Zariski pairs arising from elliptic K3 surfaces, II

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Introduction

In this article, we shall continue to study Zariski pairs which are defined by Artal Bartolo in [1] as follows:

Definition 0.1. *Let C_1 and C_2 be irreducible plane sextic curves. The pair (C_1, C_2) is called a Zariski pair of degree 6 if*

(i) C_1 and C_2 have the same set of topological types of singularities, and

(ii) $\mathbf{P}^2 \setminus C_1$ is not homeomorphic to $\mathbf{P}^2 \setminus C_2$.

The first example of a Zariski pair is found by Zariski in [Z1] and [Z2]. Since then, only a few examples have been known (see [A], [D2] and [T2]).

In the previous paper [T2], we studied Zariski pairs of degree 6 in terms of the index, the sum of subindicies of types of singularities, of a sextic with at most simple singularities. In this article, we shall focus our attention on a conjecture posed by Degtyarev in [D2].

Conjecture 0.2. (Degtyarev) *Consider irreducible sextics with a fixed singular set of the form $\alpha a_1 + \sum_d \beta_d a_{3d-1} + \gamma e_6$ with $\sum_d d\beta_d + 2\gamma = 6$. Put $\alpha_{\max} = 10 - \sum_d \beta_d [\frac{3d}{2}] - 3\gamma$, where $[x]$ denotes the maximum integer not exceeding x. Then*

(i) any two sextics with $\alpha = \alpha_{\max}$ are isotopic to each other, and

(ii) if $\alpha < \alpha_{\max}$, then there are exactly two isotopic classes.

Here, following to Degtyarev [D1], we call two irreducible sextics C_1 and C_2 are isotopic if there exists one parameter family of homeomorphisms $\varphi_t : \mathbf{P}^2 \to \mathbf{P}^2$, $t \in [0, 1]$ such that $\varphi_0 = id$ and $\varphi_1(C_1) = C_2$. Thus Conjecture 0.2 implies that there is no Zariski pair of degree 6 with $\alpha = \alpha_{\max}$, while there is a unique Zariski pair for every $\alpha < \alpha_{\max}$. We shall give a counter-example to Conjecture 0.2 (i) by using a theory of dihedral Galois coverings developed in [T1], [T2], and [T3]. This is a consequence of the following theorem.

Theorem 0.3. *For every set of singularities described below, there exists a Zariski pair of degree 6.*

Set of singularities of C
$\alpha a_1 + a_{11} + e_6 \quad (\alpha = 0, 1)$
$\alpha a_1 + 2a_5 + e_6 \quad (\alpha = 0, 1)$
$\alpha a_1 + a_2 + a_8 + e_6 \quad (\alpha = 0, 1, 2)$
$\alpha a_1 + a_5 + 2a_2 + e_6 \quad (\alpha = 0, 1, 2)$
$\alpha a_1 + 2a_2 + 2e_6 \quad (\alpha = 0, 1)$

Theorem 0.3 shows that Conjecture 0.2 (i) is false in the cases in $a_1 + a_{11} + e_6$, $a_1 + a_5 + e_6$, $2a_1 + a_2 + a_8 + e_6$ and $a_1 + 2a_1 + 2e_6$. Our proof of Theorem 0.3 is based on the following two propositions.

Proposition 0.4. *For every set of singularities as in Theorem 0.3, there exists an irreducible sextic, C, such that*

(i) C has the prescribed set of singularities, and

(ii) there exists no Galois covering of $\mathbf{P}^2$ branched along C having the third symmetric group as its Galois group.

Proposition 0.5. *For every set of singularities as in Theorem 0.3, there exists an irreducible sextic, C, such that*

(i) C has the prescribed set of singularities, and

(ii) there exists a Galois covering of $\mathbf{P}^2$ branched along C having the third symmetric group as its Galois group.

Remark 0.6. (i) Theorem 0.3 implies nothing about uniqueness for Zariski pairs. The uniqueness in Conjecture 0.2 (ii) is still unknown.

(ii) For the sets of singularities of the form $\alpha a_1 + 4a_2 + e_6$, $0 < \alpha \leq \alpha_{\max} = 3$, it is known that there exist Zariski pairs in the cases when $\alpha a_1 + 4a_2 + e_6$, $\alpha < 3$ (see [T2]).

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Notations and Conventions.

Throughout this article, the ground field will always be the complex number field $\mathbf{C}$.

$\mathbf{C}(X) :=$ the rational function field of X.

Let X be a normal variety, and let Y be a smooth variety. Let $\pi : X \to Y$ be a finite morphism from X to Y. We define the branch locus, $\Delta(X/Y)$, of f as follows:

$$\Delta(X/Y) = \{y \in Y | \sharp(\pi^{-1}(y)) < deg\pi\}.$$

For a divisor D on Y, $\pi^{-1}(D)$ denotes the set-theoretic inverse image of D, while $\pi^*(D)$ denotes the ordinary pullback. Also, $SuppD$ means the supporting set of D.

An $\mathcal{S}_3$ covering always means a Galois covering having the third symmetric group, $\mathcal{S}_3$, as it Galois group.

Let S be a finite double covering of a smooth projective surface Σ. The "*canonical resolution*" of S always means the resolution given by Horikawa in [H].

Let S be an elliptic surface over C. S is said to be *minimal* if the fibration is relatively minimal. In this paper, we always assume that an elliptic surface is minimal and has a section s_0. For singular fibers of an elliptic surface, we use the notation of Kodaira [K]. For the configuration of singular fibers, we shall use the same notations as those in [M-P2] and [P2].

We denote by $\mathbf{F}_n$ a Hirzebruch surface of degree n.

A $(-n)$-curve always means a rational curve with self-intersection number $-n$.

Let D_1, D_2 be divisors.

$D_1 \sim D_2$: linear equivalence of divisors.

$D_1 \approx D_2$: algebraic equivalence of divisors.

$D_1 \approx_{\mathbf{Q}} D_2$: $\mathbf{Q}$-algebraic equivalence of divisors.

For singularities of a plane curve, we shall use the same notation as in [P1]. Throughout the article, we assume that singularities of a plane curve are at most simple.

§1 Preliminaries

It is well-known that any elliptic K3 surface can be obtained as a double covering of $\mathbf{F}_4$ (For example, see [M2] III). In this section, we shall consider an elliptic K3 surface is obtained as a double covering of $\mathbf{P}^2$. Details may be omitted since most results in this section can be proved in a similar way as for the corresponding results of a K3 surface obtained as a double covering of $\mathbf{F}_4$. For details, see [B], [N], [M2] and [P1].

Let C be a sextic such that (i) all singularities are at most simple and (ii) C has an e_6 singularity, x. Let $f : W \to \mathbf{P}^2$ be a double covering branched along C, and let $\mu : \mathcal{E} \to W$ be its canonical resolution, which satisfies the following diagram:

$$\begin{array}{ccc} W & \stackrel{\mu}{\leftarrow} & \mathcal{E} \\ f\downarrow & & \downarrow \tilde{f} \\ \mathbf{P}^2 & \stackrel{\nu}{\leftarrow} & \Sigma \end{array} \qquad (1.1)$$

where $\nu : \Sigma \to \mathbf{P}^2$ is a succession of blowing-ups and $\tilde{f}$ is the induced finite double covering. Let σ be the covering transformation on $\mathcal{E}$ induced by $\tilde{f}$. Note that $\mathcal{E}/\langle\sigma\rangle = \Sigma$.

By our assumption, $\mathcal{E}$ has an elliptic fibration arising from a pencil of lines through x. We denote it by $\varphi_x : \mathcal{E} \to \mathbf{P}^1$. Following to Persson [P1], we call φ_x the standard fibration centered at x. Note that φ_x has a section, s_0, which is an irreducible component of the exceptional divisor of the E_6 singularity, P, of W lying over x. By looking into the canonical resolution, it is straightforward to see that other irreducible components of the exceptional divisor of P are those of a singular fiber of type I_n $(n \geq 6)$. Thus we obtain an elliptic K3 surface with a section s_0 having a singular fiber, F_0, of type I_n $(n \geq 6)$ from C in a canonical way.

We shall next look into the converse of this construction. Namely let $\varphi : \mathcal{E} \to \mathbf{P}^1$ be an elliptic K3 surface with a section s_0 having a singular fiber, F_0, of type I_n $(n \geq 6)$. We shall consider when $\mathcal{E}$ satisfies the following properties:

(i) $\mathcal{E}$ is a double covering of $\mathbf{P}^2$ branched along a sextic, C, with at most simple singularities.

(ii) $\varphi : \mathcal{E} \to \mathbf{P}^1$ is the standard fibration centered at a triple point of C.

For this question, the following proposition holds:

Proposition 1.2.(Persson) *Let $\varphi : \mathcal{E} \to \mathbf{P}^1$ be an elliptic K3 surface as above. Then there exists a degree 2 morphism $g : \mathcal{E} \to \mathbf{P}^2$ such that (i) the branch locus of f is a sextic curve, C, with at most simple singularities, and (ii) $\varphi : \mathcal{E} \to \mathbf{P}^1$ is the standard fibration at an e_6 singularity, x, of C.*

Proof of Proposition 1.1 is basically the same as that in [P1], p. 282. As φ has a section s_0, $\mathcal{E}$ is considered as a double covering of $\mathbf{F}_4$. Let σ' be the covering transformation, and let $\mathcal{E}/\langle\sigma'\rangle$ be the quotient by σ'. By looking into the action of σ' on a singular fiber of type I_n (cf. [B], §1, and [N], §2), if φ has a singular fiber of type I_n $(n \geq 6)$, we can blow down $\mathcal{E}/\langle\sigma'\rangle$ to $\mathbf{P}^2$ so that (i) the branch locus of $\mathcal{E} \to \mathcal{E}/\langle\sigma'\rangle \to \mathbf{P}^2$ is a sextic curve, C, with an e_6 singularity, x, and (ii) φ is the standard fibration centered at x. For details, see [P1], [B], and [N].

As an consequence of Proposition 1.2, we obtain a sextic with an e_6 singularity from $\mathcal{E}$ in a canonical way. By looking into the correspondence between C and $\mathcal{E}$ given above, we have the following (cf. [M2] or [MP1], Table 6.2):

Lemma 1.3. *With the notations as above, if $F_0 = I_6$ or I_7, then there is a correspondence between singular fibers of φ and singularities of C. More precisely, we have*

Type of a singular fiber	F_0	$I_n\,(n \geq 2), \neq F_0$	I_1	I_n^*
Type of a singular point	e_6	a_{n-1}	*a smooth point*	d_{n+4}

II	II^*	III	III^*	IV	IV^*
a smooth point	e_8	a_1	e_7	a_2	e_6

Remark 1.4. Consider $\mathcal{E}$ as a double covering of $\mathbf{F}_4$. Let $\bar{\mathcal{E}}$ be a Weierstrass model of $\mathcal{E}$. $\bar{\mathcal{E}}$ is a double covering of $\mathbf{F}_4$ with some rational double points, and $\mathcal{E}$ is considered as the canonical resolution of $\bar{\mathcal{E}}$. Hence we have the following diagram:

$$\begin{array}{ccc} \bar{\mathcal{E}} & \leftarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathbf{F}_4 & \leftarrow & \Sigma' \end{array}$$

where $\Sigma' \to \mathbf{F}_4$ is a succession of blowing-ups. On the other hand, let C be the sextic curve as in Proposition 0.2, and consider a double covering $f : W \to \mathbf{P}^2$ branched along C. Then the canonical resolution of W coincides with $\mathcal{E}$. Also, Σ' coincides to Σ in the diagram (1.1). Thus we have the following diagram:

$$\begin{array}{ccccc} W & \leftarrow & \mathcal{E} & \rightarrow & \bar{\mathcal{E}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{P}^2 & \leftarrow & \Sigma = \Sigma' & \rightarrow & \mathbf{F}_4 \end{array} \qquad (1.5)$$

where $\bar{\mathcal{E}}$ is a Weierstrass fibration of $\mathcal{E}$.

Let $\varphi : \mathcal{E} \to \mathbf{P}^1$ and C be as in Proposition 1.2. Let $MW(\mathcal{E})$ be the Mordell-Weil group of $\mathcal{E}$, i.e., the group of sections, with s_0 as the zero element.

Lemma 1.6. *If there is no non-trivial 2-torsion element in $MW(\mathcal{E})$, then C is irreducible.*

Proof. Consider $\mathcal{E}$ as a double covering of $\mathbf{F}_4$, and let $\Delta_0 + T$ be the branch locus of it. Then T is the image of C by the birational map from $\mathbf{P}^2$ to $\mathbf{F}_4$ in the above diagram. Hence if C is reducible, then T is also reducible. This means T contains a section component. Therefore $MW(\mathcal{E})$ has a non-trivial 2-torsion element.

We shall end this section with the following two propositions which will play a key role in proving Propositions 0.4 and 0.5.

Proposition 1.7. *Let $\mathcal{E}$ be an elliptic K3 surface as in Proposition 1.2, and let C be the corresponding sextic curve. If there exists an $\mathcal{S}_3$ covering of $\mathbf{P}^2$ branched along C, then there exists a non-trivial 3-torsion in $MW(\mathcal{E})$.*

For a proof, see [T2] Proposition 4.1.

Proposition 1.8. *Let $\mathcal{E}$ be an elliptic K3 surface as in Proposition 1.2, let C be the corresponding sextic curve, and let x be the e_6 singularity in Proposition 1.2 (ii). Suppose that*

(i) the singular fiber arising from x, F_0, is of type I_6, and

(ii) there exists a non-trivial 3-torsion in $MW(\mathcal{E})$.

Then there exists an $\mathcal{S}_3$ covering of $\mathbf{P}^2$ branched along C.

For a proof, see [T2], §5 and [T3], §3.

§2 Proof of Proposition 0.4

In this section, we shall use the notations of §1. We shall first reduce the existence of a sextic curve with singularities described in Theorem 0.3 to that of an elliptic K3 surfaces with a prescribed singular fibers.

Lemma 2.1 *Let $\mathcal{E}$ be an elliptic K3 surface as in Proposition 1.2, and let C be the corresponding sextic curve. Then we have the following table:*

Singular fibers of $\mathcal{E}$	*Set of singularities of C*
I_7, I_{12}, I_2, $3I_1$	$a_1 + a_{11} + e_6$
I_7, I_{12}, $5I_1$	$a_{11} + e_6$
I_7, $2I_6$, I_2, $3I_1$	$a_1 + 2a_5 + e_6$
I_7, $2I_6$, $5I_1$	$2a_5 + e_6$
I_7, I_9, I_3, $2I_2$, I_1	$2a_1 + a_2 + a_8 + e_6$
I_7, I_9, I_3, I_2, $3I_1$	$a_1 + a_2 + a_8 + e_6$
I_7, I_9, I_3, $5I_1$	$a_2 + a_8 + e_6$
I_7, IV^*, $2I_3$, I_2, I_1	$a_1 + 2a_2 + 2e_6$
I_7, IV^*, $2I_3$, $3I_1$	$2a_2 + 2e_6$

This is immediate by Lemma 1.4. We shall show in §4 that every elliptic K3 surface in the above table exists.

We now go on to prove Proposition 0.4. By Proposition 1.6, it is enough to show the following two lemmas.

Lemma 2.2. *Let $\varphi : \mathcal{E} \to \mathbf{P}^1$ be an arbitrary elliptic K3 surface as in Lemma 2.1. Then there is no non-trivial three torsion element in $MW(\mathcal{E})$.*

Proof. Let $\varphi : \mathcal{E} \to \mathbf{P}^1$ be any elliptic K3 surface as in Lemma 2.1. If $\mathcal{E}$ has only I_n fibers, our statement easily follows from Proposition 0.2 in [T4]. We shall go on to the case that φ has a singular fibers of type IV^*. Suppose that $MW(\mathcal{E})$ has a torsion of order 3. Let s be the corresponding section, and let $\langle\, ,\rangle$ denote Shioda's pairing defined in [S2]. Then, by Theorem 8.6 in [S2], we have

$$\langle s, s\rangle = 4 + 2ss_0 - \frac{a}{7} - \frac{b}{3} - \frac{c}{2}$$

where $a \in \{0, 6, 10, 12\}$, $b \in \{0, 2, 4, 6, 8\}$ and $c \in \{0, 1\}$.

As s corresponds to a torsion, $\langle s, s\rangle = 0$ by Theorem 8.4 in [S2]. This, however, is impossible for any possible combination of a, b, c as above.

Lemma 2.3. *Every C in Lemma 2.1 is irreducible.*

Proof. By Lemma 1.6, it is enough to show that $MW(\mathcal{E})$ of the corresponding elliptic K3 surface has no non-trivial 2-torsion element. We can show it in the same way as in the proof of Lemma 2.2, so we omit it.

§3 Proof of Proposition 0.5

In this section, we also use the notations of §1. We shall start with the following lemma.

Lemma 3.1. *Let $\mathcal{E}$ be an elliptic K3 surface as in Proposition 1.2, and let C be the corresponding sextic curve. Then we have the following table:*

Singular fibers of $\mathcal{E}$	*Set of singularities of* C
$I_6,\ I_{12},\ I_2,\ 4I_1$	$a_1 + a_{11} + e_6$
$I_6,\ I_{12},\ 6I_1$	$a_{11} + e_6$
$3I_6,\ I_2,\ 4I_1$	$a_1 + 2a_5 + e_6$
$3I_6,\ 6I_1$	$2a_5 + e_6$
$I_6,\ I_9,\ I_3,\ 2I_2,\ 2I_1$	$2a_1 + a_2 + a_8 + e_6$
$I_6,\ I_9,\ I_3,\ I_2,\ 4I_1$	$a_1 + a_2 + a_8 + e_6$
$I_6,\ I_9,\ I_3,\ 6I_1$	$a_2 + a_8 + e_6$
$I_6,\ IV^*,\ 2I_3,\ I_2,\ 2I_1$	$a_1 + 2a_2 + 2e_6$
$I_6,\ IV^*,\ 2I_3,\ 4I_1$	$2a_2 + 2e_6$

This is straightforward by Lemma 1.3. By Proposition 1.7, in order to prove Proposition 0.5, it is enough to show the following two lemmas.

Lemma 3.2. *There exists an elliptic K3 surface, $\mathcal{E}$, with a non-trivial 3-torsion element in $MW(\mathcal{E})$ for all cases in Lemma 3.1.*

Lemma 3.3. *Every C in Lemma 3.1 is irreducible.*

We shall prove Lemma 3.2 in the next section.

Proof of Lemma 3.3. By Lemma 1.6, it is enough to show that there is no non-trivial 2-torsion element in $MW(\mathcal{E})$. By Proposition 0.2 in [T4] and Remark 1.10 in [S1], there is no non-trivial 2-torsion element in $MW(\mathcal{E})$ except in the first 4 cases. We shall show that if the configuration of singular fibers of $\mathcal{E}$ is $\{I_6, I_{12}, I_6, 4I_1\}$, then there is no non-trivial 2 torsion element in $MW(\mathcal{E})$. Suppose that there is a non-trivial 2-torsion element in $MW(\mathcal{E})$, and let s be the corresponding section. Let $\langle\,,\rangle$ denote Shioda's pairing. Then, by Lemma 1.2 in [T4] and Theorem 8.6 in [S2], we have

$$\langle s, s\rangle = 4 - \frac{3}{2}a - 3b - \frac{1}{2}c$$

where $a, b, c \in \{0, 1\}$. By Theorem 8.4 in [S2], the left hand side must be 0, but this is impossible for all possible combinations of a, b, c. Similarly, we can disprove the existence of a non-trivial 2-torsion element for remaining three cases.

§4 Existence of elliptic K3 surfaces

In this section, we shall show that there exist elliptic K3 surfaces in Lemma 2.1 and Lemma 3.2. Our basic idea is to consider elliptic surfaces obtained as pull-backs of rational elliptic surfaces. To this purpose, we shall first study morphisms form $\mathbf{P}^1$ to $\mathbf{P}^1$. Let $f : \mathbf{P}^1 \to \mathbf{P}^1$ be a morphism of degree n. Let v_i $(i = 1, ..., k)$ be the branch points of f. We say that f has ramification type $(e_1^{(i)}, ..., e_k^{(i)})$ at v_i if $f^{-1}(v_i) = \{u_1, ..., u_k\}$ and the ramification index at u_j is $e_j^{(i)}$. With these notations, we have the following:

Lemma 4.1. *There exist the following morphisms, f_i, $(i = 1, ...5)$ from $\mathbf{P}^1$ to $\mathbf{P}^1$.*

	$\deg f_i$	*Branch points*	*Ramification types*
f_1	2	v_1, v_2	$v_i : (2)\ (i = 1, 2)$
f_2	3	v_1, v_2, v_3	$v_1 : (3),\ v_i : (2,1)\ (i = 2, 3)$
f_3	4	v_1, v_2, v_3	$v_i : (3,1)\ (i = 1, 2, 3)$
f_4	4	$v_1, ..., v_5$	$v_1 : (3,1),\ v_i : (2,1,1)\ (i = 2, ..., 5)$
f_5	6	v_1, v_2, v_3, v_4	$v_1 : (3,3),\quad v_2 : (3,2,1)$ $v_3 : (2,2,1,1)\quad v_4 : (2,1,1,1,1)$

Proof. For morphisms of degrees 2 and 3 in the table, it is well-known that such morphisms exist. For morphisms of degree $d \geq 4$, we denote loops around v_i by δ_i and by $\mathcal{S}_d$ the d-th symmetric group. Then, by covering space theory and the Riemann existence theorem ([F], Proposition 1.2 or [M2]), it is enough to give a map

$$\chi : \pi_1(\mathbf{P}^2 \setminus \{v_1, ..., v_r\}) \to \mathcal{S}_d,$$

such that

(i) if the ramification type over v_i is $(e_1^{(i)}, ..., e_k^{(i)})$, then $\chi(\delta_i)$ has the cycle structure $(e_1^{(i)}, ..., e_k^{(i)})$,

(ii) $\chi(\delta_i)$ $(i = 1, ..., r)$ generate a transitive subgroup of $\mathcal{S}_d$, and

(iii) $\chi(\delta_1) \cdots \chi(\delta_r) = 1$ (Here we multiply permutations from right to left).

Hence the table below shows that morphisms described in Lemma 4.1 exist.

f_3	$\chi(\delta_1) = (adb),\ \chi(\delta_2) = (abc),\ \chi(\delta_3) = (cbd)$
f_4	$\chi(\delta_1) = (abc),\ \chi(\delta_2) = (ab),\ \chi(\delta_3) = (ac),\ \chi(\delta_4) = (cd),\ \chi(\delta_5) = (cd)$
f_5	$\chi(\delta_1) = (abc)(dfe),\ \chi(\delta_2) = (acb)(de),\ \chi(\delta_3) = (ad)(ef),\ \chi(\delta_4) = (ad)$

With Lemma 4.1, we shall prove the following.

Lemma 4.2. *For every configuration of singular fibers as below, there exists an elliptic K3 surface, $\mathcal{E}$, with a non-trivial 3-torsion in $MW(\mathcal{E})$.*

1. I_6, I_{12}, I_2, $4I_1$, 2. I_6, I_{12}, $6I_1$, 3. $3I_6$, I_2, $4I_1$, 4. $3I_6$, $6I_1$,
5. $I_6, I_9, I_3, 2I_2, 2I_1$, 6. $I_6, I_9, I_3, I_2, 4I_1$, 7. $I_6, I_9, I_3, 6I_1$, 8. $I_6, IV^*, 2I_3, I_2, 2I_1$,
9. I_6, IV^*, $2I_3$, $4I_1$ 10. I_9, IV^*, $2I_3$, I_1.

Proof. Let $\psi : Y \to \mathbf{P}^1$ be a rational elliptic surface with a non-trivial 3-torsion in $MW(Y)$. Let $f_i : \mathbf{P}^1 \to \mathbf{P}^1$ be the morphism as in Lemma 4.1. Consider the pullback surface $Y \times_{\mathbf{P}^1} \mathbf{P}^1$ of Y by f_i, and denote its smooth relatively minimal model by $\mathcal{E}$. Since there exists a non-trivial 3-torsion element in $MW(Y)$, there also exists a non-trivial 3-torsion element in $MW(\mathcal{E})$. Hence our problem is to investigate when $\mathcal{E}$ becomes an elliptic K3 surface with a prescribed singular fibers. For this problem, we have the following by [MP1] Table 7.1.

f_i	Singular fibers of Y	Type of a fiber over v_i	Singular fibers of $\mathcal{E}$
f_1	I_6, I_3, $3I_1$	$v_1 : I_3$, $v_2 : I_1$	$3I_6$, I_2, $4I_1$
f_1	I_6, I_3, $3I_1$	$v_1 : I_3$, $v_2 : I_0$	$3I_6$, $6I_1$
f_2	IV, I_6, $2I_1$	$v_1 : IV$, $v_2 : I_6$, $v_3 : I_1$	I_{12}, I_6, I_2, $4I_1$
f_2	IV, I_6, $2I_1$	$v_1 : IV$, $v_2 : I_6$, $v_3 : I_0$	I_{12}, I_6, $6I_1$
f_3	IV^*, I_3, I_1	$v_1 : IV^*$, $v_2 : I_3$, $v_3 : I_1$	IV^*, I_9, $2I_3$, I_1
f_4	IV^*, I_3, I_1	$v_1 : IV^*$, $v_2 : I_3$, $v_3 : I_1$, v_4, $v_5 : I_0$	IV^*, I_6, $2I_3$, I_2, $2I_1$
f_4	IV^*, I_3, I_1	$v_1 : IV^*$, $v_2 : I_3$, $v_3, v_4, v_5 : I_0$	IV^*, I_6, $2I_3$, $4I_1$
f_5	IV^*, I_3, I_1	$v_1 : IV^*$, $v_2 : I_3$, $v_3 : I_1$, $v_4 : I_0$	I_9, I_6, I_3, $2I_2$, $2I_1$
f_5	IV^*, I_3, I_1	$v_1 : IV^*$, $v_2 : I_3$, $v_3 : I_0$, $v_4 : I_1$	I_9, I_6, I_3, I_2, $4I_1$
f_5	IV^*, I_3, I_1	$v_1 : IV^*$, $v_2 : I_3$, $v_i : I_0$, $(i = 3, 4)$	I_9, I_6, I_3, $6I_1$

For every configuration of singular fibers Y in the above table, it is known that there exists such a rational elliptic surface with a non-trivial 3-torsion in $MW(Y)$ (see [M2] and [P2]). Now Lemma 4.2 easily follows from this table.

Corollary 4.3. *There exist elliptic K3 surfaces with the following configurations of singular fibers:*
1. I_7, I_{12}, I_2, $3I_1$, 2. I_7, $I_{12}, 5I_1$, 3. $I_7, 2I_6$, I_2, $3I_1$, 4. I_7, $2I_6$, $5I_1$,
5. $I_7, I_9, I_3, 2I_2, I_1$, 6. $I_7, I_9, I_3, I_2, 3I_1$, 7. $I_7, I_9, I_3, 5I_1$, 8. $I_7, IV^*, 2I_3, I_2, I_1$,
9. I_7, IV^*, $2I_3$, $3I_1$

.

Proof. For configurations with only I_n fibers, our result is straightforward by [MP2]. For the cases 8 and 9, we obtain such elliptic K3 surfaces by applying the argument in [M2], p.205 to an elliptic K3 surface having the configuration 10 in Lemma 4.2.

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