Max-Planck-Institut für Mathematik
Bonn

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Max-Planck-Institut für Mathematik
Preprint Series 2011 (51)

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MPIM 11-51

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August 16, 2011

Abstract

We study the Yang–Mills gradient flow as a Morse function on the space $\mathcal{A}(P)$ of connection 1-forms on a principal G-bundle P over the sphere S^2. The resulting Morse homology is compared to that of the based loop group ΩG. Via a hybrid moduli space approach we obtain an isomorphism between both Morse homologies, thereby answering a question due to Atiyah.

Contents

1 Introduction

Let $\Sigma := S^2$ be the unit sphere in $\mathbb{R}^3$. Let G be a compact Lie group, $\mathfrak{g}$ its Lie algebra (on which we fix an Ad-invariant inner product), and P a principal G-bundle over Σ. In this paper we answer a question raised by Atiyah relating Yang–Mills Morse homology of the space of gauge equivalence classes of $\mathfrak{g}$-valued connections on P to heat flow homology of the group ΩG of based loops on G. The Morse complexes to be studied here are the complex generated by the L^2 gradient flow of the Yang–Mills functional $\mathcal{YM}$ on the one hand, and the complex generated by the L^2 gradient flow of the energy functional on ΩG on the other. Our goal is to establish a chain isomorphism between these two Morse complexes.

Denote by $\mathrm{ad}(P) := P \times_G \mathfrak{g}$ the adjoint Lie algebra bundle over Σ, and by $\mathcal{A}(P)$ the space of $\mathfrak{g}$-valued 1-forms on P. This is an affine space over $\Omega^1(\Sigma, \mathrm{ad}(P))$, the space of $\mathrm{ad}(P)$-valued 1-forms on Σ. The curvature of a connection is the $\mathrm{ad}(P)$-valued 2-form $F_A = dA + \frac{1}{2}[A \wedge A] \in \Omega^2(\Sigma, \mathrm{ad}(P))$. The space $\mathcal{A}(P)$ is acted on by the groups $\mathcal{G}(P)$ and $\mathcal{G}_0(P)$ of gauge, respectively based gauge transformations of P, cf. Section 2.1 for precise definitions. Each connection $A \in \mathcal{A}(P)$ induces an exterior differential $d_A : \Omega^k(\Sigma, \mathrm{ad}(P)) \to \Omega^{k+1}(\Sigma, \mathrm{ad}(P))$ via $d_A\alpha := d\alpha + [A \wedge \alpha]$. On the space $\mathcal{A}(P)$ we consider the gauge invariant Yang–Mills functional

$$\mathcal{YM}\colon \mathcal{A}(P) \to \mathbb{R}, \qquad \mathcal{YM}(A) = \frac{1}{2}\int_\Sigma \langle F_A \wedge *F_A \rangle. \tag{1}$$

The corresponding Euler-Lagrange equation is the second order partial differential equation $d_A^* F_A = 0$, called *Yang–Mills equation*. Critical points of $\mathcal{YM}$ are degenerate (due to the gauge invariance of the functional) but satisfy the so-called Morse–Bott condition, cf. the discussion in Section 2.3. The (perturbed) negative L^2 gradient flow equation associated with the Yang–Mills functional is the PDE

$$\partial_s A + d_A^* F_A + \nabla \mathcal{V}^-(A) = 0. \tag{2}$$

Cf. Section 2.2 below for the precise form of the perturbation $\mathcal{V}^-\colon \mathcal{A}(P) \to \mathbb{R}$.

Denote $S^1 := \mathbb{R}/2\pi\mathbb{Z}$. The *free loop group* of G is the space $\Lambda G := C^\infty(S^1, G)$, endowed with the group multiplication defined by $(x_1 x_2)(t) := x_1(t)x_2(t)$ for $x_1, x_2 \in \Lambda G$. The *based loop group* of G is the subgroup

$$\Omega G := \{x \in \Lambda G \mid x(0) = \mathbb{1}\}$$

of ΛG. Throughout we will identify ΩG with the quotient of ΛG modulo the free action of G defined by

$$(h \cdot x)(t) := hx(t)$$

for $h \in G$ and $x \in \Lambda G$. On ΛG there is the energy functional

$$\mathcal{E}\colon \Lambda G \to \mathbb{R}, \qquad \mathcal{E}(x) = \frac{1}{2}\int_0^1 \|\partial_t x(t)\|^2 \, dt. \tag{3}$$

It descends to a functional on ΩG by G-invariance of the metric on G. It is well-known that the critical points of $\mathcal{E}$ are precisely the closed geodesics in G. As a consequence of the invariance of the functional $\mathcal{E}$ under conjugation with elements $h \in G$, it follows that critical points of $\mathcal{E}$ are degenerate. However, also here it turns out that the Morse–Bott condition is satisfied. The (perturbed) negative L^2 gradient flow equation resulting from (3) is the PDE

$$\partial_s x - \nabla_t \partial_t x + \nabla \mathcal{V}^+(x) = 0. \tag{4}$$

For the precise form of the perturbation $\mathcal{V}^+\colon \Omega G \to \mathbb{R}$, we refer to Section 2.2 below. Morse homology groups for loop spaces of compact Lie groups and homogeneous spaces have been computed in a classical paper by Bott [4], which constitutes an interesting example of the successful application of Morse theory in the context of infinite dimensional Hilbert manifolds. For further applications to the theory of closed geodesics on general compact manifolds we refer to Klingenberg [10]. However, in both instances, Morse theory is based on a $W^{1,2}$ gradient flow, leading to an ODE in Hilbert space. In contrast, the L^2 gradient flow approach to Morse theory on loop spaces of compact Riemannian manifolds has only recently being investigated by Weber [23] and uses techniques from parabolic PDEs. In this work, we shall follow the latter approach and specialize some of the results in [23] to the case ΩG of loop groups.

Main results

In their seminal paper [3], Atiyah and Bott considered the Yang–Mills functional $\mathcal{YM}$ over a compact Riemann surface as an infinite dimensional analogue of a Morse–Bott function, and consequently gained topological information about the space of gauge equivalence classes of connections. Their discovery of a close correspondence between the Morse theoretical picture of a stratification of the space $\mathcal{A}(P)/\mathcal{G}(P)$ into stable manifolds and certain moduli spaces of semi-stable holomorphic vector bundles initiated a lot of further research in algebraic geometry, cf. the article [9] by Kirwan for a review. In [3], Atiyah and Bott pointed out that in the genus zero case the Yang–Mills critical points correspond via a so called holonomy map to closed geodesics in G. This observation was subsequently made more explicit through work by Gravesen [8] and Friedrich and Habermann [7]. In these works, a holonomy map $\Phi\colon \mathcal{A}(P) \to \Omega G$ is constructed by assigning to a connection A its holonomy along the greater arcs connecting the north and south pole in Σ, cf. Appendix B for details. The map Φ is equivariant with respect to the actions of $\mathcal{G}(P)$ by gauge transformations and of G by conjugation. It furthermore maps critical points of the Yang–Mills functional to closed geodesics in G (of a certain homotopy type, determined by the bundle P), preserving the Morse indices. The natural question whether this apparent close relation between the two sets of generators of Morse complexes extends to the full Morse theory picture, has not been resolved so far. However, there is a formal consideration, invoking an adiabatic limit of a certain deformation of the Riemannian metric on S^2, which indicates a positive answer to the following question.

Conjecture 1.1. *Let G be a compact Lie group and P be a principal G-bundle over Σ. Then there exists an isomorphism between the Morse homology $HM_*\big(\mathcal{A}(P)/\mathcal{G}_0(P)\big)$ and the Morse homology $HM_*\big(\Lambda G/G\big)$.*

For a heuristic explanation, due to Atiyah, why this should be true, we refer to the PhD thesis [5] by Davies. The aim of the present paper is to prove this conjecture. Morse homology theories based on the L^2 gradient flows associated with the functionals $\mathcal{E}$ and $\mathcal{YM}$ have been laid down by Weber [23] (for compact Riemannian manifolds, cf. also [22] for some analytical foundations, and Salamon and Weber [15] for an application to Floer homology of cotangent bundles), and the author [19] (for $\mathcal{YM}$ over arbitrary compact Riemann surfaces). The guiding idea in our proof of the above conjecture is to combine both L^2 gradient flows by studying a so-called *hybrid moduli space problem*. For a given pair $\hat{C}^{\pm}$ of critical manifolds of the functionals $\mathcal{YM}$, respectively $\mathcal{E}$, we shall consider the space of configurations

$$\begin{aligned}
&\hat{\mathcal{M}}(\hat{\mathcal{C}}^-,\hat{\mathcal{C}}^+) := \\
&\qquad\big\{(A,\Psi,x)\in C^\infty(\mathbb{R}^-,\mathcal{A}(P)\times\Omega^0(\Sigma,\mathrm{ad}(P)))\times C^\infty(\mathbb{R}^+,\Lambda G)\ \big| \\
&\quad(A,\Psi)\text{ satisfies (2)},\quad x\text{ satisfies (4)},\quad x(0)=h\Phi(A(0))\text{ for some }h\in G,\\
&\lim_{s\to-\infty}(A(s),\Psi(s))=\big(A^-,0\big)\in\hat{\mathcal{C}}^-\times\Omega^0(\Sigma,\mathrm{ad}(P)),\ \lim_{s\to+\infty}x(s)=x^+\in\hat{\mathcal{C}}^+\big\}.
\end{aligned}$$

Hence $\hat{\mathcal{M}}(\hat{\mathcal{C}}^-,\hat{\mathcal{C}}^+)$ is the moduli space consisting of tuples (A,Ψ,x) such that (A,Ψ) solves the perturbed Yang–Mills gradient flow equation (2) on the negative time interval $(-\infty,0]$, while x is a solution of the perturbed loop group gradient flow equation (4) on the positive time interval $[0,\infty)$. Both solutions are coupled under the trivialization map $\Phi\colon\mathcal{A}(P)\to\Omega G$ as introduced above. The moduli space $\mathcal{M}(\mathcal{C}^-,\mathcal{C}^+)$ to be actually studied is the quotient of $\hat{\mathcal{M}}(\hat{\mathcal{C}}^-,\hat{\mathcal{C}}^+)$ modulo the actions by gauge transformations and left translations $x\mapsto hx$ (for $h\in G$).

As indicated above, the sets of critical points of both the functionals $\mathcal{YM}$ and $\mathcal{E}$ are degenerate in a Morse–Bott sense. To overcome this difficulty we will use a certain variant of Morse theory, the so-called *Morse theory with cascades*, as introduced by Frauenfelder in [6] and described in Appendix A. Throughout we shall work on sublevel sets $\{A\in\mathcal{A}(P)\mid\mathcal{YM}(A)\leq a\}$ and $\{x\in\Lambda G/G\mid\mathcal{E}(x)\leq b\}$ (where usually $b=4a/\pi$). As an additional datum, we fix a Morse function h on the union of all critical manifolds of $\mathcal{YM}$ below the level a (respectively of $\mathcal{E}$ below the level b), the discrete set of critical points of which are the generators of two Morse complexes

$$CM_*^a\big(\mathcal{A}(P)/\mathcal{G}_0(P),\mathcal{V}^-,h\big)\qquad\text{and}\qquad CM_*^b\big(\Lambda G/G,\mathcal{V}^+,h\big).\tag{5}$$

We adapt the convention that throughout this article $\Omega G=\Lambda G/G$ shall denote the connected component of the based loop group which contains the image of $\mathcal{A}(P)$ under the map Φ, cf. Appendix B. It is determined by the equivalence class of the principal G-bundle P. Our goal is to set up a chain homomorphism Θ between the complexes in (5). It is defined for a pair of generators of equal Morse index by a count of elements in the moduli space $\mathcal{M}(\mathcal{C}^-,\mathcal{C}^+)$. The key observation, which allows us to show invertibility of the homomorphism Θ, is the property of the trivialization map $\Phi\colon \mathcal{A}(P)\to\Omega G$ to decrease energy. Namely, for any connection $A\in\mathcal{A}(P)$ there holds the inequality

$$\mathcal{YM}^{\mathcal{V}^-}(A)\geq\frac{\pi}{4}\mathcal{E}^{\mathcal{V}^+}(\Phi(A)), \tag{6}$$

cf. Lemma B.4 below. In the case of a Yang–Mills connection $A\in\operatorname{crit}(\mathcal{YM})$, equality holds in (6). The property of the map Φ to be energy decreasing is not a new result and can be found in Gravesen [8]. In our context it leads directly to the proof of invertibility of Θ and thus implies the desired isomorphism in Morse homology.

Theorem 1.2 (Main result)**.** *Let G be any compact Lie group, and P any principal G-bundle P over Σ. Let $a\geq 0$ be a regular value of $\mathcal{YM}$ and set $b:=4a/\pi$. Then, for a generic perturbation $\mathcal{V}=(\mathcal{V}^-,\mathcal{V}^+)\in Y_a$ (cf. Definition 2.4 below) the chain homomorphism*

$$\Theta_*\colon CM_*^a\big(\mathcal{A}(P)/\mathcal{G}_0(P),\mathcal{V}^-,h\big)\to CM_*^b\big(\Lambda G/G,\mathcal{V}^+,h\big)$$

induces an isomorphism

$$\boxed{[\Theta_*]\colon HM_*^a\big(\mathcal{A}(P)/\mathcal{G}_0(P),\mathcal{V}^-,h\big)\to HM_*^b\big(\Lambda G/G,\mathcal{V}^+,h\big)}$$

of Morse homology groups.

Let us point out here that the approach to define a chain isomorphism via a hybrid moduli space problem is a fairly recent one. It has successfully been employed by Abbondandolo and Schwarz [2] in proving that Floer homology $HF_*(T^*M)$ of cotangent bundles T^*M is isomorphic to singular homology of the free loop space ΛM (M a compact manifold). In their situation, an inequality similar to (6) is utilized, relating the symplectic action to the energy functional via Legendre duality.

Further directions

G-equivariant Morse homology. Let us remark that on the quotient spaces $\mathcal{G}(P)/G_0(P)$ and $\Lambda G/G$ there is a (in general not free) action of the group G. In the first case, with $G\cong\mathcal{G}(P)/\mathcal{G}_0(P)$, this is given by $g\cdot[A]=$

$[g^*A]$. In the second case it is conjugation $g[x] = [g^{-1}xg]$. In his thesis [18] the author has worked out a G-equivariant version of Theorem 1.2. This is mainly a technical extension and requires to replace the spaces $\mathcal{A}(P)/\mathcal{G}_0(P)$ and $\Lambda G/G$ by $(\mathcal{A}(P) \times E_nG)/\mathcal{G}(P)$, respectively by $(\Lambda G \times E_nG)/(G \times G)$, for a suitable finite-dimensional approximation E_nG of the classifying space EG.

Higher genus surfaces. The Morse homology $HM_*\big(\mathcal{A}(P)/\mathcal{G}_0(P), \mathcal{V}^-, h\big)$ we are dealing with in this article has more generally been defined for Riemann surfaces of arbitrary genus in [19]. It is also known from [5, 8] that Yang-Mills connections on principal G-bundles P over such surfaces correspond bijectively to certain geodesic polygons in the Lie group G. Moreover, an estimate similar to (53) relating the energy functionals $\mathcal{YM}$ and $\mathcal{E}$ in this more general situation is also known to exist. Hence one should be able to prove a version of Theorem 1.2 also for higher genus surfaces, but this is open at present.

Acknowledgements

This work is based on the author's PhD thesis [18]. He would like to express his gratitude to his advisor D. A. Salamon for all his support while working on this project. He would like to thank A. Oancea for bringing to his attention the work [2]. A discussion with M. Atiyah concerning some of the background of the question raised in the introduction is greatfully acknowledged. Many thanks also to R. Janner, M. Schwarz, M. Struwe, and J. Weber for fruitful discussions.

2 Critical manifolds, Yang–Mills gradient flow lines, and Morse complexes

2.1 Preliminaries

Let $\Sigma := S^2$ be the unit sphere in $\mathbb{R}^3$, endowed with the standard round metric. Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. On $\mathfrak{g}$ we fix an ad-invariant inner product $\langle\,\cdot\,,\,\cdot\,\rangle$, which exists by compactness of G. Let P be a principal G-bundle over Σ. A gauge transformation is a section of the bundle $\mathrm{Ad}(P) := P \times_G G$ associated to P via the action of G on itself by conjugation $(g, h) \mapsto g^{-1}hg$. Let $\mathrm{ad}(P)$ denote the Lie algebra bundle associated to P via the adjoint action

$$(g,\xi)\mapsto \frac{d}{dt}\Big|_{t=0} g^{-1}\exp(t\xi)g \qquad (\text{for } g\in G,\ \xi\in\mathfrak{g})$$

of G on $\mathfrak{g}$. We denote by $\Omega^k(\Sigma,\mathrm{ad}(P))$ the space of smooth $\mathrm{ad}(P)$-valued differential k-forms, and by $\mathcal{A}(P)$ the space of smooth connections on P. The latter is an affine space over $\Omega^1(\Sigma,\mathrm{ad}(P))$. The group $\mathcal{G}(P)$ acts on $\mathcal{A}(P)$ by gauge transformations and on $\Omega^k(\Sigma,\mathrm{ad}(P))$ by conjugation. We call a connection $A\in\mathcal{A}(P)$ *irreducible* if the stabilizer subgroup $\mathrm{Stab}\,A\subseteq\mathcal{G}(P)$ is trivial. Otherwise it is called *reducible*. It is easy to show that $\mathrm{Stab}\,A$ is a compact Lie group, isomorphic to some subgroup of G. Let $z\in\Sigma$ be arbitrary but fixed. We let $\mathcal{G}_0(P)\subseteq\mathcal{G}(P)$ denote the group of *based gauge transformation*, i.e. those gauge transformations which leave the fibre $P_z\subseteq P$ above z pointwise fixed. It is a well-known fact that $\mathcal{G}_0(P)$ acts freely on $\mathcal{A}(P)$.

The curvature of the connection A is the $\mathrm{ad}(P)$-valued 2-form $F_A=dA+\frac{1}{2}[A\wedge A]$. Covariant differentiation with respect to the Levi-Civita connection associated with the metric g and a connection $A\in\mathcal{A}(P)$ defines an operator $\nabla_A\colon\Omega^k(\Sigma,\mathrm{ad}(P))\to\Omega^1(\Sigma)\otimes\Omega^k(\Sigma,\mathrm{ad}(P))$. Its antisymmetric part is the covariant exterior differential operator

$$d_A\colon\Omega^k(\Sigma,\mathrm{ad}(P))\to\Omega^{k+1}(\Sigma,\mathrm{ad}(P)),\quad \alpha\mapsto d\alpha+[A\wedge\alpha].$$

The formal adjoints of these operators are denoted by ∇_A^* and d_A^*. The covariant Hodge Laplacian on forms is the operator $\Delta_A:=d_A^*d_A+d_Ad_A^*$, the covariant Bochner Laplacian on forms is $\nabla_A^*\nabla_A$. They are related through the Bochner–Weitzenböck formula

$$\nabla_A=\nabla_A^*\nabla_A+\{F_A,\,\cdot\,\}+\{R_\Sigma,\,\cdot\,\}.$$

Here the brackets $\{\,\cdot\,,\,\cdot\,\}$ denote C^∞-bilinear expressions with coefficients independent of A. The functional $\mathcal{YM}^\mathcal{V}:=\mathcal{YM}+\mathcal{V}$ where $\mathcal{YM}$ is as in (1) is called *perturbed Yang–Mills functional*. If $\mathcal{V}=0$, we still write $\mathcal{YM}$ and call this the *unperturbed Yang–Mills functional*. The L^2 gradient of $\mathcal{YM}^\mathcal{V}$ at the point $A\in\mathcal{A}(P)$ is

$$\nabla\mathcal{YM}^\mathcal{V}(A)=d_A^*F_A+\nabla\mathcal{V}(A)\in\Omega^1(\Sigma,\mathrm{ad}(P)).$$

Its Hessian is the second order differential operator

$$H_A\mathcal{YM}^\mathcal{V}=d_A^*d_A+*[*F_A\wedge\,\cdot\,]+H_A\mathcal{V}\colon\Omega^1(\Sigma,\mathrm{ad}(P))\to\Omega^1(\Sigma,\mathrm{ad}(P)).\quad(7)$$

We also make use of the notation $H_A := d_A^* d_A + *[*F_A \wedge \cdot]$. For a definition of Sobolev spaces of sections of vector bundles, of connections, and of gauge transformations we refer to the book [21, Appendix B]. We employ the notation $W^{k,p}(\Sigma)$ and $W^{k,p}(\Sigma, T^*\Sigma \otimes \mathrm{ad}(P))$ for the Sobolev spaces of $\mathrm{ad}(P)$-valued sections, respectively $\mathrm{ad}(P)$-valued 1-forms whose weak derivatives up to order k are in L^p. Similarly, the notation $\mathcal{A}^{k,p}(P)$ indicates the Sobolev space of connections on P of class $W^{k,p}$. We shall also use the parabolic Sobolev spaces

$$W^{1,2;p}(I \times \Sigma, \mathrm{ad}(P)) := L^p(I, W^{2,p}(\Sigma, \mathrm{ad}(P))) \cap W^{1,p}(I, L^p(\Sigma, \mathrm{ad}(P)))$$

of $\mathrm{ad}(P)$-valued sections over $I \times \Sigma$, with $I \subseteq \mathbb{R}$ an interval (and similarly for parabolic Sobolev spaces of connections and for $\mathrm{ad}(P)$-valued 1-forms). Further notation which is used frequently is $\dot{A} := \partial_s A := \frac{dA}{ds}$, etc. for derivatives with respect to time.

2.2 Perturbations

Perturbations of the Yang–Mills funcional

Our construction of a Banach space of perturbations is based on the following L^2 local slice theorem due to Mrowka and Wehrheim [12]. We fix $p > 2$ and let

$$\mathcal{S}_{A_0}(\varepsilon) := \left\{ A = A_0 + \alpha \in \mathcal{A}^{0,p}(\Sigma) \,\middle|\, d_{A_0}^* \alpha = 0, \|\alpha\|_{L^2(\Sigma)} < \varepsilon \right\}$$

denote the set of L^p-connections in the local slice of radius ε with respect to the reference connection A_0.

Theorem 2.1 (L^2 local slice theorem)**.** *Let $p > 2$. For every $A_0 \in \mathcal{A}^{0,p}(\Sigma)$ there are constants $\varepsilon, \delta > 0$ such that the map*

$$\mathfrak{m}\colon \big(\mathcal{S}_{A_0}(\varepsilon) \times \mathcal{G}^{1,p}(P)\big) / \operatorname{Stab} A_0 \to \mathcal{A}^{0,p}(\Sigma), \quad [(A_0 + \alpha, g)] \mapsto (g^{-1})^*(A_0 + \alpha)$$

is a diffeomorphism onto its image, which contains an L^2 ball,

$$B_\delta(A_0) := \left\{ A \in \mathcal{A}^{0,p}(\Sigma) \,\middle|\, \|A - A_0\|_{L^2(\Sigma)} < \delta \right\} \subseteq \operatorname{im} \mathfrak{m}.$$

Proof. For a proof we refer to [12, Theorem 1.7]. ◻

We fix the following data.

(i) A dense sequence $(A_i)_{i \in \mathbb{N}}$ of irreducible smooth connections in $\mathcal{A}(P)$.

(ii) For every A_i a dense sequence $(\eta_{ij})_{j \in \mathbb{N}}$ of smooth 1-forms in $\Omega^1(\Sigma, \mathrm{ad}(P))$ satisfying $d_{A_i}^* \eta_{ij} = 0$ for all $j \in \mathbb{N}$.

(iii) A smooth cutoff function $\rho\colon \mathbb{R} \to [0,1]$ such that $\rho = 1$ on $[-1,1]$, $\operatorname{supp}\rho \subseteq [-4,4]$, and $\|\rho'\|_{L^\infty(\mathbb{R})} < 1$. Set $\rho_k(r) := \rho(k^2 r)$ for $k \in \mathbb{N}$.

Let $\delta = \delta(A_i) > 0$ be as in Theorem 2.1 and assume that for $A \in \mathcal{A}^{0,p}(\Sigma)$ there exists $g \in \mathcal{G}^{1,p}(P)$ with $\|g^*A - A_i\|_{L^2(\Sigma)} < \delta(A_i)$. It then follows from Theorem 2.1 that there exists a unique $\alpha = \alpha(A) \in L^p(\Sigma, \operatorname{ad}(P))$ which satisfies for some $g \in \mathcal{G}^{1,p}(P)$ the conditions

$$g^*A - A_i = \alpha \qquad \text{and} \qquad d_{A_i}^*\alpha = 0. \tag{8}$$

If for $A \in \mathcal{A}(P)$ no such α exists, we formally set $\alpha(A) = -\eta_{ij}$. Hence the map

$$\mathcal{V}_\ell^- : \mathcal{A}(P) \to \mathbb{R}, \qquad A \mapsto \rho_k(\|\alpha(A)\|_{L^2(\Sigma)}^2)\langle \eta_{ij} + \alpha(A), \eta_{ij}\rangle \tag{9}$$

is well-defined for every triple $\ell = (i,j,k) \in \mathbb{N}^3$ with $k > \frac{4}{\delta(A_i)}$. Note that $\mathcal{V}_\ell^-$ is invariant under gauge transformations.

Remark 2.2. In the following we shall admit only multiindices $\ell = (i,j,k) \in \mathbb{N}^3$ with $k > \frac{4}{\delta(A_i)}$ and hence

$$\operatorname{supp}\rho_k \subseteq \Big[0, \frac{\delta(A_i)^2}{4}\Big].$$

Moreover, it is easy to see that $\langle \eta_{ij} + \alpha(A), \eta_{ij}\rangle \geq 0$ holds if $\|\alpha(A)\|_{L^2(\Sigma)} \leq \|\eta_j\|_{L^2(\Sigma)}$, and hence the map $\mathcal{V}_\ell^-$ is non-negative for sufficiently large indices k. We henceforth consider only those multiindices $\ell = (i,j,k) \in \mathbb{N}^3$ which satisfy both these two conditions, and renumber the set of such triples (i,j,k) by integers $\ell \in \mathbb{N}$.

Given $\ell \in \mathbb{N}$, we fix a constant $C_\ell > 0$ such that the following three conditions are satisfied.

(i) $\sup_{A\in\mathcal{A}(P)} |\mathcal{V}_\ell^-(A)| \leq C_\ell$,

(ii) $\sup_{A\in\mathcal{A}(P)} \|\nabla\mathcal{V}_\ell^-(A)\|_{L^2(\Sigma)} \leq C_\ell$,

(iii) $\|\nabla\mathcal{V}_\ell^-(A)\|_{L^p(\Sigma)} \leq C_\ell(1 + \|F_A\|_{L^4(\Sigma)})$ for all $1 < p < \infty$ and $A \in \mathcal{A}(P)$.

The existence of the constant C_ℓ has been shown in [19]. The *universal space of perturbations of the Yang–Mills functional* is the normed linear space

$$Y^- := \Big\{\mathcal{V}^- := \sum_{\ell=1}^\infty \lambda_\ell \mathcal{V}_\ell^- \,\Big|\, \lambda_\ell \in \mathbb{R} \text{ and } \|\mathcal{V}^-\| := \sum_{\ell=1}^\infty C_\ell|\lambda_\ell| < \infty\Big\}. \tag{10}$$

It is a separable Banach space isomorphic to the space ℓ^1 of summable real sequences. Some relevant properties of the perturbations $\mathcal{V}_\ell^-$ are discussed in [19, Appendix A].

Perturbations of the loop group energy functional

We shall follow closely Salamon and Weber [15] in our construction of a Banach space Y^+ of perturbations of the loop group energy functional $\mathcal{E}$. Let us fix the following data.

(i) A dense sequence $(x_i)_{i\in\mathbb{N}}$ of points in ΩG.

(ii) For every x_i a dense sequence $(\eta_{ij})_{j\in\mathbb{N}}$ in $T_{x_i}(\Omega G)$.

(iii) A smooth cut-off function $\rho\colon \mathbb{R}\to[0,1]$ supported in $[-4,4]$, and satisfying $\rho=1$ on $[-1,1]$ and $\|\rho'\|_{L^\infty(\mathbb{R})}<1$. Set

$$\rho_k(r):=\rho(k^2r)$$

for $k\in\mathbb{N}$.

Denote by $\iota>0$ the injectivity radius of the compact Riemannian manifold G. Fix a further cut-off function β supported in $[-\iota^2,\iota^2]$ such that $\beta=1$ on $\big[-\frac{\iota^2}{4},\frac{\iota^2}{4}\big]$. For $x_i\in\Omega G$ as fixed in (i) above and $q\in G$ within distance ι of $x_i(t)$, let $\xi_q^i(t)\in T_{x_i(t)}G$ be uniquely determined by $q=\exp_{x_i(t)}\xi_q^i(t)$. For a triple $\ell=(i,j,k)\in\mathbb{N}^3$ we define the smooth map

$$\mathcal{V}_\ell^+\colon \Omega G\to\mathbb{R},\quad x\mapsto \rho_k\big(\|x-x_i\|^2_{L^2(S^1)}\big)\int_0^1 V_{ij}(t,x(t))\,dt,$$

where

$$V_{ij}(t,q):=\begin{cases}\beta(|\xi_q^i(t)|^2)\langle\eta_{ij}(t)+\xi_q^i(t),\eta_{ij}(t)\rangle, & \text{if } |\xi_q^i(t)|<\iota,\\ 0 & \text{else.}\end{cases}$$

The L^2 distance appearing in the argument of ρ_k above refers to the L^2 distance induced after isometrically embedding the manifold G in some ambient euclidian space $\mathbb{R}^N$. Note that $\mathcal{V}_\ell^+$ extends uniquely to a map $\mathcal{V}_\ell^+\colon \Lambda G\to\mathbb{R}$, which is invariant under the free action $h\cdot x\mapsto hx$ of G on ΛG.

Remark 2.3. It is clear that $\langle\eta_{ij}+\xi_q^i,\eta_{ij}\rangle\geq 0$ holds if $\|\xi_q^i\|_{L^2(S^1)}\leq\|\eta_{ij}\|_{L^2(S^1)}$, and hence the map $\mathcal{V}_\ell$ is non-negative for sufficiently large indices k (for given pair (i,j)). We henceforth consider only those multiindices $\ell=(i,j,k)$ which satisfy this condition, and renumber the set of such triples (i,j,k) by integers $\ell\in\mathbb{N}$.

Let Y^+ denote the vector space spanned by the maps $\mathcal{V}_\ell^+$, $\ell\in\mathbb{R}^3$. As the space Y^- (cf. the previous section) it may be endowed with a norm, turning it into a separable Banach space isomorphic to the space ℓ^1 of summable real sequences (cf. [23, Section 7.1] for details).

2.3 Critical manifolds

Throughout this article we denote by $\mathrm{crit}(\mathcal{YM}) \subseteq \mathcal{A}(P)$ the set of critical points of the unperturbed Yang–Mills functional $\mathcal{YM}$, and by $\mathrm{crit}(\mathcal{E}) \subseteq \Lambda G$ the set of critical points of the unperturbed energy functional $\mathcal{E}$. We furthermore let $\widehat{\mathcal{CR}}(\mathcal{YM})$ and $\widehat{\mathcal{CR}}(\mathcal{E})$ denote the set of connected components of $\mathrm{crit}(\mathcal{YM})$, respectively of $\mathrm{crit}(\mathcal{E})$. The group $\mathcal{G}_0(P)$ of based gauge transformations acts freely on $\mathcal{A}(P)$, hence on $\mathrm{crit}(\mathcal{YM})$. Thus it makes sense to define $\mathcal{CR}(\mathcal{YM})$ as the set of connected components of $\mathrm{crit}(\mathcal{YM})/\mathcal{G}_0(P)$ in $\mathcal{A}(P)/\mathcal{G}_0(P)$. It is well known that every such connected component is a finite-dimensional submanifold of $\mathcal{A}(P)/\mathcal{G}_0(P)$ diffeomorphic to a homogeneous space G/H. Likewise, the group G acts freely on ΛG and on $\widehat{\mathcal{CR}}(\mathcal{E})$. We hence denote by $\mathcal{CR}(\mathcal{E}) = \widehat{\mathcal{CR}}(\mathcal{E})/G$ the set of connected components of $\mathrm{crit}(\mathcal{E})/G$ in $\Lambda G/G \cong \Omega G$. Furthermore, the holonomy map Φ induces a bijection between $\mathcal{CR}(\mathcal{YM})$ and $\mathcal{CR}(\mathcal{E})$ which preserves the action filtration given on both these sets, cf. Theorem B.2. Critical manifolds $\hat{\mathcal{C}}^- \in \widehat{\mathcal{CR}}(\mathcal{YM})$ and $\hat{\mathcal{C}}^+ \in \widehat{\mathcal{CR}}(\mathcal{E})$ satisfy the Morse–Bott condition, which here amounts to saying that the kernels of the Hessians $H_A\mathcal{YM}$ (for every $A \in \hat{\mathcal{C}}^-$) respectively $H_x\mathcal{E}$ (for every $x \in \hat{\mathcal{C}}^+$) are entirely due to gauge transformations in $\mathcal{G}(P)$ and transformations $x \mapsto h_1 x h_2$ for $h_1, h_2 \in G$.

In the following we shall mostly work on some sublevel set $\{A \in \mathcal{A}(P) \mid \mathcal{YM}(A) \leq a\}$ of connections, respectively $\{x \in \Lambda G \mid \mathcal{E}(x) \leq a\}$ of loops, and hence denote

$$\mathrm{crit}^a(\mathcal{YM}) := \{A \in \mathrm{crit}(\mathcal{YM}) \mid \mathcal{YM}(A) \leq a\},$$
$$\mathrm{crit}^a(\mathcal{E}) := \{x \in \mathrm{crit}(\mathcal{E}) \mid \mathcal{E}(x) \leq a\}.$$

We furthermore introduce the notation $\widehat{\mathcal{CR}}^a(\mathcal{YM})$, $\mathcal{CR}^a(\mathcal{YM})$, $\widehat{\mathcal{CR}}^a(\mathcal{E})$, and $\mathcal{CR}^a(\mathcal{E})$ for the subsets of the sets $\widehat{\mathcal{CR}}(\mathcal{YM})$ etc. as above whose elements lie in the sublevel set of value not greater than a. For a given regular value $a \geq 0$ of $\mathcal{YM}$ and each critical manifold $\hat{\mathcal{C}} \in \widehat{\mathcal{CR}}^a(\mathcal{YM})$ we fix a closed L^2 neighborhood $U_{\hat{\mathcal{C}}}$ of $\hat{\mathcal{C}}$ such that $U_{\hat{\mathcal{C}}_1} \cap U_{\hat{\mathcal{C}}_2} = \emptyset$ whenever $\hat{\mathcal{C}}_1 \neq \hat{\mathcal{C}}_2$. Because the set $\widehat{\mathcal{CR}}^a(\mathcal{YM})$ is finite for every $a < \infty$ (cf. [19] for a proof) it follows that such a choice is possible. It follows from Theorem B.5 that the holonomy map Φ induces a bijection between $\widehat{\mathcal{CR}}^a(\mathcal{YM})$ and $\mathcal{CR}^b(\mathcal{E})$ for $b = 4a/\pi$. We now choose for each $\mathcal{C} \in \mathcal{CR}^b(\mathcal{E})$ a sufficiently small closed L^2 neighborhood $U_{\mathcal{C}}$ of $\mathcal{C}$ such that $U_{\mathcal{C}_1} \cap U_{\mathcal{C}_2} = \emptyset$ if $\mathcal{C}_1 \neq \mathcal{C}_2$, and moreover it holds that $\Phi(U_{\hat{\mathcal{C}}}) \cap U_{\mathcal{C}} = \emptyset$ for all $\hat{\mathcal{C}} \in \widehat{\mathcal{CR}}^a(\mathcal{YM})$ with $\Phi(\hat{\mathcal{C}}) \neq \mathcal{C}$.

Definition 2.4. Let a regular value $a \geq 0$ of $\mathcal{YM}$ be given. We call a perturbation $\mathcal{V}^- = \sum_{\ell=1}^{\infty} \lambda_\ell \mathcal{V}_\ell^- \in Y^-$ *admissible* if it satisfies

$$\operatorname{supp} \mathcal{V}_\ell^- \cap U_{\hat{\mathcal{C}}} \neq \emptyset \quad \text{for some} \quad \hat{\mathcal{C}} \in \widehat{\mathcal{CR}}^a(\mathcal{YM}) \quad \Longrightarrow \quad \lambda_\ell = 0.$$

For the regular value $b = 4a/\pi$ of $\mathcal{E}$ we analogously define the subset of admissible perturbations $\mathcal{V}^+ \in Y^+$. We let $Y_a^- \times Y_b^+ = Y_a \subseteq Y$ denote the space of pairs $(\mathcal{V}^-, \mathcal{V}^+)$ where $\mathcal{V}^\pm$ is an admissible perturbation with respect to the regular value $a \geq 0$, respectively $b = 4a/\pi$.

It is straightforward to show that the spaces of admissible perturbations are closed subspaces of the Banach spaces $Y^\pm$, and hence Y_a is a closed subspace of Y, for every regular value a of $\mathcal{YM}$. Furthermore, it is always possible to choose a small open ball within the set of admissible perturbations in such a way that there do not arise any new critical points (below the sublevel set a). Namely, in [19] we have shown the following result.

Proposition 2.5. *For every $\varepsilon > 0$ and $a \geq 0$ there exists a constant $\delta > 0$ with the following significance. Assume the perturbation $\mathcal{V}^- \in Y^-$ satisfies the conditions $\|\mathcal{V}^-\| < \delta$ and*

$$\operatorname{supp} \mathcal{V}^- \subseteq \mathcal{A}(P) \setminus \bigcup_{A \in \widehat{\mathcal{CR}}^a(\mathcal{YM})} B_\varepsilon(A),$$

where $B_\varepsilon(A) := \{A_1 \in \mathcal{A}(P) \mid \|A_1 - A\|_{L^2(\Sigma)} < \varepsilon\}$. Then the perturbed Yang–Mills functional $\mathcal{YM}^{\mathcal{V}^-}$ has the same set of critical points as the functional $\mathcal{YM}$ below the level a, i.e. it holds that

$$\begin{aligned}\operatorname{crit}(\mathcal{YM}^{\mathcal{V}^-}) &\cap \{A \in \mathcal{A}(P) \mid \mathcal{YM}(A) < a\} \\ &= \operatorname{crit}(\mathcal{YM}) \cap \{A \in \mathcal{A}(P) \mid \mathcal{YM}(A) < a\}.\end{aligned}$$

Proof. For a proof we refer to [19, Proposition 2.7]. It relies on the fact that the functional $\mathcal{YM}$ satisfies a gauge equivariant version of the Palais–Smale condition. □

The analogous statement holds true for the energy functional $\mathcal{E}$.

Proposition 2.6. *For every $\varepsilon > 0$ and $b \geq 0$ there exists a constant $\delta > 0$ with the following significance. Assume the perturbation $\mathcal{V}^+ \in Y^+$ satisfies the conditions $\|\mathcal{V}^+\| < \delta$ and*

$$\operatorname{supp} \mathcal{V}^+ \subseteq \Lambda G \setminus \bigcup_{x \in \mathcal{CR}^b(\mathcal{E})} B_\varepsilon(x),$$

where $B_\varepsilon(x) := \{x_1 \in \Lambda G \mid \|x_1 - x\|_{L^2(S^1)} < \varepsilon\}$. Then the perturbed functional $\mathcal{E}^{\mathcal{V}^+}$ has the same set of critical points as the functional $\mathcal{E}$ below the level a, i.e. it holds that

$$\operatorname{crit}(\mathcal{E}^{\mathcal{V}^+}) \cap \{x \in \Lambda G \mid \mathcal{E}(x) < b\} = \operatorname{crit}(\mathcal{E}) \cap \{x \in \Lambda G \mid \mathcal{E}(x) < b\}.$$

Proof. For a proof which relies on the fact that $\mathcal{E}$ satisfies the Palais–Smale condition, we refer to [23, Section 6.2]. $\square$

2.4 Gradient flow lines

In the following we fix perturbations $\mathcal{V}^\pm \in Y^\pm$.

Definition 2.7. The *perturbed Yang–Mills gradient flow equation* is the nonlinear PDE

$$\partial_s A + d_A^* F_A - d_A \Psi + \nabla \mathcal{V}^-(A) = 0 \tag{11}$$

for smooth paths $A : s \mapsto A(s) \in \mathcal{A}(P)$ of connections and $\Psi : s \mapsto \Psi(s) \in \Omega^0(\Sigma, \operatorname{ad}(P))$ of $\operatorname{ad}(P)$-valued 0-forms.

The term $-d_A\Psi$ plays the role of a gauge fixing term needed to make equation (11) invariant under time-dependent gauge transformations $s \mapsto g(s) \in \mathcal{G}(P)$. These act on pairs (A, Ψ) as $g \cdot (A, \Psi) = (g^*A, g^{-1}\Psi g + g^{-1}\dot{g})$. Defining the principal G-bundle $\hat{P} := \mathbb{R}^- \times P$ over $\mathbb{R}^- \times \Sigma$ (by extending P trivially as a product), we may consider time-dependent gauge transformations as gauge transformations of $\hat{P}$ and hence denote the group of these by $\mathcal{G}(\hat{P})$. We also define

$$\mathcal{G}_0(\hat{P}) := \{g \in \mathcal{G}(\hat{P}) \mid g(s) \in \mathcal{G}_0(P) \text{ for every } s \in \mathbb{R}^-\}.$$

Definition 2.8. The *perturbed loop group gradient flow equation* is the nonlinear PDE

$$\partial_s x - \nabla_t \partial_t x + \nabla \mathcal{V}^+(x) = 0 \tag{12}$$

for a smooth path $x : s \mapsto x(s) \in \Lambda G$ of free loops.

Equation (12) is clearly invariant under the action of the group G on ΛG via $(h{\cdot}x)(t) = hx(t)$. The proper analytical setup for a study of the perturbed Yang-Mills and loop group gradient flow equations will be introduced in Section 3.1.

2.5 Morse homologies for the Yang–Mills and heat flows

For finite dimensional manifolds, the construction of a Morse homology theory from the set of critical points of a Morse functions and the isolated flow lines connecting them goes back to Thom [20], Smale [17] and Milnor [11], and had later been rediscovered by Witten [24]. Weber's construction of a heat flow homology [23] is an instance of a Morse homology theory in an infinite dimensional setting, here for the loop space of a compact manifold. The same sort of ideas underlies the author's Yang–Mills Morse homology [19], where the underlying space is the infinite dimensional manifold of gauge equivalence classes of connections over a compact Riemann surface Σ. Let us describe both Morse homology theories briefly.

Let $a \geq 0$ be a regular value of $\mathcal{YM}$. We fix an admissible perturbation $\mathcal{V}^- \in Y_a^-$ (cf. Definition 2.4). Let $h\colon \operatorname{crit}^a(\mathcal{YM})/\mathcal{G}_0(P) \to \mathbb{R}$ be a smooth Morse–Smale function (with respect to some fixed Riemannian metric on the finite-dimensional smooth manifold $\operatorname{crit}^a(\mathcal{YM})/\mathcal{G}_0(P)$). We let

$$CM_*^a(\mathcal{A}(P), \mathcal{V}^-, h)$$

denote the complex generated as a $\mathbb{Z}_2$ module by the set $\operatorname{crit}(h)$ of critical points of h. To each $x \in \operatorname{crit}(h)$ we assign the index $\operatorname{Ind}(x)$ to be the sum of the Morse indices of x as a critical point of $\mathcal{YM}$ and of the function h. For $x^-, x^+ \in \operatorname{crit}(h)$ we call the set $\mathcal{M}(x^-, x^+)$ as in [19, Section 8.2] the moduli space of Yang–Mills gradient flow lines with cascades from x^- to x^+.

Lemma 2.9. *For generic, admissible perturbation $\mathcal{V}^- \in Y_a^-$, Morse function h, and all $x^-, x^+ \in \operatorname{crit}(h)$, the set $\mathcal{M}(x^-, x^+)$ is a smooth manifold (with boundary) of dimension*

$$\dim \mathcal{M}(x^-, x^+) = \operatorname{Ind}(x^-) - \operatorname{Ind}(x^+) - 1.$$

Proof. For a proof we refer to [19, Lemma 8.3]. $\square$

For $k \in \mathbb{N}_0$ we define the Morse boundary operator

$$\partial_k^{\mathcal{YM}}\colon CM_k^a(\mathcal{A}(P), \mathcal{V}^-, h) \to CM_{k-1}^a(\mathcal{A}(P), \mathcal{V}^-, h)$$

to be the linear extension of the map

$$\partial_k^{\mathcal{YM}} x := \sum_{\substack{x' \in \operatorname{crit}(h) \\ \operatorname{Ind}(x') = k-1}} n(x, x') x', \tag{13}$$

where $x \in \mathrm{crit}(h)$ is a critical point of index $\mathrm{Ind}(x) = k$. The numbers $n(x, x')$ are given by counting modulo 2 the flow lines with cascades (with respect to $\mathcal{YM}^{\mathcal{V}^-}$ and h) from x to x', i.e.

$$n(x, x') := \#\, \mathcal{M}(x^-, x^+) \pmod 2.$$

Theorem 2.10. *Let $a \geq 0$ be a regular value of $\mathcal{YM}$. For any Morse function $h\colon \mathrm{crit}^a(\mathcal{YM})/\mathcal{G}_0(P) \to \mathbb{R}$ and generic, admissible perturbation $\mathcal{V}^- \in Y_a^-$, the map $\partial_*^{\mathcal{YM}}$ satisfies $\partial_k^{\mathcal{YM}} \circ \partial_{k+1}^{\mathcal{YM}} = 0$ for all $k \in \mathbb{N}_0$ and thus there exist well-defined homology groups*

$$HM_k^a(\mathcal{A}(P), \mathcal{V}^-, h) = \frac{\ker \partial_k^{\mathcal{YM}}}{\operatorname{im} \partial_{k+1}^{\mathcal{YM}}}.$$

*The homology $HM_*^a(\mathcal{A}(P), \mathcal{V}^-, h)$ is called **Yang–Mills Morse homology**. It is independent of the choice of admissible perturbation $\mathcal{V}^-$ and Morse function h.*

Proof. For a proof we refer to [19, Theorem 1.1]. □

Weber's heat flow homology for the loop space ΩM of a closed manifold M is based on a similar construction of a chain complex and a boundary operator. One of his main results is the following theorem (which he only states for the case where the function $\mathcal{E}^{\mathcal{V}}$ is Morse, the adaption to the present case of a Morse–Bott situation being straight-forward).

Theorem 2.11. *Let $b \geq 0$ be a regular value of $\mathcal{E}^{\mathcal{V}}$. For any Morse function $h\colon \mathrm{crit}^b(\mathcal{E})/G \to \mathbb{R}$ and generic, admissible perturbation $\mathcal{V}^+ \in Y_b^+$, the map $\partial_*^{\mathcal{E}}$ (defined in analogy to (13)) satisfies $\partial_k^{\mathcal{E}} \circ \partial_{k+1}^{\mathcal{E}} = 0$ for all $k \in \mathbb{N}_0$ and thus there exist well-defined homology groups*

$$HM_k^b(\Omega M, \mathcal{V}^+, h) = \frac{\ker \partial_k^{\mathcal{E}}}{\operatorname{im} \partial_{k+1}^{\mathcal{E}}}.$$

*The homology $HM_*^b(\Omega M, \mathcal{V}^+, h)$ is called **heat flow homology**. It is independent of the choice of admissible perturbation $\mathcal{V}^+$ and Morse function h.*

Proof. For a proof we refer to [23, Theorem 1.14]. □

2.6 Hybrid moduli spaces

We combine trajectories of the Yang–Mills and loop group gradient flows satisfying a certain coupling condition in a so-called *hybrid moduli space* as follows. For abbreviation we set $A_0 := A(0)$, $x_0 := x(0)$, and $x_{A_0} := \Phi(A(0))$. For critical manifolds $\hat{\mathcal{C}}^- \in \widehat{\mathcal{CR}}(\mathcal{YM})$ and $\hat{\mathcal{C}}^+ \in \widehat{\mathcal{CR}}(\mathcal{E})$ let us define

$$\begin{aligned}
&\hat{\mathcal{M}}(\hat{\mathcal{C}}^-, \hat{\mathcal{C}}^+) := \\
&\qquad \big\{(A,\Psi,x) \in C^\infty(\mathbb{R}^-, \mathcal{A}(P) \times \Omega^0(\Sigma, \mathrm{ad}(P))) \times C^\infty(\mathbb{R}^+, \Lambda G) \;\big|\; \\
&\qquad (A,\Psi) \text{ satisfies (11)}, \quad x \text{ satisfies (12)}, \quad x_0 = h x_{A_0} \text{ for some } h \in G, \\
&\lim_{s\to-\infty}(A(s),\Psi(s)) = \big(A^-,0\big) \in \hat{\mathcal{C}}^- \times \Omega^0(\Sigma, \mathrm{ad}(P)), \lim_{s\to+\infty} x(s) = x^+ \in \hat{\mathcal{C}}^+\big\}.
\end{aligned} \tag{14}$$

By construction, the space $\hat{\mathcal{M}}(\hat{\mathcal{C}}^-, \hat{\mathcal{C}}^+)$ is invariant under the obvious action of the group $\mathcal{G}_0(\hat{P}) \times G$ on triples (A,Ψ,x). Let us denote $\mathcal{C}^- := \hat{\mathcal{C}}^-/\mathcal{G}_0(P)$ and $\mathcal{C}^- := \hat{\mathcal{C}}^+/G$. The moduli space we shall study further on is the quotient

$$\mathcal{M}(\mathcal{C}^-, \mathcal{C}^+) := \frac{\hat{\mathcal{M}}(\hat{\mathcal{C}}^-, \hat{\mathcal{C}}^+)}{\mathcal{G}_0(\hat{P}) \times G}. \tag{15}$$

We show subsequently that $\hat{\mathcal{M}}(\hat{\mathcal{C}}^-, \hat{\mathcal{C}}^+)$ arises as the zero set $\mathcal{F}^{-1}(0)$ of an equivariant (with respect to $\mathcal{G}_0(\hat{P}) \times G$) section $\mathcal{F}$ of a suitably defined Banach space bundle $\mathcal{E}$ over a Banach manifold $\mathcal{B}$. After showing that the vertical differential $d_x\mathcal{F}$ at any such zero $x \in \mathcal{F}^{-1}(0)$ is a surjective Fredholm operator, it will follow from the implicit function theorem that the moduli space $\mathcal{M}(\mathcal{C}^-, \mathcal{C}^+)$ is a finite-dimensional smooth manifold.

3 Fredholm theory and transversality

3.1 The nonlinear setup

In this section we introduce the setup which will allow us to view the moduli space defined in (15) as the zero set of a Fredholm section of a certain Banach space bundle. These Banach manifolds are modeled on weighted Sobolev spaces in order to make the Fredholm theory work. To define these, we choose numbers $\delta > 0$ and $p > 3$, and a smooth cut-off function β such that $\beta(s) = -1$ if $s < 0$ and $\beta(s) = 1$ if $s > 1$. We define the δ-weighted

$W^{k,p}$ Sobolev norm (for $k \in \mathbb{N}_0$) of a measurable function u over some infinite or half-infinite interval to be the usual $W^{k,p}$ Sobolev norm of the function $e^{\delta\beta(s)s}u$.

For the definition of parabolic Sobolev spaces we refer to Section 2.1. We define the space $\mathcal{A}_\delta^{1,2;p}(P)$ of time-dependent connections on P which are locally of class $W^{1,2;p}$ and for which there exists a limiting connection $A^- \in \operatorname{crit}(\mathcal{YM})$ and a number $T^- \leq 0$ such that the 1-form $\alpha^- := A - A^-$ satisfies

$$\alpha^- \in W_\delta^{1,p}((-\infty, T^-], L^p(\Sigma)) \cap L_\delta^p((-\infty, T^-], W^{2,p}(\Sigma)).$$

Similarly, let $\mathcal{G}_{\delta,0}^{2,p}(\hat{P})$ denote the group of time-dependent, based gauge transformations of P which are locally of class[1] $W^{2,p}$ and in addition satisfy the following two conditions. The $\operatorname{ad}(P)$-valued 1-form $g^{-1}dg$ satisfies

$$g^{-1}dg \in L_\delta^p(\mathbb{R}^-, W^{2,p}(\Sigma, T^*\Sigma \otimes \operatorname{ad}(P))),$$

and there exists a limiting gauge transformation $g^- \in \mathcal{G}^{2,p}(P)$, a number $T^- \leq 0$, and an $\operatorname{ad}(P)$-valued 1-form

$$\gamma^- \in W_\delta^{2,p}((-\infty, T^-] \times \Sigma)$$

with

$$g(s) = g^- \exp(\gamma^-(s)) \qquad (s \leq T^-).$$

Let $\hat{\mathcal{C}}^- \in \widehat{\mathcal{CR}}(\mathcal{YM})$ be a critical manifold. We denote by $\hat{\mathcal{B}}^- := \hat{\mathcal{B}}^-(\hat{\mathcal{C}}^-, \delta, p)$ the Banach manifold of maps

$$(A, \Psi) \in \mathcal{A}_\delta^{1,2;p}(P) \times W_\delta^{1,p}(\mathbb{R}^- \times \Sigma)$$

such that the limiting connection appearing in the above definition of the space $\mathcal{A}_\delta^{1,2;p}(P)$ is contained in $\hat{\mathcal{C}}^-$. The group $\mathcal{G}_{\delta,0}^{2,p}(\hat{P})$ acts smoothly and freely on $\hat{\mathcal{B}}^-$ by gauge transformations. The resulting quotient space

$$\mathcal{B}^- := \mathcal{B}^-(\mathcal{C}^-, \delta, p) := \frac{\hat{\mathcal{B}}^-(\hat{\mathcal{C}}^-, \delta, p)}{\mathcal{G}_{\delta,0}^{2,p}(\hat{P})}$$

is again a smooth Banach manifold. The tangent space at the point $[(A, \Psi)] \in \mathcal{B}^-$ splits naturally as a direct sum

$$T_{[(A,\Psi)]}\mathcal{B}^- = T^0_{[(A,\Psi)]}\mathcal{B}^- \oplus \mathbb{R}^{\dim \mathcal{C}^-}, \tag{16}$$

[1] For a definition of Sobolev spaces of gauge transformations we refer to [21, Appendix B].

where $T^0_{[(A,\Psi)]}\mathcal{B}^-$ denotes the space of pairs

$$(\alpha, \psi) \in W^{1,2;p}_\delta(\mathbb{R}^- \times \Sigma) \oplus W^{1,p}_\delta(\mathbb{R}^- \times \Sigma)$$

which satisfy the gauge fixing condition

$$L^*_{(A,\Psi)}(\alpha, \psi) := \partial_s \psi + [\Psi, \psi] - d^*_A \alpha = 0.$$

This way, a tangent vector in $T^0_{[(A,\Psi)]}\mathcal{B}^-$ becomes identified with its unique lift to $T_{(A,\Psi)}\hat{\mathcal{B}}^-$ perpendicular to the gauge orbit through (A, Ψ).

Let $\hat{\mathcal{C}}^+ \in \widehat{\mathcal{CR}}(\mathcal{E})$. In a similar manner, we define the Banach manifold $\hat{\mathcal{B}}^+ := \hat{\mathcal{B}}^+(\hat{\mathcal{C}}^+, \delta, p)$ of maps from $\mathbb{R}^+$ to ΛG with prescribed asymptotics as $s \to \infty$. Let $\hat{\mathcal{B}}^+$ denote the Banach manifolds of maps $x\colon \mathbb{R}^+ \times S^1 \to G$ such that the condition

$$e^{\delta s}(x - x^+) \in L^p(\mathbb{R}^+, W^{2,p}(S^1, G)) \cap W^{1,p}(\mathbb{R}^+, L^p(S^1, G))$$

is satisfied for some $x^+ \in \hat{\mathcal{C}}^+$. To make sense of the difference $x - x^+$ and of the Sobolev spaces involved in this definition, we think of the Lie group G as being isometrically embedded in some euclidian space $\mathbb{R}^N$. Now put

$$\mathcal{B}^+ := \mathcal{B}^+(\mathcal{C}^+, \delta, p) := \frac{\hat{\mathcal{B}}^+}{G},$$

and define $\mathcal{B} := \mathcal{B}^- \times \mathcal{B}^+$.

Next we define the Banach bundle $\mathcal{E} = \mathcal{E}(\mathcal{C}^-, \mathcal{C}^+)$ over $\mathcal{B}$ in the following way. Let $\hat{\mathcal{E}}^-$ be the trivial Banach space bundle over $\hat{\mathcal{B}}^-$ with fibres $\hat{\mathcal{E}}^-_{(A,\Psi)} := L^p_\delta(\mathbb{R}^-, L^p(\Sigma, T^*\Sigma \otimes \mathrm{ad}(P)))$, and $\hat{\mathcal{E}}^+$ the trivial Banach space bundle over $\hat{\mathcal{B}}^+$ with fibres $\hat{\mathcal{E}}^+_x := L^p_\delta(\mathbb{R}^+, L^p(S^1, \mathfrak{g}))$. We set

$$\hat{\mathcal{E}} := \hat{\mathcal{E}}(\delta, p) := \hat{\mathcal{E}}^- \times \hat{\mathcal{E}}^+ \times \Lambda G.$$

The free action of the group $\mathcal{G}^{2,p}_{\delta,0}(\hat{P}) \times G$ on $\hat{\mathcal{B}}^- \times \hat{\mathcal{B}}^+$ lifts to a free action on $\hat{\mathcal{E}}^- \times \hat{\mathcal{E}}^+ \times \Lambda G$ via

$$(u, h) \cdot (A, \Psi, \alpha, x, \xi, x_1) := (u^*A, u^{-1}\Psi u + u^{-1}\dot{u}, u^{-1}\alpha u, hx, \xi, hx_1).$$

Let $\mathcal{E}$ denote the respective quotient space and define the smooth section $\mathcal{F}\colon \mathcal{B} \to \mathcal{E}$ of $\mathcal{E}$ by

$$\mathcal{F} : [(A, \Psi, x)] \mapsto \left[\begin{pmatrix} \partial_s A + d^*_A F_A - d_A \Psi + \nabla \mathcal{V}^-(A) \\ x^{-1}\left(\partial_s x - \nabla_t \partial_t x + \nabla \mathcal{V}^+(x)\right) \\ x(0)x^{-1}_{A(0)} \end{pmatrix} \right]. \tag{17}$$

3.2 Yang–Mills Hessian and linearized Yang–Mills gradient flow

For $A \in \mathcal{A}(P)$, we let $\mathcal{H}_A$ denote the *augmented Yang–Mills Hessian* defined by

$$\mathcal{H}_A := \begin{pmatrix} d_A^* d_A + *[*F_A \wedge \cdot\,] + H_A\mathcal{V}^- & -d_A \\ -d_A^* & 0 \end{pmatrix} : \\ \Omega^1(\Sigma, \mathrm{ad}(P)) \oplus \Omega^0(\Sigma, \mathrm{ad}(P)) \to \Omega^1(\Sigma, \mathrm{ad}(P)) \oplus \Omega^0(\Sigma, \mathrm{ad}(P)). \quad (18)$$

Here $H_A\mathcal{V}^-$ denotes the Hessian of the map $\mathcal{V}^- \colon \mathcal{A}(P) \to \mathbb{R}$. In order to find a domain which makes the subsequent Fredholm theory work, we decompose the space $\Omega^1(\Sigma, \mathrm{ad}(P))$ of smooth $\mathrm{ad}(P)$-valued 1-forms as the $L^2(\Sigma)$ orthogonal sum

$$\Omega^1(\Sigma, \mathrm{ad}(P)) = \ker\left(d_A^* : \Omega^1(\Sigma, \mathrm{ad}(P)) \to \Omega^0(\Sigma, \mathrm{ad}(P))\right) \\ \oplus\ \mathrm{im}\left(d_A : \Omega^0(\Sigma, \mathrm{ad}(P)) \to \Omega^1(\Sigma, \mathrm{ad}(P))\right).$$

Let $W_0^{2,p}$ and $W_1^{1,p}$ denote the completions of $\ker d_A^*$, respectively of $\mathrm{im}\, d_A$ with respect to the Sobolev (k,p) norm (for $k = 1, 2$). We set $W_A^{2,p}(\Sigma) := W_0^{2,p} \oplus W_1^{1,p}$ and endow this space with the sum norm. Note that this norm depends on the connection A. For $p > 1$ we consider the operator

$$\mathcal{H}_A \colon W_A^{2,p}(\Sigma) \oplus W^{1,p}(\Sigma, \mathrm{ad}(P)) \to L^p(\Sigma, T^*\Sigma \otimes \mathrm{ad}(P)) \oplus L^p(\Sigma, \mathrm{ad}(P)).$$

In the case $p = 2$ this is a densely defined symmetric operator on the Hilbert space $L^2(\Sigma, T^*\Sigma \otimes \mathrm{ad}(P)) \oplus L^2(\Sigma, \mathrm{ad}(P))$ with domain

$$\mathrm{dom}\, \mathcal{H}_A := W_A^{2,2}(\Sigma) \oplus W^{1,2}(\Sigma, \mathrm{ad}(P)). \quad (19)$$

It is shown in [19] that it is self-adjoint. For the further discussion of the operator $\mathcal{D}_A$ it will be convenient to also decompose each $\beta \in \mathrm{im}\, \mathcal{H}_A$ as $\beta = \beta_0 + \beta_1$, where $d_A^*\beta_0 = 0$ and $\beta_1 = d_A\omega$ holds for some $\omega \in \Omega^0(\Sigma, \mathrm{ad}(P))$. A short calculation shows that for $\alpha = \alpha_0 + \alpha_1 = \alpha_0 + d_A\varphi$ (with $d_A^*\alpha_0 = 0$) this decomposition is given by $\mathcal{H}_A\alpha = \beta_0 + d_A\omega$, where ω is a solution of

$$\Delta_A\omega = *[d_A * F_A \wedge \alpha]. \quad (20)$$

As Δ_A might not be injective due to the presence of Δ_A-harmonic 0-forms, the solution ω of (20) need not be unique. This ambiguity however is not relevant, as only $d_A\omega$ enters the definition of β_0 and β_1. With respect to the above decomposition of $\Omega^1(\Sigma, \mathrm{ad}(P))$ the augmented Hessian $\mathcal{H}_A$ takes the form

$$\mathcal{H}_A \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \psi \end{pmatrix} = \begin{pmatrix} \Delta_A \alpha_0 + *[*F_A \wedge \alpha_0] + [d_A^* F_A \wedge \varphi] - d_A \omega \\ -d_A \psi + d_A \omega \\ -d_A^* \alpha_1 \end{pmatrix}, \tag{21}$$

with $\alpha_1 = d_A \varphi$ and ω a solution of (20). Note that the first line of (21) does not depend on the choice of φ used to satisfy the condition $\alpha_1 = d_A \varphi$ because

$$\begin{aligned} [d_A^* F_A \wedge \varphi] &= - * d_A * [F_A \wedge \varphi] + *[*F_A \wedge d_A \varphi] \\ &= - * d_A * d_A d_A \varphi + *[*F_A \wedge d_A \varphi] = 0 \end{aligned}$$

holds for all $\varphi \in \ker d_A$.

Next we consider the linearization of the Yang–Mills gradient flow (11). Since any solution (A, Ψ) of the Yang–Mills gradient flow is gauge equivalent under $\mathcal{G}(\hat{P})$ to a solution satisfying $\Psi \equiv 0$, it suffices to consider the linearization along such trajectories only. We define for $p > 1$ the Banach spaces

$$\begin{aligned} \mathcal{Z}_A^{\delta,p,-} := &\big(W_\delta^{1,p}(\mathbb{R}^-, L^p(\Sigma, T^*\Sigma \otimes \mathrm{ad}(P))) \cap L_\delta^p(\mathbb{R}^-, W_A^{2,p}(\Sigma))\big) \\ &\oplus \big(W_\delta^{1,p}(\mathbb{R}^-, L^p(\Sigma, \mathrm{ad}(P))) \cap L_\delta^p(\mathbb{R}^-, W^{1,p}(\Sigma, \mathrm{ad}(P)))\big) \end{aligned}$$

and

$$\mathcal{L}^{\delta,p,-} := L_\delta^p(\mathbb{R}^- \times \Sigma, T^*\Sigma \otimes \mathrm{ad}(P)) \oplus L_\delta^p(\mathbb{R}^- \times \Sigma, \mathrm{ad}(P)),$$

where the number $\delta > 0$ refers to the weight function fixed at the beginning of Section 3.1. In the following we shall be concerned with the linear operator

$$\mathcal{D}_A := \frac{d}{ds} + \mathcal{H}_A \colon \mathcal{Z}_A^{\delta,p,-} \to \mathcal{L}^{\delta,p,-} \tag{22}$$

for a smooth path $s \mapsto A(s) \in \mathcal{A}(P)$, where $s \in \mathbb{R}^-$. It arises as the linearization of the Yang–Mills gradient flow (11) along a solution $(A, \Psi) = (A, 0)$. Some of its properties are collected in Appendix C.

3.3 Linearized loop group gradient flow

We discuss the linearized loop group gradient flow, following Weber [23]. The Hessian of the energy functional $\mathcal{E}$ at the loop $x \in \Lambda G$ is the linear operator

$$H_x : \xi \mapsto \nabla_t \nabla_t \xi + R(\xi, \partial_t x)\partial_t x + H_x \mathcal{V}^+ \xi \tag{23}$$

for vector fields ξ along x. Here R denotes the Riemannian curvature tensor associated with the Levi-Civita connection ∇ on TG, and $H_x\mathcal{V}^+$ denotes the Hessian of the map $\mathcal{V}^+ \colon \Lambda G \to \mathbb{R}$. Let $x \colon \mathbb{R}^+ \times S^1 \to G$ be a smooth map. We define for $p > 1$ the Banach spaces

$$\mathcal{Z}^{\delta,p,+} := W^{1,p}_\delta(\mathbb{R}^+, L^p(S^1, x^*TG)) \cap W^p_\delta(\mathbb{R}^+, W^{2,p}(S^1, x^*TG))$$

and

$$\mathcal{L}^{\delta,p,+} := L^p_\delta(\mathbb{R}^+ \times S^1, x^*TG).$$

Note that the spaces $\mathcal{Z}^{\delta,p,+}$ and $\mathcal{L}^{\delta,p,+}$ depend on x, which is suppressed in our notation. For short we will often also drop x^*TG and simply write $L^p(S^1)$ etc. The number $\delta > 0$ refers to the weight function fixed at the beginning of Section 3.1. We denote

$$\mathcal{D}_x := \frac{d}{ds} + \mathcal{H}_x \colon \mathcal{Z}^{\delta,p,+} \to \mathcal{L}^{\delta,p,+}.$$

Note that the operator $\mathcal{D}_x$ arises as the linearization of the loop group gradient flow (2.8). We discuss some of its properties in Appendix C.

3.4 Linearized moduli space problem

Let $\hat{\mathcal{C}}^- \in \widehat{\mathcal{CR}}(\mathcal{YM})$ and $\hat{\mathcal{C}}^+ \in \widehat{\mathcal{CR}}(\mathcal{E})$ be given. We define the constant $\delta_0(\mathcal{C}^-, \mathcal{C}^+)$ to be the infimum of the set

$$\big\{ |\lambda| \in \mathbb{R} \,\big|\, \lambda \neq 0 \text{ is eigenvalue of } \mathcal{H}_A \text{ for some } A \in \hat{\mathcal{C}}^- \text{ or } \lambda \neq 0 \text{ is eigenvalue of } H_x \text{ for some } x \in \hat{\mathcal{C}}^+ \big\}.$$

We remark that $\delta_0(\mathcal{C}^-, \mathcal{C}^+)$ is positive as follows from compactness of the manifolds $\mathcal{C}^\pm$. In the following we fix $p > 1$ and $0 < \delta < \delta_0(\mathcal{C}^-, \mathcal{C}^+)$. Recall the definition of the space $\hat{\mathcal{M}}(\hat{\mathcal{C}}^-, \hat{\mathcal{C}}^+)$ in (14). For $u = (A, \Psi, x) \in \hat{\mathcal{M}}(\hat{\mathcal{C}}^-, \hat{\mathcal{C}}^+)$ we define the Banach spaces

$$\mathcal{Z}^{\delta,p}_A := \mathcal{Z}^{\delta,p,-}_A \oplus \mathcal{Z}^{\delta,p,+}$$

and

$$\mathcal{L}^{\delta,p} := \mathcal{L}^{\delta,p,-} \oplus \mathcal{L}^{\delta,p,+} \oplus L^2(S^1, \mathfrak{g}).$$

Put $A_0 := A(0)$ (and analogously for α_0, x_0, and ξ_0). We use the notation

$$D\Phi_{A_0} := \Phi(A_0)^{-1}\, \mathrm{d}\Phi(A_0) \colon \Omega^1(\Sigma, \mathrm{ad}(P)) \to C^\infty(S^1, \mathfrak{g}),$$

where Φ is the holonomy map as in (49). Let us consider the linear operator $\mathcal{D}_u : \mathcal{Z}_A^{\delta,p} \to \mathcal{L}^{\delta,p}$ given by

$$(\alpha, \psi, \xi) \mapsto (\mathcal{D}_A(\alpha, \psi), \mathcal{D}_x\xi, N_A(\alpha, \xi)), \tag{24}$$

where we denote $N_A(\alpha, \xi) := x_0^{-1}\xi_0 - D\Phi_{A_0}\alpha_0$. The following remark clarifies the relation between the operator $\mathcal{D}_u$ and the linearization of the section $\mathcal{F}$ in (17).

Remark 3.1. (i) From the definition of $\mathcal{F}$ as a section $\mathcal{F}\colon \mathcal{B} \to \mathcal{E}$ (cf. (17)) it follows that its linearization $d\mathcal{F}(u)$, where $u = (A, \Psi, x)$, acts on the space of pairs (α, ψ, ξ) where $\alpha(s)$ converges exponentially to some $\alpha^- \in T_{A^-}\hat{\mathcal{C}}^-$ as $s \to -\infty$, and likewise $\xi(s) \to \xi^+ \in T_{x^+}\hat{\mathcal{C}}^+$ as $s \to \infty$. This asymptotic behaviour is in slight contrast to that required for elements of $\mathcal{Z}_A^{\delta,p}$, the domain of the operator $\mathcal{D}_u$. However, it is easy to see that $d\mathcal{F}(u)$ is Fredholm if and only if this property holds for $\mathcal{D}_u$, and that the Fredholm indices are related via the formula

$$\operatorname{ind} d\mathcal{F}(u) = \operatorname{ind} \mathcal{D}_u + \dim \mathcal{C}^- + \dim \mathcal{C}^+.$$

To see this, we view $d\mathcal{F}(u)$ as a compact perturbation of the operator $\mathcal{D}_u$, extended trivially to $\mathcal{Z}_A^{\delta,p} \oplus \mathbb{R}^{\dim \mathcal{C}^-} \oplus \mathbb{R}^{\dim \mathcal{C}^-}$.

(ii) The operator $\mathcal{D}_u$ arises as the linearization of the *unperturbed* Yang–Mills gradient flow equation (11). The Fredholm theory for general perturbations $\mathcal{V} \in Y$ can be reduced to the unperturbed case because the terms involving $\mathcal{V}$ contribute only compact perturbations to the operator $\mathcal{D}_u$.

Weighted theory

Because the Hessians $\mathcal{H}_{A^-}$ and H_{x^+} will in general (i.e. if $\dim \mathcal{C}^\pm \geq 1$) have non-trivial zero eigenspaces, we cannot directly refer to standard theorems on the spectral flow to prove Theorem 3.3. As an intermediate step, we instead use the Banach space isomorphisms

$$\nu_1^- : \mathcal{Z}_A^{\delta,p,-} \to \mathcal{Z}_A^{0,p,-} =: \mathcal{Z}_A^{p,-}, \qquad \nu_2^- : \mathcal{L}^{\delta,p,-} \to \mathcal{L}^{0,p,-} =: \mathcal{L}^{p,-}$$

and

$$\nu_1^+ : \mathcal{Z}^{\delta,p,+} \to \mathcal{Z}^{0,p,+} =: \mathcal{Z}^{p,+}, \qquad \nu_2^+ : \mathcal{L}^{\delta,p,+} \to \mathcal{L}^{0,p,+} =: \mathcal{L}^{p,+}$$

given by multiplication with the weight function $e^{\delta\beta(s)s}$, where $\beta\colon \mathbb{R} \to \mathbb{R}$ denotes the cut-off function introduced at the beginning of Section 3.1. We furthermore denote

$$\mathcal{D}_A^\delta := \nu_2^- \circ \mathcal{D}_A \circ (\nu_1^-)^{-1} \qquad \text{and} \qquad \mathcal{D}_x^\delta := \nu_2^+ \circ \mathcal{D}_x \circ (\nu_1^+)^{-1}.$$

We use the notation $\mathcal{Z}_A^p := \mathcal{Z}_A^{p,-} \times \mathcal{Z}^{p,+}$ and $\mathcal{L}^p := \mathcal{L}^{p,-} \times \mathcal{L}^{p,+} \times L^2(S^1, \mathfrak{g})$, and set

$$\mathcal{D}_u^\delta := (\mathcal{D}_A^\delta, \mathcal{D}_x^\delta, N_A)\colon \mathcal{Z}_A^p \to \mathcal{L}^p.$$

It is easy to check that the operator $\mathcal{D}_u$ is Fredholm if and only if this holds for $\mathcal{D}_u^\delta$, in which case both Fredholm indices coincide. Note also that the operator $\mathcal{D}_A^\delta$ takes the form

$$\mathcal{D}_A^\delta = \frac{d}{ds} + \mathcal{H}_A - (\beta + \beta' s)\delta, \tag{25}$$

and hence, if $\delta > 0$ is chosen sufficiently small, the operator family $s \mapsto \mathcal{H}_{A(s)} - (\beta(s) + \beta'(s)s)\delta$ converges to the invertible operator $\mathcal{H}_{A^-} + \delta$ as $s \to -\infty$. Analogously, we have that

$$\mathcal{D}_x^\delta = \frac{d}{ds} + H_x - (\beta + \beta' s)\delta. \tag{26}$$

Here the operator family $s \mapsto H_{x(s)} - (\beta(s) + \beta'(s)s)\delta$ converges to the invertible operator $H_{x^+} - \delta$ as $s \to \infty$.

3.5 Fredholm theorem

For short, we use notation like e.g. $L^p(I) := L^p(I, L^p(\Sigma, T^*\Sigma \otimes \mathrm{ad}(P))$ to denote the L^p space of $\mathrm{ad}(P)$ valued 1-forms over $I \times \Sigma$, where I is some interval.

Theorem 3.2. *Let $u = (A, \Phi, x) \in \hat{\mathcal{M}}(\hat{\mathcal{C}}^-, \hat{\mathcal{C}}^+)$. There exist positive constants $c(u)$ and $T(u)$ such that the estimate*

$$\begin{aligned} \|(\alpha, \psi, \xi)\|_{\mathcal{Z}_A^p} \leq c(u)\big(&\|\mathcal{D}_u^\delta(\alpha, \psi, \xi)\|_{\mathcal{L}^p} + \|R(\alpha, \psi)\|_{\mathcal{L}^{p,-}} \\ &+ \|(\alpha, \psi)\|_{L^p([-T(u),0])} + \|\xi\|_{L^p([0,T(u)])}\big) \end{aligned} \tag{27}$$

is satisfied for all $(\alpha, \psi, \xi) \in \mathcal{Z}_A^p$. Here R denotes the compact operator of Lemma C.2. As a consequence, the operator $\mathcal{D}_u^\delta$ has finite-dimensional kernel and closed range.

Proof. Inequality (27) is invariant under gauge transformations in $\mathcal{G}_0(\hat{P})$, and thus it suffices to prove it for $\Phi = 0$. We first estimate $\zeta := (\alpha, \psi)$. For $T = T(A) > 0$ large enough we choose a smooth cut-off function $\beta\colon (-\infty, 0] \to \mathbb{R}$ with support in $[-T, 0]$ and such that $\beta(s) = 1$ for $s \in [-T+1, 0]$. Then with R denoting the compact operator of Lemma C.2 it follows for some constant $c(A,p)$ that

$$
\begin{aligned}
&\|\zeta\|_{\mathcal{Z}_A^{p,-}} \leq \|\beta\zeta\|_{\mathcal{Z}_A^{p,-}} + \|(1-\beta)\zeta\|_{\mathcal{Z}_A^{p,-}} \\
&\leq c(A,p)\big(\|\mathcal{D}_A^\delta \zeta\|_{L^p([-T,0])} + \|\zeta\|_{L^p([-T,0])} + \|\mathcal{D}_A^\delta \zeta\|_{\mathcal{L}^{p,-}} + \|R\zeta\|_{\mathcal{L}^{p,-}}\big). \qquad (28)
\end{aligned}
$$

Here we used (58) to estimate the term $\|\beta\zeta\|_{\mathcal{Z}_A^{p,-}}$. The estimate for the term $\|(1-\beta)\zeta\|_{\mathcal{Z}_A^{p,-}}$ follows from Lemma C.2, which remains valid for any $\mathcal{D}_A^\delta$ sufficiently close (in operator norm) to $\mathcal{D}_{A^-}^\delta$. This is satisfied thanks to Lemma C.1. We next estimate ξ by applying Lemma C.4. Let $T = T(x) > 0$ be as in the lemma. Then it follows for a constant $c = c(x,p,\delta) > 0$ that

$$
\begin{aligned}
c^{-1}\|\xi\|_{\mathcal{Z}^{p,+}} &\leq \|\mathcal{D}_x^\delta \xi\|_{\mathcal{L}^{p,+}} + \|\xi\|_{L^p([0,T],L^2(S^1))} + \|\xi(0)\|_{L^2(S^1)} \\
&\leq \|\mathcal{D}_x^\delta \xi\|_{\mathcal{L}^{p,+}} + \|\xi\|_{L^p([0,T],L^2(S^1))} + \|N_A(\alpha,\xi)\|_{L^2(S^1)} + \|\,d\Phi_{A_0}\alpha(0)\|_{L^2(S^1)} \\
&\leq \|\mathcal{D}_x^\delta \xi\|_{\mathcal{L}^{p,+}} + \|\xi\|_{L^p([0,T],L^2(S^1))} + \|N_A(\alpha,\xi)\|_{L^2(S^1)} + \|\alpha(0)\|_{W^{1,2}(\Sigma)}. \qquad (29)
\end{aligned}
$$

The second line is by definition of the map N_A. The last line follows from continuity of the the map $d\Phi(A_0)\colon W^{1,2}(\Sigma) \to W^{1,2}(S^1)$, cf. [18, Lemma A.1]. To control the term $\|\alpha(0)\|_{W^{1,2}(\Sigma)}$ we use Lemma C.8 with

$$
H = W^{1,2}(\Sigma), \qquad V = W^{2,2}(\Sigma), \qquad V^* = L^2(\Sigma).
$$

Choosing the constant $\delta > 0$ in that lemma sufficiently large we obtain for a constant $c(A,p) > 0$ the estimate

$$
\begin{aligned}
\|\alpha(0)\|_{W^{1,2}(\Sigma)}^p &\leq \|\zeta(0)\|_{W^{1,2}(\Sigma)}^p \\
&\leq c(A,p)\Big(\int_{-\infty}^0 \|\mathcal{D}_A^\delta \zeta\|_{L^2(\Sigma)}^p\,ds + \int_{-\infty}^0 \|\zeta\|_{W^{1,2}(\Sigma)}^p\,ds\Big) \\
&\leq c(A,p)\big(\|\mathcal{D}_A^\delta \zeta\|_{\mathcal{L}^{p,-}}^p + \|\zeta\|_{\mathcal{Z}_A^{p,-}}^p\big). \qquad (30)
\end{aligned}
$$

The last line follows, as by definition the norm of $\mathcal{Z}_A^{p,-}$ is stronger than that of $L^p(\mathbb{R}^-, W^{1,2}(\Sigma))$, and that of $\mathcal{L}^{p,-}$ is stronger than that of $L^p(\mathbb{R}^-, L^2(\Sigma))$. Combining estimates (29) and (30) with (28) yields the claimed inequality (27) (with $T(u) := \max\{T(A), T(x)\}$). To finish the proof, we note that the operator $\mathcal{D}_u^\delta = (\mathcal{D}_A^\delta, \mathcal{D}_x^\delta, N_A)$ is bounded and that the operator R and the inclusion maps $\mathcal{Z}_A^{p,-}([-T(u),0]) \hookrightarrow L^p([-T(u),0])$ and $\mathcal{Z}^{p,+}([0,T(u)]) \hookrightarrow L^p([0,T(u)])$ are compact (the latter by Rellich's theorem). Hence the assertions on the kernel and the range follow from the abstract closed range lemma (cf. [14, p. 14]). The proof is complete. □

We now state and prove the main result concerning the linear operator $\mathcal{D}_u$.

Theorem 3.3 (Fredholm theorem)**.** *The operator $\mathcal{D}_u$ is a Fredholm operator of index*

$$\operatorname{ind} \mathcal{D}_u = \operatorname{ind} A^- - \operatorname{ind} x^+ - \dim \mathcal{C}^+. \tag{31}$$

(Here $\operatorname{ind} A^-$*, respectively* $\operatorname{ind} x^+$*, denotes the numbers of negative eigenvalues of $\mathcal{H}_{A^-}$ and H_{x^+}, counted with multiplicities).*

Proof. By what we have remarked in Section 3.4 it sufficies to prove the assertion for the operator $\mathcal{D}_u^\delta$. That the operator $\mathcal{D}_u^\delta$ has finite-dimensional kernel and closed range is part of Theorem 3.2. It remains to establish the formula for the index. Let $H := L^2(S^1, x^*TG)$ and denote

$$S := \{\xi_0 \in H \mid \exists \xi \in \mathcal{Z}^{p,+} \text{ such that } \mathcal{D}_x^\delta \xi = 0 \text{ and } \xi(0) = \xi_0\}. \tag{32}$$

Note that S is a closed subspace of H. Let T be the orthogonal complement of S in H. We denote

$$K := \{(\alpha(0), \psi(0)) \mid (\alpha, \psi) \in \ker \mathcal{D}_{(A,\Psi)}^\delta\}.$$

It follows that the kernel of the operator $\mathcal{D}_u^\delta$ has dimension

$$\dim \ker \mathcal{D}_u^\delta = \dim \ker \mathrm{d}\Phi|_K + \dim(N(K,0) \cap S). \tag{33}$$

On the other hand, the dimension of its cokernel equals the codimension of the hyperplane

$$W = \big\{N(\alpha(0), \xi(0)) \,\big|\, \exists (\alpha, \psi, \xi) \in \mathcal{Z}_A^p \text{ such that } \\ (\mathcal{D}_A^\delta(\alpha, \psi), \mathcal{D}_x^\delta \xi) = (\beta, \omega, \eta)\big\}$$

for arbitrary but fixed (β, ω, η). Let (α, ψ) vary over the space $\ker \mathcal{D}_A^\delta$ to see that this codimension is given by

$$\operatorname{codim} W = \dim T - \dim K + \dim \ker \mathrm{d}\Phi|_K + \dim(N(K,0) \cap S). \tag{34}$$

Combining (33) and (34) and using that $\dim K = \operatorname{ind} \mathcal{H}_{A^-}$ (by Lemma C.3) and $\dim T = \operatorname{ind} x^+ + \dim \mathcal{C}^+$ (by Lemma C.5), the asserted index formula follows. □

Remark 3.4. In view of Remark 3.1 and Theorem 3.3 we obtain the formula

$$\operatorname{ind} d\mathcal{F}(u) = \operatorname{ind} A^- - \operatorname{ind} x^+ + \dim \mathcal{C}^- \tag{35}$$

for the Fredholm index of the linearization of $\mathcal{F}$.

3.6 Transversality

Let a regular value $a \geq 0$ of $\mathcal{YM}$ be given. We recall the notation Y_a as introduced in Definition 2.4. Our aim is to show that for every pair $(\mathcal{C}^-, \mathcal{C}^+) \in \mathcal{CR}^a(\mathcal{YM}) \times \mathcal{CR}^b(\mathcal{E})$ (where $b = 4a/\pi$) and a dense subset of perturbations $\mathcal{V} = (\mathcal{V}^-, \mathcal{V}^+) \in Y_a$ the linearized section $d\mathcal{F}(u)$ is surjective, for all $u \in \mathcal{M}(\mathcal{C}^-, \mathcal{C}^+)$. We shall work with the so-called *universal moduli space* $\mathcal{M}^{\mathrm{univ}}(\mathcal{C}^-, \mathcal{C}^+)$, which arises as the zero set of a certain bundle section $\hat{\mathcal{F}}$ as we explain next. Let us recall the definition of the Banach manifold $\mathcal{B}$, the Banach space bundle $\mathcal{E}$, and the section $\mathcal{F}$ (cf. Section 3.1). We now change our notation slightly and let $\mathcal{M}(\mathcal{C}^-, \mathcal{C}^+; \mathcal{V})$ indicate the moduli space as in (15), i.e. defined for a fixed perturbation $\mathcal{V} \in Y_a$. Throughout the rest of this section we also replace the notation $\mathcal{F}$ by $\mathcal{F}_{\mathcal{V}}$. Let then $\hat{\mathcal{F}} \colon \mathcal{B} \times Y \to \mathcal{E}$ denote the section of the Banach space bundle $\mathcal{E}$ defined by

$$\hat{\mathcal{F}} : [(A, \Psi, x, \mathcal{V})] \mapsto \mathcal{F}_{\mathcal{V}}([(A, \Psi, x)]). \tag{36}$$

Thus the perturbation $\mathcal{V} \in Y_a$ which had previously been kept fixed is now allowed to vary over the Banach space Y_a. We define $\mathcal{M}^{\mathrm{univ}}(\mathcal{C}^-, \mathcal{C}^+) := \{w \in \mathcal{B} \times Y \mid \hat{\mathcal{F}}(w) = 0\}$. In this section, our main result is the following.

Theorem 3.5. *There exists a dense subset $Y_a^{\mathrm{reg}} \subseteq Y_a$ of perturbations such that for every $\mathcal{V} \in Y_a^{\mathrm{reg}}$ the moduli space $\mathcal{M}(\mathcal{C}^-, \mathcal{C}^+; \mathcal{V})$ is a Banach submanifold of $\mathcal{M}^{\mathrm{univ}}(\mathcal{C}^-, \mathcal{C}^+)$.*

Proof. As shown in Theorem 3.6 below, the linearized operator $d\hat{\mathcal{F}}(w)$ is surjective, for every $w \in \mathcal{M}^{\mathrm{univ}}(\mathcal{C}^-, \mathcal{C}^+)$. It hence follows from the implicit function theorem that $\mathcal{M}^{\mathrm{univ}}(\mathcal{C}^-, \mathcal{C}^+)$ is a smooth Banach manifold. The proof is now completed by an application of the Sard–Smale theorem for Fredholm maps between Banach manifolds, cf. [1, Theorem 3.6.15]. This guarantees that the set of regular values

$$Y_a^{\mathrm{reg}}(\mathcal{C}^-, \mathcal{C}^+) := \left\{\mathcal{V} \in Y_a \,\middle|\, d\pi(w) \text{ is surjective for all } w \in \mathcal{M}^{\mathrm{univ}}(\mathcal{C}^-, \mathcal{C}^+)\right\}$$

of the projection map $\pi \colon \mathcal{M}^{\mathrm{univ}}(\mathcal{C}^-, \mathcal{C}^+) \to Y_a$ is residual in Y_a. Again by the implicit function theorem, it follows that $\mathcal{M}(\mathcal{C}^-, \mathcal{C}^+; \mathcal{V})$ is a Banach submanifold of $\mathcal{M}^{\mathrm{univ}}(\mathcal{C}^-, \mathcal{C}^+)$ for every $\mathcal{V} \in Y_a^{\mathrm{reg}}$. Now define the set

$$Y_a^{\mathrm{reg}} := \bigcap_{(\mathcal{C}^-,\mathcal{C}^+) \in \mathcal{CR}^a(\mathcal{YM}) \times \mathcal{CR}^b(\mathcal{E})} Y_a^{\mathrm{reg}}(\mathcal{C}^-, \mathcal{C}^+)$$

which is the intersection of finitely many residual subsets, hence residual in Y_a. For this set Y_a^{reg}, the assertions of the theorem are satisfied. $\square$

Theorem 3.6. *The horizontal differential $d\hat{\mathcal{F}}(w)$ of the map $\hat{\mathcal{F}}$ as in* (36) *is surjective, for every pair $(\mathcal{C}^-, \mathcal{C}^+) \in \mathcal{CR}^a(\mathcal{YM}) \times \mathcal{CR}^b(\mathcal{E})$ (where $b = 4a/\pi$) and every $w \in \mathcal{M}^{\mathrm{univ}}(\mathcal{C}^-, \mathcal{C}^+)$.*

Proof. The theorem follows, combining Lemmata 3.8 and 3.9 below. $\square$

Transversality at stationary flow lines

Throughout we fix a pair $(\mathcal{C}^-, \mathcal{C}^+) \in \mathcal{CR}^a(\mathcal{YM}) \times \mathcal{CR}^b(\mathcal{E})$ such that $\mathcal{C}^+ = \Phi(\mathcal{C}^-)$ is satisfied. In this case, we show that transversality of the section $\mathcal{F}$ holds automatically.

Proposition 3.7. *Let $[u] \in \mathcal{M}(\mathcal{C}^-, \mathcal{C}^+; \mathcal{V})$ where $u = (A, 0, x)$ for some $A \in \mathcal{C}^-$ and $x = \Phi(A)$. Then $\ker \mathcal{D}_u^\delta$ is trivial.*

Proof. Let $(\alpha, \psi, \xi) \in \ker \mathcal{D}_u^\delta$ and consider the maps

$$\varphi^- \colon \mathbb{R}^- \to \mathbb{R}, \quad s \mapsto \|(\alpha(s), \psi(s))\|_{L^2(\Sigma)}^2,$$
$$\varphi^+ \colon \mathbb{R}^+ \to \mathbb{R}, \quad s \mapsto \|\xi(s)\|_{L^2(S^1)}^2.$$

As by assumption $\zeta := (\alpha, \psi)$ satisfies $\dot{\zeta} + \mathcal{H}_A \zeta = 0$ it follows that

$$\dot{\varphi}^-(s) = -2\langle \zeta, \mathcal{H}_A \zeta \rangle, \tag{37}$$
$$\ddot{\varphi}^-(s) = 4\langle \mathcal{H}_A \zeta, \mathcal{H}_A \zeta \rangle \geq 0. \tag{38}$$

Inequality (38) shows that φ^- is convex. Because $\lim_{s \to -\infty} \varphi^-(s) = 0$ it thus follows that φ^- vanishes identically or $\dot{\varphi}^- > 0$. Assume by contradiction the second case. Then (37) shows that $\langle \zeta_0, \mathcal{H}_{A_0} \zeta_0 \rangle < 0$ and from Proposition B.6 it follows that

$$\langle \xi_0, H_x \xi_0 \rangle < 0, \tag{39}$$

where we denote $\zeta_0 := \zeta(0)$ and $\xi_0 := \xi(0) = d\Phi(A)\zeta_0$. Likewise, from the assumption that ξ satisfies $\dot{\xi} + H_x \xi = 0$ it follows that

$$\dot{\varphi}^+(s) = -2\langle \xi, H_x \xi \rangle, \qquad \ddot{\varphi}^+(s) = 4\langle H_x \xi, H_x \xi \rangle \geq 0,$$

and the map φ^+ is convex. Because $\lim_{s\to\infty}\varphi^+(s)=0$ it thus follows that $\dot\varphi^+\leq 0$ and hence in particular $\dot\varphi^+(0)=-2\langle\xi_0,H_{x_0}\xi_0\rangle\leq 0$. This contradicts (39) and shows that our assumption was wrong. Hence φ^- and therefore ζ vanish identically. Then as $N_A(\alpha,\xi)=0$ by assumption, we have that $\xi_0=0$, and convexity of φ^+ shows that also ξ vanishes identically. Hence $\ker\mathcal{D}_u^\delta$ is trivial, as claimed. $\square$

Lemma 3.8. *The horizontal differential $d\mathcal{F}([u])$ is surjective, for every $[u]\in\mathcal{M}(\mathcal{C}^-,\mathcal{C}^+;\mathcal{V})$.*

Proof. Because $\mathcal{C}^+=\Phi(\mathcal{C}^-)$ by assumption, it follows from the gradient flow property that $u=(A,\Psi,x)$ is independent of s. By applying a suitable gauge transformation we can in addition assume that $\Psi=0$ and hence u satisfies the assumptions of Proposition 3.7. By formula (3.4) it follows in this case that $\operatorname{ind}d\mathcal{F}(u)=\dim\mathcal{C}^-$. We check that this number is precisely the dimension of $\ker d\mathcal{F}(u)$, which will imply the claim. Now every $[\alpha_0]\in T_{[A]}\mathcal{C}^-$ gives rise to some $(\alpha,\psi,\xi)\in\ker d\mathcal{F}(u)$ by defining $(\alpha(s),\psi(s)):=(\alpha_0,0)$ for $s\in\mathbb{R}^-$ and $\xi(s):=\xi_0:=\mathrm{d}\Phi(A)\alpha_0$. From the decomposition of $T_{[(A,0)]}\mathcal{B}^-$ in (16) it follows that a direct complement of the $\dim\mathcal{C}^-$- dimensional space of such kernel elements is given by $\mathcal{Z}_A^p$. The restriction of $d\mathcal{F}(u)$ to $\mathcal{Z}_A^p$ is the operator $\mathcal{D}_u$ which has trivial kernel as follows from Proposition 3.7. Hence indeed $\dim\ker d\mathcal{F}(u)=\dim\mathcal{C}^-$, as claimed. $\square$

Transversality in the non-stationary case

Let $(\mathcal{C}^-,\mathcal{C}^+)\in\mathcal{CR}^a(\mathcal{YM})\times\mathcal{CR}^b(\mathcal{E})$, where $b=4a/\pi$. We show surjectivity of the linearized operators in the case where $\mathcal{C}^+\neq\Phi(\mathcal{C}^-)$. Under this assumption, the following result holds true.

Lemma 3.9. *Let $w=[(A,0,x,\mathcal{V})]\in\mathcal{M}^{\mathrm{univ}}(\mathcal{C}^-,\mathcal{C}^+)$. Then the horizontal differential $d\hat{\mathcal{F}}(w)$ is surjective, for every $w\in\mathcal{M}^{\mathrm{univ}}(\mathcal{C}^-,\mathcal{C}^+)$.*

The proof is based on the following auxiliary result.

Proposition 3.10. *Under the assumptions of Lemma 3.9 the map $d\hat{\mathcal{F}}(w)$ is onto if one of the following two conditions is satisfied. (i) The linear operator*

$$\begin{aligned}\hat{\mathcal{D}}_{(A,\mathcal{V}^-)}\colon\{(\alpha,\psi)\in\mathcal{Z}_A^{p,-}\mid(\alpha(0),\psi(0))=0\}\times Y_a^-\to\mathcal{L}^{p,-},&\\ (\alpha,\psi,v^-)\mapsto\mathcal{D}_A(\alpha,\psi)+\nabla v^-(A)&\end{aligned}$$

is surjective. (ii) The linear operator

$$\hat{\mathcal{D}}_{(x,\mathcal{V}^+)}\colon\{\xi\in\mathcal{Z}^{p,+}\mid\xi(0)=0\}\times Y_b^+\to\mathcal{L}^{p,+},\quad(\xi,v^+)\mapsto\mathcal{D}_x\xi+\nabla v^+(x)$$

is surjective.

Proof. Assume case (ii). Let $(\beta, \eta, \nu) \in \mathcal{L}^{p,-} \times \mathcal{L}^{p,+} \times L^2(S^1, \mathfrak{g})$ be given. By Lemma C.3 the equation $\hat{\mathcal{D}}_{(A,\mathcal{V}^-)}(\alpha, \psi, v^-) = \beta$ admits a solution (with e.g. $v^- = 0$). Assumption (ii) implies that the equation $\hat{\mathcal{D}}_{(x,\mathcal{V}^+)}(\xi, v^+) = \eta$ can be solved for arbitrary $\xi(0)$, in particular for $\xi(0) = x(0)(\nu + D\Phi_{A(0)}\alpha(0))$, cf. (24) regarding the notation. For this $\xi(0)$, the condition $N_A(\alpha, \xi) = \nu$ is satisfied. It follows that $(\hat{\mathcal{D}}_{(A,\mathcal{V}^-)}, \hat{\mathcal{D}}_{(x,\mathcal{V}^+)})(\alpha, \psi, \xi, v^-, v^+) = (\beta, \eta, \nu)$ and hence $d\hat{\mathcal{F}}(w)$ is onto. Assuming case (i) we may argue analogously, using Lemma C.5 at the place of Lemma C.3. $\square$

Proof. **(Lemma 3.9)** From our initial assumption that $\mathcal{C}^+ \neq \Phi(\mathcal{C}^-)$ it follows that the gradient flow lines (A, Ψ) or x are not stationary. In this situation, the transversality results [19, Theorem 7.1] and [23, Proposition 7.5] apply. In the first case this yields surjectivity of $\hat{\mathcal{D}}_{(A,\mathcal{V}^-)}$, in the second case surjectivity of $\hat{\mathcal{D}}_{(x,\mathcal{V}^+)}$. The claim now follows from Proposition 3.10. $\square$

Remark 3.11. In [23, Proposition 7.5] Weber shows surjectivity of the linearized section $\hat{\mathcal{D}}_{(x,\mathcal{V}^+)}$ along a gradient flow line x defined on the infinite interval $\mathbb{R}$ by studying the kernel of the adjoint operator $\mathcal{D}_x^*$. His proof carries over almost literally to the present situation, with the only difference that we now consider the same operator over the half-infinite interval $\mathbb{R}^+$ and assume Dirichlet boundary conditions at $s = 0$. An analogous remark applies to the result [19, Theorem 7.1] used in the proof of Lemma 3.9.

4 Compactness

For a given pair $(\mathcal{C}^-, \mathcal{C}^+) \in \mathcal{CR}(\mathcal{YM}) \times \mathcal{CR}(\mathcal{E})$, we aim to show compactness of the moduli spaces $\mathcal{M}(\mathcal{C}^-, \mathcal{C}^+)$ (as defined in (15)) up to so-called convergence to broken trajectories. Let us first introduce this notion, following here the book by Schwarz [16, Definition 2.34].

Definition 4.1. A subset $K \subseteq \mathcal{M}(\mathcal{C}^-, \mathcal{C}^+)$ is called *compact up to broken trajectories of order* $\mu = (\mu^-, \mu^+) \in \mathbb{N}_0^2$ if for any sequence $[u^\nu] = [(A^\nu, \Psi^\nu, x^\nu)]$ in K the following alternative holds. Either $[u^\nu]$ possesses a C^∞ convergent subsequence, or there exist the following:

(i) numbers $0 \leq \lambda^\pm \leq \mu^\pm$ and critical manifolds

$$\mathcal{C}_0^- = \mathcal{C}^-, \ldots, \mathcal{C}_{\lambda^-}^- \subseteq \mathcal{CR}(\mathcal{YM}) \qquad \text{and} \qquad \mathcal{C}_0^+ = \mathcal{C}^+, \ldots, \mathcal{C}_{\lambda^+}^+ \subseteq \mathcal{CR}(\mathcal{E});$$

(ii) (for every $0 \leq j \leq \lambda^- - 1$) connecting trajectories $(A_j, \Psi_j) \in \hat{\mathcal{M}}(\hat{\mathcal{C}}_j^-, \hat{\mathcal{C}}_{j+1}^-)$, a sequence of gauge transformations $(g_{j,\nu})_{\nu\in\mathbb{N}} \subseteq \mathcal{G}_0(\hat{P})$ and a sequence of reparametrization times $(\tau_{j,\nu}^-)_{\nu\in\mathbb{N}} \subseteq [0,\infty)$;

(iii) a point $(A^*, \Psi^*, x^*) \in \hat{\mathcal{M}}(\hat{\mathcal{C}}_{\lambda^-}^-, \hat{\mathcal{C}}_{\lambda^+}^+)$ and a sequence of gauge transformations $(g_{\lambda^-,\nu})_{\nu\in\mathbb{N}} \subseteq \mathcal{G}_0(\hat{P})$;

(iv) (for every $1 \leq j \leq \lambda^+$) connecting trajectories $x_j \in \hat{\mathcal{M}}(\hat{\mathcal{C}}_j^+, \hat{\mathcal{C}}_{j-1}^+)$ and a sequence of reparametrization times $(\tau_{j,\nu}^+)_{\nu\in\mathbb{N}} \subseteq [0,\infty)$, with the following significance.

There exists a subsequence (again labeled by ν) such that, as $\nu \to \infty$,

$$\begin{aligned} g_{j,\nu}^*(A^\nu(\,\cdot\, - \tau_{j,\nu}^-), \Psi^\nu(\,\cdot\, - \tau_{j,\nu}^-)) \to (A_j, \Psi_j) \qquad & \text{for every } 0 \leq j \leq \lambda^- - 1, \\ x^\nu(\,\cdot\, + \tau_{j,\nu}^+) \to x_j \qquad & \text{for every } 0 \leq j \leq \lambda^+ - 1, \\ g_{\lambda^-,\nu}^*(A^\nu, \Psi^\nu) \to (A^*, \Psi^*), \qquad & x^\nu \to x^* \end{aligned}$$

holds in C^∞ on all compact domains $I \times \Sigma$, respectively $I \times S^1$, where $I \subseteq \mathbb{R}^-$ (respectively $I \subseteq \mathbb{R}^+$) is a compact interval.

Here the notation $\hat{\mathcal{M}}(\hat{\mathcal{C}}_j^\pm, \hat{\mathcal{C}}_{j\mp1}^\pm)$ refers to the moduli spaces of connecting trajectories for the gradient flows of $\mathcal{YM}^\mathcal{V}$, respectively of $\mathcal{E}^\mathcal{V}$ as introduced in [19] and [23]. As we show next, the moduli space $\mathcal{M}(\mathcal{C}^-, \mathcal{C}^+)$ is compact in the sense of Definition 4.1.

Theorem 4.2 (Compactness of moduli spaces)**.** *For every pair $(\mathcal{C}^-, \mathcal{C}^+) \in \mathcal{CR}(\mathcal{YM}) \times \mathcal{CR}(\mathcal{E})$, the moduli space $\mathcal{M}(\mathcal{C}^-, \mathcal{C}^+)$ is empty or compact up to convergence to broken trajectories of order $\mu = (\mu^-, \mu^+)$, where*

$$\mu^- + \mu^+ = \operatorname{ind} A^- + \dim \mathcal{C}^- - \operatorname{ind} x^+. \tag{40}$$

The integers $\operatorname{ind} A^-$ and $\operatorname{ind} x^+$ denote the numbers of negative eigenvalues of $\mathcal{H}_{A^-}$, respectively of H_{x^+} (for any $A^- \in \hat{\mathcal{C}}^-$ and $x^+ \in \hat{\mathcal{C}}^+$).

The statement of the theorem and its proof are in analogy to the compactness result in Abbondandolo–Schwarz [2, Theorem 3.5]. To prove it we need the following two lemmata, the first one being due to the author [19] and the second one due to Weber [23].

Lemma 4.3. *Let $\mathbb{A}^\nu = A^\nu + \Psi^\nu\, ds$, $\nu \in \mathbb{N}$, be a sequence of solutions of the gradient flow equation (11) on $\mathbb{R}^- \times \Sigma$. Assume there exists a critical manifold $\hat{\mathcal{C}}^- \in \hat{\mathcal{CR}}(\mathcal{YM})$ such that every trajectory $\mathbb{A}^\nu(s)$ converges to $\hat{\mathcal{C}}^-$ as $s \to -\infty$. Then there exists a sequence $g^\nu \in \mathcal{G}(\hat{P})$ of gauge transformations such that a subsequence of the gauge transformed sequence $(g^\nu)^*\mathbb{A}^\nu$ converges uniformly on compact sets $I \times \Sigma$ (for every compact interval $I \subseteq \mathbb{R}^-$) to a solution $\mathbb{A}^*$ of (11).*

Proof. For a proof we refer to [19, Theorem 6.2]. □

Lemma 4.4. *Let x^ν, $\nu \in \mathbb{N}$, be a sequence of solutions of the gradient flow equation* (12) *on the interval $\mathbb{R}^+$. Assume there exists a constant $C \geq 0$ such that the energy bound*

$$\sup_{s\in\mathbb{R}^+} \mathcal{E}^{\mathcal{V}^+}(x^\nu(s)) \leq C \tag{41}$$

is satisfied for all $\nu \in \mathbb{N}$. Then there exists a solution x^* of (12) *on $\mathbb{R}^+ \times S^1$ such that, after passing to a subsequence, x^ν converges uniformly to x^* on compact sets $I \times S^1$ (for every compact interval $I \subseteq \mathbb{R}^+$).*

Proof. For a proof we refer to [23, Proposition 4.14]. □

Proof. **(Theorem 4.2)** Let $u^\nu = (A^\nu, \Psi^\nu, x^\nu)$, $\nu \in \mathbb{N}$, be a sequence in $\hat{\mathcal{M}}(\hat{\mathcal{C}}^-, \hat{\mathcal{C}}^+)$. Note that the sequence x^ν satisfies condition (41). Namely, thanks to the energy inequality (54) it follows for all ν and a constant $C(\mathcal{C}^-)$ that

$$\sup_{s\in\mathbb{R}^+} \mathcal{E}^{\mathcal{V}^+}(x^\nu(s)) = \mathcal{E}^{\mathcal{V}^+}(x^\nu(0)) \leq \frac{4}{\pi}\mathcal{YM}^{\mathcal{V}^-}(A^\nu(0)) \leq C(\mathcal{C}^-).$$

Such a constant $C(\mathcal{C}^-)$ exists due to the assumption that $A^\nu(s)$ converges to $\hat{\mathcal{C}}^-$ as $s \to -\infty$. Hence Lemmata 4.3 and 4.4 apply and show that there exists a subsequence of u^ν (which we still label by ν) and a sequence g^ν of gauge transformations such that $(g^\nu)^*(A^\nu, \Psi^\nu)$ and x^ν converge in C^∞_{loc}. Furthermore, for each compact interval $I \subseteq \mathbb{R}^\pm$ it follows that the limit as $\nu \to \infty$ of $(g^\nu)^*|_{I\times\Sigma}(A^\nu|_{I\times\Sigma}, \Psi^\nu|_{I\times\Sigma})$ (respectively of $x^\nu|_{I\times S^1}$) is the restriction of a trajectory of (11) (respectively of (12)) of finite energy at most $C(\mathcal{C}^-)$. Now the exponential decay results [19, Theorem 4.1] and [23, Theorem 1.8] imply that every such finite energy solution is contained in one of the moduli spaces $\hat{\mathcal{M}}(\hat{\mathcal{C}}_0^-, \hat{\mathcal{C}}_1^-)$, $\hat{\mathcal{M}}(\hat{\mathcal{C}}_0^+, \hat{\mathcal{C}}_1^+)$, or $\hat{\mathcal{M}}(\hat{\mathcal{C}}_0^-, \hat{\mathcal{C}}_0^+)$ for some $\hat{\mathcal{C}}_j^- \in \widehat{\mathcal{CR}}^a(\mathcal{YM})$, respectively $\hat{\mathcal{C}}_j^+ \in \widehat{\mathcal{CR}}^b(\mathcal{E})$, where $j = 0, 1$ and $a = \mathcal{YM}(\hat{\mathcal{C}}^-)$, $b = 4a/\pi$. (Here we use the notation of Section 2.3). Convergence after reparametrization as required in Definition 4.1 and the relation (40) then follow from standard arguments as in [16, Proposition 2.35]. □

5 Chain isomorphism of Morse complexes

5.1 The chain map

Let $a \geq 0$ be a regular value of $\mathcal{YM}$ and set $b := 4a/\pi$. Throughout this section we fix an admissible perturbation $\mathcal{V} = (\mathcal{V}^-, \mathcal{V}^+) \in Y_a^{\mathrm{reg}}$ (with the set

Y_a^{reg} as in Theorem 3.5), which satisfies the conditions of Theorem B.5. Let $h\colon \mathrm{crit}^a(\mathcal{YM})/\mathcal{G}_0(P) \to \mathbb{R}$ be a smooth Morse–Smale function, i.e. a Morse function such that all stable and unstable manifolds $W_h^s(x)$ and $W_h^u(x)$ of h intersect transversally. We let

$$CM_*^{a,-} := CM_*^a\big(\mathcal{A}(P)/\mathcal{G}_0(P), \mathcal{V}^-, h\big), \qquad CM_*^{b,+} := CM_*^b\big(\Lambda G/G, \mathcal{V}^+, h\big)$$

denote the Morse–Bott complexes as in Section 2.5. The resulting Morse homologies will for short be denoted by

$$HM_*^{a,-} := HM_*^a\big(\mathcal{A}(P)/\mathcal{G}_0(P), \mathcal{V}^-, h\big), \qquad HM_*^{b,+} := HM_*^b\big(\Lambda G/G, \mathcal{V}^+, h\big).$$

Our construction of the moduli space $\mathcal{M}(\mathcal{C}^-, \mathcal{C}^+)$ in Section 2.6 gives rise to a chain map $\Theta\colon CM_*^{a,-} \to CM_*^{b,+}$ as we shall describe next.

Definition 5.1. Fix critical points $x^-, x^+ \in \mathrm{crit}(h)$ and integers $m^-, m^+ \geq 1$. A *hybrid flow line from x^- to x^+ with m^- upper and m^+ lower cascades* is a tuple

$$\begin{aligned}(\underline{x}^-, \underline{x}^0, \underline{x}^+, T^-, T^+) \\ = \big((x_j^-)_{j=1,\ldots,m^-}, \underline{x}^0, (x_j^+)_{j=1,\ldots,m^+}, (t_j^-)_{j=1,\ldots,m^-}, (t_j^+)_{j=0,\ldots,m^+-1}\big),\end{aligned}$$

where for each j, $x_j^-\colon \mathbb{R}^- \to \mathcal{A}(P)/\mathcal{G}_0(P)$ is a nonconstant solution of the Yang–Mills gradient flow equation (11), $x_j^+\colon \mathbb{R}^+ \to \Lambda G/G$ is a nonconstant solution of the loop group gradient flow equation (12), $t_j^\pm \in \mathbb{R}^+$, and the following conditions are satisfied.

(i) For each $1 \leq j \leq m^\pm - 1$ there exists a solution $y_j^\pm \in C^\infty(\mathbb{R}, \mathrm{crit}(h))$ of the gradient flow equation $\dot{y}_j^\pm = -\nabla h(y_j^\pm)$ such that $\lim_{s\to\infty} x_j^\pm(s) = y_j^\pm(0)$ and $\lim_{s\to-\infty} x_{j+1}^\pm(s) = y_j^\pm(t_j)$.

(ii) There exist $p^- \in W_h^u(x^-)$ and $p^+ \in W_h^s(x^+)$ such that $\lim_{s\to-\infty} x_1^-(s) = p^-$ and $\lim_{s\to\infty} x_m^+(s) = p^+$.

(iii) There exist $\mathcal{C}^- \in \mathcal{CR}(\mathcal{YM})$ and $\mathcal{C}^+ \in \mathcal{CR}(\mathcal{E})$ such that $\underline{x}^0 = [(u^-, u^+)] \in \mathcal{M}(\mathcal{C}^-, \mathcal{C}^+)$. Furthermore, there exist solutions $y_{m^-}^-$ and y_0^+ of the gradient flow equations $\dot{y}_{m^-}^- = -\nabla h(y_{m^-}^-)$ and $\dot{y}_0^+ = -\nabla h(y_0^+)$ satisfying the conditions

$$\lim_{s\to\infty} x_{m^-}^-(s) = y_{m^-}^-(0), \qquad \lim_{s\to-\infty} u^-(s) = y_{m^-}^-(t_{m^-}^-),$$
$$\lim_{s\to\infty} u^+(s) = y_0^+(0), \qquad \lim_{s\to-\infty} x_1^+(s) = y_0^+(t_0^+).$$

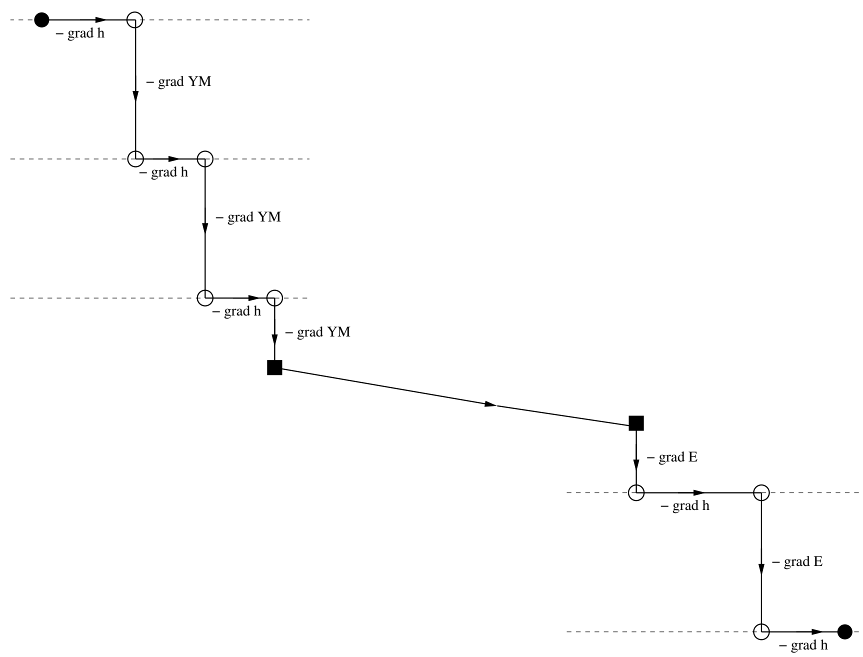

Figure 1: A hybrid flow line with 2 upper and 1 lower cascades.

A *hybrid flow line with* $m^- = 0$ *upper cascades and* $m^+ \geq 1$ *lower cascades* is a tuple

$$(\underline{x}^0, \underline{x}^+, T^+) = (\underline{x}^0, (x_j^+)_{j=1,\ldots,m^+}, (t_j^+)_{j=0,\ldots,m^+-1})$$

as before satisfying conditions (i-iii) with the following adjustment in (ii). Here we require the existence of $p^- \in W_h^u(x^-)$ such that $\lim_{s\to-\infty} u^-(s) = p^-$. Conditions involving flow lines x_j^- and times t_j^- are empty in this case. A *hybrid flow line with* $m^- \geq 0$ *upper cascades and* $m^+ = 0$ *lower cascades* is defined analogously. Conditions involving flow lines x_j^+ and times t_j^+ are then empty.

Denote by $\mathcal{M}^{\mathrm{hybr}}_{(m^-,m^+)}(x^-, x^+)$ the moduli space of hybrid flow lines from x^- to x^+ with $(m^-, m^+) \in \mathbb{N}_0^2$ upper and lower cascades (modulo the action by time-shifts of the group $\mathbb{R}^{m^-} \times \mathbb{R}^{m^+}$ on tuples $(\underline{x}^-, \underline{x}^+)$). We call

$$\mathcal{M}^{\mathrm{hybr}}(x^-, x^+) := \bigcup_{(m^-,m^+)\in\mathbb{N}_0^2} \mathcal{M}^{\mathrm{hybr}}_{(m^-,m^+)}(x^-, x^+)$$

the moduli space of *hybrid flow lines with cascades from* x^- *to* x^+.

Lemma 5.2. *The dimension of* $\mathcal{M}^{\mathrm{hybr}}(x^-, x^+)$ *is given by the formula*

$$\dim \mathcal{M}^{\mathrm{hybr}}(x^-, x^+) = \mathrm{Ind}(x^-) - \mathrm{Ind}(x^+). \tag{42}$$

Proof. The asserted formula follows from the following one for the dimension of the space $\hat{\mathcal{M}}^{\mathrm{hybr}}_{(m^-,m^+)}(x^-, x^+)$ (where we do not take into account the action of $\mathbb{R}^{m^-} \times \mathbb{R}^{m^+}$ by time shifts). Namely, by the result [6, Corollary C.15] of Frauenfelder and the index formula (35) it follows that

$$\dim \hat{\mathcal{M}}^{\mathrm{hybr}}_{(m^-,m^+)}(x^-, x^+) = \mathrm{Ind}(x^-) - \mathrm{Ind}(x^+) + m - 1,$$

where m denotes the total number of cascades of an element in $\hat{\mathcal{M}}^{\mathrm{hybr}}_{(m^-,m^+)}(x^-, x^+)$. Note that in the present setup this number is given by $m = m^- + m^+ + 1$ because we have to take into account the additional cascade coming from the configuration $\underline{x}_0$. The asserted formula for the quotient space modulo the action of $\mathbb{R}^{m^-} \times \mathbb{R}^{m^+}$ then follows. $\square$

Our construction of a moduli space $\mathcal{M}^{\mathrm{hybr}}(x^-, x^+)$ for each pair (x^-, x^+) of generators gives rise to a chain map between the Morse–Bott complexes $CM^{a,-}_*$ and $CM^{b,+}_*$.

Definition 5.3. For a pair $(x^-, x^+) \in \mathrm{crit}(h) \times \mathrm{crit}(h)$ with $\mathrm{Ind}(x^-) = k$ let

$$\Theta_k(x^-) := \sum_{\substack{x^+ \in \mathrm{crit}(h) \\ \mathrm{Ind}(x^+) = k}} \# \mathcal{M}^{\mathrm{hybr}}(x^-, x^+) \cdot x^+,$$

where $\# \mathcal{M}^{\mathrm{hybr}}(x^-, x^+)$ denotes the number (counted modulo 2) of elements of $\mathcal{M}^{\mathrm{hybr}}(x^-, x^+)$. We define the homomorphism $\Theta_k \colon CM^{a,-}_k \to CM^{b,+}_k$ of abelian groups accordingly by linear continuation, and set $\Theta := (\Theta_k)_{k \in \mathbb{N}_0}$.

Theorem 5.4. *The map* Θ *is a chain homomorphism between the Morse complexes* $CM^{a,-}_*$ *and* $CM^{b,+}_*$*. Thus for each* $k \in \mathbb{N}_0$ *it holds* $\Theta_k \circ \partial^{\mathcal{YM}}_{k+1} = \partial^{\mathcal{E}}_{k+1} \circ \Theta_{k+1} \colon CM^{a,-}_{k+1} \to CM^{b,+}_k$.

Proof. From the compactness Theorem 4.2 it follows that the number of elements of $\mathcal{M}^{\mathrm{hybr}}(x^-, x^+)$ is finite and hence the homomorphism Θ is well-defined. The proof is then completed by standard arguments as e.g. carried out in [2]. $\square$

5.2 Proof of the main theorem

The aim of this final section is to prove Theorem 1.2. Thanks to Theorem 5.4, the map Θ induces for each $k \in \mathbb{N}_0$ a homomorphism $[\Theta_k]\colon HM_k^{a,-} \to HM_k^{b,+}$ of abelian groups. It remains to show that these homomorphisms are in fact isomorphisms. In our proof we follow closely the line of argument employed by Abbondandolo and Schwarz in [2].

Proof. **(Theorem 1.2)** It suffices to prove that each of the chain homomorphisms Θ_k is in fact an isomorphism and hence induces an isomorphism in homology. Let $k \in \mathbb{N}_0$ and fix a set $\underline{p} := (p_1, \ldots, p_m)$ of generators of $CM_k^{a,-}$. We order the entries p_j of the tuple $\underline{p}$ such that the following two conditions are met. First, we require that

$$i \leq j \qquad \Longrightarrow \qquad \mathcal{YM}^{\mathcal{V}^-}(p_i) \leq \mathcal{YM}^{\mathcal{V}^-}(p_j) \tag{43}$$

holds for all $1 \leq i, j \leq m$. Secondly, in case where p_i and p_j lie on the same critical manifold $\mathcal{C}^- \in \mathcal{CR}(\mathcal{YM})$ we choose our ordering such that

$$i \leq j \qquad \Longrightarrow \qquad h(p_i) \leq h(p_j). \tag{44}$$

Set $q_j := \Phi(p_j)$ and define $\underline{q} := (q_1, \ldots, q_m)$. The m-tuple $\underline{q}$ generates $CM_k^{b,+}$ which follows from the facts that Φ induces a bijection $\mathcal{CR}^a(\mathcal{YM}) \to \mathcal{CR}^b(\mathcal{E})$ which by Theorem B.2 preserves the Morse indices, and that on both sets $\mathcal{CR}^a(\mathcal{YM})$ and $\mathcal{CR}^b(\mathcal{E})$ we use the same Morse function h. By Theorem B.5 we have for each $1 \leq j \leq m$ the identity

$$\mathcal{YM}^{\mathcal{V}^-}(p_j) = \frac{\pi}{4}\mathcal{E}^{\mathcal{V}^+}(q_j), \tag{45}$$

which by our choice of the ordering of $\underline{p}$ implies that the tuple $\underline{q}$ is ordered by non-decreasing $\mathcal{E}^{\mathcal{V}^+}$ action. Furthermore, if $\mathcal{E}^{\mathcal{V}^+}(q_i) = \mathcal{E}^{\mathcal{V}^+}(q_j)$ for some $i \leq j$ then either q_i and q_j lie on different critical manifolds in $\mathcal{CR}^b(\mathcal{E})$ or otherwise $h(q_i) \leq h(q_j)$. Let us represent the homomorphism Θ_k with respect to the ordered bases $\underline{p}$ and $\underline{q}$ by the matrix $(\Theta_{ij}^k)_{1\leq i,j\leq m} \in \mathbb{Z}_2^{m\times m}$. The following two observations are now crucial. Both are a consequence of the energy inequality

$$\mathcal{YM}^{\mathcal{V}^-}(A) \geq \frac{\pi}{4}\mathcal{E}^{\mathcal{V}^+}(\Phi(A)) \tag{46}$$

for all $A \in \mathcal{A}(P)$ as in Theorem B.5. First, $\Theta_{ii}^k = 1$ for all $1 \leq i \leq m$ because the moduli space $\mathcal{M}^{\mathrm{hybr}}(p_i, q_i)$ consists of precisely one point. It is represented by the hybrid flow line with $(m^-, m^+) = (0,0)$ upper and lower cascades and configuration $\underline{x}^0 = (p_i, q_i) = [(u^-, u^+)]$ with stationary flow lines u^- of $\mathcal{YM}^{\mathcal{V}}$, respectively u^+ of $\mathcal{E}^{\mathcal{V}}$. Note that $(m^-, m^+) \neq (0,0)$ is not possible in this case as this would contradict (45). Secondly, if $i > j$ then $\Theta^k_{ij} = 0$ because in this case $\mathcal{M}^{\text{hybr}}(p_j, q_i) = \emptyset$. Assume by contradiction that $\mathcal{M}^{\text{hybr}}(p_j, q_i)$ contains at least one element. Let $p_j \in \mathcal{C}^-$ and $q_i \in \mathcal{C}^+$ for critical manifolds $\mathcal{C}^\pm$. The gradient flow property and (46) imply that $\mathcal{YM}^{\mathcal{V}^-}(p_j) \geq \frac{\pi}{4}\mathcal{E}^{\mathcal{V}^+}(q_i)$. This inequality must in fact be an equality because otherwise there would be a contradiction to condition (43) and identity (45). Hence $\mathcal{C}^+ = \Phi(\mathcal{C}^-)$, and the gradient flow property of the function h implies that $h(q_j) = h(\Phi(p_j)) \geq h(q_i)$. With $i > j$, condition (44) and again identity (45) show that this can only be the case if $h(q_j) = h(q_i)$. It follows that $q_j = q_i$ because h is monotone decreasing. Thus $i = j$, which is a contradiction. These two observations imply that the matrix $(\Theta^k_{ij})_{1 \leq i,j \leq m}$ takes the form

$$(\Theta^k_{ij})_{1 \leq i,j \leq m} = \begin{pmatrix} 1 & * & \cdots & \cdots & * \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & * \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \in \mathbb{Z}_2^{m \times m},$$

and thus is invertible over $\mathbb{Z}_2$. This completes the proof of the theorem. □

A Morse–Bott theory

We briefly recall a version of so-called Morse–Bott theory which is due to Frauenfelder [6, Appendix C]. Let (M, g) be a Riemannian manifold. A smooth function $f \colon M \to \mathbb{R}$ is called *Morse–Bott* if the set $\operatorname{crit}(f)$ of its critical points is a submanifold of M and if for each $x \in \operatorname{crit}(f)$ the *Morse–Bott condition* $T_x \operatorname{crit}(f) = \ker H_x f$ is satisfied. Let us fix a Morse function $h \colon \operatorname{crit}(f) \to \mathbb{R}$ such that the Morse–Smale condition is satisfied, i.e. all stable and unstable manifolds $W^s_h(x)$ and $W^u_h(y)$ of h intersect transversally. We assign to a critical point $x \in \operatorname{crit}(h) \subseteq \operatorname{crit}(f)$ the index

$$\operatorname{Ind}(x) := \operatorname{ind}_f(x) + \operatorname{ind}_h(x). \tag{47}$$

Definition A.1. Let $x^-, x^+ \in \operatorname{crit}(h)$. A *flow line from x^- to x^+ with $m \geq 1$ cascades* is a tuple $(\underline{x}, T) = ((x_j)_{1 \leq j \leq m}, (t_j)_{1 \leq j \leq m-1})$ with $x_j \in C^\infty(\mathbb{R}, M)$ and $t_j \in \mathbb{R}^+$ such that the following conditions are satisfied.

(i) For each $1 \leq j \leq m$, x_j is a nonconstant solution of the gradient flow equation $\dot{x}_j = -\nabla f(x_j)$.
(ii) For each $1 \leq j \leq m-1$ there exists a solution $y_j \in C^\infty(\mathbb{R}, \mathrm{crit}(f))$ of the gradient flow equation $\dot{y}_j = -\nabla h(y_j)$ such that $\lim_{s\to\infty} x_j(s) = y_j(0)$ and $\lim_{s\to-\infty} x_{j+1}(s) = y_j(t_j)$.
(iii) There exist points $p^- \in W^u_h(x^-) \subseteq \mathrm{crit}(f)$ and $p^+ \in W^s_h(x^+) \subseteq \mathrm{crit}(f)$ such that $\lim_{s\to-\infty} x_1(s) = p^-$ and $\lim_{s\to\infty} x_m(s) = p^+$.

A *flow line with $m = 0$ cascades* simply is an ordinary flow line of $-\nabla h$ on $\mathrm{crit}(f)$ from x^- to x^+.

Denote by $\mathcal{M}_m(x^-, x^+)$ the set of flow lines from x^- to x^+ with m cascades, modulo the action by time-shifts of the group $\mathbb{R}^m$ on tuples $\underline{x} = (x_1, \dots, x_m)$. We call

$$\mathcal{M}(x^-, x^+) := \bigcup_{m\in\mathbb{N}_0} \mathcal{M}_m(x^-, x^+)$$

the set of *flow lines with cascades from x^- to x^+*. In analogy to usual Morse homology theory, a sequence of broken flow lines with cascades may converge to a limit configuration which is a connected chain of such flow lines with cascades. This limiting behaviour is captured in the following definition.

Definition A.2. Let $x^-, x^+ \in \mathrm{crit}(h)$. A *broken flow line with cascades from x^- to x^+* is a tuple $\underline{v} = (v_1, \dots, v_\ell)$ where each v_j, $j = 1, \dots, \ell$, consists of a flow line with cascades from $x^{(j-1)}$ to $x^{(j)} \in \mathrm{crit}(h)$ such that $x^{(0)} = x^-$ and $x^{(\ell)} = x^+$.

Theorem A.3. *Let $x^-, x^+ \in \mathrm{crit}(h)$. Under suitable transversality assumptions (as specified in [6, Appendix C]) the set $\mathcal{M}(x^-, x^+)$ is a smooth manifold with boundary of dimension* $\dim \mathcal{M}(x^-, x^+) = \mathrm{Ind}(x^-) - \mathrm{Ind}(x^+) - 1$. *It is compact up to convergence to broken flow lines with cascades.*

Proof. For a proof we refer to [6, Theorems C.10, C.11]. □

Define by $CM_*(M, f, h)$ the chain complex generated (as a $\mathbb{Z}$-module) by the critical points of h and graded by the index Ind. Thanks to Theorem A.3 we may define for each $k \geq 0$ a boundary operator $\partial_k \colon CM_k(M, f, h) \to CM_{k-1}(M, f, h)$ by linear extension of

$$\partial_k x := \sum_{\mathrm{Ind}(x')=k-1} n(x, x')x'$$

for $x \in \mathrm{crit}(h)$ with $\mathrm{Ind}(x) = k$. Here $n(x, x')$ denotes the (oriented) count of elements in the zero dimensional moduli space $\mathcal{M}(x, x')$. As was shown in [6] the maps ∂_k give rise to a boundary operator satisfying $\partial_* \circ \partial_* = 0$. We define the *Morse–Bott homology* $HM_*(M,f,h)$ of (M,f,h) by

$$HM_k(M,f,h) := \frac{\ker \partial_k}{\operatorname{im} \partial_{k+1}} \qquad (k \in \mathbb{N}_0).$$

B Holonomy map and critical manifolds

We think of $\Sigma = S^2 \subseteq \mathbb{R}^3$ as the unit sphere with standard round metric induced from the ambient euclidian space $\mathbb{R}^3$. Fix the pair $z^\pm = (0,0,\pm 1)$ of antipodal points and set $D^\pm := \Sigma \setminus \{z^\mp\}$. We parametrize the hypersurfaces $D^\pm \subseteq \mathbb{R}^3$ by the maps

$$u^\pm\colon [0,\pi) \times [0,2\pi) \to D^\pm, \quad (r,t) \mapsto \big(\pm\cos(t)\sin(r), \pm\sin(t)\sin(r), \pm\cos(r)\big).$$

The volume form in these coordinates is

$$\operatorname{dvol}(D^\pm) = \sin(r)\, dr \wedge dt.$$

Set $\lambda\colon [0,\pi) \to \mathbb{R}$, $r \mapsto \sin(r)$. Denote by $|\Sigma| = 4\pi$ the volume of Σ. The Hodge star operator induced by the metric on Σ acts on differential forms on $D^\pm$ as

$$*1 = \lambda\, dr \wedge dt, \qquad *\, dr \wedge dt = \lambda^{-1}, \qquad *\, dr = \lambda\, dt, \qquad *\, dt = -\lambda^{-1}\, dr.$$

We define a family γ_t $(0 \leq t \leq 2\pi)$ of paths between z^+ and z^- by

$$\gamma_t\colon [0,\pi] \to \Sigma, \qquad \gamma_t(r) := (\cos(t)\sin(r), \sin(t)\sin(r), \cos(r)). \tag{48}$$

We fix two points $p^\pm \in P_{z^\pm}$ in the fibre $P_{z^\pm} \subseteq P$ above $z^\pm$. For a connection $A \in \mathcal{A}(P)$, let $\bar{\gamma}_t^A$ denote the horizontal lift of the path γ_t with respect to A, starting at p^+.

Definition B.1. For parameter $0 \leq t \leq 2\pi$, we define $x_A(t) \in G$ by the condition

$$\bar{\gamma}_t^A(\pi) = p^- \cdot x_A(t).$$

Thus $t \mapsto x_A(t)$ is a loop in ΛG. Now define the map

$$\Phi : \mathcal{A}(P) \to \Omega G, \quad A \mapsto x_A^{-1}(0) x_A. \tag{49}$$

The map Φ is called the *holonomy map* for the principal G-bundle P.

Note that Φ is independent of the choice of p^-, while replacing p^+ by $p^+ \cdot h$ for some $h \in G$ results in the conjugate holonomy map $h\Phi h^{-1}$. Similarly, replacing A by g^*A for a gauge transformation $g \in \mathcal{G}(P)$ with $g(p^+) = h \in G$ gives $\Phi(g^*A) = h\Phi(A)h^{-1}$. Thus for the subgroup of gauge transformations $\mathcal{G}_0(P)$ based at z^+, i.e. for

$$\mathcal{G}_0(P) := \{g \in \mathcal{G}(P) \mid g(p^+) = \mathbb{1}\}, \tag{50}$$

it follows that Φ descends to a G-equivariant map $\Phi\colon \mathcal{A}(P)/\mathcal{G}_0(P) \to \Omega G$, again denoted Φ and named holonomy map. The next theorem gives an explicit description of the set of Yang–Mills connections on the principal G-bundle P. Recall that the set of isomorphism classes of principal G-bundles is in bijection with the elements of $\pi_1(G)$, as any such bundle P is determined up to isomorphism by the homotopy class of the transition map $D^+ \cap D^- \to G$ of a trivialization of P over the open sets $D^\pm$.

Theorem B.2 (Correspondence between critical points)**.** *Let P be a principal G-bundle over Σ of topological type $\alpha \in \pi_1(G)$. Then the map $\Phi : A \mapsto x_A$ induces a bijection between the set of gauge equivalence classes of Yang–Mills connections on P and the set of conjugacy classes of closed, based geodesics x on G of homotopy class $[x] = \alpha$.*

Proof. For a proof we refer to [7, Theorem 2.1]. □

Theorem B.3. *Let $[A]$ be a $\mathcal{G}_0(P)$ equivalence class of Yang–Mills connections, and let $x_A = \Phi(A) \in \Omega G$ denote the corresponding closed geodesic. Then the Hessians of $\mathcal{YM}$ at $[A]$ and of $\mathcal{E}$ at x_A have the same index and nullity.*

Proof. For a proof we refer to [7, Theorem 2.2]. □

Let $i^\pm\colon D^\pm \to \Sigma$ denote the inclusion maps and $P^\pm := (i^\pm)^*P$ the corresponding pull-back bundles. Because $D^\pm \subseteq \Sigma$ is contractible, the principal G-bundle P admits local sections over $D^\pm$ which we can use to identify any connection $(i^\pm)^*A \in \mathcal{A}(P^\pm)$ with a 1-form $u \in \Omega^1(D^\pm, \mathfrak{g})$. Given $A \in \mathcal{A}(P)$ we can choose this local sections in such a way that the pull-back of $(i^\pm)^*A \in \mathcal{A}(P^\pm)$ under these sections is of the form $u^\pm\, dt$ for maps $u^\pm \in C^\infty(D, \mathfrak{g})$ satisfying

$$u^-(r,t) = x_A(t)u^+(\pi - r, t)x_A^{-1}(t) - \partial_t x_A(t)x_A^{-1}(t) \tag{51}$$

for $0 < r < \pi$. Because the connections $(i^\pm)^*A$ are well-defined near $r = 0$ it follows that $\lim_{r\to 0} u^\pm(r,t) = 0$. Following Gravesen [8] we set

$$\ell(r) := \frac{1}{2}(1 - \cos(r)) \qquad \text{and} \qquad \xi_A := x_A^{-1}\partial_t x_A \in \Omega\mathfrak{g},$$

and split the map u^+ as

$$u^+(r,t) = \ell(r)\xi_A(t) + m_A(r,t). \tag{52}$$

($m_A \in C^\infty(D^+, \mathfrak{g})$ being defined through this equation). Because $u^+(r,t) \to 0$ as $r \to 0$ this definition implies that also $\lim_{r\to 0} m_A(r,t) = 0$. Note that by (51)

$$\begin{aligned} &u^-(r,t) = \\ &\quad \ell(\pi - r)x_A(t)\xi_A(t)x_A^{-1}(t) + x_A^{-1}(t)m_A(\pi - r, t)x_A(t) - \partial_t x_A(t)x_A^{-1}(t), \end{aligned}$$

and thus $\ell(\pi) = 1$ and the condition $u^-(r,t) \to 0$ as $r \to 0$ imply that $\lim_{r\to\pi} m(r,t) = 0$ is satisfied. The following energy identity is due to Gravesen [8, Section 2]. There is also a generalization to higher genus surfaces, cf. Davies [5, Section 4.2].

Lemma B.4 (Energy identity)**.** *For every $A \in \mathcal{A}(P)$, the identity*

$$\mathcal{YM}(A) = \frac{\pi}{4}\mathcal{E}(x_A) + \frac{1}{2}\|\lambda^{-1}\partial_r m_A(r,t)\|^2_{L^2(\Sigma, \mathrm{dvol}(\Sigma))}. \tag{53}$$

is satisfied.

Proof. We identify $A \in \mathcal{A}(P)$ locally on $D^\pm$ with 1-forms $u^\pm \in \Omega^1(D^\pm, \mathfrak{g})$ as described before, and make use of the decomposition (52). Hence the curvature F_A becomes identified locally on $D^\pm$ with the 2-form $\partial_r u^\pm\, dr \wedge dt$. Recall also the formula $*(dr \wedge dt) = \lambda^{-1}$ for the Hodge star operator on 2-forms. In addition, we use that $\partial_r \ell(r) = \frac{1}{2}\lambda(r)$. Hence it follows that

$$\begin{aligned}
\mathcal{YM}(A) &= \frac{1}{2}\int_\Sigma \langle F_A \wedge *F_A\rangle \\
&= \frac{1}{2}\int_{D^+} \lambda^{-1}\langle \partial_r u^+, \partial_r u^+\rangle\, dr \wedge dt \\
&= \frac{1}{2}\int_{D^+} \lambda^{-1}(r)\left\langle \frac{1}{2}\lambda(r)\xi_A(t) + \partial_r m_A(r,t), \frac{1}{2}\lambda(r)\xi_A(t) + \partial_r m_A(r,t)\right\rangle dr \wedge dt \\
&= \frac{1}{2}\int_{D^+} \left\langle \frac{1}{2}\xi_A(t) + \lambda^{-1}(r)\partial_r m_A(r,t), \frac{1}{2}\xi_A(t) + \lambda^{-1}(r)\partial_r m_A(r,t)\right\rangle \lambda(r) dr \wedge dt \\
&= \frac{\pi}{8}\int_0^{2\pi} \langle \xi_A(t), \xi_A(t)\rangle\, dt + \frac{1}{2}\int_{D^+} \left|\lambda^{-1}(r)\partial_r m_A(r,t)\right|^2 \lambda(r) dr \wedge dt \\
&\quad + \frac{1}{2}\int_{D^+} \partial_r \langle \xi_A(t), m_A(r,t)\rangle\, dr \wedge dt \\
&= \frac{\pi}{4}\mathcal{E}(x_A) + \frac{1}{2}\|\lambda^{-1}\partial_r m_A(r,t)\|^2_{L^2(D^+, \mathrm{dvol}(D^+))}.
\end{aligned}$$

The term in the second but last line vanishes as follows from the above stated property $\lim_{r\to 0} m_A(r,t) = \lim_{r\to\pi} m_A(r,t) = 0$. □

Theorem B.5. *Let* $\mathcal{V}^- = \sum_{\ell=1}^\infty \lambda_\ell^- \mathcal{V}_\ell^- \in Y^-$ *and* $\mathcal{V}^+ = \sum_{\ell=1}^\infty \lambda_\ell^+ \mathcal{V}_\ell^+ \in Y^+$ *be admissible perturbations (cf. Definition 2.4) with non-negative coefficients* λ_ℓ^-*, respectively non-positive coefficients* λ_ℓ^+*. Then for every* $A \in \mathcal{A}(P)$ *there holds the inequality*

$$\mathcal{YM}^{\mathcal{V}^-}(A) \geq \frac{\pi}{4}\mathcal{E}^{\mathcal{V}^+}(x_A), \tag{54}$$

with equality if A *is Yang–Mills. In this case, the loop* x_A *is a geodesic.*

Proof. In view of Remarks 2.2 and 2.3 and the assumptions on $\lambda_\ell^\pm$ it follows that $\mathcal{V}^- \geq 0$ and $\mathcal{V}^+ \leq 0$. Hence it suffices to prove the inequality in the case of vanishing perturbations $\mathcal{V}^\pm = 0$, where it follows from the energy identity (53) (note that the last term in (53) is non-negative). Equality in the case where A is a Yang–Mills connection and the assertion that then x_A is a closed geodesic follow from the discussion in [7, p. 236]. □

The following is an infinitesimal version of the energy identity (53).

Proposition B.6. *Let* $A \in \mathcal{C}^-$ *and* $x = \Phi(A) \in \mathcal{C}^+$*. Then for all* $\alpha \in \Omega^1(\Sigma, \mathrm{ad}(P))$ *with* $\langle \alpha, \mathcal{H}_A \alpha\rangle < 0$ *it follows that* $\langle \beta, H_x \beta\rangle < 0$*, where* $\beta := \mathrm{d}\Phi(A)\alpha$*.*

Proof. The claim is an immediate consequence of the energy identity (53) which implies that for sufficiently small $\varepsilon > 0$ the map $\varepsilon \mapsto \mathcal{E}(x_{A+\varepsilon\alpha})$ is strictly monotone decreasing. □

C A priori estimates

Estimates involving $\mathcal{D}_A^\delta$

Lemma C.1. *Assume* $\mathcal{V}^- = 0$ *in the definition* (18) *of* $\mathcal{H}_A$*. For every* $A^- \in \mathcal{A}(P)$ *there exists a constant* $c(A^-) > 0$ *such that the operator* $\mathcal{D}_A$ *as in* (22) *satisfies the estimate*

$$\|\mathcal{D}_A - \mathcal{D}_{A^-}\|_{\mathcal{L}(\mathcal{Z}_A^{\delta,p,-}, \mathcal{L}^{\delta,p,-})} \leq c(A^-)\|\alpha\|_{C^0(\mathbb{R}^-, C^1(\Sigma))}$$

where we denote $\alpha := A - A^-$*. Similar estimates hold for the operator* $\mathcal{D}_A^\delta$ *as in* (25) *and for domains where* $\mathbb{R}^-$ *is replaced by some subinterval* $I \subseteq \mathbb{R}^-$*.*

Proof. Consider first the upper left entry in $\mathcal{H}_A - \mathcal{H}_{A^-}$ which is

$$-*[\alpha\wedge *d_{A^-+\alpha}\cdot\,]+d_{A^-}^*[\alpha\wedge\,\cdot\,]+*[*(d_{A^-}\alpha+\frac{1}{2}[\alpha\wedge\alpha]\wedge\,\cdot\,],$$

for which clearly an estimate of the claimed type holds. The other terms follow similarly. $\square$

The following is a basic estimate involving the operator $\mathcal{D}_A^\delta$ as in (25), here for a stationary path $A(s)=A$ ($s\in\mathbb{R}^-$).

Lemma C.2. *Let $A\in\mathcal{A}(P)$ and assume for some $\delta>0$ that the operator $\mathcal{H}_A+\delta$ is injective. Then for every $p\geq 2$ there exists a constant $c(A,p,\delta)$ and a compact operator $R:\mathcal{Z}_A^{p,-}\to\mathcal{L}^{\delta,p,-}$ such that the estimate*

$$\|(\alpha,\psi)\|_{\mathcal{Z}_A^{p,-}}\leq c(A,p,\delta)\big(\|\mathcal{D}_A^\delta(\alpha,\psi)\|_{\mathcal{L}^{p,-}}+\|R(\alpha,\psi)\|_{\mathcal{L}^{p,-}}\big)\tag{55}$$

holds for all $(\alpha,\psi)\in\mathcal{Z}_A^{p,-}$. Moreover, the operator $\mathcal{D}_A^\delta\colon\mathcal{Z}_A^{p,-}\to\mathcal{L}^{p,-}$ is surjective and has finite-dimensional kernel of dimension

$$\dim\ker\mathcal{D}_A^\delta=\operatorname{ind}A.$$

(The integer $\operatorname{ind}A$ *denoting the number of negative eigenvalues of $\mathcal{H}_A+\delta$).*

Proof. The proof follows the lines of [23, Theorem 8.5] and consists of four steps. Throughout we set $\zeta:=(\alpha,\psi)$.

Step 1. *The statement on surjectivity and the kernel is true in the case $p=2$.*

The operator $\mathcal{H}_A+\delta$ with domain $\mathcal{W}^{2,A}(\Sigma)$ is an unbounded self-adjoint operator on the Hilbert space $H:=L^2(\Sigma,T^*\Sigma\otimes\operatorname{ad}(P))\oplus L^2(\Sigma,\operatorname{ad}(P))$, cf. [19, Proposition 5.1]. Denote by E^- and E^+ its negative, respectively positive eigenspaces. Since $\mathcal{H}_A+\delta$ is assumed to be injective, H splits as an orthogonal sum $H=E^-\oplus E^+$. Let $P^\pm$ denote the projections onto $E^\pm$ and set $\mathcal{H}^\pm=(\mathcal{H}_A+\delta)|_{E^\pm}$. As $\mathcal{H}^-$ and $-\mathcal{H}^+$ are negative-definite self-adjoint operators, it follows from the Hille-Yosida theorem (cf. for instance [13, Section X.8]) that they generate strongly continuous contraction semigroups $s\mapsto e^{s\mathcal{H}^-}$ on E^- respectively $s\mapsto e^{-s\mathcal{H}^+}$ on E^+, both defined for $s\geq 0$. This allows us to define the map $K\colon\mathbb{R}\to\mathcal{L}(H)$ by

$$K(s):=\begin{cases}-e^{-s\mathcal{H}^-}P^- & \text{for } s\leq 0,\\ e^{-s\mathcal{H}^+}P^+ & \text{for } s>0.\end{cases}$$

As one easily checks, K is strongly continuous in $\mathbb{R} \setminus \{0\}$ and its pointwise operator norm satisfies

$$\|K(s)\|_{\mathcal{L}(H)} \le e^{-\delta_0 |s|} \tag{56}$$

for $\delta_0 > 0$ the smallest (in absolute value) eigenvalue of $\mathcal{H}_A + \delta$. Now consider the operator $Q\colon \mathcal{L}^{2,-} \to \mathcal{Z}_A^{2,-}$ defined by

$$(Q\eta)(s) := \int_{-\infty}^{0} K(s-\sigma)\eta(\sigma)\, d\sigma.$$

It satisfies

$$\begin{aligned}
\frac{d}{ds}(Q\eta)(s) &= \frac{d}{ds}\int_{-\infty}^{s} e^{-(s-\sigma)\mathcal{H}^+} P^+\eta(\sigma)\, d\sigma - \frac{d}{ds}\int_{s}^{0} e^{-(s-\sigma)\mathcal{H}^-} P^-\eta(\sigma)\, d\sigma \\
&= P^+\eta(s) - \int_{-\infty}^{s} \mathcal{H}^+ e^{-(s-\sigma)\mathcal{H}^+} P^+\eta(\sigma)\, d\sigma \\
&\quad + P^-\eta(s) + \int_{s}^{0} \mathcal{H}^- e^{-(s-\sigma)\mathcal{H}^-} P^-\eta(\sigma)\, d\sigma \\
&= \eta(s) - (\mathcal{H}_A + \delta)(Q\eta)(s).
\end{aligned}$$

From this calculation we see that

$$\mathcal{D}_A^\delta Q\eta = \frac{d}{ds}(Q\eta) + (\mathcal{H}_A + \delta)(Q\eta) = \eta,$$

so Q is a right-inverse of $\mathcal{D}_A^\delta$. This proves surjectivity of $\mathcal{D}_A^\delta$. Now let $\zeta \in \ker \mathcal{D}_A^\delta$ such that $(\mathcal{H}_A + \delta)\zeta(0) = \lambda\zeta(0)$ for some $\lambda \in \mathbb{R}$. Then $\zeta(s) = e^{-\lambda s}\zeta(0)$, which is contained in $\mathcal{L}^{2,-}$ if and only if $\lambda < 0$. Therefore $\zeta \in \mathcal{Z}_A^{2,-}$ satisfies $\mathcal{D}_A^\delta \zeta = 0$ if and only if $\zeta(0) \in \mathcal{H}^-$. This shows that $\mathcal{H}^-$ and $\ker \mathcal{D}_A^\delta$ are isomorphic to each other.

Step 2. *For every $p \ge 2$ there exists a constant $c_1(A,p)$ such that the following holds. If $\zeta \in \mathcal{Z}_A^{2,-}$ and $\mathcal{D}_A^\delta \zeta \in \mathcal{L}^{p,-}$, then $\zeta \in \mathcal{Z}_A^{p,-}$ and*

$$\|\zeta\|_{\mathcal{Z}_A^{p,-}} \le c_1(A,p)\big(\|\mathcal{D}_A^\delta \zeta\|_{\mathcal{L}^{p,-}} + \|\zeta\|_{L^p(\mathbb{R}^-,H)}\big). \tag{57}$$

The claim follows from standard arguments based on the linear estimate

$$\|\zeta\|_{\mathcal{Z}_A^{p,-}([-1,0])} \le c(A,p)\big(\|\mathcal{D}_A^\delta \zeta\|_{L^p([-2,0])} + \|\zeta\|_{L^p([-2,0])}\big), \tag{58}$$

cf. [18, Proposition A.6]. Full details are given in [18, Lemma 3.20].

Step 3. *The operator* $Q\colon \mathcal{L}^{p,-} \to L^p(\mathbb{R}^-, H)$ *is bounded, for every* $p \geq 2$. *(In the following, we let* $c_2(A,p)$ *denote its operator norm.)*

The claim follows from Young's convolution inequality with

$$\begin{aligned}\|Q\eta\|_{L^p(\mathbb{R}^-,H)} &= \|K * \eta\|_{L^p(\mathbb{R}^-,H)} \\ &\leq \|K\|_{L^1(\mathbb{R}^-,\mathcal{L}(H))}\|\eta\|_{L^p(\mathbb{R}^-,H)} \leq \frac{1}{\delta_0}\|\eta\|_{\mathcal{L}^{p,-}}.\end{aligned}$$

In the last step we used that for $p \geq 2$ the $\mathcal{L}^{p,-}$ norm dominates the $L^p(\mathbb{R}^-, H)$ norm.

Step 4. *We prove the lemma.*

The estimates of Step 2 and Step 3 imply that

$$\begin{aligned}\|\zeta\|_{\mathcal{Z}_A^{p,-}} &\leq c_1(A,p)\big(\|\mathcal{D}_A^\delta\zeta\|_{\mathcal{L}^{p,-}} + \|\zeta\|_{L^p(\mathbb{R}^-,H)}\big) \\ &\leq c_1(A,p)\big(\|\mathcal{D}_A^\delta\zeta\|_{\mathcal{L}^{p,-}} + \|Q\mathcal{D}_A^\delta\zeta\|_{L^p(\mathbb{R}^-,H)} + \|\zeta - Q\mathcal{D}_A^\delta\zeta\|_{L^p(\mathbb{R}^-,H)}\big) \\ &\leq c_1(A,p)\big((1 + c_2(A,p))\|\mathcal{D}_A^\delta\zeta\|_{\mathcal{L}^{p,-}} + \|\zeta - Q\mathcal{D}_A^\delta\zeta\|_{L^p(\mathbb{R}^-,H)}\big).\end{aligned}$$

This shows (55) because the operator

$$R := \mathbb{1} - Q\mathcal{D}_A^\delta\colon \mathcal{Z}_A^{p,-} \to L^p(\mathbb{R}^-, H)$$

has finite rank (of dimension equal to $\dim\ker\mathcal{D}_A^\delta$) and therefore is compact. To prove surjectivity we note that the operator $\mathcal{D}_A^\delta$ has closed range by (55) and the usual abstract closed range lemma (cf. [14, p. 14]). Hence it suffices to show that $\operatorname{ran}\mathcal{D}_A^\delta$ is dense in $\mathcal{L}^{p,-}$. Hence let $\eta \in \mathcal{L}^{p,-} \cap \mathcal{L}^{2,-}$ be given. The latter is a dense subspace of $\mathcal{L}^{p,-}$ because it contains all compactly supported smooth functions. By Step 1 there exists some $\zeta \in \mathcal{Z}_A^{2,-}$ such that $\mathcal{D}_A\xi = \eta$. From (55) and the assumption $\eta \in \mathcal{L}^{p,-}$ it follows that $\xi \in \mathcal{Z}_A^{p,-}$ which implies surjectivity. Again (55) together with finiteness of the rank of R shows that the kernels of the operators $\mathcal{D}_A^\delta\colon \mathcal{Z}_A^{2,-} \to \mathcal{L}^{2,-}$ and $\mathcal{D}_A^\delta\colon \mathcal{Z}_A^{p,-} \to \mathcal{L}^{p,-}$ coincide, for all $p \geq 2$. Hence the assertion on the kernel follows from Step 1. This finishes the proof of the lemma. □

Lemma C.3. *Let* $s \mapsto A(s)$, $s \in (-\infty, 0]$, *be a smooth solution of* (11) *satisfying for a Yang–Mills connection* A^- *the asymptotic condition*

$$\lim_{s\to-\infty} A(s) = A^-$$

in $C^1(\Sigma)$. Let $\delta > 0$ be such that the operator $\mathcal{H}_A + \delta$ is injective. Then the operator $\mathcal{D}^\delta_A \colon \mathcal{Z}^{p,-}_A \to \mathcal{L}^{p,-}$ associated with A is surjective and has finite-dimensional kernel of dimension

$$\dim\ker \mathcal{D}^\delta_A = \operatorname{ind} A^-. \tag{59}$$

Proof. The proof is divided into three steps.

Step 1. *Stationary case.*

Consider the case where the path $A \equiv A^-$ is stationary. In this case Lemma C.2 applies and yields surjectivity of $\mathcal{D}^\delta_A$ and formula (59). In particular, $\mathcal{D}^\delta_A$ is a Fredholm operator.

Step 2. *Nearby case.*

Let us assume that for some sufficiently small $\varepsilon > 0$ the condition

$$\|A - A^-\|_{\mathcal{C}^0(\mathbb{R}^-,\mathcal{C}^1(\Sigma))} < \varepsilon \tag{60}$$

is satisfied. We here consider A^- as a stationary connection over $\mathbb{R}^- \times \Sigma$. Surjectivity and the Fredholm index are preserved under small perturbations with respect to the operator norm. By Lemma C.1, the operator norm of $\mathcal{D}^\delta_A$ depends continuously on A with respect to the $C^0(\mathbb{R}^-, C^1(\Sigma))$ topology. Therefore surjectivity is implied by assumption (60). As the Fredholm indices of $\mathcal{D}^\delta_{A^-}$ and $\mathcal{D}^\delta_A$ coincide it follows that (59) holds true in the nearby case.

Step 3. *General case.*

The general case can be reduced to the nearby case by a standard argument as e.g. carried out in the proof of [23, Proposition 8.3]. $\square$

Estimates involving $\mathcal{D}^\delta_x$

The following lemma gives a basic estimate for the operator $\mathcal{D}^\delta_x$ as in (26).

Lemma C.4. *Fix $p \geq 2$. Let $x \in C^\infty(\mathbb{R}^1 \times S^1, G)$ with $\lim_{s\to\infty} x(s) = x^+$ in the $C^1(S^1)$ topology for some $x^+ \in C^\infty(S^1, G)$. Then there exist positive constants c and T, which depend only on x, p, and δ, such that for every $\xi \in \mathcal{Z}^{p,+}$ the estimate*

$$\|\xi\|_{\mathcal{Z}^{p,+}} \leq c\big(\|\mathcal{D}^\delta_x \xi\|_{\mathcal{L}^{p,+}} + \|\xi\|_{L^p([0,T],L^2(S^1))} + \|\xi(0)\|_{L^2(S^1)}\big) \tag{61}$$

is satisfied.

Proof. We start with the standard parabolic estimate

$$\|\xi\|_{\mathcal{Z}^{p,+}} \le c(p,x)\big(\|\mathcal{D}^\delta_x \xi\|_{\mathcal{L}^{p,+}} + \|\xi\|_{L^p(\mathbb{R}^+,L^2(S^1))}\big)$$

as obtained (for any $p \ge 2$) in Step 3 of the proof of [23, Theorem 8.5]. To prove the lemma, it therefore remains to estimate the last term in (61), which we split into integrals over $[0,T]$ and $[T,\infty)$ for sufficiently large $T>0$. We apply Lemma C.6 to the operator $L := H_{x^+} - \delta$, which has spectrum bounded away from 0 by our choice of δ. Defining $\mathcal{D}^\delta_{x^+} := \frac{d}{ds} + L$ this yields for a constant $c(x^+,T)$ the estimate

$$\|\xi\|_{L^p([T,\infty),L^2(S^1))} \le c(x^+,T)\big(\|\mathcal{D}^\delta_{x^+}\xi\|_{L^p([T,\infty),L^2(S^1))} + \|\xi(T)\|_{L^2(S^1)}\big), \quad (62)$$

for all $\xi \in L^p([T,\infty),L^2(S^1))$. Denoting by $\mathcal{L}$ the space of bounded linear maps $W^{1,2}(S^1) \to L^2(S^1)$ we have for all $s \ge T$ the pointwise estimate

$$\begin{aligned}\|\mathcal{D}^\delta_{x^+}\xi(s) - \mathcal{D}^\delta_{x(s)}\xi(s)\|_{L^2(S^1)} &= \\ &\|H_{x^+}\xi(s) - H_{x(s)}\xi(s)\|_{L^2(S^1)} \le \|H_{x^+} - H_{x(s)}\|_{\mathcal{L}} \|\xi(s)\|_{L^2(S^1)}.\end{aligned}$$

We hence can further estimate the term $\mathcal{D}^\delta_{x^+}\xi$ in (62) as

$$\begin{aligned}&\|\mathcal{D}^\delta_{x^+}\xi\|_{L^p([T,\infty),L^2(S^1))} \\ &\quad\le\ \|\mathcal{D}^\delta_{x^+}\xi - \mathcal{D}^\delta_x\xi\|_{L^p([T,\infty),L^2(S^1))} + \|\mathcal{D}^\delta_x\xi\|_{L^p([T,\infty),L^2(S^1))} \\ &\quad\le\ \|H_{x^+} - H_x\|_{L^\infty([T,\infty),\mathcal{L})}\|\xi\|_{L^p([T,\infty),L^2(S^1))} + \|\mathcal{D}^\delta_x\xi\|_{L^p([T,\infty),L^2(S^1))}.\end{aligned}$$

Using the assumption $\lim_{s\to\infty} x(s) = x^+$ in $C^1(S^1)$, it can be checked that $\|H_{x^+} - H_x\|_{L^\infty([T,\infty),\mathcal{L})} \to 0$ as $T \to \infty$. Hence the term involving $H_{x^+} - H_x$ can be absorbed in the left-hand side of (62) for $T = T(x)$ sufficiently large. Furthermore, the term $\|\mathcal{D}^\delta_x\xi\|_{L^p([T,\infty),L^2(S^1))}$ is controlled by $\|\mathcal{D}^\delta_x\xi\|_{\mathcal{L}^{p,+}}$ as is clear from the assumption $p \ge 2$. The desired estimate now follows after applying Lemma C.7 (with $L(s) := H_{x(s)} - \delta$ and $\varepsilon > 0$ sufficienty small) to the remaining term $\|\xi(T)\|_{L^2(S^1)}$ in (62). This introduces a further term $\varepsilon\|H_x\xi\|_{L^2([0,T],L^2(S^1))}$ which can be absorbed in the left-hand side of the asserted inequality (61), and a term $\|\mathcal{D}^\delta_x\xi\|_{L^2([0,T],L^2(S^1))} + \|\xi\|_{L^2([0,T],L^2(S^1))}$ which for $p \ge 2$ is dominated by $\|\mathcal{D}^\delta_x\xi\|_{L^p([0,T],L^p(S^1))} + \|\xi\|_{L^p([0,T],L^2(S^1))}$ appearing on the right hand side of (61). $\square$

Lemma C.5. *Let $s \mapsto x(s)$, $s \in [0,\infty)$, be a smooth solution of* (4) *satisfying for a closed geodesic x^+ the asymptotic condition*

$$\lim_{s\to\infty} x(s) = x^+$$

in $C^1(S^1)$. Then the operator $\mathcal{D}_x^\delta \colon \mathcal{Z}^{p,+} \to \mathcal{L}^{p,+}$ associated with x is surjective and has finite-dimensional cokernel of dimension

$$\dim \operatorname{coker} \mathcal{D}_x^\delta = \operatorname{ind} x^+.$$

(The integer $\operatorname{ind} x^+$ *denoting the number of negative eigenvalues of $H_{x^+} - \delta$).*

Proof. The proof of an analogous result in [23, Proposition 8.3] for the backward halfcylinder $\mathbb{R}^- \times S^1$ carries over to the present situation by taking the adjoint of $\mathcal{D}_x^\delta$ and time-reversal $s \mapsto -s$. □

Further linear estimates

Lemma C.6. *Let H be a Hilbert space and $L \colon \operatorname{dom}(L) \to H$ be the infinitesimal generator of a strongly continuous one-parameter semigroup on H. We assume that the spectrum of L is contained either in the interval $(-\infty, -\lambda]$ or in the interval $[\lambda, \infty)$ for some $\lambda > 0$. Let $p \geq 1$. Then any solution $\xi \colon \mathbb{R}^+ \to H$ of the equation $\dot{\xi} + L\xi = \eta$ satisfies the estimate*

$$\|\xi\|_{L^p(\mathbb{R}^+,H)} \leq \frac{1}{\lambda} \|\eta\|_{L^p(\mathbb{R}^+,H)} + \frac{c(L,p)}{\lambda} \|\xi_0\|_H,$$

for a constant $c(L,p) \geq 0$. Here we denote $\xi_0 := \xi(0)$.

Proof. We first consider the case where the spectrum of L is contained in the interval $(-\infty, -\lambda]$. Assume also that $\xi_0 = 0$. Then ξ can be represented as the convolution $\xi(s) = (K * \eta)(s)$ with kernel

$$K(s) := \begin{cases} 0 & \text{for} \quad s \geq 0, \\ -e^{-Ls} & \text{for} \quad s < 0. \end{cases} \tag{63}$$

Our assumption on the spectrum of L implies that for $s < 0$ the operator norm of $K(s) \colon H \to H$ is bounded by $e^{\lambda s}$. Thus

$$\|K\|_{L^1(\mathbb{R},\mathcal{L}(H))} \leq \int_{-\infty}^0 e^{\lambda s}\, ds = \frac{1}{\lambda},$$

and Young's convolution inequality implies that

$$\|\xi\|_{L^p(\mathbb{R}^+,H)} \leq \|K\|_{L^1(\mathbb{R},\mathcal{L}(H))} \|\eta\|_{L^p(\mathbb{R}^+,H)} \leq \frac{1}{\lambda} \|\eta\|_{L^p(\mathbb{R}^+,H)}. \tag{64}$$

Now let $\xi_0 \in H$ be arbitrary. Defining $\xi_1 := \xi - e^{Ls}\xi_0$ for $s \geq 0$ it follows that $\xi_1(0) = 0$. Also, ξ_1 satisfies the equation

$$\dot{\xi}_1 + L\xi_1 = \eta - 2Le^{Ls}\xi_0 =: \eta_1.$$

Hence by (64),

$$\begin{aligned}
\|\xi\|_{L^p(\mathbb{R}^+,H)} &\leq \|\xi_1\|_{L^p(\mathbb{R}^+,H)} + \|e^{Ls}\xi_0\|_{L^p(\mathbb{R}^+,H)} \\
&\leq \frac{1}{\lambda}\|\eta_1\|_{L^p(\mathbb{R}^+,H)} + \frac{1}{\lambda}\|\xi_0\|_H \\
&\leq \frac{1}{\lambda}\|\eta\|_{L^p(\mathbb{R}^+,H)} + \frac{2}{\lambda}\|Le^{Ls}\xi_0\|_{L^p(\mathbb{R}^+,H)} + \frac{1}{\lambda}\|\xi_0\|_H \\
&\leq \frac{1}{\lambda}\|\eta\|_{L^p(\mathbb{R}^+,H)} + \frac{c(L,p)}{\lambda}\|\xi_0\|_H.
\end{aligned}$$

To obtain the last estimate we note that $L\colon \operatorname{dom} L \to H$ is a bounded operator, when $\operatorname{dom} L$ is endowed with the graph norm of L. It thus follows that

$$\begin{aligned}
\|Le^{Ls}\xi_0\|^p_{L^p(\mathbb{R}^+,H)} = \int_0^\infty \|Le^{Ls}\xi_0\|^p_H\, ds \leq \|L\|^p \cdot \|\xi_0\|^p_H \int_0^\infty e^{-\lambda ps}\, ds \\
= \frac{\|L\|^p \cdot \|\xi_0\|^p_H}{p\lambda}.
\end{aligned}$$

Now set $c(L,p) := 1 + \frac{2\|L\|}{\sqrt[p]{p}}$. The claim then follows. Now assume that the spectrum of L is contained in the interval $[\lambda,\infty)$. In this case we may argue as before, replacing the kernel K in (63) by

$$K(s) := \begin{cases} e^{-Ls} & \text{for} \quad s \geq 0, \\ 0 & \text{for} \quad s < 0. \end{cases}$$

The claim then follows as before. □

Lemma C.7. *Let H be a Hilbert space. Assume $\xi\colon [s_0,s_1] \to H$ satisfies the equation $\dot{\xi} + L\xi = \eta$ for a path $s \mapsto L(s)$ of (densely defined) linear operators on H. Then for every $\varepsilon > 0$ there holds the estimate*

$$\begin{aligned}
&\|\xi(s_1)\|^2_H - \|\xi(s_0)\|^2_H \\
&\qquad \leq (1+\varepsilon^{-1})\|\xi\|^2_{L^2([s_0,s_1],H)} + \varepsilon\|L\xi\|^2_{L^2([s_0,s_1],H)} + \|\eta\|^2_{L^2([s_0,s_1],H)}.
\end{aligned}$$

Proof. We integrate the equation

$$\frac{d}{ds}\frac{1}{2}\|\xi(s)\|^2_H = \langle \xi(s), \eta(s)\rangle - \langle \xi(s), L(s)\xi(s)\rangle$$

over the interval $[s_0,s_1]$ and apply to the two terms on the right-hand side the Cauchy-Schwarz inequality. This immediately yields the result. □

The following is a general interpolation lemma for operators of type $D = \frac{d}{ds} + A(s)$.

Lemma C.8 (Interpolation Lemma). *Let $V \subseteq H \subseteq V^*$ be a Gelfand triple. Assume that the family $A(s) : V \to V^*$ ($s \in \mathbb{R}^-$) of operators satisfies for all $\xi \in V$ the uniform bound*

$$\|\xi\|_V^2 \le c_1 \langle \xi, A(s)\xi \rangle_H + c_2 \|\xi\|_H^2$$

for constants $c_1, c_2 > 0$. Then for any $\delta > 0$ and $p \ge 2$ there holds the estimate

$$\begin{aligned} &\frac{1}{p}\|\xi(0)\|_H^p + \big(\frac{1}{c_1} - \frac{1}{2\delta}\big) \int_{-\infty}^0 \|\xi(s)\|_H^{p-2} \|\xi(s)\|_V^2 \, ds \\ &\qquad \le \big(\frac{\delta(p-2)}{2p} + \frac{c_2}{c_1}\big) \|\xi\|_{L^p(\mathbb{R}^-,H)}^p + \frac{\delta}{p} \|\dot{\xi}(s) + A(s)\xi(s)\|_{L^p(\mathbb{R}^-,V^*)}^p \end{aligned}$$

for all $\xi \in W^{1,p}(\mathbb{R}^-, V^*) \cap L^p(\mathbb{R}^-, V)$.

Proof. Set $\eta := \dot{\xi} + A\xi \in L^p(\mathbb{R}^-, V^*)$. For every $s \in \mathbb{R}^-$ there holds the estimate

$$\begin{aligned} &\frac{1}{p}\frac{d}{ds}\|\xi(s)\|_H^p = \|\xi(s)\|_H^{p-2} \langle \dot{\xi}(s), \xi(s) \rangle_H \\ &\quad = \ \|\xi(s)\|_H^{p-2} \langle \eta(s) - A(s)\xi(s), \xi(s) \rangle_H \\ &\quad \le \ \|\xi(s)\|_H^{p-2} \big(\|\eta(s)\|_{V^*} \|\xi(s)\|_V - \frac{1}{c_1}\|\xi(s)\|_V^2 + \frac{c_2}{c_1}\|\xi(s)\|_H^2 \big) \\ &\quad \le \ \big(\frac{1}{2\delta} - \frac{1}{c_1}\big) \|\xi(s)\|_H^{p-2}\|\xi(s)\|_V^2 + \frac{\delta}{2}\|\xi(s)\|_H^{p-2}\|\eta(s)\|_{V^*}^2 + \frac{c_2}{c_1}\|\xi(s)\|_H^p, \end{aligned}$$

for any constant $\delta > 0$. Integrating this inequality over $\mathbb{R}^-$ and applying Hölder's inequality yields

$$\begin{aligned} &\frac{1}{p}\|\xi(0)\|_H^p = \int_{-\infty}^0 \frac{1}{p}\frac{d}{ds}\|\xi(s)\|_H^p \, ds \\ &\quad \le \ \int_{-\infty}^0 \big(\frac{1}{2\delta} - \frac{1}{c_1}\big)\|\xi(s)\|_H^{p-2}\|\xi(s)\|_V^2 + \frac{\delta}{2}\|\xi(s)\|_H^{p-2}\|\eta(s)\|_{V^*}^2 + \frac{c_2}{c_1}\|\xi(s)\|_H^p \, ds \\ &\quad \le \ \int_{-\infty}^0 \big(\frac{1}{2\delta} - \frac{1}{c_1}\big)\|\xi(s)\|_H^{p-2}\|\xi(s)\|_V^2 \, ds + \frac{c_2}{c_1}\int_{-\infty}^0 \|\xi(s)\|_H^p \, ds \\ &\qquad + \frac{\delta}{2}\Big(\int_{-\infty}^0 \|\xi(s)\|_H^p \, ds\Big)^{\frac{p-2}{p}} \cdot \Big(\int_{-\infty}^0 \|\eta(s)\|_{V^*}^p \, ds\Big)^{\frac{2}{p}}. \end{aligned}$$

We now apply Young's inequality to the product term in the last line. The claim then follows. $\square$

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