THE PERIOD MAP FOR CUBIC FOURFOLDS

EDUARD LOOIJENGA

The (nontrivial) primitive cohomology of a nonsingular cubic fourfold $Y \subset \mathbb{P}^5$ sits in degree 4 and has as its nonzero Hodge numbers $h^{3,1} = h^{1,3} = 1$ and $h_o^{2,2} = 20$. If we make a Tate twist (which subtracts $(1,1)$ from the bidegrees), then this looks like the primitive cohomology of a polarized K3 surface, the difference only being that the $(1,1)$ summand is of dimension 20 instead of 19. This observation was explained by Beauville–Donagi, who showed that the Fano variety $F = F_Y$ of lines on Y is a deformation of the Hilbert scheme of length two subschemes of a polarized K3 surface. This Fano variety is in fact a polarized complex symplectic fourfold (the latter meaning that F has a holomorphic 2-form which is nondegenerate everywhere) with the property that its polarized Hodge structure in degree 2 is essentially that of Y in degree 4.

This was the point of departure for Claire Voisin's proof of the injectivity of the period map for cubic fourfolds (which amounts to the assertion that the polarized Hodge structure on the primitive cohomology of a nonsingular cubic fourfold determines the fourfold up to projective transformation). On the other hand, it is known that the period map for the H^2 of any symplectic Kähler manifold is surjective and so a continuous deformation of the polarized Hodge structure on $H_o^2(F)$ always arises from a deformation of complex structure on F.

It is however not true that a new complex structure on F comes from a new complex structure on Y. One reason is familiar: the corresponding theory for $K3$ surfaces teaches us that we must allow Y to have the kind of singularities that are hardly noticed by the period map, namely those that are simple in the sense of Arnol'd. But that will not do. Indeed, a conjecture of Brendan Hassett asserts that even if we accept such singularities, we will still miss in the period variety a hypersurface that parameterizes (the polarized Hodge structures of) $K3$ surfaces of degree 2. We outline a proof of this conjecture based on previous work with my student Swierstra and a generalization of the Baily-Borel theory that we developed a while ago; the latter even enables us to prove Hassett's conjecture and Voisin's injectivity theorem at the same time. It turns out that the missing Hodge structures are 'swallowed' by the secant variety Σ_V of the Veronese surface V in $\mathbb{P}^5$ in the sense that they only appear as the limiting Hodge structures of all possible smoothings of this variety. (This is perhaps not surprising a posteriori: a normal line to the PGL$(6,\mathbb{C})$-orbit of that Σ_V in the space of all cubics is given by the intersection of Σ_V with a cubic hypersurface; if that cubic is generic, then the double cover of V—which is in fact the singular locus of Σ_{V}—ramified along its intersection with the cubic is a $\mathsf{K3}$-surface of degree two.)

Our main result relates invariant theory with the theory of automorphic forms: we find an identification of the algebra of $\mathrm{SL}(6,\mathbb{C})$-invariant polynomials on the representation space $\mathrm{Sym}^3(\mathbb{C}^6)^*$ with a certain algebra of meromorphic automorphic forms on a symmetric domain of orthogonal type of dimension 20.

Details can be found in our paper `arXiv:0705.0951[math.AG]`. The Hassett conjecture was proved independently (and quite differently) by Radu Laza in `arXiv:0705.0949[math.AG]`.

E-mail address: `looijeng@math.uu.nl`

Mathematisch Instituut, Universiteit Utrecht, P.O. Box 80.010, NL-3508 TA Utrecht, Nederland