Jackson Integral Representations of Modified q-Bessel Functions and q-Bessel-Macdonald Functions

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Abstract

The q analog of Modified Bessel functions and Bessel-Macdonald functions, were defined in our previous work (q-alg/950913) as general solutions of a second order difference equations. Here we present a collection of their representations using the noncommutative Jackson integral.

1 Introduction

In this work we continue our investigations of Modified q-Bessel Functions (qMBF) and q-Bessel-Macdonald Functions (qBMF) introduced in [1]. Though these functions can be easily expressed through the q-Bessel functions (qBF) of Jackson [2] , as their classical counterparts, they are important in their own turn. The classical Bessel functions and the qBF arises as matrix elements in irreducible representations of the group of motion of Euclidean spaces [3] or its quantum deformations [4, 5, 6]. Similarly, the classical MBF and BMF are related to the Whittaker model of irreducible unitary representations of real simple split Lie group G, when $G = SL(2,\mathbf{R})$ [7, 8, 9, 10, 11]. Modified Bessel Functions generate a basis in common eigenspace of Laplace operators in the coordinate systems, related to the Iwaswa decomposition of $SL(2,\mathbf{R})$, while the Bessel-Macdonald Functions are bounded eigenvectors. They are special matrix elements in the irreducible principle series representations. This fact means that BMF have integral representations coming from the invariant Hermitian measure on sections of line bundles over the flag variety, where the irreducible unitary representations are realized (Borel-Weyl theorem).

The main object of present work is the analogous integral representations for qMBF and some other integral representations for both kind of functions, which have well-known classical

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forms. Though there is a clear group theoretical interpretation of these representations our derivation is pure analytic, beyond the group methods. In fact qBMF, we consider here, plays the role of the Whittaker function for $U_q(SL(2, \mathbf{C}))$ [12]. The q-integral we use very similar to the Fourier transform on the quantum line and plane [13]. Such ingredients as the horospheric projection, Whittaker vectors, Harish-Chandra function which occur in the harmonic analysis on quantum Lobachevsky space can be extracted from our integrals. They correspond to the analogous elements in the spherical model of unitary representation of $U_q(SU(1,1))$ [14], where the Jackson integral was used to describe the zonal spherical functions.

Another applications of integral representations of qMBF is Poisson kernels for quantum Lobachevsky space , or more generally, quantum hyperboloids. The details of both types of interpretations will be published elsewhere.

2 Some preliminary relations

Jackson q-integral is determined as the map from an algebra of the functions of one variable into a set of the number series

$$\int_{-1}^{1} f(x)d_q x = (1 - q) \sum_{m=0}^{\infty} q^m[f(q^m) + f(-q^m)],$$

$$\int_{0}^{\infty} f(x)d_q x = (1 - q) \sum_{m=-\infty}^{\infty} q^m f(q^m),$$

$$\int_{-\infty}^{\infty} f(x)d_q x = (1 - q) \sum_{-\infty}^{\infty} q^m[f(q^m) + f(-q^m)].$$

Define the difference operator

$$\partial_x f(x) = \frac{x^{-1}}{1 - q}[f(x) - f(qx)]. \tag{2.1}$$

The following formulas of the q-integration by parts are valid

$$\int_{-1}^{1} \phi(x)\partial_x \psi(x)d_q x = \phi(1)\psi(1) - \phi(-1)\psi(-1) - \int_{-1}^{1} \partial_x \phi(x)\psi(qx)d_q x, \tag{2.2}$$

$$\int_{0}^{\infty} \phi(x)\partial_x \psi(x)d_q x = \lim_{m \to \infty}[\phi(q^{-m})\psi(q^{-m}) - \phi(q^m)\psi(q^m)] - \int_{0}^{\infty} \partial_x \phi(x)\psi(qx)d_q x. \tag{2.3}$$

$$\int_{-\infty}^{\infty} \phi(x)\partial_x \psi(x)d_q x = \lim_{m \to \infty}[\phi(q^{-m})\psi(q^{-m}) + \phi(-q^{-m})\psi(-q^{-m})] - \int_{-\infty}^{\infty} \partial_x \phi(x)\psi(qx)d_q x. \tag{2.4}$$

The last two expressions imply the regularization of the improper integrals.

·Let z and s be noncommuting elements and

$$zs = qsz. \tag{2.5}$$

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Consider the function

$$f(x) = \sum_r a_r x^r.$$

(2.6)

The rule of q-integration in the noncommutative case is

$$\int f(zs) d_q s = \int \sum_r a_r (zs)^r d_q s = \int \sum_r a_r q^{-\frac{r(r-1)}{2}} z^r s^r d_q s,$$

$$\int d_q z f(zs) = \int d_q z \sum_r a_r (zs)^r = \int d_q z \sum_r a_r q^{-\frac{r(r-1)}{2}} z^r s^r.$$

Define the following transformation $\ddagger f \ddagger$ for functions f depending on the noncommutative variables s and z (2.5). If we have the function which has the form (2.6) and all monoms are ordered we will write

$$f(zs) = \sum_r a_r (zs)^r \rightarrow \ddagger f(zs) \ddagger = \sum_r a_r z^r s^r.$$

Definition 2.1 *The function $f(z)$ is absolutely q-integrable if the series*

$$\sum_{m=-\infty}^{\infty} q^m f(q^m)$$

converges absolutely.

It means, in particular, that

$$\lim_{m \to \pm\infty} q^m |f(q^m)| = 0$$

Let [15]

$$(a, q)_n = \begin{cases} 1 & for \quad n = 0 \\ (1-a)(1-aq)\dots(1-aq^{n-1}) & for \quad n \geq 1, \end{cases}$$

$$(a, q)_\infty = \lim_{n \to \infty} (a, q)_n, \qquad (a_1, \dots, a_k, q)_\infty = (a_1, q)_\infty \dots (a_k, q)_\infty.$$

(2.7)

and

$$\Gamma_q(\nu) = \frac{(q, q)_\infty}{(q^\nu, q)_\infty} (1-q)^{1-\nu}$$

is the q-gamma function. Consider the q-exponentials

$$e_q(\frac{1-q^2}{2} x) = \sum_{n=0}^{\infty} \frac{(1-q^2)^n x^n}{(q, q)_n 2^n}, \qquad |x| < \frac{2}{1-q^2},$$

(2.8)

$$E_q(\frac{1-q^2}{2} x) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} (1-q^2)^n x^n}{(q, q)_n 2^n},$$

(2.9)

Note that

$$e_q(\frac{1-q^2}{2} x) = \frac{1}{(\frac{1-q^2}{2} x, q)_\infty}, \quad E_q(\frac{1-q^2}{2} x) = (-\frac{1-q^2}{2} x, q)_\infty.$$

(2.10)

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and

$$e_q(q)E_q(q^{-1}) = \frac{1}{(q,q)_\infty} \sum_{k=0}^{\infty} \frac{q^{\frac{k(k+1)}{2}}}{(q,q)_k} \tag{2.11}$$

Consider also the basic hypergeometric series

$$_0\Phi_1(-;0;q,\frac{1-q^2}{2}x) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)}(1-q^2)^n x^n}{(q,q)_n 2^n}.$$

It follows immediately

Proposition 2.1

$$_0\Phi_1(-;0;q,\frac{1-q^2}{2}zs) = \ddagger E_q(\frac{1-q^2}{2}zs)\ddagger, \tag{2.12}$$

$$E_q(\frac{1-q^2}{2}zs) = \ddagger e_q(\frac{1-q^2}{2}(zs))\ddagger. \tag{2.13}$$

Now consider the q-Bessel function [2]

$$J_\nu^{(1)}((1-q^2)x;q^2) = \frac{1}{\Gamma_{q^2}(\nu+1)} \sum_{n=0}^{\infty} \frac{(-1)^n(1-q^2)^{2n}x^{\nu+2n}}{(q^2,q^2)_n(q^{2\nu+2},q^2)_n 2^{\nu+2n}}, \tag{2.14}$$

$$J_\nu^{(2)}((1-q^2)x;q^2) = \frac{1}{\Gamma_{q^2}(\nu+1)} \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n(\nu+n)}(1-q^2)^{2n}x^{\nu+2n}}{(q^2,q^2)_n(q^{2\nu+2},q^2)_n 2^{\nu+2n}}, \tag{2.15}$$

and the basic hypergeometric series [15]

$$_0\Phi_3(-;0,0,q^{2\nu+2};q^2,-(\frac{1-q^2}{2}q^{2\nu+2}x)^2) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{4n(\nu+n)}(1-q^2)^{2n}x^{2n}}{(q^2,q^2)_n(q^{2\nu+2},q^2)_n 2^{2n}}. \tag{2.16}$$

As a generalization of (2.5) we assume for any ν and μ

$$z^\nu s^\mu = q^{\nu\mu}s^\mu z^\nu. \tag{2.17}$$

Proposition 2.2

$$\frac{1}{\Gamma_{q^2}(\nu+1)}\left(q^{-1/2}\frac{zs}{2}\right)^\nu {}_0\Phi_3(-;0,0,q^{2\nu+2};q^2,-(\frac{1-q^2}{2}q^{2(\nu+1)-1/2}zs)^2) =$$

$$= q^{-\frac{\nu^2}{2}}\ddagger J_\nu^{(2)}((1-q^2)zs;q^2)\ddagger, \tag{2.18}$$

$$J_\nu^{(2)}((1-q^2)q^{-1/2}zs;q^2) = q^{-\frac{\nu^2}{2}}\ddagger J_\nu^{(1)}((1-q^2)zs;q^2)\ddagger. \tag{2.19}$$

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Proof. These formulas follow from (2.17) and (2.14)-(2.16).■

In the noncommutative case we define the difference operator similarly to (2.1)

$$\partial_s f(zs) = \frac{s^{-1}}{1-q}[f(zs) - f(qzs)].$$

Then

$$\frac{1}{1-q}[f(zs) - f(qzs)]s^{-1} = \partial_s f(q^{-1}zs).$$

As it follows from (2.1) and (2.8), (2.9) for arbitrary a

$$\frac{2\partial_z}{1+q}\ddagger e_q(\frac{1-q^2}{2}azs)\ddagger = a\ddagger e_q(\frac{1-q^2}{2}azs)\ddagger s, \tag{2.20}$$

$$\frac{2\partial_s}{1+q}\ddagger e_q(\frac{1-q^2}{2}azs)\ddagger = aqz\ddagger e_q(\frac{1-q^2}{2}aq(zs))\ddagger, \tag{2.21}$$

$$\frac{2\partial_z}{1+q}\ddagger E_q(\frac{1-q^2}{2}azs)\ddagger = a\ddagger E_q(\frac{1-q^2}{2}aq(zs))\ddagger s, \tag{2.22}$$

$$\frac{2\partial_s}{1+q}\ddagger E_q(\frac{1-q^2}{2}azs)\ddagger = aqz\ddagger E_q(\frac{1-q^2}{2}aq^2(zs))\ddagger. \tag{2.23}$$

Similarly from (2.1) and (2.14), (2.15)

$$\frac{2\partial_z}{1+q}[(z/2)^{-\nu}\ddagger J_\nu^{(1)}((1-q^2)azs;q^2)\ddagger] = -a(z/2)^{-\nu}\ddagger J_{\nu+1}^{(1)}((1-q^2)azs;q^2)\ddagger s, \tag{2.24}$$

$$\frac{2\partial_s}{1+q}[\ddagger J_\nu^{(1)}((1-q^2)azs;q^2)\ddagger(s/2)^{-\nu}] = -aqz\ddagger J_{\nu+1}^{(1)}((1-q^2)aqzs;q^2)\ddagger(s/2)^{-\nu}, \tag{2.25}$$

$$\frac{2\partial_z}{1+q}[(z/2)^{\nu}\ddagger J_\nu^{(1)}((1-q^2)azs;q^2)\ddagger] = a(z/2)^{\nu}\ddagger J_{\nu-1}^{(1)}((1-q^2)azs;q^2)\ddagger s, \tag{2.26}$$

$$\frac{2\partial_z}{1+q}[(z/2)^{-\nu}\ddagger J_\nu^{(2)}((1-q^2)azs;q^2)\ddagger] = -aq^{\nu+1}(z/2)^{-\nu}\ddagger J_{\nu+1}^{(2)}((1-q^2)aqzs;q^2)\ddagger s, \tag{2.27}$$

$$\frac{2\partial_s}{1+q}[\ddagger J_\nu^{(2)}((1-q^2)azs;q^2)\ddagger(s/2)^{-\nu}] = -aq^{\nu+2}z\ddagger J_{\nu+1}^{(2)}((1-q^2)aq^2zs;q^2)\ddagger(s/2)^{-\nu}, \tag{2.28}$$

$$\frac{2\partial_z}{1+q}[(z/2)^{\nu}\ddagger J_\nu^{(2)}((1-q^2)azs;q^2)\ddagger] = aq^{-\nu+1}(z/2)^{\nu}\ddagger J_{\nu-1}^{(2)}((1-q^2)aqzs;q^2)\ddagger s, \tag{2.29}$$

Consider the complete elliptic integrals:

$$\mathbf{K}(k) = \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1 - k^2\sin^2\alpha}}, \qquad \mathbf{K}'(k) = \int_0^{\pi/2} \frac{d\alpha}{\sqrt{\cos^2\alpha + k^2\sin^2\alpha}}$$

with

$$\ln q = -\frac{\pi\mathbf{K}'(k)}{\mathbf{K}(k)}. \tag{2.30}$$

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Then [16](5.3.6.1)

$$\mathbf{Q}_\nu = (1-q) \sum_{m=-\infty}^{\infty} \frac{1}{q^{m-\nu+1/2} + q^{-m+\nu-1/2}} = \frac{1-q}{\pi} \mathbf{K}(k) \, \mathrm{dn}[\frac{2 \ln q^{-\nu+1/2}}{\pi} \mathbf{K}'(k)], \qquad (2.31)$$

where $\mathrm{dn}\, u = \sqrt{1 - k^2 \sin^2 \phi}$, $u = \int_0^\phi \frac{d\alpha}{\sqrt{1-k^2 \sin^2 \alpha}}$, (dn u is the Jacobi elliptic function). If $u = 0$ then $\phi = 0$, and dn $u = 1$.

Proposition 2.3 *For an arbitrary* ν

$$\lim_{q \to 1-0} \mathbf{Q}_\nu = \frac{\pi}{2} \qquad (2.32)$$

Proof. Since $\mathbf{K}'(k) > 0$ for any k it follows from (2.30) that for $q \to 1-0$ $\mathbf{K}(k) \to \infty$, and hence $\lim_{q \to 1-0} k = 1 - 0$. In this case $\mathbf{K}'(1) = \pi/2$ and dn $u = 1$. Now from (2.31) we have

$$\lim_{q \to 1-0} \mathbf{Q}_\nu = -\lim_{q \to 1-0} \frac{1-q}{\ln q} \mathbf{K}'(k) \, \mathrm{dn}[\frac{2 \ln q^{-\nu+1/2}}{\pi} \mathbf{K}'(k)] = \frac{\pi}{2}.$$

3 Some properties of the q-binomial formula

There is a q-analog of the classical binomial formula [15]

$$(1-z)^{-a} = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k, \quad (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}), \qquad |z| < 1,$$

$$\frac{(q^\alpha z, q)_\infty}{(z, q)_\infty} = \sum_{k=0}^{\infty} \frac{(q^\alpha, q)_k}{(q, q)_k} z^k, \qquad |z| < 1.$$

We need in two generalizations of the q-binom

$$r(a, b, z, q) = \frac{(az, q)_\infty}{(bz, q)_\infty} \qquad (3.1)$$

$$R(a, b, \gamma, z, q^2) = \frac{(az^2, q^2)_\infty}{(bz^2, q^2)_\infty} z^\gamma \qquad (3.2)$$

Proposition 3.1 *The function* $r(a, b, z, q)$ *(3.1) satisfies the difference equation*

$$z[br(a, b, z, q) - ar(a, b, qz, q)] = r(a, b, z, q) - r(a, b, qz, q). \qquad (3.3)$$

Proof. It follows from (3.3) that

$$\frac{r(a, b, z, q)}{r(a, b, qz, q)} = \frac{1 - az}{1 - bz}.$$

Now the statement of the Proposition follows from (2.7).∎

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Proposition 3.2 *The function $R(a, b, \gamma, z, q^2)$ (3.2) satisfies the difference equation*

$$z^2[bq^\gamma R(a, b, \gamma, z, q^2) - aR(a, b, \gamma, qz, q^2)] = q^\gamma R(a, b, \gamma, z, q^2) - R(a, b, \gamma, qz, q^2). \tag{3.4}$$

Proof. Take $z^{-\gamma} R(a, b, \gamma, z, q) = r(a, b, z^2, q^2)$, and substitute it in (3.3). Then the equation

$$z^2[bz^{-\gamma} R(a, b, \gamma, z, q^2) - aq^{-\gamma}z^{-\gamma} R(a, b, \gamma, qz, q^2)] = z^{-\gamma} R(a, b, \gamma, z, q) - q^{-\gamma}z^{-\gamma} R(a, b, \gamma, qz, q^2).$$

leads directly to (3.4).∎

Lemma 3.1 *If $|a| < |b|$ the function $r(a, b, z, q)$ can be represented as the sum of the partial functions*

$$\frac{(az, q)_\infty}{(bz, q)_\infty} = \frac{1}{(q, q)_\infty} \sum_{k=0}^\infty \frac{(-1)^k q^{\frac{k(k+1)}{2}} (a/bq^{-k}, q)_\infty}{(q, q)_k (1 - zbq^k)}. \tag{3.5}$$

The series (3.5) converges absolutely for any $z \neq b^{-1}q^{-k}$, $k = 0, 1, \dots$

Proof. Let

$$r_n(a, b, z, q) = \frac{(az, q)_n}{(bz, q)_n} = (a/b)^n + \sum_{k=0}^n \frac{c_{k,n}}{1 - zbq^k}.$$

Then

$$c_{k,n} = \lim_{z \to b^{-1}q^{-k}} (1 - zbq^k) r_n(a, b, z, q) = \frac{(-1)^k q^{\frac{k(k+1)}{2}} (a/bq^{-k}, q)_n}{(q, q)_k (q, q)_{n-k}}$$

and

$$r_n(a, b, z, q) = (a/b)^n + \sum_{k=0}^n \frac{(-1)^k q^{\frac{k(k+1)}{2}} (a/bq^{-k}, q)_n}{(q, q)_k (1 - zbq^k)} \frac{1}{(q, q)_{n-k}}.$$

Since $\lim_{n \to \infty} (a/b)^n = 0$ and $\frac{1}{(q,q)_{n-k}} < \frac{1}{(q,q)_\infty}$ we obtain

$$r(a, b, z, q) = \lim_{n \to \infty} r_n(a, b, z, q) = \frac{1}{(q, q)_\infty} \sum_{k=0}^\infty \frac{(-1)^k q^{\frac{k(k+1)}{2}} (a/bq^{-k}, q)_\infty}{(q, q)_k (1 - zbq^k)}.$$

Applying the d'Alembert criterion we obtain

$$\lim_{k \to \infty} \left| \frac{q^{\frac{(k+1)(k+2)}{2}} (a/bq^{-k-1}, q)_\infty (q, q)_k (1 - zbq^k)}{(q, q)_{k+1} (1 - zbq^{k+1}) q^{\frac{k(k+1)}{2}} (a/bq^{-k}, q)_\infty} \right| = \lim_{k \to \infty} |q^{k+1} - a/b| = |a/b| < 1.$$

Hence (3.5) converges absolutely for any $|a| < |b|$, and $r(a, b, z, q)$ is the meromorphic function with the ordinary poles at the points $z = b^{-1}q^{-k}$, $k = 0, 1, \dots$ ∎

It is easy to get

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Remark 3.1 *If $0 < |a| < |b|$ then*

$$\frac{(az,q)_\infty}{(bz,q)_\infty} = \frac{(a/b,q)_\infty}{(q,q)_\infty} \sum_{k=0}^\infty \frac{(b/aq,q)_k (a/b)^k}{(q,q)_k (1 - zbq^k)}. \tag{3.6}$$

If $a = 0$ then

$$\frac{1}{(bz,q)_\infty} = \frac{1}{(q,q)_\infty} \sum_{k=0}^\infty \frac{(-1)^k q^{\frac{k(k+1)}{2}}}{(q,q)_k (1 - zbq^k)}. \tag{3.7}$$

Assume that $a = \epsilon q^{2\alpha}, b = \epsilon q^{2\beta}, \epsilon = \pm 1$ in (3.2). Then we have from (3.6)

Corollary 3.1

$$z^\gamma \frac{(\epsilon q^{2\alpha} z^2, q^2)_\infty}{(\epsilon q^{2\beta} z^2, q^2)_\infty} = z^\gamma \frac{(q^{2(\alpha-\beta)}, q^2)_\infty}{(q^2, q^2)_\infty} \sum_{k=0}^\infty \frac{(q^{2(\beta-\alpha+1)}, q^2)_k q^{2(\alpha-\beta)k}}{(q^2, q^2)_k (1 - \epsilon z^2 q^{2(\beta+k)})}. \tag{3.8}$$

Remark 3.2 *As it follows from [15](1.3.2)*

$$\frac{(\epsilon q^{2\alpha} z^2, q^2)_\infty}{(\epsilon q^{2\beta} z^2, q^2)_\infty} = \sum_{k=0}^\infty \epsilon^k q^{2\beta k} \frac{(q^{2(\alpha-\beta)}, q^2)_k}{(q^2, q^2)_k} z^{2k}, \tag{3.9}$$

which converges in the domain $|z| < q^{-\beta/2}$.

It follows from Lemma 3.1 that if $\gamma = 0$ (3.8) is the meromorphic function with the ordinary poles $z = \pm\sqrt{\epsilon} q^{-\beta-k}$, $k = 0, 1, \ldots$, and hence it is the analitic continuation of (3.9).

Corollary 3.2 *For an arbitrary real $s \neq 0$*

$$\lim_{m \to \infty} |e_q(i \frac{1-q^2}{2} q^{-m} s)| = 0.$$

Proof. Due to (2.10), Lemma 3.1 for $a = 0$, $b = i \frac{1-q^2}{2} q^{-m}$, and the evident inequality for $k \geq 0$

$$\frac{1}{1 + s^2 (\frac{1-q^2}{2})^2 q^{-2m+2k}} \leq \frac{q^{-2k}}{1 + s^2 (\frac{1-q^2}{2})^2 q^{-2m}}, \tag{3.10}$$

we have

$$|e_q(i \frac{1-q^2}{2} q^{-m} s)| \leq \frac{1}{(q,q)_\infty} \sum_{k=0}^\infty |\frac{q^{\frac{k(k+1)}{2}}}{(q,q)_k (1 - i \frac{1-q^2}{2} s q^{-m+k})}| \leq$$

$$\leq (1 + s^2 (\frac{1-q^2}{2})^2 q^{-2m})^{-1/2} \frac{1}{(q,q)_\infty} \sum_{k=0}^\infty \frac{q^{\frac{k(k-1)}{2}} q^{-k}}{(q,q)_k} = (1 + s^2 (\frac{1-q^2}{2})^2 q^{-2m})^{-1/2} E_q(q^{-1}) e_q(q).$$

∎

There are two types of q-trigonometric functions

$$\cos_q z = \frac{1}{2}[e_q(iz) + e_q(-iz)],$$

$$\text{Cos}_q z = \frac{1}{2}[E_q(iz) + E_q(-iz)], \quad \text{Sin}_q z = \frac{1}{2i}[E_q(iz) - E_q(-iz)].$$

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Corollary 3.3 *For real $s \neq 0$ and integer m*

$$|\cos_q(\frac{1-q^2}{2}q^{-m}s)| \leq \frac{E_q(q^{-1})e_q(q)}{1+(\frac{1-q^2}{2})^2q^{-2m}s^2}.$$

Proof. If $a = 0$, $b = \pm i\frac{1-q^2}{2}q^{-m}$ then from (3.7)

$$\cos_q(\frac{1-q^2}{2}q^{-m}s) = \frac{1}{(q,q)_\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{\frac{k(k+1)}{2}}}{(q,q)_k(1+(\frac{1-q^2}{2})^2q^{-2m+2k}s^2)}.$$

Using (3.10) we can put

$$|\cos(\frac{1-q^2}{2}q^{-m}s)| \leq \frac{1}{(q,q)_\infty(1+(\frac{1-q^2}{2})^2q^{-2m}s^2)}\sum_{k=0}^{\infty}\frac{q^{\frac{k(k+1)}{2}}q^{-2k}}{(q,q)_k}.$$

Then the statement follows from (2.11) ∎

Corollary 3.4 *For an arbitrary real $s > 0$*

$$\lim_{m\to\infty}e_q(-\frac{1-q^2}{2}q^{-m}s) = 0.$$

Proof. Again, due to (2.10), (2.11), Lemma 3.1 for $a = 0$, $b = -\frac{1-q^2}{2}q^{-m}$, and the inequality

$$\frac{1}{1+s\frac{1-q^2}{2}q^{-m+k}} \leq \frac{q^{-k}}{1+s\frac{1-q^2}{2}q^{-m}},$$

we have

$$|e_q(-\frac{1-q^2}{2}q^{-m}s)| \leq \frac{1}{(q,q)_\infty}\sum_{k=0}^{\infty}|\frac{q^{\frac{k(k+1)}{2}}}{(q,q)_k(1+\frac{1-q^2}{2}sq^{-m+k})}| \leq$$

$$\leq \frac{1}{|1+s\frac{1-q^2}{2}q^{-m}|}\frac{1}{(q,q)_\infty}\sum_{k=0}^{\infty}\frac{q^{\frac{k(k-1)}{2}}}{(q,q)_k} = \frac{1}{|1+s\frac{1-q^2}{2}q^{-m}|}E_q(1)e_q(q).∎$$

Corollary 3.5 *For an arbitrary real s*

$$|\mathrm{Cos}_q(\frac{1-q^2}{2}q^{-m}s)| \leq 1, \quad |\mathrm{Sin}_q(\frac{1-q^2}{2}q^{-m}s)| \leq \frac{1}{2}q^{-m}(1+q)|s|. \tag{3.11}$$

Proof. These functions are represented by the absolutely convergent alternating series

$$\mathrm{Cos}_q(\frac{1-q^2}{2}q^{-m}s) = \sum_{n=0}^{\infty}\frac{(-1)^n q^{n(2n-1-2m)}(1-q^2)^{2n}s^{2n}}{2^{2n}(q,q)_{2n}},$$

$$\mathrm{Sin}_q(\frac{1-q^2}{2}q^{-m}s) = \sum_{n=0}^{\infty}\frac{(-1)^n q^{(n-m)(2n+1)}(1-q^2)^{2n+1}s^{2n+1}}{2^{2n+1}(q,q)_{2n+1}},$$

and hence we have (3.11).∎

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Corollary 3.6 *If $\alpha > \beta + 1$ and real $z \neq 0$, then*

$$\frac{(-q^{2\alpha}z^2, q^2)_\infty}{(-q^{2\beta}z^2, q^2)_\infty} \leq \frac{C_{\alpha,\beta}}{1 + z^2 q^{2\beta}}.$$

Proof. Since for any $k \geq 0$ $\quad \frac{1}{1+z^2 q^{2\beta+2k}} \leq \frac{q^{-2k}}{1+z^2 q^{2\beta}}$ we have from (3.8)

$$\frac{(-q^{2\alpha}z^2, q^2)_\infty}{(-q^{2\beta}z^2, q^2)_\infty} \leq \frac{(q^{2(\alpha-\beta)}, q^2)_\infty}{(q^2, q^2)_\infty (1 + z^2 q^{2\beta})} \sum_{k=0}^{\infty} \left| \frac{(q^{2(\beta-\alpha+1)}, q^2)_k q^{2(\alpha-\beta-1)k}}{(q^2, q^2)_k} \right|.$$

The series in the right hand side converges due to the d'Alembert criterion and thereby produce the finite contribution in the constant $C_{\alpha,\beta}$. ■

Corollary 3.7 *If $\alpha > \beta + 1/2$*

$$\lim_{m \to \infty} \frac{(-q^{2(\alpha-m)}, q^2)_\infty}{(-q^{2(\beta-m)}, q^2)_\infty} = 0. \tag{3.12}$$

Proof. For $z = q^{-2m}, \gamma = 0, \epsilon = -1$ we have from (3.8)

$$\frac{(-q^{2(\alpha-m)}, q^2)_\infty}{(-q^{2(\beta-m)}, q^2)_\infty} = \frac{(q^{2(\alpha-\beta)}, q^2)_\infty}{(q^2, q^2)_\infty} \sum_{k=0}^{\infty} \frac{(q^{2(\beta-\alpha+1)}, q^2)_k q^{2(\alpha-\beta)k}}{(q^2, q^2)_k (1 + q^{2(\beta-m+k)})}.$$

Obviously

$$\frac{1}{1 + q^{2(\beta-m+k)}} = \frac{q^{m-k-\beta}}{q^{m-k-\beta} + q^{\beta+k-m}} \leq \frac{q^{m-k-\beta}}{2}.$$

Thus

$$\left| \frac{(-q^{2(\alpha-m)}, q^2)_\infty}{(-q^{2(\beta-m)}, q^2)_\infty} \right| \leq \frac{q^{m-\beta}}{2} \sum_{k=0}^{\infty} \left| \frac{(q^{2(\beta-\alpha+1)}, q^2)_k q^{2(\alpha-\beta-1/2)k}}{(q^2, q^2)_k} \right|.$$

The series in the right hand side converges. Due to the multiplier q^m we come to (3.12).■

Remark 3.3 *Let $a = \epsilon q^{2\alpha}, b = \epsilon q^{2\beta}$ in (3.2) and (3.4). Then if $q \to 1-0$ the difference equation (3.4) takes the form of the differential equation*

$$z(1 - \epsilon z^2) R'(z) - [\gamma + \epsilon(2\alpha - 2\beta - \gamma)z^2] R(z) = 0 \tag{3.13}$$

with solution

$$R(z) = C z^\gamma (1 - \epsilon z^2)^{\beta-\alpha}.$$

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4 The Jackson integral representation of the Modified q-Bessel Functions

We remind that modified q-Bessel function of the first kind [1]

$$I_\nu^{(1)}((1-q^2)s; q^2) = \frac{1}{\Gamma_{q^2}(\nu+1)} \sum_{n=0}^{\infty} \frac{(1-q^2)^{2n} s^{\nu+2n}}{(q^2,q^2)_n (q^{2\nu+2},q^2)_n 2^{\nu+2n}}, \quad |s| < \frac{1}{1-q^2}, \quad (4.1)$$

satisfies the difference equation

$$[1 - (\frac{1-q^2}{2})^2 q^{-2} s^2] f_\nu(q^{-1}s) - (q^{-\nu} + q^\nu) f_\nu(s) + f_\nu(qs) = 0, \quad (4.2)$$

and modified q-Bessel function of the second kind

$$I_\nu^{(2)}((1-q^2)s; q^2) = \frac{1}{\Gamma_{q^2}(\nu+1)} \sum_{n=0}^{\infty} \frac{q^{2n(\nu+n)}(1-q^2)^{2n} s^{\nu+2n}}{(q^2,q^2)_n (q^{2\nu+2},q^2)_n 2^{\nu+2n}} \quad (4.3)$$

satisfies the difference equation

$$f_\nu(q^{-1}s) - (q^{-\nu} + q^\nu) f_\nu(s) + [1 - (\frac{1-q^2}{2})^2 s^2] f_\nu(qs) = 0. \quad (4.4)$$

q-Bessel-Macdonald function (q-BMF) has been determined for noninteger ν as [1]

$$K_\nu^{(j)}((1-q^2)s; q^2) = \frac{q^{-\nu^2+1/2}}{4(a_\nu a_{-\nu})^{3/2} \sin \nu\pi} [a_\nu I_{-\nu}^{(j)}((1-q^2)s; q^2) - a_{-\nu} I_\nu^{(j)}((1-q^2)s; q^2)] \quad (4.5)$$

where

$$a_\nu = \sqrt{2/(1-q^2)} e_q(-1) \frac{I_\nu^{(2)}(2; q^2)}{{}_2\Phi_1(q^{\nu+1/2}, q^{-\nu+1/2}; -q; q, q)}, \quad (4.6)$$

and

$$a_\nu a_{-\nu} = \frac{q^{-\nu+1/2}}{2\Gamma_{q^2}(\nu)\Gamma_{q^2}(1-\nu) \sin \nu\pi}. \quad (4.7)$$

If ν is an integer n functions $K_n^{(j)}$ are well defined in the limit for $\nu \to n$ in (4.5).

Functions (4.5) satisfy (4.2) and (4.4) for $j = 1$ and $j = 2$ respectively.

Proposition 4.1 *Modified q-Bessel function (q-MBF) $I_\nu^{(1)}$ for $\nu > 0$ can be represented as the q-integral*

$$I_\nu^{(1)}((1-q^2)s; q^2) = \frac{1+q}{2\Gamma_{q^2}(\nu+1/2)\Gamma_{q^2}(1/2)} \int_{-1}^{1} d_q z \frac{(q^2 z^2, q^2)_\infty}{(q^{2\nu+1} z^2, q^2)_\infty} E_q(\frac{1-q^2}{2} zs)(s/2)^\nu. \quad (4.8)$$

Proof. Consider the q-integral

$$S_1^{(1)}(s) = \int_{-1}^{1} d_q z f_\nu^{(1)}(z) E_q(\frac{1-q^2}{2} zs), \quad (4.9)$$

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where $f_\nu^{(1)}(z)$ is a such function that it is absolutely convergent. Require that $S_1^{(1)}(s)(s/2)^\nu$ satisfies (4.2). Then $S_1^{(1)}(s)$ satisfies the equation

$$[1 - (\frac{1-q^2}{2})^2 q^{-2}s^2]q^{-\nu}S_1^{(1)}(q^{-1}s) - (q^{-\nu} + q^\nu)S_1^{(1)}(s) + q^\nu S_1^{(1)}(qs) = 0.$$

or

$$S_1^{(1)}(q^{-1}s) - S_1^{(1)}(s) - q^{2\nu}[S_1^{(1)}(s) - S_1^{(1)}(qs)] = (\frac{1-q^2}{2})^2 q^{-2}S_1^{(1)}(q^{-1}s)s^2. \qquad (4.10)$$

Substituting (4.9) in (4.10), multiplying it on $\frac{2s^{-1}}{1-q^2}$ from the right, and drawing it through E_q we obtain

$$\int_{-1}^{1} d_q z f_\nu^{(1)}(z)\frac{2s^{-1}}{1-q^2}[E_q(\frac{1-q^2}{2}q^{-2}zs) - E_q(\frac{1-q^2}{2}q^{-1}zs)]-$$

$$-q^{2\nu}\int_{-1}^{1} d_q z f_\nu^{(1)}(z)\frac{2s^{-1}}{1-q^2}[E_q(\frac{1-q^2}{2}q^{-1}zs) - E_q(\frac{1-q^2}{2}zs)] =$$

$$= \frac{1-q^2}{2}q^{-2}\int_{-1}^{1} d_q z f_\nu^{(1)}(z)E_q(\frac{1-q^2}{2}q^{-1}zs)s.$$

Due to (2.13), (2.20) and (2.21) it can be rewritten as

$$q^{-1}\int_{-1}^{1} d_q z f_\nu^{(1)}(z)z\ddagger e_q(\frac{1-q^2}{2}q^{-1}zs)\ddagger - q^{2\nu}\int_{-1}^{1} d_q z f_\nu^{(1)}(z)z\ddagger e_q(\frac{1-q^2}{2}(zs))\ddagger =$$

$$= \frac{1-q^2}{2}q^{-2}\int_{-1}^{1} d_q z f_\nu^{(1)}(z)\ddagger e_q(\frac{1-q^2}{2}q^{-1}zs)\ddagger s,$$

or

$$\int_{-1}^{1} d_q z f_\nu^{(1)}(z)z^{-2\nu+1}\frac{2z^{-1}}{1-q^2}[z^{2\nu+1}\ddagger e_q(\frac{1-q^2}{2}q^{-1}zs)\ddagger - q^{2\nu+1}z^{2\nu+1}\ddagger e_q(\frac{1-q^2}{2}(zs))\ddagger] =$$

$$= \int_{-1}^{1} d_q z f_\nu^{(1)}(z)\frac{2\partial_z}{1+q}\ddagger e_q(\frac{1-q^2}{2}q^{-1}zs)\ddagger.$$

Using the q-integration by parts (2.2) we obtain

$$-\int_{-1}^{1} d_q z \partial_z(f_\nu^{(1)}(z)z^{-2\nu+1})q^{2\nu+1}z^{2\nu+1}\ddagger e_q(\frac{1-q^2}{2}(zs))\ddagger = -\int_{-1}^{1} d_q z \partial_z f_\nu^{(1)}(z)\ddagger e_q(\frac{1-q^2}{2}(zs))\ddagger.$$

Thus we come to the difference equation for $f_\nu^{(1)}(z)$

$$q^{2\nu+1}z^2[f_\nu^{(1)}(z) - q^{-2\nu+1}f_\nu^{(1)}(qz)] = f_\nu^{(1)}(z) - f_\nu^{(1)}(qz). \qquad (4.11)$$

It coincides with (3.4) for $a = q^2, b = q^{2\nu+1}, \gamma = 0$, and hence from the Proposition 3.2

$$f_\nu^{(1)}(z) = \frac{(q^2z^2, q^2)_\infty}{(q^{2\nu+1}z^2, q^2)_\infty}. \qquad (4.12)$$

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$S_1^{(1)}(s)(s/2)^\nu$ is a solution to (4.2) and therefore it can be represented as [1]

$$S_1^{(1)}(s)(s/2)^\nu = A I_\nu^{(1)}((1 - q^2)s; q^2) + B K_\nu^{(1)}((1 - q^2)s; q^2).$$

Multiplying the both sides on $(s/2)^\nu$ and putting $s = 0$ from (4.1) and (4.5) we obtain $B = 0$. Multiplying again on $(s/2)^{-\nu}$ and assuming $s = 0$ we come to

$$\int_{-1}^1 d_q z f_\nu^{(1)}(z) = A \frac{1}{\Gamma_{q^2}(\nu + 1)}.$$

To calculate the q-integral in the left hand side we use the obvious property

$$\int_{-1}^1 f(x^2) d_q x = 2(1 - q) \sum_{m=0}^\infty q^m f(q^{2m}) = \frac{2(1 - q^2)}{1 + q} \sum_{m=0}^\infty q^{2m} f(q^{2m}) q^{-m} =$$

$$= \frac{2}{1 + q} \int_0^1 f(x) x^{-1/2} d_{q^2} x.$$

Then from [15] (1.11.7)

$$\int_{-1}^1 \frac{(q^2 z^2, q^2)_\infty}{(q^{2\nu+1} z^2, q^2)_\infty} d_q z = \frac{2}{1 + q} \int_0^1 \frac{(q^2 z, q^2)_\infty}{(q^{2\nu+1} z, q^2)_\infty} z^{-1/2} d_{q^2} z = \frac{2}{1 + q} B_{q^2}(\nu + 1/2, 1/2),$$

where $B_q(\nu; \mu) = \frac{\Gamma_q(\nu) \Gamma_q(\mu)}{\Gamma_q(\nu+\mu)}$ is the q-beta function. Hence

$$A = \frac{2}{1 + q} B_{q^2}(\nu + 1/2, 1/2) \Gamma_{q^2}(\nu + 1) = \frac{2}{1 + q} \Gamma_{q^2}(\nu + 1/2) \Gamma_{q^2}(1/2). \qquad (4.13)$$

and we come to (4.8). ∎

Proposition 4.2 *The q-MBF $I_\nu^{(2)}((1 - q^2)s; q^2)$ has the following q-integral representation*

$$I_\nu^{(2)}((1 - q^2)s; q^2) = \frac{1}{2\Gamma_{q^2}(\nu + 1) \, {}_2\Phi_1(q^{-2\nu+1}, q; q^3; q^2, 1)} \times$$

$$\times \int_{-1}^1 d_q z \frac{(q^{-2\nu+1} z^2, q^2)_\infty}{(z^2, q^2)_\infty} \, {}_0\Phi_1(-; 0; q, \frac{1 - q^2}{2} zs)(s/2)^\nu. \qquad (4.14)$$

Proof. Consider as above the absolutely convergent q-integral

$$S_1^{(2)}(s) = \int_{-1}^1 d_q z f_\nu^{(2)}(z) \, {}_0\Phi_1(-; 0; q, \frac{1 - q^2}{2} zs). \qquad (4.15)$$

and assume that $S_1^{(2)}(s)(s/2)^\nu$ satisfies (4.4). Then $S_1^{(2)}(s)$ satisfies the equation

$$q^{-\nu} S_1^{(2)}(q^{-1} s) - (q^{-\nu} + q^\nu) S_1^{(2)}(s) + [1 - (\frac{1 - q^2}{2})^2 s^2] q^\nu S_1^{(2)}(qs) = 0,$$

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or

$$q^{-2\nu}[S_1^{(2)}(q^{-1}s) - S_1^{(2)}(s)] - [S_1^{(2)}(s) - S_1^{(2)}(qs)] = (\frac{1-q^2}{2})^2 S_1^{(2)}(qs)s^2.$$

The further arguments are the same as in the proof of Preposition 4.1. Here we use (2.12),(2.22) and (2.23) and arrive to the difference equation for $f_\nu^{(2)}(z)$:

$$z^2[f_\nu^{(2)}(z) - q^{-2\nu+1}f_\nu^{(2)}(qz)] = f_\nu^{(2)}(z) - f_\nu^{(2)}(qz). \tag{4.16}$$

It coincides with (3.4) for $a = q^{-2\nu+1}, b = 1, \gamma = 0$, and hence from Proposition 3.2

$$f_\nu^{(2)}(z) = \frac{(q^{-2\nu+1}z^2, q^2)_\infty}{(z^2, q^2)_\infty}.$$

Since $S_1^{(2)}(s)(s/2)^\nu$ is a solution of (4.4) it can be represented as

$$S_1^{(2)}(s)(s/2)^\nu = AI_\nu^{(2)}((1-q^2)s; q^2) + BK_\nu^{(2)}((1-q^2)s; q^2).$$

As in the proof of Preposition 4.1 from (4.3) and (4.5) we obtain $B = 0$ and

$$A = \Gamma_{q^2}(\nu+1) \int_{-1}^1 \frac{(q^{-2\nu+1}z^2, q^2)_\infty}{(z^2, q^2)_\infty} d_q z = \frac{2\Gamma_{q^2}(\nu+1)}{1+q} \int_0^1 \frac{(q^{-2\nu+1}z, q^2)_\infty}{(z, q^2)_\infty} z^{-1/2} d_{q^2} z.$$

It follows from [15] (1.11.9) that

$$A = 2\Gamma_{q^2}(\nu+1) \, {}_2\Phi_1(q^{-2\nu+1}, q; q^3; q^2, 1),$$

and we come to (4.14).∎

Remark 4.1 *If $q \to 1 - 0$ the equations (4.11) and (4.16) take the form of the differential equation (see Remark 3.3)*

$$(1 - z^2)f_\nu'(z) + (2\nu - 1)zf_\nu(z) = 0.$$

The solution to this equation is

$$f_\nu(z) = C(1 - z^2)^{\nu-1/2},$$

which leads to the classical integral representation of Modified Bessel function [17] (7.12.10)

$$I_\nu(s) = \frac{(s/2)^\nu}{\Gamma(\nu + 1/2)\Gamma(1/2)} \int_{-1}^1 (1 - z^2)^{\nu-1/2} e^{zs} dz.$$

As it follows from

$$I_\nu^{(j)}((1 - q^2)s; q^2) = e^{-i\nu\pi/2} J_\nu^{(j)}((1 - q^2)e^{i\pi/2}s; q^2), \quad j = 1, 2.$$

Then (4.8) and (4.14) give the q-integral representations

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Corollary 4.1

$$J_\nu^{(1)}((1-q^2)s;q^2) = \frac{(1+q)}{2\Gamma_{q^2}(\nu+1/2)\Gamma_{q^2}(1/2)} \int_{-1}^1 d_qz \frac{(q^2z^2,q^2)_\infty}{(q^{2\nu+1}z^2,q^2)_\infty} E_q(-i\frac{1-q^2}{2}zs)(s/2)^\nu,$$

$$J_\nu^{(2)}((1-q^2)s;q^2) = \frac{(1+q)}{2\Gamma_{q^2}(\nu+1)\,{}_2\Phi_1(q^{-2\nu+1},q;q^3;q^2,1)} \times$$

$$\times \int_{-1}^1 d_qz \frac{(q^{-2\nu+1}z^2,q^2)_\infty}{(z^2,q^2)_\infty}\,{}_0\Phi_1(-;0;q,-i\frac{1-q^2}{2}zs)(s/2)^\nu. \tag{4.17}$$

5 The q-Fourier integral representation of q-Bessel-Macdonald Functions

Here we consider q-BMF (4.5) . We will use the following formulas from [1]:

$$I_\nu^{(2)}((1-q^2)s;q^2) = \frac{a_\nu}{\sqrt{z}}[E_q(\frac{1-q^2}{2}s)\Phi_\nu(s) + ie^{i\nu\pi}E_q(-\frac{1-q^2}{2}s)\Phi_\nu(-s)], \tag{5.1}$$

$$K_\nu^{(2)}((1-q^2)s;q^2) = \frac{q^{-\nu^2+1/2}}{2\sqrt{a_\nu a_{-\nu}s}}E_q(-\frac{1-q^2}{2}s)\Phi_\nu(s), \tag{5.2}$$

where

$$\Phi_\nu(s) = {}_2\Phi_1(q^{\nu+1/2},q^{-\nu+1/2};-q;q,\frac{2q}{(1-q^2)s}).$$

Proposition 5.1 q-BMF $K_\nu^{(1)}((1-q^2)s;q^2)$ for $\nu > 1/2$ can be represented by the q-integral

$$K_\nu^{(1)}((1-q^2)s;q^2) = \frac{q^{-\nu^2+1/2}\Gamma_{q^2}(\nu+1/2)\Gamma_{q^2}(1/2)}{4\mathbf{Q}_\nu}\sqrt{\frac{a_\nu}{a_{-\nu}}} \times$$

$$\times \int_{-\infty}^\infty d_qz \frac{(-q^2z^2,q^2)_\infty}{(-q^{-2\nu+1}z^2,q^2)_\infty} E_q(i\frac{1-q^2}{2}zs)(s/2)^{-\nu}, \tag{5.3}$$

where $\mathbf{Q}_\nu$ is defined by (2.31).

Proof. Consider the q-integral

$$S_2^{(1)}(s) = \int_{-\infty}^\infty d_qzh_\nu^{(1)}(z)E_q(i\frac{1-q^2}{2}zs), \tag{5.4}$$

and assume that it absolutely converges together with its q-derivative. According to Definition 2.1 and (2.13) it means that

$$\lim_{m\to\pm\infty} q^m|h_\nu^{(1)}(q^m)\frac{2\partial_s}{1+q}e_q(i\frac{1-q^2}{2}q^{m-1}s)| = 0.$$

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It follows from (2.21)

$$\lim_{m \to \pm\infty} q^{2m} |h_\nu^{(1)}(q^m) e_q(i\frac{1-q^2}{2}q^m s)| = 0. \tag{5.5}$$

Substitute $S_2^{(2)}(s)(s/2)^{-\nu}$ in (4.2). Then $S_2^{(1)}(s)$ satisfies the equation

$$S_2^{(1)}(q^{-1}s) - S_2^{(1)}(s) - q^{-2\nu}[S_2^{(1)}(s) - S_2^{(1)}(qs)] = (\frac{1-q^2}{2})^2 q^{-2} S_2^{(1)}(q^{-1}s)s^2. \tag{5.6}$$

Substituting (5.4) in (5.6), multiplying it on $\frac{2s^{-1}}{1-q^2}$ from the right and drawing the multiplier through E_q we obtain

$$\int_{-\infty}^{\infty} d_q z h_\nu^{(1)}(z) \frac{2s^{-1}}{1-q^2}[E_q(i\frac{1-q^2}{2}q^{-2}zs) - E_q(i\frac{1-q^2}{2}q^{-1}zs)]-$$

$$-\int_{-\infty}^{\infty} d_q z h_\nu^{(1)}(z)\frac{2s^{-1}}{1-q^2}[E_q(i\frac{1-q^2}{2}q^{-1}zs) - E_q(i\frac{1-q^2}{2}zs)] =$$

$$= \frac{1-q^2}{2}q^{-2}\int_{-\infty}^{\infty} d_q z h_\nu^{(1)}(z) E_q(i\frac{1-q^2}{2}q^{-1}zs)s.$$

Using (2.13), (2.20), and (2.21) we come to

$$\int_{-\infty}^{\infty} d_q z h_\nu^{(1)}(z) z [\ddagger e_q(i\frac{1-q^2}{2}q^{-1}zs)\ddagger - q^{-2\nu+1}\ddagger e_q(i\frac{1-q^2}{2}zs)\ddagger] =$$

$$= -i\frac{1-q^2}{2}q^{-1}\int_{-\infty}^{\infty} d_q z h_\nu^{(1)}(z)\ddagger e_q(i\frac{1-q^2}{2}q^{-1}zs)\ddagger s,$$

or

$$\int_{-\infty}^{\infty} d_q z h_\nu^{(1)}(z) z^{2\nu+1}\frac{2z^{-1}}{1-q^2}[z^{-2\nu+1}\ddagger e_q(i\frac{1-q^2}{2}q^{-1}zs)\ddagger - q^{-2\nu+1}z^{-2\nu+1}\ddagger e_q(i\frac{1-q^2}{2}zs)\ddagger] =$$

$$= \int_{-\infty}^{\infty} d_q z h_\nu^{(1)}(z)\frac{2\partial_z}{1+q}\ddagger e_q(i\frac{1-q^2}{2}q^{-1}zs)\ddagger.$$

q-Integration by parts (2.4) gives

$$\lim_{m\to\infty}[h_\nu^{(1)}(q^{-m})(q^{-2m}+1)e_q(i\frac{1-q^2}{2}q^{-m-1}s) + h_\nu^{(1)}(-q^{-m})(q^{-2m}+1)e_q(-i\frac{1-q^2}{2}q^{-m-1}s)] =$$

$$= \int_{-\infty}^{\infty} d_q z[\partial_z(h_\nu^{(1)}(z)z^{2\nu+1})q^{-2\nu+1}z^{-2\nu+1} + \partial_z h_\nu^{(1)}(z)]\ddagger e_q(i\frac{1-q^2}{2}zs)\ddagger. \tag{5.7}$$

According to (5.5) the left hand side of (5.7) vanishes. Thus we come to the difference equation for $h_\nu^{(1)}(z)$

$$z^2[-q^{-2\nu+1}h_\nu^{(1)}(z) + q^2 h_\nu^{(1)}(qz)] = h_\nu^{(1)}(z) - h_\nu^{(1)}(qz). \tag{5.8}$$

Obviously (5.8) coincides with (3.4) for $a = -q^2, b = -q^{-2\nu+1}, \gamma = 0$, and hence

$$h_\nu^{(1)}(z) = \frac{(-q^2 z^2, q^2)_\infty}{(-q^{-2\nu+1}z^2, q^2)_\infty}. \tag{5.9}$$

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It follows from Corollaris 3.2 , 3.6 that $h_\nu^{(1)}(z)$ satisfies (5.5). Then

$$S_2^{(1)}(z) = \int_{-\infty}^{\infty} d_q z \frac{(-q^2 z^2, q^2)_\infty}{(-q^{-2\nu+1} z^2, q^2)_\infty} E_q(i \frac{1-q^2}{2} zs).$$

Since $S_2^{(1)}(s)(s/2)^{-\nu}$ is a solution of (4.2) it can be represented in the form

$$\int_{-\infty}^{\infty} d_q z \frac{(-q^2 z^2, q^2)_\infty}{(-q^{-2\nu+1} z^2, q^2)_\infty} E_q(i \frac{1-q^2}{2} zs)(s/2)^{-\nu} = AI_\nu^{(1)}((1-q^2)s; q^2) + BK_\nu^{(1)}((1-q^2)s; q^2),$$

or

$$(1-q) \sum_{m=-\infty}^{\infty} q^m \frac{(-q^{2m+2}, q^2)_\infty}{(-q^{-2\nu+2m+1}, q^2)_\infty} \cos_q(\frac{1-q^2}{2} q^m s)(s/2)^{-\nu} =$$

$$= AI_\nu^{(1)}((1-q^2)s; q^2) + BK_\nu^{(1)}((1-q^2)s; q^2). \tag{5.10}$$

As it follows from Corollary 3.3, the function in the left hand side is holomorphic in the domain $\operatorname{Re} s > 0$, but $I_\nu^{(1)}((1-q^2)s; q^2)$ has ordinary poles in the points $s = \pm \frac{2q^{-r}}{1-q^2}$, $r = 0, 1, \ldots$ [1]. Hence $A = 0$. Multiplying (5.10) on $(s/2)^\nu$ and assuming $s = 0$ we find that

$$B = 8 a_{-\nu} \sqrt{a_\nu a_{-\nu}} q^{\nu^2-1/2} \sin \nu\pi \Gamma_{q^2}(-\nu+1)(1-q) \sum_{m=-\infty}^{\infty} q^m \frac{(-q^{2m+2}, q^2)_\infty}{(-q^{-2\nu+2m+1}, q^2)_\infty}.$$

Let calculate the last sum. It follows from the Corollary 3.1 for $\alpha = m+1, \beta = -\nu+m+1/2$, $\epsilon = -1, \gamma = 0$

$$\frac{(-q^{2m+2}, q^2)_\infty}{(-q^{-2\nu+2m+1}, q^2)_\infty} = \frac{(q^{2\nu+1}, q^2)_\infty}{(q^2, q^2)_\infty} \sum_{k=0}^{\infty} \frac{q^{(2\nu+1)k}(q^{-2\nu+1}, q^2)_k}{(q^2, q^2)_k(1+q^{-2\nu+2m+2k+1})},$$

and this series converges uniformly on real axis.

So

$$\sum_{m=-\infty}^{\infty} q^m \frac{(-q^{2m+2}, q^2)_\infty}{(-q^{-2\nu+2m+1}, q^2)_\infty} = \frac{(q^{2\nu+1}, q^2)_\infty}{(q^2, q^2)_\infty} \sum_{m=-\infty}^{\infty} q^m \sum_{k=0}^{\infty} \frac{q^{(2\nu+1)k}(q^{-2\nu+1}, q^2)_k}{(q^2, q^2)_k(1+q^{-2\nu+2m+2k+1})}$$

$$= \frac{q^{\nu-1/2}(q^{2\nu+1}, q^2)_\infty}{(q^2, q^2)_\infty} \sum_{k=0}^{\infty} \frac{q^{2\nu k}(q^{-2\nu+1}, q^2)_k}{(q^2, q^2)_k} \sum_{m=-\infty}^{\infty} \frac{1}{q^{\nu-m-1/2}+q^{-\nu+m+1/2}}.$$

Then using (2.7), (2.31) and (4.6), (4.7) we obtain

$$B = \frac{4q^{\nu^2+1/2} \mathbf{Q}_\nu}{\Gamma_{q^2}(\nu+1/2)\Gamma_{q^2}(1/2)} \sqrt{\frac{a_{-\nu}}{a_\nu}},$$

and (5.3).∎

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Proposition 5.2 *q-BMF* $K_\nu^{(2)}((1-q^2)s;q^2)$ *for* $\nu > 3/2$ *can be represented by the q-integral*

$$K_\nu^{(2)}((1-q^2)s;q^2) = \frac{q^{-\nu^2+\nu}\Gamma_{q^2}(\nu+1/2)\Gamma_{q^2}(1/2)}{4\mathbf{Q}_{1/2}}\sqrt{\frac{a_\nu}{a_{-\nu}}}\times$$

$$\times \int_{-\infty}^{\infty} d_q z \frac{(-q^{2\nu+1}z^2,q^2)_\infty}{(-z^2,q^2)_\infty}\,{}_0\Phi_1(-;0;q,i\frac{1-q^2}{2}zs)(s/2)^{-\nu}, \qquad (5.11)$$

where $\mathbf{Q}_{1/2}$ *is determined by (2.31).*

Proof. Consider the q-integral

$$S_2^{(2)}(s) = \int_{-\infty}^{\infty} d_q z h_\nu^{(2)}(z)\,{}_0\Phi_1(-;0;q,i\frac{1-q^2}{2}zs). \qquad (5.12)$$

Let it converges absolutely together with its q-derivative. According to Definition 2.1 and (2.12) it means that

$$\lim_{m\to\pm\infty} q^m|h_\nu^{(2)}(q^m)\frac{2\partial_s}{1+q}E_q(i\frac{1-q^2}{2}q^{m-1}s)| = 0$$

It follows from (2.23)

$$\lim_{m\to\pm\infty} q^{2m}|h_\nu^{(2)}(q^m)E_q(i\frac{1-q^2}{2}q^m s)| = 0. \qquad (5.13)$$

Require that $S_2^{(2)}(s)(s/2)^{-\nu}$ satisfies (4.4). Then $S_2^{(2)}(s)$ satisfies

$$q^{2\nu}[S_2^{(2)}(q^{-1}s) - S_2^{(2)}(s)] - [S_2^{(2)}(s) - S_2^{(2)}(qs)] = (\frac{1-q^2}{2})^2 S_2^{(2)}(qs)s^2. \qquad (5.14)$$

Using (2.12), (2.22) and (2.23) and arguing as in the proof of Proposition 5.1 we come to the equality

$$q^{2\nu-1}\int_{-\infty}^{\infty} d_q z h_\nu^{(2)}(z)z^{2\nu+1}\partial_z(z^{-2\nu+1}\ddagger E_q(i\frac{1-q^2}{2}zs)\ddagger) = -\int_{-\infty}^{\infty} d_q z h_\nu^{(2)}(z)\partial_z \ddagger E_q(i\frac{1-q^2}{2}zs)\ddagger.$$

q-Integration by parts (2.4) gives

$$\lim_{m\to\infty} q^{2\nu-1}[h_\nu^{(2)}(q^{-m})q^{-2m}E_q(i\frac{1-q^2}{2}q^{-m}s) + h_\nu^{(2)}(-q^{-m})q^{-2m}E_q(-i\frac{1-q^2}{2}q^{-m}s)] -$$

$$- \int_{-\infty}^{\infty} d_q z \partial_z(h_\nu^{(2)}(z)z^{2\nu+1})z^{-2\nu+1}\ddagger E_q(i\frac{1-q^2}{2}qzs)\ddagger =$$

$$= -\lim_{m\to\infty}[h_\nu^{(2)}(q^{-m})E_q(i\frac{1-q^2}{2}q^{-m}s) + h_\nu^{(2)}(-q^{-m})E_q(-i\frac{1-q^2}{2}q^{-m}s)] +$$

$$+ \int_{-\infty}^{\infty} d_q z \partial_z h_\nu^{(2)}(z)\ddagger E_q(i\frac{1-q^2}{2}qzs)\ddagger,$$

or

$$\lim_{m\to\infty}(q^{2\nu-1-2m} + 1)[h_\nu^{(2)}(q^{-m})E_q(i\frac{1-q^2}{2}q^{-m}s) + h_\nu^{(2)}(-q^{-m})E_q(-i\frac{1-q^2}{2}q^{-m}s)] =$$

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$$= \int_{-\infty}^{\infty} d_q z [\partial_z (h_\nu^{(2)}(z) z^{2\nu+1}) z^{-2\nu+1} + \partial_z h_\nu^{(2)}(z)] \ddagger E_q (i \frac{1-q^2}{2} qzs) \ddagger. \tag{5.15}$$

(5.13) means that the left hand side of (5.15) vanishes. Thus we come to the difference equation for $h_\nu^{(2)(z)}$

$$z^2 [h_\nu^{(2)}(z) - q^{2\nu+1} h_\nu^{(2)}(qz)] = -h_\nu^{(2)}(z) + h_\nu^{(2)}(qz). \tag{5.16}$$

It coincides with (3.4) for $a = -q^{2\nu+1}, b = -1, \gamma = 0$, and hence

$$h_\nu^{(2)}(z) = \frac{(-q^{2\nu+1} z^2, q^2)_\infty}{(-z^2, q^2)_\infty}. \tag{5.17}$$

It follows from Corollaris 3.2 and 3.6 that $h_\nu^{(2)}(z)$ satisfies (5.13). But $S_2^{(2)}(s)(s/2)^{-\nu}$ satisfies (4.4) and thereby it is represented in the form

$$\int_{-\infty}^{\infty} d_q z \frac{(-q^{2\nu+1} z^2, q^2)_\infty}{(-z^2, q^2)_\infty} {}_0\Phi_1(-; 0; q, i\frac{1-q^2}{2} zs)(s/2)^{-\nu} =$$

$$= A I_\nu^{(2)}((1-q^2)s; q^2) + B K_\nu^{(2)}((1-q^2)s; q^2),$$

or

$$2(1-q) \sum_{m=-\infty}^{\infty} q^m \frac{(-q^{2\nu+1+2m}, q^2)_\infty}{(-q^{2m}, q^2)_\infty} \mathrm{Cos}_q(\frac{1-q^2}{2} q^m s)(s/2)^{-\nu} =$$

$$= A I_\nu^{(2)}((1-q^2)s; q^2) + B K_\nu^{(2)}((1-q^2)s; q^2). \tag{5.18}$$

It follows from (5.1) and (5.2) that $\lim_{s\to\infty} I_\nu^{(2)}((1-q^2)s; q^2) = \infty$ and $\lim_{s\to\infty} K_\nu^{(2)}((1-q^2)s; q^2) = 0$.

Because the left hand side of (5.18) vanishes if $s \to \infty$, $\quad A = 0$. Multiplying (5.18) on $(s/2)^\nu$ and assuming $s = 0$ we have

$$B = 8a_{-\nu} \sqrt{a_{-\nu} a_\nu} q^{\nu^2-1/2} \sin\nu\pi\Gamma_{q^2}(-\nu+1)(1-q) \sum_{m=-\infty}^{\infty} q^m \frac{(-q^{2\nu+1+2m}, q^2)_\infty}{(-q^{2m}, q^2)_\infty}.$$

Calculate the last sum. It follows from Corollary 3.1 for $\alpha = \nu + m + 1/2, \beta = m, \epsilon = -1, \gamma = 0$ that

$$\frac{(-q^{2\nu+1+2m}, q^2)_\infty}{(-q^{2m}, q^2)_\infty} = \frac{(q^{2\nu+1}, q^2)_\infty}{(q^2, q^2)_\infty} \sum_{k=0}^{\infty} \frac{(q^{-2\nu+1}, q^2)_k q^{(2\nu+1)k}}{(q^2, q^2)_k (1+q^{2m+2k})}$$

and this series converges uniformly on real axis. So

$$\sum_{m=-\infty}^{\infty} q^m \frac{(-q^{2\nu+1+2m}, q^2)_\infty}{(-q^{2m}, q^2)_\infty} = \frac{(q^{2\nu+1}, q^2)_\infty}{(q^2, q^2)_\infty} \sum_{m=-\infty}^{\infty} q^m \sum_{k=0}^{\infty} \frac{(q^{-2\nu+1}, q^2)_k q^{(2\nu+1)k}}{(q^2, q^2)_k (1+q^{2m+2k})} =$$

$$= \frac{(q^{2\nu+1}, q^2)_\infty}{(q^2, q^2)_\infty} \sum_{k=0}^{\infty} \frac{(q^{-2\nu+1}, q^2)_k q^{2\nu k}}{(q^2, q^2)_k} \sum_{m=-\infty}^{\infty} \frac{1}{q^{-m} + q^m}$$

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Using (3.9), (2.31) and (4.6), (4.7) we obtain

$$B = \frac{4q^{\nu^2-\nu}\mathbf{Q}_{1/2}}{\Gamma_{q^2}(\nu+1/2)\Gamma_{q^2}(1/2)}\sqrt{\frac{a_{-\nu}}{a_\nu}}$$

and (5.11).∎

Remark 5.1 *If $q \to 1-0$ equations (5.8) and (5.16) take the form of the differential equation*

$$(z^2+1)h'_\nu(z) = -(2\nu+1)zh_\nu(z).$$

with solution

$$h_\nu(z) = C(z^2+1)^{-\nu-1/2},$$

Using Proposition 2.3 we obtain the classical integral representation of Bessel-Macdonald function (the Fourier integral) [17] (7.12.27)

$$K_\nu(s) = \frac{\Gamma(\nu+1/2)(s/2)^{-\nu}}{2\Gamma(1/2)}\int_{-\infty}^\infty (z^2+1)^{-\nu-1/2}e^{izs}dz.$$

6 Bessel q-integral representation of q-Bessel-Macdonald Functions

Proposition 6.1 *The q-BMF $K_\nu^{(1)}((1-q^2)s; q^2)$ for $\nu > 1/2$ can be represented as the q-integral*

$$K_\nu^{(1)}((1-q^2)s; q^2) = \frac{1}{2}q^{-\nu^2-\nu}(1+q)\Gamma_{q^2}(\nu+1)\sqrt{\frac{a_\nu}{a_{-\nu}}}\times$$

$$\times \int_0^\infty d_qz\frac{(-q^2z^2, q^2)_\infty}{(-q^{-2\nu}z^2, q^2)_\infty}zJ_0^{(2)}((1-q^2)q^{-1/2}zs; q^2)(s/2)^{-\nu}. \tag{6.1}$$

Proof. Consider the absolutely convergent q-integral

$$S_3^{(1)}(s) = \int_0^\infty d_qzg_\nu^{(1)}(z)J_0^{(2)}((1-q^2)q^{-1/2}zs; q^2) \tag{6.2}$$

together with its q-derivative. According to Definition 2.1 and (2.2) it means

$$\lim_{m\to\pm\infty}q^m|g_\nu^{(1)}(q^m)\frac{2\partial_s}{1+q}J_0^{(1)}((1-q^2)q^{m-1}s, q^2)| = 0$$

It follows from (2.25)

$$\lim_{m\to\pm\infty}q^{2m}|g_\nu^{(1)}(q^m)J_1^{(1)}((1-q^2)q^ms)| = 0. \tag{6.3}$$

Assume that $S_3^{(1)}(s)(s/2)^{-\nu}$ satisfies (4.2). Then $S_3^{(1)}(s)$ satisfies (5.6). Substituting (6.2) in (5.6), multiplying it on $\frac{2s^{-1}}{1-q^2}$ from the right and drawing the multiplier through $J_0^{(2)}$ on left we obtain

$$\int_0^\infty d_qzg_\nu^{(1)}(z)\frac{2s^{-1}}{1-q^2}[J_0^{(2)}((1-q^2)q^{-5/2}zs; q^2) - J_0^{(2)}((1-q^2)q^{-3/2}zs; q^2)]-$$

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$$-q^{-2\nu}\int_0^\infty d_q z\, g_\nu^{(1)}(z)\frac{2s^{-1}}{1-q^2}[J_0^{(2)}((1-q^2)q^{-3/2}zs;q^2) - J_0^{(2)}((1-q^2)q^{-1/2}zs;q^2)] =$$

$$= \frac{1-q^2}{2}q^{-2}\int_0^\infty d_q z\, g_\nu^{(1)}(z)J_0^{(2)}((1-q^2)q^{-3/2}zs;q^2)s.$$

Using (2.2), (2.24) and (2.25) we transform it as

$$-q^{-1}\int_0^\infty d_q z\, g_\nu^{(1)}(z)z\ddagger J_1^{(1)}((1-q^2)q^{-1}zs;q^2)\ddagger + q^{-2\nu}\int_0^\infty d_q z\, g_\nu^{(1)}(z)z\ddagger J_1^{(1)}((1-q^2)zs;q^2)\ddagger =$$

$$= \frac{1-q^2}{2}q^{-2}\int_0^\infty d_q z\, g_\nu^{(1)}(z)\ddagger J_0^{(1)}((1-q^2)q^{-1}zs;q^2)\ddagger s,$$

or

$$\int_0^\infty d_q z\, g_\nu^{(1)}(z)z^{2\nu+1}\frac{2z^{-1}}{1-q^2}[z^{-2\nu+1}\ddagger J_1^{(1)}((1-q^2)q^{-1}zs;q^2)\ddagger - q^{-2\nu+1}z^{-2\nu+1}\ddagger J_1^{(1)}((1-q^2)zs;q^2)\ddagger] =$$

$$= -q^{-1}\int_0^\infty d_q z\, g_\nu^{(1)}(z)\ddagger J_0^{(1)}((1-q^2)q^{-1}zs;q^2)\ddagger s.$$

It follows from (2.26) that

$$\int_0^\infty d_q z\, g_\nu^{(1)}(z)z^{2\nu+1}\partial_z(z^{-2\nu+1}\ddagger J_1^{(1)}((1-q^2)q^{-1}zs;q^2)\ddagger) =$$

$$= -\int_0^\infty d_q z\, g_\nu^{(1)}(z)(z/2)^{-1}\partial_z(z/2\ddagger J_1^{(1)}((1-q^2)q^{-1}zs;q^2)\ddagger).$$

Using the q-integration by parts (2.3) we obtain

$$\lim_{m\to\infty}[g_\nu^{(1)}(q^{-m})q^{-2m}J_1^{(1)}((1-q^2)q^{-m-1}s;q^2) - g_\nu^{(1)}(q^m)q^{2m}J_1^{(1)}((1-q^2)q^{m-1}s;q^2)]-$$

$$-\int_0^\infty d_q z\,\partial_z(g_\nu^{(1)}(z)z^{2\nu+1})q^{-2\nu+1}z^{-2\nu+1}\ddagger J_1^{(1)}((1-q^2)zs;q^2)\ddagger =$$

$$= -\lim_{m\to\infty}[g_\nu^{(1)}(q^{-m})J_1^{(1)}((1-q^2)q^{-m-1}s;q^2) - g_\nu^{(1)}(q^m)J_1^{(1)}((1-q^2)q^{m-1}s;q^2)]+$$

$$+\int_0^\infty d_q z\,\partial_z(p_\nu^{(1)}(z)(z/2)^{-1})qz/2\ddagger J_1^{(1)}((1-q^2)zs;q^2)\ddagger,$$

or

$$\lim_{m\to\infty}[g_\nu^{(1)}(q^{-m})(q^{-2m}+1)J_1^{(1)}((1-q^2)q^{-m-1}s;q^2) - g_\nu^{(1)}(q^m)(q^{2m}+1)J_1^{(1)}((1-q^2)q^{m-1}s;q^2)] =$$

$$= \int_0^\infty d_q z[\partial_z(g_\nu^{(1)}(z)(z/2)^{-1})qz/2 + \partial_z(g_\nu^{(1)}(z)z^{2\nu+1})q^{-2\nu+1}z^{-2\nu+1}]\ddagger J_1^{(1)}((1-q^2)zs;q^2)\ddagger. \quad (6.4)$$

It follows from (6.3) that the left hand side of (6.4) vanishes. Thus we come to the difference equation for $g_\nu^{(1)}(z)$

$$z^2[q^{-2\nu+1}g_\nu^{(1)}(z) - q^2 g_\nu^{(1)}(qz)] = -q g_\nu^{(1)}(z) + g_\nu^{(1)}(qz). \quad (6.5)$$

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But (6.5) coincides with (3.4) for $a = -q^2, b = -q^{-2\nu}, \gamma = 1$ and hence

$$g_\nu^{(1)}(z) = \frac{(-q^2 z^2, q^2)_\infty}{(-q^{-2\nu} z^2, q^2)_\infty} z. \tag{6.6}$$

Since $S_3^{(1)}(s)(s/2)^{-\nu}$ satisfies (4.2) it can be represented in the form

$$\int_0^\infty d_q z \frac{(-q^2 z^2, q^2)_\infty}{(-q^{-2\nu} z^2, q^2)_\infty} z J_0^{(2)}((1-q^2)q^{-1/2} z s; q^2)(s/2)^{-\nu} = A I_\nu^{(1)}((1-q^2)s; q^2) + B K_\nu^{(1)}((1-q^2)s; q^2). \tag{6.7}$$

The function $I_\nu^{(1)}((1-q^2)s; q^2)$ is meromorphic with ordinary poles $s = \pm \frac{2q^{-r}}{1-q^2}, r = 0, 1, \ldots$. On the other hand the function $K_\nu^{(1)}((1-q^2)s; q^2)$ and the left hand side of (6.7) are the holomorphic functions in the domain $\operatorname{Re} s > 0$. Thereby $A = 0$. Multiplying (6.7) on $(s/2)^\nu$ and taking $s = 0$ we obtain

$$\int_0^\infty d_q z \frac{(-q^2 z^2, q^2)_\infty}{(-q^{-2\nu} z^2, q^2)_\infty} z = B \frac{q^{-\nu^2 + 1/2}}{4 a_\nu \sqrt{a_\nu a_{-\nu}} \sin \nu \pi \Gamma_{q^2}(-\nu + 1)}.$$

Rewriting the q-integral by means of (2.10) as

$$\int_0^\infty d_q z \frac{(-q^2 z^2, q^2)_\infty}{(-q^{-2\nu} z^2, q^2)_\infty} z = \frac{1}{1+q} \int_0^\infty d_{q^2} z \frac{(-q^2 z, q^2)_\infty}{(-q^{-2\nu} z, q^2)_\infty} =$$

$$= \frac{1}{1+q} \int_0^\infty d_{q^2} z E_{q^2}(q^2 z) e_{q^2}(-q^{-2\nu} z)$$

and defining

$$D_z f(z) = \frac{f(z) - f(q^2 z)}{(1 - q^2)z},$$

$$D_z E_{q^2}(z) = \frac{1}{1-q^2} E_{q^2}(q^2 z), \quad D_z e_{q^2}(-q^{-2\nu} z) = -\frac{q^{-2\nu}}{1-q^2} e_{q^2}(-q^{-2\nu} z).$$

we obtain by means of (2.3)

$$\int_0^\infty d_{q^2} z E_{q^2}(q^2 z) e_{q^2}(-q^{-2\nu} z) = (1 - q^2) \int_0^\infty d_{q^2} z D_z E_{q^2}(z) e_{q^2}(-q^{-2\nu} z) =$$

$$= (1 - q^2) \lim_{m \to \infty} [E_{q^2}(q^{-2m}) e_{q^2}(-q^{-2\nu - 2m}) - E_{q^2}(q^{2m}) e_{q^2}(-q^{-2\nu + 2m})] -$$

$$- (1 - q^2) \int_0^\infty d_{q^2} z E_{q^2}(q^2 z) D_z e_{q^2}(-q^{-2\nu} z) =$$

$$= (1 - q^2) [\lim_{m \to \infty} \frac{(-q^{-2m}, q^2)_\infty}{(-q^{-2\nu - 2m}, q^2)_\infty} - 1] + q^{-2\nu} \int_0^\infty d_{q^2} z E_{q^2}(q^2 z) e_{q^2}(-q^{-2\nu} z).$$

Due to Corollary 3.7 for $\alpha = 0, \beta = -\nu$ the limit vanishes and we obtain

$$\int_0^\infty d_z q^2 z E_{q^2}(q^2 z) e_{q^2}(-q^{-2\nu} z) = -\frac{1 - q^2}{1 - q^{-2\nu}}.$$

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Hence
$$\int_0^\infty d_q z \frac{(-q^2 z^2, q^2)_\infty}{(-q^{-2\nu} z^2, q^2)_\infty} z = -\frac{1-q}{1-q^{-2\nu}},$$

and
$$B = -\frac{1-q}{1-q^{-2\nu}} 4 q^{\nu^2 - \frac{1}{2}} a_{-\nu} \sqrt{a_\nu a_{-\nu}} \sin \nu\pi \Gamma_{q^2}(-\nu+1) = \frac{2q^{\nu^2+\nu}}{(1+q)\Gamma_{q^2}(\nu+1)} \sqrt{\frac{a_\nu}{a_{-\nu}}}.$$

Then from (6.7) we obtain (6.1) . $\blacksquare$

Proposition 6.2 q-BMF $K_\nu^{(2)}((1-q^2)s; q^2)$ for $\nu > 3/2$ can be represented as the q-integral

$$K_\nu^{(2)}((1-q^2)s; q^2) = \frac{1}{2} q^{-\nu^2+\nu}(1+q)\Gamma_{q^2}(\nu+1)\sqrt{\frac{a_\nu}{a_{-\nu}}} \times$$

$$\times \int_0^\infty d_q z \frac{(-q^{2\nu+2} z^2, q^2)_\infty}{(-z^2, q^2)_\infty} z \; {}_0\Phi_3(-; 0, 0, q^2; q^2, -(\frac{1-q^2}{2} q^{3/2} zs)^2)(s/2)^{-\nu}. \tag{6.8}$$

Proof. Consider as before the absolutely convergent q-integral

$$S_3^{(2)}(s) = \int_0^\infty d_q z g_\nu^{(2)}(z) \; {}_0\Phi_3(-; 0, 0, q^2; q^2, -(\frac{1-q^2}{2} q^{3/2} zs)^2). \tag{6.9}$$

It means according to Definition 2.1 and (2.18) that

$$\lim_{m \to \pm\infty} q^m |g_\nu^{(2)}(q^m) \frac{2\partial_s}{1+q} J_0^{(2)}((1-q^2)q^{m-1}s, q^2)| = 0$$

It follows from (2.28) that

$$\lim_{m \to \pm\infty} q^{2m} |g_\nu^{(2)}(q^m) J_1^{(2)}((1-q^2)q^m s)| = 0. \tag{6.10}$$

Substitute $S_3^{(2)}(s)(s/2)^{-\nu}$ in (4.4). Then $S_3^{(2)}(s)$ satisfies (5.14). Acting as in in the proof of Proposition 6.1 and using (2.18), (2.27), (2.28), (2.29) we come to the equality

$$-q^{2\nu} \int_0^\infty d_q z g_\nu^{(2)}(z) z^{2\nu+1} \frac{2z^{-1}}{1-q^2} [z^{-2\nu+1} \ddagger J_1^{(2)}((1-q^2)zs; q^2)\ddagger - q^{-2\nu+1} z^{-2\nu+1} \ddagger J_1^{(2)}((1-q^2)qzs; q^2)\ddagger] =$$

$$= \int_0^\infty d_q z g_\nu^{(2)}(z)(z/2)^{-1} \frac{2\partial_z}{1+q} (z/2 \ddagger J_1^{(2)}((1-q^2)zs; q^2)\ddagger).$$

Then by means of (2.3) we obtain

$$\lim_{m \to \infty} [g_\nu^{(2)}(q^{-m})(q^{2\nu-2m}+1)J_1^{(2)}((1-q^2)q^{-m}s; q^2) - g_\nu^{(2)}(q^m)(q^{2\nu+2m}+1)J_1^{(2)}((1-q^2)q^m s; q^2)] =$$

$$\int_0^\infty d_q z [\partial_z(g_\nu^{(2)}(z)(z/2)^{-1})qz/2 - \partial_z(g_\nu^{(2)} z^{2\nu+1})qz^{-2\nu+1}] \ddagger J_1^{(2)}((1-q^2)qzs; q^2))\ddagger. \tag{6.11}$$

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It follows from (6.10) that the left hand side of (6.11) is equal to zero. Then we have

$$z^2[qg_\nu^{(2)}(z) - q^{2\nu+2}g_\nu^{(2)}(qz)] = -qg_\nu^{(2)}(z) + g_\nu^{(2)}(qz) \tag{6.12}$$

Note that (6.12) is the equation (3.4) with $a = -q^{2\nu+2}, b = 1, \gamma = 1$ and hence

$$g_\nu^{(2)}(z) = \frac{(-q^{2\nu+2}z^2, q^2)_\infty}{(-z^2, q^2)_\infty} z. \tag{6.13}$$

Again $S_3^{(2)}(s)(s/2)^{-\nu}$ as a solution of (4.4) can be represented in the form

$$\int_0^\infty d_q z \frac{(-q^{2\nu+2}z^2, q^2)_\infty}{(-z^2, q^2)_\infty} z \,_0\Phi_3(-; 0, 0, q^2; q^2, -(\frac{1-q^2}{2}q^{3/2}zs)^2)(s/2)^{-\nu} =$$

$$= AI_\nu^{(2)}((1-q^2)s; q^2) + BK_\nu^{(2)}((1-q^2)s; q^2). \tag{6.14}$$

It can be derived from (5.1) and (5.2), that $\lim_{s\to\infty} I_\nu^{(2)}((1-q^2)s; q^2) = \infty$ and $\lim_{s\to\infty} K_\nu^{(2)}((1-q^2)s; q^2) = 0$. Because the left hand side of (6.14) vanishes if $s \to \infty$, $A = 0$. Multiplying (6.14) on $(s/2)^\nu$ and assuming $s = 0$ we get

$$\int_0^\infty d_q z \frac{(-q^{2\nu+2}z^2, q^2)_\infty}{(-z^2, q^2)_\infty} z = B \frac{q^{-\nu^2+1/2}}{4a_{-\nu}\sqrt{a_\nu a_{-\nu}}\sin\nu\pi\Gamma_{q^2}(-\nu+1)}.$$

Using the same recipe as in Proposition 6.1 we calculate this q-integral:

$$\int_0^\infty d_q z \frac{(-q^{2\nu+2}z^2, q^2)_\infty}{(-z^2, q^2)_\infty} z = \frac{1-q}{1-q^{2\nu}}.$$

Thus

$$B = \frac{1-q}{1-q^{2\nu}} 4q^{\nu^2-1/2} a_{-\nu}\sqrt{a_\nu a_{-\nu}}\sin\nu\pi\Gamma_{q^2}(-\nu+1) = \frac{2q^{\nu^2-\nu}}{(1+q)\Gamma_{q^2}(\nu+1)}\sqrt{\frac{a_{-\nu}}{a_\nu}}$$

and we obtain (6.8) from (6.14).∎

Remark 6.1 *If $q \to 1 - 0$ the equations (6.5) and (6.12) take the form of the differential equation*

$$z(1 + z^2)g_\nu'(z) - [1 - (2\nu + 1)z^2]g_\nu(z) = 0.$$

The solution to this equation is

$$g_\nu(z) = Cz(1 + z^2)^{-\nu-1},$$

and therefore we come to the classical integral representation of Bessel-Macdonald function [17]

$$K_\nu(s) = \Gamma(\nu + 1)(s/2)^{-\nu} \int_0^\infty (1 + z^2)^{-\nu-1}zJ_0(zs)dz$$

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7 The representation of q-BMF by the double q-integral

Here we combine results from Sections 4 and 6.

Introduce new variable ζ with the commutations

$$\zeta z = z\zeta, \qquad \zeta s = qs\zeta \tag{7.1}$$

Proposition 7.1 *The q-BMF $K_\nu^{(1)}((1-q^2)s; q^2)$ for $\nu > 1/2$ can be represented as the double q-integral*

$$K_\nu^{(1)}((1-q^2)s; q^2) = \frac{q^{-\nu^2-\nu}(1+q)^2\Gamma_{q^2}(\nu+1)}{4\,_2\Phi_1(q,q;q^3;q^2,1)}\sqrt{\frac{a_\nu}{a_{-\nu}}}\times$$

$$\times \int_0^\infty d_q z \int_{-1}^1 d_q\zeta \frac{(-q^2z^2, q\zeta^2, q^2)_\infty}{(-q^{-2\nu}z^2, \zeta^2, q^2)_\infty} z\,_0\Phi_1(-; 0; q, -i\frac{1-q^2}{2}\zeta zs)(s/2)^{-\nu} \tag{7.2}$$

Proof. This result follows from (6.1) and (4.17). In view of of (7.1) the result of q-integrations is independent on their order

$$K_\nu^{(1)}((1-q^2)s; q^2) = \frac{q^{-\nu^2-\nu}(1+q)^2\Gamma_{q^2}(\nu+1)}{4\,_2\Phi_1(q,q;q^3;q^2,1)}\sqrt{\frac{a_\nu}{a_{-\nu}}}\times$$

$$\times \int_0^\infty d_q z \int_{-1}^1 d_q\zeta \frac{(-q^2z^2, q\zeta^2, q^2)_\infty}{(-q^{-2\nu}z^2, \zeta^2, q^2)_\infty} zE_q(-i\frac{1-q^2}{2}(: (\zeta z)s :))(s/2)^{-\nu} =$$

$$= \frac{q^{-\nu^2-\nu}(1-q^2)^2\Gamma_{q^2}(\nu+1)}{2\,_2\Phi_1(q,q;q^3;q^2,1)}\sqrt{\frac{a_\nu}{a_{-\nu}}}\times$$

$$\times \sum_{m=-\infty}^\infty q^{2m}\sum_{l=0}^\infty \frac{(-q^{2m+2}, q^{2l+1}, q^2)_\infty}{(-q^{2m-2\nu}, q^{2l}, q^2)_\infty} \operatorname{Cos}_q(\frac{1-q^2}{2}q^{m+l}s)(s/2)^{-\nu}.$$

Inner series converges uniformly with respect to m, and we can change the order of summation.∎

Remark 7.1 *If $q \to 1 - 0$ representation (7.2) takes the form*

$$K_\nu(s) = \frac{1}{\pi}\Gamma(\nu+1)(s/2)^{-\nu}\int_0^\infty (1+z^2)^{-\nu-1}zdz\int_{-1}^1 (1-\zeta^2)^{-\frac{1}{2}}e^{-i\zeta zs}d\zeta. \tag{7.3}$$

Assume $\zeta = \cos\phi$ in (7.3). Then

$$K_\nu(s) = \frac{1}{\pi}\Gamma(\nu+1)(s/2)^{-\nu}\int_0^\infty (1+z^2)^{-\nu-1}zdz\int_0^\pi e^{-izs\cos\phi}d\phi \tag{7.4}$$

It is known that BMF is the two-dimensional Fourier transform of $f(x,y) = (1+x^2+y^2)^{-\nu-1}$:

$$\int\int (1+x^2+y^2)^{-\nu-1}e^{-i(x\xi+y\eta)}dxdy = \frac{2^{-\nu}\pi}{\Gamma(\nu+1)}K_\nu(\sqrt{\xi^2+\eta^2})(\xi^2+\eta^2)^{\nu/2}.$$

The change of variables $x = z\cos\alpha, y = z\sin\alpha, \quad \xi = s\cos\beta, \eta = s\sin\beta$, and $\alpha - \beta = \phi$ leads to (7.4).

Thus (7.2) can be considered as the q-analog of the Fourier transform of

$$f(z,\zeta) = \frac{(-q^2 z^2, q\zeta^2, q^2)_\infty}{(-q^{-2\nu} z^2, \zeta^2, q^2)_\infty}$$

on the quantum plane in the polar coordinate representation (7.1).

8 The special case of commuting variables

In this section we assume that

$$zs = sz, \qquad \zeta s = s\zeta, \qquad z\zeta = \zeta z. \tag{8.1}$$

Then we have the q-integral representations which we will give without proof.

$$I_\nu^{(1)}((1 - q^2)s; q^2) = \frac{(1 + q)(s/2)^\nu}{2\Gamma_{q^2}(\nu + 1/2)\Gamma_{q^2}(1/2)} \int_{-1}^1 \frac{(q^2 z^2, q^2)_\infty}{(q^{2\nu+1} z^2, q^2)_\infty} e_q(\frac{1 - q^2}{2} zs) d_q z, \tag{8.2}$$

$$I_\nu^{(2)}((1 - q^2)s; q^2) =$$

$$= \frac{(s/2)^\nu}{2\Gamma_{q^2}(\nu + 1)\, {}_2\Phi_1(q^{-2\nu+1}, q; q^3; q^2, 1)} \int_{-1}^1 \frac{(q^{-2\nu+1} z^2, q^2)_\infty}{(z^2, q^2)_\infty} E_q(\frac{1 - q^2}{2} zs) d_q z, \tag{8.3}$$

$$K_\nu^{(1)}((1 - q^2)s; q^2) =$$

$$= \frac{q^{-\nu^2+1/2}\Gamma_{q^2}(\nu + 1/2)\Gamma_{q^2}(1/2)}{4Q_\nu} \sqrt{\frac{a_\nu}{a_{-\nu}}} (s/2)^{-\nu} \int_{-\infty}^\infty \frac{(-q^2 z^2, q^2)_\infty}{(-q^{-2\nu+1} z^2, q^2)_\infty} e_q(i\frac{1 - q^2}{2} zs) d_q z, \tag{8.4}$$

$$K_\nu^{(2)}((1 - q^2)s; q^2) =$$

$$= \frac{q^{-\nu^2+\nu}\Gamma_{q^2}(\nu + 1/2)\Gamma_{q^2}(1/2)}{4Q_{1/2}} \sqrt{\frac{a_\nu}{a_{-\nu}}} (s/2)^{-\nu} \int_{-\infty}^\infty \frac{(-q^{2\nu+1} z^2, q^2)_\infty}{(-z^2, q^2)_\infty} E_q(i\frac{1 - q^2}{2} zs) d_q z, \tag{8.5}$$

$$K_\nu^{(1)}((1 - q^2)s; q^2) =$$

$$= \frac{1}{2} q^{-\nu^2-\nu}(1 + q)\Gamma_{q^2}(\nu + 1) \sqrt{\frac{a_\nu}{a_{-\nu}}} (s/2)^{-\nu} \int_0^\infty \frac{(-q^2 z^2, q^2)_\infty}{(-q^{-2\nu+1} z^2, q^2)_\infty} z J_0^{(1)}((1 - q^2)zs; q^2) d_q z, \tag{8.6}$$

$$K_\nu^{(2)}((1 - q^2)s; q^2) =$$

$$= \frac{1}{2} q^{-\nu^2+\nu}(1 + q)\Gamma_{q^2}(\nu + 1) \sqrt{\frac{a_\nu}{a_{-\nu}}} (s/2)^{-\nu} \int_0^\infty \frac{(-q^{2\nu+2} z^2, q^2)_\infty}{(-z^2, q^2)_\infty} z J_0^{(2)}((1 - q^2)zs; q^2) d_q z, \tag{8.7}$$

$$K_\nu^{(1)}((1 - q^2)s; q^2) = \frac{q^{-\nu^2-\nu}(1 + q)^2\Gamma_{q^2}(\nu + 1)}{4\Gamma_{q^2}^2(1/2)} \sqrt{\frac{a_\nu}{a_{-\nu}}} (s/2)^{-\nu} \times$$

$$\times \int_0^\infty z d_q z \int_{-1}^1 \frac{(-q^2 z^2, q^2 \zeta^2, q^2)_\infty}{(-q^{-2\nu} z^2, q\zeta^2, q^2)_\infty} e_q(-i\frac{1-q^2}{2} z\zeta s) d_q \zeta, \tag{8.8}$$

$$K_\nu^{(2)}((1-q^2)s; q^2) = \frac{q^{-\nu^2+\nu}(1+q)^2 \Gamma_{q^2}(\nu+1)}{4 \, _2\Phi_1(q, q; q^3; q^2, 1)} \sqrt{\frac{a_\nu}{a_{-\nu}}}(s/2)^{-\nu} \times$$

$$\times \int_0^\infty z d_q z \int_{-1}^1 \frac{(-q^{2\nu+2} z^2, q\zeta^2, q^2)_\infty}{(-z^2, \zeta^2, q^2)_\infty} E_q(-i\frac{1-q^2}{2} z\zeta s) d_q \zeta, \tag{8.9}$$

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