq d \leq c(X) - 3. \tag{4.66}$$

Indeed, if $c(X) - 3$ is even, then we may choose $m = \delta/2$ to obtain the above relation with $d = 0$ and, as we explain shortly, the relation for $d = 0$ also holds when $c(X) - 3$ is odd. Hence, the degree-d terms in the Taylor expansion of $\mathbf{SW}_X^{w_0}(h)e^{-\langle\Lambda,h\rangle}$ about $h = 0$ are zero for $0 \leq d \leq c(X) - 3$ and so the same holds for $\mathbf{SW}_X^{w_0}(h)$.

If w is any integral lift of $w_2(X)$, write $w = w + \Lambda_0 - \Lambda_0$ and observe that

$$\mathbf{SW}_X^{w}(h) = (-1)^{\frac{1}{2}(\Lambda_0^2-2w\cdot\Lambda_0)}\mathbf{SW}_X^{w+\Lambda_0}(h).$$

Thus $\mathbf{SW}_X^{w}(h)$ vanishes to the same order as $\mathbf{SW}_X^{w_0}(h)$ with $w_0 = w+\Lambda_0$ and this completes the proof, aside from the remark below on the case of odd $c(X) - 3$.

When $c(X) - 3$ is odd, it only remains to show that the relation (4.66) still holds when $d = 0$. We choose $\Lambda_1 \in B^\perp$ with $\Lambda_1^2 = 6 - (\chi + \sigma)$ and $r(\Lambda_1) = c(X) - 6$, so that (4.65) allows $\delta_1 = c(X) - 4$, which must be even and thus, taking $m = \delta_1/2$ yields

$$\sum_{\mathfrak{s}\in\operatorname{Spin}^c(X)} (-1)^{\frac{1}{2}(w_1^2+c_1(\mathfrak{s})\cdot w_1)} SW_X(\mathfrak{s}) = 0,$$

for any $w_1 \in H_2(X;\mathbb{Z})$ with $w_1 - \Lambda_1 \equiv w_2(X)$. Writing $w_1 = w_0 + \Lambda_1 - \Lambda_0$ and combining

$$\mathbf{SW}_X^{w_1}(h) = (-1)^{\frac{1}{2}((\Lambda_1-\Lambda_0)^2+2w_0\cdot(\Lambda_1-\Lambda_0))}\mathbf{SW}_X^{w_0}(h).$$

with the previous vanishing result yields the relation (4.66) when $d = 0$. □

Proof of Theorem 1.1. We assume without loss that $c(X) > 0$. From Theorem 1.4 we know that $D_X^w(x^m h^{\delta-2m}) = 0$ if $\delta < r(\Lambda)$ and that the first potentially non-zero invariant is given by equation (1.19) when $\delta = r(\Lambda)$. The cobordism method constrains Λ^2 by requiring that $\delta < i(\Lambda)$. Hence, from the graphs of $r(\Lambda)$ and $i(\Lambda)$ as functions of Λ^2 (see Figure 1 in [8]) one sees that these two lines meet for $\Lambda_0 \in H^2(X;\mathbb{Z})$ with $\Lambda_0^2 = -(\chi + \sigma)$, at which point $r(\Lambda_0) = c(X) = i(\Lambda_0)$. Therefore, we choose Λ^2 to give the largest possible $\delta = r(\Lambda) < c(X)$. We also take $\Lambda \in B^\perp$, to simplify the formula (1.19) and as the SW-basic classes $c_1(\mathfrak{s})$ are characteristic, this gives (for B non-empty) $\Lambda \cdot c_1(\mathfrak{s}) = 0$ and $\Lambda \cdot c_1(\mathfrak{s}) \equiv \Lambda^2 \pmod 2$, so that Λ^2 is even. Thus we want to choose $\Lambda \in B^\perp$ with smallest even value of $\Lambda^2 > -(\chi + \sigma)$, namely

$$\Lambda^2 = 2 - (\chi + \sigma), \tag{4.67}$$

because $\chi + \sigma$ is even (in fact, divisible by four since $b_1(X) = 0$ and $b_2^+(X)$ is odd). By hypothesis, Λ exists and the formula (1.17) for $r(\Lambda)$ and the definition of $c(X)$ yield

$$\begin{aligned}\delta = r(\Lambda) &= -\Lambda^2 + c(X) - (\chi + \sigma)\\ &= c(X) - 2.\end{aligned}$$

Therefore Theorem 1.4 and the fact that $\delta = c(X) - 2 = r(\Lambda)$ yield

$$D_X^w(x^m h^{d-2m}) = 0, \quad 0 \leq d < \delta \quad \text{and} \quad 0 \leq m \leq [d/2]. \tag{4.68}$$

When $m = 0$, the power of (-1) in equation (1.19) simplifies to

$$(-1)^{\frac{1}{2}(w^2+c_1(\mathfrak{s})\cdot w)},$$

as $\frac{1}{2}(\sigma-w^2)\equiv 1 \pmod 2$. Indeed, to see this note that $w-\Lambda$ is characteristic, so $(w-\Lambda)\cdot\Lambda\equiv \Lambda^2 \pmod 2$, and as $\chi+\sigma\equiv 0 \pmod 4$, we have

$$
\begin{aligned}
w^2 &= (w-\Lambda)^2 + 2(w-\Lambda)\cdot\Lambda + \Lambda^2 \\
&\equiv \sigma + \Lambda^2 \pmod 4 \\
&\equiv \sigma + 2 \pmod 4 \quad \text{(by Equation (4.67))}.
\end{aligned} \tag{4.69}
$$

The power of 2 in equation (1.19), when $\delta = c(X)-2$, becomes

$$2^{2-c(X)},$$

as we expect from Witten's formula (1.6).

From equation (4.68), the invariants $D^w_X(h^d)$ and $D^w_X(xh^{d-2})$ are zero when $d<\delta = c(X)-2$ (while the method of this article does not allow us to compute the invariants when $d\geq \delta+4$), so (compare equation (1.5))

$$
\begin{aligned}
&\mathbf{D}^w_X(h) \equiv 0 \pmod{h^\delta}, \\
&\mathbf{D}^w_X(h) \equiv \tfrac{1}{\delta!}D^w_X(h^\delta) + \tfrac{1}{2(\delta-2)!}D^w_X(xh^{\delta-2}) \pmod{h^{\delta+2}}.
\end{aligned} \tag{4.70}
$$

For a monomial $z\in\mathbb{A}(X)$, the invariant $D^w_X(z)$ is zero unless $\deg(z)\equiv -2w^2-\frac{3}{2}(\chi+\sigma) \pmod 8$. Therefore, as $\delta\equiv w^2-\frac{3}{4}(\chi+\sigma) \pmod 4$, the invariants $D^w_X(h^{\delta+2})$ and $D^w_X(xh^\delta)$ are necessarily zero and the next potentially non-zero invariant of higher degree in h would be $D^w_X(xh^{\delta+2})$.

For the terms $D^w_X(h^\delta)$, equation (1.19) yields

$$\tfrac{1}{\delta!}D^w_X(h^\delta) = 2^{2-c(X)}\sum_{\mathfrak{s}\in\mathrm{Spin}^c(X)} (-1)^{\frac{1}{2}(w^2+c_1(\mathfrak{s})\cdot w)} SW_X(\mathfrak{s})\tfrac{1}{\delta!}\langle c_1(\mathfrak{s})-\Lambda,h\rangle^\delta. \tag{4.71}$$

Since $\mathbf{SW}^{w-\Lambda}_X(h)\equiv 0 \pmod{h^\delta}$ by Corollary 1.5, we have

$$\mathbf{SW}^w_X(h) = (-1)^{\frac{1}{2}(\Lambda^2+2(w-\Lambda)\cdot\Lambda)}\mathbf{SW}^{w-\Lambda}_X(h)\equiv 0 \pmod{h^\delta}. \tag{4.72}$$

Therefore, using equation (4.71) and noting that the terms in $e^{\frac{1}{2}Q(h,h)}$ and $e^{\langle -\Lambda,h\rangle}$ of lowest degree in h are both 1 and the lowest-degree non-zero term in $\mathbf{SW}^w_X(h)$ has degree δ in h by equation (4.72), we see that

$$\tfrac{1}{\delta!}D^w_X(h^\delta) = \left[2^{2-c(X)}\mathbf{SW}^w_X(h)e^{\langle -\Lambda,h\rangle}\right]_\delta = \left[2^{2-c(X)}e^{\frac{1}{2}Q(h,h)}\mathbf{SW}^w_X(h)\right]_\delta, \tag{4.73}$$

where $[\,\cdot\,]_\delta$ denotes the term of degree δ in h in the power series.

For the term $D^w_X(xh^{\delta-2})$, equation (1.19) yields

$$
\begin{aligned}
D^w_X(xh^{\delta-2}) &= -2^{2-c(X)}\sum_{\mathfrak{s}\in\mathrm{Spin}^c(X)} (-1)^{\frac{1}{2}(w^2+c_1(\mathfrak{s})\cdot w)} SW_X(\mathfrak{s})\langle c_1(\mathfrak{s})-\Lambda,h\rangle^{\delta-2} \\
&= -\left[2^{2-c(X)}\mathbf{SW}^w_X(h)e^{\langle -\Lambda,h\rangle}\right]_{\delta-2}\cdot(\delta-2)! \\
&= 0,
\end{aligned} \tag{4.74}
$$

where the final equality follows from the fact that the term in $e^{\langle -\Lambda,h\rangle}$ of lowest degree in h is 1 and the terms in $\mathbf{SW}^w_X(h)$ of degree $\delta-2$ or lower in h are zero. Combining equations (4.70), (4.73), and (4.74) thus completes the proof. □

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