A Characterization of the Singular Green Operators in Boutet de Monvel's Calculus via Wedge Sobolev Spaces

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Abstract

The singular Green operators in Boutet de Monvel's calculus are characterized by the behavior of their iterated commutators with differentiations and vector fields tangential to the boundary on wedge Sobolev spaces.

Key Words: Boundary value problems, Boutet de Monvel's calculus, (singular) Green operators, wedge Sobolev spaces

AMS Subject Classification: 35 S 15, 47 D 25, 46 E 35

Introduction

In 1977, Richard Beals showed that the pseudodifferential operators on $\mathbf{R}^n$ can be characterized in terms of the mapping properties of their iterated commutators with multiplications and vector fields [1]. Specialized to the standard symbol classes, his result says the following:

Theorem. *Let $m \in \mathbf{R}, 0 \le \delta \le \rho \le 1, \delta < 1$, and let $P : \mathcal{S}(\mathbf{R}^n) \longrightarrow \mathcal{S}'(\mathbf{R}^n)$ be a continuous operator. Then P is a pseudodifferential operator with a symbol in $S^m_{\rho,\delta}(\mathbf{R}^n \times \mathbf{R}^n)$, if and only if for all $s \in \mathbf{R}$, and all multi-indices α, β, the iterated commutators $\mathrm{ad}^\alpha(-ix)\mathrm{ad}^\beta(D_x)P$ have bounded extensions*

$$\mathrm{ad}^\alpha(-ix)\mathrm{ad}^\beta(D_x)P : H^s(\mathbf{R}^n) \longrightarrow H^{s-m+\rho|\alpha|-\delta|\beta|}(\mathbf{R}^n). \tag{1}$$

For the notation see section 1. At about the same time, H.-O. Cordes independently obtained a similar characterization for the class $S^0_{0,0}$ with different methods, cf. [6]. A new proof for Beals' result in the above formulation was given by Ueberberg [33]; the manifold case may be found in Coifman and Meyer's monograph [5].
Characterizations of families of operators via the properties of their iterated commutators have been used by Cordes and various associates in the theory of operator algebras, [7], [9], [22].
Beals' characterization was the crucial step towards the proof of the spectral invariance of the algebra of pseudodifferential operators of order zero in $\mathcal{L}(L^2(\mathbf{R}^n))$, [1], theorem 3.2; it also turned out to be useful in many other situations, cf. Schrohe [23], Leopold and Schrohe [18], [19].
Moreover, a characterization like (1) allows to introduce a topology on the algebra of pseudodifferential operators of order zero which makes it topologically an intersection of Banach algebras, a 'submultiplicative' Fréchet algebra, a fact observed by Gramsch, Ueberberg, and Wagner, [13], and of particular interest in connection with the results of C. Phillips [20] on K-theory and Gramsch [11] on non-abelian cohomology and Oka principle.
In the present paper, a similar description is given for the singular Green operators of type zero in Boutet de Monvel's calculus on the half-space $\mathbf{R}^n_+$. In spite of the rather complicated usual description of these elements in terms of estimates on the singular Green symbols or symbol kernels, cf. definition 1.5, below, the characterization in theorem 2.1 is surprisingly simple and close to (1).
It shows that these operators have a natural connection to the wedge Sobolev spaces introduced by B.-W. Schulze for the analysis on manifolds with singularities. Locally, a wedge is of the form {cone } $\times\, \mathbf{R}^q$. The half-space $\mathbf{R}^n_+$ is a particularly simple wedge where the cone is $\mathbf{R}_+$. Wedge Sobolev spaces differ from the usual ones in that one has a group action along the cone.
Already in 1991, Schulze has pointed out that there is a close relation between singular Green operators and wedge spaces, [29] theorem 3.1. His observation was an important motivation for this paper.
In a related paper [27], a characterization of the pseudodifferential operators with the transmission property via commutators and wedge Sobolev spaces will be given. This partly extends the 1990 results of Grubb and Hörmander on the transmission property, [15]. Using these characterizations I then show in joint work with B. Gramsch [12] that

the algebra of operators of order and type zero in Boutet de Monvel's calculus also is an intersection of Banach algebras, making it accessible to the analysis in [20] and [11].

1 Notation. Pseudodifferential Operators and Singular Green Operators

1.1 Definition.
(a) For $m \in \mathbf{R}, 0 \leq \delta \leq \rho \leq 1, S^m_{\rho,\delta} = S^m_{\rho,\delta}(\mathbf{R}^n \times \mathbf{R}^n)$ denotes the set of all smooth functions p on $\mathbf{R}^n \times \mathbf{R}^n$ satisfying the estimates

$$|D^\alpha_\xi D^\beta_x p(x,\xi)| \leq C_{\alpha\beta} \langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}. \tag{1}$$

Here, $\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$. The choice of best constants in (1) gives the Fréchet topology for $S^m_{\rho,\delta}$.
In general, the symbols will take values in matrices over $\mathbf{C}$. I also admit the case that $p(x,\xi) \in \mathcal{L}(E,F)$ with Hilbert spaces E and F; for clarity I will then speak of operator-valued symbols.
(b) A symbol $p \in S^m_{\rho,\delta}$ defines a pseudodifferential operator $\mathrm{Op}\, p$ or $p(x,D)$ by

$$[p(x,D)u](x) = [\mathrm{Op}\, p\, u](x) = (2\pi)^{-n/2} \int e^{ix\xi} p(x,\xi)\hat{u}(\xi) d\xi, \tag{2}$$

where u is a rapidly decreasing function, $u \in \mathcal{S}(\mathbf{R}^n)$, or – in the case of operator-valued symbols – $u \in \mathcal{S}(\mathbf{R}^n, E)$, and $\hat{u}$ is the Fourier transform of u.
(c) For $s \in \mathbf{R}, H^s(\mathbf{R}^n)$ denotes the usual Sobolev space on $\mathbf{R}^n$, cf. [17], ch. 3, def. 2.1. For $s,t \in \mathbf{R}$, let

$$H^{s,t}(\mathbf{R}^n) = \{\langle x\rangle^{-t} u : u \in H^s(\mathbf{R}^n_x)\}.$$

$H^{s,t}(\mathbf{R}^n, E), E$ a Hilbert space, denotes the vector-valued analog.
(d) For multi-indices $\alpha, \beta \in \mathbf{N}^n_0$ and an operator T acting on functions or distributions on $\mathbf{R}^n$, let

$$\mathrm{ad}^\alpha(-ix)\mathrm{ad}^\beta(D_x)T = \mathrm{ad}^{\alpha_1}(-ix_1)\cdots \mathrm{ad}^{\alpha_n}(-ix_n)\mathrm{ad}^{\beta_1}(D_{x_1})\cdots \mathrm{ad}^{\beta_n}(D_{x_n})T.$$

Here, $\mathrm{ad}^0(-ix_j)T = T$, and $\mathrm{ad}^k(-ix_j)T = [-ix_j, \mathrm{ad}^{k-1}(-ix_j)T], k = 1,2,\ldots$; the iterated commutators $\mathrm{ad}^{\beta_j}(D_{x_j})T$ are defined correspondingly. Of course, we are assuming for the moment that all compositions involved make sense.

1.2 Remark. Given a pseudodifferential operator $P = \mathrm{Op}\, p$ with $p \in S^m_{\rho,\delta}$ it is easily seen that $\mathrm{ad}^\alpha(-ix)\mathrm{ad}^\beta(D_x)P$ is the pseudodifferential operator with the symbol $\partial^\alpha_\xi D^\beta_x p(x,\xi)$.

1.3 Notation on the half-space. We will write $\mathbf{R}^n_+ = \{(x_1,...,x_n) : x_n > 0\}$ and $x = (x', x_n), \xi = (\xi', \xi_n)$ with $x' = (x_1,\ldots,x_{n-1}), \xi' = (\xi_1,\ldots,\xi_{n-1})$.
(a) For a function or distribution f on $\mathbf{R}^n$ let $\mathrm{r}^+ f$ denote its restriction to $\mathbf{R}^n_+$; for a function g on $\mathbf{R}^n_+$ denote by $\mathrm{e}^+ g$ its extension to $\mathbf{R}^n$ by zero.
(b) Let $\mathcal{S}(\mathbf{R}^n_+) = \{\mathrm{r}^+ f : f \in \mathcal{S}(\mathbf{R}^n)\}$, and $H^{s,t}(\mathbf{R}^n_+) = \{\mathrm{r}^+ f : f \in H^{s,t}(\mathbf{R}^n)\}, s,t \in \mathbf{R}$. $H^{s,t}_0(\mathbf{R}^n_+)$ is the closure of $C^\infty_0(\mathbf{R}^n_+)$ in the topology of $H^{s,t}(\mathbf{R}^n)$.

1.4 Green operators and Singular Green Operators. A *Green operator of order and type zero* in Boutet de Monvel's calculus on $\mathbf{R}^n_+$ is an operator of the form

$$A = \begin{bmatrix} P_+ + \mathrm{G} & \mathrm{K} \\ \mathrm{T} & \mathrm{S} \end{bmatrix} : \begin{array}{c} C_0^\infty(\bar{\mathbf{R}}^n_+) \\ \oplus \\ C_0^\infty(\mathbf{R}^{n-1}) \end{array} \longrightarrow \begin{array}{c} C^\infty(\bar{\mathbf{R}}^n_+) \\ \oplus \\ C^\infty(\mathbf{R}^{n-1}) \end{array},$$

where P is a pseudodifferential operator with the transmission property of order zero, $P_+ = \mathrm{r}^+ P \mathrm{e}^+, G$ is a singular Green operator of order and type zero, i.e. with a symbol kernel in $\tilde{\mathcal{B}}^{-1,0}$, the precise definition being given in 1.5, K is a Poisson operator, T a trace operator, and S is a pseudodifferential operator with a symbol in $S^0_{1,0}(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1})$.
The most interesting part within this setting is the algebra

$$\mathcal{A} = \{A : A = P_+ + G\}$$

of the elements in the upper left corner.
For the details concerning the calculus, I refer to [2], [14], or [21].
While pseudodifferential operators with the transmission property are the object of [27], this paper focuses on a description of the singular Green operators G within this algebra. For the description of these elements I am using symbol *kernels* rather than the singular Green *symbols*, because it makes things slightly easier.

1.5 Definition. Let $\mu \in \mathbf{R}$. The class $\tilde{\mathcal{B}}^{\mu,0}$ consists of all smooth functions g on $\mathbf{R}^{n-1}_{x'} \times \mathbf{R}^{n-1}_{\xi'} \times \mathbf{R}_{+x_n} \times \mathbf{R}_{+y_n}$ (symbol kernels) satisfying the estimates

$$\|x_n^k D_{x_n}^{k'} y_n^m D_{y_n}^{m'} D_{\xi'}^\alpha D_{x'}^\beta g(x', \xi', x_n, y_n)\|_{L^2(\mathbf{R}_{+x_n} \times \mathbf{R}_{+y_n})} = \mathrm{O}(\langle \xi' \rangle^{\mu+1-k+k'-m+m'-|\alpha|}) \qquad (1)$$

for every fixed choice of $k, k', m, m', \alpha, \beta$.
Such a symbol kernel g induces the singular Green operator $\mathrm{Op}_G\, g$ by

$$[\mathrm{Op}_G\, g(f)](x) = (2\pi)^{\frac{n-1}{2}} \iint_0^\infty e^{ix'\xi'} g(x', \xi', x_n, y_n)(\mathcal{F}_{x' \to \xi'} f)(\xi', y_n) dy_n d\xi', \qquad (2)$$

$f \in \mathcal{S}(\mathbf{R}^n_+)$; g is called the symbol kernel of $\mathrm{Op}_G\, g$.
For fixed x', ξ' let the operator $g(x', \xi', D_n)$ be defined on $\mathcal{S}(\mathbf{R}_+)$ by

$$[g(x', \xi', D_n)f](x_n) = \int_0^\infty g(x', \xi', x_n, y_n) f(y_n) dy_n;$$

then

$$\mathrm{Op}_G\, g = \mathrm{Op}' g(x', \xi', D_n),$$

where Op' denotes the usual pseudodifferential action with respect to the x', ξ'−variables for operator-valued symbols.

We finally need the concept of wedge Sobolev spaces, cf. [28], section 3.1.

1.6 Definition. (a) For $f \in L^2(\mathbf{R}_+), \lambda > 0$, let

$$(\kappa_\lambda f)(t) = \lambda^{\frac{1}{2}} f(\lambda t). \qquad (1)$$

This defines a unitary map

$$\kappa_\lambda : L^2(\mathbf{R}_+) \longrightarrow L^2(\mathbf{R}_+)$$

with respect to the sesquilinear form

$$\langle f, g\rangle = \int_{\mathbf{R}_+} f(t)\bar{g}(t)dt.$$

(b) More generally let E be a Banach space and suppose that $\{\kappa_\lambda : \lambda \in \mathbf{R}_+\}$ is a strongly continuous group of operators on E, i.e. $\lambda \mapsto \kappa_\lambda \in C(\mathbf{R}_+, \mathcal{L}_\sigma(E))$, and $\kappa_\lambda\kappa_\rho = \kappa_{\lambda\rho}$.
The *wedge Sobolev space* modelled on E, $\mathcal{W}^s(\mathbf{R}^q, E), s \in \mathbf{R}, q \in \mathbf{N}_0$ is defined as the completion of $\mathcal{S}(\mathbf{R}^q, E) = \mathcal{S}(\mathbf{R}^q)\hat{\otimes}_\pi E$ with respect to the norm

$$\|u\|_{\mathcal{W}^s(\mathbf{R}^q,E)} = \left(\int \langle\eta\rangle^{2s} \|\kappa_{\langle\eta\rangle^{-1}}\mathcal{F}_{y\to\eta}u(\eta)\|_E^2 d\eta\right)^{\frac{1}{2}}.$$

Here, $\mathcal{F}_{y\to\eta}u$ denotes the Fourier transform of the E-valued function or distribution u,

$$\mathcal{F}_{y\to\eta}u(\eta) = (2\pi)^{-q/2}\int e^{-iy\eta}u(y)dy.$$

(c) For $s, t \in \mathbf{R}$, let

$$\mathcal{W}^{s,t}(\mathbf{R}^q, E) = \{\langle y\rangle^{-t} u : u \in \mathcal{W}^s(\mathbf{R}^q, E)\}.$$

In general, the wedge Sobolev space will depend on the choice of the group action on E. Here, however, we will only deal with the usual weighted Sobolev spaces on $\mathbf{R}_+$, cf. 1.3(b), and there we will always use the group defined by (1).
(d) For convenience we introduce the following notation for inductive and projective limits. Let $\{E_k : k \in \mathbf{N}\}$ be a sequence of Banach spaces with $E_{k+1} \hookrightarrow E_k$ and $E = \text{proj-lim } E_k$. Suppose the group action is the same on all spaces. Then let

$$\mathcal{W}^{s,t}(\mathbf{R}^q, E) = \text{proj-lim } \mathcal{W}^{s,t}(\mathbf{R}^q, E_k).$$

Vice versa, if $E_k \hookrightarrow E_{k+1}$, $E = \text{ind-lim } E_k$, and the group action coincides, then let

$$\mathcal{W}^{s,t}(\mathbf{R}^q, E) = \text{ind-lim } \mathcal{W}^{s,t}(\mathbf{R}^q, E_k).$$

1.7 Remark.

$$\begin{array}{lrcl}
\text{(a)} & \mathcal{S}(\mathbf{R}^n_+) & = & \text{proj-lim}_{\,s,t\to\infty} H^{s,t}(\mathbf{R}^n_+). \\
\text{(b)} & \mathcal{S}'(\mathbf{R}^n_+) & = & \text{ind-lim}_{\,s,t\to\infty} H_0^{-s,-t}(\mathbf{R}^n_+). \\
\text{(c)} & \mathcal{W}^s(\mathbf{R}^q, H^s(\mathbf{R}_+)) & = & H^s(\mathbf{R}^{q+1}_+), s \geq 0. \\
\text{(d)} & \mathcal{W}^s(\mathbf{R}^q, H_0^s(\mathbf{R}_+)) & = & H_0^s(\mathbf{R}^{q+1}_+), s \leq 0.
\end{array}$$

For (c) and (d), cf. [28], section 3.1.1, (17) and (18).

1.8 Lemma. $\text{proj}-\lim_{s_1,s_2,\sigma_1,\sigma_2\to\infty} \mathcal{W}^{s_1,s_2}(\mathbf{R}^q, H^{\sigma_1,\sigma_2}(\mathbf{R}_+)) = \mathcal{S}(\mathbf{R}^{q+1}_+)$.

Proof. Write $\mathbf{R}^{q+1}_+ = \{x = (y,t) : y \in \mathbf{R}^q, t > 0\}$, and write the covariable in the form $\xi = (\eta, \tau)$. Let $s, N \geq 0, u \in \mathcal{S}(\mathbf{R}^{q+1}_+)$, and let U denote an arbitrary extension of u to a distribution in $H^{s,N}(\mathbf{R}^{q+1})$. Then

$$\begin{aligned}
\|u\|_{H^{s,N}(\mathbf{R}^{q+1}_+)} &= \inf_U \|\langle D_x\rangle^s \langle x\rangle^N U\|_{L^2(\mathbf{R}^{q+1})} \\
&\leq \inf_U C\|\langle x\rangle^N \langle D_x\rangle^s U\|_{L^2(\mathbf{R}^{q+1})}
\end{aligned}$$

with $C = C(s,N)$ in view of the boundedness of the operator $\langle D_x\rangle^s \langle x\rangle^N \langle D_x\rangle^{-s} \langle x\rangle^{-N}$ on $L^2(\mathbf{R}^{q+1})$. So the last expression is

$$\begin{aligned} &\le \inf_U C' \| \langle y\rangle^N \langle t\rangle^N \langle D_x\rangle^s U\|_{L^2(\mathbf{R}^{q+1})} \\ &\le \inf_U C'' \| \langle D_x\rangle^s \langle y\rangle^N \langle t\rangle^N U\|_{L^2(\mathbf{R}^{q+1})} \end{aligned}$$

in view of the boundedness of the operator $\langle y\rangle^N \langle t\rangle^N \langle D_x\rangle^s \langle t\rangle^{-N} \langle y\rangle^{-N} \langle D_x\rangle^{-s}$ on $L^2(\mathbf{R}^{q+1})$, cf. [24]. The last expression equals

$$\begin{aligned} &= C'' \| \langle y\rangle^N \langle t\rangle^N u\|_{H^s(\mathbf{R}^{q+1}_+)} \\ &\le C''' \| \langle y\rangle^N \langle t\rangle^N u\|_{\mathcal{W}^s(\mathbf{R}^q, H^s(\mathbf{R}_+))} \\ &= C''' \left\{ \int \langle\eta\rangle^{2s} \|\kappa_{\langle\eta\rangle^{-1}} \mathcal{F}_{y\to\eta}(\langle y\rangle^N \langle t\rangle^N u)(\eta)\|^2_{H^s(\mathbf{R}_+)} d\eta \right\}^{\frac{1}{2}} \quad (1) \end{aligned}$$

where the inequality is justified by 1.7(c). Let $v = \langle y\rangle^N u$. Then

$$\|\kappa_{\langle\eta\rangle^{-1}} \mathcal{F}_{y\to\eta}(\langle y\rangle^N \langle t\rangle^N u)(\eta)\|^2_{H^s(\mathbf{R}_+)} \le \|\kappa_{\langle\eta\rangle^{-1}}\|^2_{\mathcal{L}(H^s(\mathbf{R}_+))} \| \langle t\rangle^N \mathcal{F}v(\eta)\|^2_{H^s(\mathbf{R}_+)} \quad (2)$$

By [28], 3.1.2 lemma 1, there are c, M such that

$$\|\kappa_{\langle\eta\rangle^{-1}}\|_{\mathcal{L}(H^s(\mathbf{R}_+))} \le c\langle\eta\rangle^M,$$

so (2) can be estimated by

$$\begin{aligned} & c^2 \langle\eta\rangle^{2M} \| \langle t\rangle^N \mathcal{F}v(\eta)\|^2_{H^s(\mathbf{R}_+)} \\ &= c^2 \langle\eta\rangle^{2M} \|\mathcal{F}v(\eta)\|^2_{H^{s,N}(\mathbf{R}_+)} \\ &\le d^2 \langle\eta\rangle^{2M'} \|\kappa_{\langle\eta\rangle^{-1}} \mathcal{F}v(\eta)\|^2_{H^{s,N}(\mathbf{R}_+)} \end{aligned}$$

by the same argument as before. We can therefore estimate (1) by

$$\begin{aligned} & D\left\{ \int \langle\eta\rangle^{2s+2M'} \|\kappa_{\langle\eta\rangle^{-1}} \mathcal{F}(\langle y\rangle^N u)\|^2_{H^{s,N}(\mathbf{R}_+)} \right\}^{\frac{1}{2}} \\ &= D\| \langle y\rangle^N u\|_{\mathcal{W}^{s+M'}(\mathbf{R}^q, H^{s,N}(\mathbf{R}_+))} \\ &= D\|u\|_{\mathcal{W}^{s+M',N}(\mathbf{R}^q, H^{s,N}(\mathbf{R}_+))}. \end{aligned}$$

Hence the topology induced by the $\mathcal{W}$-spaces is stronger than the Sobolev topology. On the other hand, we have $\langle t\rangle^N \le \langle\eta\rangle \left\langle \langle\eta\rangle^{-1} t\right\rangle^N$, and therefore

$$\| \langle t\rangle^N \kappa_{\langle\eta\rangle^{-1}} \mathcal{F}u(\eta)\|_{H^\sigma(\mathbf{R}_+)} \le \langle\eta\rangle^N \|\kappa_{\langle\eta\rangle^{-1}} \mathcal{F}_{y\to\eta}(\langle t\rangle^N u)(\eta)\|_{H^\sigma(\mathbf{R}_+)}.$$

This implies

$$\begin{aligned} \|u\|^2_{\mathcal{W}^{s-N,N}(\mathbf{R}^q, H^{s,N}(\mathbf{R}_+))} &= \int \langle\eta\rangle^{2s-2N} \| \langle t\rangle^N \kappa_{\langle\eta\rangle^{-1}} \mathcal{F}_{y\to\eta}(\langle y\rangle^N u)\|^2_{H^s(\mathbf{R}_+)} d\eta \\ &\le \int \langle\eta\rangle^{2s} \|\mathcal{F}_{y\to\eta}(\langle y\rangle^N \langle t\rangle^N u)\|^2_{H^s(\mathbf{R}_+)} d\eta \\ &\le \| \langle y\rangle^N \langle t\rangle^N u\|_{\mathcal{W}^s(\mathbf{R}^q, H^s(\mathbf{R}_+))} \\ &\le C\| \langle y\rangle^N \langle t\rangle^N u\|_{H^s(\mathbf{R}^{q+1}_+)} \\ &= C \inf_U \| \langle y\rangle^N \langle t\rangle^N u\|_{H^s(\mathbf{R}^{q+1})} \\ &\le C' \inf_U \| \langle x\rangle^{2N} u\|_{H^s(\mathbf{R}^{q+1})} \\ &= C' \|u\|_{H^{s,2N}(\mathbf{R}^{q+1})}. \end{aligned}$$

Here, the infimum is always taken over all $U \in H^s(\mathbf{R}^{q+1})$ with $U|_{\mathbf{R}^{q+1}_+} = u$, and the last inequality is due to the fact that multiplication with $\langle y\rangle^N \langle t\rangle^N \langle x\rangle^{-2N}$ is bounded on $H^s(\mathbf{R}^{q+1})$; this in turn follows from the fact that $D_x^\alpha \langle y\rangle^N$ and $D_x^\alpha \langle t\rangle^N$ are both $O(\langle x\rangle^{[N-|\alpha|]_+})$.
Therefore, each weighted Sobolev space is embedded in one of the weighted wedge spaces. Altogether, we have equivalence of the $\mathcal{W}$-semi-norm system and the H-semi-norm system. In connection with 1.7(a) this gives the assertion.

1.9 Definition and Lemma, cf. [28], section 3.1.2, proposition 10. Let $E = H^{\sigma,\tau}(\mathbf{R}_+)$ for some choice of $\sigma, \tau \in \mathbf{R}$, and let $E' = H_0^{-\sigma,-\tau}(\mathbf{R}_+)$ denote its dual with respect to the extension of the sesquilinear form

$$(u, v)_{E,E'} = \int u(x)\bar{v}(x)dx$$

defined for $u \in C_0^\infty(\bar{\mathbf{R}}_+), v \in C_0^\infty(\mathbf{R}_+)$.
We obtain a natural duality $\mathcal{W}^{s,t}(\mathbf{R}^q, E), \mathcal{W}^{-s,-t}(\mathbf{R}^q, E')$ and a non-degenerate sesquilinear form by

$$\begin{aligned} \langle f, g\rangle_{\mathcal{W}^{s,t}(\mathbf{R}^q,E),\mathcal{W}^{-s,-t}(\mathbf{R}^q,E')} &= \int (\kappa_{\langle\eta\rangle^{-1}} \mathcal{F}_{y\to\eta} f, \kappa_{\langle\eta\rangle^{-1}} \mathcal{F}_{y\to\eta} g)_{E,E'}\, d\eta \qquad (1) \\ &= \int (\mathcal{F}_{y\to\eta} f, \mathcal{F}_{y\to\eta} g)_{E,E'}\, d\eta, \end{aligned}$$

noting that the group $\{\kappa_\lambda : \lambda \in \mathbf{R}_+\}$ of 1.6 (a) is unitary with respect to $(\cdot,\cdot)_{E,E'}$, i.e.

$$(\kappa_\lambda u, \kappa_\lambda v)_{E,E'} = (u, v)_{E,E'}.$$

(1) extends the usual Sobolev space sesquilinear form on $\mathbf{R}^{q+1}_+$.

1.10 Corollary. $\mathcal{S}'(\mathbf{R}^{q+1}_+) = \text{ind-lim}_{\,s,t,\sigma,\tau\to-\infty}\, \mathcal{W}^{s,t}(\mathbf{R}^q, H_0^{\sigma,\tau}(\mathbf{R}_+))$.

Proof. This follows from 1.7(a) together with the fact that $H_0^{s,t}(\mathbf{R}^{q+1}_+) = (H^{-s,-t}(\mathbf{R}^{q+1}_+))'$, and the proof of lemma 1.8.

2 Characterization of Singular Green Operators

2.1 Theorem. *Let $G : \mathcal{S}(\mathbf{R}^n_+) \longrightarrow \mathcal{S}'(\mathbf{R}^n_+)$ be a continuous linear operator. Then the following is equivalent*
(i) $G = \mathrm{Op}_G g$ *for some* $g \in \tilde{\mathcal{B}}^{-1,0}$.
(ii) *For all multi-indices* $\alpha, \beta \in \mathbf{N}_0^{n-1}$, *all* $s, t \in \mathbf{R}$, *the operator* $\mathrm{ad}^\alpha(-ix')\mathrm{ad}^\beta(D_{x'})G$ *has a continuous extension*

$$\mathrm{ad}^\alpha(-ix')\mathrm{ad}^\beta(D_{x'})G : \mathcal{W}^{s,t}(\mathbf{R}^{n-1}, \mathcal{S}'(\mathbf{R}_+)) \longrightarrow \mathcal{W}^{s-|\alpha|,t}(\mathbf{R}^{n-1}, \mathcal{S}(\mathbf{R}_+)). \qquad (1)$$

(iii) *G has the mapping properties* (1) *for* $t = 0$.

The proof will be given in a series of lemmas. I will first show that a singular Green operator has indeed the mapping properties (1). The idea is to reduce the task to standard vector valued Sobolev spaces. The crucial step in this direction is lemma 2.7.

2.2 Remark. In [28], section 3.1.2, B.-W. Schulze has introduced the following notation: Let $\mu \in \mathbf{R}$ and let E, F be Banach spaces with group actions $\kappa_\lambda, \tilde{\kappa}_\lambda$ as in 1.6. Define

$$S^\mu(\mathbf{R}^q \times \mathbf{R}^q \times \mathbf{R}^q, E, F) = \{a \in C^\infty(\mathbf{R}^q_y \times \mathbf{R}^q_{y'} \times \mathbf{R}^q_\eta, \mathcal{L}(E, F)) :$$
$$\sup_{y \in K, y' \in K} \|\tilde{\kappa}_{\langle\eta\rangle^{-1}} D^\alpha_\eta D^\beta_y D^\gamma_{y'} a(y, y', \eta) \kappa_{\langle\eta\rangle}\|_{\mathcal{L}(E,F)} \leq C_{\alpha,\beta,\gamma,K} \langle\eta\rangle^{\mu-|\alpha|}, K \subset\subset \mathbf{R}^q\}.$$

Supposing that $C_0^\infty(\mathbf{R}^q)$ acts continuously on $\mathcal{W}^s(\mathbf{R}^q, E)$ and $\mathcal{W}^s(\mathbf{R}^q, E)$ – a condition which turned out to be always fulfilled, cf. [16] – he has then proven that, given $a \in S^\mu(\mathbf{R}^q \times \mathbf{R}^q \times \mathbf{R}^q, E, F)$,

$$\mathrm{Op}\, a : \mathcal{W}^s_{comp}(\mathbf{R}^q, E) \longrightarrow \mathcal{W}^{s-\mu}_{loc}(\mathbf{R}^q, F) \tag{1}$$

is bounded. If the symbol is independent of y and y', then the *comp* and *loc* may be omitted in (1).

2.3 Definition. Let $0 \leq \rho \leq 1, \mu \in \mathbf{R}$. By $\mathcal{H}^\mu_\rho$ denote all $h \in C^\infty(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1} \times \mathbf{R}^{n-1} \times \bar{\mathbf{R}}_+ \times \bar{\mathbf{R}}_+)$ such that

$$\|x_n^k D^{k'}_{x_n} y_n^m D^{m'}_{y_n} D^\alpha_\eta D^\beta_y D^\gamma_{y'} g(y, y', \eta, x_n, y_n)\|_{L^2(\mathbf{R}_{+x_n} \times \mathbf{R}_{+y_n})} \leq C_{k,k',m,m',\alpha,\beta,\gamma} \langle\eta\rangle^{\mu-\rho|\alpha|} \tag{1}$$

This is a Fréchet space with the topology of the best constants in (1).
Write $\mathbf{R}^2_{++}$ instead of $\mathbf{R}_+ \times \mathbf{R}_+$.

2.4 Lemma. (a) *Replacing* $\|\cdot\|_{L^2(\mathbf{R}^2_{++})}$ *by* $\|\cdot\|_{\sup_{x_n, y_n}}$ *in* 2.3(1) *yields the same topology.*
(b) $H^\mu_\rho \cong \mathcal{S}(\mathbf{R}_+ \times \mathbf{R}_+, S^\mu_{\rho,0})$ *topologically by* $(x_n, y_n) \mapsto h(\cdot, \cdot, \cdot, x_n, y_n)$.
(c) $\mathcal{S}(\mathbf{R}_+ \times \mathbf{R}_+, S^\mu_{\rho,0}(\mathbf{R}^{n-1})) = \mathcal{S}(\mathbf{R}_+ \times \mathbf{R}_+) \hat{\otimes}_\pi S^\mu_{\rho,0}(\mathbf{R}^{n-1})$.
Here, $S^\mu_{\rho,0}(\mathbf{R}^{n-1})$ *denotes* $S^\mu_{\rho,0}(\mathbf{R}^{n-1}_y \times \mathbf{R}^{n-1}_{y'} \times \mathbf{R}^{n-1}_\eta)$.

Proof. (a) is obvious, since the topology generated on $\mathcal{S}(\mathbf{R}_+ \times \mathbf{R}_+)$ by the semi-norm systems $\|x_n^k D^{k'}_{x_n} y_n^m D^{m'}_{y_n} f\|_{L^2(\mathbf{R}^2_{++})}$ and $\|x_n^k D^{k'}_{x_n} y_n^m D^{m'}_{y_n} f\|_{\sup_{x_n, y_n}}$ is the same.
(b) In view of (a) and the fact that the supremum can be taken in any order, this is evident.
(c) Follows from the nuclearity of $\mathcal{S}(\mathbf{R}_+ \times \mathbf{R}_+)$, cf. [31], p.533.

2.5 Remark. Obviously, the same result holds for the subspace of y'-independent functions in $\mathcal{H}^\mu_\rho$.

2.6 Definition. Let κ_λ be a group action on the Banach space E as in 1.6(b). Define

$$T : \mathcal{W}^s(\mathbf{R}^q, E) \longrightarrow H^s(\mathbf{R}^q, E) \tag{1}$$

by

$$T = \mathcal{F}^{-1}_{\eta \to y} \kappa_{\langle\eta\rangle^{-1}} \mathcal{F}_{y \to \eta}.$$

By [28], 3.1.2, proposition 3, T extends to an isometric isomorphism.
Let F be another Banach space with a group action $\tilde{\kappa}_\lambda$ and the associated isomorphism $\tilde{T}$. Given an operator

$$A : \mathcal{W}^s(\mathbf{R}^q, E) \longrightarrow \mathcal{W}^{s'}(\mathbf{R}^q, F) \tag{2}$$

define the operator

$$A^\kappa := \tilde{T} A T^{-1} : H^s(\mathbf{R}^q, E) \longrightarrow H^{s'}(\mathbf{R}^q, F). \tag{3}$$

Clearly, A in (2) is continuous if and only if A^κ in (3) is continuous.

In the following we will use the above definition with $E = H_0^{-\sigma,-\tau}(\mathbf{R}_+)$ and $F = H^{\sigma',\tau'}(\mathbf{R}_+)$ for some choice of $\sigma, \tau, \sigma', \tau' \geq 0$. In all cases we shall use the group action of 1.6(a), and we will not discern between T and $\tilde{T}$.

2.7 Lemma. *Let $g \in \tilde{\mathcal{B}}^{\mu,0}$ be independent of y'. Then $(\mathrm{Op}_G\, g)^\kappa = \mathrm{Op}_G\, h$, where*

$$h(y,\eta,x_n,y_n) = \mathrm{Os-}\iint e^{-iw\sigma} g(y+w,\eta,\frac{x_n}{\langle\eta+\sigma\rangle},\frac{y_n}{\langle\eta\rangle})\,\langle\eta+\sigma\rangle^{-\frac{1}{2}}\,\langle\eta\rangle^{-\frac{1}{2}}\,dw\,d\sigma, \qquad (1)$$

and $h \in \mathcal{H}_1^{\mu+1}$.

Proof. A direct calculation shows the particular form (1) of the symbol kernel. Now for the estimates. We have

$$\begin{aligned} x_n^k D_{x_n}^{k'} y_n^m D_{y_n}^{m'} D_y^\beta h(y,\eta,x_n,y_n) &= \mathrm{Os-}\iint e^{-iw\sigma}\left(\frac{x_n}{\langle\eta+\sigma\rangle}\right)^k\left(\frac{y_n}{\langle\eta\rangle}\right)^m \\ &\cdot\left(D_{x_n}^{k'} D_{y_n}^{m'} D_y^\beta g\right)(y+w,\eta,\frac{x_n}{\langle\eta+\sigma\rangle},\frac{y_n}{\langle\eta\rangle})\,\langle\eta\rangle^{m-m'-\frac{1}{2}}\,\langle\sigma+\eta\rangle^{k-k'-\frac{1}{2}}\,dwd\sigma. \end{aligned}$$

If $g \in \tilde{\mathcal{B}}^{\mu,0}$, then

$$\langle\eta\rangle^{-1-\mu+m-m'-\frac{1}{2}}\, x_n^k D_{x_n}^{k'} y_n^m D_{y_n}^{m'} D_y^\beta g \in \tilde{\mathcal{B}}^{-k+k'-\frac{3}{2},0} \qquad (2)$$

In order to prove the estimate, it is therefore sufficient to show that for $k \in \tilde{\mathcal{B}}^{-m-\frac{3}{2}}, q \in S_{1,0}^{\tilde{m}-\frac{1}{2}}(\mathbf{R}^{n-1})$

$$D_\eta^\alpha\,\mathrm{Os-}\iint e^{-iw\sigma} k(y+w,\eta,\frac{x_n}{\langle\eta+\sigma\rangle},\frac{y_n}{\langle\eta\rangle})q(\sigma+\eta)dwd\sigma = \mathrm{O}(\langle\eta\rangle^{-m+\tilde{m}-|\alpha|}). \qquad (3)$$

Now,

$$\begin{aligned} &\partial_{\eta_j}\{k(y+w,\eta,\frac{x_n}{\langle\sigma+\eta\rangle},\frac{y_n}{\langle\eta\rangle})\} \\ =\ &(\partial_{\eta_j}k)(y+w,\eta,\frac{x_n}{\langle\sigma+\eta\rangle},\frac{y_n}{\langle\eta\rangle}) \\ &+\frac{x_n}{\langle\sigma+\eta\rangle}(\partial_{x_n}k)\left(y+w,\eta,\frac{x_n}{\langle\sigma+\eta\rangle}\frac{y_n}{\langle\eta\rangle}\right)\langle\sigma+\eta\rangle\,\partial_{\eta_j}(\langle\sigma+\eta\rangle^{-1}) \\ &+\frac{y_n}{\langle\eta\rangle}(\partial_{y_n}k)\left(y+w,\eta,\frac{x_n}{\langle\sigma+\eta\rangle},\frac{y_n}{\langle\eta\rangle}\right)\langle\eta\rangle\,\partial_{\eta_j}(\langle\eta\rangle^{-1}). \end{aligned}$$

In order to establish (3), it is therefore sufficient to consider the case $\alpha = 0$, using (2). To this end we will use a procedure developed by Kumano-go for pseudodifferential operators, cf. [17], chapter 2, proof of lemma 2.4. Write

$$h(y,\eta,x_n,y_n) = \int\int e^{-iw\sigma} r(y,\eta,x_n,y_n;w,\sigma)dwd\sigma,$$

where

$$r(y,\eta,x_n,y_n;w,\sigma) = \langle w\rangle^{-2l_0}(1-\Delta_\sigma)^{l_0}\left\{k(y+w,\eta,\frac{x_n}{\langle\sigma+\eta\rangle},\frac{y_n}{\langle\eta\rangle})q(\sigma+\eta)\right\}.$$

Let $\Omega_1 = \{|\sigma| \leq \frac{1}{2}\}, \Omega_2 = \{\frac{1}{2} \leq |\sigma| \leq \frac{1}{2}\langle\eta\rangle\}, \Omega_3 = \{|\sigma| \geq \frac{1}{2}\langle\eta\rangle\}$, and write for $j = 1, 2, 3$:

$$I_j = \int_{\Omega_j}\int_{\mathbf{R}^{n-1}} e^{-iw\sigma} r(y, \eta, x_n, y_n; w, \sigma) dw d\sigma.$$

Then

$$\frac{1}{2}\langle\eta\rangle \leq \langle\sigma+\eta\rangle \leq \frac{3}{2}\langle\eta\rangle \text{ on } \Omega_1 \cup \Omega_2. \tag{4}$$

By Leibniz' rule

$$\begin{aligned} & \|r(y, \eta, x_n, y_n; w, \sigma)\|_{L^2(\mathbf{R}^2_{++})} \qquad (5) \\ \leq \ & \langle w\rangle^{-2l_0} \sum_{|\alpha_1+\alpha_2|\leq 2l_0} c_{\alpha_1,\alpha_2} \left\| D_\sigma^{\alpha_1}\{k(y+w, \eta, \frac{x_n}{\langle\sigma+\eta\rangle}, \frac{y_n}{\langle\eta\rangle})\} \right\|_{L^2(\mathbf{R}^2_{++})} \\ & \cdot |q^{(\alpha_2)}(\sigma+\eta)|. \end{aligned}$$

Note that

$$\begin{aligned} & \partial_{\sigma_j}\{k(y+w, \eta, \frac{x_n}{\langle\sigma+\eta\rangle}, \frac{y_n}{\langle\eta\rangle})\} \\ = \ & \left(\frac{x_n}{\langle\sigma+\eta\rangle}\right)(\partial_{x_n}k)\left(y+w, \eta, \frac{x_n}{\langle\sigma+\eta\rangle}, \frac{y_n}{\langle\eta\rangle}\right)\langle\sigma+\eta\rangle\, \partial_{\sigma_j}(\langle\sigma+\eta\rangle^{-1}) \end{aligned}$$

For a function $\tilde{k} \in \tilde{\mathcal{B}}^{\mu,0}$ we have

$$\|x_n^k D_{x_n}^{k'} D_y^\beta \tilde{k}(y, \eta, x_n, y_n)\|_{L^2(\mathbf{R}^2_{++})} = \mathrm{O}(\langle\eta\rangle^{\mu+1-k+k'}),$$

and correspondingly

$$\begin{aligned} & \left\| \left(\frac{x_n}{\langle\sigma+\eta\rangle}\right)^k (D_{x_n}^{k'} D_y^\beta \tilde{k})\left(y+w, \eta, \frac{x_n}{\langle\sigma+\eta\rangle}, \frac{y_n}{\langle\eta\rangle}\right) \right\|_{L^2(\mathbf{R}^2_{++})} \\ = \ & \mathrm{O}(\langle\sigma+\eta\rangle^{\frac{1}{2}}\langle\eta\rangle^{\frac{1}{2}+\mu+1-k+k'}) \end{aligned} \tag{6}$$

We may therefore estimate (5) with the help of (4) and (6) by

$$C\langle w\rangle^{-2l_0}\sum_{\alpha_1,\alpha_2}\left(\langle\sigma+\eta\rangle^{\frac{1}{2}}\langle\eta\rangle^{-m}\langle\sigma+\eta\rangle^{-|\alpha_1|}\right)\langle\sigma+\eta\rangle^{\tilde{m}-\frac{1}{2}-|\alpha_2|}$$

$$\leq c'\langle w\rangle^{-2l_0}\langle\eta\rangle^{\tilde{m}-m}.$$

This shows that

$$\|I_1\|_{L^2(\mathbf{R}^2_{++})} \leq c_1 \int_{\Omega_1}\int_{\mathbf{R}^{n-1}} \langle w\rangle^{-2l_0} dw\, d\sigma\, \langle\eta\rangle^{\tilde{m}-m} \leq c_2\langle\eta\rangle^{\tilde{m}-m}.$$

Next let us use that for $l \in \mathbf{N}_0$

$$\iint e^{-iw\sigma} r\, dw\, d\sigma = \iint e^{-iw\sigma}|\sigma|^{-2l}\Delta_y^l r\, dw\, d\sigma. \tag{7}$$

Hence

$$\begin{aligned}
\|I_2\|_{L^2(\mathbf{R}^2_{++})} &\leq \int_{\Omega_2} |\sigma|^{-2l} \int \langle w\rangle^{-2l_0} \sum_{|\alpha_1+\alpha_2|\leq 2l_0} c_{\alpha_1,\alpha_2} \\
&\qquad \left\| D_\sigma^{\alpha_1} \Delta_y^l \{k(y+w,\eta,\frac{x_n}{\langle\sigma+\eta\rangle},\frac{y_n}{\langle\eta\rangle})\}\right\|_{L^2(\mathbf{R}^2_{++})} |q^{(\alpha_2)}(\sigma+\eta)| dw d\sigma \\
&\leq c_3 \int_{\Omega_2} |\sigma|^{-2l} \int \langle w\rangle^{-2l_0} \sum \langle\sigma+\eta\rangle^{\frac{1}{2}-|\alpha_1|+\tilde{m}-\frac{1}{2}-|\alpha_2|} \langle\eta\rangle^{-m} dw d\sigma \\
&\leq c_4 \int_{\Omega_2} |\sigma|^{-2l} d\sigma \langle\eta\rangle^{\tilde{m}-m} = c_5 \langle\eta\rangle^{\tilde{m}-m},
\end{aligned}$$

provided $2l > n-1$. For the last inequality, estimate (4) has been employed, for that before, I have used (6).
Now consider Ω_3. Here,

$$\langle\sigma+\eta\rangle \leq 3|\sigma| \text{ and } \langle\eta\rangle \leq 2|\sigma|,$$

and (7) gives

$$\begin{aligned}
\|I_3\|_{L^2(\mathbf{R}^2_{++})} &\leq \int_{\Omega_3} |\sigma|^{-2l} \int \langle w\rangle^{-2l_0} \sum c_{\alpha_1,\alpha_2} \left\| D_\sigma^{\alpha_1} \Delta_y^l \{k(y+w,\eta,\frac{x_n}{\langle\sigma+\eta\rangle},\frac{y_n}{\langle\eta\rangle})\}\right\|_{L^2(\mathbf{R}^2_{++})} \\
&\qquad \cdot |q^{(\alpha_2)}(\sigma+\eta)|\, dw\, d\sigma \\
&\leq c_6 \int_{\Omega_3} |\sigma|^{-2l} \int \langle w\rangle^{-2l_0} \langle\sigma+\eta\rangle^{\frac{1}{2}-|\alpha_1|+\tilde{m}-\frac{1}{2}-|\alpha_2|} \langle\eta\rangle^{-m}\, dw\, d\sigma \\
&\leq c_7 \int_{\Omega_3} |\sigma|^{-2l+2|\tilde{m}|} d\sigma \langle\eta\rangle^{\tilde{m}-m} = c_8 \langle\eta\rangle^{\tilde{m}-m}.
\end{aligned}$$

provided we choose $2l > n-1+2|\tilde{m}|$. This completes the proof.

2.8 Lemma. *Let $g \in \tilde{\mathcal{B}}^{\mu,0}$. Then for all $s,t \in \mathbf{R}$, all $\sigma,\sigma',\tau,\tau'$*

$$\mathrm{Op}_G\, g : \mathcal{W}^{s,t}(\mathbf{R}^{n-1},E) \longrightarrow \mathcal{W}^{s-\mu-1,t}(\mathbf{R}^{n-1},F)$$

is bounded, where $E = H_0^{-\sigma,-\tau}(\mathbf{R}_+)$, $F = H^{\sigma',\tau'}(\mathbf{R}_+)$. In other words:

$$\mathrm{Op}_G\, g : \mathcal{W}^{s,t}(\mathbf{R}^{n-1},\mathcal{S}'(\mathbf{R}_+)) \longrightarrow \mathcal{W}^{s-\mu-1,t}(\mathbf{R}^{n-1},\mathcal{S}(\mathbf{R}_+))$$

is bounded.

Proof. In view of the composition rules for Green operators and by using commutator identities like

$$\langle y\rangle^N A = \sum_{l=0}^{N} \binom{l}{N} \mathrm{ad}^l \langle y\rangle (A) \langle y\rangle^{N-l} \tag{1}$$

as in [24], lemma 2.6, we may assume $s=t=0$ and $\mu=-1$. Let us show equivalently that

$$(\mathrm{Op}_G\, g)^\kappa : H^0(\mathbf{R}^{n-1},E) \longrightarrow H^0(\mathbf{R}^{n-1},F)$$

is bounded. By lemma 2.7, $(\mathrm{Op}_G\, g)^\kappa = \mathrm{Op}_G h$ with an $h \in H^0$. We know from lemma 2.4 that we may write

$$h(y,\eta,x_n,y_n) = \sum_{j=0}^{\infty} \lambda_j a_j(y,\eta)\psi_j(x_n,y_n) \tag{2}$$

with $(\lambda_j) \in l^1, a_j \in S^0_{1,0}(\mathbf{R}^{n-1})$, and $\psi_j \in \mathcal{S}(\mathbf{R}_+ \times \mathbf{R}_+)$ tending to zero.
The integral operators with kernels in $\mathcal{S}(\mathbf{R}_+ \times \mathbf{R}_+)$ have bounded norm in $\mathcal{L}(E,F)$ which can be estimated by the $\mathcal{S}$-semi-norms of the kernel. Therefore the boundedness of the operators associated with each summand in (2) is a consequence of an extension of Calderón and Vaillancourt's theorem to the case of pseudodifferential symbols with values in $\mathcal{L}(E,F)$, noting that E, F are Hilbert spaces. The operators associated with the partial sums in (2) converge to an operator which is $(\mathrm{Op}_G\, g)^\kappa$ by Lebesgue's theorem on dominated convergence.

2.9 Conclusion. This ends the first part of the proof of 2.1, since

$$\mathrm{ad}^\alpha(-iy)\mathrm{ad}^\beta(D_y)G$$

is the operator with the symbol kernel $\partial_\eta^\alpha D_y^\beta g(y,\eta,x_n,y_n) \in \tilde{\mathcal{B}}^{-1-|\alpha|}$, and therefore has the desired mapping properties.

Let us now prove that 2.1(iii) implies 2.1(i).

2.10 Definition. Fix a function $\phi \in C_0^\infty(\mathbf{R}^{n-1})$, equal to 1 in a neighborhood of the origin. For $\epsilon > 0$ let $\phi_\epsilon(y) = \phi(\epsilon y)$, let 1 denote the identity mapping on $\mathcal{S}'(\mathbf{R}_+)$ and let

$$P_\epsilon := \mathrm{Op}\ (\phi_\epsilon(y)\phi_\epsilon(\eta))\, 1;$$

$$p_\epsilon(y,\eta) := \phi_\epsilon(y)\phi_\epsilon(\eta)\, 1;$$
$$G_\epsilon := P_\epsilon G_\epsilon P_\epsilon.$$

2.11 Lemma. *Let $E, F \in \{H^{\sigma,\tau}(\mathbf{R}_+), H_0^{\sigma,\tau}(\mathbf{R}_+) : \sigma, \tau \in \mathbf{R}\}, s, t, \bar{s}, \bar{t} \in \mathbf{R}, 0 < \epsilon \le 1$. Then*
(a) $P_\epsilon : \mathcal{W}^{s,t}(\mathbf{R}^{n-1}, E) \longrightarrow \mathcal{W}^{\bar{s},\bar{t}}(\mathbf{R}^{n-1}, E)$ *is bounded;*
(b) $P_\epsilon : \mathcal{W}^{s,t}(\mathbf{R}^{n-1}, E) \longrightarrow \mathcal{W}^{s,t}(\mathbf{R}^{n-1}, E)$ *is uniformly bounded,* $0 < \epsilon \le 1$.
(c) $G_\epsilon : \mathcal{W}^{s}(\mathbf{R}^{n-1}, E) \longrightarrow \mathcal{W}^{s}(\mathbf{R}^{n-1}, F)$ *is uniformly bounded,* $0 < \epsilon \le 1$.
(d) G_ϵ *is an integral operator with kernel function in* $\mathcal{S}(\mathbf{R}^n_+ \times \mathbf{R}^n_+)$.

Proof. For fixed $(y,\eta), p_\epsilon(y,\eta)$ simply is a constant multiplication operator on $\mathcal{S}(\mathbf{R}_+)$. We have

$$\|\kappa_{\langle\eta\rangle^{-1}} D_\eta^\alpha D_y^\beta p_\epsilon(y,\eta)\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(E)} = |D_\eta^\alpha D_y^\beta p_\epsilon(y)\phi_\epsilon(\eta))|.$$

(a) now follows from 2.2, since $P_\epsilon = \phi_\epsilon(y)\mathrm{Op}\ \phi_\epsilon(\eta)$ and since ϕ has compact support.
(b) For $t = 0$, this follows from 2.2, using that $\phi_\epsilon(y)$ is uniformly bounded in C_b^∞ for $0 < \epsilon \le 1$, and $\phi_\epsilon(\eta)$ is uniformly bounded in $S^0_{1,0}(\mathbf{R}^{n-1}_\eta)_{const}$, the space of symbols that are independent of y and y'. For small $|t|$, the commutator $[\langle y\rangle^t, P_\epsilon]$ is uniformly bounded, so we also obtain the assertion; finally use commutator identities for the case of large $|t|$, cf. 2.8(1) and [24], lemma 2.6.
(c) is immediate from (b).
(d) For every $\epsilon > 0$ and all $s, t, \bar{s}, \bar{t} \in \mathbf{R}$,

$$G_\epsilon : \mathcal{W}^{s,t}(\mathbf{R}^{n-1}, E) \longrightarrow \mathcal{W}^{\bar{s},\bar{t}}(\mathbf{R}^{n-1}, F)$$

is bounded for arbitrary E, F. By 1.8, G_ϵ therefore is bounded from $\mathcal{S}'(\mathbf{R}^n_+)$ to $\mathcal{S}(\mathbf{R}^n_+)$ and thus has its kernel in $\mathcal{S}(\mathbf{R}^n_+ \times \mathbf{R}^n_+)$.

2.12 Definition. Let $G_\epsilon^\kappa = TG_\epsilon T^{-1}$. In view of the mapping properties 2.11,

$$G_\epsilon^\kappa : \mathcal{S}'(\mathbf{R}_+^n) \longrightarrow \mathcal{S}(\mathbf{R}_+^n)$$

is bounded, thus G_ϵ^κ has a kernel $t_\epsilon^\kappa \in \mathcal{S}(\mathbf{R}_+^n \times \mathbf{R}_+^n)$. Fix a function $\gamma \in \mathcal{S}(\mathbf{R}^{n-1})$ such that $\gamma(0) = 1$ and $\gamma(-y) = \gamma(y)$. Let $\gamma_{y'}(y) = \gamma(y - y')$. Then define for fixed $y, y', \eta \in \mathbf{R}^{n-1}$

$$g_\epsilon^\kappa(y, y', \eta, D_n) : \mathcal{S}(\mathbf{R}_+) \longrightarrow \mathcal{S}(\mathbf{R}_+)$$

by

$$[g_\epsilon^\kappa(y, y', \eta, D_n)v](x_n) = e^{-iy\eta} G_\epsilon^\kappa(e^{i\cdot\eta}\gamma_{y'} \otimes v)(y, x_n).$$

We will also apply $g_\epsilon^\kappa(y, y', \eta, D_n)$ to functions on $\mathbf{R}_+^n$, understanding that then D_n acts with respect to the variable in $\mathbf{R}_+$ only.

2.13 Lemma. $\mathrm{Op}_{y,y',\eta} g_\epsilon^\kappa(y, y', \eta, D_n) = G_\epsilon^\kappa$.

Proof. I am using ideas of Ueberberg [33], pp. 465, 466. Let $u \in \mathcal{S}(\mathbf{R}^{n-1})$ be arbitrary. Then

$$u(y) = \mathrm{Os}\text{-}\iint e^{i(y-y')\eta} u(y')\gamma_{y'}(y)dy'đ\eta.$$

Therefore we obtain for $v \in \mathcal{S}(\mathbf{R}_+)$

$$\begin{aligned} G_\epsilon^\kappa(u \otimes v)(y, x_n) &= \int\int_0^\infty t_\epsilon^\kappa(y, x_n, z, y_n) \\ &\quad \cdot \mathrm{Os}\text{-}\iint e^{i(z-y')\eta} u(y')\gamma_{y'}(z)dy'đ\eta \quad v(y_n)dy_n dz \qquad (1) \\ &= \iint_\infty^0 t_\epsilon^\kappa(y, x_n, z, y_n) \\ &\quad \cdot \left[\lim_{\alpha\to 0} \iint e^{i(z-y')\eta} u(y')\gamma_{y'}(z)\chi(\alpha\eta)dy'đ\eta\right] \quad v(y_n)dy_n dz \end{aligned}$$

by definition of the oscillatory integral. Here $\chi \in C_0^\infty(\mathbf{R}^{n-1})$ is a function with $\chi(0) = 1$ and $\alpha > 0$. For fixed y, z define

$$r_\alpha(z) = \iint e^{i(z-y')\eta} u(y')\gamma_{y'}(z)\chi(\alpha\eta)dy'đ\eta.$$

Then $z \mapsto r_\alpha(z)$ is a measurable function, and for fixed z,

$$r_\alpha(z) \to r(z) := \mathrm{Os}\text{-}\iint e^{i(z-y')\eta} u(y')\gamma_{y'}(z)dy'đ\eta$$

as $\alpha \to 0^+$. Taking $n_0 = \mathrm{entier}(\frac{n-1}{2}) + 1$, partial integration shows that

$$\begin{aligned} |r_\alpha(z)| &\le \left| \int \frac{1}{1+|\eta|^{2n_0}} e^{i(z-y')\eta}(1 + (-\Delta_{y'})^{n_0})\{u(y')\gamma_{y'}(z)\}\chi(\alpha\eta)dy'đ\eta \right| \\ &\le \text{const., independent of } \alpha. \end{aligned}$$

Since $t_\epsilon^\kappa(y, x_n, z, y_n) \in L^1(\mathbf{R}_z^{n-1} \times \mathbf{R}_{+y_n})$, independent of α, Lebesgue's theorem on dominated convergence shows that

$$G_\epsilon^\kappa(u \otimes v)(y, x_n) \qquad (2)$$

$$
\begin{aligned}
&= \lim_{\alpha\to 0}\iint_0^\infty t_\epsilon^\kappa(y,x_n,z,y_n)\iint e^{i(z-y')\eta}u(y')\gamma_{y'}(z)\chi(\alpha\eta)\,dy'đ\eta v(y_n)dy_ndz\\
&= \lim_{\alpha\to 0}\iint e^{-iy'\eta}\chi(\alpha\eta)\iint_0^\infty t_\epsilon^\kappa(y,x_n,z,y_n)e^{iz\eta}\gamma_{y'}(z)v(y_n)dy_ndz\ u(y')dy'đ\eta\\
&= \lim_{\alpha\to 0}\iint e^{-iy'\eta}\chi(\alpha\eta)G_\epsilon^\kappa(e^{i\cdot\eta}\gamma_{y'}\otimes v)u(y')dy'đ\eta\\
&= \mathrm{Os}-\iint e^{i(y-y')\eta}g_\epsilon^\kappa(y,y',\eta,D_{y_n})(u\otimes v)dy'đ\eta\\
&= [\mathrm{Op}\ g_\epsilon^\kappa(y,y',\eta,D_n)(u\otimes v)]\,(y,x_n)
\end{aligned}
$$

Since $\mathcal{S}(\mathbf{R}^n_+)=\mathcal{S}(\mathbf{R}^{n-1})\hat{\otimes}_\pi\mathcal{S}(\mathbf{R}_+)$, we obtain the assertion.

2.14 Corollary. Since

$$G_\epsilon^\kappa:\mathcal{S}'(\mathbf{R}^n_+)\longrightarrow\mathcal{S}(\mathbf{R}^n_+),$$

$g_\epsilon^\kappa(y,y',\eta,D_n)$ actually maps $\mathcal{S}'(\mathbf{R}_+)$ to $\mathcal{S}(\mathbf{R}_+)$, and is an integral operator with a kernel function $g_\epsilon^\kappa(y,y',\eta,x_n,y_n)$, which is rapidly decreasing with respect to x_n and y_n. We obtain it by

$$g_\epsilon^\kappa(y,y',\eta,x_n,y_n)=\langle g_\epsilon^\kappa(y,y',\eta,D_n)\delta_{y_n},\delta_{x_n}\rangle_{\mathcal{S}(\mathbf{R}_+),\mathcal{S}'(\mathbf{R}_+)}\,. \tag{1}$$

In fact we may replace the above duality by any of the $H^{\sigma,\tau}(\mathbf{R}_+),H_0^{-\sigma,-\tau}(\mathbf{R}_+)$-dualities for sufficiently large σ,τ.

2.15 Lemma. (a) *The following identities can be verified by a direct computation.*

(i) $x_n\partial_{x_n}T^{\pm1} = T^{\pm1}x_n\partial_{x_n}$

(ii) $\mathrm{Op}\ p(\eta)T^{\pm1} = T^{\pm1}\mathrm{Op}\ p(\eta)$ *for any* $p\in S^0_{0,0}(\mathbf{R}^{n-1})$ *which is independent of* y.

For $i=1,\dots,n-1$,

(iii) $\mathrm{ad}(-iy_j)T = -\frac{1}{2}\mathrm{Op}\ (\frac{\eta_j}{\langle\eta\rangle^2})T-x_n\partial_{x_n}\mathrm{Op}\ (\frac{\eta_j}{\langle\eta\rangle^2})T$

(iv) $\mathrm{ad}(-iy_j)T^{-1} = \frac{1}{2}\mathrm{Op}\ (\frac{\eta_j}{\langle\eta\rangle^2})T+\mathrm{Op}\ (\frac{\eta_j}{\langle\eta\rangle^2})Tx_n\partial_{x_n}$

(v) $\mathrm{ad}(D_{y_j})T = \mathrm{ad}(D_{y_j})T^{-1}=0.$

(b) *For every choice of* $\alpha,\beta,s,\sigma,\tau$,

$$\mathrm{ad}^\alpha(-iy)\mathrm{ad}^\beta(D_y)G_\epsilon^\kappa:H^s(\mathbf{R}^{n-1},H_0^{-\sigma,-\tau}(\mathbf{R}_+))\longrightarrow H^{s+|\alpha|}(\mathbf{R}^{n-1},H^{\sigma,\tau}(\mathbf{R}_+))$$

is bounded uniformly in $\epsilon, 0<\epsilon\le 1$.

Proof of (b). By Leibniz' rule

$$
\begin{aligned}
\mathrm{ad}^\alpha(-iy)\mathrm{ad}^\beta(D_y)G_\epsilon^\kappa &= \sum_{\substack{\alpha_1+\alpha_2+\alpha_3=\alpha\\ \beta_1+\beta_2+\beta_3=\beta}} c_{\alpha_1,\dots,\beta_3}\ \ \mathrm{ad}^{\alpha_1}(-iy)\mathrm{ad}^{\beta_1}(D_y)T\cdots\\
&\quad\cdots\mathrm{ad}^{\alpha_2}(-iy)\mathrm{ad}^{\beta_2}(D_y)G_\epsilon\,\mathrm{ad}^{\alpha_3}(-iy)\mathrm{ad}^{\beta_3}(D_y)T^{-1}.
\end{aligned}
$$

In connection with 2.11, the identities of (a) give the assertion, noting that the mappings

$$
\begin{aligned}
x_n\partial_{x_n}\ &:\ H^s(\mathbf{R}^{n-1},H^{\sigma,\tau}(\mathbf{R}_+))\to H^s(\mathbf{R}^{n-1},H^{\sigma-1,\tau-1}(\mathbf{R}_+)),\\
x_n\partial_{x_n}\ &:\ H^s(\mathbf{R}^{n-1},H_0^{-\sigma,-\tau}(\mathbf{R}_+))\to H^s(\mathbf{R}^{n-1},H_0^{-\sigma-1,-\tau-1}(\mathbf{R}_+)),\\
\mathrm{Op}\ (\frac{\eta_j}{\langle\eta\rangle^2})\ &:\ H^s(\mathbf{R}^{n-1},H^{\sigma,\tau}(\mathbf{R}_+))\to H^{s+1}(\mathbf{R}^{n-1},H^{\sigma,\tau}(\mathbf{R}_+)),\\
\mathrm{Op}\ (\frac{\eta_j}{\langle\eta\rangle^2})\ &:\ H^s(\mathbf{R}^{n-1},H_0^{-\sigma,-\tau}(\mathbf{R}_+))\to H^{s+1}(\mathbf{R}^{n-1},H_0^{-\sigma,-\tau}(\mathbf{R}_+))
\end{aligned}
$$

are all bounded, provided we ask that $\sigma>\frac{1}{2}$ for the first mapping.

2.16 The rough estimate. Let $g^\kappa_\epsilon(y,y',\eta,D_n)$ and $g^\kappa_\epsilon(y,y',\eta,x_n,y_n)$ be as in 2.13, 2.14. For fixed $k,k',m,m',\alpha,\beta,\beta'$ consider

$$E_\epsilon(y,y',\eta,x_n,y_n) = x_n^k D_{x_n}^{k'} y_n^m D_{y_n}^{m'} \partial_\eta^\alpha D_y^\beta D_{y'}^{\beta'} g^\kappa_\epsilon(y,y',\eta,x_n,y_n).$$

Then by 2.14(1)

$$\begin{aligned} & |E_\epsilon(y,y',\eta,x_n,y_n)\langle x_n\rangle\langle y_n\rangle| & (1)\\ = & \left|\left\langle \langle u_n\rangle u_n^k D_{u_n}^{k'} \partial_\eta^\alpha D_y^\beta D_{y'}^{\beta'} g^\kappa_\epsilon(y,y',\eta,D_n)(\langle z_n\rangle z_n^m D_{z_n}^{m'}\delta_{y_n}),\delta_{x_n}\right\rangle_{H^{\sigma,\tau}(\mathbf{R}_{+u_n}),H_0^{-\sigma,\tau}(\mathbf{R}_{+u_n})}\right| \\ \le & \|\partial_\eta^\alpha D_y^\beta D_{y'}^{\beta'} g^\kappa_\epsilon(y,y',\eta,D_n)(\langle z_n\rangle z_n^m D_{z_n}^{m'}\delta_{y_n})\|_{H^{\sigma,\tau}(\mathbf{R}_{+x_n})} \\ & \cdot \|D_{z_n}^{k'}\{z_n^k\langle z_n\rangle\delta_{x_n}\}\|_{H_0^{-\sigma,-\tau}(\mathbf{R}_{+x_n})}. \end{aligned}$$

For $\sigma > k'+\frac{1}{2}, \tau > k+\frac{3}{2}$, $\|D_{z_n}^{k'}\{z_n^k\langle z_n\rangle\delta_{x_n}\}\|_{H_0^{-\sigma,-\tau}(\mathbf{R}_{+x_n})} \le C < \infty$, independent of x_n. Moreover, for $v\in\mathcal{S}'(\mathbf{R}_+)$ a calculation shows that

$$\begin{aligned} &\left[\partial_\eta^\alpha D_y^\beta D_{y'}^{\beta'} g^\kappa_\epsilon(y,y',\eta,D_n)v\right](x_n) = \\ &= e^{-iy\eta}\sum_{\beta_1+\beta_2=\beta} c_{\beta_1,\beta_2}\mathrm{ad}^\alpha(-iy)\mathrm{ad}^{\beta_1}(D_y)\left(G^\kappa_\epsilon(e^{i\cdot\eta}D_y^{\beta_2+\beta'}\gamma_{y'}\otimes v)\right)(x_n). \qquad (2) \end{aligned}$$

Since for measurable f,

$$\int |f(x_n,y_n|^2 dx_n dy_n \le |f(x_n,y_n)\langle x_n\rangle\langle y_n\rangle|^2_{\sup_{x_n,y_n}}\int\langle x_n\rangle^{-2}\langle y_n\rangle^{-2}dx_n dy_n$$

we conclude that

$$\begin{aligned} & \|E_\epsilon(y,y',\eta,x_n,y_n)\|^2_{L^2(\mathbf{R}^{n-1}_y\times\mathbf{R}_+\times\mathbf{R}_+)} \\ = & \int \|E_\epsilon(y,y',\eta,x_n,y_n)\|^2_{L^2(\mathbf{R}^2_{++})}dy \\ \le & C\int\|E_\epsilon(y,y',\eta,x_n,y_n)\langle x_n\rangle\langle y_n\rangle\|^2_{\sup_{x_n,y_n}}dy. \qquad (3) \end{aligned}$$

Using (1) and (2), (3) can be estimated by

$$\le C'\sum_{\beta_1+\beta_2=\beta}\int\|\mathrm{ad}^\alpha(-iy)\mathrm{ad}^\beta(D_y)G^\kappa_\epsilon(e^{i\cdot\eta}D_y^{\beta_2+\beta'}\gamma_{y'}\otimes v)\|^2_{H^{\sigma,\tau}(\mathbf{R}_+)}dy, \qquad (4)$$

where $v(z_n) = \langle z_n\rangle z_n^m D_{z_n}^{m'}\delta_{y_n}$. We may estimate (4) by

$$\begin{aligned} \le\ & C''\sum_{\beta_1+\beta_2=\beta}\|\mathrm{ad}^\alpha(-iy)\mathrm{ad}^{\beta_1}(D_y)G^\kappa_\epsilon\|^2_{\mathcal{L}(L^2(\mathbf{R}^{n-1},H_0^{-\sigma,-\tau}(\mathbf{R}_+)),L^2(\mathbf{R}^{n-1},H^{\sigma,\tau}(\mathbf{R}_+)))} \\ & \cdot\|e^{iy\eta}D_y^{\beta_2+\beta'}\gamma_{y'}(y)z_n^m D_{z_n}^{m'}\delta_{x_n}\|^2_{L^2(\mathbf{R}^{n-1}_y,H_0^{-\sigma,-\tau}(\mathbf{R}_+))}. \end{aligned}$$

The first norm in each of these summands is bounded independent of ϵ for $0<\epsilon\le 1$ by lemma 2.15, the second equals

$$\|D^{\beta_2+\beta'}\gamma_{y'}\|^2_{L^2(\mathbf{R}^{n-1})}\|\langle z_n\rangle z_n^m D_{z_n}^{m'}\delta_{x_n}\|^2_{H_0^{-\sigma,-\tau}(\mathbf{R}_+)}.$$

Here, the first factor is clearly bounded, independent of y', the second is bounded independent of x_n, if we choose $\sigma > m'+\frac{1}{2}, \tau > m+\frac{3}{2}$.

Altogether, we have shown that the L^2-norm of the mapping

$$(y, x_n, y_n) \mapsto x_n^k D_{x_n}^{k'} y_n^m D_{y_n}^{m'} \partial_\eta^\alpha D_y^\beta D_{y'}^{\beta'} g_\epsilon^\kappa(y, y', \eta, x_n, y_n)$$

is bounded independent of ϵ and y', and so is – with the same considerations – the L^2-norm of any of its derivatives. By Sobolev's lemma,

$$|x_n^k D_{x_n}^{k'} y_n^m D_{y_n}^{m'} \partial_\eta^\alpha D_y^\beta D_{y'}^{\beta'} g_\epsilon^\kappa(y, y', \eta, x_n, y_n)| \le C(k, k', m, m', \alpha, \beta, \beta'),$$

independent of ϵ and y', or equivalently,

$$\|x_n^k D_{x_n}^{k'} y_n^m D_{y_n}^{m'} \partial_\eta^\alpha D_y^\beta D_{y'}^{\beta'} g_\epsilon^\kappa(y, y', \eta, x_n, y_n)\|_{L^2(\mathbf{R}^2_{++})} \le D(k, k', m, m', \alpha, \beta, \beta'). \quad (5)$$

2.17 Corollary. The estimates in 2.16 show that

$$\{g_\epsilon^\kappa : 0 < \epsilon \le 1\} \text{ is uniformly bounded in } \mathcal{H}_0^0.$$

The symbol kernels g_ϵ^κ depend on y'. However, we can find symbol kernels $\tilde{g}_\epsilon^\kappa$ such that
(i) $\tilde{g}_\epsilon^\kappa$ is independent of y',
(ii) $\{\tilde{g}_\epsilon^\kappa : 0 < \epsilon \le 1\}$ is uniformly bounded in $\mathcal{H}_0^0$,
(iii) $\mathrm{Op}_G\, \tilde{g}_\epsilon^\kappa = \mathrm{Op}_G\, g_\epsilon^\kappa = G_\epsilon^\kappa$.

Proof. By 2.4(b),(c) we can write

$$g_\epsilon^\kappa(y, y', \eta, x_n, y_n) = \sum \lambda_j^{(\epsilon)} \psi_j^{(\epsilon)}(x_n, y_n) a_j^{(\epsilon)}(y, y', \eta) \quad (1)$$

with $\lambda_j^{(\epsilon)} \in l^1$, and $\psi_j^{(\epsilon)}, a_j^{(\epsilon)}$ null sequences in $\mathcal{S}(\mathbf{R}_+ \times \mathbf{R}_+)$ and $S_{0,0}^0(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1})$, respectively.
By standard pseudodifferential methods, [17], chapter 2, we can find $b_j^{(\epsilon)} \in S_{0,0}^0(\mathbf{R}_y^{n-1} \times \mathbf{R}_\eta^{n-1})$ such that Op $b_j^{(\epsilon)}$ = Op $a_j^{(\epsilon)}$; moreover, the mapping $a_j^{(\epsilon)} \mapsto b_j^{(\epsilon)}$ is continuous with respect to the symbol topology. Now define $\tilde{g}_\epsilon^\kappa$ by replacing $a_j^{(\epsilon)}$ in (1) by $b_j^{(\epsilon)}$. Since the topology on $\mathcal{H}_{0,0}^0$ coincides with the tensor product topology, (i) and (ii) are obvious.
(iii) Consider the associated operators. The corresponding sum (1) converges in operator norm and the operators given by the partial sums coincide by construction, so we get the operator identity.

2.18 Corollary. The estimate 2.17(ii) may be rephrased in the following way: For all $N, N', k, m, \alpha, \beta$ there is a $C > 0$ such that

$$|D_{x_n}^k D_{y_n}^m D_\eta^\alpha D_y^\beta D^{\beta'} \tilde{g}_\epsilon^\kappa(y, y', \eta, x_n, y_n)| \le C \langle x_n \rangle^{-N} \langle y_n \rangle^{-N'}. \quad (1)$$

By the theorem of Arzelà and Ascoli, there is a sequence $\epsilon_j \to 0$ and a function $g^\kappa \in C^\infty(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1} \times \mathbf{R}^{n-1} \times \bar{\mathbf{R}}_+ \times \bar{\mathbf{R}}_+)$ such that $g_{\epsilon_j}^\kappa \to g^\kappa$ in all derivatives, uniformly on compact subsets of $\mathbf{R}^{n-1} \times \mathbf{R}^{n-1} \times \mathbf{R}^{n-1} \times \bar{\mathbf{R}}_+ \times \bar{\mathbf{R}}_+$. In particular, the pointwise limit g^κ also satisfies the estimates (1).

2.19 Lemma. $\mathrm{Op}_G\,(g^\kappa) = G^\kappa$.

Proof. It is sufficient to show that

$$[\operatorname{Op}_G(g^\kappa)f](x) = [G^\kappa f](x) \tag{1}$$

for every $f \in \mathcal{S}(\mathbf{R}^n_+)$ and every $x \in \mathbf{R}^n_+$; we may even restrict ourselves to f of the form $f(y, x_n) = f_1(y) f_2(x_n)$, where $f_1 \in \mathcal{F}^{-1}(C_0^\infty(\mathbf{R}^{n-1}))$ and $f_2 \in C_0^\infty(\mathbf{R}_+)$: this follows from the fact that $\mathcal{S}(\mathbf{R}^n_+) = \mathcal{S}(\mathbf{R}^{n-1}) \hat{\otimes}_\pi \mathcal{S}(\mathbf{R}_+)$, the density of $C_0^\infty(\mathbf{R}^{n-1})$ in $\mathcal{S}(\mathbf{R}^{n-1})$ and the fact that $\mathcal{F} : \mathcal{S}(\mathbf{R}^{n-1}) \longrightarrow \mathcal{S}(\mathbf{R}^{n-1})$ is an isomorphism.
For such f we conclude that

$$\operatorname{Op}_G(g^\kappa_{\epsilon_j}) f(y, x_n) = (2\pi)^{\frac{1-n}{2}} \int e^{iy\eta} \int_0^\infty g^\kappa_{\epsilon_j}(y, \eta, x_n, y_n) f_2(y_n) dy_n \ \hat{f}_1(\eta) d\eta$$

tends to

$$(2\pi)^{\frac{1-n}{2}} \int e^{iy\eta} \int_0^\infty g^\kappa(y, \eta, x_n, y_n) f_2(y_n) dy_n \hat{f}_1(\eta) d\eta$$

by Lebesgue's theorem on dominated convergence, because on the compact support of $\hat{f}_1 \otimes f_2, g^\kappa_{\epsilon_j}$ converges uniformly.
On the other hand, let us show that $\operatorname{Op}_G g^\kappa_{\epsilon_j} f(x) \to G^\kappa f(x)$: We have

$$\operatorname{Op}_G g^\kappa_{\epsilon_j} = G^\kappa_{\epsilon_j} = T P_{\epsilon_j} G P_{\epsilon_j} T^{-1}. \tag{2}$$

By 1.8, $T^{-1} : \mathcal{S}(\mathbf{R}^n_+) \to \mathcal{S}(\mathbf{R}^n_+)$ is an isomorphism. For $f \in \mathcal{S}(\mathbf{R}^n_+)$, we have $P_{\epsilon_j} f \to f$ in $\mathcal{S}(\mathbf{R}^n_+)$ as $j \to \infty$. Therefore $P_{\epsilon_j} T^{-1} f \to T^{-1} f$ in $\mathcal{S}(\mathbf{R}^n_+)$, *a forteriori* in $H^s(\mathbf{R}^n_+) = \mathcal{W}^s(\mathbf{R}^{n-1}, H^s(\mathbf{R}_+))$, and $G P_{\epsilon_j} T^{-1} f \to G T^{-1} f$ in view of the mapping properties 2.1(iii). The family $\{P_{\epsilon_j} : j = 1, 2, \ldots\}$ is uniformly bounded on $H^s(\mathbf{R}^n_+)$ by 2.11. For $\tilde{f} \in H^s(\mathbf{R}^n_+)$ we thus have $P_{\epsilon_j} \tilde{f} \to \tilde{f}$ in $H^s(\mathbf{R}^n_+)$. We conclude that $G^\kappa_{\epsilon_j} f \to G^\kappa f$ in $H^s(\mathbf{R}^{n-1}, H^s(\mathbf{R}_+))$. For large s we obtain (1).

2.20 Definition. Let $m \in \mathbf{R}$. By Ψ^m denote the space of all $H \in \mathcal{L}(\mathcal{S}(\mathbf{R}^n_+), \mathcal{S}'(\mathbf{R}^n_+))$ such that for all multi-indices α, β, and all $s \in \mathbf{R}, \sigma, \sigma', \tau, \tau' \in \mathbf{N}_0$, $\operatorname{ad}^\alpha(-iy)\operatorname{ad}^\beta(D_y)H$ has a bounded extension to an operator from $H^s(\mathbf{R}^{n-1}_y, H_0^{-\sigma,-\tau}(\mathbf{R}_+))$ to $H^{s-m+|\alpha|}(\mathbf{R}^{n-1}_y, H^{\sigma',\tau'}(\mathbf{R}_+))$

2.21 Corollary. We have so far shown that, given $H \in \Psi^0$, there is an $h \in \mathcal{H}^0_0$ such that $H = \operatorname{Op}_G(h)$. Let us now show that indeed $h \in \mathcal{H}^0_1$.
From what we have shown we deduce that for every $K \in \Psi^m$, there is a $k \in \mathcal{H}^m_0$ with $\operatorname{Op}_G k = K, m \in \mathbf{R}$; this is a consequence of the fact that

$$\langle D_y \rangle^{-m} : H^{s-m}(\mathbf{R}^{n-1}_y, H^{\sigma',\tau'}(\mathbf{R}_+)) \longrightarrow H^s(\mathbf{R}^{n-1}_y, H^{\sigma',\tau'}(\mathbf{R}_+))$$

is an isomorphism.
So, for the above $H, \operatorname{ad}^\alpha(-iy)\operatorname{ad}^\beta(D_y)H \in \Psi^{-|\alpha|}$, thus has a symbol kernel $h_{\alpha,\beta} \in \mathcal{H}_0^{-|\alpha|}$. But $h_{\alpha,\beta} = \partial^\alpha_\eta D^\beta_y h$, and we conclude that

$$\|x_n^k D^{k'}_{x_n} y_n^m D^{m'}_{y_n} \partial^\alpha_\eta D^\beta_y h(y, \eta, x_n, y_n)\|_{L^2(\mathbf{R}^2_{++})} \leq C(k, \ldots, \beta) \langle \eta \rangle^{-|\alpha|},$$

and this is what we had to show.
In other words, G^κ has a symbol kernel $g^\kappa \in \mathcal{H}^0_1$.

2.22 Conclusion. Let $g^\kappa \in \mathcal{H}_1^0$ be the symbol kernel for G^κ. Then G is the operator with the symbol kernel

$$g(y,\eta,x_n,y_n) = \text{Os}-\iint e^{-iw\sigma} g(y+w,\eta,\langle\sigma+\eta\rangle x_n, \langle\eta\rangle y_n) \langle\sigma+\eta\rangle^{\frac{1}{2}} \langle\eta\rangle^{\frac{1}{2}} \, dw d\sigma,$$

and $g \in \tilde{\mathcal{B}}^{-1,0}$.

Proof. A direct computation shows that the symbol kernel has the asserted form. The analysis of the estimates then is as in 2.7.

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