RESOLUTION OF SINGULARITIES IN DENJOY-CARLEMAN CLASSES

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Abstract. We show that a version of the desingularization theorem of Hironaka holds for certain classes of $\mathcal{C}^\infty$ functions (essentially, for subrings that exclude flat functions and are closed under differentiation and the solution of implicit equations). Examples are quasianalytic classes, introduced by E. Borel a century ago and characterized by the Denjoy-Carleman theorem. These classes have been poorly understood in dimension > 1. Resolution of singularities can be used to obtain many new results; for example, topological Noetherianity, Łojasiewicz inequalities, division properties.

1. Introduction

We show that a version of the desingularization theorem of Hironaka [Hi1] holds for certain classes of infinitely differentiable functions – essentially, for subrings that exclude flat functions and are closed under differentiation and the solution of equations satisfying the conditions of the implicit function theorem. Examples are "quasianalytic classes", introduced by E. Borel a century ago [Bo1] and characterized (following questions of Hadamard in studies of linear partial equations [Ha]) by the Denjoy-Carleman theorem [De], [Ca]. (See Section 2 below.)

Quasianalytic classes in one variable play an important part in harmonic analysis and other areas. (See, for instance, [HJ], [Ko], [T].) In several variables, there are beautiful modern developments of E.M. Dyn'kin [Dy1], [Dy2], but the subject is much less understood, perhaps because of a lack of the standard techniques of local analytic geometry. For example, the Weierstrass preparation theorem fails

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(Childress [Ch]) and it seems unknown (and unlikely) that, in general, a ring of germs of functions in a Denjoy-Carleman class in Noetherian.

It may seem surprising that desingularization theorems nevertheless hold for Denjoy-Carleman classes. Our proof of resolution of singularities in [BM2], however, (at least in the case of a hypersurface or "principalization of an ideal" [BM2, Thm. 1.10]) uses only elementary "differential calculus" properties that are satisfied in these classes. This was pointed out in [BM2, (0.1)] and the analytic details are provided here. A simple version of resolution of singularities (as in [BM1, Sect. 4]) for quasianalytic classes has already been used by Rolin, Speissegger and Wilkie in their study of o-minimality of Denjoy-Carleman classes [RSW].

In this article, we isolate the properties of a class of $\mathcal{C}^\infty$ functions needed for resolution of singularities (Section 3). We formulate the most general version of desingularization known for these classes in Section 5 (Theorems 5.9 and 5.10; see also Remark 7.10). These theorems require algebraic arguments that can be found in [BM2], but we include a complete proof of a simpler version (Theorem 5.12; cf. [BM1, Sect. 4]) that in general suffices for applications, in a language that makes it clear that only the properties of Section 3 are involved. The proof (presented in Section 7) is meant at the same time to serve as an introduction to two further articles, *Desingularization algorithms I. Role of exceptional divisors* and *Desingularization algorithms II. Binomial varieties*, that we plan to publish shortly.

The properties of Section 3 are known for Denjoy-Carleman classes. (See references in Section 4.) We include proofs in Section 4 because the minimal hypotheses required are not always clear in the literature, nor is the elementary nature of the results and the background required to prove them.

Resolution of singularities is of course a powerful tool; it can be used to prove several other new results about Denjoy-Carleman classes. Many of the geometric properties of semialgebraic sets, for example, are satisfied by o-minimal structures in general [vdDM]. The following are discussed in Section 6 below: (1) Topological

Noetherianity (Theorem 6.1). (2) Łojasiewicz inequalities (Theorem 6.2). Proofs of Łojasiewicz's inequalities depending only on a simple version of resolution of singularities were already given in [BM2, Sect.2]. These inequalities were known previously for Denjoy-Carleman classes only in dimension 2, under more restrictive hypotheses (Vol'berg [V]). (3) Division properties (Theorem 6.3).

Several unresolved questions about Denjoy-Carleman classes are raised in the text. We are grateful to Vincent Thilliez for clarifying many points about quasianalytic functions.

2. Quasianalytic functions

A "quasianalytic" function means (roughly speaking) a $\mathcal{C}^\infty$ function that is determined by its Taylor expansion at any point. Quasianalytic functions originate in E. Borel's ideas on generalization of the principal of analytic continuation. Borel showed that, if a sequence of complex number $\{A_k\}$ converges to 0 fast enough, then a series $\sum_{k=1}^\infty A_k/(z-a_k)$ converges normally together with all its derivatives on a "big" set of real line segments in a compact set. If the poles a_k accumulate everywhere on such a line segment, we get a quasianalytic function on the line segment that is nowhere analytic [Bo2].

Let $\mathcal{C}^\infty(U)$ denote the ring of $\mathcal{C}^\infty$ functions on an open subset U of $\mathbb{R}^n$. Let $f \in \mathcal{C}^\infty(U)$. For every $\alpha \in \mathbb{N}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, we write

$$f_\alpha \ := \ \frac{1}{\alpha!} D^\alpha f \ ,$$

where $\alpha! = \alpha_1! \cdots \alpha_n!$ and D^α denotes the partial derivative $\partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$. ($\mathbb{N}$ denotes the nonnegative integers.)

Let $m = \{m_0, m_1, \ldots\}$ denote a sequence of positive numbers.

Definition 2.1. $\mathcal{C}_m(U) := \{f \in \mathcal{C}^\infty(U)$: for every compact $K \subset U$, there are constants A, $B > 0$ such that

$$|f_\alpha(x)| \ \leq \ AB^{|\alpha|} m_{|\alpha|} \ ,$$

for all $\alpha \in \mathbb{N}^n$ and $x \in K$.

Hadamard raised the question of characterizing the sequences m such that the class $\mathcal{C}_m$ is *quasianalytic*; i.e., such that, if U is connected, then the Taylor series homomorphism

$$f \;\mapsto\; \widehat{f}_a(x) \;:=\; \sum_{\alpha\in\mathbb{N}^n} f_a(a)x^\alpha$$

from $\mathcal{C}_m(U)$ to the ring of formal power series in n indeterminates, in injective for any $a \in U$ [Ha, Bk. I, Ch. II]. The Denjoy-Carleman theorem [De], [Ca] is a solution of Hadamard's problem.

We assume that $m = \{m_k\}$ satifies the hypothesis

(2.2) $$\{m_k\} \text{ is logarithmically convex,}$$

or, equivalently,

$$\left\{\frac{m_{k+1}}{m_k}\right\} \text{ is increasing.}$$

(By "increasing", we mean "nondeceasing"; i.e., "$\leq$".) The hypothesis (1.1) implies that

$$m_j m_k \;\leq\; m_o m_{j+k}\ , \qquad \text{for all } j,k\ ,$$

and that

$$\{m_k^{1/k}\} \text{ is increasing.}$$

The first of these conditions implies that $\mathcal{C}_m(U)$ is a ring, and the second that $\mathcal{C}_m(U)$ contains the ring $\mathcal{O}(U)$ of real-analytic functions on U, for all open $U \in \mathbb{R}^n$.

Under the hypothesis (2.2), the Denjoy-Carleman theorem (see [Hö, Thm. 1.3.8], [Ru, Thm. 19.11]) asserts that $\mathcal{C}_m$ is quasianalytic if and only if

(2.3) $$\sum_{k=0}^{\infty} \frac{m_k}{(k+1)m_k} \;=\; \infty\ .$$

Note. In the literature, $\mathcal{C}_m$ is more usually denoted $\mathcal{C}_M$, where $M = \{M_k\}$ and each $M_k = k!m_k$. The use of $\{m_k\}$ instead of $\{M_k\}$ and $f_\alpha = D^\alpha f/\alpha!$ instead of

$D^\alpha f$ will be convenient for the estimates on derivatives that we make in Section 3 below.

If $m = \{m_k\}$ satisfies the hypotheses (2.2) and (2.3), then $\mathcal{C}_m$ is called a *Denjoy-Carleman class.*

If $f \in \mathcal{C}_m(U)$, then each partial derivative $f_{(j)} = \partial f/\partial x_j \in \mathcal{C}_{m^{+1}}(U)$, where m^{+1} denotes the shifted sequence

$$m^{+1} \ := \ \{m_{k+1}\}_{k\in\mathbb{N}} \ .$$

Clearly, if m satisfies (2.2) (respectively, (2.3)), then m^{+1} satisfies (2.2) (respectively, (2.3)). If $m = \{m_k\}$ satisfies the hypothesis

$$\sup \left(\frac{m_{k+1}}{m_k}\right)^{1/k} \ < \ \infty \ , \tag{2.4}$$

then $\mathcal{C}_{m^{+1}} = \mathcal{C}_m$, so that $\mathcal{C}_m$ is closed under differentiation. (Conversely, if $\mathcal{C}_m$ is closed under differentiation, then (2.4) holds (S. Mandelbrojt [M])).

3. $\mathcal{C}^\infty$ CLASSES THAT ADMIT RESOLUTION OF SINGULARITIES

Suppose that, for every open subset U of $\mathbb{R}^n$, $n \in \mathbb{N}$, we have an $\mathbb{R}$-subalgebra $\mathcal{C}(U)$ of $\mathcal{C}^\infty(U)$. Our desingularization thoerems require only the following assumptions (3.1)–(3.6) on $\mathcal{C}(U)$, for any open $U \in \mathbb{R}^n$.

(3.1) $\mathcal{P}(U) \subset \mathcal{C}(U)$, where $\mathcal{P}(U)$ denotes the algebra of restrictions to U of polynomial functions on $\mathbb{R}^n$.

(3.2) *$\mathcal{C}$ is closed under composition.* Suppose that V is an open subset of $\mathbb{R}^p$ and that $\varphi = (\varphi_1, \ldots, \varphi_p) : U \to V$ is a mapping such that each $\varphi_i \in \mathcal{C}(U)$. Then $g \circ \varphi \in \mathcal{C}(U)$, for all $g \in \mathcal{C}(V)$.

A mapping $\varphi : U \to V$ will be called a *$\mathcal{C}$-mapping* if $g \circ \varphi \in \mathcal{C}(U)$, for every $g \in \mathcal{C}(V)$. Write $\varphi = (\varphi_1, \ldots, \varphi_p)$. It follows from (3.1) and (3.2) that φ is a $\mathcal{C}$-mapping if and only if $\varphi_i \in \mathcal{C}(U)$, $i = 1, \ldots, p$.

(3.3) *$\mathcal{C}$ is closed under differentiation.* For all $f \in \mathcal{C}(U)$,

$$\frac{\partial f}{\partial x_i} \ \in \ \mathcal{C}(U) \ , \quad i = 1, \ldots, n.$$

(3.4) *$\mathcal{C}$ is quasianalytic.* If $f \in \mathcal{C}(U)$ and $\widehat{f}_a = 0$, where $a \in U$, then f vanishes in a neighbourhood of a.

Since $\{x : \widehat{f}_x = 0\}$ is closed in U, (3.4) is equivalent to the following property: If U is connected, then, for all $a \in U$, the Taylor series homomorphism $\mathcal{C}(U) \ni f \mapsto \widehat{f}_a \in \mathbb{R}[\![x]\!]$ is injective. ($\mathbb{R}[\![x]\!] = \mathbb{R}[\![x_1, \ldots, x_n]\!]$ denotes the ring of formal power series in $x = (x_1, \ldots, x_n)$ with coefficients in $\mathbb{R}$.

(3.5) *$\mathcal{C}$ is closed under division by a coordinate.* If $f \in \mathcal{C}(U)$ and

$$f(x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_n) \equiv 0 ,$$

then $f(x) = (x_i - a_i)h(x)$, where $h \in \mathcal{C}(U)$.

(3.6) *$\mathcal{C}$ is closed under inverse.* Let $\varphi : U \to V$ be a $\mathcal{C}$-mapping between open subsets U, V of $\mathbb{R}^n$. Suppose that $a \in U$, $\varphi(a) = b$ and the Jacobian matrix

$$\frac{\partial \varphi}{\partial x}(a) := \frac{\partial(\varphi_1, \ldots, \varphi_n)}{\partial(x_1, \ldots, x_n)}(a)$$

is invertible. Then there are neighbourhoods U' of a, V' of b, and a $\mathcal{C}$-mapping $\psi : V' \to U'$ such that $\psi(b) = a$ and $\varphi \circ \psi$ is the identity mapping of V'.

Property (3.6) is equivalent to the following *implicit function theorem in* $\mathcal{C}$: Suppose that U is open in $\mathbb{R}^n \times \mathbb{R}^p$ (with product coordinates $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_p)$. Suppose that $f_1, \ldots, f_p \in \mathcal{C}(U)$, $(a, b) \in U$, $f(a, b) = 0$ and $(\partial f/\partial y)(a, b)$ is invertible, where $f = (f_1, \ldots, f_p)$. Then there is a product neighbourhood $V \times W$ of (a, b) in U and a $\mathcal{C}$-mapping $g : V \to W$ such that $g(a) = b$ and

$$f\big(x, g(x)\big) = 0 , \quad x \in V.$$

Property (3.6) implies that $\mathcal{C}$ *is closed under reciprocal*; i.e., if $f \in \mathcal{C}(U)$ vanishes nowhere in U, then $1/f \in \mathcal{C}(U)$.

Under the conditions (3.1)-(3.6), we can use open subsets U of $\mathbb{R}^n$ and the algebras of functions $\mathcal{C}(U)$ as local models in order to define a category $\underline{\mathcal{C}}$ of *$\mathcal{C}$-manifolds* and *$\mathcal{C}$-mappings*. The dimension theory of $\underline{\mathcal{C}}$ follows from that of $\mathcal{C}^\infty$ manifolds. We will need two fundamental properties of such a category $\underline{\mathcal{C}}$:

Proposition 3.7. *A* smooth *subset of a $\mathcal{C}$-manifold is a $\mathcal{C}$-submanifold. In other words: Let M be a $\mathcal{C}$-manifold. Suppose that U is open in M, $g_1, \ldots, g_p \in \mathcal{C}(U)$, and the gradients* $\operatorname{grad} g_i$ *are linearly independent at every point of the zero set*

$$X \ := \ \{x \in U : \ g_i(x) = 0 \ , \quad i = 1, \ldots, p\} \ .$$

Then X is a closed $\mathcal{C}$-submanifold of U, of codimension p.

Proposition 3.7 is of course a consequence of the implicit function property (3.6).

Proposition 3.8. *$\underline{\mathcal{C}}$ is closed under blowing up with centre a closed $\mathcal{C}$-submanifold.*

Definition 3.9. A *blowing-up* of a $\mathcal{C}^\infty$ manifold M with *centre* a smooth closed subset C is a $\mathcal{C}^\infty$ mapping $\sigma : M' \to M$ from a $\mathcal{C}^\infty$ manifold M', that can be described in local coordinates as follows. (See also [BM2, p. 236].) First suppose that U is an open neighbourhood of 0 in $\mathbb{R}^n$, and that C is a coordinate subspace

$$C \ = \ \{x_i = 0 \ , \quad i \in I\} \ ,$$

where $I \subset \{1, \ldots, n\}$. The blowing-up $\sigma : U' \to U$ with centre C is a mapping where U' can be covered by coordinate charts U_i, $i \in I$, and each U_i has a coordinate system $(y_1, \ldots, y_n)$ in which σ is given by the formulas

$$\begin{aligned} &x_i = y_i \ , \\ &x_j = y_i y_j \ , \quad && j \in I \backslash \{i\} \ , \\ &x_j = y_j \ , && j \notin I \ . \end{aligned}$$

In general, if M is a $\mathcal{C}^\infty$ manifold and C a closed $\mathcal{C}^\infty$ submanifold of M, then every point of C admits a coordinate neighbourhood U in which C is a coordinate subspace as above; over this neighbourhood, the blowing-up $\sigma : M' \to M$ identifies with the mapping $U' \to U$ defined above, On the other hand, σ is a diffeomorphism over $M \backslash C$. The preceding conditions determine $\sigma : M' \to M$ uniquely, up to a diffeomorphism of M' commuting with the projections to M.

It is easy to see that if M is a $\mathcal{C}$-manifold and C is a closed $\mathcal{C}$-submanifold of M, then the blowing up $\sigma : M' \to M$ with centre C is a $\mathcal{C}$-mapping. This is the assertion of Proposition 3.8.

4. Denjoy-Carleman classes

Let $m = \{m_k\}_{k\in\mathbb{N}}$ denote a sequence of positive numbers satisfying (2.2); i.e., m is logarithmically convex. Since $\mathcal{O}(U) \subset \mathcal{C}_m(U)$ for all open subsets U of $\mathbb{R}^n$, $n \in \mathbb{N}$, then $\mathcal{C} = \mathcal{C}_m$ satisfies property (3.1). We will show that $\mathcal{C} = \mathcal{C}_m$ satisfies properties (3.2) and (3.6) below (Theorems 4.7 and 4.10). The following weaker version of (3.3) is obvious.

(3.3′) If U is open in $\mathbb{R}^n$ and $f \in \mathcal{C}_m(U)$, then each partial derivative $\partial f/\partial x_i \in \mathcal{C}_{m^{+1}}(U)$.

By the standard integral formula, we get the following weaker version of (3.5).

(3.5′) If $f \in \mathcal{C}_m(U)$ and $f(x_1,\ldots,x_{i-1},a_i,x_{i+1},\ldots,x_n) \equiv 0$, then $f(x) = (x_i - a_i)h(x)$, where $h \in \mathcal{C}_{m^{+1}}(U)$.

Therefore, if $\mathcal{C} = \mathcal{C}_m$, where m satisfies (2.4), or, more generally, if

$$\mathcal{C} \;=\; \bigcup_{j=0}^{\infty} \mathcal{C}_{m^{+j}} \;, \tag{4.1}$$

where m^{+j} denotes the shifted sequence

$$m^{+j} \;=\; \{m_{k+j}\}_{k\in\mathbb{N}} \;,$$

then $\mathcal{C}$ satisfies (3.1), (3.2), (3.3), (3.5) and (3.6). Of course, if m satisfies the Denjoy-Carleman condition (2.3), then $\mathcal{C} = \mathcal{C}_m$ and $\mathcal{C} = \bigcup \mathcal{C}_{m^{+j}}$ satisfy property (3.4).

Our proofs of properties (3.2) and (3.6) are based on a several-variable version of Faà de Bruno's formula [FdB]. Consider a composite function $h = f \circ g$, where $g(x) = \big(g_1(x),\ldots,g_p(x)\big)$, $x = (x_1,\ldots,x_n)$, and $f(y) = f(y_1,\ldots,y_p)$. Recall that $f_\alpha(y)$ denotes $D^\alpha f(y)/\alpha!$, $\alpha \in \mathbb{N}^p$. Write $g_\gamma := (g_{1,\gamma},\ldots,g_{p,\gamma})$, $\gamma \in \mathbb{N}^n$. Thus the $h_\gamma(x) = (f\circ g)_\gamma(x)$, $\gamma \in \mathbb{N}^n$, are the coefficients of the power series in u,

$$\sum_{\gamma\in\mathbb{N}^n} h_\gamma(x) u^\gamma \;,$$

obtained by substituting the power series

$$\sum_{\delta\in\mathbb{N}^n\setminus\{0\}} g_\delta(x)u^\delta$$

for $z=(z_1,\ldots,z_p)$ in the power series

$$\sum_{\alpha\in\mathbb{N}^p} f_\alpha\big(g(x)\big)z^\alpha \ .$$

Lemma 4.2. *Let $a_i\in\mathbb{R}^p$, $i=1,\ldots,\ell$, and let $\alpha=(\alpha_1,\ldots,\alpha_p)\in\mathbb{N}^p$. Then*

$$(a_1+a_2+\cdots+a_\ell)^\alpha \ = \ \sum \frac{\alpha!}{k_1!\cdots k_\ell!}a_1^{k_1}\cdots a_\ell^{k_\ell} \ ,$$

where the sum is taken over all $(k_1,\ldots,k_\ell)\in(\mathbb{N}^p)^\ell$ such that $\alpha=\sum_{i=1}^\ell k_i$.

Proof.

$$\begin{aligned}(a_1+a_2+\cdots+a_\ell)^\alpha \ &= \ \prod_{j=1}^p (a_{1j}+\cdots+a_{\ell j})^{\alpha_j} \\ &= \ \prod_{j=1}^p \left(\sum_{\substack{\alpha_j=\sum k_{ij},\\ k_{ij}\in\mathbb{N}}} \frac{\alpha_j!}{k_{1j}!\cdots k_{\ell j}!}a_{1j}^{k_{1j}}\cdots a_{\ell j}^{k_{\ell j}} \right) \ .\end{aligned}$$

In the expansion of the latter product, each term is a unique product of terms, one from each of the p factors in the product. □

Proposition 4.3. *(Faà de Bruno's formula in several variables.) For all $\gamma\in\mathbb{N}^n\setminus\{0\}$,*

$$h_\gamma(x) \ = \ \sum \frac{\alpha!}{k_1!\cdots k_\ell!}f_\alpha\big(g(x)\big)g_{\delta_1}(x)^{k_1}\cdots g_{\delta_\ell}(x)^{k_\ell} \ ,$$

where $\alpha=k_1+\cdots+k_\ell$ and the sum is taken over all sets $\{\delta_1,\ldots,\delta_\ell\}$ of ℓ distinct elements of $\mathbb{N}^n\setminus\{0\}$ and all ordered ℓ-tuples $(k_1,\ldots,k_\ell)\in(\mathbb{N}^p\setminus\{0\})^\ell$, $\ell=1,2,3,\ldots$, such that

$$\gamma \ = \ \sum_{i=1}^\ell |k_i|\delta_i \ .$$

Proof.

$$\sum_{\gamma\in\mathbb{N}^n} h_\gamma(x)u^\gamma \;=\; \sum_{\alpha\in\mathbb{N}^p} f_\alpha\big(g(x)\big)\left(\sum_{\delta\in\mathbb{N}^n\setminus\{0\}} g_\delta(x)u^\delta\right)^\alpha ,$$

so the result follows immediately from Lemma 4.2. □

In the remainder of this section, $m = \{m_k\}_{k\in\mathbb{N}}$ denotes a logarithmically convex sequence of positive numbers. The following inequality of Childress [Ch] is obviously connected to the Faà de Bruno formula in one variable.

Proposition 4.4. *Let $k_1,\ldots,k_n$ be nonnegative integers such that $k_1+2k_2+\cdots+nk_n = n$. Set $k = k_1+\cdots+k_n$. Then*

$$m_k m_1^{k_1}\cdots m_n^{k_n} \;\le\; m_1^k m_n .$$

Proof. The result is trivial if $k_n = 1$; we can therefore assume that $k_n = 0$.

Case I. $k_1 \neq 0$. Let $k_1' = k_1 - 1$, $k' = k-1$. Then $k' = k_1' + k_2 + \cdots + k_{n-1}$ and $n-1 = k_1' + 2k_2 + \cdots + (n-1)k_{n-1}$. By induction on n,

$$m_{k-1}m_1^{k_1'}m_2^{k_2}\cdots m_{n-1}^{k_{n-1}}m_n^{k_n} \;\le\; m_1^{k'}m_{n-1}$$

(remember $m_n^{k_n} = 1!$); thus

$$\begin{aligned} m_k m_1^{k_1}\cdots m_n^{k_n} &= m_1\frac{m_k}{m_{k-1}}m_{k-1}m_1^{k_1'}m_2^{k_2}\cdots m_n^{k_n} \\ &= m_1\frac{m_n}{m_{n-1}}m_1^{k'}m_{n-1} \\ &= m_1^k m_n . \end{aligned}$$

Case II. $k_1 = 0$. We have

$$n-k \;=\; k_2 + 2k_3 + \cdots + (n-k)k_{n-k+1} + \cdots ;$$

thus $k_j = 0$ if $j > n-k+1$, and $k = k_2 + \cdots + k_{n-k+1}$. By induction,

$$m_{k+1}m_2^{k_2}\cdots m_{n-k+1}^{k_{n-k+1}} \;\le\; m_2^k m_{n-k+1} ;$$

in other words,

$$m_{k+1}m_1^{k_1}\cdots m_n^{k_n} \;\le\; m_2^k m_{n-k+1}\ .$$

Therefore,

$$\begin{aligned} m_k m_1^{k_1}\cdots m_n^{k_n} &\le \frac{m_k}{m_{k+1}} m_2^k m_{n-k+1} \\ &\le m_1 m_2^{k-1} m_{n-k+1} \\ &\le m_1^2 m_2^{k-2} m_{n-k+2} \\ &\le \cdots \le m_1^k m_n\ . \end{aligned}$$

□

Corollary 4.5. *Let $k_1,\dots k_\ell \in \mathbb{N}^p\backslash\{0\}$ and $\delta_1,\dots,\delta_\ell \in \mathbb{N}^n\backslash\{0\}$. Set $\alpha = k_1 + \cdots + k_\ell$ and $\gamma = |k_1|\delta_1 + \cdots + |k_\ell|\delta_\ell$. Then*

$$m_{|\alpha|}m_{|\delta_1|}^{|k_1|}\cdots m_{|\delta_\ell|}^{|k_\ell|} \;\le\; m_1^{|\alpha|}m_{|\gamma|}\ . \tag{4.6}$$

Proof. This is a special case of Childress's inequality because the latter applies with some $k_i = 0$. (We can assume that all $|\delta_i|$ are distinct because if $|\delta_i| = |\delta_j|$ for some i and $j \ne i$, then we can replace $m_{|\delta_i|}^{|k_i|}m_{|\delta_j|}^{|k_j|}$ in the left-hand side of (4.6) by $m_{|\delta_i|}^{|k_i|+|k_j|}$.) □

Theorem 4.7. *(Composition; cf.* [Rou]*.) Let U and V denote open subsets of $\mathbb{R}^n$ and $\mathbb{R}^p$, respectively. Let $f \in \mathcal{C}_m(V)$ and let $g = (g_1,\dots,g_p) : U \to V$, where each $g_j \in \mathcal{C}_m(U)$. Then $f\circ g \in \mathcal{C}_m(U)$.*

Proof. Let K be a compact subset of U. Then there are constants a, b, c, $d > 0$ such that

$$\begin{aligned} |f_\alpha(y)| &\le ab^{|\alpha|}m_{|\alpha|}\ , &&\text{for all } y \in g(K), \alpha \in \mathbb{N}^p, \\ |g_{j,\delta}(x)| &\le cd^{|\delta|}m_{|\delta|}\ , &&\text{for all } x \in K, \delta \in \mathbb{N}^p, j = 1,\dots,p. \end{aligned}$$

Let $h = f\circ g$. Let $\gamma \in \mathbb{N}^n\backslash\{0\}$. By Proposition 4.3 and Corollary 4.5, if $x \in K$, then

$$\begin{aligned} |h_\gamma(x)| &\le a\sum \frac{\alpha!}{k_1!\cdots k_\ell!}(bc)^{|\alpha|}\delta^{|\gamma|}m_{|\alpha|}m_{|\delta_1|}^{|k_1|}\cdots m_{|\delta_\ell|}^{|k_\ell|} \\ &\le ad^{|\gamma|}m_{|\gamma|}\sum \frac{\alpha!}{k_1!\cdots k_\ell!}(bcm_1)^{|\alpha|}\ . \end{aligned}$$

By Lemma 4.8 following, there are constants C, D depending only on bcm_1, n and p, such that

$$\sum \frac{\alpha!}{k_1! \cdots k_\ell!}(bcm_1)^{|\alpha|} \;\leq\; CD^{|\gamma|} \;,$$

for all $\gamma \in \mathbb{N}^n \backslash \{0\}$. (The summation is always as in Proposition 4.3.) Thus,

$$|h_\gamma(x)| \;\leq\; aC(dD)^{|\gamma|} m_{|\gamma|} \;.$$

$\square$

Lemma 4.8. *Let $\lambda > 0$. Set $H(u) = \sum_{\gamma \in \mathbb{N}^n} H_\gamma u^\gamma$, $u = (u_1, \ldots, u_n)$, where $H_0 = 1$ and, for each $\gamma \in \mathbb{N}^n \backslash \{0\}$,*

$$H_\gamma = \sum \frac{\alpha!}{k_1! \cdots k_\ell!} \lambda^{|\alpha|}$$

(summation as in Proposition 4.3). Then H is a convergent power series.

Proof. Define

$$\begin{aligned} G_j(u_1, \ldots, u_n) \;&:=\; \prod_{i=1}^{n} \left(\frac{1}{1 - u_i} \right) - 1 \;, \quad j = 1, \ldots, p, \\ F(z_1, \ldots, z_p) \;&:=\; \prod_{j=1}^{p} \left(\frac{1}{1 - \lambda z_j} \right) \;, \end{aligned}$$

so that

$$\begin{aligned} G_j(u) \;&=\; \prod_{i=1}^{n} (1 + u_i + u_i^2 + \cdots) - 1 \\ &=\; \sum_{\delta \in \mathbb{N}^n \backslash \{0\}} u^\delta \;, \\ F(z) \;&=\; \sum_{\alpha \in \mathbb{N}^p} (\lambda z)^\alpha \;. \end{aligned}$$

Then $H = F \circ G$, by Proposition 4.3. $\square$

Remark 4.9. In the 1-variable case of Theorem 4.7 ($n = p = 1$), we can use Faà de Bruno's formula to show that the constants C and D in the proof can be taken more precisely as $C = bcm_1$, $D = 1 + bcm_1$ (cf, [KP, Proposition 1.3.3]).

Theorem 4.10. *(Inverse function theorem; cf.* [Kom]*.) Let $f : U \to V$ denote a $\mathcal{C}_m$-mapping between open subsets U, V of $\mathbb{R}^n$. Let $x_0 \in U$. Suppose that the Jacobian matrix $(\partial\varphi/\partial x)(x_0)$ is invertible. Then there are neighbourhoods U' of x_0, V' of $y_0 := f(x_0)$ and a $\mathcal{C}_m$-mapping $g : V' \to U'$ such that $g(y_0) = x_0$ and $f \circ g$ is the identity mapping of V'.*

Proof. Write $f = (f_1, \ldots, f_n)$. We can assume that f has a $\mathcal{C}^\infty$ inverse. Let K be a compact subset of U. Then there are constants a, $b > 0$ such that, for all $\alpha \in \mathbb{N}^n$, $x \in K$ and $i = 1, \ldots, n$,

$$|f_{i,\alpha}(x)| \;\le\; ab^{|\alpha|} m_{|\alpha|} \; .$$

Let $x_0 \in K$. Consider the solution $x = g(y)$ of the equation

$$f(x_0 + x) \;=\; f(x_0) + y \; .$$

We want to show there are constants c, $d > 0$ *independent of* $x_0 \in K$, such that

$$|g_{j,\beta}(0)| \;\le\; cd^{|\beta|} m_{|\beta|} \; ,$$

for all $\beta \in \mathbb{N}^n$ and $j = 1, \ldots, n$.

Write

$$f(x_0 + x) - f(x_0) \;=\; \frac{\partial f}{\partial x}(x_0) \cdot x - \frac{\partial f}{\partial x}(x_0) \cdot \varphi(x) \; ;$$

in other words,

$$\varphi(x) \;=\; x - \frac{\partial f}{\partial x}(x_0)^{-1} \big(f(x_0 + x) - f(x_0) \big) \; .$$

Set $\Theta(x) := (\partial f/\partial x)(x)^{-1} = \big(\theta_{ij}(x)\big)$. Thus,

$$g(y) \;=\; \Theta(x_0) \cdot y + \varphi\big(g(y)\big) \; ,$$

where $\varphi(0) = 0$, $(\partial\varphi/\partial x_k)(0) = 0$, $k = 1, \ldots, n$, and

$$\varphi_{i,\alpha}(0) \;=\; -\sum_{j=1}^{n} \theta_{ij}(x_0) f_{j,\alpha}(x_0) \; ,$$

for all $i = 1, \ldots, n$ and $|a| \geq 2$. Choose $r > 0$ such that

$$|\theta_{ij}(x)| \ \leq \ r$$

for all i, $j = 1, \ldots, n$ and $x \in K$. Then, for all i, α,

$$|\varphi_{i,\alpha}(0)| \ \leq \ nrab^{|\alpha|}m_{|\alpha|} \ .$$

By Proposition 4.3, if $|\gamma| \geq 2$, then

$$g_\gamma(0) \ = \ \sum \frac{\alpha!}{k_1! \cdots k_\ell!}\varphi_\alpha(0)g_{\delta_1}(0)^{k_1} \cdots g_{\delta_\ell}(0)^{k_\ell} \ ,$$

where only terms with $|\alpha| \geq 2$ are nonzero, so only $g_{\delta_j}(0)$ with $|\delta_j| < |\gamma|$ occur. (The latter remark is also clear from the method of undetermined coefficients applied to (4.12).)

Let $\Phi(x) = \sum \Phi_\alpha x^\alpha$ denote the *convergent* power series

$$\Phi(x) \ = \ \sum_{|\alpha| \geq 2} nra(m_1 b)^{|\alpha|}x^\alpha \ ,$$

and consider the system of equations

$$G_i(y) \ = \ \frac{r}{m_1}(y_1 + \cdots + y_n) + \Phi(G(y)) \ , \qquad i = 1, \ldots, n \ ; \tag{4.13}$$

write $G_i(y) = \sum G_{i,\gamma}y^\gamma$ and $G = (G_1, \ldots, G_n)$. Then necessarily $G_1 = \cdots = G_n$; the solution of (4.13) is convergent, so there are constants c, d depending only on n, m_1, r, a and b, such that

$$|G_{i,\gamma}| \ \leq \ cd^{|\gamma|}$$

for all i, γ. Note that *all* $G_{i,\gamma}$ *are nonnegative* (recursively by the Faà de Bruno formula, or by (4.13)). We claim that

$$|g_{i,\gamma}(0)| \ \leq \ G_{i,\gamma}m_{|\gamma|} \ , \qquad |\gamma| \geq 1 \ ; \tag{4.14}$$

this gives (4.11).

We prove (4.14) by induction on $|\gamma|$. To begin, consider $\gamma = (j)$ (where (j) denotes the multiindex with 1 in the j'th place and 0 elsewhere):

$$|g_{i,(j)}(0)| \;=\; |\theta_{ij}(x_0)| \;\leq\; r \;=\; G_{i,(j)}m_1 \;.$$

By Proposition 4.3, Corollary 4.5 and the induction hypothesis, if $|\gamma| \geq 2$, then

$$\begin{aligned}
|g_{i,\gamma}(0)| \;&\leq\; \sum \frac{\alpha!}{k_1!\cdots k_\ell!}|\varphi_{i,\alpha}(0)||g_{\delta_1}(0)^{k_1}|\cdots|g_{\delta_\ell}(0)^{k_\ell}| \\
&\leq\; \sum \frac{\alpha!}{k_1!\cdots k_\ell!} nrab^{|\alpha|}m_{|\alpha|}G_{\delta_1}^{k_1}\cdots G_{\delta_\ell}^{k_\ell}m_{|\delta_1|}^{|k_1|}\cdots m_{|\delta_\ell|}^{|k_\ell|} \\
&=\; \sum \frac{\alpha!}{k_1!\cdots k_\ell!}\Phi_\alpha G_{\delta_1}^{k_1}\cdots G_{\delta_\ell}^{k_\ell}\frac{m_{|\alpha|}}{m_1^{|\alpha|}}m_{|\delta_1|}^{|k_1|}\cdots m_{|\delta_\ell|}^{|k_\ell|} \\
&\leq\; m_{|\gamma|}\sum \frac{\alpha!}{k_1!\cdots k_\ell!}\Phi_\alpha G_{\delta_1}^{k_1}\cdots G_{\delta_\ell}^{k_\ell} \\
&=\; G_{i,\gamma}m_{|\gamma|} \;,
\end{aligned}$$

where the last equality is the Faà de Bruno formula for the coefficients of G_i, from (4.13). □

Remark 4.15. We can again get a more precise estimate in the 1-variable case. If $f \in \mathcal{C}_m$ and $f(g(x)) = x$, choose a, $b > 0$ such that

$$\begin{aligned}
|f_{(1)}| \;&\geq\; \frac{1}{am_1} \\
|f_{(n+1)}| \;&\geq\; \frac{1}{(n+1)am_1^2}b^n m_{n+1} \;, \qquad n \geq 1
\end{aligned}$$

(on a compact subset of $\mathbb{R}$). Then

$$|g_{(n)}| \;\leq\; (-1)^{(n-1)}\binom{1/2}{n}2^n a^n c^{n-1} m_n \;,$$

where $c = 2bm_1$ (cf. [KP, Theorem 1.4.3]).

5. Resolution of singularities

In this section, $\mathcal{C}$ denotes a class of $\mathcal{C}^\infty$ functions satisfying the hypotheses (3.1)-(3.6).

Spaces of class $\mathcal{C}$. Let M be a manifold of class $\mathcal{C}$. Let $\mathcal{O}^{\mathcal{C}} = \mathcal{O}^{\mathcal{C}}_M$ denote the sheaf of germs of functions of class $\mathcal{C}$ at points of M. We regard M as a local-ringed space $M = (|M|, \mathcal{O}^{\mathcal{C}}_M)$, where $|M|$ denotes the underlying topological space of M. (Each stalk $\mathcal{O}^{\mathcal{C}}_a$ of $\mathcal{O}^{\mathcal{C}}$ is a local ring.) We usually do not distinguish in notation between M and $|M|$. If $\dim_a M = n$, then the completion of $\mathcal{O}^{\mathcal{C}}_a$ in the Krull topology can be identified with the ring of formal power series in n indeterminates.

Let $\mathcal{I} \subset \mathcal{O}^{\mathcal{C}}$ denote a sheaf of ideals. We say that $\mathcal{I}$ is of *finite type* if, for each $a \in M$, there is an open neighbourhood U of a and finitely many sections $f_1, \ldots, f_q \in \mathcal{O}^{\mathcal{C}}(U) = \mathcal{C}(U)$ such that, for all $b \in U$, the stalk $\mathcal{I}_b$ is generated by the germs $f_{i,b}$ of the f_i at b.

Suppose that $\mathcal{I}$ is an ideal of finite type in $\mathcal{O}^{\mathcal{C}}$ (i.e., a subsheaf of ideals of finite type). Let

$$|X| \ := \ \operatorname{supp} \frac{\mathcal{O}^{\mathcal{C}}}{\mathcal{I}}\ , \qquad \mathcal{O}^{\mathcal{C}}_X \ := \ \left(\frac{\mathcal{O}^{\mathcal{C}}}{\mathcal{I}}\right)\Big| |X|\ .$$

We call $X = (|X|, \mathcal{O}^{\mathcal{C}}_X)$ a (*closed*) $\mathcal{C}$-*subspace* of M, and $|X|$ a (*closed*) $\mathcal{C}$-*subset*. (We again usually do not distinguish in notation between X and $|X|$.) Write $\mathcal{I} = \mathcal{I}_X$.

A closed $\mathcal{C}$-subspace X of M is a *hypersurface* if $\mathcal{I}_X$ is a principal ideal (i.e., a sheaf of principal ideals).

We say that a closed $\mathcal{C}$-subspace X of M is *smooth at* at point $a \in X$ (or that a is a *smooth point* of X) if $\mathcal{I}_{X,a}$ is generated by elements $f_1, \ldots, f_q$ whose gradients are linearly independent at a. Let $\operatorname{Sing} X \subset |X|$ denote the complement of the set of smooth points. By Proposition 3.7, a smooth $\mathcal{C}$-subspace of M is a $\mathcal{C}$-submanifold.

Suppose that $\varphi : N \to M$ is a $\mathcal{C}$-mapping of $\mathcal{C}$-manifolds. If $\mathcal{I} \subset \mathcal{O}^{\mathcal{C}}_M$ is an ideal of finite type, let $\varphi^{-1}(\mathcal{I}) \subset \mathcal{O}^{\mathcal{C}}_N$ denote the ideal sheaf $\varphi^*(\mathcal{I}) \cdot \mathcal{O}^{\mathcal{C}}_N$ whose stalk at each point $b \in N$ is generated by the ring of pull-backs $\varphi^*(\mathcal{I})_b$ of all elements of $\mathcal{I}_{\varphi(b)}$. If X is a closed $\mathcal{C}$-subspace of M, let $\varphi^{-1}(X)$ denote the closed $\mathcal{C}$-subspace of N determined by the ideal $\varphi^{-1}(\mathcal{I}_X)$.

Transformations by blowing up. Let M denote a $\mathcal{C}$-manifold, and C a closed $\mathcal{C}$-submanifold of M. Let $\sigma : M' \to M$ denote the blowing-up of M with centre C (Definition 3.9). Then $\sigma^{-1}(C)$ is a smooth closed hypersurface in M'; we write

y_{exc} to denote a generator of $\mathcal{I}_{\sigma^{-1}(C),a'}$, at any point $a' \in M'$. Let $\mathcal{I} \subset \mathcal{O}_M^{\mathcal{C}}$ be a sheaf of ideals of finite type.

Definition 5.1. If $a \in M$, then the *order of* $\mathcal{I}$ *at* a,

$$\mu_a(\mathcal{I}) \;:=\; \max\{k \in \mathbb{N} :\ \mathcal{I}_a \subseteq \underline{m}_a^k\}\ ,$$

where $\underline{m}_a = \underline{m}_{M,a}$ denotes the maximal ideal of $\mathcal{O}_{M,a}^{\mathcal{C}}$. If $a \in C$, then the *order of* $\mathcal{I}$ *along* C *at* a,

$$\mu_{C,a}(\mathcal{I}) \;:=\; \max\{k \in \mathbb{N} :\ \mathcal{I}_a \subset \mathcal{I}_{C,a}^k\}\ .$$

(If g is a germ of a function of class $\mathcal{C}$ at a, we define $\mu_a(g)$ and $\mu_{C,a}(g)$ in the same way; $\mu_a(g) = \mu_a(\mathcal{I})$ and $\mu_{C,a}(g) = \mu_{C,a}(\mathcal{I})$, where $\mathcal{I}$ is the ideal generated by g.)

Lemma 5.2. *Each point of* M *admits a neighbourhood* U *in which* $\mu_x(\mathcal{I})$ *takes only finitely many values and, for any* $d \in \mathbb{N}$, $Z_d := \{x \in U : \mu_x(\mathcal{I}) \geq d\}$ *is a closed* $\mathcal{C}$*-subset of* U.

(We say that $\mu_x(\mathcal{I})$ is *Zariski-semicontinuous (relative to the class* $\mathcal{C}$*)*; cf. [BM2, Lemma 3.10].)

Proof. Let $a \in M$. Let U be an open neighbourhood of a for which there are $g_1, \ldots, g_q \in \mathcal{I}(U)$ that generate $\mathcal{I}_x$, for all $x \in U$. Then, for any $d \in \mathbb{N}$, $Z_d = \{x \in U : D^\alpha g_i(x) = 0,\ |\alpha| < d,\ i = 1, \ldots, q\}$. After shrinking U if necessary, $\mu_x(\mathcal{I}) \leq \mu_a(\mathcal{I})$, for all $x \in U$. □

Lemma 5.3. *If* $a \in C$, *then*

$$\mu_{C,a}(\mathcal{I}) \;=\; \min\{\mu_x(\mathcal{I}) :\ x \in C \ \textit{near}\ a\}\ .$$

In particular, $\mu_{C,x}(\mathcal{I})$ *is locally constant on* C.

This is a simple exercise.

Definitions and Remarks 5.4. Let $\mathcal{I} \subset \mathcal{O}_M^{\mathcal{C}}$ be a sheaf of ideals of finite type. The *(weak) transform* $\mathcal{I}'$ *of* $\mathcal{I}$ *by* the blowing-up σ is a sheaf of ideals of finite type in

$\mathcal{O}^{\mathcal{C}}_{M'}$ defined as follows: Let $a' \in M'$ and $a = \sigma(a')$. Then $\mathcal{I}'_{a'}$ is the ideal generated by

$$g' \ := \ y_{\mathrm{exc}}^{-d}\, g \circ \sigma \ , \quad g \in \mathcal{I}_{X,a} \ , \tag{5.5}$$

where d is the largest power of y_{exc} that factors from all $g \in \mathcal{I}_{X,a}$. (If $a' \notin \sigma^{-1}(C)$, then we can take $y_{\mathrm{exc}} = 1$ at a', and $g' = g \circ \sigma$ in (5.5).)

It is easy to see that if $a' \in \sigma^{-1}(C)$, then $d = \mu_{C,a}(\mathcal{I})$; it follows from Lemma 5.3 that $\mathcal{I}'$ is of finite type.

Let $X \subset M$ denote a closed $\mathcal{C}$-subspace and let $X' \subset M'$ denote the closed $\mathcal{C}$-subspace of M' given by $\mathcal{I}_{X'} := \mathcal{I}'$, where $\mathcal{I}'$ is the above transform of $\mathcal{I} := \mathcal{I}_X$; X' is called the *weak transform of* X *by* σ.

Suppose that X *is a hypersurface.* In this case, X' is also called the *strict transform of* X *by* σ. If $a \in C$ and g denotes a generator of $\mathcal{I}_{X,a}$, then, for all $a' \in \sigma^{-1}(a)$, $\mathcal{I}_{X',a'}$ is the ideal generated by

$$g' \ := \ y_{\mathrm{exc}}^{-d}\, g \circ \sigma \ ,$$

where d is the largest power of y_{exc} that factors from $g \circ \sigma$.

If $\operatorname{codim} C = 1$, then the blowing-up σ is the identity mapping, but the transforms above still make sense. For example, if $\mathcal{I} = \mathcal{I}_C$ (or if $X = C$) , then $\mathcal{I}' = \mathcal{O}^{\mathcal{C}}_{M'}$ (or $X' = \emptyset$).

Remark 5.6. In general, the notions of weak and strict transform do not coincide; see [BM2, Section 3]. For Denjoy-Carleman classes (or spaces of class $\mathcal{C}$), it is not clear that the strict transform X' is, in general, even well-defined as a closed $\mathcal{C}$-subspace of M'. One proves that the strict transform is well-defined in a category of schemes or analytic spaces, for example, using Noetherianity of the local rings and "Oka-Cartan theory" of coherent sheaves. (See [BM2, Prop. 3.13 ff.].)

Desingularization theorems. Le M denote a $\mathcal{C}$-manifold. Let $\mathcal{I} \subset \mathcal{O}^{\mathcal{C}}_M$ denote a sheaf of ideals of finite type. We consider sequences of transformations

$$
\begin{array}{ccccccccccc}
\longrightarrow & M_{j+1} & \xrightarrow{\sigma_{j+1}} & M_j & \longrightarrow & \cdots & \longrightarrow & M_1 & \xrightarrow{\sigma_1} & M_0 & = & W \\
 & \mathcal{I}_{j+1} & & \mathcal{I}_j & & & & \mathcal{I}_1 & & \mathcal{I}_0 & = & \mathcal{I}|W \\
 & E_{j+1} & & E_j & & & & E_1 & & E_0 & = & \emptyset
\end{array}
\tag{5.7}
$$

where W is an open subset of M and, for each j:

(1) $\sigma_{j+1} : M_{j+1} \to M_j$ is a blowing-up with smooth centre $C_{j+1} \subset M_j$, $\mathcal{I}_{j+1}$ is the transform of $\mathcal{I}_j$ by σ_{j+1}, and E_{j+1} is the collection of *exceptional hypersurfaces*

$$E_{j+1} \; := \; E_j' \cup \{\sigma_{j+1}^{-1}(C_{j+1})\} \; ,$$

where E_j' denotes the collection of strict transforms H' by σ_{j+1} of all hypersurfaces $H \in E_j$.

(2) C_{j+1} and E_j *simultaneously have only normal crossings* (i.e., locally, we can choose coordinates with respect to which C_{j+1} is a coordinate subspace and E_j is a collection of coordinate hyperplanes.

We say that the blowing-up σ_{j+1} (or the centres C_{j+1}) in (5.7) are *admissible* (or *μ-admissible*) if, in addition, $\mu_a(\mathcal{I}_j)$ is locally constant on C_{j+1}, for each j.

If X is a closed $\mathcal{C}$-hypersurface in M and $\mathcal{I} = \mathcal{I}_X$, then each $\mathcal{I}_{j+1}$ in (5.7) is the ideal sheaf $\mathcal{I}_{X_{j+1}}$ of the strict transform X_{j+1} of X_j (where $X_0 = X|W$). In this case, we also write $\mu_{X,a} := \mu_a(\mathcal{I}_X)$; $\mu_{X,a}$ is called the *order of X at a*.

The condition (2) in (5.7) guarantees inductively that each E_{j+1} has only normal crossings (i.e., locally, we can choose coordinates with respect to which every element of E_{j+1} is a coordinate subspace), according to the following simple lemma.

Lemma 5.8. *Let $H_1, \ldots, H_q$ denote smooth $\mathcal{C}$-hypersurfaces in M that simultaneously have only normal crossings. Let $\sigma : M' \to M$ denote a blowing-up with centre C a smooth $\mathcal{C}$-subspace, such that C, $H_1, \ldots, H_q$ simultaneously have only normal crossings. Then the strict transforms $H_1', \ldots, H_q'$ together with $H_{q+1}' := \sigma^{-1}(C)$ simultaneously have only normal crossings.*

Theorem 5.9. *Let M denote a $\mathcal{C}$-manifold and let $\mathcal{I} \subset \mathcal{O}_M^{\mathcal{C}}$ be a sheaf of ideals of finite type. Let K be a compact subset of M. Then there is a neighbourhood W of K and a finite sequence (5.7) of admissible blowings-up σ_j, $j = 1, \ldots, k+1$, such that $\mathcal{I}_{k+1} = \mathcal{O}_{M_{k+1}}^{\mathcal{C}}$ and $\sigma^{-1}(\mathcal{I})$ is a normal-crossings divisor, where $\sigma : M_{k+1} \to W \subset M$ denotes the composite of the σ_j.*

In fact, there is a finite sequence (5.7) satisfying the preceding assertions and the additional condition that, if $\mathcal{J}_\sigma$ denotes the ideal generated by the Jacobian determinant of σ (with respect to any local coordinate systems), then $\mathcal{J}_\sigma \cdot \sigma^{-1}(\mathcal{I})$ is a normal-crossings divisor.

(*Normal-crossings divisor* means a principal ideal of finite type, generated locally by a monomial in suitable coordinates.)

Suppose that X is a closed $\mathcal{C}$-hypersurface in M. Clearly then, $\operatorname{Sing} X$ is a closed $\mathcal{C}$-subset of M. (It is defined locally by a generator g of $\mathcal{I}_X$ together with all first-order partial derivatives of g.)

Theorem 5.10. *Let M denote a $\mathcal{C}$-manifold, and let X be a closed $\mathcal{C}$ hypersurface in M. Set $\mathcal{I} = \mathcal{I}_X$. Let K be a compact subset of M. Then there is an open neighbourhood W of K and a finite sequence (5.7) of admissible blowings-up σ_j, $j = 1, \ldots, k$, such that:*

(1) for each $j = 0, \ldots, k-1$, either $C_{j+1} \subset \operatorname{Sing} X_j$ or X_j is smooth and $C_{j+1} \subset X_j \cap E_j$;

(2) X_k is smooth;

(3) X_k, E_k and the Jacobian ideal $\mathcal{J}_\sigma$ (cf. Theorem 5.9) simultaneously have only normal crossings.

Theorem 5.9 in the case of a sheaf of principal ideals is an immediate consequence of Theorem 5.10. (Let X be the $\mathcal{C}$-hypersurface determined by the ideal $\mathcal{I}$. Then X_k (from Theorem 5.10) is smooth and of codimension 1. Let σ_{k+1} be the blowing-up with centre $C_{k+1} = X_k$. Then σ_{k+1} is the identity, but the strict transform X_{k+1} of X_k is empty; i.e., $\mathcal{I}_{k+1} = \mathcal{O}_{M_{k+1}}^{\mathcal{C}}$.) Theorems 5.9 and 5.10 are, in fact, proved in [BM2] using the same *desingularization algorithm*; we refer to [BM2] for details, but the idea is very roughly as follows: There is an invariant

$$\mathrm{inv}_{\mathcal{I}}(a) \;=\; \big(\nu_1(a), s_1(a); \ldots; \nu_{t+1}(a)\big)\ , \qquad a \in M_j\ , \tag{5.11}$$

defined recursively over a sequence of transformations (5.7) whose successive centres are "$\mathrm{inv}_{\mathcal{I}}$-admissible", where $\nu_1(a) = \mu_a(\mathcal{I}_j)$ and $t \le \dim_a M_j$. Sequences of the form (5.11) can be ordered lexicographically; $\mathrm{inv}_c I(\cdot)$ takes only finitely many values in a neighbourhood W_j of the compact subset $K_j = \sigma_j^{-1}(K_{j-1})$ (where $K_0 = K$), and the maximum locus of $\mathrm{inv}_c I$ is a union of smooth closed $\mathcal{C}$-subsets of W_j having only normal crossings. The desingularization algorithm is given by taking as each successive centre C_{j+1} one of the components of the maximum locus of $\mathrm{inv}_{\mathcal{I}}$ on W_j. Theorems 5.9 and 5.10 follow from the basic properties of $\mathrm{inv}_{\mathcal{I}}$ (given in [BM2, Theorem 1.14]).

The desingularization theorems are proved in [BM2] in a language common to algebraic schemes and analytic spaces (in characteristic zero), or hypersurfaces of class $\mathcal{C}$ as here. (See [BM2, (0.1)].) In Section 7, we present a complete proof of a simple version of Theorem 5.10 (Theorem 5.12 below) that suffices for applications of the kind considered in Section 6 (or, for example, in [RSW]). The proof is similar to that of [BM1, Theorem 4.4], which is the source of the desingularization algorithms in [BM2], but is presented in a language that clearly involves only the properties (3.1)–(3.6) of a class $\mathcal{C}$.

Theorem 5.12. *Let M denote a $\mathcal{C}$-manifold, and let X denote a closed $\mathcal{C}$-hypersurface in M. Let K be a compact subset of M. Then there is a neighbourhood W of K and a surjective mapping $\varphi : W' \to W$ of class $\mathcal{C}$, such that:*
(1) φ is a composite of finitely many $\mathcal{C}$-mappings, each of which is either a blowing-up with smooth centre (that is nowhere dense in the smooth points of the strict transform of X) or a surjection of the form

$$\coprod_j U_j \;\to\; \bigcup_j U_j\ ,$$

where the latter is a finite covering of the target space by coordinate charts and $\coprod$ means disjoint union.

(2) The final strict transform X' of X is smooth, and $\varphi^{-1}(X)$ has only normal crossings. (In fact $\varphi^{-1}(X)$ and $\det d\varphi$ simultaneously have only normal crossings, where $d\varphi$ is the Jacobian matrix of φ with respect to any local coordinate system.)

We note two immediate consequences of Theorem 5.12. Let M denote a $\mathcal{C}$-manifold, and let X denote any closed $\mathcal{C}$-subset of M.

Corollary 5.13. *(Rectilinearization theorem.) Suppose that M is of (pure) dimension n. Let K be a compact subset of M. Then there are finitely many mappings of class $\mathcal{C}$, $\varphi_i : U_i \to M$, where each U_i is an open neighbourhood of the origin in $\mathbb{R}^n$, such that:*
(1) There is a compact subset L_i of U_i, for each i, such that $\bigcup \varphi_i(L_i)$ is a neighbourhood of K in M.
(2) For each i, $\varphi_i^{-1}(X)$ is a union of coordinate subspaces.

We can obtain Corollary 5.13 by applying Theorem 5.12 locally to the hypersurface defined by the product of local defining equations of X.

Corollary 5.14. *(Uniformization theorem.) There is a manifold N of class $\mathcal{C}$ and a proper $\mathcal{C}$-mapping $\varphi : N \to M$ such that $\varphi(N) = X$.*

Remark 5.15. The *dimension* of a closed $\mathcal{C}$-subset X of M is well-defined (for example, by Corollary 5.14 and invariance of domain). If X is a hypersurface, then Theorem 5.12 implies Corollary 5.14 with $\dim N = \dim X$. In general, Corollary 5.14 follows from Corollary 5.13, but without the equality of dimensions. (Theorem 5.12 does imply that, if X is a proper $\mathcal{C}$-subset of M, then each $\varphi_i^{-1}(X)$ is a union of proper coordinate subspaces of $\mathbb{R}^n$, in Corollary 5.13, and $\dim N < \dim M$, in Corollary 5.14.) In order to deduce Corollary 5.14 with equality of dimensions, in general, from Theorem 5.12 (by the argument of [BM1, Proof of Theorem 5.1], for example), we would need a positive answer to the following question:

Let X, $Y \subset M$ denote closed $\mathcal{C}$-subsets such that $\dim(X\backslash Y) = k$. Is there a closed $\mathcal{C}$-set Z such that $X\backslash Y \subset Z$ and $\dim Z = k$ (for example, when Y is a smooth hypersurface)?

6. Applications

In this section, we note three applications of resolution of singularities (or, more precisely, of the weaker version, Theorem 5.12, and its Corollaries). These results seem to be new for Denjoy-Carleman classes. Let $\mathcal{C}$ denote a class of $\mathcal{C}^\infty$ functions satisfying the hypotheses (3.1)–(3.6).

Topological Noetherianity. Let M be a manifold of class $\mathcal{C}$, and let $\mathcal{O}_a^{\mathcal{C}}$ denote the local ring of germs of functions of class $\mathcal{C}$ at a point $a \in M$. The completion of $\mathcal{O}_a^{\mathcal{C}}$ can be identified with the ring of formal power series over $\mathbb{R}$ in n indeterminates, where $n = \dim_a M$. The following are important questions for Denjoy-Carleman classes:

Is $\mathcal{O}_a^{\mathcal{C}}$ Noetherian? Or, equivalently, is the formal power series ring flat over $\mathcal{O}_a^{\mathcal{C}}$? (Or, is every finitely generated ideal closed in $\mathcal{O}_a^{\mathcal{C}}$?)

The following theorem is the topological version of Noetherianity.

Theorem 6.1. *Any decreasing sequence of closed $\mathcal{C}$-subsets of M, $X_1 \supseteq X_2 \supseteq \cdots$, stabilizes in some neighbourhood of a compact set K. (In other words, there exists k such that, in a neighbourhood of K, $X_j = X_k$ for all $j \geq k$.)*

Proof. We can assume that $X_1 \neq M$. By Corollary 5.14 (and Remark 5.15), there is a proper $\mathcal{C}$-mapping $\varphi : M_1 \to M$ such that $\dim M_1 < \dim M$ and $\varphi(M_1) = X_1$. Then $\varphi^{-1}(X_2) \supseteq \varphi^{-1}(X_3) \supseteq \cdots$ is a decreasing sequence of closed $\mathcal{C}$-subsets of M_1, so that the result follows by induction on the dimension of the ambient manifold. □

Łojasiewicz inequalities. Proofs of Łojasiewicz's inequalities (Theorem 6.2 below) depending only on Theorem 5.12 were given in [BM2, Section 2]. For Denjoy-Carleman classes, only more restrictive results, in dimension 2, were previously known [V].

Theorem 6.2. *I. Let M denote a manifold of class $\mathcal{C}$, and let $f,\ g \in \mathcal{C}(M)$. Suppose that $\{x : g(x) = 0\} \subseteq \{x : f(x) = 0\}$ in a neighbourhood of a compact set*

K. Then there exist c, $\lambda > 0$ such that

$$|g(x)| \;\geq\; c|f(x)|^{\lambda}$$

in a neighbourhood of K. (The infimum of such λ is a positive rational number.)

II. Let $f \in \mathcal{C}(U)$, where U is open in $\mathbb{R}^n$. Suppose that K is a compact subset of U on which $\mathrm{grad}\, f(x) = 0$ only if $f(x) = 0$. Then there exist $c > 0$ and μ, $0 < \mu \leq 1$, such that

$$|\mathrm{grad}\, f(x)| \;\geq\; c|f(x)|^{1-\mu}$$

in a neighbourhood of K. (Sup μ is rational.)

III. Let $f \in \mathcal{C}(U)$, where $U \subset \mathbb{R}^n$ is open. Set $Z = \{x \in U : f(x) = 0\}$. Suppose that $K \subset U$ is compact. Then there are $c > 0$ and $\nu \geq 1$ such that

$$|f(x)| \;\geq\; cd(x, Z)^{\nu}$$

in a neighbourhood of K. ($d(\cdot, Z)$ is the distance to Z. Inf ν is rational.)

Division.

Theorem 6.3. *Let W be an open subset of $\mathbb{R}^n$ (or a manifold of class $\mathcal{C}$) and let $\xi \in \mathcal{C}(W)$. Let $f \in \mathcal{C}^\infty(W)$. Suppose f is formally divisible by ξ (i.e., for all $a \in W$, $\widehat{f}_a$ is divisible by $\widehat{\xi}_a$ in the ring of formal power series). Then there exists $g \in \mathcal{C}^\infty(W)$ such that $f = \xi \cdot g$.*

Proof. We follow Atiyah's proof of the division theorem of Hormander and Łojasiewicz. (See [A], [Hö1], [Ł].) Let $\varphi : W' \to W$ be a mapping of class $\mathcal{C}$ as in Theorem 5.12, such that the pull-back $\varphi^*(\xi) := \xi \circ \varphi$ is locally a monomial times an invertible factor (in suitable coordinates). Since $\varphi^*(f)$ is formally divisible by $\varphi^*(\xi)$, it follows from property (3.5) that there is a $\mathcal{C}^\infty$ function g' on W' such that $\varphi^*(f) = \varphi^*(\xi) \cdot g'$.

Since f is formally divisible by ξ and $\widehat{\varphi}^*_b$ is injective, for all $b \in W'$ (where $\widehat{\varphi}^*_b$ denotes the formal pull-back homomorphism from Taylor series centred at $a = \varphi(b)$ to Taylor series centred at b), it follows that g' is *formally a composite with* φ; i.e., for all $a \in W$, there is a formal power series G_a at a such that $\widehat{g}'_b = \widehat{\varphi}^*_b(G_a)$, for all $b \in \varphi^{-1}(a)$. Moreover, G_a is uniquely determined since $\widehat{\varphi}^*_b$ is injective. It is enough to show there is a $\mathcal{C}^\infty$ function g on W such that $\widehat{g}_a = G_a$, for all a.

Arguing inductively over the tower of mappings of which φ is composed, it suffices to prove the following assertion: Let U denote a coordinate chart of class $\mathcal{C}$ in W, and let $\sigma : U' \to U$ denote a blowing-up of U with centre a coordinate subspace. If $\eta \in \mathcal{C}^\infty(U')$ is formally a composite with σ, then there exists $\zeta \in \mathcal{C}^\infty(U)$ such that $\eta = \sigma^*(\zeta)$. This assertion is a special case of Glaeser's composite function theorem [G] since σ is a very simple rational mapping. $\square$

7. Proof of the desingularization Theorem 5.12

We begin with a simple but important lemma on transformation of differential operators by blowing up (cf. [H2, Section 8, (1.1)], [EV, Lemma 4.5]). Consider a blowing-up $\sigma : U' \to U$, where U is an open neighbourhood of 0 in $\mathbb{R}^n$, with centre a coordinate subspace $C = \{x_i = 0\ ,\ i \in I\}$, where $I \subset \{1, \ldots, n\}$. We use the notation of Definition 3.9. Note that, for each $i \in I$, $y_{\text{exc}} = y_i$ generates $\mathcal{I}_{\sigma^{-1}(C)}$ in the chart U_i, and

$$U' \backslash \bigcup_{j \neq i} U_j \;=\; \{y \in U_i :\ y_j = 0\ ,\ j \in I\backslash\{i\}\}\ .$$

The following is an easy calculation.

Lemma 7.1. *Let f be a germ of a function of class $\mathcal{C}$ at a point $a \in C$. Let $e \in \mathbb{N}$. Suppose that $\mu_{C,a}(f) \geq e$. Then, for each $i \in I$:*

(1) If $j \notin I$, then

$$\frac{1}{y_i^{e-1}}\left(\frac{\partial f}{\partial x_j} \circ \sigma\right) \;=\; y_i \frac{\partial}{\partial y_j}\left(\frac{f \circ \sigma}{y_i^e}\right)\ .$$

(2) If $j \in I\backslash\{i\}$, then

$$\frac{1}{y_i^{e-1}}\left(\frac{\partial f}{\partial x_j} \circ \sigma\right) \;=\; \frac{\partial}{\partial y_j}\left(\frac{f \circ \sigma}{y_i^e}\right)\ .$$

(3) (If $j = i$, then)

$$\frac{1}{y_i^{e-1}}\left(\frac{\partial f}{\partial x_i} \circ \sigma\right) \;=\; e\frac{f \circ \sigma}{y_i^e} + y_i \frac{\partial}{\partial y_i}\left(\frac{f \circ \sigma}{y_i^e}\right) - \sum_{j \in I\backslash\{i\}} y_j \frac{\partial}{\partial y_j}\left(\frac{f \circ \sigma}{y_i^e}\right)\ .$$

Proof of Theorem 5.12. Our aim is to define the finite sequence of transformations comprising the mapping φ. At an intermediate step, we have both the strict transform of X and the accumulated exceptional hypersurfaces $H_1, \ldots, H_r$. Hence we consider this more general situation from the beginning:

Let M be a manifold of class $\mathcal{C}$. Let X denote a closed $\mathcal{C}$-hypersurface in M, and let $H_1, \ldots, H_r$ be smooth "exceptional" hypersurfaces in M, *which we do not necessarily assume have only normal crossings.* Let $a \in M$. Suppose that s exceptional hypersurfaces pass through a, say $H_1, \ldots, H_s$. There is a local coordinate chart U containing a, with coordinates $(x_1, \ldots, x_n)$ in which $a = 0$, such that $X|U$ is defined by an equation of class $\mathcal{C}$,

$$g(x_1, \ldots, x_n) \;=\; 0$$

(i.e., $g \in \mathcal{C}(U)$ generates $\mathcal{I}_{X,x}$, for all $x \in U$). Write

$$d(x) \;:=\; \mu_{X,x} \;=\; \mu_x(g)\;, \qquad x \in U\;.$$

Set $d := d(a)$. We can assume that $d(x) \leq d$, for all $x \in U$, and that no exceptional hypersurfaces other than $H_1, \ldots, H_s$ intersect U. After a linear coordinate change, we can assume that

$$\frac{\partial^d g}{\partial x_n^d} \;\neq\; 0 \quad \text{in } U$$

and that, for each $p = 1, \ldots, s$, H_p is defined in U by an equation $\lambda_p(x) = 0$, of class $\mathcal{C}$, and

$$\frac{\partial \lambda_p}{\partial x_n} \;\neq\; 0 \quad \text{in } U\;.$$

Let $z := \partial^{d-1} g / \partial x_n^{d-1}$. Then $\partial z / \partial x_n \neq 0$ in U, so that $\{z = 0\}$ defines a submanifold N of U, of class $\mathcal{C}$. y the implicit function theorem (property (3.6)), we cansolve $z = 0$ locally as

$$x_n \;=\; \varphi(x_1, \ldots, x_{n-1})\;,$$

where φ is of class $\mathcal{C}$. In fact, then, we can assume that

$$z \;=\; u\,(x_n - \varphi(x_1, \ldots, x_{n-1}))$$

in U, where u is nonvanishing and u is of class $\mathcal{C}$ (by property (3.5)). On course, $(x_1, \ldots, x_{n-1})$ restricts to a coordinate system on N; we write $\tilde{x} = (x_1, \ldots, x_{n-1})$ to denote this coordinate system.

After a coordinate change $x'_n = x_n - \varphi(x_1, \ldots, x_{n-1})$, $x'_j = x_j$, $j < n$, we can assume that $\varphi = 0$. In other words, we can assume that

$$N \ = \ \{z = 0\} \ , \quad \text{where } z \ = \ \frac{\partial^{d-1} g}{\partial x_n^{d-1}} \ ,$$

and that

$$z \ = \ u \cdot x_n \ ,$$

where u does not vanish in U. In particular:

(7.2) For all $x \in U$, $\mu_x(g) \geq d$ if and only if $x \in N$ (i.e., $x_n = 0$) and

$$\mu_{\tilde{x}} \left(\frac{\partial^q g}{\partial x_n^q} \Big| N \right) \ \geq \ d - q \ , \quad q = 0, \ldots, d-2 \ .$$

Consider also the exceptional hypersurfaces $H_1, \ldots, H_s$. Write

$$\begin{aligned} c_q(g) \ &:= \ \frac{\partial^q g}{\partial x_n^q} \Big| N \ , \quad q = 0, \ldots, d-2 \ , \\ b(\lambda_p) \ &:= \ \lambda_p \big| N \quad , \quad p = 1, \ldots, s \ . \end{aligned} \tag{7.3}$$

If $x \in U$, set

$$s(x) \ := \ \#\{p : \ x \in H_p \ , \ p = 1, \ldots, s\}$$

(so that $s(a) = s$). Extending (7.2), we have

$$\begin{aligned} &\{x \in U : \ \big(d(x), s(x)\big) = (d, s)\} \\ &\qquad = \{x \in U : \ \mu_x(g) \geq d \ , \ \mu_x(\lambda_p) \geq 1 \ , \ p = 1, \ldots, s\} \\ &\qquad = \{\tilde{x} \in N : \ \mu_{\tilde{x}}(c_q) \geq d - q \ , \ q = 0 \ , \ldots, d-2 \ , \\ &\qquad\qquad\qquad \mu_{\tilde{x}}(b_p) \geq 1 \ , \ p = 1 \ , \ldots, s\} \ , \end{aligned} \tag{7.4}$$

where each $c_q = c_q(g)$, $b_p = b(\lambda_p)$.

Claim. We can assume that every c_q or b_p that is not identically zero satisfies

$$(7.5)\qquad \begin{aligned} c_q^{d!/(d-q)} &= (\tilde{x}^{\Omega(q)})^{d!} c_q^* , \quad q = 0, \ldots, d-2 ,\\ b_p^{d!} &= (\tilde{x}^{\tau(p)})^{d!} b_p^* , \quad p = 1, \ldots, s , \end{aligned}$$

where each $\Omega(q)$, $\tau(p) \in \mathbb{Q}^{n-1}$, all $c_q^*(\tilde{x})$, $b_p^*(\tilde{x})$ are nonvanishing, and the collection of multiindices $\{\Omega(q), \tau(p)\}$ is totally ordered with respect to the natural partial ordering of $\mathbb{N}^{n-1}$. (If $\Omega, \tau \in \mathbb{N}^{n-1}$, then $\Omega \le \tau$ means $\Omega_j \le \tau_j$, $j = 1, \ldots, n-1$. $\tilde{x}^\Omega$ denotes the monomial $x_1^{\Omega_1} \cdots x_{n-1}^{\Omega_{n-1}}$.)

When the assumptions (7.5) are satisfied, we will say we are in the "monomial case". We will prove the claim below, by induction on dimension. But first we calculate the effect on our local equations of blowing up with suitable centre, since this calculation provides both the motivation for making the claim, and tools that we will need to complete the proof of the theorem once we reduce to the monomial case.

Effect of blowing up. Consider a blowing-up $\sigma : U' \to U$ with centre *a* $\mathcal{C}$-submanifold C of U in the *equimuiltiple locus of* $a = 0$ (i.e., in $\{x \in U : d(x) = d := d(a)\}$). Then $C \subset N$, by (7.2), so we can assume that

$$(7.6)\qquad C = \{\tilde{x} = (x_1, \ldots, x_{n-1}) \in N : \ x_i = 0 , \ i \in I\} ,$$

where $I \subset \{1, \ldots, n-1\}$. Then U' is covered by coordinate charts U_i, $i \in I$, and U_n, as in Definition 3.9.

Since $N = \{x_n = 0\}$, the strict transform N' of N lies in $\bigcup_{i\in I} U_i$. For each $i \in I$,

$$N' \cap U_i = \{y = (y_1, \ldots, y_n) : \ y_n = 0\} .$$

(We use the notation of Definition 3.9.)

Consider $i \in I$. Then the strict transform X' of X by σ is defined in U_i by $g'(y_1, \ldots, y_n) = 0$, where

$$g' = y_i^{-d} g \circ \sigma .$$

By Lemma 7.1 (2), for all $q = 0, \dots, d$,

$$\frac{\partial^q g'}{\partial y_n^q} \;=\; \frac{1}{y_i^{d-q}} \frac{\partial^q g}{\partial x_n^q} \circ \sigma \;.$$

In particular, $\mu_y(g') \le d$ if and only if $y \in N' = \{y_n = 0\}$ and

$$\mu_{\tilde{y}}\left(\frac{\partial^q g'}{\partial x_n^q} \Big| N' \right) \;\ge\; d - q \;, \quad q = 0, \dots, d-2 \;.$$

Moreover, writing $c_q(g') := (\partial^q g'/\partial y_n^q)|N'$, $q = 0, \dots, d-2$ (cf. (7.3)), we have

$$c_q(g') \;=\; y_{\mathrm{exc}}^{-(d-q)} c_q(g) \circ \tilde{\sigma} \;,$$

where $\tilde{\sigma} := \sigma|N' : N' \to N$; the latter is the blowing-up of N with centre C.

On the other hand, consider $a' \in U' \backslash \bigcup_{i \in I} U_i$. Then $d(a') < d(a)$ (where $d(a') := \mu_{a'}(g')$): Since $a' \notin N'$, $d(a') = d$ only if $a' \in \sigma^{-1}(C)$. But in the chart U_n, the intersection of $\sigma^{-1}(C)$ with the complement of $\bigcup_{i \in I} U_i$ is given by

$$\{y : \; y_i = 0 \;, \; i \in I \cup \{n\}\} \;.$$

Since $\partial^d g'/\partial y_n^d \ne 0$, it follows, by Lemma 7.1 (3) (applied successively with $(f, e) = \big(\partial^{d-1} g/\partial x_n^{d-1}\,, 1\big)$, $\big(\partial^{d-2} g/\partial x_n^{d-2}\,, 2\big), \dots, (g, d))$ that if $a' \in \sigma^{-1}(C)$, then $g'(a') \ne 0$.

Now consider also the exceptional hypersurfaces $H_1, \dots, H_s$. Suppose that the centre C of the blowing-up σ lies in the "equimultiple locus of a for the pair $\big(d(\cdot), s(\cdot)\big)$"; i.e., $C \subset \{x \in U : \big(d(x), s(x)\big) = (d, s)\}$. Define $s(x')$, $x' \in U'$, analogously to $s(x)$, using the strict transforms H'_p of the H_p by σ; i.e., $s(x') = \#\{p : x' \in H'_p, p = 1, \dots, s\}$. For each p, H'_p is defined locally by $\lambda'_p(y) = 0$, where

$$\lambda'_p \;=\; y_{\mathrm{exc}}^{-1} \lambda_p \circ \sigma \;.$$

Consider any chart U_i, $i \in I$ (under the assumption (7.6) above). Write $b'_p = b(\lambda'_p) := \lambda'_p \big| N'$ (cf. (7.3)). Then

$$b'_p \;=\; y_{\mathrm{exc}}^{-1} b_p \circ \tilde{\sigma} \;, \quad p = 1, \dots, s \;,$$

and $\big(d(x'), s(x')\big) = (d, s)$, where $x' \in U_i$, if and only if

$$\begin{aligned} \mu_{\tilde{x}'}(c'_q) \;&\ge\; d - q \;, \quad q = 0, \dots, d-2 \;, \\ \mu_{\tilde{x}'}(b'_p) \;&\ge\; 1 \qquad\;\;, \quad p = 1, \dots, s \;. \end{aligned}$$

To reduce to the monomial case. We apply the assertion of Theorem 5.12 by induction on dimension to (the hypersurface defined by) the function of class $\mathcal{C}$ on the $\mathcal{C}$-manifold N given by the product of all nonzero $c_q^{d!/(d-q)}$, all nonzero $b_p^{d!}$, and all nonzero differences between two functions from this list. The claim (7.5) is then a consequence of the following elementary lemma [BM1, Lemma 4.7].

Lemma 7.7. *Let $y = (y_1, \ldots, y_m)$. Let α, β, $\gamma \in \mathbb{N}^m$ and let $a(y)$, $b(y)$, $c(y)$ be nonvanishing germs of functions of class $\mathcal{C}$ at the origin of $\mathbb{R}^m$. If*

$$a(y)y^\alpha - b(y)y^\beta = c(y)y^\gamma \ ,$$

then either $\alpha \le \beta$ or $\beta \le \alpha$.

Remark 7.8. Suppose that $\tilde{\sigma}$ is a blowing-up of $N = \{x_n = 0\}$ with smooth centre $\tilde{C}$. We can assume that $\tilde{C} = \{x \in N : \ x_i = 0, \ i \in I\}$, where $I \subset \{1, \ldots, n-1\}$. Then $\tilde{\sigma}$ induces a blowing-up σ of U with centre $C = \{x \in U : x_i = 0, \ i \in I\}$. In each coordinate chart U_i , $i \in I$ (as defined above), the pull-back of g (which coincides with the strict transform) and the $\sigma^{-1}(H_p)$ (which coincide with the strict tranforms H'_p of the H_p) will retain the forms described above; in particular, the analogue of (7.4) still holds. (Each H'_p is smooth because C has only normal crossings with respect to each H_p, although C does not necessarily simultaneously have only normal crossings with respect to the collection $\{H_p\}$.) Note also that the centre C of the induced blowing-up σ does not lie in the equimultiple locus of $a = 0$. In these ways, Theorem 5.12 is weaker than Theorem 5.10 – this is the price we pay to get a much simpler proof.

The effect of reducing to the monomial case by the inductive argument above is that, in addition to (the strict transforms of) the "old" exceptional hypersurfaces $H_1, \ldots, H_s$, we have introduced "new" exceptional hypersurfaces corresponding to the blowings-up needed in the reduction. This means that, in addition to $H_1, \ldots, H_s$, we have a collection of "new" exceptional hypersurfaces that can be assumed each to be a coordinate subspace $x_j = 0$, where $1 \le j \le n-1$.

The monomial case. We assume (7.5) (and admit the possibility of other "new" exceptional hypersurfaces, each of the form $x_j = 0$, $1 \le j \le n-1$. We consider lexicographic ordering of pairs (d, s). Let S denote the equimultiple locus of $a = 0$ for the pair $\big(d(\cdot), s(\cdot)\big)$; i.e., $S := \{x \in U : \big(d(x), s(x)\big) = (d, s)\}$. Suppose that $d > 1$. Choose coordinates as above.

Remark 7.9. If all $c_q \equiv 0$, then $N = \{x \in U : \ d(x) = d\}$ and it follows from property (3.5) that $g(x) = v(x)z^d$ in a neighbourhood of N, where v is nonvanishing.

Consider the case that all c_q and b_p vanish identically. Let $\sigma : \ U' \to U$ be the blowing-up with centre $C = N$. If X' denotes the strict transform of X by σ, then $X' \cap \sigma^{-1}(C) = \emptyset$; i.e., $d(x') = 0$, for all $x' \in \sigma^{-1}(C)$.

Now suppose that not all c_q and b_p vanish identically. Then, by (7.4) and (7.5),

$$S \;=\; \{\tilde{x} = (x_1, \ldots, x_{n-1}) \in N : \ \mu_{\tilde{x}}(\tilde{x}^{\Omega}) \ge 1\} \ ,$$

where $\Omega := \min\{\Omega(q), \tau(p)\}$. (The meaning of the order of a monomial with rational powers is clear.) Then

$$S \;=\; \bigcup_I Z_I \ ,$$

where

$$Z_I \;:=\; \{\tilde{x} \in N : \ x_i = 0 \ , \quad i \in I\} \ ,$$

and I runs over the *minimal* subsets of $\{1, \ldots, n-1\}$ such that $\sum_{j\in I} \Omega_j \ge 1$; i.e., I runs over the subsets of $\{1, \ldots, n-1\}$ such that

$$0 \;\le\; \sum_{j\in I} \Omega_j - 1 \;<\; \Omega_i \ , \qquad \text{for all } i \in I \ .$$

Consider the blowing-up σ of U with centre $C = Z_I$, for any such I (so that U' is covered by coordinate charts U_i, $i \in I \cup \{n\}$, as before). In any chart U_i, $i \in I$, we have

$$\begin{aligned} c'_q(\tilde{y})^{d!/(d-q)} \;&=\; (\tilde{y}^{\Omega(q)'})^{d!}(c^*_q \circ \tilde{\sigma})(\tilde{y}) \ , \\ b'_p(\tilde{y})^{d!} \;&=\; (\tilde{y}^{\tau(p)'})^{d!}(b^*_p \circ \tilde{\sigma})(\tilde{y}) \ , \end{aligned}$$

where, for each $\zeta = \Omega(q)$ or $\tau(p)$,

$$\tilde{y}^{\zeta'} = y_1^{\zeta_1} \cdots y_i^{\sum_{j\in I} \zeta_j - 1} \cdots y_{n-1}^{\zeta_{n-1}} .$$

In particular, if $a' \in \sigma^{-1}(a) \in U_i$ and $\big(d(a'), s(a')\big) = (d, s) = \big(d(a), s(a)\big)$, then

$$1 \le |\Omega'| < |\Omega|$$

(where $|\Omega| := \Omega_1 + \cdots + \Omega_{n-1}$). (Recall that if $d(a') = d$, then necessarily $a' \in U_i$, for some $i \in I$.) In particular, $\big(d(a'), s(a')\big) < \big(d(a), s(a)\big) = (d, s)$ (throughout U') after at most $d!|\Omega|$ blowings-up (each with centre given by a coordinate subspace of a chart occuring as above. Note that, by the "monomial" assumption, in each chart, $d(a') = d$ (or $s(a') = s$) at some point a' only if $d(0) = d$ (or $s(0) = d$, respectively).

If (in some chart) we have $d(a') = d(a)$ but $s(a') < s(a)$, then we can simply continue: Some H_p' does not intersect N' near a'. We repeat the argument of the monomial case above without this exceptional divisor, using the new $\Omega = \min\{\Omega(q)', \tau(p)'\}$. Finally, then, after finitely many such blowings-up, we have $d(a') < d(a)$ throughout each coordinate chart over U.

At each step in the process, the strict transform H_p' of each original hypersurface H_p remains smooth, and each new exceptional hypersurface is a coordinate subspace $y_j = 0$, $1 \le j \le n-1$. (Thus N' and the collection of new exceptional hypersurfaces simultaneously have only normal crossings.) Moreover, each H_p' and the collection of new exceptional hypersurfaces simultaneously have only normal crossings.

If $d(a) = d = 1$, then $N = X$ and we can use the argument above to blow up until $s(a') = 0$, if $a' \in N'$. Since the new exceptional divisors simultaneously have only normal crossings with respect to N', we have the conclusion of the theorem except perhaps at points over U that are outside $N' = X'$.

It remains therefore to consider the case that $X = \emptyset$ and we have simply s smooth hypersurfaces $H_1, \ldots, H_s$. Locally, we can choose coordinates so that H_s is given by $x_n = 0$ and each H_p, $1 \le p < s$, is defined as before. (In fact, by the implicit function condition (3.6), we can assume that, for each $p = 1, \ldots, s-1$, H_p is defined by an equation of the form $x_n + b_p(x_1, \ldots, x_{n-1}) = 0$.) The theorem now follows essentially by repeating the argument above, with $N = H_s$. (For details, we refer to [BM1, Proof of Thm. 4.4, Case 2, p. 2727].)

Remark 7.10. There are variants of Theorems 5.9 and 5.12 in which we avoid blowing up with centre along which the space is "geometrically smooth" (or smooth with respect to the "reduced" structure). Let M be a manifold of class $\mathcal{C}$ and let X denote a closed $\mathcal{C}$-hypersurface in M. Let $a \in X$, and let g denote a generator of $\mathcal{I}_{X,a}$. Say $\mu_a(g) = d$. We say that X is *geometrically smooth at a* if

$$g(x) \;=\; v(x)h(x)^d \;,$$

where $v(x), h(x)$ are of class $\mathcal{C}$ and $v(a) \neq 0$. (Otherwise we say that a is a *geometrically singular* point.)

In terms of local coordinates as in the proof of Theorem 5.12 above, X is geometrically smooth at a if and only if

$$\frac{\partial^q g}{\partial x_n^q}\Big| N \;=\; 0 \;, \quad q = 0, \ldots, d-2 \;;$$

moreover, in this case we can take

$$h(x) \;=\; z \;:=\; \frac{\partial^{d-1} g}{\partial x_n^{d-1}}$$

(or $h(x) = x_n$); cf. Remark 7.9. It follows that if a is geometrically singular, then $\{x : \; \mu_x(g) \geq d\}$ contains no geometrically smooth point near a.

We can obtain the following variants of Theorems 5.9 and 5.12: In the statement of Theorem 5.9, replace the condition "$C_{j+1} \subset \operatorname{Sing} X_j$" in (1) by "$C_{j+1}$ lies in the geometrically singular locus", and replace (2) by "X_k is geometrically smooth". In the statement of Theorem 5.12, replace "centre (that is nowhere dense in the smooth points of the strict transform of X)" in (1) by "centre (that is nowhere dense in the geometrically smooth points of the strict transform of X)", and replace "The final strict transform X' of X is smooth" in (2) by "The final strict transform X' of X is geometrically smooth".

The only change needed in the proofs is to "stop the process sooner"; for example, in the proof above, we simply do not blow up with centre $C = N$ in the case that all c_q and b_p vanish identically (following Remark 7.9).

It would be interesting to show that *the geometrically singular locus of X is a closed $\mathcal{C}$-subset*.

References

[A] M.F. Atiyah, *Resolution of singularities and division of distributions*, Comm. Pure Appl. Math. **23** (1970), 145–150.

[BM1] E. Bierstone and P.D. Milman, *Semianalytic and subanalytic sets*, Inst. Hautes Études Sci. Publ. Math. **67** (1988), 5–42.

[BM2] E. Bierstone and P.D. Milman, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Inventiones Math. **128** (1997), 207–302.

[Bo1] E. Borel, *Sur la généralisation du prolongement analytique*, C.R. Acad. Sci Paris **130** (1900), 1115–1118.

[Bo2] E. Borel, *Sur les séries de polynômes et de fractions rationelles*, Acta. Math. **24** (1901), 309–387.

[Ca] T. Carleman, *Les fonctions quasi-analytiques*, Collection Borel, Gauthier-Villars, Paris, 1926.

[Ch] C.L. Childress, *Weierstrass division in quasianalytic local rings*, Can. J. Math. **28** (1976), 938–953.

[De] A. Denjoy, *Sur les fonctions quasi-analytiques de variable réelle*, C.R. Acad. Sci. Paris **173** (1921), 1320–1322.

[vdDM] L. van den Dries and C. Miller, *Geometric categories and o-minimal structures*, Duke Math. J. **84** (1996), 497–540.

[Dy1] E.M. Dyn'kin, *Pseudoanalytic extensions of smooth functions. The uniform scale*, Amer. Math. Soc. Translations (2) **115** (1980), 33–58.

[Dy2] E.M. Dyn'kin, *The pseudoanalytic extension*, J. Analyse Math. **60** (1993), 45–70.

[EV] S. Encinas and O. Villamayor, *Good points and constructive resolution of singularities*, Acta. Math. **181** (1998), 109–158.

[FdB] Faà de Bruno, *Note sur une nouvelle formule de calcul*, Quart. J. Pure Appl. Math. **1** (1857), 359–360.

[G] G. Glaeser, *Fonctions composées différentiables*, Ann. of Math. **77** (1963), 193–209.

[Ha] J.Hadamard, *Lectures on Cauchy's problem in linear partial differential equations*, Yale Univ. Press, New Haven, 1923.

[HJ] V. Havin and B. Jöricke, *The uncertainty principle in harmonic analysis*, Ergebnisse der Mathematik, Springer-Verlag, Berlin Heidelberg New York, 1994.

[Hi1] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero: I, II*, Ann. of Math. **79** (1964), 109–326.

[Hi2] H. Hironaka, *Idealistic exponents of singularity*, Algebraic geometry, J.J. Sylvester Sympos., Johns Hopkins Univ., Baltimore 1976, Johns Hopkins Univ. Press, Baltimore, 1977, pp. 52–225.

[Hö] L. Hörmander, *The analysis of linear partial differential operators I*, Springer-Verlag, Berlin Heidelberg New York, 1983.

[Kom] H. Komatsu, *The implicit function theorem for ultradifferentiable mappings*, Proc. Japan Acad. **55** (1979), 69–72.

[Koo] P. Koosis, *The logarithmic integral*, Vols. 1, 2, Cambridge Univ. Press, Cambridge, 1988, 1990.

[KP] S.G. Krantz and H.R. Parks, *A primer of real analytic functions*, Birkhäuser, Basel, 1992.

[Ł] S. Łojasiewicz, *Ensembles semi-analytiques*, Inst. Hautes Études Sci., Bures-sur-Yvette, 1964.

[M] S. Mandelbrojt, *Séries adhérentes, régularisation des suites, applications*, Collection Borel, Gauthiers-Villars, Paris, 1952.

[RSW] J.-P. Rolin, P. Speissegger and A.J. Wilkie, *Quasianalytic Denjoy-Carleman classes and o-minimality*, to appear (2001).

[Rou] C. Roumieu, *Ultradistributions définies sur $\mathbb{R}^n$ et sur certaines classes de variétés différentiables*, J. Analyse Math. **10** (1962–63), 153-192.

[Ru] W. Rudin, *Real and complex analysis*, McGraw-Hill, New York, 1966.

[T] V. Thilliez, *Quelques propriétés de quasi-analyticité*, Gaz. Math. **70** (1996), 49–68.

[V] A.L. Vol'berg, *The Łojasiewicz inequality for very smooth functions*, Soviet Math. Dokl. **41** (1990), 170–174.

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