On the homotopy category of Moore spaces and an old result of Barratt

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Let $n \geq 1$, in this paper we describe a minimal algebraic model for the homotopy category $\underline{\underline{P}}_n/\simeq$ of Moores spaces $M(\mathbb{Z}/f,n)$ of cyclic groups $\mathbb{Z}/f$, $f \in \mathbb{N}$. For $n = 1$ we obtain the isomorphism of categories

$$(1) \qquad \underline{\underline{P}}_1/\simeq \xrightarrow{\cong} \underline{\underline{R}}/\simeq$$

where $\underline{\underline{R}}$ is a category derived from group rings of cyclic groups, see (1.6) and (1.7). This seems to be the most elegant description of the category $\underline{\underline{P}}_1/\simeq$; results of Rutter [10] are immediate consequences of the isomorphism (1).

For $n \geq 2$ we show that the category $\underline{\underline{P}}_n/\simeq$ is a split linear extension of the category $\underline{\underline{F\ Cyc}}$ of cyclic groups, see (2.5). Moreover we compute the suspension functors

$$(2) \qquad \underline{\underline{P}}_1/\simeq \xrightarrow{\Sigma} \underline{\underline{P}}_2/\simeq \xrightarrow{\Sigma} \underline{\underline{P}}_3/\simeq \xrightarrow{\Sigma} \dots$$

on these categories, see (2.7) and (2.5). Using these functors Σ we obtain a canonical splitting functor B_n of the homology functor H_n:

$$(3) \qquad \underline{\underline{P}}_n/\simeq \underset{B_n}{\overset{H_n}{\rightleftarrows}} \underline{\underline{F\ Cyc}}, \quad n \geq 2,$$

compare (2.3). We determine the additive structure of $P_n/\simeq$, $n \geq 2$, by computing the term

$$(4) \qquad \Delta(\varphi,\varphi') = B_n(\varphi + \varphi') - B_n(\varphi) - B_n(\varphi')$$

for $\varphi,\varphi' \in \mathrm{Hom}(\mathbb{Z}/f,\mathbb{Z}/g)$, see (2.10). Using this formula we obtain a new proof of an old result of Barratt [2] on the homotopy groups

$$(5) \qquad [M(\mathbb{Z}/f,n),M(\mathbb{Z}/g,n)], \quad n \geq 2,$$

see (2.13) and (2.14). Our method for the computation of the group (5) is algebraic and very different from Barratt's highly involved geometrical techniques, compare the remark following (2.14). We also derive from (1) and (3) an algebraic description of the group of homotopy equivalences

(6) $$\mathrm{Aut}(M(\mathbb{Z}/f,n))^*, \quad n \geq 1.$$

For $n = 1$ this yields an easy proof of a result of Olum [9], see (1.15). For $n \geq 2$ the description of the group (6) gives us the result of Sieradski [11], see (2.8). Our computation of the homotopy category $P_n/\simeq$, $n \geq 2$, also solves a problem of Barratt [1], compare the remark following (2.14).

In the first two sections § 1, § 2 we describe the main results of this paper. In section § 3 we recall some basic facts on crossed chain complexes which are the crucial tools in our proofs in section § 4. In particular we derive from the tensor product for crossed chain complexes (due to Brown–Higgins [6]) a formula for the crossed chain complex of the James construction $J(X)$ of a CW–complex X, see (3.5). This formula is essential in our computation of (4) and (5) above, see (4.5) and (4.9).

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§ 1. The homotopy category of pseudo projective planes

Pseudo projective planes, $P_f = M(\mathbb{Z}/f,1)$, are the most elementary 2–dimensional CW–complexes. They are obtained by attaching a 2–cell e^2 to a 1–sphere S^1 by an attaching map $f: S^1 \longrightarrow S^1$ of degree $f \geq 1$, that is

(1.1) $$P_f = S^1 \cup_f e^2 = D/\sim_f .$$

Here D is the unit disk of complex numbers with boundary $S^1 = \partial D$ and with basepoint $* = 1$. The equivalence relation $\sim_f$ is generated by the relations $x \sim_f y \Leftrightarrow x^f = y^f$ with $x,y \in S^1$. Clearly $P_2 = \mathbb{R}P_2$ is the real projective plane. Let $\underline{\underline{P}}$ be the category consisting of pseudo projective planes P_f and of cellular maps. We consider the quotient functors

(1.2) $$\underline{\underline{P}} \longrightarrow \underline{\underline{P}}/\overset{1}{\simeq} \longrightarrow \underline{\underline{P}}/\simeq$$

where we use 0–homotopies ($\overset{0}{\simeq}$) running through cellular maps and homotopies ($\simeq$) relative $*$. Moreover, there is a canonical functor

$$\tau : \underline{\underline{\mathrm{Pair}}}(\mathbb{N}) \longrightarrow \underline{\underline{\mathrm{P}}} \tag{1.3}$$

where $\underline{\underline{\mathrm{Pair}}}(\mathbb{N})$ is the category of pairs in the monoid $\mathbb{N}$ of natural numbers. Objects are elements $f \in \mathbb{N}$ and morphisms $f \longrightarrow g$ are pairs $(\xi,\eta) \in \mathbb{N} \times \mathbb{N}$ with $g\xi = \eta f$. Let $[f,g]$ be the set of such morphisms $(\xi,\eta) : f \longrightarrow g$. The functor τ carries f to P_f and (ξ,η) to the map $\tau_\xi : P_f \longrightarrow P_g$ with $\tau_\xi\{x\} = \{x^\xi\}$ for $x \in D$, see (1.1). The induced homomorphism

$$\pi_1(\xi,\eta) = \pi_1(\tau_\xi) : \pi_1(P_f) = \mathbb{Z}/f \longrightarrow \pi_1(P_g) = \mathbb{Z}/g \tag{1.4}$$

on fundamental groups is given by the number $\eta = g\xi/f$ which carries the generator $1 \in \mathbb{Z}/f$ to $\eta \cdot 1 \in \mathbb{Z}/g$. Clearly τ above is a faithful functor. We now introduce the natural equivalence relation $\simeq$ on $\underline{\underline{\mathrm{Pair}}}(\mathbb{N})$ which is generated by the relations

$$(\xi,\eta) \simeq (\xi',\eta') \qquad \Leftrightarrow \eta,\eta' \equiv 0 \bmod g\,,$$
$$\Leftrightarrow \pi_1(\xi,\eta) = \pi_1(\xi',\eta') = 0.$$

(1.5) <u>Theorem</u>: The functor τ induces faithful functors

$$\tau : \underline{\underline{\mathrm{Pair}}}(\mathbb{N}) \rightarrowtail \underline{\underline{\mathrm{P}}}/\overset{0}{\simeq}\,, \quad \text{and} \quad \tau : \underline{\underline{\mathrm{Pair}}}(\mathbb{N})/\simeq \rightarrowtail \underline{\underline{\mathrm{P}}}/\simeq .$$

The image category of τ in $\underline{\underline{\mathrm{P}}}/\simeq$ is the subcategory of principal maps in the sense of (V.§3) in Baues [3]. We now define a category $\underline{\underline{\mathrm{R}}}$ which is actually a simple algebraic model of the category $\underline{\underline{\mathrm{P}}}/\overset{0}{\simeq}$.

(1.6) <u>Definition</u>: The objects of the category $\underline{\underline{\mathrm{R}}}$ are the elements $f \in \mathbb{N}$. A morphism $\lambda \in \underline{\underline{\mathrm{R}}}(f,g)$ is an element $\lambda \in \mathbb{Z}[\mathbb{Z}/g]$ for which there is $\eta \in \mathbb{Z}$ with $g \cdot \epsilon(\lambda) = f \cdot \eta$. Here $\epsilon : \mathbb{Z}[\mathbb{Z}/g] \longrightarrow \mathbb{Z}$ is the augmentation of the group ring. Composition $\lambda \circ \mu$ for $\mu \in \underline{\underline{\mathrm{R}}}(h,f)$ is defined by

$$\lambda \circ \mu = \lambda \cdot \lambda_{\#}(\mu) \tag{1}$$

where the right hand side is a product in the group ring $\mathbb{Z}[\mathbb{Z}/g]$. The homomorphism $\lambda_{\#} : \mathbb{Z}[\mathbb{Z}/f] \longrightarrow \mathbb{Z}[\mathbb{Z}/g]$ with $\lambda_{\#}[x] = [\eta x]$ is induced by the homomorphism $\pi_1(\lambda) = \eta : \mathbb{Z}/f \longrightarrow \mathbb{Z}/g$. Let

$$\partial_f = \sum_{x \in \mathbb{Z}/f} [x] \tag{2}$$

be the norm element in $\mathbb{Z}[\mathbb{Z}/f]$. We introduce a natural equivalence relation $\simeq$ on the category $\underline{\underline{\mathrm{R}}}$ as follows $(\lambda,\mu \in \underline{\underline{\mathrm{R}}}(f,g))$:

(3) $$\lambda \simeq \mu \Leftrightarrow \pi_1(\lambda) = \pi_1(\mu) \text{ and } \exists\ \beta \in \mathbb{Z}[\mathbb{Z}/g] \text{ with } \lambda - \mu = \lambda_{\#}(\partial_f) \cdot \beta ,$$

(1.7) Theorem: There are isomorphisms of categories

$$\rho : \underline{P}/\simeq \xrightarrow{\sim} \underline{R}, \text{ and } \rho : \underline{\underline{P}} \simeq \xrightarrow{\sim} \underline{\underline{R}} /\simeq .$$

Various results of Olum [9] and Rutter [10] are immediate consequences of this theorem. For $\varphi \in \mathrm{Hom}(\mathbb{Z}/f,\mathbb{Z}/g)$ let $[f,g]_\varphi$ and $[P_f,P_g]_\varphi$ be the set of all morphisms in $[f,g]$ and $[P_f,P_g]$ respectively which induce φ on fundamental groups, see (1.4). By (1.5) the function

(1.8) $$\tau : [f,g]_\varphi \longrightarrow [P_f,P_g]_\varphi$$

is injective for $\varphi \neq 0$ and is identically 0 if $\varphi = 0$. The group of integers $\mathbb{Z}$ acts freely on $[f,g]$ by $(\xi,\eta) + k = (\xi + kf,\ \eta + kg)$ and $[f,g]_\varphi$ is the orbit of (ξ,η) with $\pi_1(\xi,\eta) = \varphi$. On the other hand the coaction $P_f \longrightarrow P_f \vee S^2$ induces an action $+$ of the cohomology group

(1.9) $$E_\varphi = \hat{H}^2(P_f,\varphi^*\pi_2 P_g) = \pi_2 P_g/(\pi_2 P_g)\cdot \varphi_{\#}\partial_f ,$$

on the set $[P_f,P_g]_\varphi$ which is transitive and effective. The group $\pi_2 P_g$ can be described by each of the following equations

(1.10) $$\begin{aligned} \pi_2 P_g = H_2 \hat{P}_g &= \{x \in \mathbb{Z}[\mathbb{Z}/g] \mid \partial_g \cdot x = 0\} \\ &= \text{kernel } (\epsilon : \mathbb{Z}[\mathbb{Z}/g] \longrightarrow \mathbb{Z}) \\ &= ([0] - [1]) \cdot \mathbb{Z}[\mathbb{Z}/g] . \end{aligned}$$

Let $t : \mathbb{Z} \longrightarrow E_\varphi$ be the homomorphism mapping 1 to the class of

$$t_\varphi = f \cdot [0] - \varphi_{\#}\partial_f \in \text{kernel } (\epsilon) = \pi_2 P_g .$$

(1.11) Proposition: τ in (1.8) is t–equivariant or equivalently

$$\tau(\xi,\eta) + k \cdot t_\varphi = \tau(\xi + kf,\ \eta + kg) \text{ in } [P_f,P_g].$$

This result follows easily from (1.7).

We next derive from (1.7) a result on the group of homotopy equivalences $\mathrm{Aut}(P_f)^*$, in the category $\underline{\underline{P}}/\simeq$. Let I be the ideal generated by the norm element ∂_f in $\mathbb{Z}[\mathbb{Z}/f]$ and let U_f be the group of units in the quotient ring $\mathbb{Z}[\mathbb{Z}/f]/I$. Moreover let $\tilde{U}_f$ be the group whose elements are those of U_f but with a multiplication

$$\{\lambda\} \circ \{\mu\} = \{\lambda \cdot \lambda_{\#}(\mu)\} .$$

Here $\{\lambda\}$ denotes the class of $\lambda \in \mathbb{Z}[\mathbb{Z}/f]$ modulo I .

(1.12) Proposition: There is an isomorphism of groups

$$\mathrm{Aut}(P_f)^* \xrightarrow{\cong} \tilde{U}_f .$$

Proof: Let E(f) be the group of equivalences of the object f in $\underline{\underline{R}}/\simeq$. Then we have $\{\lambda\} \in E(f)$ iff there is μ with $\lambda\mu \simeq [0]$, $\mu\lambda \simeq [0]$. This is equivalent to $\mu \cdot (\mu_{\#}\lambda) \simeq [0]$ and $\pi_1\mu = (\pi_1\lambda)^{-1}$. This is the case iff

$$\exists \beta \text{ with } \mu \cdot (\mu_{\#}\lambda) = [0] + \beta \cdot \mu_{\#}\partial_f$$

$$\Leftrightarrow \quad \exists \beta \text{ with } (\lambda_{\#}\mu) \cdot \lambda = [0] + (\lambda_{\#}\beta) \cdot \partial_f$$

$$\Leftrightarrow \quad \{\lambda\} \in U_f .$$

Since the composition in $\mathrm{Aut}(P_f)^*$ corresponds to the composition in $\underline{\underline{R}}$, we get the isomorphism for $\mathrm{Aut}(P_f)^*$.

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We do not know whether the functor

$$\pi_1 : \underline{\underline{P}}/\simeq \xrightarrow{\sim} \underline{\underline{R}}/\simeq \twoheadrightarrow \underline{\underline{FCyc}}$$

admits a splitting where $\underline{\underline{FCyc}}$ is the category of finite cyclic groups $\mathbb{Z}/f$, $f \in \mathbb{N}$. However a splitting of the homomorphism

$$\pi_1 : \mathrm{Aut}(P_f)^* \cong \tilde{U}_f \twoheadrightarrow \mathrm{Aut}(\mathbb{Z}/f)$$

can be constructed as follows. For this we consider the commutative diagram

$$(1.13) \qquad \begin{array}{ccc} [f,g] & \xrightarrow{\quad\Gamma\quad} & \underline{\underline{R}}\,[f,g] \\ \downarrow \pi_1 & & \downarrow q \\ \mathrm{Hom}(\mathbb{Z}/f,\mathbb{Z}/g) & \dashrightarrow{\tilde{\Gamma}} & \underline{\underline{R}}\,[f,g]/\simeq \end{array}$$

Here Γ is defined for $(\xi,\eta) \in [f,g]$ by

$$\Gamma(\xi,\eta) = \sum_{j=0}^{\xi-1} [j \cdot \varphi 1] \in \mathbb{Z}[\mathbb{Z}/g]$$

with $\varphi = \pi_1(\xi,\eta)$ and q is the quotient map.

(1.14) Lemma: The function Γ induces a function $\tilde{\Gamma}$ such that (1.13) commutes.

Proof: We have to check that $\pi_1(\xi,\eta) = \pi_1(\xi',\eta') = \varphi$ implies $\Gamma(\xi,\eta) \simeq \Gamma(\xi',\eta')$:

$$\begin{aligned} \Gamma(\xi+f,\eta+g) &= \sum_{j=0}^{\xi+f-1} [j \cdot \varphi 1] = \Gamma(\xi,\eta) + \sum_{j=\xi}^{\xi+f-1} [j \cdot \varphi 1] \\ &= \Gamma(\xi,\eta) + \varphi_{\#}([\xi \cdot 1] \cdot \sum_{j=0}^{f-1} [j \cdot 1] \\ &= \Gamma(\xi,\eta) + \varphi_{\#}([\xi \cdot 1] \cdot \varphi_{\#} \partial_f \end{aligned}$$

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(1.15) Proposition: Let U_f^1 be the group of units x in the quotient ring $\mathbb{Z}[\mathbb{Z}/f]/I$ with $\epsilon(x) = 1$. Then we have the split short exact sequence of groups

$$0 \longrightarrow U_f^1 \longrightarrow \tilde{U}_f \xrightarrow{\pi_1} \mathrm{Aut}(\mathbb{Z}/f) \longrightarrow 0$$

where $\tilde{U}_f \cong \mathrm{Aut}(P_f)^*$ by (1.12). The splitting carries $\varphi \in \mathrm{Aut}(\mathbb{Z}/f)$ to $(\varphi^{-1})_{\#}\tilde{\Gamma}(\varphi)$.

(1.16) Remark: Proposition (1.15) is proved by different methods in (3.5) of Olum. It is known that there is an isomorphism of abelian groups

$$U_f^1 \cong \mathbb{Z}^{\chi} \oplus \mathbb{Z}/f\,,$$

where χ is the number of all $i \in \mathbb{N}$, $1 \leq i \leq f/2$, for which i is not a divisor of f. It is, however, a deep number theoretic problem to determine the action of $\mathrm{Aut}(\mathbb{Z}/f)$ on $\mathbb{Z}^{\chi} \oplus \mathbb{Z}/f$ in terms of basis elements. This action is defined by the split exact sequence (1.15).

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§ 2. The homotopy category of suspended pseudo projective planes

We consider the suspensions

$$\Sigma^{n-1}P_f = M(\mathbb{Z}/f,n) = S^n \cup_f e^n \tag{2.1}$$

of pseudo projective planes, $n \geq 1$, which are Moore spaces of cyclic groups. Let $\underline{\underline{P}}_n$ be the category consisting of the spaces $\Sigma^{n-1}P_f$, $f \geq 1$, and of cellular maps. In section 1 above we studied the category $\underline{\underline{P}} = \underline{\underline{P}}_1$ of pseudo projective planes and its homotopy category $\underline{\underline{P}}/\simeq$. We here compute the suspension functor $\Sigma : \underline{\underline{P}}_n/\simeq \longrightarrow \underline{\underline{P}}_{n+1}/\simeq$, $n \geq 1$, which is an isomorphism of categories for $n \geq 3$. For this we consider the commutative diagram of functors

$$\begin{array}{ccccc} \underline{\underline{\mathrm{Pair}}}(\mathbb{N}) \xrightarrow{\ \tau\ } & \underline{\underline{P}}/\simeq & \xrightarrow{\Sigma} & \underline{\underline{P}}_2/\simeq & \xrightarrow{\Sigma} \underline{\underline{P}}_3/\simeq \\ & \downarrow \pi_1 & & H_2 & \qquad H_3 \\ & \underline{\underline{FCyc}} & & & \end{array} \tag{2.2}$$

where H_2 and H_3 are the homology functors. The next result seems to be new; recall that $[f,g]$ is the set of morphisms $f \longrightarrow g$ in $\underline{\underline{\mathrm{Pair}}}(\mathbb{N})$, $f,g, \in \mathbb{N}$, see (1.3).

(2.3) Theorem: Let $\varphi \in \mathrm{Hom}(\mathbb{Z}/f,\mathbb{Z}/g)$. Then there is a unique element $\overline{\varphi} = B_2(\varphi)$ in the image of

$$\Sigma\tau : [f,g] \longrightarrow [\Sigma P_f,\Sigma P_g]$$

with $H_2\overline{\varphi} = \varphi$. Moreover there is a unique element $\overline{\overline{\varphi}} = B_3(\varphi)$ in the image of

$$\Sigma\Sigma : [P_f,P_g] \longrightarrow [\Sigma^2 P_f,\Sigma^2 P_g]$$

with $H_3\overline{\overline{\varphi}} = \varphi$.

(2.4) Corollary: The functors H_n $(n = 2,3)$ in (2.2) admit a splitting functor

$$B_n : \underline{\underline{FCyc}} \longrightarrow \underline{\underline{P}}_n/\simeq$$

with $H_n B_n = 1$

This follows immediately from (2.3) since the definition of $B_n(\varphi)$ is compatible with compositions. The splitting functor B_n, however, is not additive; below we describe the distributivity law for $B_n(\varphi + \varphi')$. The functors H_n in (2.2) are part of the following commutative diagram in which the rows are split linear extensions of categories (compare IV. § 3 and V § 3a in [3])

$$\begin{array}{ccccc} E^2 & +\!\!> & \underline{\underline{P}}_2/\simeq & \xrightarrow{H_2} & \underline{\underline{FCyc}} \\ \downarrow \sigma_* & & \downarrow \Sigma & & \| \\ E^3 & +\!\!> & \underline{\underline{P}}_3/\simeq & \xrightarrow[H_3]{} & \underline{\underline{FCyc}} \end{array} \tag{2.5}$$

Here E^n is the bifunctor on $\underline{\underline{FCyc}}$ given by

(1) $$E^n(\mathbb{Z}/f,\mathbb{Z}/g) = \mathrm{Ext}(\mathbb{Z}/f,\Gamma^1_n\mathbb{Z}/g)$$

where Γ^1_n is Whitehead's functor Γ for $n = 2$ and the functor $-\otimes\mathbb{Z}/2$ for $n \geq 3$. The group (1) is a cyclic group of order $(f,2g,g^2)$ for $n = 2$ and $(f,g,2)$ for $n \geq 3$ where the bracket $(\dots)$ denotes the greatest common divisor. The natural transformation σ_* in (2.5) is induced by the surjection

(2) $$\sigma : \Gamma(\mathbb{Z}/g) \longrightarrow \mathbb{Z}/g \otimes \mathbb{Z}/2$$

compare (IX. 4.4) [3]. The action of E^n on $\underline{\underline{P}}_n/\simeq$ is given by the well known central extension of groups

(3) $$\mathrm{Ext}(\mathbb{Z}/f,\pi_{n+1}U) \stackrel{i}{\rightarrowtail} [\Sigma^{n-1}P_f,U] \xrightarrow{\pi_n} \mathrm{Hom}(\mathbb{Z}/f,\pi_n U)$$

which is known as the '*universal coefficient sequence*' compare Hilton [7] or (V.3a) in Baues [3]. For $U = \Sigma^{n-1}P_g$ we have $\pi_{n+1}U = \Gamma^1_n(\mathbb{Z}/g)$. The splitting B_n gives us an identification $(n \geq 2)$

(4) $$\mathrm{Hom}(\mathbb{Z}/f,\mathbb{Z}/g) \times E^n(\mathbb{Z}/f,\mathbb{Z}/g) = [\Sigma^{n-1}P_f,\Sigma^{n-1}P_g]$$

which carries (φ,α) to $B_n(\varphi) + i(\alpha)$. The composition in $\underline{\underline{P}}_n/\simeq$ then satisfies the simple formula

$$(5) \qquad (\varphi,\alpha) \circ (\Psi,\beta) = (\varphi\Psi, \varphi_* \alpha + \Psi^* \beta).$$

This indeed yields a very simple algebraic description of the category $\underline{\underline{P}}_n/\simeq$. The suspension functor in (2.5) is given by $\Sigma(\varphi,\alpha) = (\varphi,\sigma_*\alpha)$. We now consider the image category of the functor $\Sigma : \underline{\underline{P}}/\simeq \longrightarrow \underline{\underline{P}}_2/\simeq$. Recall that $1 \in \mathbb{Z}/f$ denotes the canonical generator.

(2.6) Definition: For maps $u,v : P_f \longrightarrow P_g$ in $\underline{\underline{P}}$ we set $u \equiv v$ if $\Sigma f \simeq \Sigma g$. Whence the quotient category $\underline{\underline{P}}/\equiv$ is the same as the image category $\Sigma(\underline{\underline{P}}/\simeq)$. For morphisms $\lambda,\mu \in \underline{\underline{R}}(f,g)$, see (1.6), we set $\lambda \equiv \mu$ if $\pi_1(\lambda) = \pi_1(\mu)$ and if for some $\beta \in \mathbb{Z}[\mathbb{Z}/g]$ with

$$\lambda - \mu - \epsilon(\lambda - \mu)[0] = ([0] - [1]) \cdot \beta$$

the greatest common divisor $(f,g^2,2g)$ divides $g \cdot \epsilon(\beta)$.

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The following result shows that the image category $\Sigma(\underline{\underline{P}}/\simeq)$ is surprisingly small. By (2.3) we know that the image category $\Sigma\Sigma(\underline{\underline{P}}/\simeq)$ is isomorphic to $\underline{\underline{FCyc}}$.

(2.7) Theorem: The isomorphism ρ in (1.7) induces an isomorphism of categories

$$\rho : \Sigma(\underline{\underline{P}}/\simeq) = \underline{\underline{P}}/\equiv \xrightarrow{\cong} \underline{\underline{R}}/\equiv .$$

Moreover one has a split linear extension of categories

$$\tilde{E} + \!\!\!>\longrightarrow \underline{\underline{P}}/\equiv \xrightarrow[\pi_1]{} \!\!\!\!\rightarrow \underline{\underline{FCyc}}$$

where $\tilde{E}$ is the quotient of E^2 above with $\tilde{E}(\mathbb{Z}/f,\mathbb{Z}/g) = g \cdot \mathrm{Ext}(\mathbb{Z}/f,\Gamma(\mathbb{Z}/g))$. This group is $\mathbb{Z}/2$ if $(f,g^2,2g) = 2g$ and in 0 otherwise. The splitting is given by B_2 in (2.3).

We derive from (2.6) and (2.7) the following commutative diagram in which the rows are split extensions of groups. Here $\mathrm{Aut}(X)^*$ denotes the group of homotopy classes of basepoint preserving homotopy equivalences of X .

(2.8)

$$\begin{array}{ccccc} & & \Sigma\mathrm{Aut}(P_f)^* & \cong & \mathrm{Aut}(\mathbb{Z}/f) \\ & & \downarrow & & \| \\ \mathbb{Z}/f & \rightarrowtail & \mathrm{Aut}(\Sigma P_f)^* & \twoheadrightarrow & \mathrm{Aut}(\mathbb{Z}/f) \\ \downarrow & & \downarrow & & \| \\ \mathbb{Z}/(f,2) & \rightarrowtail & \mathrm{Aut}(\Sigma^2 P_f)^* & \twoheadrightarrow & \mathrm{Aut}(\mathbb{Z}/f) \end{array}$$

Using different methods the split extension for $\mathrm{Aut}(\Sigma P_f)^*$ was obtained by Sieradski [11].

The morphism sets $[\Sigma P_f, \Sigma P_g]$ in $\underline{\underline{P}}_2/\simeq$ are groups since the suspension ΣP_f is a co–H–group. As pointed out in (2.4) the splitting

(2.9) $$B_2 : \mathrm{Hom}(\mathbb{Z}/f, \mathbb{Z}/g) \longrightarrow [\Sigma P_f, \Sigma P_g]$$

is not additive. We now describe the distributivity law for $B_2(\varphi + \varphi')$. Let

$$\Delta : \mathrm{Hom}(\mathbb{Z}/f, \mathbb{Z}/g) \times \mathrm{Hom}(\mathbb{Z}/f, \mathbb{Z}/g) \longrightarrow \mathrm{Ext}(\mathbb{Z}/f, \Gamma\mathbb{Z}/g) \cong \mathbb{Z}/(f,2g,g^2)$$

be the linear map which carries the pair (φ, φ') to the element

$$\Delta(\varphi, \varphi') = (f(f-1)/2)\varphi_1 \cdot \varphi_1' \cdot 1$$

where $\varphi(1) = \varphi_1 1$, $\varphi'(1) = \varphi_1' 1$. Then we get

(2.10) <u>Theorem</u>: $B_2(\varphi + \varphi') = B_2(\varphi) + B_2(\varphi') + \Delta(\varphi, \varphi')$

The splitting B_n, $n \geq 3$, satisfies the addition law $B_n(\varphi, \varphi') = B_n(\varphi) + B_n(\varphi') + \sigma_* \Delta(\varphi, \varphi')$. This follows from (2.5). The formula for Δ yields the following property.

(2.11) <u>Lemma</u>: Let $f = 2^a f_0$, $g = 2^b g_0$ where f_0 and g_0 are odd. Then we have $\Delta \neq 0$ iff $a = b \geq 1$ or $a = b + 1 \geq 2$ and we have $\sigma_* \Delta \neq 0$ iff $a = b = 1$.

Using the identification (2.5)(4) we can describe the group structure $+$ of the group $[\Sigma^{n-1} P_f, \Sigma^{n-1} P_g]$, $n \geq 2$, by the formula

(2.12) $$(\varphi, \alpha) + (\varphi', \alpha') = (\varphi + \varphi', \alpha + \alpha' + \Delta_n(\varphi, \varphi'))$$

where $\Delta_n = \Delta$ for $n = 2$ and $\Delta_n = \sigma_* \Delta$ for $n \geq 3$. This formula describes completely the additive structure of the category $\underline{\underline{P}}_n/\simeq$. Since $\Delta(\varphi,\varphi') = \Delta(\varphi',\varphi)$ we see that also the group $[\Sigma P_f, \Sigma P_g]$ is abelian for all $f,g \in \mathbb{N}$. The cyclic summands and explicit generators of this group are described in the next result.

(2.13) Corollary: Let $f = 2^a f_0$ and $g = 2^b g_0$ where f_0 and g_0 are odd. Then the homomorphism

$$H_2 : [\Sigma P_f, \Sigma P_g] \longrightarrow \mathrm{Hom}(\mathbb{Z}/f, \mathbb{Z}/g) \cong \mathbb{Z}/d$$

has an additive splitting of abelian groups if and only if $(a,b) \neq (1,1)$. Moreover for the greatest common divisors $d = (f,g)$ and $c = (f,g^2,2g)$ one has

$$[\Sigma P_f, \Sigma P_g] = \begin{cases} \mathbb{Z}/d \oplus \mathbb{Z}/c & \text{for } (a,b) \neq (1,1), \\ \mathbb{Z}/2d \oplus \mathbb{Z}/(c/2) & \text{for } (a,b) = (1,1). \end{cases}$$

The generator of the first summand is $(\varphi_0, (f/4)1)$ if $a > b = 1$ and $(\varphi_0, 0)$ otherwise where φ_0 is a generator of $\mathrm{Hom}(\mathbb{Z}/f, \mathbb{Z}/g)$. The generator of the second summand is $(0,1)$ if $(a,b) \neq (1,1)$ and is $(0, 2\cdot 1)$ if $(a,b) = (1,1)$. Here we use again the identification in (2.5)(4).

(2.14) Addendum: The homomorphism

$$H_3 : [\Sigma^2 P_f, \Sigma^2 P_g] \longrightarrow \mathrm{Hom}(\mathbb{Z}/f, \mathbb{Z}/g) = \mathbb{Z}/d$$

has an additive splitting if and only if $(a,b) \neq (1,1)$. Moreover for $e = (f,g^2,2)$ one has

$$[\Sigma^2 P_f, \Sigma^2 P_g] = \begin{cases} \mathbb{Z}/d \oplus \mathbb{Z}/e & \text{for } (a,b) \neq (1,1), \\ \mathbb{Z}/2d & \text{for } (a,b) = (1,1). \end{cases}$$

The generator of the first summand is $(\varphi_0, 0)$ and the generator of the second summand $\mathbb{Z}/e$ is $(0,1)$.

Remark: The result in (2.13), (2.14) is due to Barratt [2], (table 2 in 10.6). Barratt uses Whitney's tube system for proving this result; his arguments are highly geometrical and totally different from our method. Hilton (p. 125) presents a different approach for the stable groups $[\Sigma^2 P_f, \Sigma^2 P_g]$ and points out that a more simple minded proof of Barratt's result is needed. A further improvement in the results above is the fact that we describe explicitly generators of the cyclic summands. The algebraic description of the category $\underline{\underline{P}}_n/\simeq$ by (2.5)(5) and (2.12) solves a problem of Barratt [1] who used generators and relations for the description of $\underline{\underline{P}}_n/\simeq$, $n \geq 3$. Our algebraic model of $\underline{\underline{P}}_n/\simeq$ is simpler and also available for $n = 2$.

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Proof of (2.13): H_2 has a splitting if and only if there is α such that (φ_0,α) has order d. By the group law in (2.12) we obtain the formula

(1) $$(\varphi_0,\alpha)\cdot d = (0,\alpha\cdot d + \sum_{t=1}^{d-1} \Delta(\varphi_0,t\varphi_0)) .$$

We choose the generator $\varphi_0 = \pi_1(\xi,\eta)$ with $\eta = g/d$, see (1.4). Then we have

(2) $$\Delta(\varphi_0,t\varphi_0) = (f(f-1)/2)\eta\cdot t\eta\cdot 1$$

and therefore we have $(\varphi_0,\alpha)\cdot d = 0$ iff

(3) $$\alpha d = (f(f-1)/2)\eta\eta(d(d-1)/2)\cdot 1 .$$

If $a > b = 1$ we see that η and $d/2$ are odd. Thus $\alpha = (f/4)1$ satisfies the equation (3). Otherwise $\alpha = 0$ satisfies (3) for $(a,b) \neq (1,1)$. For $(a,b) = (1,1)$ we have $\alpha d = 0$ for all α, however, the right hand side of (3) is a non trivial element of order 2 in this case. This proves the proposition. If we reduce equation (3) modulo 2 then both sides of (3) are zero for $a > b = 1$. This shows that $B_3(\varphi)$ yields an additive splitting for if $(a,b) \neq (1,1)$.

//

Finally we consider the group structure of the homotopy groups with coefficients in $\mathbb{Z}/f$. As in (2.5)(3) we have the central extension of groups

(2.14) $$\mathbb{Z}/f \otimes \pi_{n+1}U \rightarrowtail [\Sigma^{n-1}P_f,U] \xrightarrow{\pi_n} \text{Hom}(\mathbb{Z}/f,\pi_n U)$$

where we identify $\text{Ext}(\mathbb{Z}/f,\pi_{n+1}U) = \mathbb{Z}/f \otimes \pi_{n+1}U$. This extension is completely determined by the following proposition which completes the partial results on the extension (2.14) in Hilton [7] (page 125–128).

(2.15) Proposition: For $x,y \in [\Sigma P_f,U]$ we have the commutator rule $(v = f(f-1)/2)$

$$-x-y+x+y = v1 \otimes [i^*x,i^*y]$$

where $i : S^2 \subset \Sigma P_f$ is the inclusion and where $[i^*x, i^*y] \in \pi_3 U$ is the Whitehead product. Moreover let $\mathbb{Z}\varphi$ be the subgroup of $\mathrm{Hom}(\mathbb{Z}/f, \pi_n U)$ generated by an element φ. Then there is a function $T : \mathbb{Z}\varphi \longrightarrow [\Sigma^{n-1} P_f, U]$ with $\pi_n T(x) = x$ for $x \in \mathbb{Z}\varphi$ and with $(r, s \in \mathbb{Z})$

$$-T((r+s)\varphi) + T(r\varphi) + T(s\varphi) = rtv1 \otimes (\eta^* \varphi(1))$$

where $\eta : S^{n+1} \longrightarrow S^n$ is the Hopf element.

Proof: The property of the commutator follows from the definition of the Whitehead product and the lemma on the reduced diagonal $\Delta : P_f \longrightarrow P_f \wedge P_f$ in (4.10) below, see for example II.1.12 in Baues [4]. Next let $\mathbb{Z}/g$ be the cyclic group generated by $\varphi(1)$ in $\pi_n U$. Then we can choose a map $F : \Sigma^{n-1} P_g \longrightarrow U$ with $i^* F = \varphi(1)$. Moreover an element $t\varphi \in \mathbb{Z}\varphi$ corresponds to a homomorphism $t\varphi : \mathbb{Z}/f \longrightarrow \mathbb{Z}/g$. Now we define T in (D.22) by $T(t\varphi) = F_* B_n(t\varphi)$ where B_n is the splitting in (2.4) and (2.10).

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§ 3 Crossed chain complexes

Let $\underline{\underline{CW}}$ be the category of CW–complexes X with $X^0 = *$ and of cellular maps. Our main tool for the proofs of the results in § 1 and § 2 is the functor

(3.1) $$\rho : \underline{\underline{CW}} \longrightarrow \underline{\underline{H}}$$

which carries X to the crossed chain complex $\rho(X)$. Here $\underline{\underline{H}}$ is the category of totally free crossed chain complexes which are called homotopy systems in Whitehead [12], compare also (VI. § 1) [3] where we set $D = *$ and $G = 0$. The crossed chain complex $\rho(X)$ is given by the sequence of boundary homomorphisms

$$\cdots \xrightarrow{d_4} \pi_3(X^3, X^2) \xrightarrow{d_3} \pi_2(X^2, X^1) \xrightarrow{d_2} \pi_1(X^1)$$

with $d_{n-1} d_n = 0$. The cells of X form a basis of the totally free crossed chain complex ρX, that is $\pi_1(X^1)$ is a free group generated by the 1–cells of X and d_2 is a free crossed module generated by the 2–cells of X, moreover $\pi_n(X^n, X^{n-1})$, $n \geq 3$, is a free $\pi_1(X)$–module generated by the n–cells of X. There is a notion of homotopy $\simeq$ for morphisms in $\underline{\underline{H}}$ such that ρ induces a functor $\rho : \underline{\underline{CW}}/\simeq \longrightarrow \underline{\underline{H}}/\simeq$ between homotopy categories. Here we use basepoint preserving homotopies for maps in $\underline{\underline{CW}}$ denoted by $f \simeq g$. Let $f \overset{o}{\simeq} g$ be a homotopy running through cellular maps.

(3.2) Theorem: The functor ρ induces equivalences of categories

$$\begin{cases} \rho : \underline{\underline{CW}}^2/\overset{o}{\simeq} \xrightarrow{\sim} \underline{\underline{H}}^2 , \\ \rho : \underline{\underline{CW}}^2/\simeq \xrightarrow{\sim} \underline{\underline{H}}^2/\simeq \end{cases}$$

where $\underline{\underline{CW}}^2$ and $\underline{\underline{H}}^2$ denote the full subcategories of 2–dimensional objects. Moreover $\rho : \underline{\underline{CW}}/\simeq \longrightarrow \underline{\underline{H}}/\simeq$ induces the map

$$\rho : [X,Y] \longrightarrow [\rho X, \rho Y]$$

between homotopy sets which is a bijection if $\dim X \leq 2$, $X,Y \in \underline{\underline{CW}}$.

This theorem is an old result of J.H.C. Whitehead [12], it is as well proved in chapter VI of [3], (compare (VI.3.5) and (VI.6.5)). We use the theorem as the main tool in the proofs of § 4. We shall also use the tensor product of Brown–Higgins [6] which gives us a functor $\otimes : \underline{\underline{H}} \times \underline{\underline{H}} \longrightarrow \underline{\underline{H}}$ such that there is a natural isomorphism

$$\rho(X \times Y) = \rho(X) \otimes \rho(Y) . \tag{3.3}$$

Here $X \times Y$ is the product with the CW–topology given by product cells $e \times f$. The crossed chain complex $A \otimes B$ is generated by elements $a \otimes b$, $a \otimes *$, $* \otimes b$ where $a \in A$, $b \in B$ with the following defining relations (plus, of course, the laws for crossed chain complexes):

(1) $|a \otimes b| = |a| + |b|$, $|a \otimes *| = |a|$, $|* \otimes b| = |b|$.

(2) $(*\otimes b)^{*\otimes t} = *\otimes(b^t)$ for $|t|=1$ and $(a\otimes b)^{*\otimes t} = a\otimes(b^t)$ for $|t|=1$, $|b|\geq 2$,
$(a\otimes *)^{s\otimes *} = (a^s)\otimes *$ for $|s|=1$ and $(a\otimes b)^{s\otimes *} = (a^s)\otimes b$ for $|s|=1$, $|a|\geq 2$.

(3) $(a + a') \otimes * = a \otimes * + a' \otimes *$,
$(a + a') \otimes b = a \otimes b + a' \otimes b$ for $|a| \geq 2$,
$(a + a') \otimes b = (a \otimes b)^{a' \otimes *} + a' \otimes b$ for $|a| = 1$.

(4) $* \otimes (b + b') = * \otimes b + * \otimes b'$,
$a \otimes (b + b') = a \otimes b + a \otimes b'$ for $|b| \geq 2$,
$a \otimes (b + b') = a \otimes b' + (a \otimes b)^{*\otimes b'}$ for $|b| = 1$.

(5) $d(a \otimes *) = (da) \otimes *$, $d(* \otimes b) = * \otimes (db)$ and $d(a \otimes b) =$

$$\begin{cases} -a \otimes * - * \otimes b + a \otimes * + * \otimes b & \text{for } |a| = |b| = 1 \quad , \\ [-(* \otimes b)^{a\otimes *} + * \otimes b] - a \otimes db & \text{for } |a| = 1 \ , \quad |b| \geq 2 \, , \\ (da) \otimes b + (-1)^{|a|}[-(a \otimes *)^{*\otimes b} + a \otimes *] & \text{for } |a| \geq 2 \ , \quad |b| = 1 \, , \\ (da) \otimes b + (-1)^{|a|} a \otimes (db) & \text{for } |a| \geq 2 \ , \quad |b| \geq 2 \, . \end{cases}$$

Moreover $A \otimes B$ is totally free if A and B are totally free, a basis of $A \otimes B$ is given by the elements $* \otimes b$, $a \otimes *$, $a * b$ where a and b are basis elements of A and B respectively. The isomorphism (3.3) carries $e \times f$ to $e \otimes f$.

Next we consider the James construction which is a functor $J : \underline{\underline{CW}} \longrightarrow \underline{\underline{CW}}$ given by the direct limit $JX = \underrightarrow{\lim} J_n X$ where $J_n X = (X \times \ldots \times X)/\sim$ is given by the relations $(x_1,\ldots,x_{n-1},*) \sim (x_1,\ldots,x_{t-1},*,x_t,\ldots,x_{n-1})$ for $t = 1,\ldots,n$. It is a classical result of James [8] that there is a natural homotopy equivalence

(3.4) $$J(X) \simeq \Omega\Sigma X$$

for X in $\underline{\underline{CW}}$. Using (3.3) we obtain a functor $J : \underline{\underline{H}} \longrightarrow \underline{\underline{H}}$ together with a natural isomorphism

(3.5) $$\rho J(X) = J\rho(X) \, .$$

Let A be a crossed chain complex. The crossed chain complex JA is generated by all words $a_1 \ldots a_n$ $(a_i \in A,\ i = 1,\ldots,n$ and $n \geq 1)$ with the following defining relations (plus, of course, the laws of crossed chain complexes). Let u,v be such words or empty words ϕ and let $a,a',t \in A$. Then (a) denotes the word given by $a \in A$.

(1) $|a_1 \ldots a_n| = |a_1| + \ldots + |a_n|$.

(2) $(uav)^t = u(a^t)v$ for $|t| = 1$, $|a| \geq 2$ and $(a)^t = (a^t)$ for $|t| = 1$.

(3) $$u(a + a')v = \begin{cases} uav + ua'v & \text{for } |a| \geq 2 \\ ua'v + (uav)^{a'} & \text{for } |a| = 1,\ |u| \geq 1 \\ (uav)^{a'} + ua'v & \text{for } |a| = 1,\ |v| \geq 1 \\ (a) + (a') & \text{for } u = \phi = v \end{cases}$$

(4) $d(a) = (da)$ and $d(uv) =$

$$\begin{cases} -u-v+u+v & \text{for } |u| = |v| = 1, \\ -v^{u}+v-u(dv) & \text{for } |u| = 1,\ |v| \geq 2, \\ (du)v + (-1)^{|u|}(-u^{v}+u) & \text{for } |u| \geq 2,\ |v| = 1, \\ (du)v + (-1)^{|u|}u(dv) & \text{for } |u| \geq 2,\ |v| \geq 2. \end{cases}$$

For a map $F: A \longrightarrow B$ in $\underline{\underline{H}}$ the induced map $JF: JA \longrightarrow JB$ is defined by $(JF)(a_1 \ldots a_n) = (Fa_1)\ldots(Fa_n)$. There is a well defined natural map

(5) $$\mu: JA \otimes JA \longrightarrow JA$$

given by $\mu(u\otimes v) = uv$, $\mu(*\otimes v) = v$, $\mu(u\otimes *) = u$. Therefore JA is an example of a '*crossed chain algebra*'.

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One can check that JA is totally free if A is totally free. In fact, if Z is a basis of A then $Mon(Z) - *$ is a basis of JA: Here $Mon(Z)$ is the free monoid generated by Z. As an application of (3.2) we get:

(3.6) Corollary: Let X,Y be CW–complexes in $\underline{\underline{CW}}$ with $\dim(X) \leq 2$. Then one has the binatural isomorphism of groups

$$[\Sigma X, \Sigma Y] \cong [X, JY] \cong [\rho X, J\rho Y]$$

where the group structure in $[\rho X, J\rho Y]$ is induced by μ in (3.5) (5). As a special case one gets $\pi_3(\Sigma Y) = \pi_2(J\rho Y)$.

A more detailed study of the James construction of crossed chain complexes can be found in Baues [5].

§ 4. Proofs

We here prove the main results of § 1 and § 2. Using (3.2) and (3.6) these proofs turn out to be purely algebraic. This indeed is an advantage compared with the longwinded sequence of geometric arguments of Barratt [2]. For a more detailed discussion of the following proofs see Baues [5].

We first observe that the group $\pi_2(P_f,S^1)$ is abelian. Let e_2 be the 2–cell of P_f and let e be the 1–cell of P_f. Then the elements e_2^{ne}, $n \in \mathbb{Z}$, generate the group $\pi_2(P_f,S^1)$. The commutators satisfy the formula

$$(4.1) \qquad -e_2^{ne} - e_2^{me} + e_2^{ne} + e_2^{me} = \langle e_2^{ne}, e_2^{me} \rangle - \langle e_2^{me}, e_2^{me} \rangle .$$

Here $\langle x,y \rangle = -x-y+x+y^{\partial x}$ is the Peiffer commutator which is trivial in the crossed module $d_2 : \pi_2(P_f,S^1) \longrightarrow \pi_1(S^1)$.

(4.2) <u>Proof of</u> (1.7): Since $\pi_2(P_f,S^1)$ is abelian we have an isomorphism

$$h_2 : \pi_2(P_f,S^1) \cong \mathbb{Z}[\mathbb{Z}/f] ,$$

this follows from a result of J.H.C. Whitehead [12], compare for example (VI.1.12) in Baues [3]. As a special case of diagram (3) in (VI.1.14) [3] we obtain the commutative diagram

$$\begin{array}{ccc} \pi_2(P_f,S^1) & \xrightarrow{d_2} & \pi_1(S^1) = \mathbb{Z} \\ {\scriptstyle h_2}\downarrow & f\cdot\epsilon & \downarrow{\scriptstyle h_1} \\ \mathbb{Z}[\mathbb{Z}/f] & \xrightarrow[d]{} & \mathbb{Z}[\mathbb{Z}/f] \end{array}$$

where $d(x) = x \cdot \vartheta_f$ is given by the norm element ϑ_f in (1.6). The boundary d describes the cellular chain complex of the universal covering of P_f and h_1 is a $(\mathbb{Z} \longrightarrow \mathbb{Z}/f)$–crossed homomorphism. Using the isomorphism h_2 we can identify the crossed module $d_2 = \rho(P_f)$ with the map $f\cdot\epsilon$ where ϵ is the augmentation of the group ring $\mathbb{Z}[\mathbb{Z}/f]$. We now restrict the functor ρ in (3.1) to the subcategory $\underline{\underline{P}} \subset \underline{\underline{CW}}$. The functor ρ carries P_f to the totally free crossed module $f\cdot\epsilon$ and carries a map $F : P_f \longrightarrow P_g$ to a map $(\xi,\eta) : f\cdot\epsilon \longrightarrow g\cdot\epsilon$ which is given by a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}[\mathbb{Z}/f] & \xrightarrow{f\cdot\epsilon} & \mathbb{Z} \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \eta} \\ \mathbb{Z}[\mathbb{Z}/g] & \xrightarrow[g\cdot\epsilon]{} & \mathbb{Z} \end{array}$$

We identify the full subcategory of $\underline{\underline{H}}^2$ consisting of the objects $f\cdot\epsilon$, $f \in \mathbb{N}$, with the category $\underline{\underline{R}}$ defined in (1.6). The identification carries the morphism (ξ,η) in $\underline{\underline{H}}^2$ to the morphism

$\lambda = \xi[1]$ in $\underline{\underline{R}}$. This proves that $\rho : \underline{P}/\underline{\underline{Q}} \xrightarrow{\sim} \underline{\underline{R}}$ is the restriction of the first equivalence in (3.2). A homotopy $\alpha : (\xi,\eta) \simeq (\xi',\eta')$ in $\underline{\underline{H}}^2$ is an η–crossed homomorphism $\alpha : \mathbb{Z} \longrightarrow \mathbb{Z}[\mathbb{Z}/g]$ which is determined by an element $\alpha(1) = \beta$ as in (1.6) (3). The equation $-\xi + \xi' = \alpha(f\epsilon)$ is equivalent to the equation

$$-\lambda + \lambda' = -\xi[1] + \xi'[1] = \alpha(f) = \alpha(1) \cdot (\eta_{\#}\partial_f)$$

which is equivalent to the equation in (1.6) (3).

//

(4.3) Proof of (1.5): The functor ρ in (1.7) carries τ_ξ to the element $\xi \cdot [0] \in \underline{\underline{R}}(f,g) \subset [\mathbb{Z}/g]$ where $[0]$ is the unit of the ring $\mathbb{Z}[\mathbb{Z}/g]$. By (1.7) we know

$$\tau(\xi,\eta) \simeq \tau(\xi',\eta') \Leftrightarrow \varphi = \pi_1(\xi,\eta) = \pi_1(\xi',\eta') \text{ and } \exists \beta \in \mathbb{Z}[\mathbb{Z}/g] \text{ with } (\xi-\xi')[0] = \beta \cdot \varphi_{\#}(\partial_f).$$

This implies $\eta - \eta' = \epsilon(\beta) \cdot g$. We now observe

(1) $$\varphi_{\#}\partial_f = \sum_{x \in \mathbb{Z}/f} [\varphi x] = t \cdot \sum_{y \in \varphi\mathbb{Z}/f} [y]$$

where t is the number of elements in the kernel of φ. For $\beta = \sum_{y \in \mathbb{Z}/g} a_y[y]$ in $\mathbb{Z}[\mathbb{Z}/g]$ we have

(2) $$\beta \cdot \varphi_{\#}\partial_f = t \cdot \sum_{y \in \mathbb{Z}/g,\ v \in \varphi\mathbb{Z}/f} a_y[y+v],$$

$$= t \cdot \sum_{u \in \mathbb{Z}/g} \Big(\sum_{y \in u+\varphi\mathbb{Z}/f} a_y \Big)[u].$$

Now $\beta \cdot \varphi_{\#}\partial_f = (\xi'-\xi)[0]$ with $\varphi = \pi_1\tau_\xi = \pi_1\tau_{\xi'}$ implies

(3) $$0 = \sum_{y \in u+\varphi\mathbb{Z}/f} a_y \quad \text{for } u \in \mathbb{Z}/g,\ u \neq 0.$$

The number of elements in $\varphi\mathbb{Z}/f$ is $g/\gcd(\eta,g)$. If $\gcd(\eta,g) < g$ we add up the equations in (3) for $u \in U = \{x \cdot 1; 1 \leq y \leq \gcd(\eta,g)\} \subset \mathbb{Z}/g$. Since the union of all $u + \varphi\mathbb{Z}/f$, $u \in U$, is $\mathbb{Z}/g$ we get $\epsilon(\beta) = \sum_y a_y = 0$ for $\gcd(\eta,g) < g$. Whence in this case $\eta = \eta'$ and thus $(\xi,\eta) = (\xi',\eta')$. If $\gcd(\eta,g) = g$, that is, if $\varphi = 0$ we see by (3) that $a_y = 0$ for $y \neq 0$ and $a_0 = \xi' - \xi$, so that in this case β exists.

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The crucial step for the proof of (2.3) and (2.7) is the computation of the suspension homomorphism Σ on $\pi_2 P_g$. For this we consider the diagram

(4.4)
$$\begin{array}{ccc} \pi_2(P_g) & \xrightarrow{\quad\Sigma\quad} & \pi_3 \Sigma P_g \\ \wr\| & & \wr\| \\ ([0]-[1])\mathbb{Z}[\mathbb{Z}/g] & \xrightarrow{\quad\Sigma'\quad} & \mathbb{Z}/(g^2,2g) \end{array}$$

where Σ' is defined by the formula $\Sigma'(([0]-[1])\beta) = g \cdot \epsilon(\beta) \cdot 1$ for $\beta \in \mathbb{Z}[\mathbb{Z}/g]$. Here ϵ is the augmentation. The right hand isomorphism carries the generator 1 to the Hopf element $S^3 \longrightarrow S^2 \subset \Sigma P_g$. The left hand side isomorphism is described in (1.12).

(4.5) Lemma: Diagram (4.4) commutes. This shows that $\Sigma\pi_2(P_g)$ is of order 2 if g is even, and is trivial if g is odd. Moreover the subgroup $\Sigma^2\pi_2(P_g) = 0$ is trivial in $\pi_4(\Sigma^2 P_g)$ for all g.

Proof: We use (3.6) so that Σ in (4.4) corresponds to the map

(1)
$$i_* : \pi_2(\rho) \longrightarrow \pi_2 J(\rho) = H_2 CJ(\rho)$$

where $\rho = \rho(P_g)$ and where $i : \rho \subset J(\rho)$ is the inclusion. Let $e = e_1$ and e_2 be the cells of P_g which are the generators of ρ with $d(e_2) = g \cdot e = e+\ldots+e$. The boundary $d : J(\rho)_3 \longrightarrow J(\rho)_2$ is given on generators by the following formulas.

(2)
$$\begin{aligned} d(ee_2) &= -e_2^e + e_2 - e(g \cdot e) \\ &= -e_2^e + e_2 - \sum_{i=0}^{g-1} (ee)^{ie} \end{aligned}$$

(3)
$$\begin{aligned} d(e_2 e) &= (g \cdot e)e - e_2^e + e_2 \\ &= (\sum_{i=1}^{g} (ee)^{(g-i)e}) - e_2^e + e_2 \end{aligned}$$

(4)
$$d(eee) = -(ee)^e + ee$$

These equations are simple applications of (3.5) (4) since $d(ee) = 0$. The same equation hold in $CJ(\rho)$, this is the cellular chain complex of the universal covering of $J(P_g)$, compare the definition of the functor C in (VI.1.2) of Baues [3]. Let $B = d(CJ\rho)_3$ be the group of boundaries. Then we get modulo B the congruences (see (1.6)(2))

(5) $$e_2[0]-[1])\equiv(ee)\cdot\partial_g\equiv(ee)(g\cdot[0])\,.$$

The Hopf element in (2.9) corresponds to the cycle ee in $CJ(\rho)$. Whence (5) shows that Σ' is defined correctly for $\beta=[0]$. For general β we can choose $\gamma\in\mathbb{Z}[\mathbb{Z}/g]$ such that $\beta=\epsilon(\beta)[0]+\gamma([0]-[1])$. Then we get

(6) $$e_2([0]-[1])\beta\equiv(ee)\partial_g\beta=(ee)\partial_g\epsilon(\beta)\equiv(ee)(g\cdot\epsilon(\beta)[0])$$

since $([0]-[1])\partial_g=0$ for the norm element ∂_g. This proves (2.10).

//

(4.6) Proof of (2.3) and (2.7): The second part of (2.3) and also (2.7) follow immediately from lemma (2.10) since Σ is compatible with the coaction on P_f. For the first part of (2.3) we have to check that $\Sigma' t_\varphi$ is trivial if considered as an element in the group

(1) $$\mathrm{Ext}(\mathbb{Z}/f,\Gamma(\mathbb{Z}/g))=\mathbb{Z}/(f,2g,g^2)\,.$$

Here we use (1.13). By (1.9)(1) we know

(2) $$t_\varphi=f\cdot[0]-\varphi_{\#}\partial_f=t\cdot(v[0]-\sum_{x\in V}[x])$$

where $t=|\ker\varphi|$, $t\cdot v=f$, $V=\mathrm{image}(\varphi)$ with $|V|=v$. Since $t_\varphi\in\ker(\epsilon)$ there is β with

(3) $$v[0]-\sum_{x\in V}[x]=([0]-[1])\beta\,.$$

This shows that there is an integer b with

(4) $$\epsilon(\beta)=g\cdot b-u(v(v-1))/2$$

where $u\cdot v=g$. Whence $\Sigma'(t_\varphi)$ is given by the element

(5) $$\Sigma'(t_\varphi)=t\cdot g\cdot\epsilon(\beta)\cdot 1=-(tg^2(v-1)/2)1$$

in $\mathbb{Z}/(g^2,2g)$. Now it is clear that (5) represents the trivial element in the group (1).

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The proof of (2.10) is based on the following lemma on commutators $(a,b)=-a-b+a+b$.

(4.7) Lemma: Let $G = \langle a,b \rangle$ be the free group generated by elements a and b and let $f \in \mathbb{N}$. Then there exists elements $\xi_i \in G$ ($i = 1,2,\dots,f(f-1)/2$) such that

$$(a+b)\cdot f = a\cdot f + b\cdot f - \sum_{i=1}^{f(f-1)/2} (a,b)^{\xi_i}.$$

Here we set $x\cdot f = x+\dots+x$ (f–times x) and we set $x^y = -y + x + y$. The sum is the ordered sum in the non commutative group G.

Proof: We show inductively that there are $\alpha_i \in G$ with

$$(1) \qquad (a+b)\cdot f = a\cdot f + b \cdot f - \sum_{i=1}^{f-1} (a,b\cdot i)^{\alpha_i}.$$

This is true for $f = 1$. Now we get for $(a+b)(f+1)$:

$$\begin{aligned}(a+b)f+(a+b) &= a\cdot f+b\cdot f - \sum_{i=1}^{f-1} (a,bi)^{\alpha_i} + (a+b) \\ &= a\cdot f+b\cdot f+(a+b) - \sum_{i=1}^{f-1} (a,bi)^{\alpha_i+a+b} \\ &= a\cdot(f+1)+b(f+1) - (a,bf)^b - \sum_{i=1}^{f-1} (a,bi)^{\alpha_i+a+b}.\end{aligned}$$

This proves (1). Moreover there are $\beta_j \in G$ with

$$(2) \qquad (a,b\cdot i) = \sum_{j=1}^{i} (a,b)^{\beta_j}.$$

This follows inductively from $(a,y+z) = (a,z) + (a,y)^z$ where we set $y = b\cdot i$ and $z = b$. From (1) and (2) we derive the proposition.

//

We derive from (4.7) the following algebraic description of the diagonal $\Delta : P_f \longrightarrow P_f \times P_f$.

(4.8) Corollary: Let e,e_2 be the generators of $\rho = \rho(P_f)$. Then $a = e\otimes *$, $b = * \otimes e$ are generators of $\rho \otimes \rho$ so that $(\rho \otimes \rho)_1 = \langle a,b \rangle$. Moreover we obtain a map $\Delta : \rho \longrightarrow \rho\otimes\rho$ by $\Delta(e) = a+b$ and

$$\Delta(e_2) = e_2 \otimes * + * \otimes e_2 - \sum_{i=1}^{f(f-1)/2} (e \otimes e)^{\xi_i}$$

where the ξ_i are the elements in (4.7). The map Δ satisfies $p_1\Delta = 1$ and $p_2\Delta = 1$ where p_1, p_2 are the projections of the tensor product $\rho \otimes \rho$.

Since we have $[P_f, P_f \times P_f] = [\rho, \rho \otimes \rho]$ by (3.2) we see that Δ in (4.8) represents the homotopy class of the topological diagonal of P_f.

(4.9) <u>Proof of</u> (2.10): The group addition in the group

(1) $$[\Sigma P_f, \Sigma P_g] = [\rho P_f, J\rho P_g]$$

can be described by the composition $F + F' = G$,

(2) $$G : \rho P_f \xrightarrow{\Delta} \rho P_f \otimes \rho P_f \xrightarrow{F \otimes F'} J\rho P_g \otimes J\rho P_g \xrightarrow{\mu} J\rho P_g ,$$

where Δ is the diagonal in (4.8). Now we choose (ξ, η), resp. $(\xi', \eta') \in [f,g]$ which induces φ, resp. φ', in (2.10). We may set $\eta = \varphi_1, \eta' = \varphi_1'$. Moreover let F, resp. F', be given by

(3) $$\begin{cases} F(e) = \eta e, \ F(e_2) = \xi e_2 , \\ F'(e) = \eta' e, \ F'(e_2) = \xi' e_2 . \end{cases}$$

Then the composition G in (2) represents $B_2(\varphi) + B_2(\varphi')$. Explicitly we get G by the formulas:

(4) $$\begin{cases} G(e) = (\eta + \eta')e \ , \\ G(e_2) = (\xi + \xi')e_2 - \sum_{i=1}^{v} (\eta e) \cdot (\eta' e)^{\mu(\xi_i)} \end{cases}$$

where $v = f(f-1)/2$ and $\eta e = e + \dots e = \eta$–fold sum of e. On the other hand $B_2(\varphi + \varphi')$ is represented by the map $G' : \rho P_f \longrightarrow J\rho P_g$ with $G'(e) = (\eta + \eta')e$ and $G'(e_2) = (\xi + \xi')e_2$. This shows that $\Delta(\varphi, \varphi')$ is represented by the element

(5) $$\alpha = \sum_{i=1}^{v} (\eta e)(\eta' e)^{\mu(\xi_i)} \in \text{kernel}(d_2)$$

where d_2 is the boundary of $J\rho(Pg)$. We know that $\pi_2 J\rho P_g = \text{kernel}(d_2)/\text{image}(d_3)$ is generated by the cycle ee. Whence we have to show

$$(6) \qquad \alpha \equiv v \cdot \eta \cdot \eta'(ee) \ \text{ modulo image } (d_3) \, .$$

This is easily checked by use of (4.5) (4) and (3.5) (3). Therefore the proof of (2.19) is complete. //

We also derive from (4.8) the following well known result on the reduced diagonal of P_f.

(4.10) Corollary: The following diagram homotopy commutes.

$$\begin{array}{ccc} P_f & \xrightarrow{\ \Delta\ } & P_f \times P_f \\ \downarrow q & & \downarrow q \\ S^2 \xrightarrow{\ v\ } & S^1 \wedge S^2 & \subset P_f \wedge P_j \end{array}$$

Here v is a map of degree $f(f-1)/2$ and q denotes the quotient maps. Recall that the smash product $A \wedge B$ is defined by the quotient $A \wedge B = A \times B/(A \vee B)$. We obtain (4.10) directly from (4.8) since $\rho(P_f \wedge P_f) = p(P_f) \otimes \rho(P_f)/\rho(P_f \vee P_f)$.

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