On the relative polynomial construction and torsionfree nilpotent groups

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In this preprint we present a new approach to the study of torsionfree nilpotent groups (= $\mathcal{T}$-groups for short). As its key point we construct 'abelian models' for central extensions of such groups; this amounts to a functorial equivalence of $\mathcal{T}$-groups with certain iterated singular ring extensions. The concept follows along the line to establish an 'integral' substitute of Mal'cev's (or Lazard's) Lie algebra models for (sufficiently) divisible nilpotent groups in the nondivisable situation. We obtain applications of our model theorems to construction, classification and automorphisms of $\mathcal{T}$-groups.

In the first part we start with a detailed study of a relative version of Passi's polynomial construction $P_n(G)$, i.e. of certain quotients of the integral group ring. The results obtained include integral analogues of some wellknown results of Quillen on the rational group algebra.

This is used in part II to establish the following model theorem: The category of central extensions $B \overset{i}{\rightarrowtail} E \overset{\pi}{\twoheadrightarrow} G$ of (finitely generated) torsionfree nilpotent *groups* of class $\leq n$ is canonically equivalent to the category of central extensions $B \rightarrowtail M \twoheadrightarrow P_n^{\surd}(G)$ of **Z**-torsionfree (finitely generated **Z**-free) nilpotent *G-modules* of class $\leq n$, with $P_n^{\surd}(G) = P_n(G)/\mathbf{Z} - torsion$. As an immediate application we derive a description of $H^2(G,B)$ for $\mathcal{T}$-groups G and torsionfree trivial modules B in terms of polynomial cocycles. This improves a result of Passi, Sucheta and Tahara for $B = \mathbf{Q}$; moreover, it leads to a new method of computing $H^2(G,\mathbf{Z})$ in terms of cocycles given by rational integer valued polynomials, starting with a presentation of G and using integral matrix calculus. We point out that this result also admits a short reduction of the isomorphism problem for $\mathcal{T}$-groups to an orbit problem of arithmetic group actions which was shown to be decidable in [Gr-Se80a]. We thus obtain a much simpler proof of the decidability of the first problem than it was given in [Gr-Se80b]. This will be described in detail in [Ha92k] where we also give an inductive construction of all finitely generated $\mathcal{T}$-groups in terms of free abelian groups.

In the third part of the present paper we determine the automorphism groups of $\mathcal{T}$-groups in terms of iterated group extensions. Here all terms, in particular the corresponding obstruction operators and extension classes are described in terms of the polynomial construction $P_n^{\surd}(G)$; this ensures that they are computable from a presentation of G by linear algebra, using proposition E in part II.

We conclude with the remark that the results of this paper also admit applications to *simplicial* $\mathcal{T}$-groups and thus to homotopy theory as well as to group cohomology with respect to varieties of nilpotent groups [Ha92h].

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Contents

Part I: On the relative polynomial construction

The polynomial construction $P_n(E)$ for groups E is due to the classical work of Passi [Pa68a]. The relative polynomial construction defined below is only implicit, however, in Passi's studies of the dimension subgroup problem [Pa68b], [Pa79] or in Roquette's proof of the Golod-Šafarevič inequality [Roq67]. We here determine the structure of the relative polynomial construction $P_n(E,B)$ *modulo torsion*. Moreover, the torsion subgroup of $P_n(E,B)$ is identified in terms of the groups $B \lhd E$, and generalizations of theorems of Witt and Quillen are derived concerning certain Lie algebras associated with groups. Our results below are also key steps in our investigation of torsionfree nilpotent groups in parts two and three of this paper.

1 Results

1.1 Definitions: Let E be a group, R be a commutative ring and let $\Delta_R(E)$ be the augmentation ideal of the group algebra $R(E)$. For a normal subgroup $B \lhd E$ we define the quotient ring

$$P_{n,R}(E,B) = \Delta_R(E)\big/\left(\Delta_R(B)\Delta_R(E) + \Delta_R^{n+1}(E)\right) .$$

Thus $P_{n,R}(E) = P_{n,R}(E,\{1\})$ is the polynomial construction of Passi. For an abelian group A let $\tau A =$ *torsion of* A and $\bar{\tau} A = A/\tau A$. We define the quotient rings

$$P_n^{\surd}(E,B) = \bar{\tau}\, P_{n,\mathbf{Z}}(E,B) , \quad P_n^{\surd}(E) = P_n^{\surd}(E,\{1\}) .$$

Via left multiplication, $P_{n,R}(E,B)$, $P_n^{\surd}(E,B)$ and $P_n^{\surd}(E/B)$ are left nilpotent E/B-modules of class $\leq n$. Moreover, the maps

$$p_{n,R} : E \to P_{n,R}(E,B) , \quad p_n^{\surd} : E \to P_n^{\surd}(E,B)$$

sending $a \in E$ to the coset of $a-1$ are $(E \twoheadrightarrow E/B)$-derivations; they are universal with respect to nilpotent $R(E/B)$-modules of class $\leq n$ or to nilpotent $\mathbf{Z}$-torsionfree E/B-modules of class $\leq n$, respectively. A canonical isomorphism $P_{n,R}(E,B) \cong R(E/B) \otimes_E P_{n,\mathbf{Z}}(E)$ of $R(E/B)$-modules is induced by this property of $p_{n,R}$.

We write ρ for the canonical quotient map from $\Delta_R(E)$ to $P_{n,R}(E)$ or to $P_{n,R}(E,B)$.

Let $E = \gamma_1(E) \supset \gamma_2(E) \supset \ldots$ denote the lower central series of E, and for $U < E$ let $\sqrt{U} = \{a \in E \mid \exists m \in \mathbf{Z} : a^m \in U\}$, the isolator of U, which is a subgroup if E is nilpotent.

Theorem A: *Let $B \overset{i}{\rightarrowtail} E \overset{\pi}{\twoheadrightarrow} G$ be a central extension of groups and let $n \geq 1$. Suppose that the relation $\sqrt{B\,\gamma_{n+1}(E)} \subset B\sqrt{\gamma_{n+1}(E)}$ holds in E or, equivalently, that the relation $\sqrt{\gamma_{n+1}(G)} \subset \pi\sqrt{\gamma_{n+1}(E)}$ holds in G (this is the case, for example, if G is torsionfree nilpotent of class $\leq n$). Then the following is true:*

(1) The sequence of homomorphisms between **Z**-torsionfree nilpotent G-modules of class $\leq n$

$$0 \longrightarrow \bar{\tau}(B/B \cap \gamma_{n+1}(E)) \overset{\bar{\tau}(p_n^{\surd} i)}{\longrightarrow} P_n^{\surd}(E,B) \overset{P_n^{\surd}(\pi)}{\longrightarrow} P_n^{\surd}(G) \longrightarrow 0$$

is **exact**. Here the G-action on the left hand term is trivial. If E is finitely generated all terms are finitely generated free abelian groups, so the sequence splits (nonnaturally).

(2) The **torsion subgroup** of $P_n(E,B)$ is generated by the elements $p_n(a_1)\cdots p_n(a_k)$, $k \geq 1$, with $a_1 \in \sqrt{\gamma_{n+1}(E)}$ if $k = 1$ and with $a_i \in \sqrt{B\,\gamma_{s_i}(E)}$, $1 \leq i \leq k$ and $s_1 + \ldots + s_k > n$. In particular, taking $B = \{1\}$, one obtains a canonical isomorphism

$$P_n^{\surd}(G) \cong \Delta(G)/\Delta_{\surd}^{n+1}(G)$$

where the ideal filtration $\{\Delta_{\surd}^{i}(G)\}$ of $\mathbf{Z}(G)$ is induced by the N-series $\{\sqrt{\gamma_i(G)}\}$ of G, see [Pa79].

(3) If E and G are torsionfree nilpotent groups of class $\leq n$ then the derivation

$$p_n^{\surd} : E \to P_n^{\surd}(E,B)$$

is an **injective** map.

The structure of the group $P_n^{\surd}(G)$ is derived from the following integral analogue of Quillen's result on the structure of $\mathrm{Gr}(\mathbf{Q}(G))$ [Qu68].

The graded Lie ring $\mathrm{L}^{\surd}(G)$ is defined by

$$\mathrm{L}^{\surd}(G) = \sum_{i \geq 1} \sqrt{\gamma_i(G)}\Big/\sqrt{\gamma_{i+1}(G)}\,,$$

$$\left[a\sqrt{\gamma_{i+1}(G)}\,, b\sqrt{\gamma_{j+1}(G)}\,\right] = (a,b)\sqrt{\gamma_{i+j+1}(G)}\,,$$

and the grading of the envelopping algebra $\mathrm{UL}^{\surd}(G)$ is induced by the grading of $\mathrm{L}^{\surd}(G)$.

Theorem B: *For all groups G there is a* **natural isomorphism**

$$\theta_{\mathbf{Z}} : \mathrm{U\,L}^{\surd}(G) \overset{\cong}{\longrightarrow} \mathrm{Gr}^{\surd}(\mathbf{Z}(G))$$

of **Z-torsionfree graded rings**. *Here the associated graded ring $\mathrm{Gr}^{\surd}(\mathbf{Z}(G))$ is taken with respect to the filtration $\{\Delta_{\surd}^{i}(G)\}$ of $\mathbf{Z}(G)$, cf. theorem A (2). The map $\theta_{\mathbf{Z}}$ is induced by the inclusions $\sqrt{\gamma_i(G)} \subset \Delta_{\surd}^{i}(G)$.* □

Corollary C: *Let G be a group and $n \geq 1$. Then the following statements hold:*

(1) There is a natural isomorphism

$$\mathrm{Gr}^{\surd}\left(P_n^{\surd}(G)\right) \cong \mathrm{UL}^{\surd}(G) \Big/ \sum_{i>n} \mathrm{U}_i\mathrm{L}^{\surd}(G)$$

of graded rings induced by $\theta_{\mathbf{Z}}^{-1}$, cf. theorem B. If G is finitely generated then there is a (nonnatural) isomorphism of groups

$$P_n^{\surd}(G) \cong \bigoplus_{i=1}^{n} \mathrm{U}_i\mathrm{L}^{\surd}(G)\,.$$

In particular, the rank of $P_n^{\surd}(G)$ is determined by the numbers $r_i = rank\left(\sqrt{\gamma_i(G)}\big/\sqrt{\gamma_{i+1}(G)}\right)$, $1 \leq i \leq n$, according to the Birckhoff-Witt-theorem.

(2) The following sequence consists of G-linear homomorphisms and is exact:

$$0 \longrightarrow \sqrt{\gamma_n(G)}\ \big/\ \sqrt{\gamma_{n+1}(G)} \xrightarrow{p_n^{\surd}} P_n^{\surd}(G) \xrightarrow{P_n^{\surd}(\pi)} P_n^{\surd}(G/\sqrt{\gamma_n(G)}) \longrightarrow 0$$

with $\pi\colon G \twoheadrightarrow G\big/\sqrt{\gamma_n(G)}$.

(3) The divisability in $P_n^{\surd}(G)$ of elements in $p_n^{\surd}(\gamma_n(G))$ is given by the identity

$$\sqrt[+]{p_n^{\surd}(\gamma_n(G))} = p_n^{\surd}(\sqrt{\gamma_n(G)})\,.$$

Here the symbol $\sqrt[+]{\ }$ indicates that the corresponding isolator is taken with respect to the *additive* structure of $P_n^{\surd}(G)$. If K_n denotes the subgroup generated by the n-fold ring commutators of $P_n^{\surd}(G)$ we have the following isomorphism and identities in $P_n^{\surd}(G)$:

$$\sqrt{\gamma_n(G)}\big/\sqrt{\gamma_{n+1}(G)} \cong p_n^{\surd}\sqrt{\gamma_n(G)} = \sqrt[+]{p_n^{\surd}(\gamma_n(G))} = \sqrt[+]{K_n}\,.$$

□

Part (2) may be viewed as a generalization of Witt's theorem that the commutator Lie ring $L(G)$ of a free group G is free. In fact, let T denote the noncommutative polynomial ring generated by a basis of G and let $\bar{T}$ be its augmentation ideal. Then Witt's theorem says that $L(G)$ is isomorphic with the sub-Lie algebra of T generated by homogenous elements of degree 1. This is a special case of corollary B (2) since we have $\sqrt{\gamma_n(G)} = \gamma_n(G)$, $P_n^{\surd}(G) \cong \bar{T}/\bar{T}^{n+1}$ and $\sqrt[+]{K_n} = K_n$.

Theorem D: *Let G be a group. Then the graded Lie ring $L^{\surd}(G)$ is determined by the group ring $\mathbf{Z}(G)$ up to isomorphism.* In fact, let $\mathrm{Gr}^{\tau}(\Delta_{\mathbf{Z}}(G))$ be the associated graded ring of $\Delta_{\mathbf{Z}}(G)$ with respect to the filtration $\left\{ \sqrt[+]{\Delta^n_{\mathbf{Z}}(G)} \right\}$. Let $\mathrm{L}^{\tau}(G)$ be the sub-Lie ring of $\mathrm{Gr}^{\tau}(\Delta_{\mathbf{Z}}(G))$ which is the additive isolator of the sub-Lie ring generated by $\mathrm{Gr}^{\tau}_1(\Delta_{\mathbf{Z}}(G))$. *Then the inclusion* $G \subset \mathbf{Z}(G)$ *induces a* **natural isomorphism of Lie rings**

$$\mathrm{L}^{\surd}(G) \cong \mathrm{L}^{\tau}(G). \qquad \square$$

This is an integral analogue of Quillen's result in [Qu68] that the Lie algebra $L^{\surd}(G) \otimes \mathbf{Q}$ is determined by the ring $\mathbf{Q}(G)$.

2 Proofs

We start by showing that our statements hold after tensoring with $\mathbf{Q}$. The 'descent to $\mathbf{Z}$' then turns out to be much simpler than in comparable situations in the literature; e.g., lattice subgroups do not play a distinguished role.

2.1 First note that we have for all groups the elementary relations $\mathbf{Q} \otimes \left(\gamma_i(G) / \gamma_{i+1}(G)\right) \cong \mathbf{Q} \otimes \left(\sqrt{\gamma_i(G)} / \sqrt{\gamma_{i+1}(G)}\right)$, $\Delta_{\mathbf{Q}}\left(\sqrt{\gamma_i(G)}\right) \subset \Delta^i_{\mathbf{Q}}(G)$ and hence $\Delta^i_{\surd,\mathbf{Q}}(G) = \Delta^i_{\mathbf{Q}}(G)$.

Theorem 2.2 ([Qu68]) *For all groups there is a* **natural isomorphism**

$$\theta_{\mathbf{Q}} : \mathrm{U}(\mathbf{Q} \otimes \mathrm{L}^{\surd}(G)) \xrightarrow{\cong} \mathrm{Gr}(\mathbf{Q}(G))$$

of graded algebras induced by the inclusion $\Delta_{\mathbf{Q}}\left(\sqrt{\gamma_i(G)}\right) \subset \Delta^i_{\mathbf{Q}}(G)$. *Here the associated graded of* $\mathbf{Q}(G)$ *is taken with respect to the augmentation filtration. In particular, the map* $\mathrm{L}^{\surd}(G) \longrightarrow \mathrm{Gr}(\mathbf{Q}(G))$ *and hence also the map*

$$p_{n,\mathbf{Q}} : G / \sqrt{\gamma_{n+1}(G)} \to P_{n,\mathbf{Q}}(G)$$

are **injective**.

As a generalization of the latter statemant we obtain

Theorem 2.3 *Let* $\underline{E} = (\, B \overset{i}{\rightarrowtail} E \overset{\pi}{\twoheadrightarrow} G \,)$ *be a central extension of groups. Then the following sequence of* G*-linear homomorphisms is* **exact***:*

$$0 \longrightarrow \mathbf{Q} \otimes (B \,/\, B \cap \gamma_{n+1}(E)) \xrightarrow{\mathbf{Q}\otimes p_n i} P_{n,\mathbf{Q}}(E,B) \xrightarrow{P_{n,\mathbf{Q}}(\pi)} P_{n,\mathbf{Q}}(G) \longrightarrow 0$$

Proof:

(1) Using 2.1 one replaces $\underline{E}$ by the extension

$$B \,/\, B \cap \gamma_{n+1}(G) \overset{\bar{\imath}}{\rightarrowtail} E \,/\, \gamma_{n+1}(E) \overset{\bar{\pi}}{\twoheadrightarrow} G/\gamma_{n+1}(G)\,.$$

Now the only non elementary part is to prove injectivity of $\mathbf{Q} \otimes p_n \tilde{\imath}$. Write $\tilde{B} = B \,/\, B \cap \gamma_{n+1}(G)$, $\tilde{E} = E \,/\, \gamma_{n+1}(E)$ and $\tilde{G} = G/\gamma_{n+1}(G)$.

(2) Let $0 \neq x \in \mathbf{Q} \otimes \tilde{B}$; we have to show that $p_{n,\mathbf{Q}}\tilde{\imath}(x) \neq 0$ in $P_{n,\mathbf{Q}}(\tilde{E}, \tilde{B})$.

Choose a linear retraction $r : \mathbf{Q} \otimes \tilde{B} \twoheadrightarrow \mathbf{Q} \cdot x$ of the inclusion $\mathbf{Q} \cdot x \subset \mathbf{Q} \otimes \tilde{B}$. Let

$$\mathbf{Q}\, x \overset{\bar{\imath}}{\rightarrowtail} \bar{E} \overset{\bar{\pi}}{\twoheadrightarrow} G$$

be the central extension induced by the homomorphism

$$f :\ \tilde{B} \cong \mathbf{Z} \otimes \tilde{B} \to \mathbf{Q} \otimes \tilde{B} \overset{r}{\twoheadrightarrow} \mathbf{Q}x\,.$$

Then

(3) $\bar{E}$ is nilpotent of class $\leq n$, and

(4) the number $k = -1 + \min\{l \mid \bar{\imath}\,\mathbf{Q}x \cap \sqrt{\gamma_l(\bar{E})} = \{1\}\}$ is defined and satisfies $1 \leq k \leq n$ since

$$\bar{\imath}\,\mathbf{Q}x \cap \sqrt{\gamma_{n+1}(\bar{E})} = \bar{\imath}\,\mathbf{Q}x \cap \sqrt{\{1\}} = \{1\}$$

by (3) and the fact that $\mathbf{Q}$ is torsionfree. We obtain the following commutative diagram

(5)
$$\begin{array}{ccccc}
\mathbf{Q} \otimes \tilde{B} & \overset{r}{\twoheadrightarrow} & \mathbf{Q}\,x & \overset{\alpha}{\rightarrowtail} & \bar{E} \big/ \sqrt{\gamma_{k+1}(\bar{E})} \\
\downarrow{\scriptstyle p_{n,\mathbf{Q}}\tilde{\imath}} & & & & \downarrow{\scriptstyle p_{k,\mathbf{Q}}} \\
P_{n,\mathbf{Q}}(\tilde{E}, \tilde{B}) & & & & P_{k,\mathbf{Q}}(\bar{E}) \\
\downarrow{\scriptstyle \rho} & & & & \downarrow{\scriptstyle \beta} \\
P_{k,\mathbf{Q}}(\tilde{E}, \tilde{B}) & & \overset{P_{k,\mathbf{Q}}(\bar{f})}{\longrightarrow} & & P_{k,\mathbf{Q}}(\bar{E}, \bar{\imath}\,\mathbf{Q}\,x)
\end{array}$$

Here α is induced by $\bar{\imath}$, β is the canonical quotient, and $\bar{f} :\ \tilde{E} \to \bar{E}$ is induced by f.

(6) We have $\bar{\imath}\,\mathbf{Q}x \subset \sqrt{\gamma_k(\bar{E})}$: By definition of k there is $0 \neq y \in \mathbf{Q}x$ with $\bar{\imath}y \in \sqrt{\gamma_k(\bar{E})}$. For $z \in \mathbf{Q}x$ there are $m, n \in \mathbf{Z}$ with $\bar{\imath}(mz) = \bar{\imath}(ny) \in \sqrt{\gamma_k(\bar{E})}$ which implies $\bar{\imath}z \in \sqrt{\gamma_k(\bar{E})}$.

By (6) and 2.1 one has $\Delta_{\mathbf{Q}}(\bar{\imath}\,\mathbf{Q}x)\,\Delta_{\mathbf{Q}}(\bar{E}) \subset \Delta_{\mathbf{Q}}^{k+1}(\bar{E})$; hence β is isomorphic. Furthermore $p_{k,\mathbf{Q}}$ and α in (5) are injective by 2.2 and by definition of k, resp. Now by commutativity of (5) it follows that $p_{n,\mathbf{Q}}\,\bar{\imath}(x) \neq 0$ which concludes the proof by (1) and (2). □

Now our 'descent to $\mathbf{Z}$' rests on the following

Lemma 2.4 *Let L be a graded Lie ring which is torsionfree as an abelian group. Then the envelopping algebra $\mathrm{U}L$ over $\mathbf{Z}$ is also torsionfree as an abelian group.*

Proof: The Lie ring L is the colimit of its sub-Lie rings L_α which are locally (i.e. in each degree) finitely generated as abelian groups. Since the envelopping algebra as a left adjoint functor preserves colimits we get

$$*) \qquad \mathrm{U}L \cong \mathrm{colim}_\alpha \mathrm{U}L_\alpha = \bigsqcup_\alpha \mathrm{U}L_\alpha \,/ \sim$$

where $x \sim y$, $x \in \mathrm{U}L_\alpha$, $y \in \mathrm{U}L_\beta$, if there is an inclusion $i_{\alpha\beta}\colon L_\alpha \subset L_\beta$ in L such that $\mathrm{U}(i_{\alpha\beta})(x) = y$.

Since L is supposed to be additively torsionfree each L_α is locally and hence globally a free $\mathbf{Z}$-module. The Birckhoff-Witt-theorem then provides a *natural* ring isomorphism $\mathrm{Gr}(\mathrm{U}L_\alpha) \cong \mathrm{S}(L_\alpha)$ of the associated graded of $\mathrm{U}L_\alpha$ with respect to the filtration by word length with the symmetric $\mathbf{Z}$-Algebra over L_α. It follows that $\mathrm{U}L_\alpha$ is additively free abelian and whence torsionfree, and that the homomorphisms $\mathrm{Gr}(U(i_{\alpha\beta}))$ induced by the inclusions $i_{\alpha\beta}\colon L_\alpha \subset L_\beta$ are injective. Thus also the maps $\mathrm{U}(i_{\alpha\beta})$ are injective which implies the assertion by $*$). □

For the proof of theorem A we introduce a relative version of the filtration $\{\Delta^i_{\surd}(E)\}$ of the group ring $\mathbf{Z}(E)$.

Definition 2.5 *Let E be a group and $B \lhd E$. Then the subgroup*

$$\Delta^i_{\surd}(E,B) \subset \Delta_{\mathbf{Z}}(E)$$

is generated by the elements $(a_1-1)\cdots(a_k-1)$, $k \geq 1$, with $a_1 \in \sqrt{\gamma_{n+1}(E)}$ if $k=1$ and with $a_i \in \sqrt{B\,\gamma_{s_i}(E)}$, $1 \leq i \leq k$ and $s_1 + \ldots + s_k > n$. Note that $\Delta^i_{\surd}(E,\{1\}) = \Delta^i_{\surd}(E)$. Since $\Delta^i_{\surd}(E,B)$ is a two-sided ideal we have the quotient rings

$$\tilde{P}_n^{\surd}(E,B) = \Delta_{\mathbf{Z}}(E)\big/\Delta_{\surd}^{n+1}(E,B)\,, \quad \tilde{P}_n^{\surd}(E) = \tilde{P}_n^{\surd}(E,\{1\})\ .$$

Proof of theorem A and B.

For an abelian group A let ν or $\nu(A)\colon A \cong \mathbf{Z}\otimes A \to \mathbf{Q}\otimes A$ be the canonical map. Let G be a group.
Using 2.1 and 2.2 we obtain the following factorization of $\nu(\mathrm{UL}^{\surd}(G))$:

$$\mathrm{UL}^{\surd}(G) \xrightarrow{\theta_{\mathbf{Z}}} \mathrm{Gr}^{\surd}(\mathbf{Z}(G)) \xrightarrow{\bar{\nu}} \mathrm{Gr}(\mathbf{Q}(G)) \overset{\theta}{\cong} \mathrm{U}(\mathbf{Q}\otimes \mathrm{L}^{\surd}(G)) \cong \mathbf{Q}\otimes \mathrm{UL}^{\surd}(G)$$

Here $\bar{\nu}$ is induced by $\nu: \mathbf{Z}(G)\to\mathbf{Q}(G)$. Since $\mathrm{L}^{\surd}(G)$ and thus by 2.4 also $\mathrm{UL}^{\surd}(G)$ are torsionfree as abelian groups we obtain

(1) injectivity of $\nu(\mathrm{UL}^{\surd}(G))$ and whence

(2) injectivity of $\theta_{\mathbf{Z}}$, i.e. theorem B

(3) injectivity of $\bar{\nu}$.

By 2.1 we have a factorization

$$\nu\colon P_n(G) \xrightarrow{\omega} \tilde{P}_n^{\surd}(G) \xrightarrow{\nu_2} P_{n,\mathbf{Q}}(G) \cong \mathbf{Q}\otimes P_n(G),$$

where ω is the canonical quotient map. But

(4) $$\mathrm{Gr}^{\surd}(\nu_2) = \bar{\nu}\,|\bigoplus_{i\le n} \mathrm{Gr}_i^{\surd}(\mathbf{Z}(G))$$

where the associated graded map $\mathrm{Gr}^{\surd}(\nu_2)$ is taken with respect to the filtration $\{\frac{\Delta^k_{\surd}(G)}{\Delta^{n+1}_{\surd}(G)}\}$ of $\tilde{P}_n^{\surd}(G)$ and with respect to the augmentation filtration of $P_{n,\mathbf{Q}}(G)$. Now (3) entrains injectivity of ν_2 which implies

(5) $$\tau P_n(G) = \mathrm{Ker}(\nu) = \mathrm{Ker}(\omega) = \Delta^n_{\surd}(G)\,/\,\Delta^n(G)$$

This proves theorem A (2) in case $B=\{1\}$. To study the relative case we first consider

(6) $$\tilde{P}_n^{\surd}(E,B) = P_n(E,B)\big/\rho\Delta^{n+1}_{\surd}(E,B)\,,$$

see 1.1, 2.5. Observe that by centrality of B in E we have a natural factorization

$$P_{n,R}(E,B)\otimes P_{n,R}(E,B) \xrightarrow{\sigma P_n(\pi)\otimes\sigma P_n(\pi)} P_{n-1}(G)\otimes_G P_{n-1}(G) \xrightarrow{\mu} P_{n,R}(E,B)$$

of the multiplication map of $P_{n,R}(E,B)$ where $\sigma: P_{n,R}(G)\twoheadrightarrow P_{n-1,R}(G)$ is the canonical quotient map. Using this and the relation $\sqrt{B\,\gamma_i(G)} = \pi^{-1}\sqrt{\gamma_i(H)}$ we may rewrite (6) by

(7) $$\rho\Delta^{n+1}_{\surd}(E,B) = p_n\sqrt{\gamma_{n+1}(E)} + \mu\left(\textstyle\sum_{i+j>n}\rho\Delta^i_{\surd}(G)\otimes\rho\Delta^j_{\surd}(G)\right)$$

Consider the following commutative diagram with exact bottom row by 2.3.

$$
(8)\quad
\begin{array}{ccccc}
\tau(B\,/\,B\cap\gamma_{n+1}(G)) & \xrightarrow{\tilde p_n^{\surd} i} & \tilde P_n^{\surd}(E,B) & \xrightarrow{\tilde P_n^{\surd}(\pi)} & \tilde P_n^{\surd}(G) \\
\big\downarrow & & \big\downarrow{\scriptstyle\alpha} & & \big\downarrow{\scriptstyle\beta} \\
\mathbf{Q}\otimes(B\,/\,B\cap\gamma_{n+1}(G)) & \xrightarrow{p_{n,\mathbf{Q}} i} & P_{n,\mathbf{Q}}(E,B) & \xrightarrow{P_{n,\mathbf{Q}}(\pi)} & P_{n,\mathbf{Q}}(G)
\end{array}
$$

The homomorphism $\tilde p_n^{\surd} i$ is welldefined since $\tau(B/B\cap\gamma_{n+1}(G)) = B\big/\sqrt{B\cap\gamma_{n+1}(G)}$ and $\tilde p_n^{\surd} i\sqrt{B\cap\gamma_{n+1}(G)}\subset \tilde p_n^{\surd} i\sqrt{\gamma_{n+1}(E)}=0$ by (7). The maps α and β are induced by $\nu(P_n(E,B))$ and $\nu(P_n(G))$, resp. and are welldefined since we have

$$
\begin{aligned}
\nu\rho\,\Delta_{\surd}^{n+1}(E,B) &\subset \rho\nu\left(\Delta_{\surd}^{n+1}(E)\right)+\mu\sum_{i+j>n}\rho\nu\Delta_{\surd}^{i}(G)\otimes\rho\nu\Delta_{\surd}^{j}(G)\\
&\subset \rho\Delta_{\mathbf{Q}}^{n+1}(E)+\mu\sum_{i+j>n}\rho\Delta_{\mathbf{Q}}^{n}(G)\otimes\rho\Delta_{\mathbf{Q}}^{i+j-n}(G)\\
&= 0
\end{aligned}
$$

Hence $\tilde p_n^{\surd} i$ is injective by commutativity of the left hand square in (8). Moreover, the top row is exact in $\tilde P_n^{\surd}(E,B)$ since $P_n(\pi)\left(\rho\Delta_{\surd}^{n+1}(E,B)\right)=\rho\Delta_{\surd}^{n+1}(G)$ by (7) and the relation $\pi\sqrt{\gamma_{n+1}(E)}=\sqrt{\gamma_{n+1}(G)}$ which holds by assumption. Now the injectivity of β by (5) implies injectivity of α. From the factorization

$$\nu\colon\ P_n(E,B)\ \twoheadrightarrow\ \tilde P_n^{\surd}(E,B)\ \overset{\alpha}{\rightarrowtail}\ P_{n,\mathbf{Q}}(E,B)$$

we conclude $\tau P_n(E,B)=\operatorname{Ker}\nu=\rho\Delta_{\surd}^{n+1}(E,B)$ which is assertion (2). Whence $P_n^{\surd}(E,B)\cong\tilde P_n^{\surd}(E,B)$ and the short exactness of the top row in (8) already shown proves assertion (1). For the proof of (3) let E and G be $\mathcal{T}_n$-groups. Then the map

$$p_{n,\mathbf{Q}}\colon G\xrightarrow{p_n^{\surd}}P_n^{\surd}(G)\xrightarrow{\nu}P_{n,\mathbf{Q}}(G)$$

is injective by 2.2 whence so is $p_n^{\surd}$. Now let $a,b\in E$ with $p_n^{\surd}(a)=p_n^{\surd}(b)$ in $P_n^{\surd}(E,B)$. Then $\pi(a)=\pi(b)$ by naturality of $p_n^{\surd}$ and by injectivity of $p_n^{\surd}$ on G. Thus we have $a^{-1}b\in i(B)$ and

$$p_n^{\surd}(a)=p_n^{\surd}(b)=p_n^{\surd}(a\,i(i^{-1}(a^{-1}b)))=p_n^{\surd}(a)+p_n^{\surd}i(i^{-1}(a^{-1}b))\,.$$

This implies $a^{-1}b=1$ by injectivity of $p_n^{\surd}i$ proved above. □

Proof of Corollary C:

(1): We have isomorphisms of graded rings

$$\mathrm{Gr}^{\surd}\left(P_n^{\surd}(G)\right)\cong\mathrm{Gr}^{\surd}\left(\Delta_{\mathbf{Z}}(G)/\Delta_{\surd}^{n+1}(G)\right)\cong \mathrm{UL}^{\surd}(G)\Big/\sum_{i>n}\mathrm{U}_i\mathrm{L}^{\surd}(G)$$

by theorem A (2) and theorem B. If G is finitely generated then $\mathrm{L}_i^{\surd}(G) = \sqrt{\gamma_i(G)}\big/\sqrt{\gamma_{i+1}(G)}$ is a finitely generated free abelian group, thus so is $\mathrm{U}_i\mathrm{L}^{\surd}(G)$ and the second assertion follows.

(2): By theorem A (1), (2) and 2.5 it remains to show that the quotient maps

$$\tilde{P}_n^{\surd}(G) \xrightarrow{\pi_1} \tilde{P}_n^{\surd}\left(G\,/\,\sqrt{\gamma_{n+1}(G)}\right) \xrightarrow{\pi_2} \tilde{P}_n^{\surd}\left(G\big/\sqrt{\gamma_{n+1}(G)}\,,\ \sqrt{\gamma_n(G)}\,\big/\,\sqrt{\gamma_{n+1}(G)}\right)$$

are injective. This follows readily from the following relations in $\tilde{G} = G\big/\sqrt{\gamma_{n+1}(G)}$ which are easily verified: For $i \geq n$ one has $\sqrt{\gamma_i(\tilde{G})} = \sqrt{\gamma_i(G)}\big/\sqrt{\gamma_{n+1}(G)}$ and

$$\sqrt{\left(\sqrt{\gamma_n(G)}\big/\sqrt{\gamma_{n+1}(G)}\right)\gamma_i(\tilde{G})} = \sqrt{\sqrt{\gamma_n(\tilde{G})}\,\gamma_i(\tilde{G})} \subset \sqrt{\gamma_i(\tilde{G})}\,.$$

(3): In order to apply (2) we calculate

$$P_n^{\surd}(\pi)\ \sqrt[+]{p_n^{\surd}(\gamma_n(G))} \subset \sqrt[+]{p_n^{\surd}\left(\gamma_n\left(G\big/\sqrt{\gamma_n(G)}\right)\right)} = \sqrt[+]{0} = 0$$

Now (2) implies

$$\sqrt[+]{p_n^{\surd}(\gamma_n(G))} \subset p_n^{\surd}\ \sqrt{\gamma_n(G)}\,.$$

The converse inclusion follows from the linearity of $p_n^{\surd}|\sqrt{\gamma_n(G)}$. □

Proof of theorem D: The ring structure of $\Delta_{\mathbf{Z}}(G)$ is determined by the ring structure of $\mathbf{Z}(G)$ [Pm77] p. 664, whence also the Lie ring $\mathrm{L}^{\tau}(G)$. The asserted isomorphism with $\mathrm{L}^{\surd}(G)$ is provided by corollary C (3) as an isomorphism *of abelian groups.*

Now let $a \in \sqrt{\gamma_k(G)}$ und $b \in \sqrt{\gamma_l(G)}$. Then $(a,b) \in \sqrt{\gamma_{k+l}(G)}$, and we have

$$*)\quad p_{k+l}^{\surd}(a,b) = \left[p_{k+l}^{\surd}(a)\,,\,p_{k+l}^{\surd}(b)\right] + p_{k+l}^{\surd}(a^{-1}b^{-1})\left[p_{k+l}^{\surd}(a)\,,\,p_{k+l}^{\surd}(b)\right]$$

by the relation $(a,b) - 1 = a^{-1}b^{-1}[a-1\,,b-1]$ in $\mathbf{Z}(G)$. But the last term in $*$) lies in the image of $\Delta_{\surd}^{k+l+1}(G)$ in $P_{k+l}^{\surd}(G)$ and thus vanishes by theorem A (2). This proves the theorem. □

Part II: On central extensions of torsionfree nilpotent groups

It is a classical result [Re-Ro79] that for a group G the category of *group* extensions $B \rightarrowtail E \twoheadrightarrow G$ is equivalent with the category of *G-module* extensions $B \rightarrowtail M \twoheadrightarrow \Delta(G)$ where $\Delta(G)$ is the augmentation ideal of the group ring $\mathbf{Z}(G)$. We here show that for *central* extensions of *torsionfree nilpotent* groups G and E an analogous 'model theorem' holds using $\mathbf{Z}$-torsionfree *nilpotent* modules and replacing $\Delta(G)$ by the polynomial quotient $P_n^{\surd}(G) = \left(\Delta(G)/\Delta^{n+1}(G)\right)/\mathbf{Z} - torsion$. From this result we obtain inductive models for torsionfree nilpotent groups ($\mathcal{T}$-groups for short) in terms of nilpotent rings. Further applications are concerned with the cohomology group $H^2(G,B)$ of $\mathcal{T}$-groups G and $\mathbf{Z}$-torsionfree trivial G-modules B: We give a description in terms of polynomial cocycles and of polynomial coboundaries in the sense of Passi. This result generalizes a theorem of [Pa-Su-Ta87] for $B = \mathbf{Q}$ to nondivisable coefficients. Moreover, it amounts to a computation of $H^2(G,B)$ in terms of explicit integer valued rational polynomials, starting from a presentation of G and using matrix calculus. Compared with corresponding classical results these polynomials are of a particular simple form and of minimal degree.

The results of this paper form also the basis of our inductive description of automorphism groups of $\mathcal{T}$-groups in part three of this paper.

1 Abelian models of torsionfree nilpotent groups

We consider the category $\mathcal{G}_\mathbf{n}$ of central group extensions $\underline{E} = (\, B \overset{i}{\rightarrowtail} E \overset{\pi}{\twoheadrightarrow} G\,)$ where G and E are $\mathcal{T}$-groups of class $\leq n$ ($\mathcal{T}_n$-groups for short). Morphisms from $\underline{E}_1$ to $\underline{E}_2$ in $\mathcal{G}_\mathbf{n}$ are tripels (f, F, f') of homomorphisms which make the following diagram commutative:

$$\begin{array}{ccccc} B_1 & \overset{i_1}{\rightarrowtail} & E_1 & \overset{\pi_1}{\twoheadrightarrow} & G_1 \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle f} \\ B_2 & \overset{i_2}{\rightarrowtail} & E_2 & \overset{\pi_2}{\twoheadrightarrow} & G_2 \end{array}$$

For the construction of an '*n-model*' of a group extension $\underline{E} \in \mathcal{G}_\mathbf{n}$ we recall the definition of the relative polynomial construction in part I: Let $\Delta(E)$ denote the augmentation ideal of the group ring $\mathbf{Z}(E)$. We define the quotient rings

$$P_n(E,B) = \Delta(E)\Big/\left(\Delta(B)\Delta(E) + \Delta^{n+1}(E)\right)$$

$$P_n^{\surd}(E,B) = P_n(E,B)\big/\mathbf{Z} - torsion\,, \quad P_n^{\surd}(E) = P_n^{\surd}(E,\{1\})\,.$$

Via the map π and left multiplication, $P_n(E,B)$, $P_n^{\surd}(E,B)$ and $P_n^{\surd}(G)$ are left nilpotent G-modules of class $\leq n$, and the map

$$p_n^{\surd} : E \to P_n^{\surd}(E,B) \ , \quad p_n^{\surd}(a) = \{a-1\} \ ,$$

is a π-derivation which is universal with respect to $\mathbf{Z}$-torsionfree modules of this type.

Now an *n-model* is a pair $(G, \underline{M})$ where G is a $\mathcal{T}_n$-group and where $\underline{M} = (B \rightarrowtail M \twoheadrightarrow P_n^{\surd}(G))$ is an extension of left G-modules such that B is a $\mathbf{Z}$-torsionfree trivial module. We assign to a group extension $\underline{E} = (\ B \overset{i}{\rightarrowtail} E \overset{\pi}{\twoheadrightarrow} G\ )$ in $\mathcal{G}_{\mathbf{n}}$ the n-model

$$P_n(\underline{E}) = \left(G,\ B \overset{p_n^{\surd} i}{\rightarrowtail} P_n^{\surd}(E,B) \overset{P_n^{\surd}(\pi)}{\twoheadrightarrow} P_n^{\surd}(G)\right) .$$

The term on the right has the required properties of an n-model by theorem A in part I of this paper.

1.1 Conversely, an n-model $(G, \underline{M})$ gives rise to a group extension

$$Pull_n(G, \underline{M}) = (B \overset{i'}{\rightarrowtail} E_o \overset{pr_2}{\twoheadrightarrow} G)$$

in $\mathcal{G}_{\mathbf{n}}$ defining

$$E_o = \{\, (x,a) \in M \times G \,|\, \pi(x) = p_n^{\surd}(a) \,\} \ ,$$
$$(x,a)\cdot(y,b) = (x + a\cdot y \,, ab)$$

and $i'x = (x,1)$.

We also want to compare the automorphism groups of group extensions in $\mathcal{G}_{\mathbf{n}}$ with those of their n-models. For this we define the category $\mathcal{M}_{\mathbf{n}}$ of n-models by taking as morphisms from $(G_1, \underline{M}_1)$ to $(G_2, \underline{M}_2)$ all tripels (f, F, f') with the following properties:

- $f : G_1 \to G_2$ and $f' : B_1 \to B_2$ are homomorphisms of groups;
- $F : M_1 \to M_2$ is an f-equivariant homomorphism, i.e. $F(ax) = f(a)\,F(x)$ for $x \in M_1$, $a \in G_1$;
- the following diagram is commutative:

$$\begin{array}{ccccc} B_1 & \overset{i_1}{\rightarrowtail} & M_1 & \overset{\pi_1}{\twoheadrightarrow} & P_n^{\surd}(G_1) \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle P_n^{\surd}(f)} \\ B_2 & \overset{i_2}{\rightarrowtail} & M_2 & \overset{\pi_2}{\twoheadrightarrow} & P_n^{\surd}(G_2) \end{array}$$

The constructions P_n and $Pull_n$ above are actually functors

$$\mathcal{G}_{\mathrm{n}} \underset{Pull_n}{\overset{P_n}{\rightleftarrows}} \mathcal{M}_{\mathrm{n}}$$

by defining $P_n(f,F,f') = (f,\ P_n^{\sqrt{}}(F),f')$ and $Pull_n(f,F,f') = (f,F \times f,f')$.

Model theorem A: *A group extension* $\underline{E} \in \mathcal{G}_{\mathrm{n}}$ *is uniquely determined by its n-model up to congruence and conversely. Furthermore, the automorphism group of* $\underline{E}$ *is mapped isomorphicly onto the automorphism group of its n-model by the functor* P_n. *Actually, the functors* P_n *and* $Pull_n$ *are mutually inverse equivalences of categories.*

Proof: Natural transformations $P_n \circ Pull_n \longrightarrow id$ and $id \longrightarrow Pull_n \circ P_n$ are readily established by the universal properties of $p_n^{\sqrt{}}$ and of the pullback used in the definition of $Pull_n$, resp. These are isomorphisms by the five-lemma. □

Thus theorem A relies essentially on the fact that the functor P_n is welldefined, i.e. on our study of the relative polynomial construction in part I.

Remark: In comparison with the general model theorem on arbitrary group extensions in [Re-Ro79] our special result for $\mathcal{T}$-groups has several advantages:

- For finitely generated groups E the models $P_n(\underline{E})$ consist of *finitely generated free abelian groups.*
- $P_n^{\sqrt{}}(G)$ is computable by abelian generators and relations if G is given by a finite presentation, cf. proposition E below. This is due to the fact that for free groups F the ring $P_n^{\sqrt{}}(F)$ is a truncated noncommuative polynomial ring, i.e. it is free in the category of rings of index $n+1$ while the ring $\mathbf{Z}(G)$ itself is *not* free but has a much more complicated structure.
- The modules involved in our models are *nilpotent* which is well adapted to inductive constructions [Ha92k].

From Theorem A we obtain the following inductive abelian models for torsionfree nilpotent groups and their homomorphisms.

For a group E and $U < E$ let $\sqrt{U} = \{a \in E \,|\, \exists m \in \mathbf{Z} : a^m \in U\}$, the isolator of U, which is a subgroup if E is nilpotent. In particular, $E\big/\sqrt{\gamma_{n+1}(G)}$ is a $\mathcal{T}_n$-group. Now let S be a ring (without unit) of index $n+1$, i.e. $S^{n+1} = 0$. Then $K_n(S)$ denotes the additive subgroup of S generated by the n-fold ring commutators of S and $\sqrt[+]{K_n(S)}$ denotes its isolator taken with respect to the *additive* structure of S.

We define the category $\mathcal{R}_{\mathrm{n}}$ of models for $\mathcal{T}_n$-groups as follows: Objects are pairs $\underline{S} = (G,\ \pi : S \twoheadrightarrow P_n^{\sqrt{}}(G)\ )$ where G is a $\mathcal{T}_{n-1}$-group, S is a ring of index $n+1$ whose additive group is torsionfree and where π is a surjective ring homomorphism such that $\mathrm{Ker}(\pi) = \sqrt[+]{K_n(S)}$. Morphisms from $\underline{S}_1$ to $\underline{S}_2$ in $\mathcal{R}_n$ are pairs (f, F) with $f \in \mathrm{Hom}(G_1, G_2)$ and where $F: S_1 \to S_2$ is a ring homomorphism such that $P_n^{\sqrt{}}(f)\,\pi_1 = \pi_2 F$.

Now we assign to a $\mathcal{T}_n$-group E the model

$$R_n(E) = \left(E/\sqrt{\gamma_n(E)}\,,\ \ P_n^{\sqrt{}}(\pi)\ :\ P_n^{\sqrt{}}(E) \twoheadrightarrow P_n^{\sqrt{}}\left(E/\sqrt{\gamma_n(E)}\right)\right)$$

where $\pi : E \twoheadrightarrow G/\sqrt{\gamma_n(E)}$ is the canonical quotient map. This is in fact an object in $\mathcal{R}_n$ by Corollary C in part I. Conversely, an arbitrary object $\underline{S} = (G,\ \pi : S \twoheadrightarrow P_n^{\sqrt{}}(G)\ ) \in \mathcal{R}_n$ gives rise to a group

$$G_n(\underline{S}) = \{(x,a) \in S \times G \mid \pi(x) = p_n^{\sqrt{}}(a)\} \quad \text{with} \quad (x,a)(y,b) = (x + \bar{a}y, ab)$$

where $\bar{a}$ is some element in S with $\pi(\bar{a}) = p_n^{\sqrt{}}(a)$. The product $\bar{a}y$ is welldefined since $\sqrt[+]{K_n(S)}\,S \subset \sqrt[+]{S^{n+1}} = 0$ by our assumptions on S.

Model theorem B: *A $\mathcal{T}_n$-group E is determined by its model $R_n(G)$ up to isomorphism, and its automorphism group is canonically isomorphic to that of $R_n(G)$. Actually, the constructions R_n and G_n are mutually inverse equivalences of the category of $\mathcal{T}_n$-groups and the category $\mathcal{R}_n$.*

In some sense, this result is a substitute of the Mal'cev correspondence between rational nilpotent groups and rational nilpotent Lie algebras in the nondivisable case. It may be used for the construction of 'small' models for simplicial $\mathcal{T}$-groups and thus for integral homotopy types, in analogy with Quillen's use of the Mal'cev correspondence as a key ingredient in his modelization of rational homotopy theory by rational differential graded Lie algebras. As a first step in this direction we derive from theorem B a Dold-Kan theorem for simplicial groups of class 2 [Ha92h]. Applications to group cohomology with respect to the variety of nilpotent groups of class n are also to be exspected.

Theorem B is an immediate consequence of theorem A using the following

Remark 1.2 An extension of G-modules $B \rightarrowtail M \xrightarrow{\pi} P_n^{\sqrt{}}(G)$ can be viewed as a singular ring extension with $B \cdot M = 0$ and conversely. In fact, the module M can be endowed with the multiplication

$$M \otimes M \xrightarrow{\pi \otimes M} P_n^{\sqrt{}}(G) \otimes M \xrightarrow{\mu} M$$

where μ is induced by the given G-action on M. The converse is clear.

2 The second homology and cohomology of torsionfree nilpotent groups

We proceed to apply model theorem A to the classification of group extensions.

Let G be group and B be a trivial G-module. Let

$$\mathrm{Ext}^1_n(\, P_n^{\surd}(G), B) \subset \mathrm{Ext}^1_G(\, P_n^{\surd}(G), B)$$

be the subgroup of congruence classes of G-module extensions $B \rightarrowtail M \twoheadrightarrow P_n^{\surd}(G)$ for which the module M is nilpotent of class $\leq n$. Similarly, consider the ring

$$\tilde{P}_n^{\surd}(G) = \big(\mathbf{Z}(G)/\Delta^{n+1}(G)\big)/\mathbf{Z} - torsion$$

and let

$$H^2_n(\tilde{P}_n^{\surd}(G), B) \subset H^2_H(\tilde{P}_n^{\surd}(G), B)$$

be the subgroup of the Hochschild cohomology of $\tilde{P}_n^{\surd}(G)$ consisting of congruence classes of singular ring extensions $B \rightarrowtail S \overset{\pi}{\twoheadrightarrow} \tilde{P}_n^{\surd}(G)$ for which $\bar{S} = \mathrm{Ker}(\epsilon\,\pi)$ satisfies $\bar{S}^{n+1} = 0$. Then theorem A implies

Corollary C: *Let G be a $\mathcal{T}_n$-group and let B be a $\mathbf{Z}$-torsionfree G-module. Then the constructions P_n and $Pull_n$ provide mutually inverse natural isomorphisms*

(1) $$H^2(G, B) \cong \mathrm{Ext}^1_{n+1}(P_{n+1}^{\surd}(G), B)$$

(2) $$H^2(G, B) \cong H^2_{n+1}(\tilde{P}_{n+1}^{\surd}(G), B)$$

where for (2) we suppose in addition that G is finitely generated.

The latter condition assures that the group $\tilde{P}_{n+1}^{\surd}(G)$ is finitely generated and whence free abelian. Thus (2) follows from (1) via 1.2.

Now we derive from corollary C an explicit formula for $H^2(G, B)$. One has natural homomorphisms

$$\mathbf{Z} \otimes_G B_2(G) \xrightarrow{\psi} P_n(G) \otimes_G P_n(G) \xrightarrow{\mu_{n+1}} P_{n+1}(G)\,.$$

Here $B_2(G)$ is the term of degree 2 in the normalized bar resolution of $\mathbf{Z}$ over $\mathbf{Z}(G)$, $\psi[a|b] = p_n(a) \otimes p_n(b)$, and μ_{n+1} is the canonical factorization of the multiplication map of $P_{n+1}(G)$.

Theorem D: *Let G be a finitely generated $\mathcal{T}_n$-group and let B be a $\mathbf{Z}$-torsionfree trivial G-module. Then ψ induces natural isomorphisms*

$$H^2(G, B) \cong \mathrm{Hom}_{\mathbf{Z}}\big(P_n(G) \otimes_G P_n(G)\,,\, B\big)/\mu^*_{n+1}\,\mathrm{Hom}_{\mathbf{Z}}\big(P_{n+1}(G), B\big)$$

$$H_2(G)/torsion \cong \mathrm{Ker}(\mu_{n+1})/torsion\,.$$

In [Ha91] we show that the assertion on $H_2(G)$ actually holds without assuming G to be finitely generated, and we also determine $H^2(G,B)$ as a bifunctor in this general case.

Theorem D may be viewed as a convergence theorem on the polynomial approximations of H^2 and H_2 defined in [Pa74] and [Ha92d], respectively. In fact, the statement on $H^2(G,B)$ improves the following result of Passi, Sucheta and Tahara in two directions: As a main result in [Pa-Su-Ta87] they prove that each cohomology class in $H^2(G,\mathbf{Q})$ is representable by a polynomial cocycle of degree $\leq n$ (in Passi's sense [loc.cit]) for a $\mathcal{T}$-group G and the trivial G-action on $\mathbf{Q}$. Theorem A extends the result to arbitrary torsionfree coefficient groups (in particular, to $B = \mathbf{Z}$) and, moreover, gives a precise computation of the group $H^2(G,B)$ in terms of polynomial cocycles and polynomial coboundaries. In the following we make this computation explicit and give an example before proving the theorem.

Let $G = \langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l \rangle$ be a presentation of a $\mathcal{T}_n$-group. Then one can compute the groups $H^2(G,\mathbf{Z})$, $H_2(G)/torsion$ and the Kronecker pairing by matrix calculus using theorem D and the next propositions E and F.

Let T be the noncommutative polynomial ring in variables $x_1, \ldots, x_k$, and let $T_n = T/\bar{T}^{n+1}$. Write $r_i = x_{j_1}^{\pm 1} \cdots x_{j_{s_i}}^{\pm 1}$ in the free group $F = \langle x_1, \ldots, x_k \rangle$. Let

$$\tilde{r}_i = (1 + x_{j_1})^{\pm 1} \cdots (1 + x_{j_{s_i}})^{\pm 1} \; - \; 1 \quad \text{in} \quad \bar{T}_n$$

with $(1+x)^{-1} = 1 + \sum_{\nu=1}^{n} (-1)^\nu x^\nu$. Let $\delta \colon R = \langle r_1, \ldots, r_l \rangle \to F$ be the homomorphism defined by $\delta(r_i) = r_i$, and let $R_n = span\{\tilde{r}_1, \ldots, \tilde{r}_l\}$ and $S_n = T_n R_n \bar{T}_n + \bar{T}_n R_n T_n \subset \bar{T}_n^2$.

Proposition E: *We have the following commutative diagram with isomorphisms m_1, m_2 defined below and with exact rows*

$$\begin{array}{ccccc}
\operatorname{Ker} \mu_n & \hookrightarrow & P_{n-1}(G) \otimes_G P_{n-1}(G) & \xrightarrow{\mu_n} & P_n(G) \\
\uparrow & & \cong \uparrow m_1 & & \cong \uparrow m_2 \\
\operatorname{Ker}(\delta^{ab} \colon R^{ab} \to F^{ab}) & \xrightarrow{\bar{\delta}} & \bar{T}_n^2 \,/\, S_n & \xrightarrow[\mu_T]{} & \bar{T}_n \big/ T_n R_n T_n
\end{array}$$

Here m_1 and m_2 are induced by

$$m_1(x_{s_1} \cdots x_{s_m}) = p_{n-1}\{x_{s_1}\} \otimes p_{n-1}\{x_{s_2}\} \cdots p_{n-1}\{x_{s_m}\}$$

$$m_2(x_{s_1} \cdots x_{s_m}) = p_{n-1}\{x_{s_1}\} \cdots p_{n-1}\{x_{s_m}\}$$

where $\{x_i\}$ denotes the coset of x_i in G. Furthermore, μ_T and $\bar{\delta}$ are induced by the inclusion $\bar{T}_n^2 \subset \bar{T}_n$ and by the map $r_i \mapsto \tilde{r}_i$, respectively.

The proof is easy [Ha91] 1.2.9.

Proposition F: *Let $x_1, \ldots, x_k$ be a set of generetors of a $\mathcal{T}_n$-group G (e.g., a Mal'cev basis of G) and let $d\colon\ P_n(G) \otimes_G P_n(G) \to B$ be a homomorphism into some abelian group B. Then the corresponding polynomial cocycle $D:\ G \times G \to B$ is given by*

$$D\left(x_{k_1}^{n_1} \cdots x_{k_r}^{n_r}\,,\, x_{l_1}^{m_1} \cdots x_{l_s}^{m_s}\right) \ =$$

$$\sum_I \binom{n_1}{\mu_1} \cdots \binom{n_r}{\mu_r} \binom{m_1}{\nu_1} \cdots \binom{m_s}{\nu_s}\, d\left(p_n(x_{k_1})^{\mu_1} \cdots p_n(x_{k_r})^{\mu_r} \otimes p_n(x_{l_1})^{\nu_1} \cdots p_n(x_{l_s})^{\nu_s}\right)$$

with $I = \left\{(\mu_1, \ldots, \mu_r, \nu_1, \ldots, \nu_s) \,\middle|\, 0 \le \mu_i, \nu_j \le n\,,\ 1 \le \sum \mu_i, \sum \nu_j\,,\ \sum \mu_i + \sum \nu_j \le n+1\right\}$ *and where* $\binom{s}{\mu}$, $s \in \mathbf{Z}$, $\mu \ge 0$, *is defined to be the coefficient of* x^μ *in the expansion of* $(1+x)^s$ *in* $\mathbf{Z}[[x]]$.

Note that $\binom{s}{\mu}$ is an integer valued rational polynomial of degree μ in s for fixed μ and fixed sign of s. *Hence the coefficients in the sum expansion of D above are integer valued rational polynomials of degree $\le n$ in the variables n_i and m_j* **and of total degree** $\le n+1$.

Example: Consider the **Heisenberg groups** $G_q = \langle a, b, c \mid (a,b) = c^q,\ (a,c) = (b,c) = 1 \rangle$, $q \in \mathbf{Z}$. Here a computation by the procedure indicated above ([Ha91] 2.3.20) gives isomorphisms

$$\bar{\tau} H_2(G) \ \cong\ \mathbf{Z} \cdot \bar{\tau}\bar{p}_3\{(a,c)\} \oplus \mathbf{Z} \cdot \bar{\tau}\bar{p}_3\{(b,c)\}$$

$$H^2(G, \mathbf{Z}) \ \cong\ \mathbf{Z}/q\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}\,,$$

where the tupel (u, w, z) is represented by the polynomial cocycle

$$D(u,w,z)(a^{a_1} b^{b_1} c^{c_1}\,,\, a^{a_2} b^{b_2} c^{c_2}) =$$

$$u\, b_1 a_2 + w\left(q b_1 \binom{a_2}{2} - c_1 a_2\right) + z\left(q b_1 a_2 b_2 + q\binom{b_1}{2} a_2 - c_1 b_2\right)\,,$$

$a_i, b_i, c_i \in \mathbf{Z}$. Moreover, the Kronecker product

$$<\,,\,>\colon H^2(G, \mathbf{Z}) \times \bar{\tau} H_2(G) \longrightarrow \mathbf{Z}$$

is given by

$$<(u,w,z)\,,\, k\{(a,c)\} + l\{(b,c)\}> \,=\, wk + zl\,.$$

Proof of theorem D:

(1) First observe that a central extension $\underline{M} = (B \overset{i}{\rightarrowtail} M \overset{\pi}{\twoheadrightarrow} P^{\surd}_{n+1}(G))$ of $\mathbf{Z}$-torsionfree nilpotent modules of class $\le n$ is equivalently described by a diagram

$$\begin{array}{ccccc} & & P_n^{\surd}(G) \otimes_G P_n^{\surd}(G) & & \\ & & \downarrow W & & \\ B & \stackrel{i}{\rightarrowtail} & M & \stackrel{\pi}{\twoheadrightarrow} & P_{n+1}^{\surd}(G) \end{array}$$

where the row is the underlying extension of abelian groups and where W is a homomorphism of abelian groups lifting the map

$$\mu_{n+1}^{\surd}: \ P_n^{\surd}(G) \otimes_G P_n^{\surd}(G) \to P_{n+1}^{\surd}(G)$$

induced by the multiplication map of the ring $P_{n+1}^{\surd}(G)$. In fact, W is obtained by the following factorization of the action map

$$\Delta(G) \otimes M \xrightarrow{\rho \otimes M} P_n(G) \otimes M \xrightarrow{\bar{\tau} \otimes \pi} P_n^{\surd}(G) \otimes_G P_n^{\surd}(G) \xrightarrow{W} M$$

where ρ and $\bar{\tau}$ are the canonical quotient maps.

(2) The terms $P_n(G) \otimes_G P_n(G)$ and $P_{n+1}(G)$ in theorem D can be replaced by $P_n^{\surd}(G) \otimes_G P_n^{\surd}(G)$ and $P_{n+1}^{\surd}(G)$, resp., since the group B is torsionfree by assumption.

Using (1) define a map

$$\chi: \ \mathrm{Hom}(\ P_n^{\surd}(G) \otimes_G P_n^{\surd}(G)\ ,\ B) \to \mathrm{Ext}_n^1(P_{n+1}^{\surd}(G), B)$$

by

$$\begin{array}{c} P_n^{\surd}(G) \otimes_G P_n^{\surd}(G) \\ \qquad\qquad\qquad\qquad \downarrow (d, \mu_{n+1}^{\surd})^t \\ \chi(d) \ = \ \left\{ \ B \xrightarrow{(1,0)^t} B \oplus P_{n+1}^{\surd}(G) \xrightarrow{(0,1)} P_{n+1}^{\surd}(G) \right\} \end{array}$$

where brackets $\{\}$ denote the congruence class.

One readily checks that χ is a homomorphism using the Baer sum construction of module extensions [ML63]. Moreover, the identity

(3) $$Pull_n \circ \chi = \psi^*$$

is readily verified using the transversal $a \mapsto (0, p_{n+1}^{\surd}(a))$, $a \in G$, for B in E_o, cf. 1.1.

Since G is supposed to be finitely generated the group $P_{n+1}^{\surd}(G)$ is finitely generated torsionfree, whence *free abelian*; this shows that χ is **surjective**. To determine

$\mathrm{Ker}(\chi)$ suppose that $\chi(d) = 0 = \chi(0)$. This is equivalent with the existence of an automorphism α of $B \oplus P^{\surd}_{n+1}(G)$ with matrix

$$\begin{pmatrix} 1 & \alpha' \\ 0 & 1 \end{pmatrix},$$

$\alpha' \in \mathrm{Hom}(P^{\surd}_{n+1}(G), B)$, such that

$$(d, \mu^{\surd}_{n+1})^t = \alpha(0, \mu^{\surd}_{n+1})^t = (\alpha' \mu^{\surd}_{n+1}, \mu^{\surd}_{n+1}),$$

i.e. $d = {\mu^{\surd}_{n+1}}^* \alpha'$. By corollary C together with (3) we thus have proved the first part of the theorem. For the second one first note that by theorem A (2) in part I we have for $G \in \mathcal{T}_{n+1}$ the relations

(4) $$\tau(P_{n+1}(G)) = \mu_{n+1}\Big(\tau(P_n(G)) \otimes \tau(P_n(G))\Big) \subset P_{n+1}(G)^2$$

where $\tau(-)$ denotes the torsion subgroup.

(5) Thus the sequence of homomorphisms

$$P^{\surd}_n(G)^2 \hookrightarrow P^{\surd}_n(G) \overset{\rho}{\twoheadrightarrow} P_1(G) \cong G^{ab}$$

is welldefined and is a *free resolution* of the free abelian group G^{ab}. Let σ denote the canonical quotient maps from $\Delta(G)$ to $P^{\surd}_n(G)$ or to $P^{\surd}_{n+1}(G)$. Consider the following commutative diagram of short exact sequences where the top row is the universal coefficient sequence for $H^2(G, B)$ and where both terms on the left equal $\mathrm{Ext}^1_{\mathbf{Z}}(G^{ab}, \mathbf{Q}) = 0$.

$$\begin{array}{ccccc}
\dfrac{\mathrm{Hom}(P^{\surd}_{n+1}(G)^2, \mathbf{Q})}{j^*\mathrm{Hom}(P^{\surd}_{n+1}(G), \mathbf{Q})} & \overset{-{\mu^{\surd}_{n+1}}^*}{\rightarrowtail} & \dfrac{\mathrm{Hom}(P^{\surd}_n(G) \otimes_G P^{\surd}_n(G), \mathbf{Q})}{(j\mu^{\surd}_{n+1})^*\mathrm{Hom}(\Delta(G), \mathbf{Q})} & \overset{\iota^*}{\twoheadrightarrow} & \mathrm{Hom}(\mathrm{Ker}(\mu^{\surd}_{n+1}), \mathbf{Q}) \\
\downarrow{\scriptstyle \sigma^*} & & \downarrow{\scriptstyle (\sigma\otimes\sigma)^*} & & \downarrow{\scriptstyle (\sigma\otimes\sigma)^*} \\
\dfrac{\mathrm{Hom}(\Delta^2(G), \mathbf{Q})}{j^*\mathrm{Hom}(\Delta(G), \mathbf{Q})} & \overset{-\mu^*}{\rightarrowtail} & \dfrac{\mathrm{Hom}(\Delta(G) \otimes_{\mathbf{Z}(G)} \Delta(G), \mathbf{Q})}{(j\mu)^*\mathrm{Hom}(\Delta(G), \mathbf{Q})} & \overset{\iota^*}{\twoheadrightarrow} & \mathrm{Hom}(\mathrm{Ker}(\mu), \mathbf{Q})
\end{array}$$

The vertical map in the middle is already proved to be isomorphic whence so is σ^* on the right. This shows that the map

$$H_2(G) \cong \mathrm{Ker}\Big(\mu : \Delta(G) \otimes_{\mathbf{Z}(G)} \Delta(G) \to \Delta(G)\Big) \overset{\sigma\otimes\sigma}{\twoheadrightarrow} \mathrm{Ker}(\mu^{\surd}_{n+1})$$

is **injective mod torsion**. It is also **surjective** by surjectivity of the map $\rho \otimes \rho$: $\mathrm{Ker}(\mu) \longrightarrow \mathrm{Ker}(\mu_{n+1})$ (which is easily checked) and by (4) above. This proves the theorem. □

Part III: On automorphism groups of torsion-free nilpotent groups

We describe an inductive computation of the automorphism group of torsionfree nilpotent groups (= $\mathcal{T}$-groups) in terms of iterated group extensions. All terms involved are computable by use of linear algebra in case G is given by a finite presentation.

In particular we consider the automorphism groups of free nilpotent groups which are of particular interest in algebraic K-theory; for nilpotency class 2 they are related to the exotic element of $K_3(\mathbf{Z})$ as was pointed out by Baues-Dreckmann [Ba-Dr89].

Let G be a **free nilpotent group** of class n, i.e. $G = F/\gamma_{n+1}(F)$ for some free group F, where $\gamma_i(F)$ is given by the lower central series of F. For the free abelian group $A = G^{ab} = G/\gamma_2(G)$ let $T(A) = \mathbf{Z} \oplus \bigoplus_{i>1} A^{\otimes i}$ be the tensor ring over A, $\bar{T}$ its augmentation ideal, $\bar{T}_n = \bar{T}/\bar{T}^{n+1}$ the quotient ring, and let $L_n(A)$ be the subgroup of the free Lie-ring over A which is generated by the commutators of length n. Then we have the following two exact sequences of abelian groups:

$$L_n(A) \xrightarrow{l_n} \bar{T}_n^2 \xrightarrow{\mu} \bar{T}_n^2/L_n(A) \overset{j}{\hookrightarrow} \bar{T}_n/L_n(A) \xrightarrow{\rho} A$$

Here l_n is the restriction of the canonical inclusion for the universal envelopping algebra $\mathrm{UL}_*(A) \cong T(A)$.

Note that $G_n \overset{def}{=} G/\gamma_n(G)$ is a free nilpotent group of class n − 1. The sequences above are sequences of left $\mathrm{Aut}(G_n)$ - modules: Let $f \in \mathrm{Aut}(G_n)$. Then f acts on $A \cong G_n^{ab}$ and on $L_n(A)$ by the induced map $f_\sharp = f^{ab}$ and $f_\sharp = L_n(f^{ab})$ respectively. Next choose a basis $\{x_i\}$ of G_n and write

$$f(x_i) = x_{i_1}^{\pm 1} \cdots x_{i_s}^{\pm 1}\, \gamma_n(G)\,.$$

Note that $\bar{T}_n$ is isomorphic with the truncated (non commutative) polynomial ring with generators $\{x_i\}$. Ring isomorphisms $f_\sharp$ on $\bar{T}_n$ and $\bar{T}_n^2$, respectively, are defined by sending the generator x_i to the coset of the element

$$(1 + x_{i_1})^{\pm 1} \cdots (1 + x_{i_s})^{\pm 1} \;-\; 1$$

with $(1+x)^{-1} = 1 + \sum_{\nu=1}^{n}(-1)^\nu x^\nu$. The isomorphisms $f_\sharp$ now yield the action of f on $\bar{T}_n^2$, $\bar{T}_n^2/L_n(A)$ and $\bar{T}_n/L_n(A)$.

The exact sequences above induce the following short exact sequences with $\mathrm{Hom} = \mathrm{Hom}_{\mathbf{Z}}$.

$$\text{I:}\mathrm{Hom}\big(A, L_n(A)\big) \overset{\rho^*}{\rightarrowtail} \mathrm{Hom}\big(\bar{T}_n/L_n(A)\,,\, L_n(A)\big) \overset{j^*}{\twoheadrightarrow} \mathrm{Hom}\big(\bar{T}_n^2/L_n(A)\,,\, L_n(A)\big)$$

$$\text{II:}\quad \mathrm{Hom}\big(\bar{T}_n^2/L_n(A)\,,\, L_n(A)\big) \overset{l_{n*}}{\rightarrowtail} \mathrm{Hom}\big(\bar{T}_n^2/L_n(A)\,,\, \bar{T}_n^2\big)$$

$$\overset{\mu_*}{\twoheadrightarrow} \mathrm{Hom}\big(\bar{T}_n^2/L_n(A)\,,\, \bar{T}_n^2/L_n(A)\big)$$

These are actually sequences of left $\mathrm{Aut}(G_n)$-modules by defining $f \cdot \alpha(x) = f_\sharp\alpha(f_\sharp^{-1}x)$ where α is an element in one of the Hom-modules in I or II.

Theorem A: *Let G be a free nilpotent group of class n with $A = G^{ab}$. Then the classifying cohomology class of the group extension*

$$0 \longrightarrow \mathrm{Hom}(A\,,\, L_n(A)) \overset{1^+}{\longrightarrow} \mathrm{Aut}(G) \overset{\Pi}{\longrightarrow} \mathrm{Aut}(G/\gamma_n(G)) \longrightarrow 1$$

coincides with the element

$$\beta_I\beta_{II}(\epsilon) \in H^2\big(\mathrm{Aut}(G/\gamma_n(G))\,,\, \mathrm{Hom}(A\,,\, L_n(A)\big)$$

Here β_I and β_{II} are the Bockstein operators associated with the sequences I and II respectively and ϵ is the 0-dimensional cohomology class given by the identity of $\bar{T}_n^2\big/L_n(A)$. .

The homomorphism Π in the theorem is reduction mod $\gamma_n(G)$ and the homomorphism 1^+ is defined for $x \in G$ by $1^+(\alpha)(x) = x \cdot y$ where $y = i\,\alpha(x\,\gamma_2(G)) \in G$ by the isomorphism of Witt $i: L_n(A) \cong \gamma_n(G)$.

Next we consider the more general case of a **finitely generated torsionfree nilpotent group** G of class n.

Let $\sqrt{\gamma_n(G)} = \{a \in G \mid a^k \in \gamma_n(G) \text{ for some k}\}$ be the isolator of $\gamma_n(G)$ so that $G_n \overset{def}{=} G\big/\sqrt{\gamma_n(G)}$ is a $\mathcal{T}$-group of class n − 1. Now the group $\mathrm{Aut}(G)$ can be computed from the group $\mathrm{Aut}(G_n)$ in a similar way as in Theorem A. Here, however, an additional obstruction operator arises, and the polynomial ring $\bar{T}_n$ above has to be replaced by the following more general construction: Define the ring

$$P_n^{\surd}(G) = \big(\Delta(G)\big/\Delta^{n+1}(G)\big)\big/\mathbf{Z} - torsion$$

where $\Delta(G)$ denotes the augmentation ideal of the group ring $\mathbf{Z}(G)$, and let

$$\mu_n = \mu_n(G): \; P_{n-1}^{\surd}(G_n) \otimes_{G_n} P_{n-1}^{\surd}(G_n) \to \; P_n^{\surd}(G)$$

be the canonical factorization for the multiplication map of $P_n^{\surd}(G)$. All these terms can be computed from the truncated polynomial ring $\bar{T}_n$ above in case G is given by a free presentation $R \hookrightarrow F \twoheadrightarrow G$ with $A = F^{ab}$.

Let K_n denote the isolator in $P_n^{\surd}(G)$ of the additive subgroup of $P_n^{\surd}(G)$ which is generated by the n-fold ring commutators of $P_n^{\surd}(G)$. We recall from corollary C (3) in part I the following

Proposition: *Let G be a group. Then for $n \geq 1$ there is a natural isomorphism*

$$k_n : \sqrt{\gamma_n(G)} \big/ \sqrt{\gamma_{n+1}(G)} \;\cong\; K_n$$

induced by the map $a \mapsto \{a-1\} \in P_n^{\surd}(G)$.

This may be viewed as a generalization of Witt's isomorphism $\gamma_n(G) \cong L_n(A)$ used in theorem A to arbitrary groups since in case G is free we have isomorphisms $P_n^{\surd}(G) \cong \bar{T}_n$ and $K_n \cong L_n(A) \overset{l_n}{\rightarrowtail} \bar{T}_n$.

Now for $f \in \operatorname{Aut}(G_n)$ the induced isomorphism $\bigotimes^2 P_{n-1}^{\surd}(f)$ of $P_{n-1}^{\surd}(G_n) \otimes_{G_n} P_{n-1}^{\surd}(G_n)$ restricts to an isomorphism f_μ of the subgroup $M = \mu_n(G)^{-1}K_n$ since one has the identity $M = \mu_n(G)^{-1}K_n = \operatorname{Ker} \mu_n(G_n)$ by I. Corollary C (2). Let

$$\operatorname{cAut}(G_n) \subset \operatorname{Aut}(G)$$

be the subgroup of all automorphisms f for which there exists $f_\sharp \in \operatorname{Aut}(K_n)$ such that

$$f_\sharp(\mu_n|M) = (\mu_n|M) f_\mu .$$

The map $f_\sharp$ is uniquely determined since the span of the n-fold ring commutators of $P_n^{\surd}(G)$ is contained in $K_n \cap P_n^{\surd}(G)^2 = K_n \cap \operatorname{Im} \mu_n$ and has finite index in K_n where K_n is torsionfree. Hence for $f \in \operatorname{cAut}(G_n)$ one obtains a unique automorphism of

$$\Lambda_n \overset{def}{=} K_n + P_n^{\surd}(G)^2 ,$$

also denoted by $f_\sharp$, satisfying

$$f_\sharp \mu_n = \mu_n \bigotimes^2 P_{n-1}^{\surd}(f) .$$

Thus K_n and Λ_n are left $\operatorname{cAut}(G_n)$-modules by use of $f \mapsto f_\sharp$.

The following lemma is an immediate consequence of theorem A (2) in part I.

Lemma: *One has the following short exact sequence of homorphisms between abelian groups:*

$$P_n^{\surd}(G_n)^2 \overset{j}{\hookrightarrow} P_n^{\surd}(G_n) \overset{\rho}{\twoheadrightarrow} G_n^{ab} ,$$

where ρ sends the coset of $a-1$, $a \in G$, to the element $a\gamma_2(G)$.

Therefore the following sequences are short exact sequences of $\operatorname{cAut}(G_n)$-bimodules.

$$I' : \quad \operatorname{Hom}(G_n^{ab}, K_n) \overset{\rho^*}{\rightarrowtail} \operatorname{Hom}(P_n^{\surd}(G_n), K_n) \overset{j^*}{\longrightarrow} \operatorname{Hom}(P_n^{\surd}(G_n)^2, K_n)$$
$$\overset{E^*}{\twoheadrightarrow} \operatorname{Ext}^1_{\mathbb{Z}}(G_n^{ab}, K_n)$$

II': $\operatorname{Hom}(\, P_n^{\surd}(G_n)^2\,,\, K_n) \overset{\iota_*}{\rightarrowtail} \operatorname{Hom}(\, P_n^{\surd}(G_n)^2\,,\, \Lambda_n) \overset{\mu_n(G_n)_*}{\twoheadrightarrow}$

$$\operatorname{Hom}(\, P_n^{\surd}(G_n)^2\,,\; P_n^{\surd}(G_n)^2)$$

with $\iota\colon K_n \hookrightarrow \Lambda_n$. As above, these sequences are sequences of *left* $\mathrm{cAut}(G_n)$-modules by setting $f\cdot\alpha = f_* f^{-1*}\alpha$.

Now a *derivation*

$$\mathcal{O}\colon \mathrm{cAut}(G_n) \to \operatorname{Hom}(\, P_n^{\surd}(G_n)^2, K_n)$$

with respect to the left $\mathrm{cAut}(G_n)$-action is given as follows: Since $P_n^{\surd}(G)$ is a finitely generated free abelian group one can choose a $\mathbf{Z}$-linear splitting t of the surjection

$$P_n^{\surd}(\pi) \;:\; P_n^{\surd}(G) \twoheadrightarrow P_n^{\surd}(G_n)$$

where $\pi : G \twoheadrightarrow G_n$ is the canonical quotient map. Here t carries $P_n^{\surd}(G_n)^2$ to Λ_n, so one gets the restriction

$$l\colon\; P_n^{\surd}(G_n)^2 \to \Lambda_n \quad \text{of } t.$$

Thus we obtain for $f \in \mathrm{cAut}(G_n)$ the welldefined homomorphism

$$\mathcal{O}(f) = \iota^{-1}\Big(f_\sharp \circ l \circ (\, P_n^{\surd}(f^{-1}) \,|\, P_n^{\surd}(G_n)^2) \;-\; l\Big)$$

Hence the operator $\mathcal{O}$ satisfies $\mathcal{O}(f) = \iota^{-1}(f\cdot l - l)$ so that $\mathcal{O}$ is a derivation.

Theorem A is a special case of the next result.

Theorem B: *Let G be a finitely generated $\mathcal{T}$-group of class n.*

(i) *The following sequence is exact:*

$$1 \longrightarrow \operatorname{Hom}(G_n^{ab}, K_n) \xrightarrow{1^+} \mathrm{Aut}(G) \xrightarrow{\Pi} \mathrm{cAut}(G_n) \xrightarrow{E^*\mathcal{O}} \operatorname{Ext}^1_{\mathbf{Z}}(G_n^{ab}, L_n^{\surd}(G))$$

Here the homomorphism Π in the theorem is reduction mod $\sqrt{\gamma_n(G)}$ and the homomorphism 1^+ is defined for $x \in G_n$ by $1^+(\alpha)(x) = x\cdot y$ where $y = k_n^{-1}\,\alpha(x\,\gamma_2(G_n)) \in G$ by the isomorphism $k_n\colon \sqrt{\gamma_n(G)} \cong K_n$ in the proposition above. The derivation $E^*\mathcal{O}$ is given by E in I'. Moreover, the action of $\operatorname{Ker} E^*\mathcal{O}$ on

$$\operatorname{Hom}(G_n^{ab}, K_n) \overset{k_n^{-1}{}_*}{\cong} \operatorname{Hom}\left(G_n^{ab}, \sqrt{\gamma_n(G)}\right)$$

defined by conjugation in $Aut(G)$ coincides with the restriction of the left $cAut(G_n)$-action defined above.

(ii) *The classifying cohomology class of the group extension*

$$\mathrm{Hom}(G_n^{ab}, K_n) \overset{1^+}{\rightarrowtail} \mathrm{Aut}(G) \overset{\Pi}{\twoheadrightarrow} \mathrm{Ker}(E^*\mathcal{O})$$

obtained from (i) coincides with the element

$$\beta_I'\{\mathcal{O}|\mathrm{Ker}(E^*\mathcal{O})\} \in H^2\big(\ , \mathrm{Hom}(G_n^{ab}, K_n)\big)\,,$$

where

$$\beta_I' : H^1\big(\mathrm{Ker}(E^*\mathcal{O})\,,\, \mathrm{Ker}E^*\big) \to H^2\big(\mathrm{Ker}(E^*\mathcal{O})\,,\, \mathrm{Hom}(G_n^{ab}, K_n)\big)$$

is the Bockstein operator associated with the extension I' of $\mathrm{Ker}(E^*\mathcal{O})$*-modules.*

(iii) *Let $(G_n)^{ab}$ be torsionfree (this, in particular, holds if G^{ab} is torsionfree, i.e. $\sqrt{\gamma_2(G)} = \gamma_2(G)$). Then the* Ext*-term in (i) vanishes so that one has the group extension*

$$\mathrm{Hom}(G_n^{ab}, K_n) \overset{1^+}{\rightarrowtail} \mathrm{Aut}(G) \overset{\Pi}{\twoheadrightarrow} \mathrm{cAut}(G_n)\,.$$

The classifying cohomology class of this extension coincides with the element

$$\beta_I'\beta_{II}'(\epsilon) \in H^2\big(\mathrm{cAut}(G_n)\,,\, \mathrm{Hom}(G_n^{ab}, K_n)\big)\,.$$

Here β_{II}' is the Bockstein operator associated with extension II' of $\mathrm{Ker}\,E^*\bar{\mathcal{O}}$*-modules defined above and ϵ is the 0-dimensional cohomolgy class given by the identity of $P_n^{\surd}(G_n)^2$.*

Theorem A and B are corollaries of more general results on the category $\mathcal{T}_{\mathrm{n}}$ of $\mathcal{T}$-groups of class $\leq n$. For this one needs the notions of 'linear extension of categories', 'exact sequences for functors' and the 'cohomology of categories' as established by Baues [Ba88]. Using these new concepts we proved results on the categories $\mathcal{T}_{\mathrm{n}}$ which are almost literally of the same nature as theorem A and B. This in fact shows that not only automorphism groups but also all diagrams in $\mathcal{T}_{\mathrm{n}}$ can be computed by results as above.

The proofs are contained in [Ha91]. There also automorphism groups of nilpotent groups with dimension property are described.

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