QUATERNION MINIVERSALE DEFORMATIONS AND NONCOMMUTATIVE FROBENIUS MANIFOLDS.

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Abstract. We extend the classical topological Landau-Ginsburg model to a quaternion Landau-Ginsburg model, that satisfy the axioms of open-closed topological field theory. Later we prove, that its deformations form a space of quaternion miniversale deformation for the singularity A_n, and, moreover, it is non-commutative Frobenius manifold in means of [15].
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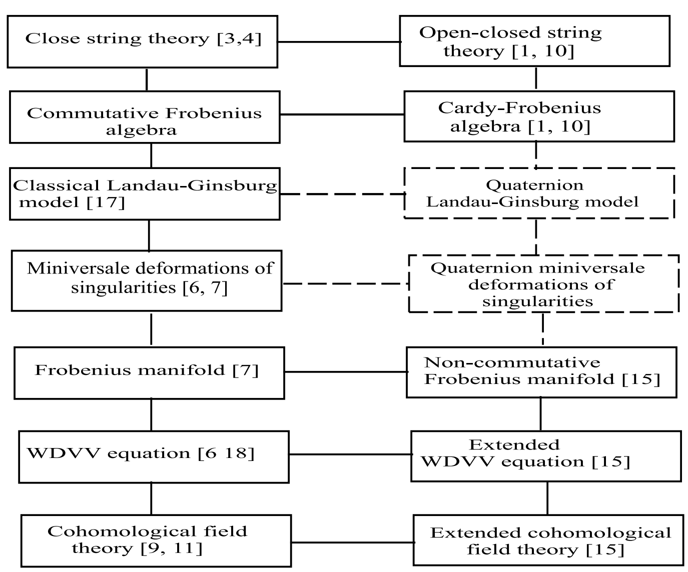
1. Introduction

In this paper we construct the dashed cells and dashed lines in the next scheme:



Explain this scheme in detail. The classical topological Landau-Ginsburg model was found by Vafa [17](see also Dijkgraaf- Witten [5]). It generates an algebra over $\mathbb{C}$ on the tangent space to a polynomial $p(z) = z^{n+1} + a_1z^{n-1} + a_2z^{n-2} + ... + a_n$ in the space $Pol(n)$ of all such polynomials. This algebra A_p is associative, with the unity and a linear functional $l_p : A_p \to \mathbb{C}$, such that the bilinear form $(d_1, d_2) = l_p(d_1d_2)$ is non degenerates. We call by Frobenius pairs all pairs (A_p, l_p) with such algebraic properties. Algebra A_p is commutative for classical topological Landau-Ginsburg model.

Commutative Frobenius pairs one-to-one correspond to topological field theories that appear from closed topological strings [3, 4, 7, 16]. These topological field theories naturally extend up to open-closed topological field theories, describing strings with boundary [10, 12], and even up to Kleinian topological field theories, describing strings with arbitrary world sheets [1]. In their turn, open-closed topological field theories one-to-one correspond to combinations from one commutative and one unrestricted Frobenius pairs, connected by Cardy condition [1]. We call such algebraic structure by Cardi-Frobenius algebra. Some classifications of Cardi-Frobenius algebras is contained in [1].

In the present paper we construct some extension of classical topological Landau-Ginsburg model to a Cardy-Frobenius algebra with a quaternion structure. Next we prove, that the set of such quaternion Landau-Ginsburg models over all polynomials $p(z) = z^{n+1} + a_1z^{n-1} + a_2z^{n-2} + ... + a_n$ form a non-commutative Frobenius manifold in means of [15].

Let us explain this result in more detail. The moduli space of the classical topological Landau-Ginsburg models coincides with the space $Pol(n)$ of miniversale deformation for the singularity of type A_n [2]. The metrics $(d_1, d_2) = l_p(d_1d_2)$ on the algebras A_p turn $Pol(n)$ into a Riemannian manifold with some additional properties [6, 7]. The differential-geometric structure, arising here, is an important example of Frobenius manifolds [7, 11]. The theory of Frobenius manifolds has a lot of applications in different areas of mathematics (integrable systems, singularity theory, topology of symplectic manifolds, geometry of moduli spaces of algebraic curves etc.).

The Dubrovin's theory of Frobenius manifolds [7] is a theory of flat deformations of commutative Frobenius pairs. As we discussed, Frobenius pairs are extended up to Cardy-Frobenius algebras. This suggests on extension of Frobenius manifolds to Cardy-Frobenius manifolds. An approach to this problem, is contained in [15]. It is based on Kontsevich-Manin cohomological field theory [9].

In this paper we define Cardy-Frobenius bundles as spaces of some flat deformations of Cardy-Frobenius algebras and prove, that they are

Cardy-Frobenius (noncommutative) manifolds as regards to [15]. Moreover we prove that the family of all the quaternion Landau-Ginsburg models form a Cardy-Frobenius bundle with quaternion structure.

2. Cardy-Frobenius algebras.

Following [10,12] describe the algebraic structure (Cardy-Frobenius algebra), connected with open-closed topological field theory. It is follow from [1], that Cardy-Frobenius algebras one-to-one correspond to open-closed topological field theories.

2.1.Frobenius pairs.

Definition. By *Frobenius pair* over a field $\mathbb{K}$ we call a set (D, l^D) where

1. D is an associative algebra over $\mathbb{K}$ with the unity element 1^D.
2. $l^D : D \to \mathbb{K}$ is a $\mathbb{K}$ - linear functional such that the bilinear form $(d_1, d_2)^D = l^D(d_1 d_2)$ is non degenerated.

In this case D is a Frobenius algebra [8].

Definition. By *orthogonal sum* of Frobenius pairs (D_1, l^{D_1}) and (D_2, l^{D_2}) we call the Frobenius pair $(D, l^D) = (D_1, l^{D_1}) \oplus (D_2, l^{D_2})$, where $D = D_1 \oplus D_2$ is the direct sum of algebras ($d_1 d_2 = 0$ for $d_i \in D_i$) and $l^D = l^{D_1} \oplus l^{D_2}$.

Exampele 2.1. $\mathbb{K}$ - numbers.

$\mathbb{K}(\lambda) = (\mathbb{K}, l_\lambda)$, where $\lambda \neq 0 \in \mathbb{K}$ and $l_\lambda(z) = \lambda z$ for $z \in \mathbb{K}$.

Exampele 2.2. Matrixes over $\mathbb{K}$.

$\mathbb{M}(n, \mathbb{K})(\mu) = (\mathbb{M}(n, \mathbb{K}), l_\mu)$, where $\mathbb{M}(n, \mathbb{K})$ is the algebra of $n \times n$ $\mathbb{K}$-matrixes, $\mu \neq 0 \in \mathbb{K}$ and $l_\mu(z) = \mu \mathrm{tr}(z)$ for $z \in \mathbb{M}(n, \mathbb{K})$.

Exampele 2.3. Quaternions over $\mathbb{K}$.

Let $\mathbb{R}_\mathbb{K} \in \mathbb{K}$ be a subfield, isomorphic to the field of real numbers. Let $\mathbb{H}_\mathbb{R}$ be the algebra of quaternions, that is the algebra over $\mathbb{R}$, generated by vectors $1^\mathbb{H}$, I, J, K, where $IJ = K, JK = I, KI = J$. Use the isomorphism $\mathbb{R}_\mathbb{K} \cong \mathbb{R}$ for define $\mathbb{H}_\mathbb{K} = \mathbb{H}_\mathbb{R} \bigotimes_\mathbb{R} \mathbb{K}$. We will consider $1^\mathbb{H}$, I, J, K also as a basis of $\mathbb{H}_\mathbb{K}$ over $\mathbb{K}$. Put $\mathbb{H}_\mathbb{K}(\rho) = (\mathbb{H}_\mathbb{K}, l_\rho)$ where $\rho \neq 0 \in \mathbb{K}$ and $l_\rho : \mathbb{H}_\mathbb{K} \to \mathbb{K}$ be the $\mathbb{K}$-linear functional, defined by $l_\rho(1^\mathbb{H}) = 2\rho$, $l_\rho(I) = l^\mathbb{H}(J) = l^\mathbb{H}(K) = 0$.

The correspondance

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mapsto 1^\mathbb{H}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \mapsto I, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mapsto J, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \mapsto K$$

defines an isomorphism between $\mathbb{M}(2, \mathbb{K})(\rho)$ and $\mathbb{H}_\mathbb{K}(\rho)$.

2.2 Cardy-Frobenius algebras.

Definition. By *Cardy-Frobenius algebra* over $\mathbb{K}$ we call a set $\{(A, l^A), (B, l^B), \phi\}$, where

1. (A, l^A) and (B, l^B) are Frobenius pairs over $\mathbb{K}$ and A is a commutative algebra;

2. $\phi : A \to B$ is a homomorphism of algebras, such that $\phi(A)$ belong to centre of B;

3. For any $x, y \in B$ is fulfilled $(\phi^*(x), \phi^*(y))^A = tr(W_{x,y})$, where $\phi^* : B \to A$ is the linear operator, such that $(a, \phi^*(b))^A = (\phi(a), b)^B$, for $a \in A, b \in B$, and $W_{x,y} : B \to B$ is the linear operator, defined by $W_{x,y}(b) = xby$.

The condition of this type first appear in works of Cardy and usually has him name. Let us give a coordinate representation of Cardy condition, using bases $(\alpha_1, ...\alpha_n) \subset A$ and $(\beta_1, ...\beta_n) \subset B$. Let $\{F^{\alpha_i,\alpha_j}\}$ and $\{F^{\beta_i,\beta_j}\}$ be matrixes inverse to $F_{\alpha_i,\alpha_j} = (\alpha_i, \alpha_j)^A$ and $F_{\beta_i,\beta_j} = (\beta_i, \beta_j)^B$. Then condition 3. is equivalent to

3'. For any $x, y \in B$ is fulfilled $F^{\alpha_i,\alpha_j} l^B(\phi(a_i)x) l^B(\phi(a_j)y) = F^{\beta_k,\beta_j} l^B(x\beta_j y\beta_k) = F^{\beta_k,\beta_j} l^B(\beta_j x \beta_k y)$.

Proof of equivalent. The equality $(\phi^*(x), \phi^*(y))^A = F^{\alpha_i,\alpha_j} l^B(\phi(a_i)x) l^B(\phi(a_j)y)$ is obviously. Consider a basis of B such that $(\beta_k, \beta_j) = \delta_{kj}$. Then $F^{\beta_k,\beta_j} l^B(x\beta_j y\beta_k) = F^{\beta_k,\beta_k} l^B(x\beta_k y\beta_k) = \sum_{k=1}^n (W_{x,y}(\beta_k), \beta_k)^B = tr(W_{x,y})$.

Exampele 2.4. $\mathbb{K}$ - numbers and matrixes.

$\{\mathbb{K}(\mu^2), \mathbb{M}(n, \mathbb{K})(\mu), \phi_M\}$, where $\phi_M : \mathbb{K} \to \mathbb{M}(n, \mathbb{K})$ is the naturel homomorphism of numbers to diagonal matrixes.

Check the axiom 3. Choice the elementary matrixes E_{ij} as a base of $\mathbb{M}(n, \mathbb{K})$. Left and right parts of axiom 3' are equal to 0, if one of the matrixes x or y are not diagonal. Left and right and parts of axiom 3' are equal to 1 for $x = E_{ii}$, $y = E_{jj}$.

Definition. By *orthogonal sum* of Cardy-Frobenius algebras $\{(A_1, l^{A_1}), (B_1, l^{B_1}), \phi_1\}$ and $\{(A_2, l^{A_2}), (B_2, l^{B_2}), \phi_2\}$ we call the Cardy-Frobenius algebra $\{(A, l^A), (B, l^B), \phi\} = \{(A_1, l^{A_1}), (B_1, l^{B_1}), \phi_1\} \oplus \{(A_2, l^{A_2}), (B_2, l^{B_2}), \phi_2\}$, where $(A, l^A) = (A_1, l^{A_1}) \oplus (A_2, l^{A_2})$, $(B, l^B) = (B_1, l^{B_1}) \oplus (B_2, l^{B_2})$ and $\phi = \phi_1 \oplus \phi_2$.

A Cardy-Frobenius algebra is called *semisimple*, if A and B are semisimple algebras. It is follow from [1], that any semisimple Cardy-Frobenius algebra is isomorphic to orthogonal sum of some numbers of algebras of $\{\mathbb{K}(\mu_i^2), \mathbb{M}(n, \mathbb{K})(\mu_i), \phi_M\}$ and some numbers of algebras $\{\mathbb{K}(\lambda_i), 0, 0\}$.

Exampele 2.5. $\mathbb{K}$ - numbers and quaternions.

$\{\mathbb{K}(\rho^2), \mathbb{H}_{\mathbb{K}}(\rho), \phi_{\mathbb{H}}\}$, where homomorphism $\phi_{\mathbb{H}} : \mathbb{K} \to \mathbb{H}$ is defined by $\phi_{\mathbb{H}}(1) = 1^{\mathbb{H}}$.

The isomorphism between $\mathbb{M}(2, \mathbb{K})(\rho)$ and $\mathbb{H}_{\mathbb{K}}(\rho)$ generate the isomorphism between $\{\mathbb{K}(\rho^2), \mathbb{M}(2, \mathbb{K})(\rho), \phi_M\}$ and $\{\mathbb{K}(\rho^2), \mathbb{H}_{\mathbb{K}}(\rho), \phi_{\mathbb{H}}\}$.

3. Quaternion Landau-Ginsburg models.

The classical topological Landau-Ginsburg model [17] of degree n is generated by a complex polynomial p in the form $p(z) = z^{n+1} + a_1 z^{n-1} + a_2 z^{n-2} + ... + a_n$, such that all roots $\alpha_1, ..., \alpha_n$ of its derivative $p'(z)$ are simple.

The set of all such polynomials form a complex manifold $Pol(n)$ of complex dimension n. Its tangent space A_p in a point p consists of all polynomials of degree $n-1$. The Landau-Ginsburg model generates on A_p a structure of algebra, where the multiplication $q = q_1 *_p q_2$ for polynomials $q_1 = q_1(z)$ and $q_2 = q_2(z) \in A_p$ is defined by condition $q(z) = q_1(z)q_2(z)(\mathtt{mod} p'(z))$.

Moreover, the Landau-Ginsburg model generates on A_p a non-degenerated bilinear form $(q_1, q_2)_p = l_p(q_1 q_2) = l_p(q_1 *_p q_2)$, where $l_p(q) = \frac{1}{2\pi i}\oint \frac{q(z)dz}{p'(z)}$. (Here and late the formula $\frac{1}{2\pi i}\oint$ means "minus residue in ∞"). Thus A_p has a structure of Frobenius algebra.

The polynomials $e_{p,\alpha_i}(z) = \prod_{j\neq i}\frac{z-\alpha_j}{\alpha_i-\alpha_j}$ form a basis of idempotents of A_p, that is $e_{p,\alpha_i}e_{p,\alpha_j} = \delta_{ij}e_{p,\alpha_i}$. Thus A_p is a semi-simple algebra. Put $\mu_{p,\alpha_i} = l_p(e_{p,\alpha_i}) = \frac{1}{n+1}\prod_{j\neq i}\frac{1}{\alpha_i-\alpha_j}$. Let A_{p,α_i} be the complex vector space generated by e_{p,α_i} and $l^A_{p,\alpha_i} = l_p|_{A_{p,\alpha_i}}$. Then $(A_{p,\alpha_i}, l^A_{p,\alpha_i})$ is a Frobenius pair, isomorphic to $\mathbb{C}(\mu_{p,\alpha_i})$.

Consider the Frobenius pair $(B_{p,\alpha_i}, l^B_{p,\alpha_i})$, where $B_{p,\alpha_i} = A_{p,\alpha_i}\otimes_{\mathbb{C}} \mathbb{H}_{\mathbb{C}} \cong \mathbb{H}_{\mathbb{C}}$ and l^B_{p,α_i} is defined by $l^B_{p,\alpha_i}(1^{\mathbb{H}}) = 2\rho_{p,\alpha_i}$, $\rho^2_{p,\alpha_i} = \mu_{p,\alpha_i}$, $l^B_{p,\alpha_i}(I) = l^B_{p,\alpha_i}(J) = l^B_{p,\alpha_i}(K) = 0$. Let us define the homomorphism $\phi_{p,\alpha_i} : A_{p,\alpha_i} \to B_{p,\alpha_i}$ by $\phi_{p,\alpha_i}(e_{p,\alpha_i}) = e_{p,\alpha_i}\otimes_{\mathbb{C}}\mathbb{H}_{\mathbb{C}}$. Then $\{(A_{p,\alpha_i}, l^A_{p,\alpha_i}), (B_{p,\alpha_i}, l^B_{p,\alpha_i}), \phi_{p,\alpha_i}\}$ is a Cardy-Frobenius algebra, isomorphic to $\{\mathbb{C}(\rho^2_{p,\alpha_i}), \mathbb{H}_{\mathbb{C}}(\rho_{p,\alpha_i}), \phi_{\mathbb{H}}\}$.

Put $B_p = \bigoplus_{i=1}^n B_{p,\alpha_i} = A_p\otimes_{\mathbb{C}}\mathbb{H}_{\mathbb{C}}$. Let $\phi_p : A_p \to A_p\otimes_{\mathbb{C}}\mathbb{H}_{\mathbb{C}} = B_p$ be the natural homomorphism. By *quaternion Landau-Ginsburg model* we call the Cardy-Frobenius algebra $\{(A_p, l^A_p), (B_p, l^B_p), \phi_p\} = \bigoplus_{i=1}^n\{(A_{p,\alpha_i}, l^A_{p,\alpha_i}), (B_{p,\alpha_i}, l^B_{p,\alpha_i}), \phi_{p,\alpha_i}\}$.

4. Frobenius manifolds.

4.1. Frobenius manifolds and WDVV equations.

In middle of 90 years of the last century B.Dubrovin found and investigated a class of "flat" deformation of commutative Frobenius algebras that appear in different domain of mathematics. He calls this structure Frobenius manifolds [7]. *Frobenius manifold* is a manifold with a Dubrovin connection. Present some equivalent definitions of the Dubrovin connection.

Definition.([7]) Let M be a smooth (real or complex) manifold. By *Dubrovin connection* on M is called is a family of commutative Frobenius pairs (M_p, θ_p) on the tangent spaces $M_p = T_pM$ for any point $p \in M$ such that

1. Tensors $\theta = \{\theta_p | p \in M\}$, $g(a, b) = \theta(ab)$, $c(a, b, d) = \theta(abd)$ are smooth and $\mathbf{d}\theta = 0$.

2. The Levi-Civita connection ∇ of the matric g is flat and such that $\nabla e = 0$, where e is the field of unity elements of the algebras M_p.

3. Tensors $c(a,b,d)$ and $\nabla_f c(a,b,d)$ are symmetrical by variables a,b,d,f.

4. There exists a vector field E (it is called *Euler field*), such that $\nabla(\nabla E)=0$.

A Frobenius manifold M is called semi-simple, if M_p is a semi-simple algebra for any point $p\in M$.

If $\theta(e)=0$, then Frobenius manifold has a special *flat quasi-homogeneous coordinate system* $t=(t^1,...,t^n)$, such that $g=\sum_{ij}\delta_{i+j,n+1}\mathbf{d}t^i\bigotimes\mathbf{d}t^j$, $e=\partial/\partial t^1$; $d_n=1$, $d_ir_i=0$, $d_i+d_{n+1-i}=\upsilon+2$ for all i; $E=\sum_{i=1}^n(d_it^i+r_i)(\partial/\partial t^i)$ for some constants d^i,r_i. We call such Dubrovin connection *anti-diagonale*.

Recall, that any semi-simple commutative Frobenius algebra is a direct sum of one-dimensional one [8]. Thus it has a *canonical basis* $e_1,...,e_n$. This is a basis with the properties $e_ie_j=\delta_{ij}e_i$, $l(e_i)\neq 0$. The canonical basis is defined uniquely up to enumeration of its elements.

Definition.([13]) A semi-simple anti-diagonale Dubrovin connection on a smooth (real or complex) manifold M is a family of commutative Frobenius pairs (M_p,θ_p) on the tangent spaces $M_p=T_pM$ for any point $p\in M$ such that

1. Tensors $\theta=\{\theta_p|p\in M\}$, $g(a,b)=\theta(ab)$, $c(a,b,d)=\theta(abd)$ are smooth and $\mathbf{d}\theta=0$

2. There exists a covering $M=\bigcup_\alpha U_\alpha$ by coordinate maps $(x^1_\alpha,...,x^n_\alpha):U_\alpha\to\mathbb{K}^n$, such that: a) the vectors $(\partial/\partial x^1_\alpha,...,\partial/\partial x^n_\alpha)$ form a canonical basis in M_p for any $p\in U_\alpha$; b) the field $E=\sum_{i=1}^n x^i_\alpha(\partial/\partial x^i_\alpha)$ don't depend from α; c)$L_E\theta=(\upsilon+1)\theta$, where L_E is the Li derivative by E. Such coordinate system is call *canonical*.

3. The Levi-Civita connection ∇ of the matric g is flat. It exists a coordinate system $t=(t^1,...,t^n)$ on M, such that $g=\sum_{ij}\delta_{i+j,n+1}\mathbf{d}t^i\bigotimes\mathbf{d}t^j$, $e=\partial/\partial t^1$ and $E=\sum_{i=1}^n(d_it^i+r_i)(\partial/\partial t^i)$, where d_i,r_i are constants.

By *potential* of Dubrovin connection (M_p,θ_p) on flat quasi-homogeneous coordinates $t=(t^1,...,t^n)$, is called a function $F(t^1,...,t^n)$ such that $\theta_p(\frac{\partial}{\partial t^i}\frac{\partial}{\partial t^j}\frac{\partial}{\partial t^k})=\frac{\partial F}{\partial t^i\partial t^j\partial t^k}$.

Definition. Let $F(t^1,...,t^n)$ be a function on a set $U\subset\mathbb{C}^n=(t^1,...,t^n)$. Let $E=\sum_{i=1}^n(d_it^i+r_i)(\partial/\partial t^i)$ be a vector field ,such that $d_n=1$, $d_ir_i=0$ $\ d_i+d_{n+1-i}=\upsilon+2$ for all i. Then the pair (F,E) is a *solution of WDVV equations* if:

1.$\sum_{q=1}^n\frac{\partial^3F}{\partial t^i\partial t^j\partial t^q}\ \frac{\partial^3F}{\partial t^k\partial t^l\partial t^{n+1-q}}=\sum_{q=1}^n\frac{\partial^3F}{\partial t^k\partial t^j\partial t^q}\ \frac{\partial^3F}{\partial t^i\partial t^l\partial t^{n+1-q}}$
(associativity equations);

2.$\frac{\partial^3F}{\partial t^i\partial t^j\partial t^n}=\delta_{i+j,n+1}$
(normalization condition);

3.$L_EF=(\upsilon+3)F+\sum_{ij}A_{ij}t^it^j+\sum_iB_it^i+C$, where L_E is the Li derivative and A_{ij},B_i,C are constants.
(quasi-homogeneous conditions).

This equations was found in works of Witten [18] and Dijkgraaf, E.Verlinde, H.Verlinde [6] for a description of spaces of deformations of topological fields theories.

According to [7], any solution of WDVV equations is a potential of some anti-diagonale Dubrovin connection and moreover the correspondance (potential)$\mapsto$(anti-diagonale Dubrovin connection) form one-to-one correspondance between solutions of WDVV equations and anti-diagonale Dubrovin connections.

If a solution F of WDVV equations has a representation in the form of Tailor series $F(t) = \sum c(i_1, i_2, ..., i_k)t^{i_1}t^{i_2}...t^{i_k}$, then associativity equations are equivalent to some relations between the coefficients $c(i_1, i_2, ..., i_k)$. M.Kontsevich, Yu.Manin [9, 11] presented a family of coefficients with these relations by a special system of homomorphisms $\mathbb{C}^{\otimes l} \to H^*(\overline{M}_{0,l})$, where $M_{0,l}$ is the moduli space of spheres with l pictures. This gives some other method of description for Frobenius manifolds. It is called *Cohomological Field Theory.*

4.2. Classical space of miniversale deformation for the singularity A_n.

The first and very important example of a complex Frobenius manifold is the the moduli space $Pol(n)$ of classical topological Landau-Ginsburg models. This space appear also in the theory of singularity, the theory of Coxeter groups, the theory of moduli spaces of Riemann surfaces, in matrix models of mathematical physics and in integrable systems. Let us describe this example more detailed, following on the whole [7].

Theorem 4.1.[6,7] The structure of Frobenius pairs (A_p, l_p) for $p \in Pol(n)$ generates a complex Dubrovin connection on the space $Pol(n)$ of polynomials $p(z) = z^{n+1} + a_1 z^{n-1} + a_2 z^{n-2} + ... + a_n$.

Proof. Axiom 1 directly follow from the definitions. Let $(\alpha_1, ..., \alpha_n)$ be the set of roots of p'. Consider the functions $x^i(p) = p(\alpha_i)$ as coordinates on $Pol(n)$.

Lemma 4.1. The coordinates $(x^1, ..., x^n)$ are canonical, $\upsilon = \frac{2}{n+1} - 1$ and $E = \sum_{i=k}^{n} \frac{k+1}{n+1} a_k \partial/\partial a_k$.

Proof. Prove, that $\frac{\partial}{\partial x^j} = e_{p,\alpha_j}$. Really, $\delta_{ij} = \frac{\partial x^i}{\partial x^j} = \frac{\partial((\alpha_i)^{n+1} + a_1(\alpha_i)^{n-2} + ... + a_n)}{\partial x^j} = ((n+1)(\alpha_i)^n + (n-1)a_1(\alpha_i)^{n-2} + ... + a_{n-1})\frac{\partial \alpha_i}{\partial x^j} + \frac{\partial a_1}{\partial x^j}(\alpha_i)^{n-1} + ... + \frac{\partial a_n}{\partial x^j} = p'(\alpha_i)\frac{\partial \alpha_i}{\partial x^j} + \frac{\partial p}{\partial x^j}(\alpha_i) = \frac{\partial p}{\partial x^j}(\alpha_i)$. Thus $\frac{\partial p}{\partial x^j}(\alpha_i) = \delta_{ij} = e_{p,\alpha_j}(\alpha_i)$ and therefore $\frac{\partial p}{\partial x^j}(z) = e_{p,\alpha_j}(z)$. By definition, this means that the polynomial e_{p,α_j} correspond to the tangent vector $\frac{\partial}{\partial x^j}$. Thus $(\partial/\partial x^1_\alpha, ..., \partial/\partial x^n_\alpha)$ form a canonical basis.

Prove now, that $l_p = \frac{\mathbf{d}a_1}{n+1}$. Really, considering the coefficients for z^{n-1} in $\frac{\partial a_1}{\partial x^i} z^{n-1} + ... + \frac{\partial a_n}{\partial x^i} = \frac{\partial p}{\partial x^i}(z) = e_{p,\alpha_i}(z)$ we find that $l_p(\partial/\partial x^i_\alpha) = l_p(e_{p,\alpha_i}) = \frac{1}{n+1}\frac{\partial a_1}{\partial x^i}$. Thus $l_p = \sum_{i=1}^{n} l_p(\partial/\partial x^i_\alpha)\mathbf{d}x^i_\alpha = \sum_{i=1}^{n} \frac{1}{n+1}\frac{\partial a_1}{\partial x^i}\mathbf{d}x^i_\alpha = \frac{\mathbf{d}a_1}{n+1}$.

Prove, that $E=\sum_{i=1}^n x^i_\alpha(\partial/\partial x^i_\alpha)$ don't depend from α and $L_E l_p = (v+1)l_p$. Prove at first, that $L_E p = p - \frac{z}{n+1}p'$. Really, the polynomials $L_E p$ and $p-\frac{z}{n+1}p'$ have the same degree $n-1$ and they are the same values in the points $\alpha_1,...,\alpha_n$, because $L_E(p)(\alpha_k) = (\sum_{i=1}^n x^i_\alpha(\partial/\partial x^i_\alpha)(p))(\alpha_k) = x^k = p(\alpha_k) = p(\alpha_k) - \frac{z}{n+1}p'(\alpha_k)$. Consider now the vector field $F=\sum_{k=1}^n \frac{k+1}{n+1}a_k\partial/\partial a_k$. Then $L_F(p)(z) = \sum_{k=1}^n \frac{k+1}{n+1}a_k z^{n-k} = \sum_{k=1}^n(1-\frac{n-k}{n+1})a_k z^{n-k} = p(z)-\frac{z}{n+1}p'(z) = L_E(p)(z)$. Thus $L_E(l_p) = L_F(\frac{\mathbf{d}a_1}{n+1}) = \frac{1}{n+1}\frac{2}{n+1}\mathbf{d}a_1 = \frac{2}{n+1}l_p$.

□

Construct now a flat coordinate systems on $Pol(n)$. Consider a function $\omega=\omega(p,z)$ on $Pol(n)\times\mathbb{C}$ such that $\omega^{n+1}=p(z)$. Put $z=\omega+\frac{\tilde t^1}{\omega}+\frac{\tilde t^2}{\omega^2}+\frac{\tilde t^3}{\omega^3}+....$ The equality $\omega^{n+1} = p(\omega+\frac{\tilde t^1}{\omega}+\frac{\tilde t^2}{\omega^2}+\frac{\tilde t^3}{\omega^3}+...)$ gives a possibility to find $\tilde t^i$ as a quasihomogenous polynomial of $\{a_j\}$. In particularity $\tilde t^1 = -\frac{a_1}{n+1}$ $\tilde t^2 = -\frac{a_2}{n+1}$. Put $t^1=-(n+1)\tilde t^n$, $t^n=-\tilde t^1$ and $t^i = -\sqrt{n+1}\tilde t^{n+1-i}$ for $i=2,...,n-1$.

Lemma 4.2. The coordinates $(t^1,...,t^n)$ are flat quasi-homogeneous and $d_i=\frac{n+2-i}{n+1}$.

Proof. Consider the set of polynomials $p(z,\tilde t) = z^{n+1}+a_1(\tilde t)z^{n-1} + a_2(\tilde t)z^{n-2}+...+a_n(\tilde t)$, where $\tilde t = (\tilde t^1,...,\tilde t^n)$. Consider the function $z(\omega,\tilde t) = \omega+\frac{\tilde t^1}{\omega}+\frac{\tilde t^2}{\omega^2}+\frac{\tilde t^3}{\omega^3}+...$, such that $\omega^{n+1}=p(z(\omega,\tilde t),\tilde t)$. We will consider ω and $\{\tilde t^1,\dots,\tilde t^n\}$ as independent variables. Then $0 = \frac{\mathbf{d}\omega^{n+1}}{\mathbf{d}\tilde t^i} = \frac{\partial p}{\partial \tilde t^i} + \frac{\partial p}{\partial z}\frac{\partial z}{\partial \tilde t^i} = \frac{\partial p}{\partial \tilde t^i} + p'\frac{1}{\omega^i}$. Thus $\frac{\partial p}{\partial \tilde t^i} = -p'\frac{1}{\omega^i}$. Therefore $g(\partial/\partial\tilde t^i,\partial/\partial\tilde t^j) = l_p(\partial/\partial\tilde t^i\partial/\partial\tilde t^j) = -\mathbf{Res}_{z=\infty}\frac{\frac{\partial p}{\partial\tilde t^i}\frac{\partial p}{\partial\tilde t^j}}{p'}\mathbf{d}z = -\mathbf{Res}_{z=\infty}\frac{p'\mathbf{d}z}{\omega^{i+j}} = -\mathbf{Res}_{\omega=\infty}\frac{\mathbf{d}p}{\omega^{i+j}} = -\mathbf{Res}_{\omega=\infty}\frac{\mathbf{d}\omega^{n+1}}{\omega^{i+j}} = (n+1)\delta_{i+j,n+1}$. Thus, $g(\partial/\partial t^i,\partial/\partial t^j) = \delta_{i+j,n+1}$.

Let $e=\sum_{i=1}^n\rho_i\frac{\partial}{\partial\tilde t^i}$ be the field of unity elements of the algebras A_p. Recall, that $\tilde t^1 = -\frac{a_1}{n+1}$. Using this fact and lemma 4.1, we find that $\delta_{\beta,1} = \mathbf{d}\tilde t^1(\partial/\partial\tilde t^\beta) = -\frac{a_1}{n+1}\mathbf{d}a_1(\partial/\partial\tilde t^\beta) = -l_p(\partial/\partial\tilde t^\beta) = -g(e,\partial/\partial\tilde t^\beta) = -g(\sum_{i=1}^n\rho_i\frac{\partial}{\partial\tilde t^i},\frac{\partial}{\partial\tilde t^\beta}) = -\sum_{i=1}^n\rho_i g(\frac{\partial}{\partial\tilde t^i},\frac{\partial}{\partial\tilde t^\beta}) = -(n+1)\sum_{i=1}^n\rho_i\delta_{i+\beta,n+1} = -(n+1)\rho_{n+1-\beta}$. Therefore, $e = -\frac{1}{n+1}\frac{\partial}{\partial\tilde t^n} = \frac{\partial}{\partial t^1}$

It is follow from lemma 4.1. that $L_E a_i = \frac{i+1}{n+1}a_i$. Thus, by the definition of $\tilde t^i$, we find that $L_E\tilde t^i = \frac{i+1}{n+1}\tilde t^i$. Therefore $L_E t^i = \frac{n+2-i}{n+1}t^i$, and $E=\sum_{i=1}^n\frac{n+2-i}{n+1}t^i(\partial/\partial t^i)$.

□

Example 4.1. Find the numbers μ_{p,α_i} as some functions of flat quasi-homogeneous coordinates for $n=2$. If $p(z)=z^3+a_1z+a_2$, then $t^2=-\tilde t^1=\frac{a_1}{3}$. Moreover $p'(z)=3z^2+a_1$ and $\alpha_i=\pm\sqrt{-\frac{a_1}{3}}=\pm\sqrt{-t^2}$. Thus, $\mu_{p,\alpha_i}=\pm\frac{1}{6\sqrt{-t^2}}$.

Example 4.2. The potential of the Frobenius manifold $Pol(n)$ is a polynomial F_n [7]. Its coefficients was found in [14]. In particulary $F_2=\frac{1}{2}(t^1)^2t^2+\frac{1}{24}(t^2)^4$.

5. Non-commutative Frobenius manifolds.

5.1. Extended WDVV equations.

A theory of deformations of closed strings is one of sours for the theory of Frobenius manifolds. Its mathematical equivalent is flat deformations of commutative Frobenius pairs. But the theory of closed strings is only part of more general open-closed topological field theory. It is follow from [1,10,12], that a mathematica equivalent of a open-closed topological field theory is a Cardy-Frobenius algebra. A theory of flat deformation for Cardy-Frobenius algebras was suggested in [15]. It continue the Kontsevich and Manin approach [9,11] and it gives some extension of WDVV equations to differential equations on series of non-commutative variables.

Describe more detail these equations. Let $t = (t^1, ..., t^n)$ (respectively $s = (s^1, ..., s^m)$ be the standard coordinates on $A \cong \mathbb{C}^n$ (respectively on $B \cong \mathbb{C}^m$). Consider the algebras of formal tensor series $F = \sum c(i_1, i_2, ..., i_k|j_1, j_2, ..., j_l)t^{i_1} \otimes t^{i_2}...t^{i_k} \otimes s^{j_1} \otimes s^{j_2}...s^{j_k}$, $c(i_1, i_2, ..., i_k|j_1, j_2, ..., j_l) \in \mathbb{C}$. Let F_A be the part of the series F that consists of all monomial without s^i .

Partial derivatives of F are defined by partial derivatives of monomials.

We consider that $\frac{\partial(t^{i_1}\otimes t^{i_2}\otimes...\otimes t^{i_k}\otimes s^{j_1}\otimes s^{j_2}\otimes...\otimes s^{j_k})}{\partial t^i}$ is the sum of monomials $t^{i_1} \otimes t^{i_2} \otimes ... \otimes t^{i_{p-1}} \otimes t^{i_{p+1}} ... \otimes t^{i_k} \otimes s^{j_1} \otimes s^{j_2} \otimes ... \otimes s^{j_k})$, such that $i_p = i$.

Reciprocally $\frac{\partial(t^{i_1}\otimes t^{i_2}\otimes...\otimes t^{i_k}\otimes s^{j_1}\otimes s^{j_2}\otimes...\otimes s^{j_k})}{\partial s^j}$ is the sum of monomials $t^{i_1} \otimes t^{i_2} \otimes ... \otimes t^{i_k} \otimes s^{j_1} \otimes s^{j_2} \otimes ... \otimes s^{j_{p-1}} \otimes s^{j_{p+1}} ... \otimes s^{j_k}$, such that $j_p = j$.

Put $\frac{\partial^2}{\partial t^i \partial t^j} = \frac{\partial}{\partial t^i}\frac{\partial}{\partial t^j}$, $\frac{\partial^2}{\partial t^i \partial s^j} = \frac{\partial}{\partial t^i}\frac{\partial}{\partial s^j}$, $\frac{\partial^2}{\partial s^i \partial s^j} = \frac{\partial}{\partial s^i}\frac{\partial}{\partial s^j}$, $\frac{\partial^3}{\partial t^i \partial t^j \partial t^k} = \frac{\partial}{\partial t^i}\frac{\partial}{\partial t^j}\frac{\partial}{\partial t^k}$.

The definition of the partial derivatives $\frac{\partial^3(t^{i_1}\otimes\cdots\otimes t^{i_k}\otimes s^{j_1}\otimes\cdots\otimes s^{j_\ell})}{\partial s^i \partial s^j \partial s^r}$ is more complicated. These partial derivatives are the sum of monomials $t^{i_1} \otimes \cdots \otimes t^{i_k} \otimes s^{k_2} \otimes \cdots \otimes s^{k_{p-1}} \otimes s^{k_{p+1}} \otimes \cdots \otimes s^{k_{q-1}} \otimes s^{k_{q+1}} \otimes s^{k_\ell}$ such that the sequences $s^i, s^{k_2}, \cdots, s^{k_{p-1}}, s^j, s^{k_{p+1}}, \cdots, s^{k_{q-1}}, s^r, s^{k_{q+1}}, \cdots, s^{k_\ell}$ and $(s^{j_1}, \cdots, s^{j_\ell})$ are the same after an cyclic transposition.

We consider that a monomials $t^{i_1}\otimes\cdots\otimes t^{i_k}\otimes s^{j_1}\otimes\cdots\otimes s^{j_\ell}$ and $t^{\widetilde{i}_1}\otimes\cdots\otimes t^{\widetilde{i}_k}\otimes s^{\widetilde{j}_1}\otimes\cdots\otimes s^{\widetilde{j}_\ell}$ are *equivalent*, if $\cup_{r=1}^k i_r = \cup_{r=1}^k \widetilde{i}_r$ and $\cup_{r=1}^l j_r = \cup_{r=1}^l \widetilde{j}_r$. Let $[t^{i_1} \otimes \cdots \otimes s^{j_\ell}]$ be the equivalent class of $t^{i_1} \otimes \cdots \otimes s^{j_\ell}$. The tensor series $F = \sum c(i_1, ..., i_k|j_1, ..., j_\ell)a_{i_1} \otimes \cdots \otimes b_{j_\ell}$ generate the tensor series $[F] = \sum c[i_1, ..., i_k|j_1, ..., j_\ell][a_{i_1} \otimes \cdots \otimes b_{j_\ell}]$, where the sum is the sum by all equivalent classes of monomials and $c[i_1, ..., i_k|j_1, ..., j_\ell]$ is the sum of all coefficients $c(\widetilde{i}_1, ..., \widetilde{j}_\ell)$, corresponding monomials from equivalent class $[a_{i_1} \otimes \cdots \otimes b_{j_\ell}]$.

We say that a tensor series $F = \sum c(i_1 \cdots i_k|j_1 \cdots j_\ell)t^{i_1} \otimes \cdots \otimes t^k \otimes s^{j_1} \otimes \cdots \otimes s^{j_\ell}$ satisfy *extended WDVV equations* on a space $H = A \oplus B$, if the following conditions hold

1. The coefficients $c(i_1\cdots i_k|j_1\cdots j_\ell)$ are invariant under all permutation of $\{i_r\}$.

2. The coefficients $c(i,j|)$ $c(|i,j)$ generate nondegenerate matrices. By $F_a^{t^it^j}$ $F_b^{s^is^j}$ denote the inverse matrices of $c(i,j|)$ and $c(|i,j)$ respectively.

3.
$$[\sum_{p,q=1}^{n}\frac{\partial^3F_A}{\partial t^i\partial t^j\partial t^p}\otimes F_a^{t^pt^q}\frac{\partial^3F_A}{\partial t^q\partial t^k\partial t^\ell}]=[\sum_{p,q=1}^{n}\frac{\partial^3F_A}{\partial t^k\partial t^j\partial t^p}\otimes F_a^{t^pt^q}\frac{\partial^3F_A}{\partial t^q\partial t^i\partial t^\ell}].$$

4.
$$[\sum_{p,q=1}^{m}\frac{\partial^3F}{\partial s^i\partial s^j\partial s^p}\otimes F_b^{s^ps^q}\frac{\partial^3F}{\partial s^q\partial s^k\partial s^\ell}]=\sum_{p,q=1}^{m}[\frac{\partial^3F}{\partial s^\ell\partial s^i\partial s^p}\otimes F_b^{s^ps^q}\frac{\partial^3F}{\partial s^q\partial s^j\partial s^k}].$$

5.
$$[\sum\frac{\partial^2F}{\partial t^k\partial s^p}\otimes F_b^{s^ps^q}\frac{\partial^3F}{\partial s^q\partial s^i\partial s^j}]=[\sum\frac{\partial^2F}{\partial t^k\partial s^p}\otimes F_b^{s^ps^q}\frac{\partial^3F}{\partial s^q\partial s^j\partial s^i}].$$

6.
$$[\sum\frac{\partial^2F}{\partial s^k\partial t^p}\otimes F_a^{t^pt^q}\frac{\partial^3F}{\partial t^q\partial t^i\partial t^j}]=[\sum\frac{\partial^2F}{\partial t^i\partial s^p}\otimes F_b^{s^ps^q}\frac{\partial^3F}{\partial s^q\partial s^k\partial s^r}\otimes F_b^{s^rs^\ell}\frac{\partial^2F}{\partial s^\ell\partial t^j}].$$

7.
$$[\sum\frac{\partial^2F}{\partial s^u\partial t^p}\otimes F_a^{t^pt^q}\frac{\partial^2F}{\partial t^q\partial s^v}]=[\sum\frac{\partial^3F}{\partial s^u\partial s^p\partial s^r}F_b^{s^rs^l}\otimes F_b^{s^ps^q}\frac{\partial^3F}{\partial s^l\partial s^v\partial s^q}].$$

In [15] are demonstrated that solutions of extended WDVV equations one-to-one correspond to potentials of some extension of Cohomological Field Theory and moreover they describe some class of deformations of Cardy-Frobenius algebras. Thus it is natural to consider solutions of extended WDVV equations as (non commutative) extension of Frobenius manifolds, that we call *Cardy-Frobenius manifolds*. Late we prove that quaternion Landau-Ginsburg models generate a Cardy-Frobenius manifold.

5.2. Cardy-Frobenius bundles.

Definition. Let M be a , were $\mathbb{K}$ is the real or the complex field. By *Cardy-Frobenius bundle* on smooth (real or complex) manifold M we call a pair of bundles $\varphi_A : A\to M$ and $\varphi_B : B\to M$ with a flat connection ∇_B, and a set of Cardy-Frobenius algebras $\{(A_p,l_p^A),(B_p,l_p^B),\phi_p\}$, where $A_p=\varphi_A^{-1}(p)$, $B_p=\varphi_B^{-1}(p)$, such that:

1. The algebras $\{(A_p,l_p^A)\}$ form a Dubrovin connection.

2. The connection ∇_B conserve the family of bilinear forms $\{(b_1,b_2)_p=l_p^B(b_1b_2)|p\in M\}$.

Call by *a flat system of coordinates* on B a family of linear coordinates systems $s=\{s_p=(s_p^1,...,s_p^m)|p\in M\}$ on bands B_p, that is invariant by ∇_B. It generates a basis $(\frac{\partial}{\partial s_p^1},...,\frac{\partial}{\partial s_p^m})$ on any vector space

B_p. Axiom 2. say that values $(\frac{\partial}{\partial s^i_p}, \frac{\partial}{\partial s^j_p})_p = l^B_p(\frac{\partial}{\partial s^i_p}\frac{\partial}{\partial s^j_p})$ are constants on M.

3. Let $s = (s^1, ..., s^m)$ be a flat system of coordinates on B. Then the tensor fields $c^B(\frac{\partial}{\partial s^i}, \frac{\partial}{\partial s^j}, \frac{\partial}{\partial s^k}) = l^B_p(\frac{\partial}{\partial s^i}\frac{\partial}{\partial s^j}\frac{\partial}{\partial s^k})$, are smooth as functions on M. We call B-*structures tensors.*

4. The natural map $\phi = \{\bigcup \phi_p | p \in M\} : A \to B$ is smooth. It define smooth *transition tensors field* $c^{AB}(a, b) = l^B_p(\phi(a)b)$.

Theorem 5.1. Let M be a semi-simple Frobenius manifold with a Dubrovin connection $\{(A_p, l^A_p)|p \in M\}$. Then there exist Cardy-Frobenius bundles $\{(A_p, l^A_p), (B_p, l^B_p), \phi\}$.

Proof. Let $(x^1_\alpha, ..., x^n_\alpha)$ be canonical coordinates on M. Put $\lambda_{p,i} = l^A_p(\partial/\partial x^i_\alpha(p))$. Let $\mu_{p,i}$ be a smooth function on M, such that $\mu^2_{p,i} = \lambda_{p,i}$. Consider the family of Frobenius pair $(B_{p,i}, l^B_{p,i}) = \mathbb{M}(m, \mathbb{C})(\mu_{p,i})$ from example 2.2. Put $(B_p, l^B_p) = \bigoplus_i (B_{p,i}, l^B_{p,i})$ and $B = \bigoplus_{p\in M} B_p$. Describe a connection ∇_B. The standard basis $\{E^{kr}\}$ of $\mathbb{M}(m, \mathbb{C})$ generate the basis the $\{E^{kr}_{p,i}\}$ of $B_{p,i}$. We will be consider, that the connection ∇_B generate the transfer $E^{kr}_{p,i}$ to $\frac{\mu_{p,i}}{\mu_{q,i}} E^{kr}_{q,i}$ for q from a neighborhood of p. Define a structure of smooth manifold on B considering that the projection $\varphi_B(B_p) = p$ is smooth. Define the homomorphism $\phi_p : A_p \to B_p$, considering that $\phi_p(\partial/\partial x^i_\alpha)$ is the unit element of the algebra $B_{p,i}$. According to example 2.4. the structure, that we constructed, is Cardy-Frobenius bundles.

$\square$

Definition. Let $\varphi_A : A \to M$, $\varphi_B : B \to M$, ∇_B, $\{(A_p, l^A_p), (B_p, l^B_p), \phi_p | p \in M\}$ be a Cardy-Frobenius bundle on M. Let $t = (t^1, ..., t^n)$ be a system of flat quasi-homogenies coordinates of the Dubrovin connection (A_p, l^A_p) and let $s = (s^1, ..., s^m)$ be a system of flat coordinates on B. By *potential* of this Cardy-Frobenius bundle is called the formal tensor series $F(t|s) = \sum c(i_1, i_2, ..., i_k | j_1, j_2, ..., j_l) t^{i_1} \otimes t^{i_2} ... t^{i_k} \otimes s^{j_1} \otimes s^{j_2} ... s^{j_k}$, where $c(i_1, i_2, ..., i_k | j_1, j_2, ..., j_l) \in \mathbb{C}$, such that

1. The matrixes $c(i, j|) = l^A_p(\frac{\partial}{\partial t^i}\frac{\partial}{\partial t^j})$ $\;c(|i, j) = l^B_p(\frac{\partial}{\partial s^i}\frac{\partial}{\partial s^j})$ are nondegenerated. Let $F^{t^i t^j}_a$ and $F^{s^i s^j}_b$ be the matrixes inverted to $c(i, j|)$ and $c(|i, j)$ respectively.

2. If F_A is the part of F that don't depend from s, that it pass to the potential of Dubrovin connection (M_p, l^A_p) after changing the tensor multiplication to the ordinary multiplication.

3. The formal tensor series $\frac{\partial^3 F}{\partial s^i \partial s^j \partial s^r}$ are not depend from s and, after changing the tensor multiplication to the ordinary multiplication, coincide with the B-structure tensors of the bundle.

4. The formal tensor series $\frac{\partial^2 F}{\partial t^i \partial s^j}$ are not depend from s and , after changing the tensor multiplication to the ordinary multiplication, coincide with the transition function of the bundle.

Theorem 5.2. Let a Dubrovin connection (A_p, l^A_p) on M has a potential in form of Taylor series. Then any Cardy-Frobenius bundle

$\varphi_A : A \to M$, $\varphi_B : B \to M$, ∇_B, $\{(A_p, l_p^A), (B_p, l_p^B), \phi_p | p \in M\}$ also has a potential.

Proof. Let $t = (t^1, ..., t^n)$ be a flat quasi-homogeneous coordinates of the Dubrovin connection $\{(A_p, l_p^A) | p \in M\}$. Let $s = (s^1, ..., s^m)$ be be a flat coordinates system on B. Consider the potential of the Dubrovin connection $\{(A_p, l_p^A) | p \in M\}$. Changing the ordinary multiplication to the tensor multiplication we obtain a tensor series F_A. Let F_A^i be the tensor series, such that $\frac{\partial F_A^i}{\partial t^i} = F_A$. Then the tensor series $F = F_A + \frac{1}{2} \sum l^A(\frac{\partial}{\partial t^i} \frac{\partial}{\partial t^j}) t^i \otimes t^j + \sum l^B(\frac{\partial}{\partial s^i} \frac{\partial}{\partial s^j}) s^i \otimes s^j + \sum F_A^i t^i \otimes s^i + \frac{1}{3} \sum l^B(\frac{\partial}{\partial s^i} \frac{\partial}{\partial s^j} \frac{\partial}{\partial s^k}) s^i \otimes s^j \otimes s^k$ is the potential of the Cardy-Frobenius bundle.

□

Theorem 5.3. The potential of any Cardy-Frobenius bundle satisfy the extended WDVV equations.

Proof. All relation follow from properties of Cardy-Frobenius algebras $\{(A_p l_p^A), (B_p, l_p^B), \phi_p\}$. In particulary relation 1 follow from the commutativity of A. Relation 2 follow from the non-degeneracy the bilinear forms on A and B. Relations 3 and 4 follow from the associativity of algebras A and B. Relation 5. is true because $\phi_p(A_p)$ belong to center of algebra B_p. Relation 6 is true because the map ϕ_p is a homomorphism. Relation 7 follow from Cardy axiom.

□

5.3. Quaternion miniversale deformation for the singularity A_n.

Let us use the construction from theorem 5.1. for quaternion Landau-Ginsburg models. Then we have

Corollary 5.1. Family of quaternion Landau-Ginsburg models $\{(A_p, l_p^A), (B_p, l_p^B,), \phi_p | p \in Pol(n)\}$ form a Cardy-Frobenius bundles over $Pol(n)$. The space B has a natural quaternion structure that is invariant by the connection ∇_B.

A flat system of coordinates $t = (t^1, ..., t^n)$ for A is , described in section 4. A basis on B_p is $\{1^{\mathbb{H}} e_{p,\alpha_i}, I e_{p,\alpha_i}, J e_{p,\alpha_i}, K e_{p,\alpha_i} | i = 1, ..., n\}$ from section 3. In a neighborhood of p it generate a flat coordinate system $s_{q,i} = (s_{q,i}^{1^{\mathbb{H}}}, s_{q,i}^I, s_{q,i}^J, s_{q,i}^K) = \frac{\rho_{p,\alpha_i}}{\rho_{q,\alpha_i}} (1^{\mathbb{H}} e_{p,\alpha_i}, I e_{p,\alpha_i}, J e_{p,\alpha_i})$, where $\rho_{p,\alpha_i}^2 = \mu_{p,\alpha_i}$.

Example 5.1. For any $V, W \in \{1^{\mathbb{H}}, I, J, K\}$, $\frac{\partial}{\partial s_{h,i}^V} \frac{\partial}{\partial s_{q,i}^W} = \delta_{h,q} \delta_{i,j}$ $\frac{\rho_{p,\alpha_i}}{\rho_{q,\alpha_i}} \frac{\partial}{\partial, s_{q,i}^{VW}}$

Thus we can to find the bilinear form $(\frac{\partial}{\partial s^{ij}}, \frac{\partial}{\partial s^{kl}})^B = l^B(\frac{\partial}{\partial s^{ij}} \frac{\partial}{\partial s^{kl}})$ and the structure tensor $c^B(\frac{\partial}{\partial s^{ij}}, \frac{\partial}{\partial s^{kl}}, \frac{\partial}{\partial s^{uv}}) = l^B(\frac{\partial}{\partial s^{ij}} \frac{\partial}{\partial s^{kl}} \frac{\partial}{\partial s^{uv}})$, by values ρ_{q,α_i}. Example 4.1. contain an algorithm for these calculations for $n = 2$. In this case $\rho_{q,\alpha_i}^2 = \pm \frac{1}{6\sqrt{-t^2}}$.

Example 5.2. Coupling between canonical $x = (x^1, ..., x^n)$ and flat quasi-homogeneous coordinates $t = (t^1, ..., t^n)$ generate the transition tensors. Demonstrate this for $n = 2$.

According to our definitions, $\frac{\partial}{\partial t^1} = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}$ $\frac{\partial}{\partial t^2} = R_1 \frac{\partial}{\partial x^1} + R_2 \frac{\partial}{\partial x^2}$. Thus $\frac{\partial}{\partial t^2}\frac{\partial}{\partial t^2} = R_1^2 \frac{\partial}{\partial x^1} + R_2^2 \frac{\partial}{\partial x^2}$. On the other hand, according to example 4.2., $\frac{\partial}{\partial t^2}\frac{\partial}{\partial t^2} = t^2 \frac{\partial}{\partial t^1} = t^2(\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2})$. Thus $R_i = \pm\sqrt{t^2}$.

It is follow from example 4.2., that $F_A = \frac{1}{2}(t^1)^2 t^2 + \frac{1}{24}(t^2)^4$. Therefore $F_A^1 = \frac{1}{6}(t^1)^3 t^2 + \frac{1}{24} t^1 (t^2)^4$ and $F_A^2 = \frac{1}{4}(t^1)^2 (t^2)^2 + \frac{1}{120}(t^2)^5$.

Thus, we can to find the potential of the Cardy-Frobenius bundle for $n = 2$. According to theorem 5.2 it is the tensor series

$F = F_A + \frac{1}{2}\sum l^A(\frac{\partial}{\partial t^i}\frac{\partial}{\partial t^j}) t^i \otimes t^j + \sum l^B(\frac{\partial}{\partial s^i}\frac{\partial}{\partial s^j}) s^i \otimes s^j + \sum F_A^i t^i \otimes s^i + \frac{1}{3}\sum l^B(\frac{\partial}{\partial s^i}\frac{\partial}{\partial s^j}\frac{\partial}{\partial s^k}) s^i \otimes s^j \otimes s^k$.

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References

1. A.Alekseevskii, S.Natanzon, Noncommutative two-dimensional topological field theories and Hurwitz numbers for real algebraic curves. http://xxx.lanl.org/math.GT/0202164.

2. V.Arnold, S. Gusein-Zade, A.Varchenko. Singularities of differentiable maps, vols I,II. Birkhauser,Boston,1985 and 1988.

3. M.Atiyah, Topological quantum field theories, Inst.Hautes Etudes Sci. Publ. Math., 68(1988), 175-186.

4. R.Dijkgraaf, Geometrical approach to two-dimensional conformal field theory, Ph.D. Thesis (Utrecht, 1989).

5. R.Dijkgraaf, E.Witten, Mean field theory theory, topological field theory theory and multi-matrix models, Nucl.Phys., B342 (1990), 486.

6. R.Dijkgraaf, E.Verlinde, H.Verlinde, Note on topological string theory and 2D quantum gravity, Nucl. Phys.B352(1991) 59.

7. B.Dubrovin, Geometry of 2D topological field theories In: LNM, 1620 (1996), 120-348.

8. C.Faith, Algebra: rings, modules and categories I. Springer-Verlag, Berlin Heidelberg New York, 1973.

9. M.Kontsevich, Yu.Manin, Gromov-Witten classes, quantum cohomology, and enumerative geometry, Commun. Math. Phys, 164 (1994), 525-562.

10. C.I.Lazorau, On the structure of open-closed topological field theory in two-dimension Nucl.Phys. B603(2001),497-530.

11. Yu.Manin, Frobenius Manifolds, Quantum Cohomology, and Moduli Spaces, vol.47, American Mathematical Society, Colloquium Publication, 1999.

12. G.Moor, D-branes, RR-Fields and K-theory. http://online.itp.ucsb.edu/online/mp01/moore2/

13. S.Natanzon, Structures de Dubrovin. Preprint 1997/26 IRMA Strasbourg, 1997.

14. S.Natanzon, Formulas for A_n and B_n - solutions of WDVV equations,J. Geom. Phys, 39 (2001), 323-336.

15. S.Natanzon, Extended cohomological field theories and noncommutative Frobenius manifolds, J. Geom. Phys, 51 (2003), 387-403.

16. V. Turaev, Homotopy field theory in dimension 2 and group-algebras. http://xxx.lanl.org/math.GT/9910010.

17. C.Vafa, Topological Landau-Ginsburg models. Modern Physics Letters A, Vol 6, N 4 (1991), 337-349.

18. E.Witten, On the structure of the topological phase of two-dimensional gravity, Nucl.Phys., B340 (1990), 281

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