ON A RESULT OF MIYANISHI-MASUDA

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1. Introduction

Let X be a smooth affine surface over $\mathbb{C}$ with an affine ruling (an $\mathbb{A}^1$-fibration) $\rho : X \to \mathbb{A}^1_{\mathbb{C}}$. Assume that ρ is surjective, has a unique degenerate fiber, and this fiber is irreducible. In [3] such a surface X is called *affine pseudo-plane*. It is *of class* ML_1 if ρ is unique up to an automorphism of $\mathbb{A}^1_{\mathbb{C}}$. In [3] the following classification result is obtained.

Theorem 1.1. (Miyanishi-Masuda) *Suppose that X is an affine pseudo-plane of class ML_1. If X admits an effective $\mathbb{C}^*$-action then the following hold.*

(i) This $\mathbb{C}^*$-action is necessarily hyperbolic.
(ii) The universal covering $\tilde{f} : \tilde{X} \to X$ is a cyclic covering of degree d, where d is the multiplicity of the unique degenerate fiber of ρ.
(iii) $\tilde{X}$ is an affine hypersurface in $\mathbb{A}^3_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y, z]$ with equation $x^m y = z^d - 1$ for some $m > 1$.
(iv) The Galois group $\mathbb{Z}_d = \langle\zeta\rangle$ of the covering $\tilde{f} : \tilde{X} \to X$, where $\zeta = \zeta_d$ is a primitive d-th root of unity, acts on $\tilde{X}$ via $\zeta.(x, y, z) = (\zeta x, \zeta^{-m} y, \zeta^e z)$, where $\gcd(e, d) = 1$.
(v) The $\mathbb{C}^*$-action $\lambda.(x, y, z) := (\lambda x, \lambda^{-m} y, z)$ $(\lambda \in \mathbb{C}^*)$ on $\tilde{X}$ descends to the given $\mathbb{C}^*$-action on X, up to replacing λ by λ^{-1}.

Our interest in this result is explained by our previous study [1, 2] of normal affine surfaces admitting a $\mathbb{C}^*$-action and an affine ruling. We give an alternative proof of Theorem 1.1 based on these results. We deduce it from an abstract description of a certain subclass of such surfaces realized as hypersurfaces in $\mathbb{A}^3_{\mathbb{C}}$ (see Lemma 2.2 below).

Let us add some remarks. An affine ruling on X induces an affine ruling $\tilde{\rho} : \tilde{X} \to \mathbb{A}^1_{\mathbb{C}}$ with a unique degenerate fiber consisting of d disjoint components isomorphic to $\mathbb{A}^1_{\mathbb{C}}$. In case $m > 1$ there is an essentially unique such affine ruling on $\tilde{X}$, defined by the restriction $x|\tilde{X}$. However, for $m = 1$, $y|\tilde{X}$ gives a second independent affine ruling, which also descends to $X = \tilde{X}/\mathbb{Z}_d$. Thus in this case X cannot be a ML_1 surface.

If we want the $\mathbb{Z}_d$-action on $\tilde{X}$ to be free, the exponents e and d above must be coprime. Indeed, otherwise $\zeta^{eb} = 1$ for some b with $0 < b < d$, and we would have $\zeta^b.(0, 0, z) = (0, 0, z)$ for every d-th root of unity z.

On the other hand, for every triple (d, e, m) with $d \geq 1, m \geq 2$ and $\gcd(e, d) = 1$, (iii)-(v) determine a smooth affine pseudo-plane X of class ML_1 with an effective $\mathbb{C}^*$-action. Thus Theorem 1.1 provides indeed a complete classification of these surfaces.

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2. The proof

Under the assumptions of Theorem 1.1 $X \not\cong \mathbb{A}^2_{\mathbb{C}}$, since otherwise X would admit another affine ruling $\rho' : X \to \mathbb{A}^1_{\mathbb{C}}$ with general fibers different from those of ρ, which contradicts the condition ML_1.

A smooth affine surface X with an elliptic $\mathbb{C}^*$-action is always isomorphic to $\mathbb{A}^2_{\mathbb{C}}$, so this case is impossible. If X is smooth and the $\mathbb{C}^*$-action on X is parabolic then according to Proposition 3.8(b) in [1], $X = \operatorname{Spec} A_0[D]$ for an integral divisor D on a smooth affine curve $C = \operatorname{Spec} A_0$. The existence of an affine ruling ρ on X with the base $\mathbb{A}^1_{\mathbb{C}}$ implies that $C \cong \mathbb{A}^1_{\mathbb{C}}$. Hence D is a principal divisor. By Theorem 3.2(b) in [1], we have again $X \cong \mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} A_0[0]$ with $A_0 = \mathbb{C}[t]$, which is impossible.

Thus the $\mathbb{C}^*$-action on $X = \operatorname{Spec} A$ is necessarily hyperbolic. Accordingly we can write

$$A = A_0[D_+, D_-] \tag{1}$$

with a pair of $\mathbb{Q}$-divisors $D_\pm$ on a smooth affine curve $C = \operatorname{Spec} A_0$ satisfying $D_+ + D_- \le 0$, see Theorem 4.3 in [1]. The remainder of the proof is based on Lemmas 2.1 and 2.2 below.

Lemma 2.1. *Under the assumptions of Theorem 1.1, $A \cong A_0[D_+, D_-]$, where $A_0 = \mathbb{C}[t]$ and*

$$D_+ = -\frac{e'}{d}[0], \qquad D_- = \frac{e'}{d}[0] - \frac{1}{m}[1]\,.$$

Proof of Lemma 2.1. By Lemmas 1.6 and 2.1 in [2], X admits an affine ruling over an affine base if and only if it admits a non-trivial $\mathbb{C}_+$-action defined by a non-zero homogeneous locally nilpotent derivation $\partial \in \operatorname{Der}(A)$. Moreover, $A_0 = \mathbb{C}[t]$ in (1) and, up to an automorphism $\lambda \longmapsto \lambda^{-1}$ of $\mathbb{C}^*$ (thus switching $(D_+, D_-) \longmapsto (D_-, D_+)$) we may assume that $e = \deg \partial \ge 0$. By Lemma 3.5 and Corollary 3.27 in [2], $e = 0$ implies that $X \cong \mathbb{A}^1_{\mathbb{C}} \times \mathbb{C}^*$, so the induced affine ruling $X \to \mathbb{C}^*$ is essentially unique and has the base $\mathbb{C}^*$, which contradicts our assumption. Thus $e > 0$.

According to Corollary 3.23 in [2], the latter implies that the fractional part $\{D_+\} = D - \lfloor D \rfloor$ is zero or is supported on one point, and we can choose this point to be $0 \in \mathbb{A}^1_{\mathbb{C}}$. Such a surface $X = \operatorname{Spec} A$ is of class ML_1 if and only if the fractional part $\{D_-\}$ is supported on at least 2 points, see [2, Theorem 4.5].

Replacing (D_+, D_-) by the equivalent pair $(\{D_+\}, D_- + \lfloor D_+ \rfloor)$ (see Theorem 4.3(b) in [1]) we may suppose that $D_+ = \{D_+\} = -e'/d[0]$, where $\gcd(e', d) = 1$ and $d > 0$.

For any affine pseudo-plane X, the Picard group $\operatorname{Pic} X$ is a torsion group [4, Ch. 3, 2.4.4]. On the other hand, for a $\mathbb{C}^*$-surface X as above, $\operatorname{rk}_{\mathbb{Q}}(\operatorname{Pic} X \otimes \mathbb{Q}) \ge l - 1$, where l is the number of points $b_j \in \mathbb{A}^1_{\mathbb{C}}$ such that $(D_+ + D_-)(b_j) < 0$, see Corollary 4.24 in [2]. Hence $l \le 1$ and so, $\exists p \in \mathbb{A}^1_{\mathbb{C}} : (D_+ + D_-)(q) = 0 \ \forall q \ne p$.

Since $D_+(q) = 0 \ \forall q \ne 0$ we have $D_-(q) = 0 \ \forall q \ne 0, p$. It follows that $\operatorname{supp}(D_-) = \operatorname{supp}(\{D_-\}) = \{0, p\}$ with $p \ne 0$. After an automorphism of $\mathbb{A}^1_{\mathbb{C}}$ we may assume that $p = 1$. Thus finally

$$D_\pm(0) = \mp e'/d, \qquad D_+(1) = 0, \quad D_-(1) = a/m \notin \mathbb{Z} \qquad \text{and} \quad D_\pm(q) = 0 \ \ \forall q \ne 0, 1\,,$$

where $\gcd(a, m) = 1$ and $m > 0$. The smoothness of X forces $a = -1$, see Theorem 4.15 in [1]. This proves Lemma 2.1. □

Next we use the following description [2, Corollary 3.30], where for a $\mathbb{Q}$-divisor D, $d(D)$ denotes the minimal positive integer d such that dD is integral.

Lemma 2.2. *We let* $A = \mathbb{C}[t][D_+, D_-]$, *where* $D_+ + D_- \le 0$, $d(D_+) = d$, $d(D_-) = k$. *We assume that* $D_+ = -\frac{e'}{d}[0]$ *and* $D_-(0) = -\frac{l}{k}$, *and we let* $\partial \in \mathrm{Der}(A)$ *be a homogeneous locally nilpotent derivation with* $e = \deg \partial > 0$. *Then there exists a unitary polynomial* $Q \in \mathbb{C}[t]$ *with* $Q(0) \neq 0$ *and* $\mathrm{div}(t^l Q(t)) = -kD_-$ *such that, if* $A' = A_{k,P}$ *is the normalization of*

$$B_{k,P} = \mathbb{C}[u, v, s]/\left(u^k v - P(s)\right), \quad \textit{where} \quad P(s) = Q(s^d)s^{ke'+dl}, \tag{2}$$

then the group $\mathbb{Z}_d = \langle \zeta \rangle$ *acts on* $B_{k,P}$ *and also on* A' *via*

$$\zeta.(u, v, s) = (\zeta^{e'} u,\, v,\, \zeta s), \tag{3}$$

so that $A \cong A'^{\mathbb{Z}_d}$. *Furthermore,* $ee' \equiv 1 \mod d$ *and* $\partial = cu^e \frac{\partial}{\partial s}|A$ *for some constant* $c \in \mathbb{C}^*$.

With this result we can complete the proof of Theorem 1.1 as follows. We may assume that $A = A_0[D_+, D_-]$ with $A_0 = \mathbb{C}[t]$ and (D_+, D_-) as in Lemma 2.1. With $k := \mathrm{lcm}(d, m)$ let us write $k = mm' = dd'$ and $l = -e'd'$, so that

$$D_+ = -\frac{e'}{d}[0] = \frac{l}{k}[0], \qquad D_- = \frac{e'}{d}[0] - \frac{1}{m}[1] = -\frac{l}{k}[0] - \frac{m'}{k}[1].$$

Thus Lemma 2.2 can be applied in our setting with $Q = (t-1)^{m'}$. By this lemma, $A = A'^{\mathbb{Z}_d}$, where A' is the normalization of

$$B = \mathbb{C}[u, v, s]/(u^k v - (s^d - 1)^{m'}),$$

with the action of $\mathbb{Z}_d$ as in (3) and with the $\mathbb{C}^*$-action $\lambda.(u, v, s) = (\lambda u, \lambda^{-k} v, s)$.

The element $w = \frac{s^d - 1}{u^m} \in \mathrm{Frac}(B)$ satisfies $w^{m'} = v$ and so is integral over B, hence

$$A' \cong \mathbb{C}[u, w, s]/(u^m w - (s^d - 1)).$$

Because of (3) we have $\zeta.w = \zeta^{-me'} w$. Thus after applying an automorphism $\zeta \mapsto \zeta^{e'}$ of $\mathbb{Z}_d$, both the $\mathbb{Z}_d$-action and the $\mathbb{C}^*$-action on $\tilde{X} = \mathrm{Spec}\, A' \subseteq \mathbb{A}^3_{\mathbb{C}} = \mathrm{Spec}\, \mathbb{C}[u, w, s] \cong \mathrm{Spec}\, \mathbb{C}[x, y, z]$ have the claimed form

$$\zeta.(u, w, s) = (\zeta u, \zeta^{-m} w, \zeta^e s) \qquad \text{respectively,} \qquad \lambda.(u, w, s) = (\lambda u, \lambda^{-m} w, s).$$

This proves the theorem. □

References

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