FACTORIZABLE $\mathcal{D}$-MODULES

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1. Introduction

1.1. Let $\mathfrak{g}$ be a complex semisimple Lie algebra; let $\mathrm{Mod}(\mathfrak{g})$ be the category of finite dimensional representations of $\mathfrak{g}$. In the paper [D] Drinfeld introduced a remarkable structure of a braided tensor category on $\mathrm{Mod}(\mathfrak{g})$. In fact, the tensor product is the usual one but the commutativity and associativity isomorphisms are deformed; they depend on a "quantization" parameter h (a sufficiently small complex number or a formal variable) and are defined using the Knizhnik-Zamolodchikov differential equations (cf. Section 5 below).

In this paper we propose a geometric construction of the Drinfeld's braided tensor category. Namely, we introduce a category $\mathcal{FM}_\kappa$ (depending on a non-zero complex parameter $\kappa \notin \mathbb{Q}$) of *factorizable $\mathcal{D}$-modules* on the space $\mathrm{Div}^+(\mathbf{A}^1; Y)$ of non-negative Y-valued divisors on the affine line $\mathbf{A}^1$, Y being the coroot lattice of $\mathfrak{g}$.

An object of $\mathcal{FM}_\kappa$ is a certain collection of regular holonomic $\mathcal{D}$-modules on various symmetric powers of $\mathbf{A}^1$ connected by *factorization isomorphisms* (for the precise definition see Section 2 below). The category $\mathcal{FM}_\kappa$ comes equipped with a braided tensor structure defined using the Malgrange — Kashiwara specialization functors, [K], [M]. The main result of this paper is Theorem 5.2 which provides a tensor equivalence

(a) $\Phi_\kappa : \mathcal{FM}_\kappa \xrightarrow{\sim} \mathrm{Mod}(\mathfrak{g})_\kappa$

where $\mathrm{Mod}(\mathfrak{g})_\kappa$ denotes the Drinfeld's tensor category corresponding to $h = \kappa^{-1}$.

The definition of the category $\mathcal{FM}_\kappa$ is completely parallel to that of $\mathcal{FS}_q$ from [FS1], [FS2]; the only difference is that we replace perverse sheaves by $\mathcal{D}$-modules. By [FS1]III Thm. 18.4, we have a tensor equivalence

(b) $\Phi_q : \mathcal{FS}_q \xrightarrow{\sim} \mathrm{Mod}(\mathrm{U}_q\,\mathfrak{g})$

where the right hand side denotes the category of finite dimensional representations of the quantized enveloping algebra $\mathrm{U}_q\,\mathfrak{g}$, $q \in \mathbb{C}^*$ not a root of unity.

On the other hand, the Riemann-Hilbert correspondence induces the tensor equivalence

(c) $RH : \mathcal{FM}_\kappa \xrightarrow{\sim} \mathcal{FS}_q$

where $q = \exp(1/2\pi i\kappa)$. Combining (a), (b) and (c) we get the tensor equivalence

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(d) $\mathrm{Mod}(\mathfrak{g})_\kappa \xrightarrow{\sim} \mathrm{Mod}(\mathrm{U}_q\,\mathfrak{g})$.

The existence of an equivalence (d) was one of the main results of [D] (with $h = 1/\kappa$ formal). In [KL] it was established in a different manner, assuming $\kappa \notin \mathbb{Q}$. Thus, our results provide the third, "Riemann-Hilbert", proof of the equivalence (d), together with its explicit construction.

Thus, we have a square of equivalences

$$\begin{array}{ccc} \mathcal{FS}_q & \overset{RH}{\xrightarrow{\sim}} & \mathcal{FM}_\kappa \\ \Phi_q \downarrow & & \downarrow \Phi_\kappa \\ \mathrm{Mod}(\mathrm{U}_q\,\mathfrak{g}) & \xrightarrow{\sim} & \mathrm{Mod}(\mathfrak{g})_\kappa \end{array}$$

In a sense, the tensor categories in the left (resp., right) column may be regarded as a "multiplicative" (resp., "additive") incarnations of the same tensor category[1].

Our proof of the equivalence (a) differs from the method used in [FS1] for proving (b); in a sense it is more direct. It is based on the methods of [Kh1], [Kh2] which provide a full "quiver" description of the categories of $\mathcal{D}$-modules involved (cf. also Remark 5.5).

As a byproduct, we get a natural explanation of the *ad hoc* formulas for the solutions of KZ equations from [SV], cf. 5.3. Roughly speaking, these formulas are contained in the *inverse* to the functor (a). The KZ equations themselves appear as a result of an explicit computation of the Malgrange — Kashiwara specialization of factorizable $\mathcal{D}$-modules, the crucial result being Theorem 4.12.

The proofs are omitted or sketched in this paper. A detailed account will appear later on.

1.2. **Open questions.** At the moment, our methods give the result only for an irrational κ, while the main result of [FS1], [FS2] establishes an equivalence (b) in the most interesting case *q equal to a root of unity* as well. In this case the category $\mathrm{Mod}(\mathrm{U}_q\,\mathfrak{g})$ should be replaced by the category $\mathcal{C}$ studied by Andersen — Jantzen — Soergel. It would be tempting to find a version of the equivalence (a) in this case. The first problem is, that nobody (to our knowledge) knows what category should appear in the lower right corner of the square above.

The Drinfeld's equivalence for an irrational κ is a starting point for the Kazhdan — Lusztig remarkable theorem [KL] establishing an analogue of this equivalence for the case of a rational (non-positive) κ. Here the category $\mathrm{Mod}(\mathrm{U}_q\,\mathfrak{g})$ is replaced by an appropriate category of representations of the quantum group *with divided powers*, and $\mathrm{Mod}(\mathfrak{g})_\kappa$ — by an appropriate category of representations of the affine Lie algebra $\hat{\mathfrak{g}}$ with central charge

[1]This situation resembles the isomorphism, given by the Chern character, between K-theory and cohomology. This resemblance is supported by the "Riemann-Roch type" commutation formulas of the "Riemann-Hilbert" isomorphism for vanishing cycles with the maps u (canonical) and v (variation), (these maps being analogous to the inverse (resp., direct) image for a closed codimension 1 embedding), cf. [K], Theorem 2 (2).

$\kappa - h$ (h being the dual Coxeter number of $\mathfrak{g}$). It would be interesting to find a geometrical interpretation of the Kazhdan-Lusztig equivalence.

1.3. A few words about the notations. We will work over $\mathbb{C}$.

We will deal with smooth complex algebraic varieties A equipped with finite algebraic Whitney stratifications $\mathcal{S}_A$ (with smooth strata). We will denote by $\mathcal{M}(A)$ the category of regular holonomic algebraic $\mathcal{D}$-modules, *lisse along* $\mathcal{S}_A$. It is an artinian abelian category with a duality $D : \mathcal{M}(A)^{\mathrm{opp}} \xrightarrow{\sim} \mathcal{M}(A)$. We allow ourselves to drop $\mathcal{S}_A$ from the notation since in each case a variety A *will be equipped with an explicitly specified stratification.*

If A, B are two stratified varieties, we have an exact exterior tensor product functor $\boxtimes : \mathcal{M}(A) \times \mathcal{M}(B) \longrightarrow \mathcal{M}(A \times B)$. Here $A \times B$ is equipped with the product stratification. This functor induces an equivalence $\mathcal{M}(A) \otimes \mathcal{M}(B) \xrightarrow{\sim} \mathcal{M}(A \times B)$.

$*$ will denote a one-element set, as well as its unique element.

Throughout the paper we fix a non-zero complex parameter $\kappa \notin \mathbb{Q}$.

For the convenience of the reader, we list below the main notations concerning configuration spaces, together with the places where they are introduced.

$\mathbb{A}^J$: 2.2. We will use two stratifications on these spaces: $\mathcal{S}$, defined in 2.2 and $\mathcal{S}_{diag}$, defined in 4.1. $\mathbb{A}^{J\bullet}, \mathbb{A}^{Jo}$: 2.2.

$\mathbb{A}^\nu, \mathbb{A}^{\nu\bullet}, \mathbb{A}^{\nu o}$: 2.2.

$\mathbb{A}(K)$: 3.1.

$\mathbb{A}^J(K), \mathbb{A}^J(K)^\bullet, \mathbb{A}^J(K)^o$: 3.3, see also 4.8.

$\mathbb{A}^\nu(K), \mathbb{A}^\nu(K)^\bullet, \mathbb{A}^\nu(K)^o$: 3.3.

1.4. We are grateful to M.Finkelberg who has explained to us that the Malgrange — Kashiwara specialization serves as a $\mathcal{D}$-module counterpart of the topological construction from [FS1].

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2. Definition of the category $\mathcal{FM}_\kappa$

This section is devoted to the definition of the category $\mathcal{FM}_\kappa$.

2.1. We will use the Lusztig's notations for root systems, cf. [L]. Throughout this section, we fix a Cartan datum $(I, \cdot)$ of finite type. Let $(Y, X, \dots)$ be the simply connected root datum of type $(I, \cdot)$. Thus, I is a finite set with a symmetric bilinear form $\nu_1, \nu_2 \mapsto \nu_1 \cdot \nu_2$ on the free abelian group $\mathbb{Z}[I]$ satisfying the known properties, $Y = \mathbb{Z}[I]$, $X =$

$\mathrm{Hom}(Y, \mathbb{Z})$. The brackets $\langle \cdot, \cdot \rangle : Y \times X \longrightarrow \mathbb{Z}$ will denote the obvious pairing; the obvious embedding $I \hookrightarrow Y$ will be denoted $i \mapsto i$, we have an embedding $Y \subset X$ given by $i \mapsto i' := 2i \cdot i/i \cdot j$; we will denote by the same letter $\nu \in Y$ and its image in X. We set $Y^+ := \mathbb{N}[I] \subset Y$.

We will use the following partial order on X: for $\mu_1, \mu_2 \in X$ we write $\mu_1 \leq \mu_2$ iff $\mu_2 - \mu_1 \in Y^+$.

2.2. **Divisor spaces.** Let $\mathbf{A}^1$ denote the complex affine line. For a finite set J, let $\mathbf{A}^J$ denote the J-fold cartesian power of $\mathbf{A}^1$. We fix a coordinate t on $\mathbf{A}^1$; this provides $\mathbf{A}^J$ with the coordinates $\{t_j\}, j \in J$. For $\nu = \sum \nu_i i \in Y^+$, let us call an *unfolding* of ν a map $\pi : J \longrightarrow I$ such that $|\pi^{-1}(i)| = \nu_i$ for each i; we define a group $\Sigma_\pi := \{\sigma : J \xrightarrow{\sim} J | \ \pi \circ \sigma = \pi\}$.

We define the space $\mathbf{A}^\nu := \mathbf{A}^J/\Sigma_\pi$. Here π is an unfolding of ν, the group Σ_π operates on the space $\mathbf{A}^J$ by permutations of coordinates. We will denote by the same letter π the canonical projection $\mathbf{A}^J \longrightarrow \mathbf{A}^\nu$.

The space $\mathbf{A}^\nu$ does not depend on the choice of an unfolding. Points of $\mathbf{A}^\nu$ are finite formal linear combinations $\sum \nu_a x_a, \nu_a \in Y^+, \ x_a \in \mathbf{A}^1$ with $\sum \nu_a = \nu$. Therefore one can think of the disjoint union

$$\mathrm{Div}^+(\mathbf{A}^1; Y) := \coprod_{\nu \in Y^+} \mathbf{A}^\nu$$

as of the space of nonnegative Y-valued divisors on $\mathbf{A}^1$.

We equip $\mathbf{A}^J$ with a stratification $\mathcal{S}$. By definition, the closures of its strata are all nonempty intersections of the hyperplanes $t_i = 0$, $t_i = t_j$ $(i \neq j$ in $I)$. Define the open subspaces

$$\mathbf{A}^{J\bullet} := \{(t_j) \in \mathbf{A}^J | \ t_j \neq 0 \text{ for all } j \in J\}$$

and

$$\mathbf{A}^{Jo} := \{(t_j) \in \mathbf{A}^{J\bullet} | \ t_{j'} \neq t_{j''} \text{ for all } j' \neq j'' \text{ in } J\}.$$

The last space is the unique open stratum of $\mathcal{S}$.

If $\pi : J \longrightarrow I$ is an unfolding of ν, we will denote by $\mathbf{A}^{\nu\bullet}$ (resp., by $\mathbf{A}^{\nu o}$) the image of $\mathbf{A}^{J\bullet}$ (resp., of $\mathbf{A}^{Jo}$) under the canonical projection; these subspaces do not depend on the choice of an unfolding.

We will equip the spaces $\mathbf{A}^{J\bullet}$, $\mathbf{A}^\nu$, etc. with the induced stratifications.

2.3. Let $\pi : \mathbf{A}^J \longrightarrow \mathbf{A}^\nu$ be the projection corresponding to an unfolding of ν. The morphism π is finite, surjective and flat; therefore we have two exact adjoint functors

$$\pi_* : \mathcal{M}(\mathbf{A}^J) \rightleftharpoons \mathcal{M}(\mathbf{A}^\nu) : \pi^*$$

For $M \in \mathcal{M}(\mathbf{A}^\nu)$, the adjunction morphism $M \longrightarrow \pi_*\pi^* M$ identifies M with the submodule of Σ_π-invariants $(\pi_*\pi^* M)^{\Sigma_\pi}$.

Note that all our varieties $\mathbf{A}^J, \mathbf{A}^\nu$, etc. are affine, hence $\mathcal{D}$-affine.

2.4. **Specialization functors.** Suppose that a finite set J is represented as a disjoint union $J = J_1 \coprod J_2$. Then we have a functor

$$\mathrm{Sp}_{J_1,J_2} : \mathcal{M}(\mathbb{A}^J) \longrightarrow \mathcal{M}(\mathbb{A}^{J_1} \times \mathbb{A}^{J_2 \bullet})$$

It is defined as a composition of the Kashiwara specialization along the closed submanifold given by the equations $\{t_j = 0\}$ $(j \in J_1)$ (cf. [K]) which will live on the product $\mathbb{A}^{J_1} \times \mathbb{A}^{J_2}$, and the restriction to the open subspace. In the analytical picture, $\mathrm{Sp}_{J_1,J_2}(M)$ is just the restriction of M to the "asymptotic zone" $|t_{j_1}| << |t_{j_2}|$ $(j_i \in J_i)$.

These functors enjoy the following fundamental properties.

(a) (*Unit*) $\mathrm{Sp}_{J,\emptyset} = \mathrm{Id}$; $\mathrm{Sp}_{\emptyset,J}$ coincides with the restriction.

(b) (*Associativity*, or 2-*cocycle property*) If $J = J_1 \coprod J_2 \coprod J_3$, we have a natural isomorphism

$$\alpha_{J_1,J_2,J_3} : \mathrm{Sp}_{J_1,J_2} \circ \mathrm{Sp}_{J_1 \coprod J_2,J_3} \xrightarrow{\sim} \mathrm{Sp}_{J_2,J_3} \circ \mathrm{Sp}_{J_1,J_2 \coprod J_3}.$$

of functors $\mathcal{M}(\mathbb{A}^J) \longrightarrow \mathcal{M}(\mathbb{A}^{J_1} \times \mathbb{A}^{J_2 \bullet} \times \mathbb{A}^{J_3 \bullet})$.

(c) (3-*cocycle property for* α*'s*) For $J = J_1 \coprod J_2 \coprod J_3 \coprod J_4$, we have an equality

$$\alpha_{J_2,J_3,J_4} \circ \alpha_{J_1,J_2 \coprod J_3,J_4} \circ \alpha_{J_1,J_2,J_3} = \alpha_{J_1 \coprod J_2,J_3,J_4} \circ \alpha_{J_1,J_2,J_3 \coprod J_4}$$

of the natural isomorphisms

$$\mathrm{Sp}_{J_1,J_2} \circ \mathrm{Sp}_{J_1 \coprod J_2,J_3} \circ \mathrm{Sp}_{J_1 \coprod J_2 \coprod J_3,J_4} \xrightarrow{\sim} \mathrm{Sp}_{J_3,J_4} \circ \mathrm{Sp}_{J_2,J_3 \coprod J_4} \circ \mathrm{Sp}_{J_1,J_2 \coprod J_3 \coprod J_4}.$$

2.4.1. *Remark.* The interested reader may try to draw the full diagrams involved. He would encounter the first *permutoedra*, cf. [MS]. □

The property (c) guarantees that all possible iterations of the specialization functors are canonically equivalent (*MacLane's coherence*). Thus, we can, and will, simply identify them, i.e. pretend that the isomorphisms α are identities.

The functors Sp_{J_1,J_2} induce the functors

$$\mathrm{Sp}_{\nu_1,\nu_2} : \mathcal{M}(\mathbb{A}^{\nu_1+\nu_2}) \longrightarrow \mathcal{M}(\mathbb{A}^{\nu_1}) \times \mathcal{M}(\mathbb{A}^{\nu_2 \bullet})$$

satisfying the similar associativity property.

Iterating, we get the functors

$$\mathrm{Sp}_{\nu_1,\dots,\nu_n} : \mathcal{M}(\mathbb{A}^{\nu_1+\dots+\nu_n}) \longrightarrow \mathcal{M}(\mathbb{A}^{\nu_1}) \times \prod_{i=2}^{n} \mathcal{M}(\mathbb{A}^{\nu_i \bullet}).$$

2.5. **Cartan $\mathcal{D}$-modules.** Let $\mu \in X, \nu \in Y^+$. Choose an unfolding $\pi : J \longrightarrow I$ of ν. Let $\tilde{\mathcal{I}}^{\pi}_{\mu}$ be the lisse $\mathcal{D}$-module over $\mathbb{A}^{Jo}$ given by an integrable connection on $\mathcal{O}_{\mathbb{A}^J}$ with the connection form

$$\sum_{j \in J} -\frac{\pi(j) \cdot \mu}{\kappa} \mathrm{dlog}\, t_j + \sum_{j' \neq j''} \frac{\pi(j') \cdot \pi(j'')}{\kappa} \mathrm{dlog}(t_{j'} - t_{j''})$$

The $\mathcal{D}$-module $\tilde{\mathcal{I}}^{\pi}_{\mu}$ admits an obvious Σ_{π}-equivariant structure. We define a $\mathcal{D}$-module $\mathcal{I}^{\nu}_{\mu}$ over $\mathbb{A}^{\nu o}$ as

$$\mathcal{I}^{\nu}_{\mu} := (\pi_* \tilde{\mathcal{I}}^{\pi}_{\mu})^{\Sigma_{\pi},-}.$$

Here the superscript $\Sigma_{\pi}, -$ denotes the submodule of skew Σ_{π}-invariants.

These $\mathcal{D}$-modules enjoy the following basic *factorization property.*

2.5.1. *For all $\nu_1, \nu_2 \in Y^+$, there are canonical* factorization isomorphisms

$$\phi_{\mu}(\nu_1, \nu_2): \ \mathrm{Sp}_{\nu_1,\nu_2} \mathcal{I}^{\nu_1+\nu_2}_{\mu} \xrightarrow{\sim} \mathcal{I}^{\nu_1}_{\mu} \boxtimes \mathcal{I}^{\nu_2}_{\mu-\nu_1}.$$

These isomorphisms satisfy the **associativity property***: for all $\nu_1, \nu_2, \nu_3 \in Y^+$, we have the equality*

$$\phi_{\mu}(\nu_1, \nu_2)\phi_{\mu}(\nu_1 + \nu_2, \nu_3) = \phi_{\mu-\nu_1}(\nu_2, \nu_3)\phi_{\mu}(\nu_1, \nu_2 + \nu_3)$$

of isomorphisms

$$\mathrm{Sp}_{\nu_1,\nu_2,\nu_3} \mathcal{I}^{\nu_1+\nu_2+\nu_3}_{\mu} \xrightarrow{\sim} \mathcal{I}^{\nu_1}_{\mu} \boxtimes \mathcal{I}^{\nu_2}_{\mu-\nu_1} \boxtimes \mathcal{I}^{\nu_3}_{\mu-\nu_1-\nu_2}$$

Let $\mathcal{I}^{\nu\bullet}_{\mu}$ denote the Deligne-Goresky-MacPherson extension of $\mathcal{I}^{\nu}_{\mu}$ to the space $\mathbb{A}^{\nu\bullet}$. By functoriality, we have the factorization isomorphisms (to be denoted by the same letters)

$$\phi_{\mu}(\nu_1, \nu_2): \ \mathrm{Sp}_{\nu_1,\nu_2} \mathcal{I}^{\nu_1+\nu_2\bullet}_{\mu} \xrightarrow{\sim} \mathcal{I}^{\nu_1\bullet}_{\mu} \boxtimes \mathcal{I}^{\nu_2\bullet}_{\mu-\nu_1}$$

which enjoy the associativity property.

2.6. **Factorizable $\mathcal{D}$-modules.** Let us fix a coset $c \in X/Y$. A *factorizable $\mathcal{D}$-module supported at c* is a collection $\mathcal{M}$ of data (a), (b), (c) below.

(a) An element $\mu = \mu(\mathcal{M}) \in c$.

(b) $\mathcal{D}$-modules $\mathcal{M}^{\nu} \in \mathcal{M}(\mathbb{A}^{\nu})$ $(\nu \in Y^+)$.

(c) Isomorphisms $\psi(\nu_1, \nu_2): \ \mathrm{Sp}_{\nu_1,\nu_2} \mathcal{M}^{\nu_1+\nu_2} \xrightarrow{\sim} \mathcal{M}^{\nu_1} \boxtimes \mathcal{I}^{\nu_2}_{\mu-\nu_1}$ $(\nu_1, \nu_2 \in Y^+)$.

These isomorphisms are called *factorization isomorphisms.* They must satisfy the *associativity property* (d) below.

(d) For all $\nu_1, \nu_2, \nu_3 \in Y^+$, $\psi(\nu_1, \nu_2)\psi(\nu_1 + \nu_2, \nu_3) = \phi_{\mu-\nu_1}(\nu_2, \nu_3)\psi(\nu_1, \nu_2 + \nu_3)$.

2.7. Let $\iota_{\nu_1,\nu_2}: \mathbb{A}^{\nu_1} \hookrightarrow \mathbb{A}^{\nu_1+\nu_2}$ denote the closed embedding adding ν_2 points sitting at the origin $O \in \mathbb{A}^1$.

Let $\mathcal{M} = (\mu, \mathcal{M}^{\nu}, \ldots)$ be a factorizable $\mathcal{D}$-module supported at $c \in X/Y$. For each $\mu' \geq \mu$, $\nu \in Y^+$, define a $\mathcal{D}$-module $\mathcal{M}^{\nu}_{\mu'} \in \mathcal{M}(\mathbb{A}^{\nu})$ by

$$\mathcal{M}^{\nu}_{\mu'} = \begin{cases} \iota_{\nu-\mu'+\mu,\mu'-\mu *} \mathcal{M}^{\nu-\mu'+\mu} & \text{if } \nu - \mu' + \mu \in Y^+ \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{N} = (\mu', \mathcal{N}^\nu, \dots)$ be another factorizable $\mathcal{D}$-module supported at c. For $\lambda \in X$, $\lambda \geq \mu$, $\lambda \geq \mu'$ and $\nu \geq \nu'$ in Y^+, consider the composition

$$\tau_\lambda^{\nu,\nu'} : \mathrm{Hom}(\mathcal{M}_\lambda^\nu, \mathcal{N}_\lambda^\nu) \xrightarrow{\mathrm{Sp}} \mathrm{Hom}(\mathrm{Sp}_{\nu',\nu'-\nu}\mathcal{M}_\lambda^\nu, \mathrm{Sp}_{\nu',\nu'-\nu}\mathcal{N}_\lambda^{\nu'})$$

$$\xrightarrow{\psi}_{\sim} \mathrm{Hom}(\mathcal{M}_\lambda^{\nu'} \boxtimes \mathcal{I}_{\lambda-\nu'}^{\nu-\nu'\bullet}, \mathcal{N}_\lambda^{\nu'} \boxtimes \mathcal{I}_{\lambda-\nu'}^{\nu-\nu'\bullet}) = \mathrm{Hom}(\mathcal{M}_\lambda^{\nu'}, \mathcal{N}_\lambda^{\nu'}).$$

Let us define the space $\mathrm{Hom}(\mathcal{M}, \mathcal{N})$ as the double limit

$$\mathrm{Hom}(\mathcal{M}, \mathcal{N}) := \lim_{\rightarrow\,\lambda} \lim_{\leftarrow\,\nu} \mathrm{Hom}(\mathcal{M}_\lambda^\nu, \mathcal{N}_\lambda^\nu).$$

Here the inverse limit is taken over Y^+ with its partial order, the transtition maps being $\tau_\lambda^{\nu,\nu'}$, and the direct limit is taken over the set of all $\lambda \in X$ such that $\lambda \geq \mu$, $\lambda \geq \mu'$, the transition maps being induced by the obvious isomorphisms

$$\mathrm{Hom}(\mathcal{M}_\lambda^\nu, \mathcal{N}_\lambda^\nu) = \mathrm{Hom}(\mathcal{M}_{\lambda+\nu'}^{\nu+\nu'}, \mathcal{N}_{\lambda+\nu'}^{\nu+\nu'}).$$

With this definition of morphisms, factorizable $\mathcal{D}$- modules supported at c form a category, to be denoted by $\widetilde{\mathcal{FM}}_{\kappa;c}$. The composition of morphisms is defined in the obvious manner. We define the category $\widetilde{\mathcal{FM}}_\kappa$ as the direct product $\prod_{c\in X/Y} \widetilde{\mathcal{FM}}_{\kappa;c}$.

2.8. **Finite modules.** Let us call a factorizable $\mathcal{D}$-module $\mathcal{M} = (\mathcal{M}^\nu, \dots) \in \widetilde{\mathcal{FM}}_{\kappa;c}$ *finite* if there exists only a finite number of $\nu \in Y^+$ such that the conormal bundle of the origin $O \in \mathbb{A}^\nu$ is contained in the singular support of $\mathcal{M}^\nu$. Let $\mathcal{FM}_{\kappa;c} \subset \widetilde{\mathcal{FM}}_{\kappa;c}$ be the full subcategory of finite factorizable modules. We define the category $\mathcal{FM}_\kappa$ by $\mathcal{FM}_\kappa := \prod_{c\in X/Y} \mathcal{FM}_{\kappa;c}$.

Obviously, $\mathcal{FM}_\kappa$ is an additive category. In fact, it is an abelian artinian category.

2.9. The duality functor for holonomic $\mathcal{D}$-modules induces an equivalence $D : \mathcal{FM}_\kappa^{\mathrm{opp}} \xrightarrow{\sim} \mathcal{FM}_{-\kappa}$.

3. Tensor structure

In this section the braided tensor structure on the category $\mathcal{FM}_\kappa$ is introduced. At the beginning we recall the Deligne's definition of a braided tensor category, using the language of specialization.

3.1. For a finite set K, set $\mathbb{A}(K) := \{(z_k) \in \mathbb{A}^K \mid z_k \neq z_l \text{ for all } k \neq l \text{ in } K\}$. We equip this space with the trivial stratification; thus, $\mathcal{M}(\mathbb{A}(K))$ will consist of lisse $\mathcal{D}$-modules.

For a surjective map $\rho : K \longrightarrow L$, we have a specialization functor

$$\mathrm{Sp}_\rho : \mathcal{M}(\mathbb{A}(K)) \longrightarrow \mathcal{M}(\mathbb{A}(L) \times \prod_L \mathbb{A}(K_l)).$$

Here $K_l := \rho^{-1}(l)$.

(a) If $\rho:\ K \longrightarrow L$, $\tau:\ L \longrightarrow M$ are two surjective maps, we have a natural isomorphism

$$\alpha_{\rho,\tau}:\ \mathrm{Sp}_\tau \mathrm{Sp}_\rho \xrightarrow{\sim} (\prod_M \mathrm{Sp}_{\rho_m})\mathrm{Sp}_{\tau\rho}$$

Here $\rho_m := \rho|\ K_m$. The LHS denotes the composition

$$\mathcal{M}(\mathbb{A}(K)) \longrightarrow \mathcal{M}(\mathbb{A}(L) \times \prod_L \mathbb{A}(K_l)) \longrightarrow \mathcal{M}(\mathbb{A}(M) \times \prod_M \mathbb{A}(L_m)\ \times \prod_L \mathbb{A}(K_l))$$

and the RHS — the composition

$$\mathcal{M}(\mathbb{A}(K)) \longrightarrow \mathcal{M}(\mathbb{A}(M) \times \prod_M \mathbb{A}(K_m)) \longrightarrow \mathcal{M}(\mathbb{A}(M) \times \prod_M (\mathbb{A}(L_m) \times \prod_{L_m} \mathbb{A}(K_l))).$$

Note that $\prod_M\ (\mathbb{A}(L_m) \times \prod_{L_m}\ \mathbb{A}(K_l)) = \prod_M\ \mathbb{A}(L_m) \times \prod_L\ \mathbb{A}(K_l)$.

The isomorphisms satisfy a "two-cocycle" property; we leave it to the reader to write it down. Due to this property, we will identify the both sides of (a), as in 2.4.

In other words, the categories $\mathcal{M}(\mathbb{A}(K))$ for various K form a "2-operad". We have borrowed the operadic notations from [BD].

3.2. **Braided tensor structures.** The formalism below is due to Deligne, [De].

Let $\mathcal{C}$ be a category; let T be a topological space. It is clear what a *presheaf* $\mathcal{F}$ *on* T *with values in* C is. For each $X \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, \mathcal{F})$ is a presheaf of sets on T. A presheaf $\mathcal{F}$ is called a *sheaf* if for every $X \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, \mathcal{F})$ is a sheaf.

Assume that T is locally connected and locally simply connected. Each object $X \in \mathcal{C}$ defines a constant sheaf $X_{\mathcal{C}}$; by definition, for a connected open $U \subset T$, $\Gamma(U; X_{\mathcal{C}}) = X$. A *local system* on T with values in $\mathcal{C}$ is a sheaf locally isomorphic to a constant sheaf.

For local systems on the spaces $\mathbb{A}(K)$ with values in $\mathcal{C}$, we have the same formalism of specialization as in the previous subsection, with $\mathcal{M}(\mathbb{A}(K))$ replaced by the categories $\mathcal{M}_{\mathcal{C}}(\mathbb{A}(K))$ of local systems on $\mathbb{A}(K)$ with values in $\mathcal{C}$.

A *braided tensor structure* on $\mathcal{C}$ consists of the data (a), (b) below.

(a) A local system $\otimes_K\ X_k\ \in \mathcal{M}_{\mathcal{C}}(\mathbb{A}(K))$, given for any finite set K and a K-tuple $\{X_k\}$ $(X_k \in \mathcal{C})$.

(b) *Factorization isomorphisms.* A natural isomorphism $\psi_\rho:\ \mathrm{Sp}_\rho(\otimes_K X_k) \xrightarrow{\sim} \otimes_L(\otimes_{K_l} X_k)$, given for any surjective map $\rho:\ K \longrightarrow L$ and a K-tuple as above.

The isomorphisms ψ_ρ must satisfy the

(c) **Associativity axiom.** For any pair of surjective maps $K \xrightarrow{\rho} L \xrightarrow{\tau} M$ and a K-tuple as above, the two compositions

$$\mathrm{Sp}_\tau \mathrm{Sp}_\rho(\otimes_K X_k) \xrightarrow{\psi_\rho} \mathrm{Sp}_\tau(\otimes_L(\otimes_{K_l} X_k)) \xrightarrow{\mathrm{Sp}_\tau} \otimes_M(\otimes_{L_m}(\otimes_{K_l} X_k))$$

and

$$\mathrm{Sp}_\tau \mathrm{Sp}_\rho(\otimes_K X_k) = \prod_M \mathrm{Sp}_{\rho_m} \circ \mathrm{Sp}_{\tau\rho}(\otimes_K X_k) \xrightarrow{\psi_{\tau\rho}} \prod_M \mathrm{Sp}_{\rho_m}(\otimes_M(\otimes_{K_m} X_k)) =$$

$$= \otimes_M \mathrm{Sp}_{\rho_m}(\otimes_{K_m} X_k) \xrightarrow{\otimes\psi_{\rho_m}} \otimes_M(\otimes_{L_m}(\otimes_{K_l} X_k))$$

coincide.

Now we are going to introduce a setup which is a common generalization of 2.4 and 3.2.

3.3. For a finite set K and $\nu \in Y^+$, consider a manifold $\mathbb{A}^\nu(K) := \mathbb{A}^\nu \times \mathbb{A}(K)$. Let $\pi : J \longrightarrow K$ be an unfolding of ν. We set $\mathbb{A}^J(K) := \mathbb{A}^J \times \mathbb{A}(K)$.

We equip $\mathbb{A}^J(K)$ with a stratification $\mathcal{S}$ the closures of whose strata are all nonempty intersections of hyperplanes $t_j = z_k$, $t_{j'} = t_{j''}$ $(j' \neq j'')$; we denote by the same letter $\mathcal{S}$ the image of $\mathcal{S}$ under the projection $\pi : \mathbb{A}^J(K) \longrightarrow \mathbb{A}^\nu(K)$.

We set $\mathbb{A}^J(K)^\bullet := \{(t_j) |\ t_j \neq z_k \text{ for all } j,k\}$; $\mathbb{A}^\nu(K)^\bullet := \pi(\mathbb{A}^J(K)^\bullet) \subset \mathbb{A}^\nu(K)$. Let $\mathbb{A}^J(K)^o \subset \mathbb{A}^J(K)^\bullet$ (resp., $\mathbb{A}^\nu(K)^o \subset \mathbb{A}^\nu(K)^\bullet$) be the open strata of $\mathcal{S}$.

3.4. (a) If $\rho : K \longrightarrow L$ is a surjective map of finite sets we have the specialization functor

$$\mathrm{Sp}_\rho :\ \mathcal{M}(\mathbb{A}^\nu(K); \mathcal{S}) \longrightarrow \mathcal{M}(\mathbb{A}^\nu(L) \times \prod_L \mathbb{A}(K_l); \mathcal{S}).$$

These functors satisfy the operadic associativity property, as in 3.1 (a).

In particular, we have a functor

$$\mathrm{Sp}_{e_K} :\ \mathcal{M}(\mathbb{A}^\nu(K); \mathcal{S}) \longrightarrow \mathcal{M}(\mathbb{A}^\nu(*) \times \mathbb{A}(K); \mathcal{S})$$

corresponding to the map $K \longrightarrow *$.

(b) If $\vec{\nu} = (\{\nu_k\} \in Y^{+K}$ and $\nu_0 \in Y^+$ we have the specialization functor

$$\mathrm{Sp}_{\vec{\nu};\nu_0} :\ \mathcal{M}(\mathbb{A}^\nu(K); \mathcal{S}) \longrightarrow \mathcal{M}(\prod_K \mathbb{A}^{\nu_k} \times \mathbb{A}^{\nu_0}(K)^\bullet; \mathcal{S}).$$

Here $\nu = \sum_K \nu_k + \nu_0$. These functors satisfy the associativity property, as in 2.4.

3.5. For a K-tuple $\vec{\mu} \in X^K$, define a lisse $\mathcal{D}$- module $\mathcal{I}^\nu_{\vec{\mu}}$ over $\mathbb{A}^\nu(K)^o$ as follows. Choose an unfolding $\pi : J \longrightarrow I$ of ν; let $\mathcal{I}^\pi_{\vec{\mu}}$ be the $\mathcal{D}$-module over $\mathbb{A}^J(K)^o$ corresponding to the integrable connection on its structure sheaf given by the form

$$\sum_{k' \neq k''} \frac{\mu_{k'} \cdot \mu_{k''}}{\kappa} \mathrm{dlog}(z_{k'} - z_{k''}) + \sum_{j \in J,\ k \in K} -\frac{\pi(j) \cdot k}{\kappa} \mathrm{dlog}(t_{\pi(j)} - z_k) + \sum_{j' \neq j''} \frac{\pi(j') \cdot \pi(j'')}{\kappa} \mathrm{dlog}(t_{j'} - t_{j''})$$

We set by definition,

$$\mathcal{I}^\nu_{\vec{\mu}} := (\pi_* \mathcal{I}^\pi_{\vec{\mu}})^{\Sigma_\pi, -}.$$

Here π denotes the projection $\mathbb{A}^J(K) \longrightarrow \mathbb{A}^\nu(K)$.

These $\mathcal{D}$-modules satisfy the two *factorization properties* corresponding to the specializations (a) and (b) above.

(a) We have isomorphisms

$$\phi_\rho : \mathrm{Sp}_\rho(\mathcal{I}^\nu_{\vec{\mu}}) \xrightarrow{\sim} \mathcal{I}^\nu_{\rho_*\vec{\mu}} \boxtimes \boxed{\times}_L \mathcal{I}^0_{\vec{\mu}_l}.$$

Here $\rho_*\vec{\mu}$ denotes the L-tuple $\{\mu_l\}$ with $\mu_l = \sum_{K_l} \mu_k$, and $\vec{\mu}_l := \{\mu_k\}_{k\in K_l}$.

(b) We have isomorphisms

$$\phi_{\vec{\nu};\nu_0} : \mathrm{Sp}_{\vec{\nu};\nu_0}\mathcal{I}^\nu_{\vec{\mu}} \xrightarrow{\sim} \boxed{\times}_K \mathcal{I}^{\nu_k}_{\mu_k} \boxtimes \mathcal{I}^{\nu_0}_{\vec{\mu}-\vec{\nu}}.$$

The isomorphisms (a) and (b) satisfy the associativity properties.

We denote by $\mathcal{I}^{\nu\bullet}_{\vec{\mu}}$ the Deligne-Goresky-MacPherson extension of $\mathcal{I}^\nu_{\vec{\mu}}$ to the space $\mathbb{A}^\nu(K)$. These $\mathcal{D}$-modules are also connected by factorization isomorphisms.

3.6. Let $\{\mathcal{FM}_k = (\mu_k, \mathcal{FM}^\nu_k, \dots)\}$ $(k \in K)$ be a K-tuple of factorizable $\mathcal{D}$-modules.

(a) *Exterior tensor product* $\boxed{\times}_K \mathcal{M}_k$ is by definition a collection of $\mathcal{D}$-modules $(\boxed{\times}_K \mathcal{M}_k)^\nu \in \mathcal{M}(\mathbb{A}^\nu(K); \mathcal{S})$ $(\nu \in Y^+)$. Namely, $(\boxed{\times}_K \mathcal{M}_k)^\nu$ is the unique $\mathcal{D}$-module such that for all decompositions $\nu = \sum_K \nu_k + \nu_0$, one has isomorphisms

$$\mathrm{Sp}_{\vec{\nu};\nu_0}(\boxed{\times}_K \mathcal{M}_k)^\nu \equiv \boxed{\times}_K \mathcal{M}^{\nu_k} \boxtimes \mathcal{I}^{\nu_0}_{\vec{\mu}-\vec{\nu}}(K),$$

these isomorphisms satisfying a cocycle condition. This is the same as the gluing of a sheaf from sheaves given on an open covering, together with the isomorphisms on double intersections.

These $\mathcal{D}$-modules for different ν are connected by the obvious factorization isomorphisms satisfying the associativity property.

(b) *The K-fold tensor product* $\otimes_K \mathcal{M}_k$ is the collection of $\mathcal{D}$-modules $(\otimes_K \mathcal{M}_k)^\nu \in \mathcal{M}(\mathbb{A}^\nu(*) \times \mathbb{A}(K); \mathcal{S})$ where

$$(\otimes_K \mathcal{M}_k)^\nu := \mathrm{Sp}_{e_K}(\boxed{\times}_K \mathcal{M}_k)^\nu.$$

These $\mathcal{D}$-modules are connected by the factorization isomorphisms. After taking the sheaf of solutions along $\mathbb{A}(K)$, this collection maybe regarded as a local system of factorizable $\mathcal{D}$-modules over $\mathbb{A}(K)$. When the set K varies, these local systems satisfy in turn a factorization property, thus defining a tensor structure on $\mathcal{FM}$.

4. $\mathcal{D}$-MODULES AND QUIVERS

This section contains the description of certain $\mathcal{D}$-module categories on configuration spaces, in terms of linear algebra data. The main results are theorems 4.6, and 4.12, complemented by 4.4. These results (which may be of interest for their own sake) are the main technical tools in our proof of the Equivalence theorem 5.2 below.

4.1. Let us consider an affine space $\mathbb{A}^I$ with fixed coordinates $\{t_i\}$ $(i \in I)$, I being a finite set. Let $\mathcal{S}_{diag}$ be a stratification of $\mathbb{A}^I$ the closures of whose strata are all non-empty intersections of the diagonal hyperplanes $t_i = t_j$ $(i \neq j)$. Throughout this section, we will imply this stratification when we speak about the category $\mathcal{M}(\mathbb{A}^I)$. (Pay attention that $\mathcal{S}_{diag}$ is different from the stratification $\mathcal{S}$ used in Section 2.)

Let $Q(I)$ denote the set of *quotients* of I, i.e. of classes of surjective maps $\alpha : I \longrightarrow J$, the two maps $\alpha : I \longrightarrow J$ and $\alpha' : I \longrightarrow J'$ defining the same element of $Q(I)$ iff there exists a bijection $\sigma : J \xrightarrow{\sim} J'$ such that $\alpha' = \sigma\alpha$. Abusing the notation, we will not distinguish between a map α and its class in $Q(I)$. The set $Q(I)$ is equipped with the following partial order: for $\alpha : I \longrightarrow J$, $\beta : I \longrightarrow K$ we write $\beta \leq \alpha$ iff there exists a surjective map $\gamma : J \longrightarrow J'$ such that $\beta = \gamma\alpha$.

Let assign to an element $\alpha \in Q(I)$ a stratum S_α whose closure is equal to $\{(t_i) \in \mathbb{A}^I \mid t_i = t_j \text{ if } \alpha(i) = \alpha(j)\}$. This way we get a bijection between $Q(I)$ and the set of all strata. We have $\dim S_\alpha = |J|$ if $\alpha : I \longrightarrow J$, and $\alpha \leq \beta$ iff $S_\alpha \subset \overline{S}_\beta$.

For $(\alpha : I \longrightarrow J) \in Q(I)$ and $i \neq j$ in J, set $J_{ij} := J - \{i, j\} \coprod *$; let $\gamma_{ij} : J \longrightarrow J_{ij}$ be map sending i and j to $*$ and k to k for every k not equal to i and j. We will regard J_{ij} as J with i and j identified, and the image of i and j in J_{ij} will also be denoted by either i or j. Set $\alpha_{ij} := \gamma_{ij}\alpha : I \longrightarrow J_{ij}$. The stratum $S_{\alpha_{ij}}$ has codimension one in $\overline{S}_\alpha$, and this way we get all codimension one adjunctions.

Let us define the category $\mathrm{Qui}(\mathbb{A}^I)$ as follows. Its objects are the collections of data (a), (b) below.

(a) For each stratum S_α, a finite dimensional vector space V_α.

(b) For each pair of codimension one adjacent strata $(S_\alpha,\ S_{\alpha_{ij}})$ and an ordering (i, j) of the set $\{i, j\}$, a pair of linear operators $a^\alpha_{ij} : V_\alpha \longrightarrow V_{\alpha_{ij}}$ and $b^\alpha_{ij} : V_{\alpha_{ij}} \longrightarrow V_\alpha$.

These operators should satisfy the relations (c) — (h) below.

(c) $a^\alpha_{ij} = -a^\alpha_{ji}$; $b^\alpha_{ij} = -b^\alpha_{ji}$.

(d) For $(\alpha : I \longrightarrow J) \in Q(I)$ and pairwise distinct i, j, k, l in J,

$$a_{ij}a^\alpha_{kl} = a_{kl}a^\alpha_{ij};\ b^\alpha_{ij}b_{kl} = b^\alpha_{kl}b_{ij}.$$

(e) For α as above, and pairwise distinct i, j, k in J,

$$a_{ij}a^\alpha_{jk} + a_{jk}a^\alpha_{ki} + a_{ki}a^\alpha_{ij} = 0;\ b^\alpha_{jk}b_{ij} + b^\alpha_{ki}b_{jk} + b^\alpha_{ij}b_{ki} = 0.$$

(f) For α, i, j, k, l as in (d),

$$a_{ij}b^\beta_{kl} = b_{kl}a^\beta_{ij}.$$

Here $\beta = \alpha_{kl}$.

(g) For α, i, j, k as in (e),

$$a_{jk}b^\beta_{ij} = b_{ij}a^\beta_{jk}.$$

Here $\beta = \alpha_{ij}$.

In these relations, we have suppressed the upper indexes in all but one the operators in the products, the remaining ones being restored uniquely.

To formulate the last relation, we need some more notation. Let $\alpha,\ i,j,k.l$ be as in (d). Let $J_{ij;kl}$ (resp., J_{ijkl}) denote the quotient of J obtained by the identifications $i=j;\ k=l$ (resp., $i=j=k=l$); let $\alpha_{ij;kl}$ (resp., α_{ijkl}) denote the composition of α with the quotient map $J \longrightarrow J_{ij;kl}$ (resp., $J \longrightarrow J_{ijkl}$). We have the corresponding operators $a_{ij;kl}:\ V_{ij;kl} \longrightarrow V_{ijkl}$ and $b_{ik;jl}:\ V_{ijkl} \longrightarrow V_{ik;jl}$.

(h) For $\alpha,\ i,j,k,l$ as in (d), $b_{ik;jl}a_{ij;kl} = 0$.

These objects will be called *quivers* (corresponding to the stratification $\mathcal{S}_{diag}$). Morphisms in the category $\mathrm{Qui}(\mathbb{A}^I)$ are defined in the natural way.

4.2. Let us define a *duality functor* $D:\ \mathrm{Qui}(\mathbb{A}^I)^{\mathrm{opp}} \xrightarrow{\sim} \mathrm{Qui}(\mathbb{A}^I)$. For a quiver $\mathcal{V} = (V_\alpha, a^\alpha_{ij}, b^\alpha_{ij})$, we define $D\mathcal{V} = (W_\alpha, {}'a^\alpha_{ij}, {}'b^\alpha_{ij})$ by $W_\alpha = V^*_\alpha$ (the dual vector spaces), ${}'a^\alpha_{ij} = b^{\alpha^*}_{ij}$, ${}'b^{\alpha^*}_{ij} = a^{\alpha^*}_{ij}$. Obviously, D is an equivalence and is involutive.

4.3. We are going to define a functor

$$G:\ \mathrm{Qui}(\mathbb{A}^I) \longrightarrow \mathcal{M}(\mathbb{A}^I).$$

For $(\alpha:\ I \longrightarrow J) \in Q(I)$ and $j \in J$, define a vector field ∂^α_j on $\mathbb{A}^I$ by

$$\partial^\alpha_j = \sum_{i \in \alpha^{-1}(j)} \partial_{t_i}.$$

For a quiver $\mathcal{V} = (V_\alpha, \dots) \in \mathrm{Qui}(\mathbb{A}^I)$, $G(\mathcal{V})$ is by definition the quotient of the free $\mathcal{D}_{\mathbb{A}^I}$-module $\mathcal{D}_{\mathbb{A}^I} \otimes (\oplus_\alpha V_\alpha)$ be the left ideal generated by the relations (a) and (b) below.

(a) For all $(\alpha:\ I \longrightarrow J) \in Q(I),\ j \in J$,

$$\partial^\alpha_j x_\alpha = \sum_{i \neq j} a_{ij}(x_\alpha)\ (x_\alpha \in V_\alpha).$$

(b) For all $(\beta:\ I \longrightarrow K) \in Q(I),\ k \in K,\ p,q \in \beta^{-1}(k)$,

$$(t_p - t_q)x_\beta = \sum b_{ij}(x_\beta)\ (x_\beta \in V_\beta).$$

Here the summation is taken over all $\alpha:\ I \longrightarrow K$ such that $\beta = \alpha_{ij}$ for some $i \neq j$ in J, k is equal to the image of i in K, $p \in \alpha^{-1}(i),\ q \in \alpha^{-1}(j)$.

4.4. **Theorem.** *One has natural isomorphisms* $G(D\mathcal{V}) = DG(\mathcal{V})\ (\mathcal{V} \in \mathrm{Qui}(\mathbb{A}^I))$.

Here D in the RHS is the duality on holonomic $\mathcal{D}$-modules.

4.5. Assume that we are given a set of complex numbers $\vec{\lambda} = \{\lambda_{ij}\}$ $(i \neq j$ in $I)$ such that $\lambda_{ij} = \lambda_{ji}$. For $(\alpha : I \longrightarrow K) \in Q(I)$, set

$$\lambda_\alpha = \frac{1}{2} \sum_{k \in K} (\sum_{i,j \in \alpha^{-1}(k)} \lambda_{ij}).$$

In other words, we have a collection of numbers $\lambda_{ij} = \lambda_H$ assigned to all hyperplanes $H : t_i = t_j$ of our stratification, and

$$\lambda_\alpha = \sum_{H \supset S_\alpha} \lambda_H.$$

For two adjacent strata S_α, S_β $(\alpha : I \longrightarrow J,\ \beta = \alpha_{ij}\ (i \neq j)$ in $J)$, set

$$\lambda^\alpha_{ij} = \lambda_{\alpha\beta} := \sum_{p \in \alpha^{-1}(i),\ q \in \alpha^{-1}(j)} \lambda_{pq}.$$

Let $\mathrm{Qui}_{\vec{\lambda}}(\mathbb{A}^I)$ denote the full subcategory of $\mathrm{Qui}(\mathbb{A}^I)$ consisting of all $\mathcal{V} = (V_\alpha, a^\alpha_{ij}, b^\alpha_{ij})$ such that all the operators $b^\alpha_{ij} a^\alpha_{ij} - \lambda^\alpha_{ij} \mathrm{Id}_{V_\alpha}$ are nilpotent.

The duality induces an equivalence $D : \mathrm{Qui}_{\vec{\lambda}}(\mathbb{A}^I)^{\mathrm{opp}} \xrightarrow{\sim} \mathrm{Qui}_{-\vec{\lambda}}(\mathbb{A}^I)$.

Let us pick some most generic (with respect to our stratification) functions f_α such that $f^{-1}(0) \supset S_\alpha$. Let $\mathcal{M}_{\vec{\lambda}}(\mathbb{A}^I)$ denote the full subcategory of $\mathcal{M}(\mathbb{A}^I)$ consisting of all $\mathcal{D}_{\mathbf{A}^I}$-modules $\mathcal{M}$ such that for all α, the $\mathcal{D}_{\overline{S}_\alpha}$-module $\Phi_{f_\alpha}(\mathcal{M})$ restricted to S_α is isomorphic to the lisse $\mathcal{D}_{S_\alpha}$-module given by an integrable connection on a trivial vector bundle with the connection form

$$\sum \Lambda_{\alpha\beta} \mathrm{dlog}\ f_\beta$$

where $\Lambda_{\alpha\beta}$ are constant linear operators with the unique eigenvalue $\lambda_{\alpha\beta}$ $(S_\beta \subset \overline{S}_\alpha$ of codimension one). This condition does not depend on the choice of functions f_α.

The functor G induces the functor

$$G_{\vec{\lambda}} : \mathrm{Qui}_{\vec{\lambda}}(\mathbb{A}^I) \longrightarrow \mathcal{M}_{\vec{\lambda}}(\mathbb{A}^I).$$

4.6. **Theorem.** *Assume that the* **non-resonance** *assumption* (NR) *below holds true.*

(NR) *For all* $\alpha \in Q(I)$, $\lambda_\alpha \notin (\mathbb{Z} - \{0\})$.

Then the functor $G_{\vec{\lambda}}$ *is an equivalence of categories.*

This theorem is proved by the methods of [Kh1], [Kh2].

4.7. **Remark.** Set $I^* = I \amalg *$. Let us consider the space $\mathbf{A}^{I^*}$ with the diagonal stratification $\mathcal{S}_{diag}$ as above, and the space $\mathbb{A}^I$ with the stratification $\mathcal{S}$ defined in 2.2. We have a closed embedding of stratified varieties $(\mathbf{A}^I, \mathcal{S}) \hookrightarrow (\mathbf{A}^{I^*}, \mathcal{S}_{diag})$ given by the equation $t_* = 0$. It is evident that the pullback induces an equivalence $\mathcal{M}(\mathbf{A}^{I^*}; \mathcal{S}_{diag}) \xrightarrow{\sim} \mathcal{M}(\mathbf{A}^I; \mathcal{S})$.

4.8. Let I, J be finite sets, and set $K := I \coprod J$. Let us consider the stratified space $\mathbb{A}^I(J)$, cf. 3.3. Its strata are numbered by the subset $Q(I;J) := \{(\gamma : K \longrightarrow K' \text{ such that } \gamma|_J \text{ is injective }\}$.

Let us denote by $\mathrm{Qui}(\mathbb{A}^I(J))$ the category whose objects are the collections of data (a) — (c) below.

(a) For each $\gamma : K \longrightarrow K' \in Q(I;J)$, a finite dimensional vector space V_γ.

(b) For each γ as above, and $p \neq q$ in K' such that $\{p,q\} \not\subset \gamma(J)$, a pair of linear operators

$$a^\gamma_{pq} : V_\gamma \longrightarrow V_{\gamma_{pq}} : b^\gamma_{pq}.$$

(c) For each γ as above, and $i \neq j$ in J, an operator $c^\gamma_{ij} : V_\gamma \longrightarrow V_\gamma$.

The relations (d) — (i) below should hold.

(d) The operators a, b must satisfy the relations 4.1 (c) — (h).

(e) For each γ, p, q as in (b), and $i \neq j$ in J such that $\{p,q\} \cap \{\gamma(i), \gamma(j)\} = \emptyset$,

$$a_{pq}c^\gamma_{ij} = c_{ij}a^\gamma_{pq},\ b_{pq}c^\gamma_{ij} = c_{ij}b^\gamma_{pq}\ .$$

(f) For each γ, p, q as in (b), and $i \neq j$ in J such that $p = \gamma(i)$ (then automatically $q \neq \gamma(j)$),

$$c_{ij}a^\gamma_{pq} = a_{pq}(c^\gamma_{ij} + b_{jq}a^\gamma_{jq}),\ (c^\gamma_{ij} + b^\gamma_{jq}a_{jq})b_{pq} = b^\gamma_{pq}c_{ij}\ .$$

(g) For each γ, i, j as in (c),

$$c^\gamma_{ij} = -c^\gamma_{ji}\ .$$

(h) For each $\gamma \in Q(I;J)$ and pairwise distinct i, j, k, l in J,

$$c_{ij}c^\gamma_{kl} = c_{kl}c^\gamma_{ij}\ .$$

(i) For each $\gamma \in Q(I;J)$ and pairwise distinct i, j, k in J,

$$[c^\gamma_{ij}, c^\gamma_{ik} + c^\gamma_{jk}] = 0\ .$$

Morphisms are defined in the natural way.

In particular (setting $I = \emptyset$), the category $\mathrm{Qui}(\mathbf{A}(J))$ is defined. Its object is a vector space V together with endomorphisms $\gamma_{ij} : V \longrightarrow V$ ($i \neq j$ in J) satisfying the *infinitesimal braid relations* (g) — (i) above (one should omit the upper index γ in them).

We have a *restriction functor*

$$r : \mathrm{Qui}(\mathbf{A}^K) \longrightarrow \mathrm{Qui}(\mathbf{A}^I(J))$$

which assignes to a quiver $\mathcal{V} = (V_\alpha, a^\alpha_{ij}, b^\alpha_{ij}) \in \mathrm{Qui}(\mathbf{A}^K)$ a quiver $r(\mathcal{V}) = (W_\gamma, ...)$ with $W_\gamma = V_\gamma$ the operators a, b for $r(\mathcal{V})$ coinciding with the corresponding operators for $\mathcal{V}$, and $c^\gamma_{jj'} := b^\gamma_{\gamma(j)\gamma(j')}a^\gamma_{\gamma(j)\gamma(j')}$.

For a collection of numbers $\vec{\lambda} = \{\lambda_{kk'}\}$ $(k \neq k'$ in $K)$, we define $\mathrm{Qui}_{\vec{\lambda}}(\mathbb{A}^I(J))$ as the full subcategory $r(\mathrm{Qui}_{\vec{\lambda}}(\mathbb{A}^K))$.

4.9. We have a gluing functor

$$G: \ \mathrm{Qui}(\mathbb{A}^I(J)) \longrightarrow \mathcal{M}(\mathbb{A}^I(J)).$$

The restriction functor r above corresponds to the restriction of $\mathcal{D}$- modules.

For $\vec{\lambda}$ as above, G induces a functor $G_\lambda: \ \mathrm{Qui}_{\vec{\lambda}}(\dots) \longrightarrow \mathcal{M}_{\vec{\lambda}}(\dots)$. If $\vec{\lambda}$ satisfies (NR) then $G_{\vec{\lambda}}$ is an equivalence.

4.10. If $\mathrm{Qui}(A_1)$ and $\mathrm{Qui}(A_2)$ are quiver categories corresponding to stratified spaces as above, we define their *tensor product* $\mathrm{Qui}(A_1) \otimes \mathrm{Qui}(A_2)$, also to be denoted by $\mathrm{Qui}(A_1 \times A_2)$, as follows.

Objects of $\mathrm{Qui}(A_i)$ are collections $\mathcal{V}_i = (V_{\alpha_i}, a^{\alpha_i}_{\beta_i}: \ V_{\alpha_i} \longrightarrow V_{\beta_i})$ where α_i numerate strata of $\mathcal{S}_i$, and pairs (α_i, β_i) numerate pairs of (adjacent) strata.

By definition, an object of $\mathrm{Qui}(A_1 \times A_2)$ is a collection of

(a) finite dimensional vector spaces $V_{\alpha_1\alpha_2}$ indexed by strata of the product stratification;

(b) linear operators

$$'a^{\alpha_1\alpha_2}_{\beta_1}: \ V_{\alpha_1\alpha_2} \longrightarrow V_{\beta_1\alpha_2}$$

and

$$''a^{\alpha_1\alpha_2}_{\beta_2}: \ V_{\alpha_1\alpha_2} \longrightarrow V_{\alpha_1\beta_2}$$

The morphisms $'a$ (resp., $''a$) must satisfy the relations in $\mathrm{Qui}(A_1)$ (resp., in $\mathrm{Qui}(A_2)$).

Morphisms in $\mathrm{Qui}(A_1 \times A_2)$ are defined in the obvious manner.

A pair of gluing functors $G_i: \ \mathrm{Qui}(A_i) \longrightarrow \mathcal{M}(A_i)$ $(i = 1, 2)$ defines the functor $G = G_1 \otimes G_2: \ \mathrm{Qui}(A_1 \times A_2) \longrightarrow \mathcal{M}(A_1 \times A_2)$.

4.11. As an example, we will need the categories $\mathrm{Qui}(\mathbb{A}^I(*) \times \mathbb{A}(J))$. Thus, an object of this category is a collection $\mathcal{W} = (W_\alpha, a^\alpha, b^\alpha; d^\alpha_{jj'})$ where $(W_\alpha, a^\alpha, b^\alpha) \in \mathrm{Qui}(\mathbb{A}^I(*))$ and $d^\alpha_{jj'}: \ W_\alpha \longrightarrow W_\alpha$ $(j \neq j'$ in $J)$. The operators d must commute with the operators a, b.

We have a *specialization functor*

$$\mathrm{Sp}_J: \ \mathrm{Qui}(\mathbb{A}^I(J)) \longrightarrow \mathrm{Qui}(\mathbb{A}^I(*) \times \mathbb{A}(J))$$

which assigns to $\mathcal{V} = (V_\gamma, a^\gamma_{pq}, b^\gamma_{pq}, c^\gamma_{jj'})$ a quiver $\mathcal{W} = \mathrm{Sp}_J(\mathcal{V}) = (W_\alpha, a^\alpha_{pq}, b^\alpha_{pq}; d^\alpha_{jj'})$. Namely, for $(\alpha: \ I^* \longrightarrow I') \in Q(I^*)$ (we denote $I^* := I \coprod *$), set $I_\alpha := \alpha^{-1}(\alpha(*)) - * \subset I$ and $I'' := I' - \{\alpha(*)\}$. For a map $\delta: \ I_\alpha \longrightarrow J$, define a map

$$\gamma_\delta: \ I \coprod J \longrightarrow I'' \coprod J$$

as follows. On $I - I_\alpha$ it coincides with $\alpha|_{I-I_\alpha}: \ I - I_\alpha \longrightarrow I''$; on I_α it coincides with δ and on J it is identical.

By definition,

(a) $W_\alpha = \oplus_\delta\ V_{\gamma_\delta}$,

the sum over all maps $\delta:\ I_\alpha \longrightarrow J$.

Let $p \neq q$ in I'. If $p, q \neq \alpha(*)$ then the operators $a^\alpha_{pq}, b^\alpha_{pq}$ in $\mathcal{W}$ are induced by the operators $a^{\gamma_\delta}_{pq}, b^{\gamma_\delta}_{pq}$ in $\mathcal{V}$. If, say $q = \alpha(*)$, then a^α_{pq} is induced by the sum

$$\oplus_{j\in J}\ a^{\gamma_\delta}_{pj};$$

the same story with the operators b; the case $p = \alpha(*)$ is completely similar.

Finally, the operators $d^\alpha_{jj'}$ may be expressed as a sum $d^\alpha_{jj'} = {}'d^\alpha_{jj'} + {}''d^\alpha_{jj'}$. The part ${}'d^\alpha_{jj'}$ (the "diagonal" part with respect to the decomposition (a)) is the sum of the operators $c^{\gamma_\delta}_{jj'}$. The "off diagonal" part ${}''d^\alpha_{pq}$ is the sum of the operators $b_{jj'}a^{\gamma_\delta}_{jj'}$.

4.12. **Theorem.** *For $\mathcal{V} \in \mathrm{Qui}(\mathbb{A}^I(J))$, one has a natural isomorphism $\mathrm{Sp}_J G(\mathcal{V}) = G\mathrm{Sp}_J(\mathcal{V})$.*

Here Sp_J in the LHS is the specialization of $\mathcal{D}$- modules, cf. 3.4 (a).

5. Drinfeld's tensor category and Equivalence theorem

In this section we recall the definition of the Drinfeld's tensor category $\mathrm{Mod}(\mathfrak{g})_\kappa$, and state the Equivalence theorem 5.2, with the sketch of the proof.

5.1. Let us return to the setup of Section 2. Let $\mathfrak{g}$ be the complex semisimple Lie algebra corresponding to the Cartan datum $(I, \cdot)$, with Chevalley generators $\{e_i, f_i, h_i\}$ $(i \in I)$. By definition, $\mathfrak{g}$ comes equipped with a fixed invariant symmetric form extending the given scalar product on $\mathfrak{h} := \oplus\ \mathbb{C}h_i = \mathbb{C}\otimes Y$. Let $\mathrm{Mod}(\mathfrak{g})$ be the category of finite dimensional $\mathfrak{g}$-modules. An object of this category may be described as a finite dimensional X-graded vector space $M = \oplus_X\ M_\lambda$ together with operators $e_i:\ M_\lambda \longrightarrow M_{\lambda+i'}$, $f_i:\ M_\lambda \longrightarrow M_{\lambda-i'}$ satisfying the usual relations.

The Drinfeld's tensor structure on $\mathrm{Mod}(\mathfrak{g})$ is defined as follows. We have to specify for each finite set J and a J-tuple $\{M_j\}$ $(j \in J, M_j \in \mathrm{Mod}(\mathfrak{g}))$, a local system or, what is the same, a lisse regular $\mathcal{D}$-module $\otimes_J\ M_j$ over the space $\mathbb{A}(J)$. By definition, this $\mathcal{D}$-module is given by the integrable connection on the trivial vector bundle on $\mathbb{A}(J)$ with a fiber $M := \otimes_J M_j$ (the product of M_j as vector spaces), with the Knizhnik-Zamolodchikov connection form

$$\omega_{KZ} = \frac{1}{2\kappa}\sum_{i\neq j}\Omega_{ij}\mathrm{dlog}(t_i - t_j).$$

Here $\Omega \in \mathfrak{g}\otimes\mathfrak{g}$ is the symmetric tensor correponding to the bilinear form on $\mathfrak{g}$ which induces the linear operators Ω_{ij} $(i \neq j$ in $J)$ on M in the usual way.

The factorization isomorphisms (and their associativity property) have been defined (resp., proved) by Drinfeld in [D] (in a slightly different language; in fact, the Drinfeld's construction was a starting point for the Deligne's definition of braided tensor structures).

Let $\mathrm{Mod}(\mathfrak{g})_\kappa$ denote $\mathrm{Mod}(\mathfrak{g})$ together with the above tensor structure. Recall that we assume that κ is irrational.

5.2. **Equivalence theorem.** *One has a tensor equivalence*

$$\Phi_\kappa : \mathcal{FM}_\kappa \xrightarrow{\sim} \mathrm{Mod}(\mathfrak{g})_\kappa.$$

One has natural isomorphism $D\Phi_\kappa = \Phi_{-\kappa}D$.

Here D in the RHS is the duality functor 2.9, and D in the LHS is the *contravariant duality* functor $D : \mathrm{Mod}(\mathfrak{g})_\kappa^{\mathrm{opp}} \xrightarrow{\sim} \mathrm{Mod}(\mathfrak{g})_{-\kappa}$.

Let us explain briefly how the proof goes. Using the main Theorem 4.6, together with the remarks 2.3 and 4.7, one obtains a quiver description of the categories $\mathcal{M}_{\vec{\lambda}}(\mathbb{A}^\nu)$, for non-resonance collections $\vec{\lambda}$. Now, the notion of a factorizable module translates into that of a *factorizable quiver* which is a collection of quivers $\{\mathcal{V}^\nu\}$ over the spaces $\mathcal{A}^\nu$ connected by factorization isomorphisms. The irrationality assumption on κ guarantees that the corresponding monodromies $\vec{\lambda}$ are non-resonance. After that, it is a matter of an algebraic reformulation to show that the category of factorizable quivers is equivalent to $\mathrm{Mod}(\mathfrak{g})$. This provides an equivalence Φ_κ.

Let $\mathcal{M} = (\mu, \mathcal{M}^\nu, \dots) \in \mathcal{FM}_\kappa$, $\Phi_\kappa(\mathcal{M}) = M$ and $\mathcal{V}^\nu$ be the quiver corresponding to $\mathcal{M}^\nu$. The "a-part" of $\mathcal{V}$ essentially coincides with the homogeneous part of the Lie algebra chain complex $C_\bullet(\mathfrak{n}_-; M)_{\mu-\nu}$. The "$b$-part" is restored in a similar way from the $\mathfrak{n}_+$-module structure, cf. [S]. Here $\mathfrak{n}_-$ (resp., $\mathfrak{n}_+$) is the Lie subalgebra of $\mathfrak{g}$ generated by f_i (resp., by e_i).

Theorem 4.12 implies that Φ_κ is a tensor functor. Commutation with the duality is a consequence of Theorem 4.4.

5.3. Let us look at the quasi-inverse to Φ_κ. It assigns to a module M a factorizable module $\mathcal{M} = (\mu, \mathcal{M}^\nu, \dots)$ where each $\mathcal{M}^\nu$ given by a gluing functor, as in 4.3. In particular the spaces $M_{\mu-\nu}$ are the subspaces of the spaces $\Gamma(\mathbb{A}^\nu; \mathcal{M}^\nu)$. For example, if M is the contragradient to a Verma module then $\mathcal{M}^\nu$ is the $*$-extension of $\mathcal{I}_\mu^{\nu\bullet}$, so the space of its global sections is a subspace of rational functions on $\mathbb{A}^\nu$. If we pass from left to right $\mathcal{D}$-modules as usual — by multiplication by the the sheaf of volume forms, we get a map from M_ν to the space of the top degree rational differential forms on $\mathbb{A}^\nu$ which coincides with the map defined in [SV].

5.4. **Corollary.** *Let L_μ be the irreducible finite dimensional representation of $\mathfrak{g}$ with the highest weight μ. Set $\mathcal{L}_\mu^\nu := j_{!*}\mathcal{I}_\mu^\nu$ where $j : \mathbb{A}^{\nu o} \hookrightarrow \mathbb{A}^\nu$ is the open embedding. We have an isomorphism*

$$H^\bullet(\mathbb{A}^\nu; \mathcal{L}_\mu^\nu) = H_{-\bullet}(\mathfrak{n}_-; L_\mu)^{\mu-\nu}.$$

Here the superscript in the RHS denotes the weight component. In other words, the complexes of "flag forms" (the image of the map S) from [SV] (cf. *op. cit.* 6.5.3) compute the intersection cohomology for an irrational κ.

5.5. **Remark.** If we drop the finiteness assumption from the definition of the category $\mathcal{FS}_\kappa$, cf.2.8, we get the category equivalent to the category $\mathcal{O}$ of Bernstein-Gelfand-Gelfand (the proof is the same).

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