FINITENESS OF COHERENT COHOMOLOGY FOR PROPER FPPF STACKS

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1. Introduction

The purpose of this note is to prove the coherence of direct images for proper maps of Noetherian fppf stacks. It was inspired by reading the book [3] where the result is shown (Th.15.6) under some restrictive hypotheses (called $(*)$). These become necessary because of the use of Chow's lemma which is completely adequate for schemes or algebraic spaces but has some deficiencies for stacks. For full generality it is natural to look for other methods of proof. There is no need to look far since there is only one other wellknown choice. The classical method for analytic spaces consists in shrinking an acyclic covering and using the fact that the induced map on Cech-cohomology is a compact isomorphism. The same procedure works for rigid spaces, see [2]. We show that it also works here. I thank the referee for his careful reading of the manuscript which lead to many clarifications.

Theorem 1. *Suppose $Spec(R)$ is an affine noetherian scheme,*

$$f : \mathcal{X} \to Spec(R)$$

a separated proper map of algebraic fppf-stacks, $\mathcal{F}$ a coherent sheaf on $\mathcal{X}$. Then all direct images $\mathbb{R}^i f_*(\mathcal{F})$ are coherent sheaves on $Spec(R)$.

Remark Recall that an algebraic stack over $Spec(R)$ can be given by two algebraic $Spec(R)$-spaces X_0, X_1 which are objects and arrows of a representable groupoid over $Spec(R)$. That we have an fppf-stack means that the two projections (source and target) $pr_i : X_1 \to X_0$ are flat and locally of finite presentation. That it is separated means that $(pr_0, pr_1) : X_1 \to X_0 \times_R X_0$ is proper. That $\mathcal{X}$ is proper means for us that X_0 is of finite type over R and that $\mathcal{X}$ satisfies the valuative criterion:
Assume V is an R-algebra which is a valuation ring, with field of fractions K. Assume furthermore given an R-map $x : Spec(K) \to X_0$ with image a generic point of X_0. Then there exists an extension $K' \supseteq K$, a valuation ring $V' \subset K'$ dominating V, a map $x' : Spec(V') \to X_0$ equivalent to x, that is (x, x') lifts to $Spec(K') \to X_1$. (I show at the end that it suffices to require this for discrete valuation rings, again eliminating condition $(*)$ used in [3], Prop. 7.12).
Also a coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ is given by a coherent sheaf $\mathcal{F}_0$ on X_0, together with a descent datum (satisfying a cocycle condition)

$$pr_1^*(\mathcal{F}) \cong pr_2^*(\mathcal{F}).$$

It then induces a cartesian coherent sheaf on the simplicial space $X_\bullet$ defined by $\mathcal{X}$, with

$$X_n = (X_1/X_0)^n.$$

In principle we would have to specify the various projections from X_1 to X_0 involved in the product. However we think of X_n as the $(n+1)$-fold fibre-product of X_0 over $\mathcal{X}$, or as classifying "$n+1$ objects with compatible isomorphisms between them". Thus for each way to connect $n+1$ dots without cycles we get a product, and all of them are canonically isomorphic. The direct images $\mathbb{R}^i f_*(\mathcal{F})$ are quasicoherent sheaves on S, corresponding to the R-modules $H^i(X_\bullet, \mathcal{F})$. They do not depend on the choice of presentation of $\mathcal{X}$ as X_0/X_1.

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2. Preliminaries

Consider the R-algebra $S = R[t]$ obtained by adjoining one indeterminate t. All algebraic spaces will be assumed to be of finite type over S.

Definition 2. *A map*

$$f : M \to N$$

of R-modules is called a weak isomorphism of its kernel and cokernel are annihilated by some power t^n. It is called weakly surjective if only the cokernel is annihilated by t^n.

For finitely generated S-modules weakly isomorphic means that f becomes an isomorphism if we invert t, but in general being weakly isomorphic is stronger. We use this property to prove finiteness of cohomology via the following lemma.

Lemma 3. *Suppose K^* is a complex of R-modules, $M^* = K^*[t]$ the induced complex over S, N^* another complex of S-modules, $f : N^* \to M^*$ a map of complexes of S-modules. Assume that for some index i*

- *The image of N^i in $M^i/t^nM^i = K^i$ is finitely generated (over R), for any n.*
- *$H^i(N^*) \to H^i(M^*)$ is weakly surjective.*

Then $H^i(K^*)$ is finitely generated over R.

Proof of lemma Choose an integer n such that the image of $H^i(N^*)$ contains $t^{n-1}H^i(M^*)$. Then $H^i(K^*)$ is a subquotient of the image of N^i in M^i/t^nM^i, which is finitely generated over R.
Next we need some ideas from rigid analytic geometry. Since our setup does not quite conform to that theory we have to explain what we mean by that. For simplicity we denote for any S-scheme X its "generic fibre" by

$$X^\circ = X \otimes_S S[t^{-1}].$$

Definition 4. *The category of rigid spaces over S is the category of algebraic S-spaces X, where we invert all proper maps $Y \to X$ which are "generic isomorphisms" $X^\circ \tilde{\to} Y^\circ$.*

In fact it suffices to invert blow-ups of closed subspaces in the complement of the "generic fibre" (use [1], Th. 5.2.2 to make a proper modification flat). Obviously in the definition of rigid spaces we can restrict to algebraic spaces X with X° scheme theoretically dense in X.

Definition 5. *A map $f : X \to Y$ of rigid spaces is called surjective if the following holds:*

Suppose V is a discrete valuation ring which is an S-algebra, and such that t does not map to zero in V. Then for any point $y \in Y(V)$ there exists an extension (of valuation rings) $W \supseteq V$ and a point $x \in X(W)$ mapping to y.
f is called a covering if it is surjective and

$$f^\circ : X^\circ \to Y^\circ$$

is flat.

By [1], Th.5.2.2 f is a covering if and only if it can be represented by a faithfully flat map of algebraic spaces (the reference talks about schemes, but the method also applies to algebraic spaces: Choose a maximal Zariski open set over which the assertion is true, and pass to the henselisation in a maximal point of the complement, etc.).

Definition 6. *A map of coherent sheaves on an algebraic S-space X is called a weak isomorphism if its kernel and cokernel are annihilated by a fixed power t^n.*
The (fibered) category of coherent sheaves on rigid S-spaces is the category of coherent sheaves on algebraic S-spaces, where we invert weak isomorphisms (and also the same blow-ups as before).

Of course one can check that the inverted blow-ups induce equivalences on the categories of coherent sheaves, up to weak isomorphism. Also the cohomology functors $H^i(X, \mathcal{F})$ with values in S-modules/weak isomorphisms are well defined. A covering $f : X \to Y$ satisfies cohomological descent, that is the cohomology of Y is equal to that of the simplicial rigid space $X_\bullet$. It also satisfies effective descent for coherent sheaves. Namely represent it by a faithfully flat map $f : X \to Y$. For a coherent sheaf $\mathcal{F}$ on X a descent datum is a weak isomorphism (on $X \times_Y X$)

$$pr_1^*(\mathcal{F}) \cong pr_2^*(\mathcal{F}),$$

satisfying the usual transitivity. We then may divide $\mathcal{F}$ by its t-torsion, and obtain a "real" descent datum on

$$\mathcal{G} = \mathcal{F} \cap pr_{1,*}pr_2^*(\mathcal{F}).$$

That is we obviously first get a descente-datum on the direct image $j_*(\mathcal{F})$, $j : X^\circ \subseteq X$ the inclusion of the generic fibre, which is thus pullback of a quasicoherent sheaf on Y. Then $\mathcal{G}$ corresponds to the maximal subsheaf whose preimage is contained in $\mathcal{F}$.
On one occasion we will encounter a "rigid fppf-algebraic space", that is an algebraic scheme X/Y and a closed subscheme of $X \times_Y X$ which defines a flat equivalence relation over Y^0. As we do not know how to extend this flat equivalence relation over Y, there is no obvious definition of a quotient. However we can as usual define (via descent data) the category of coherent sheaves on the fictional quotient, up to weak isomorphism. Also the higher direct images (on Y) of such a sheaf are well defined, again up to weak isomorphism.
Also our "rigid fppf-algebraic space" will be "proper", because after basechange via a covering $Y' \to Y$ it will be the rigid space defined by a proper algebraic space over Y'. This implies coherence (up to weak isomorphisms) of higher direct images: As this holds for a faithfully flat covering, we are reduced to the case where Y' is proper over Y, and isomorphic over Y^0. But then the higher direct images on Y are the direct images of their counterparts on Y', by Leray.

Finally we need the notation of enlargement. Suppose X is an affine algebraic S-scheme, and embed it into an affine space:

$$X \subseteq \mathbb{A}^n_S$$

.

Definition 7. *The r-enlargement $X^{\flat,r}$ is the scheme theoretic closure of the image of X under multiplication by t^r:*

$$t^r : \mathbb{A}^n_S \to \mathbb{A}^n_S .$$

Obviously this depends on the choice of embedding, but it seems too burdensome to include this into the notation. One easily sees that for a map $f : X \to Y$ and a given r, there exists an s such that f extends to $f : X^{\flat,r} \to Y^{\flat,s}$. Also if V is a discrete valuation ring over S, with t nonzero in V, then the V-points of $X^{\flat,r}$ are the K-points of X all whose affine coordinates lie in $t^{-r}V \subset K$.
Finally to deal with mappings we need a variant involving varying factors on the coordinates. Suppose $f : X \to Y$ is an S-map of affine schemes, and choose closed embeddings

$$X \subseteq \mathbb{A}^m_S , Y \subseteq \mathbb{A}^n_S .$$

Definition 8. *$X^{\flat,r,s}$ is the scheme theoretic closure of the image of X under the map*

$$(t^r, t^s f) : X \to \mathbb{A}^{m+n}_S .$$

Thus we have an extension

$$f : X^{\flat,r,s} \to Y^{\flat,s}.$$

Our main result about these is the following:

Lemma 9. *Suppose $f : Y \to X \times X'$ is a map of affine algebraic S-schemes such that the projection from Y^0 to X^0 is flat. Furthermore we require that for any valuation ring V over S, with t nonzero in V, and any point $x \in X(V[t^{-1}])$, there exists a valuation ring $W \supseteq V$ dominating V, a point $x' \in X'(W)$, and a point $y \in Y(W[t^{-1}])$ mapping to (x, x'). Then for any s there exists an $r \gg 0$ with*

$$g = pr_1 \circ f : Y^{\flat,r,s} \to X^{\flat,s}$$

a covering. Here the enlargement $Y^{\flat,r,s}$ is the closure of the image of the scaled embedding of Y° into an affine space over $X^{\flat,s} \times X'$.

Proof We obviously may assume that $s = 0$, and that X° is dense in X. By [1], Th. 5.2.2 we may choose for any r a blow-up (with center in the special fibre) $X_r \to X$ such that the proper transform of $Y^{\flat,r,0}$ is a flat $g_r : Y_r \to X_r$. Here "proper transform" means fibered product divided by t-torsion. Furthermore we may assume that X_{r+1} dominates X_r, so that the X_r form a projective system of algebraic S-schemes, with proper transition maps. Also the canonical maps $Y^{\flat,r,0} \to Y^{\flat,r+1,0}$ define

$$Y_r \times_{X_r} X_{r+1} \to Y_{r+1}.$$

(Note that the fibered product has no t-torsion as it is flat over X_{r+1}.)
Furthermore the image of g_r is open. Let $Z_r \subseteq X_r$ denote its complement, with the reduced structure. It is contained in the "special fibre", that is its structure sheaf is annihilated by t. Also Z_{r+1} projects into Z_r, so the Z's also form a projective system with proper transition maps. The assertion of the lemma means that they

are empty for big r.
So assume that this is not true. For each r the images of $Z_s \to Z_r$ $(s \geq r)$ form a decreasing sequence of subschemes which must stabilise to a $\bar{Z}_r \subseteq Z_r$. For the $\bar{Z}$'s the transition maps are dominant, so we can lift a generic point of $\bar{Z}_r$ to a generic point of $\bar{Z}_{r+1}$, and thus obtain a projective system of points $z_r \in \bar{Z}_r$.
Denote by A_r the local rings of Y_r in z_r. These form an inductive system of local rings, with direct limit

$$A = \lim_{\rightarrow} A_r.$$

It is a local ring in which t is a regular element, and it dominates all A_r. Choose a maximal ideal $\mathfrak{p} \subset A[t^{-1}]$, which we identify with a prime $\mathfrak{p} \subset A$. Denote its residue field by $\kappa(\mathfrak{p})$.
Now choose a valuation ring $V \subset \kappa(\mathfrak{p})$ dominating $A/\mathfrak{p}$. The map from A into V defines a compatible system of points of X_r with values in V and thus a $x \in X(V)$.
By assumption we can find W dominating V and $x' \in X'(W)$ such that (x, x') lifts to $y \in Y(W[t^{-1}])$.
But if we choose r big enough so that multiplication by t^r makes the affine coordinates of this point y integral then y defines a W-point of $Y^{\flat,r,0}$ and of Y_r.
Applying g_r sends the maximal ideal of W to the point $z_r \in Z_r$, which contradicts the definition of Z_r. Thus the lemma is shown.

3. Proof of the theorem

Suppose given a separated proper stack

$$\mathcal{X} = X_0/X_1$$

over R, as in the introduction. We may assume that X_0 is affine. Furthermore there is an étale covering $Y \to X_1$ with Y affine. We extend the base to $S = R[t]$ and use the same notation. Apply lemma 2 to the map

$$Y \to X_1 \to X_0^{\flat,1} \times X_0.$$

It satisfies the condition because each $V[t^{-1}]$-point y of X_0 is equivalent to a V'-point $(V' \supseteq V)$ y' of X_0, that is (y, y') lifts to a K'-point x of X_1 which extends to a $V'[t^{-1}]$-point by properness (the valuative criterion for $\mathcal{X}$), and after passing to a finite extension of K' to a $V'[t^{-1}]$-point of Y (which is an étale cover of X_1).
Thus we obtain

$$Y^\flat = Y^{\flat,r,0} \to X_0^{\flat,1} \times X_0$$

covering

$$X_0^\flat = X_0^{\flat,1}$$

and also the second factor X_0. Furthermore the fibered product $Y^\flat \times_{X_0^\flat} Y^\flat$ maps (in the category of rigid spaces) to X_1:
Over the "generic fibres" such a map exists because two elements $y_1, y_2 \in Y$ with the same first projection x define isomorphisms (in $\mathcal{X}$) between x and their second projections x_1, x_2, and thus also an isomorphism between x_1 and x_2. However as X_1 is proper over $X_0 \times X_0$ the closure of the generic graph defines a rigid map.
This means that the second projection from $Y^\flat$ to X_0 descends to a map of $X_0^\flat$ into the rigid fppf stack $\mathcal{X}$, which extends the map from X_0 (same argument as before).
This map is a covering, and thus in the rigid category $\mathcal{X}$ becomes the quotient of $X_0^\flat$ by

$$X_1^\flat = X_0^\flat \times_{\mathcal{X}} X_0^\flat.$$

There is a slight inconvenience that $X_1^\flat$ is only a "rigid fppf-algebraic space", that is we have a flat cover of it ($Y^\flat \times_{\mathcal{X}} Y^\flat$) and a flat equivalence relation, but we do not know how to construct the quotient. However it suffices to define cohomology. Also it is true and makes sense to say that $X_1^\flat$ is proper over $X_0^\flat \times X_0^\flat$, because this holds after basechange to $Y^\flat \times Y^\flat$.
Now from $X_0^\flat$ and $X_1^\flat$ we can as usual define a simplicial "rigid fppf-algebraic space" $X_\bullet^\flat$, which can be used to compute the cohomology of $\mathcal{X}$. It is proper over the simplicial affine scheme $Cosk_0(X_0^\flat)$ made up from powers $(X_0^\flat)^{n+1}$. Thus the direct images of coherent sheaves on $X_n^\flat$ are (that is can be represented by) coherent sheaves on $(X_0^\flat)^{n+1}$). Namely after basechange to $Cosk_0(Y^\flat)$ the map can be represented by the "real" proper map of algebraic spaces

$$Cosk_0^{\mathcal{X}}(Y^\flat) \to Cosk_0(Y^\flat),$$

and one can use flat descent. Finally the shrinking-argument should be applied to $Cosk_0(X_0) \to Cosk_0(X_0^\flat)$.
After this outline of strategy let us be more explicit: Consider the bisimplicial scheme $Cosk_0(Cosk_0^{X_0^\flat}(Y^\flat))$ which in bidegree (m,n) is the product of $m+1$ copies of the $n+1$-fold fibered product of $Y^\flat$ over $X_0^\flat$. It projects (in the m-direction) to $Cosk_0(X_0^\flat)$ and is a rigid hypercovering of $Cosk_0(X_0^\flat)$. Pulling back (the problematic) $Cosk_0^{\mathcal{X}}(X_0^\flat)$ gives a rigid bisimplicial algebraic space $Cosk_0^{\mathcal{X}}(Cosk_0^{X_0^\flat}(Y^\flat))$ proper over $Cosk_0(Cosk_0^{X_0^\flat}(Y^\flat))$. Namely in bidegree (m,n) choose any degeneracy $Cosk_0(Cosk_0^{X_0^\flat}(Y^\flat))_{m,n} \to Cosk_0(Cosk_0^{X_0^\flat}(Y^\flat))_{m,0} = Cosk_0(Y^\flat)_m \to Cosk_0(X_0)_m$ (second projection of $Y^\flat$ to X_0) and pullback $X_\bullet \to Cosk_0(X_0)$. Up to admissible blowups the result is independant of the choice of degeneracies (Since we pull back a proper map it suffices to check this over generic fibres. There in bidegree (m,n) we obtain a covering of $X_{mn+2m+n+1}$, a fibered product of X_1's which classifies $(m+1)(n+2)$ objects with compatible isomorphisms between them. The covering is defined by replacing $(m+1)(n+1)$ of the factors factors X_1 by Y) . Call the resulting bisimplicial (rigid) algebraic space $A_{\bullet\bullet}$. There are variants $B_{\bullet\bullet}$ obtained by pullback (on Coskeletons) $X_0 \to X_0^\flat$ and $C_{\bullet\bullet}$ where $X_0^\flat$, $Y^\flat$ are replaced by X_0 and Y, respectively. So $C_{\bullet\bullet} = Cosk_0^{\mathcal{X}}(Cosk_0^{X_0}(Y))$, but to write down $B_{\bullet\bullet}$ requires even more complicated notation.
Finally define a bisimplicial space $D_{\bullet\bullet}$ with $D_{m,n}$ the fibered product over $\mathcal{X}$ of $(m+1)\cdot(n+1)$ copies of X_0, that is X_{mn+m+n}, or $D_{\bullet\bullet} = Cosk_{0,0}^{\mathcal{X}}(X_0)$. The second projections from $Y^\flat$ to X_0 define a map from $Cosk_0(Cosk_0^{X_0^\flat}(Y^\flat))$ to $Cosk_{0,0}(X_0)$ which lifts on the generic fibre canonically to a map from $A_{\bullet\bullet}$ into $D_{\bullet\bullet}$. As $D_{\bullet\bullet}$ is proper over $Cosk_{0,0}(X_0)$ this map extends after admissible blowups. So finally we have a sequence of rigid maps

$$C_{\bullet\bullet} \to B_{\bullet\bullet} \to A_{\bullet\bullet} \to D_{\bullet\bullet}.$$

Note that $C_{\bullet\bullet}$, $D_{\bullet\bullet}$, and the map between them actually belong to the category of bisimplicial algebraic spaces (no blowups necessary). Also $C_{\bullet\bullet}$ maps to $X_\bullet$ (respecting first indices) inducing an isomorphism on cohomology of cartesian quasicoherent sheaves.
Now assume given a coherent sheaf $\mathcal{F}$ on $\mathcal{X}$. It pulls back to coherent sheaves $\mathcal{F}$ on $X_\bullet$, $A_{\bullet\bullet}$, $B_{\bullet\bullet}$, $C_{\bullet\bullet}$ and $D_{\bullet\bullet}$. The map from $C_{\bullet\bullet}$ to $D_{\bullet\bullet}$ induces an isomorphism on cohomology: By the usual spectralsequence it suffices to check for the simplicial

spaces where one fixes the second index n. But then both simplicial schemes are fibered products (coskeletons) over $\mathcal{X}$, namely of $(Y/X_0)^{n+1}$ and X_n, respectively. Hence both times we obtain the cohomology of $\mathcal{X}$.
To compute cohomology for the rigid bisimplicial spaces $A_{\bullet\bullet}$ and $B_{\bullet\bullet}$ replace them by affine hypercoverings, and uses the usual complexes defined by global sections. Note that these complexes only use the simplicial faces (corresponding to injections of finite ordered sets), not the degeneracies. It is then possible to find models (for the spaces and for the sheaf $\mathcal{F}$ on them) such that all face-maps are "real". By direct image we get complexes of modules

$$K^*(Cosk_0(X_0)), K^*(Cosk_0(X_0^\flat))$$

over the (cosimplicial) rings defining the affine simplicial schemes $Cosk_0(X_0)$ and $Cosk_0(X_0^\flat)$. These are not cosimplicial modules because the the degeneracies are not "real" maps, but "half-cosimplicial" as the faces are. We can replace them by weakly isomorphic complexes with finitely generated cohomology: We obtain (by properness) coherent direct images over $Cosk_{0,0}^{X_0^\flat}(Y^\flat)$, and they are cartesian up to weak isomorphism. By descent (which means direct image) this remains true over $Cosk_0(X_0^\flat)$, and over $Cosk_0(X_0)$ by pullback. If we truncate at some sufficiently high degree (i.e. apply $\tau_{\leq N}$) we can replace them by quasi-isomorphic complexes of finitely generated modules, using the following rather obvious result:

Lemma 10. *Suppose A is a noetherian ring, K^* a bounded above complex of A-modules, with finitely generated cohomology. Given a subcomplex $L^* \subseteq K^*$ with each L^n finitely generated over A, there exists another such subcomplex $\tilde{L}^*$ with $L^* \subseteq \tilde{L}^*$, and $\tilde{L}^*$ quasi-isomorphic to K^*.*

Finally using the same lemma again we can represent the map

$$\tau_{\leq N}\mathbb{R}\Gamma(A_{\bullet\bullet}, \mathcal{F}) \to \tau_{\leq N}\mathbb{R}\Gamma(B_{\bullet\bullet}, \mathcal{F})$$

by the map of total complexes associated to compatible (with faces) systems of maps

$$L^*(Cosk_0(X_0^\flat)) \to L^*(Cosk_0(X_0))$$

of complexes of finitely generated modules over the affine rings of $Cosk_0(X_0^\flat)$ respectively $Cosk_0(X_0)$. However these then satisfy the conditions in lemma 1: Namely the affine rings are quotients of polynomial-rings, and the map between them replaces each indeterminate T_i by tT_i. So modulo a fixed t-power t^n almost all monomials map to zero, and thus any semilinear map of finitely generated modules has its image modulo t^n finitely generated over R. The same then holds for the map

$$\tau_{\leq N}\mathbb{R}\Gamma(A_{\bullet\bullet}, \mathcal{F}) \to \tau_{\leq N}\mathbb{R}\Gamma(C_{\bullet\bullet}, \mathcal{F}),$$

that is it can be represented by a map of complexes satisfying the condition in lemma 3. However it is weakly surjective on cohomology, with one sided inverse induced by $A_{\bullet\bullet} \to D_{\bullet\bullet}$. So finally applying lemma 3 proves the theorem.

4. Concluding remarks

1. One should also try to extend the proper basechange theorem in étale cohomology, as well as the construction of direct images with compact support.

2. There seems to be no easy way to extend to non Noetherian schemes. Although everything comes by basechange from a Noetherian situation, the cohomological basechange involves derived tensor products. It does not mix well with cohomology because the cohomological dimension may be infinite.

3. I promised to relax the properness condition. So assume X_0 affine, $X_1 \to X_0 \times X_0$ proper, and the projections to X_0 flat. Also everything is of finite type over a noetherian R. We say that the extension property holds for a valuation ring V (over R) if for any $y \in X_0(K)$, there exists $V' \supseteq V$ and $y' \in X_0(V')$ with (y, y') lifting to $x \in X_1(K')$. Then we claim that the extension property holds for all valuation rings V if it holds for discrete valuations. For the proof consider a minimal closed subscheme $Z \subseteq X_0$ such that the extension property fails for some point in $Z(K)$. Then Z is irreducible. Choose a projective scheme $P \to Z \times X_0$ dominating $Z \times_{X_0} X_1$. Embed $Z \subset \bar{Z}$ into a projective R-scheme, and extend P to a projective $\bar{P} \to \bar{Z} \times X_0$. Then for any discrete valuation ring V and any K-point $y \in Z(K)$ we can extend to $y \in \bar{Z}(V)$. Also after $V' \supseteq V$ we find $y' \in X_0(V')$ with (y, y') lifting to $x \in X_1(K')$ and after further extension to $x' \in P(K')$, because P dominates the pullback of X_1. Finally x' lifts to a point in $\bar{P}(V')$ because of properness. Now by [1] Th. 5.2.2 there exists a modification $\tilde{Z} \to \bar{Z}$ such that the proper transform $\tilde{P}$ of $\bar{P}$ becomes flat over $\tilde{Z}$. As any point z of $\tilde{Z}$ is a specialisation of the generic point, the exists a discrete valuation ring V and a point $y \in \tilde{Z}(V)$ such that the special point of $Spec(V)$ maps to z, and the generic point $Spec(K)$ to the generic point of Z. By the extension property we can lift to $x' \in \tilde{P}(V')$, so z lies in the image of $\tilde{P}$. Thus $\tilde{P}$ is faithfully flat over $\tilde{Z}$, and for any valuation ring V any point $y \in Z(K)$ extends to $y \in \tilde{Z}(V)$ (properness) and (for suitabe $V' \supseteq V$) to $x' \in \tilde{P}(V')$ because of faithful flatness. This projects to $y' \in X_0(V')$ and $x' \in P(K')$.

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