ON A RESULT OF MIYANISHI-MASUDA

HUBERT FLENNER AND MIKHAIL ZAIDENBERG

1. Introduction

Let X be a smooth affine surface over $\mathbb{C}$ with an affine ruling (an $\mathbb{A}^1$-fibration) $\rho : X \to \mathbb{A}^1_{\mathbb{C}}$. Assume that ρ is surjective, has a unique degenerate fiber, and this fiber is irreducible. In [3] such a surface X is called *affine pseudo-plane*. It is *of class* ML_1 if ρ is unique up to an automorphism of $\mathbb{A}^1_{\mathbb{C}}$. In [3] the following classification result is obtained.

Theorem 1.1. (Miyanishi-Masuda) *Suppose that X is an affine pseudo-plane of class ML_1. If X admits an effective $\mathbb{C}^*$-action then the following hold.*

(i) *This $\mathbb{C}^*$-action is necessarily hyperbolic.*
(ii) *The universal covering $\tilde{f} : \tilde{X} \to X$ is a cyclic covering of degree d, where d is the multiplicity of the unique degenerate fiber of ρ.*
(iii) *$\tilde{X}$ is an affine hypersurface in $\mathbb{A}^3_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y, z]$ with equation $x^m y = z^d - 1$ for some $m > 1$.*
(iv) *The Galois group $\mathbb{Z}_d = \langle\zeta\rangle$ of the covering $\tilde{f} : \tilde{X} \to X$, where $\zeta = \zeta_d$ is a primitive d-th root of unity, acts on $\tilde{X}$ via $\zeta.(x, y, z) = (\zeta x, \zeta^{-m} y, \zeta^e z)$, where $\gcd(e, d) = 1$.*
(v) *The $\mathbb{C}^*$-action $\lambda.(x, y, z) := (\lambda x, \lambda^{-m} y, z)$ ($\lambda \in \mathbb{C}^*$) on $\tilde{X}$ descends to the given $\mathbb{C}^*$-action on X, up to replacing λ by λ^{-1}.*

Our interest in this result is explained by our previous study [1, 2] of normal affine surfaces admitting a $\mathbb{C}^*$-action and an affine ruling. We give an alternative proof of Theorem 1.1 based on these results. We deduce it from an abstract description of a certain subclass of such surfaces realized as hypersurfaces in $\mathbb{A}^3_{\mathbb{C}}$ (see Lemma 2.2 below).

Let us add some remarks. An affine ruling on X induces an affine ruling $\tilde{\rho} : \tilde{X} \to \mathbb{A}^1_{\mathbb{C}}$ with a unique degenerate fiber consisting of d disjoint components isomorphic to $\mathbb{A}^1_{\mathbb{C}}$. In case $m > 1$ there is an essentially unique such affine ruling on $\tilde{X}$, defined by the restriction $x|\tilde{X}$. However, for $m = 1$, $y|\tilde{X}$ gives a second independent affine ruling, which also descends to $X = \tilde{X}/\mathbb{Z}_d$. Thus in this case X cannot be a ML_1 surface.

If we want the $\mathbb{Z}_d$-action on $\tilde{X}$ to be free, the exponents e and d above must be coprime. Indeed, otherwise $\zeta^{eb} = 1$ for some b with $0 < b < d$, and we would have $\zeta^b.(0, 0, z) = (0, 0, z)$ for every d-th root of unity z.

On the other hand, for every triple (d, e, m) with $d \geq 1, m \geq 2$ and $\gcd(e, d) = 1$, (iii)-(v) determine a smooth affine pseudo-plane X of class ML_1 with an effective $\mathbb{C}^*$-action. Thus Theorem 1.1 provides indeed a complete classification of these surfaces.

This research was done during visits of the first author at the Institute Fourier of the University of Grenoble and of both authors at the Max Planck Institute of Mathematics at Bonn. We thank these institutions for their hospitality and support.

2000 Mathematics Subject Classification: 14R05; 14R20; 14R25.

Key words: affine pseudo-plane, $\mathbb{C}^*$-action, $\mathbb{C}_+$-action, affine surface.

2. The proof

Under the assumptions of Theorem 1.1 $X \not\cong \mathbb{A}^2_{\mathbb{C}}$, since otherwise X would admit another affine ruling $\rho' : X \to \mathbb{A}^1_{\mathbb{C}}$ with general fibers different from those of ρ, which contradicts the condition ML_1.

A smooth affine surface X with an elliptic $\mathbb{C}^*$-action is always isomorphic to $\mathbb{A}^2_{\mathbb{C}}$, so this case is impossible. If X is smooth and the $\mathbb{C}^*$-action on X is parabolic then according to Proposition 3.8(b) in [1], $X = \operatorname{Spec} A_0[D]$ for an integral divisor D on a smooth affine curve $C = \operatorname{Spec} A_0$. The existence of an affine ruling ρ on X with the base $\mathbb{A}^1_{\mathbb{C}}$ implies that $C \cong \mathbb{A}^1_{\mathbb{C}}$. Hence D is a principal divisor. By Theorem 3.2(b) in [1], we have again $X \cong \mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} A_0[0]$ with $A_0 = \mathbb{C}[t]$, which is impossible.

Thus the $\mathbb{C}^*$-action on $X = \operatorname{Spec} A$ is necessarily hyperbolic. Accordingly we can write

$$A = A_0[D_+, D_-] \tag{1}$$

with a pair of $\mathbb{Q}$-divisors $D_\pm$ on a smooth affine curve $C = \operatorname{Spec} A_0$ satisfying $D_+ + D_- \le 0$, see Theorem 4.3 in [1]. The remainder of the proof is based on Lemmas 2.1 and 2.2 below.

Lemma 2.1. *Under the assumptions of Theorem 1.1, $A \cong A_0[D_+, D_-]$, where $A_0 = \mathbb{C}[t]$ and*

$$D_+ = -\frac{e'}{d}[0], \qquad D_- = \frac{e'}{d}[0] - \frac{1}{m}[1]\,.$$

Proof of Lemma 2.1. By Lemmas 1.6 and 2.1 in [2], X admits an affine ruling over an affine base if and only if it admits a non-trivial $\mathbb{C}_+$-action defined by a non-zero homogeneous locally nilpotent derivation $\partial \in \operatorname{Der}(A)$. Moreover, $A_0 = \mathbb{C}[t]$ in (1) and, up to an automorphism $\lambda \longmapsto \lambda^{-1}$ of $\mathbb{C}^*$ (thus switching $(D_+, D_-) \longmapsto (D_-, D_+)$) we may assume that $e = \deg \partial \ge 0$. By Lemma 3.5 and Corollary 3.27 in [2], $e = 0$ implies that $X \cong \mathbb{A}^1_{\mathbb{C}} \times \mathbb{C}^*$, so the induced affine ruling $X \to \mathbb{C}^*$ is essentially unique and has the base $\mathbb{C}^*$, which contradicts our assumption. Thus $e > 0$.

According to Corollary 3.23 in [2], the latter implies that the fractional part $\{D_+\} = D - \lfloor D \rfloor$ is zero or is supported on one point, and we can choose this point to be $0 \in \mathbb{A}^1_{\mathbb{C}}$. Such a surface $X = \operatorname{Spec} A$ is of class ML_1 if and only if the fractional part $\{D_-\}$ is supported on at least 2 points, see [2, Theorem 4.5].

Replacing (D_+, D_-) by the equivalent pair $(\{D_+\},\, D_- + \lfloor D_+ \rfloor)$ (see Theorem 4.3(b) in [1]) we may suppose that $D_+ = \{D_+\} = -e'/d[0]$, where $\gcd(e', d) = 1$ and $d > 0$.

For any affine pseudo-plane X, the Picard group $\operatorname{Pic} X$ is a torsion group [4, Ch. 3, 2.4.4]. On the other hand, for a $\mathbb{C}^*$-surface X as above, $\operatorname{rk}_{\mathbb{Q}}(\operatorname{Pic} X \otimes \mathbb{Q}) \ge l - 1$, where l is the number of points $b_j \in \mathbb{A}^1_{\mathbb{C}}$ such that $(D_+ + D_-)(b_j) < 0$, see Corollary 4.24 in [2]. Hence $l \le 1$ and so, $\exists p \in \mathbb{A}^1_{\mathbb{C}} : (D_+ + D_-)(q) = 0 \ \forall q \ne p$.

Since $D_+(q) = 0 \ \forall q \ne 0$ we have $D_-(q) = 0 \ \forall q \ne 0, p$. It follows that $\operatorname{supp}(D_-) = \operatorname{supp}(\{D_-\}) = \{0, p\}$ with $p \ne 0$. After an automorphism of $\mathbb{A}^1_{\mathbb{C}}$ we may assume that $p = 1$. Thus finally

$$D_\pm(0) = \mp e'/d, \qquad D_+(1) = 0, \quad D_-(1) = a/m \notin \mathbb{Z} \qquad \text{and} \quad D_\pm(q) = 0 \ \ \forall q \ne 0, 1\,,$$

where $\gcd(a, m) = 1$ and $m > 0$. The smoothness of X forces $a = -1$, see Theorem 4.15 in [1]. This proves Lemma 2.1. □

Next we use the following description [2, Corollary 3.30], where for a $\mathbb{Q}$-divisor D, $d(D)$ denotes the minimal positive integer d such that dD is integral.

Lemma 2.2. *We let $A = \mathbb{C}[t][D_+, D_-]$, where $D_+ + D_- \le 0$, $d(D_+) = d$, $d(D_-) = k$. We assume that $D_+ = -\frac{e'}{d}[0]$ and $D_-(0) = -\frac{l}{k}$, and we let $\partial \in \mathrm{Der}(A)$ be a homogeneous locally nilpotent derivation with $e = \deg \partial > 0$. Then there exists a unitary polynomial $Q \in \mathbb{C}[t]$ with $Q(0) \neq 0$ and $\mathrm{div}(t^l Q(t)) = -kD_-$ such that, if $A' = A_{k,P}$ is the normalization of*

$$B_{k,P} = \mathbb{C}[u, v, s]/\left(u^k v - P(s)\right), \quad \text{where} \quad P(s) = Q(s^d)s^{ke'+dl}, \tag{2}$$

then the group $\mathbb{Z}_d = \langle \zeta \rangle$ acts on $B_{k,P}$ and also on A' via

$$\zeta.(u, v, s) = (\zeta^{e'} u,\, v,\, \zeta s), \tag{3}$$

so that $A \cong A'^{\mathbb{Z}_d}$. Furthermore, $ee' \equiv 1 \mod d$ and $\partial = cu^e \frac{\partial}{\partial s}|A$ for some constant $c \in \mathbb{C}^*$.

With this result we can complete the proof of Theorem 1.1 as follows. We may assume that $A = A_0[D_+, D_-]$ with $A_0 = \mathbb{C}[t]$ and (D_+, D_-) as in Lemma 2.1. With $k := \mathrm{lcm}(d, m)$ let us write $k = mm' = dd'$ and $l = -e'd'$, so that

$$D_+ = -\frac{e'}{d}[0] = \frac{l}{k}[0], \qquad D_- = \frac{e'}{d}[0] - \frac{1}{m}[1] = -\frac{l}{k}[0] - \frac{m'}{k}[1].$$

Thus Lemma 2.2 can be applied in our setting with $Q = (t-1)^{m'}$. By this lemma, $A = A'^{\mathbb{Z}_d}$, where A' is the normalization of

$$B = \mathbb{C}[u, v, s]/(u^k v - (s^d - 1)^{m'}),$$

with the action of $\mathbb{Z}_d$ as in (3) and with the $\mathbb{C}^*$-action $\lambda.(u, v, s) = (\lambda u, \lambda^{-k} v, s)$.

The element $w = \frac{s^d - 1}{u^m} \in \mathrm{Frac}(B)$ satisfies $w^{m'} = v$ and so is integral over B, hence

$$A' \cong \mathbb{C}[u, w, s]/(u^m w - (s^d - 1)).$$

Because of (3) we have $\zeta.w = \zeta^{-me'} w$. Thus after applying an automorphism $\zeta \mapsto \zeta^{e'}$ of $\mathbb{Z}_d$, both the $\mathbb{Z}_d$-action and the $\mathbb{C}^*$-action on $\tilde{X} = \mathrm{Spec}\, A' \subseteq \mathbb{A}^3_{\mathbb{C}} = \mathrm{Spec}\, \mathbb{C}[u, w, s] \cong \mathrm{Spec}\, \mathbb{C}[x, y, z]$ have the claimed form

$$\zeta.(u, w, s) = (\zeta u, \zeta^{-m} w, \zeta^e s) \qquad \text{respectively,} \qquad \lambda.(u, w, s) = (\lambda u, \lambda^{-m} w, s).$$

This proves the theorem. $\square$

References

[1] H. Flenner, M. Zaidenberg: *Normal affine surfaces with $\mathbb{C}^*$-actions.* Osaka J. Math. 40, 2003, 981–1009.

[2] H. Flenner, M. Zaidenberg: *Locally nilpotent derivations on affine surfaces with a $\mathbb{C}^*$-action.* Prépublication de l'Institut Fourier de Mathématiques, 638, Grenoble 2004; math.AG/0403215; to appear in Osaka J. Math.

[3] M. Miyanishi, K. Masuda: *Affine Pseudo-planes with torus actions.* Preprint, 2005.

[4] M. Miyanishi: *Open algebraic surfaces.* CRM Monograph Series, 12. Amer. Math. Soc., Providence, RI, 2001.

Fakultät für Mathematik, Ruhr Universität Bochum, Geb. NA 2/72, Universitätsstrasse 150, 44780 Bochum, Germany

E-mail address: Hubert.Flenner@rub.de

Université Grenoble I, Institut Fourier, UMR 5582 CNRS-UJF, BP 74, 38402 St. Martin d'Hères cédex, France

E-mail address: zaidenbe@ujf-grenoble.fr