ALMOST RIEMANNIAN SPACES

by

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A complete metric space (X, d) is called *almost Riemannian* if X is finite dimensional and d is a geodesically complete inner metric of (metric) curvature locally bounded below. Our main result is the following:

Theorem 1. *If (X, d) is almost Riemannian, then X is a topological manifold. For each $p \in X$ there exist an n-dimensional vector space $\overline{T}_p$ ($n = \dim X$) with inner product $\langle \cdot,\cdot \rangle_p$; a function $\exp_p : \overline{T}_p \to X$ which is continuous in a neighborhood of 0; and a dense subset T_p of $\overline{T}_p$, having the following properties:*

a) if $v \in T_p$, then $tv \in T_p$ for all $t \in \mathbb{R}$,

b) the restriction $\exp_p : T_p \to X$ is surjective,

c) the correspondence $v \leftrightarrow \gamma_v(t) = \exp_p(tv)$ is a one-to-one correspondence between unit vectors in $(T_p, \langle \cdot,\cdot \rangle_p)$ and unit geodesics starting at p.

Exp_p need not be locally one-to-one (so there may not be "normal coordinates"), but very short geodesics are "almost minimal" in the sense that the ratio of their length to the distance between p and their endpoint is uniformly close to 1 (Lemma 12). In particular, there are not arbitrarily short geodesic loops starting at p.

Theorem 1 represents the last "manifold theorem" having as its hypothesis only finite dimensionality and some combination of the three fundamental metric conditions, 1) geodesic completeness, 2) curvature locally bounded below, and 3) curvature locally bounded above. In [Be] and [N1] (cf. also [ABN]) it is shown that a space satisfying 1), 2), and 3) is a smooth manifold with a $C^{1,\alpha}$ Riemannian metric. This theorem leads to a short, entirely "metric" proof of the Convergence Theorem for Riemannian manifolds ([P], [GW]). The main theorem of [P1] is that a space satisfying 2) and 3) is a smooth manifold with boundary, with failure of geodesic completeness occuring precisely on the boundary. Theorem 1 covers the case of 1) and 2), and examples show that there are finite dimensional non-manifolds satisfying any other combination of the above properties.

Theorem 1 is also a little progress toward solving the conjecture that limits in the Grove-Petersen-Wu class of Riemannian manifolds ([GPW]) are topological manifolds. These spaces have curvature bounded below, but are not geodesically complete; Theorem 1 reduces the problem to considering neighborhoods of "geodesic terminals" (points where geodesic completeness fails).

Finally, Theorem 1 gives rise to the question of whether almost Riemannian spaces admit smooth structures. If some do not, then one must ask how large the class of topological manifolds admitting an almost Riemannian structure is, and whether the structure produced in Theorem 1 has useful topological applications.

For basic definitions, see [P1] or [R]. We confine ourselves to a few background comments. Since a finite dimensional, metrically complete metric space is locally compact, *we will assume, for the remainder of this paper, that (X, d) is a metrically complete, locally compact inner metric space having curvature locally bounded below.* Then every pair of points in X is joined by a minimal curve and there is a notion of angle between geodesics. We denote the angle between geodesics γ, β by $\alpha(\gamma, \beta)$. Every sufficiently small (open) ball $B = B(x, r)$ is a *region of curvature* $\geq k$ for some k, in which the following hold:

T1. *For any geodesic triangle $(\gamma_{ab}, \gamma_{ac}, \gamma_{bc})$ in B such that γ_{ab} and γ_{bc} are minimal and $L(\gamma_{ac}) \leq \pi/\sqrt{k}$, there exists a representative triangle $(\Gamma_{AB}, \Gamma_{AC}, \Gamma_{BC})$ in S_k (i.e., with same side lengths), and $\alpha((\Gamma_{AB}, \Gamma_{AC}) \leq \alpha(\gamma_{ab}, \gamma_{ac})$.*

T2. *For any geodesic wedge $(\gamma_{ab}, \gamma_{ac})$ in B such that γ_{ab} is minimal and $L(\gamma_{ac}) \leq \pi/\sqrt{k}$, there is a representative wedge $(\Gamma_{AB}, \Gamma_{AC})$ in S_k (i.e., with same side lengths and angle), and $d(B, C) \geq d(b, c)$.*

The above comparisons for *minimal* wedges and triangles follow easily from the definition of bounded curvature (cf.

[P1]); the more general T1 and T2 can then be proved as in the final step in the proof of Toponogov's Theorem ([CE]).

The *space of directions* at a point $p \in X$ is the metric space (S_p, α) of all unit geodesics starting at p. If S_p has at most two points, it is easy to show that X is homeomorphic to an interval or a circle. Some of the lemmas below fail for this trivial case, and to avoid special exceptions in the statements, the direction space at each point will be assumed, when necessary, to have at least three elements. The *tangent space* T_p at a point $p \in X$ is the metric space obtained from $S_p \times R^+$ by identifying all points of the form $(\gamma, 0)$ (and denoting the resulting point by 0) with the following metric, where the class of (γ, t) in the identification space is denoted $t\gamma$:

$$\delta(t\gamma, s\beta) = (t^2 + s^2 - 2st \cdot \cos \alpha(\gamma, \beta))^{1/2}.$$

The *exponential map* is defined by $\exp_p(s \cdot \gamma) = \gamma(s)$; if X is geodesically complete, then $\exp_p$ is defined on all of T_p, and is, by T2, continuous on any $B(0, r)$ such that $\exp_p(B(0, r))$ is contained in a region of curvature $\geq k$. Exp_p then has a continuous extension to the metric completion $\overline{T}_p$ of T_p. Furthermore, $\exp_p$ is (locally) a radial isometry, and preserves the angle between radial geodesics (i.e., starting at p). The *cut radius map* $C : S_p \to R^+ \cup \infty$ is defined by

$$C(\gamma) = \sup \{t : \gamma|_{[0,t]} \text{ is minimal}\}.$$

C is clearly upper semicontinuous. For $v \in T_p$, we let $C(v) = C(v/\|v\|)$.

Let $\bar{S}_p$ be the metric completion of S_p; then elements of $\bar{T}_p$ can clearly be written in the form $t\bar{\gamma}$, where $\bar{\gamma} \in \bar{S}_p$, $t \in R^+$, and $0\bar{\gamma} = 0$. For any $\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3 \in \bar{S}_p$, $\bar{\gamma}_2$ is said to be *between* $\bar{\gamma}_1$ and $\bar{\gamma}_3$ if $\alpha(\bar{\gamma}_1, \bar{\gamma}_3) = \alpha(\bar{\gamma}_1, \bar{\gamma}_2) + \alpha(\bar{\gamma}_2, \bar{\gamma}_3)$. For any distinct $\bar{\gamma}_1, \bar{\gamma}_2 \in \bar{S}_p$, the *span* $sp\ (\bar{\gamma}_1, \bar{\gamma}_2) \subseteq \bar{T}_p$ of $\bar{\gamma}_1, \bar{\gamma}_2$ is the set of all $t\bar{\gamma}$ such that one of $\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}$ is between the other two. In general, given distinct $\bar{\gamma}_1, \ldots, \bar{\gamma}_k \in \bar{S}_p$, $k > 1$, the span of $\bar{\gamma}_1, \ldots, \bar{\gamma}_k$ is the smallest subset $sp\ (\bar{\gamma}_1, \ldots, \bar{\gamma}_k) \subseteq \bar{T}_p$ containing $\bar{\gamma}_1, \ldots, \bar{\gamma}_k$ such that if $\bar{\alpha}, \bar{\gamma} \in sp\ (\bar{\gamma}_1, \ldots, \bar{\gamma}_k)$, then $sp\ (\bar{\alpha}, \bar{\gamma}) \subset sp\ (\bar{\gamma}_1, \ldots, \bar{\gamma}_k)$. The elements $\bar{\gamma}_1, \ldots, \bar{\gamma}_k$ are said to be *independent* if $\bar{\gamma}_{j+1}$ does not lie in $sp\ (\bar{\gamma}_1, \ldots, \bar{\gamma}_j)$ for any j. The notions of angle (not as a metric!), betweeness, etc., can be generalized to the space T_p in the obvious way; e.g., for $t_1, \ldots, t_k > 0$, $sp\ (t_1\bar{\gamma}_1, \ldots, t_k\bar{\gamma}_k) = sp\ (\bar{\gamma}_1, \ldots, \bar{\gamma}_k)$.

Finally, a *geodesic terminal* is a point in X beyond which some geodesic cannot be extended. An open subset U of X is *geodesically complete* if U has no geodesic terminals.

Lemma 2. *Let $\epsilon > 0$ and k be arbitrary. Then*

a) there exists a number $\delta > 0$ such that if γ_{xa}, γ_{xb} are minimal curves in S_k of length $L \leq 1$ with $d(a, b) / L < \delta$, then $\alpha(\gamma_{xa}, \gamma_{xb}) \leq \epsilon$, and

b) there exists a $\nu > 0$ such that if γ_{xa}, γ_{xb} are minimal curves in S_k of length $L \leq 1$ with $\alpha(\gamma_{xa}, \gamma_{xb}) \leq \nu$, then $d(a, b) / L < \epsilon$.

Proof. For $a \neq x$, let $\psi(a)$ be the smallest number such that if $d(a, b) / d(a, x) = \psi(a)$, $\alpha(\gamma_{xa}, \gamma_{xb}) = \epsilon$. The map ψ is easily seen to be continuous (in fact dependent only on $d(x, a)$) and positive, with $\lim_{a \to x} \psi(a) = 2 \cdot \sin(\epsilon/2)$, and so has some positive minimum δ on $\bar{B}(x, 1)$. This proves part a), and the proof of part b) is similar.

Lemma 3. *Suppose $B = B(p, r)$ is a region of curvature $\geq k$ in X. Let (γ_i) and (η_i) be Cauchy sequences in S_p. For any positive $s_i \to 0$ and $t_i \to 0$ such that $s_i \leq C(\gamma_i)$, $t_i \leq C(\eta_i)$, and $c_1 \leq s_i/t_i \leq c_2$ for some c_1, $c_2 \in (0, \infty)$, if $d_i = d(\gamma_i(s_i), \eta_i(t_i))$, then*

$$\lim_{i \to \infty} \alpha(\gamma_i, \eta_i) = \lim_{i \to \infty} \cos^{-1} [(s_i^2 + t_i^2 - d_i^2) / 2s_i t_i].$$

Proof. For any positive $s \leq C(\gamma_i)$ and $t \leq T(\eta_i)$, define $d_i(s, t) = d(\gamma_i(s), \eta_i(t))$ and

$$\varphi_i(s, t) = \cos^{-1} [(s^2 + t^2 - d_i(s, t))^2) / 2st].$$

It φ_i is continuously extended to $(0, 0)$, then $\varphi_i(0, 0) =$

$\alpha(\gamma_i, \eta_i)$. We have $\cos \varphi_j(s, t) - \cos \varphi_i(s, t) =$

$$[(d_i(s, t) - d_j(s, t))(d_i(s, t) + d_j(s, t)] / 2st.$$

Assuming $0 < c_1 \le s/t \le c_2 < \infty$, we have $(d_i(s, t) - d_j(s, t)) / s$

$$\le d(\gamma_i(s), \gamma_j(s)) / s + d(\eta_i(t), \eta_j(t)) / s$$

$$\le d(\gamma_i(s), \gamma_j(s)) / s + d(\eta_i(t), \eta_j(t)) / c_1 t.$$

By Lemma 2.b) and T2, the last quantity is arbitrarily small for sufficiently large i and j, *independent of* s and t. By a similar argument we obtain that $(d_i(s, t) + d_j(s, t)) / t$ is bounded, and conclude that for any $\zeta > 0$ there exists an m such that for all i, j > m, $|\varphi_i(s, t) - \varphi_j(s, t)| \le \zeta/2$. If m is also chosen large enough that $|\varphi_j(0, 0) - \lim_{i \to \infty} \alpha(\gamma_i, \eta_i)| \le \zeta/2$ for all j > m, then for $s \le C(\gamma_j)$, $t \le C(\eta_j)$, $|\varphi_j(s, t) - \lim_{i \to \infty} \alpha(\gamma_i, \eta_i)| \le \zeta$, and the lemma follows. □

Notation. For results 4-8, 10-15, and 17 below, let $B = B(p, r)$ be a geodesically complete region of curvature $\ge k$ in X.

Lemma 4. *For every distinct* $\bar{\eta}_1, \bar{\eta}_2 \in \bar{S}_p$ *and* $\alpha_3 \in [\alpha(\bar{\eta}_1, \bar{\eta}_2), \pi]$, *there exists a unique* $\bar{\eta}_3 \in \bar{S}_p$ *such that* $\bar{\eta}_2$ *is between* $\bar{\eta}_1$ *and* $\bar{\eta}_3$, *and* $\alpha(\bar{\eta}_3, \bar{\eta}_1) = \alpha_3$.

Proof. For the proof of the existence, see the Addendum at the end of this paper.

Let $\alpha_1 = \alpha(\bar{\eta}_1, \bar{\eta}_2)$ and $\alpha_2 = \alpha_3 - \alpha_1$. Suppose, contrary to uniqueness, there exist sequences $\{\gamma_{1i}\}$ and $\{\gamma_{2i}\}$ in S_p of unit minimal curves such that $\alpha(\gamma_{1i}, \gamma_{2i}) > \delta > 0$ for all i, and, for $k = 1, 2$,

$$|\alpha(\bar{\eta}_2, \gamma_{ki}) - \alpha_2| \le 2^{-i}, \text{ and}$$

$$|\alpha(\bar{\eta}_1, \gamma_{ki}) - \alpha_3| \le 2^{-i}.$$

Let $\eta_{1i}, \eta_{2i} \in S_p$ such that $\eta_{1i} \to \bar{\eta}_1$ and $\eta_{2i} \to \bar{\eta}_2$. In the plane, choose points X, A, B, and T such that A, B, and T are collinear, $XA = 1$, $XB = 1$, $\alpha(\overline{XA}, \overline{XB}) = \alpha_1$ and $\alpha(\overline{XB}, \overline{XT}) = \alpha_2$. Choose $t_i \to 0$ such that $t_i \le \min\{C(\gamma_{1i}), C(\gamma_{2i})\}$, $r_i = t_i \cdot XA / XT \le C(\eta_{1i})$, and $s_i = t_i \cdot XB / XT \le C(\eta_{2i})$. Let β_i, ζ_{1i}, and ζ_{2i} be a minimal curves from $\eta_{2i}(s_i)$ to $\eta_{1i}(r_i)$, $\gamma_{1i}(t_i)$, and $\gamma_{2i}(t_i)$, respectively. By Lemma 3 for $k = 1, 2$ and any $\lambda > 0$ there exists a j such that for all $i > j$, $L(\beta_i) + L(\zeta_{ki}) \le (1+\lambda)\cdot d(\eta_{1i}(r_i), \gamma_{ki}(t_i))$; it follows that the angle of a wedge W_i in S_K representing the wedge formed by β_i and ζ_{ki} tends to π. T1 then implies that $\lim_{i\to\infty} \alpha(\beta_i, \zeta_{ki}) = \pi$. On the other hand, if β'_i is a minimal curve beginning at $\eta_{2i}(s_i)$ and extending β_i as a geodesic beyond $\eta_{2i}(s_i)$, then $\lim_{i\to\infty} \alpha(\beta'_i, \zeta_{ki}) = 0$. This, in turn, implies $\lim_{i\to\infty} \alpha(\zeta_{1i}, \zeta_{2i}) = 0$. Let Z_{1i}, Z_{2i} be unit minimal curves in S_K, with common endpoint y and other endpoints z_{1i} and z_{2i}, respectively, such that $L(Z_{1i}) = L(\zeta_{1i})$, $L(Z_{2i}) = L(\zeta_{2i})$, and $\alpha(Z_{1i}, Z_{2i}) = \alpha(\zeta_{1i}, \zeta_{2i})$. Then

$$0 = \lim_{i\to\infty} d(z_{1i}, z_{2i}) / L(Z_{1i})$$

$$\ge \lim_{i\to\infty} d(\gamma_{1i}(t_i), \gamma_{2i}(t_i)) / L(\zeta_{1i})$$

$$= \lim_{i\to\infty} d(\gamma_{1i}(t_i),\ \gamma_{2i}(t_i))\cdot(XT/BT)\ /\ t_i.$$

This last limit being 0 implies that $\lim_{i\to\infty} \alpha(\gamma_{1i},\ \gamma_{2i}) = 0$, a contradiction.

Lemma 5. *For $\bar\eta_1, \bar\eta_3 \in \bar S_p$ and $\alpha_1 \in [0, \alpha(\bar\eta_1, \bar\eta_3)]$, there exists an $\bar\eta_2 \in \bar S_p$ such that $\bar\eta_2$ is between $\bar\eta_1$ and $\bar\eta_3$, and $\alpha(\bar\eta_1, \bar\eta_2) = \alpha_1$. Furthermore, if $\alpha(\bar\eta_1, \bar\eta_3) < \pi$, then $\bar\eta_2$ is the unique such element.*

Proof. The existence part of the proof is again contained in the proof of Lemma 3.4 in [P1]. If $\alpha(\bar\eta_1, \bar\eta_3) < \pi$, then the element $\bar\eta_4 \in \bar S_p$ such that $\alpha(\bar\eta_1, \bar\eta_4) = \pi$ is distinct from $\bar\eta_3$, and uniqueness follows from uniqueness in Lemma 4 (applied to $\bar\eta_4$ and $\bar\eta_3$ to obtain $\bar\eta_2$).

Lemma 6. *Let $\bar\gamma_1, \ldots, \bar\gamma_4 \in \bar S_p$ be distinct and, setting $\alpha_{ij} = \alpha(\bar\gamma_i, \bar\gamma_j)$, suppose $\alpha_{12} + \alpha_{23} = \alpha_{13} < \pi$. Then there exist unit vectors $v_i \in R^3$ such that $\alpha(v_i, v_j) = \alpha_{ij}$, and a choice of v_4 any two of v_1, v_2, v_3 determines the remaining v_i.*

Proof. The lemma is trivial if $\bar\gamma_4 \in sp\ (\bar\gamma_1, \bar\gamma_2)$; assume the contrary. There exist $X_i \in R^3\setminus 0$ such that X_1, X_2, and X_3 are colinear, with $\alpha(\overline{0X}_1, \overline{0X}_2) = \alpha_{12}$, $\alpha(\overline{0X}_2, \overline{0X}_3) = \alpha_{23}$, $\alpha(\overline{0X}_1, \overline{0X}_4) = \alpha_{14}$, and $\alpha(\overline{0X}_3, \overline{0X}_4) = \alpha_{34}$. The lemma will be proved if it is shown that $\alpha_{24} = \alpha(\overline{0X}_2, \overline{0X}_4)$; the proof begins with the inequality $\alpha_{24} \geq \alpha(\overline{0X}_2, \overline{0X}_4)$.

Choose $\gamma_{ij} \in S_p$ such that $\gamma_{ij} \to \bar\gamma_i$, $i = 1, \ldots, 4$, and positive $t_j \to 0$ such that $s_{ij} = \|t_j \cdot X_i\| \le C(\gamma_{ij})$. Let $\beta^j_{ik} : [0, 1] \to B$ be minimal from $x_{ij} = \gamma_{ij}(s_{ij})$ to $x_{kj} = \gamma_{kj}(s_{kj})$, let γ'_{2j} be minimal from p to $x'_{2j} = \beta^j_{13}(\alpha_{12}/\alpha_{13})$, and let β'^j_{12} be minimal from x_{1j} to x'_{2j}. To prove the above inequality, it suffices, by Lemma 3 and the uniqueness of Lemma 4, to show that $\lim \alpha(\alpha'_{2j}, \alpha_{4j}) \ge \alpha(\overline{0X}_2, \overline{0X}_4)$; i.e., $\lim d(x'_{2j}, x_{4j}) / t_j \ge X_2X_4$. Let T_j denote the representative in S_K of the triangle formed by β^j_{13}, β^j_{14}, and β^j_{34}. By Lemma 3 and the definition of angle (applied to $\alpha(\overline{X_1X_3}, \overline{X_1X_4})$), the angle in T_j corresponding to $\alpha(\beta^j_{13}, \beta^j_{14})$ tends to $\alpha(\overline{X_1X_3}, \overline{X_1X_4})$. In particular, T_j could be chosen so that the sides corresponding to β^j_{13}, and those corresponding to β^j_{14}, both are Cauchy sequences. Lemma 3 now implies that, if D_j is the distance in S_K from the point corresponding to x'_{2j} to that corresponding to x_{4j}, $\lim D_j / t_j = X_2X_4$. But by the definition of curvature $\ge k$, $\lim d(x'_{2j}, x_{4j}) / t_j \ge \lim D_j / t_j$.

To complete the proof of the lemma, note that the above argument can be applied to $\overline{\gamma}'_1$, $\overline{\gamma}'_2$, $\overline{\gamma}'_3$ and $\overline{\gamma}_4$, where $\alpha(\overline{\gamma}'_i, \overline{\gamma}_i) = \pi$, for $i = 1, 2, 3$. One then obtains that both $\alpha(\overline{\gamma}'_2, \overline{\gamma}_4)$ and $\alpha(\overline{\gamma}_2, \overline{\gamma}_4)$ must be $\ge$ their Euclidean counterparts. Since $\alpha(\overline{\gamma}'_2, \overline{\gamma}_4) + \alpha(\overline{\gamma}_2, \overline{\gamma}_4) = \pi$, this is only possible if both equal their Euclidean counterparts. The last part of the lemma is elementary linear algebra.

Lemmas 4, 5, and 6 correspond exactly to Lemmas 3.5, 3.4, and 3.6 in [P1]; the proof of the next proposition is similar to that of Proposition 3.7, [P1].

Proposition 7. *If* $\overline{\gamma}_1, \ldots, \overline{\gamma}_n \in \overline{S}_p$ *are independent, then* $sp\ (\overline{\gamma}_1, \ldots, \overline{\gamma}_n)$ *is isometric to* R^n.

Sp $(\overline{\gamma}_1, \ldots, \overline{\gamma}_n)$ will now be identified with R^n. Any infinite independent subset of S_p has no convergent subsequence; in other words:

Corollary 8. *If* S_p *is precompact, then* $\overline{T}_p$ *is isometric to* R^n *for some n.*

Definition 9. A minimal curve is called *strictly minimal* if it is the unique minimal curve between its endpoints.

Lemma 10. *Let b, c* $\in B$ *such that* γ_{pb} *and* γ_{pc} *are strictly minimal. Then for any* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that for all* $a \in B(p, \delta)$ *and minimal curves* γ_{ab} *and* γ_{ac},

$$|\alpha(\gamma_{pb}, \gamma_{pc}) - \alpha(\gamma_{ab}, \gamma_{ac})| < \epsilon.$$

Proof. Let $a_i \to p$ and suppose γ_i and η_i are minimal curves from a_i to b and c, respectively; we will show first that $\underline{\lim}_{i\to\infty} \alpha(\gamma_i, \eta_i) \geq \alpha(\gamma_{pb}, \gamma_{pc})$. Let $\zeta > 0$. Choose T > 0 so that if γ_{PB}, γ_{PC} are minimal curves in S_k with d(P, B) = d(P, C) = T and d(B, C) = $d(\gamma_{pb}(T), \gamma_{pc}(T))$, then $\alpha(\gamma_{pb}, \gamma_{pc}) - \alpha(\gamma_{PB}, \gamma_{PC}) \leq \zeta$. Since γ_{bp} and γ_{cp} are strictly minimal, $\lim_{i\to\infty} \alpha(\gamma_i, \gamma_{bp}) = \lim_{i\to\infty} \alpha(\eta_i, \gamma_{cp}) = 0$; by T2, $\lim_{i\to\infty} d(\gamma_{pb}(T), \gamma_i(T)) =$

$\lim_{i\to\infty} d(\gamma_{pc}(T), \eta_i(T)) = 0$. If C_i is the point closest to C in S_k such that $d(P, C_i) = T$ and $d(B, C_i) = d(\gamma_i(T), \eta_i(T))$, then applying T1 and Lemma 2, we obtain $\lim_{i\to\infty} \alpha(\gamma_i, \eta_i) \geq \lim_{i\to\infty} \alpha(\gamma_{PB}, \gamma_{PC_i}) = \alpha(\gamma_{PB}, \gamma_{PC}) \geq \alpha(\gamma_{pb}, \gamma_{pc}) - \zeta$. Since ζ was arbitrary, the first inequality follows.

Let d be a point on the geodesic extension of γ_{bp} beyond p such that γ_{pd} is strictly minimal. If β_i is a minimal curve from a_i to d, and ν_i is minimal starting at a_i such that $\alpha(\gamma_i, \nu_i) = \pi$, then by the above argument, $\lim_{i\to\infty} \alpha(\gamma_i, \beta_i) = \pi$, hence $\lim_{i\to\infty} \alpha(\beta_i, \nu_i) = 0$. Also by the above argument, $\lim_{i\to\infty} \alpha(\beta_i, \eta_i) \geq \alpha(\gamma_{pd}, \gamma_{pc}) = \pi - \alpha(\gamma_{pb}, \gamma_{pc})$. By the triangle inequality, $\alpha(\gamma_i, \eta_i) \leq \pi - \alpha(\eta_i, \beta_i) + \alpha(\beta_i, \nu_i)$, and we obtain the desired inequality by passing to the limit.

Lemma 11. *Let A_i, B_i, $C_i \in S_k$, with A_i, C_i distinct, $\lim_{i\to\infty} d(A_i, C_i) = 0$, and $d(A_i, B_i) \geq D$ for some $D > 0$ and all i. Suppose γ_i is minimal from A_i to B_i and β_i is minimal from A_i to C_i. Then $\varphi = \lim_{i\to\infty} \alpha(\gamma_i, \beta_i)$ exists if and only if $L = \lim_{i\to\infty} [d(A_i, B_i) - d(B_i, C_i)] / d(A_i, C_i)$ exists. If φ and L exist, $L = \sin(\pi/2 - \varphi)$.*

Proof. If $d(A_i, B_i) = d(B_i, C_i)$ for all i, $\lim_{i\to\infty} \alpha(\gamma_i, \beta_i) = \pi/2$ follows from the Cosine Laws for S_k. In the general case, let α'_i be unit minimal of length max $\{d(A_i, B_i), d(A_i, C_i)\}$ starting at B_i and containing the point C_i; let $D_i = \alpha'_i(d(A_i, B_i))$ and α_i be the segment of α'_i from D_i to B_i. If $D_i =$

A_i for all large i, then α_i and γ_i coincide, and the lemma is trivial. Otherwise, applying the above special case we obtain that if ζ_i is minimal from D_i to A_i, $\lim_{i\to\infty} \alpha(\zeta_i, \beta_i) = \lim_{i\to\infty} \alpha(\zeta_i, \gamma_i) = \pi/2$. The Lemma now follows from the Cosine Laws and the definition of angle.

Lemma 12. *Suppose $B(p, r)$ is a geodesically complete region of curvature $\geq k$, and let $\gamma_1, \ldots, \gamma_k \in T_p$ be independent. Then for every small $\epsilon > 0$ there exists a $\rho > 0$ such that for all $\gamma \in sp(\gamma_1, \ldots, \gamma_k) \cap S_p$ and $t < \rho$, $1-\epsilon \leq d(p, \gamma(t)) / t \leq 1$.*

Proof. Let $\delta = \sin^{-1}(1 - \epsilon/2)$, $\alpha_1, \ldots, \alpha_M \in S_p$ be δ-dense in $sp\ (\gamma_1, \ldots, \gamma_k) \cap S_p$, and $R > 0$ be small enough that $\alpha_i|_{[0,R]}$ is strictly minimal for all i. Let Γ_{ab} be minimal in S_k of length R and Γ_{ac} be unit minimal, with $\alpha(\Gamma_{ab}, \Gamma_{ac}) = \delta$. Then by Lemma 11, $\lim_{t\to 0} (R - d(b, \Gamma_{ac}(t))) / t = 1 - \epsilon/2$; let $\rho > 0$ be such that for all $t < \rho$, $(R - d(b, \Gamma_{ac}(t))) / t \geq 1-\epsilon$. For any $\gamma \in sp\ (\gamma_1, \ldots, \gamma_k) \cap S_p$, there exists some α_i such that $\alpha(\gamma, \alpha_i) < \delta$. By the triangle inequality, $d(p, \gamma(t)) \geq R - d(\gamma(t), \alpha_i(R))$, and the lemma follows from T2.

Lemma 13. *Suppose $B(p, r)$ is a geodesically complete region of curvature $\geq k$, and let $\gamma_1, \ldots, \gamma_k \in T_p$ be independent. Then there exists an* L > 0 *such that*

$$exp_p^{-1}(p) \cap B(0, L) \cap sp\ (\gamma_1, \ldots, \gamma_k) = (0).$$

In particular, there are not arbitrarily small geodesic loops at p in $exp_p(sp\ (\gamma_1, \ldots, \gamma_k) \cap S_p)$.

Proof. By Lemma 12 there exists an $L > 0$ such that for all $\gamma \in sp\ (\gamma_1, \ldots, \gamma_k) \cap S_p$ and $t < L$, $d(p, \gamma(t)) / t > 1/2$. Now suppose there exists some $v \in exp_p^{-1}(p) \cap sp\ (\gamma_1, \ldots, \gamma_k)$ with $0 < \|v\| = L' < L$. Then there are $v_i \in T_p$ with $v_i \to v$. But then the continuity of the exponential map implies that $d(exp_p(L' \cdot v_i), p) \to 0$, a contradiction.

Lemma 14. *Suppose $B(p, r)$ is a geodesically complete region of curvature $\geq k$, and let $\gamma_1, \ldots, \gamma_k \in T_p$ be independent. Then for any $\epsilon > 0$ there exist δ, $R > 0$ such that if $\alpha, \beta \in sp\ (\gamma_1, \ldots, \gamma_k) \cap S_p$ and $d(\alpha(t), \beta(s)) / s < \delta$ for some $0 < s < t < R$, then $\alpha(\alpha, \beta) < \epsilon$.*

Proof. Suppose, to the contrary, there exist $\alpha_i, \beta_i \in sp\ (\gamma_1, \ldots, \gamma_k) \cap S_p$ and $0 < s_i < t_i < 2^{-i}$, with $\alpha(\alpha_i, \beta_i) > \epsilon$ and, letting $d_i = d(\alpha_i(s_i), \beta_i(t_i))$, $d_i / s_i < 2^{-i}$. Choosing a subsequence if necessary, we can assume that both $\{\alpha_i\}$ and $\{\beta_i\}$ are Cauchy. Let $\zeta, \eta \in S_p$ be such that for all sufficiently large i, $\alpha(\alpha_i, \zeta) < \epsilon/4$ and $\alpha(\beta_i, \eta) < \epsilon/4$. In S_k, let Γ_{xa}, Γ_{xb},

Γ_{xc} be unit minimal such that $\alpha(\Gamma_{xa}, \Gamma_{xb}) = \alpha(\Gamma_{xb}, \Gamma_{xc}) = \epsilon/4$ and $\alpha(\Gamma_{xa}, \Gamma_{xc}) = \epsilon/2$. Define

$$a'_i = d(\Gamma_{xa}(s_i), \Gamma_{xb}(s_i)),$$
$$b'_i = d(\Gamma_{xb}(t_i), \Gamma_{xc}(t_i)),$$
$$c_i = d(\zeta(s_i), \eta(t_i)), \text{ and}$$
$$c'_i = d(\Gamma_{xa}(s_i), \Gamma_{xc}(t_i)).$$

By T2 and the triangle inequality,

$$c_i \le a'_i + b'_i + d_i \le c'_i + (t_i - s_i) + d_i.$$

Lemma 12 implies that if $\delta > 0$, then for all sufficiently large i,

$$* \qquad 1 - \delta \le d(p, \beta_i(t_i) / t_i \le (s_i + d_i) / t_i$$
$$\Leftrightarrow t_i - s_i \le \delta \cdot t_i + d_i \le \delta \cdot (s_i + (t_i - s_i)) + d_i$$
$$\Leftrightarrow (t_i - s_i) / s_i \le (\delta + d_i/s_i) / (1 - \delta).$$

Combining these inequalities we obtain $\lim_{i->\infty} (c'_i - c_i) / s_i \ge 0$. From * we obtain that $\lim_{i->\infty} s_i/t_i = 1$. By Lemma 3,

$$\begin{aligned}\cos \alpha(\zeta, \eta) &= \lim_{i->\infty} (s_i^2 + t_i^2 - c_i^2) / 2s_it_i \\ &= \lim_{i->\infty} [(s_i^2 + t_i^2 - c'^2_i) + (c'^2_i - c_i^2)] / 2s_it_i \\ &= \cos(\epsilon/2) + \lim_{i->\infty} (c'_i + c_i)(c'_i - c_i) / 2s_it_i \\ &\ge \cos(\epsilon/2),\end{aligned}$$

since $\lim_{i->\infty} c'_i/t_i$ is bounded. From the triangle inequality we have, for all sufficiently large i, $\alpha(\alpha_i, \beta_i) < \epsilon$, a contradiction.

Proposition 15. *Suppose $B(p, R)$ is a geodesically complete region of curvature $\geq k$, and let $\gamma_1, \ldots, \gamma_m \in T_p$ be independent. Then $B(p, r) \cap \exp (sp (\gamma_1, \ldots, \gamma_m))$ has dimension m for all sufficiently small $r > 0$.*

Proof. We first prove that $\dim B(p, R) \geq m$. Let $B = B(0, 1) \subset sp (\gamma_1, \ldots, \gamma_m) = R^m$ and consider the maps $\varphi_r : B \to B(p, r) \cap \exp (sp (\gamma_1, \ldots, \gamma_m))$ given by $\varphi_r(v) = \exp_p(rv)$. We claim that for any $\epsilon > 0$ there exists an $r > 0$ such that φ_r is an ϵ-mapping; i.e., for all $x \in B(p, r)$, $\mathrm{dia}\, (\varphi_r^{-1}(x)) < \epsilon$. Let $\zeta = \cos^{-1}(1 - \epsilon^2/2)$ and choose r, $\delta > 0$ by Lemma 14 for the number $\zeta/3$, and choose r even smaller, if necessary, to satisfy the conclusion of Lemma 12 for the number $\epsilon/4$. Let $x \in B(p, r)$ and suppose $v, w \in \varphi_r^{-1}(x)$, with, say, $\|v\| \leq \|w\|$. Choose v', $w' \in B \cap T_p$ such that $\|v'\| = \|v\|$, $\|w'\| = \|w\|$, $\alpha(v, v') < \zeta/3$, $\alpha(w, w_i) < \zeta/3$, and both $d(\exp_p(v_i), x))$ and $d(\exp_p(w_i), x))$ are $\leq \min \{\delta\|v\|/2, \epsilon r/8\}$. By Lemma 14 and the triangle inequality, $\alpha(v, w) < \zeta$. On the other hand, we have by Lemma 12 and the triangle inequality,

$$\|v\| - \|w\| \leq (\epsilon/4)\|v\| + (d(\exp_p(v_i), x)) + d(\exp_p(w_i), x)) / r$$
$$\leq \epsilon/2.$$

We now apply the triangle inequality to the points v, $(\|w\|/\|v\|)v$, and w to obtain $d(v, w) \leq \epsilon$.

Since B has dimension m and there exist ϵ-maps from B onto $B(p, r)$ for arbitrarily small $\epsilon > 0$, it follows (cf. [Na], IV.5.

A) that B(p, r) has dimension $\geq$ m.

On the other hand for $r < \pi/\sqrt{k}$, the set $B' = B(0, r) \subset$ sp $\{\gamma_1, \ldots, \gamma_m\}$ possesses a metric with which it is isometric to an open ball of radius r in S^m_k. If

$$K = \{v \in B' \cap T_p : C(v) \geq \|v\|\}$$

then $\exp_p|_K$ is surjective onto B(p, r) and distance decreasing by T2. Since K is a closed subset of B', K has Hausdorff dimension $\leq$ m, and since a distance decreasing map cannot increase Hausdorff dimension, dim B(p, r) $\leq$ m.

Corollary 16. *If X is geodesically complete, the following are equivalent:*

a) X has dimension $n < \infty$,

b) at one point $p \in X$, S_p *is precompact, with dim* $\overline{T}_p = n$,

c) at every point $q \in X$, *dim* $\overline{T}_p = n$.

Proof. By Proposition 15, we need only show that the mapping $q \longmapsto \dim \overline{T}_q$ is uppersemicontinuous. Let $\{\gamma_1, \ldots, \gamma_m\} \in S_q$ be independent. If $R > 0$ is such that $\gamma_i|_{[0,R]}$ is minimal for all i, then by Lemma 10, for all z sufficiently close to q, if α_i is minimal from z to $\gamma_i(R)$, then $\{\alpha_1, \ldots, \alpha_m\}$ is independent in T_z, and it follows that $\dim \overline{T}_z \geq \dim \overline{T}_q$.

Lemma 17. *Let $B = B(p, r)$ be a geodesically complete region of curvature $\geq k$, γ be strictly minimal in B from x to z, and x_j, y_j, $\in B$ with $x_j \to x$. If α_j is minimal from x_j to z and β_j is minimal from x_j to y_j then $\varphi = \lim_{j\to\infty} \alpha(\alpha_j, \beta_j)$ exists if and only if $L = \lim_{j\to\infty} [d(z, x_j) - d(z, y_j)] / d(x_j, y_j)$ exists. If L and φ exist, then $L = \sin(\pi/2 - \varphi)$.*

Proof. Let q lie on an extension γ' of γ as a minimal curve past x, and define $A_{1j} = d(x_j, z)$, $A_{2j} = d(x_j, q)$, $B_{1j} = d(y_j, z)$, $B_{2j} = d(y_j, q)$, and $C_j = d(x_j, y_j)$. We relabel α_j as α_{1j}, let α_{2j} be minimal from x_j to q, ζ_{1j} be minimal from y_j to z and ζ_{2j} be minimal from y_j to q. Since γ' is minimal, $\lim_{j\to\infty} \alpha(\alpha_{1j}, \alpha_{2j}) = \pi$, and T1 and Lemma 11 imply, for $i = 1, 2$, and $*i = (i - 2)^2 + 1$.

$$\begin{aligned}\liminf_{j\to\infty} (A_{ij} - B_{ij}) / C_j &\geq \liminf_{j\to\infty} \sin(\pi/2 - \alpha(\alpha_{ij}, \beta_j)) \\ &= \sin(\liminf_{j\to\infty} \alpha(\alpha_{*ij}, \beta_j) - \pi/2) \\ &= -\sin(\pi/2 - \liminf_{j\to\infty} \alpha(\alpha_{*ij}, \beta_j)). \\ &\geq -\limsup_{j\to\infty} (A_{*ij} - B_{*ij}) / C_j),\end{aligned}$$

since by T1 and Lemma 11,

$$\liminf_{j\to\infty} \alpha(\alpha_{ij}, \beta_j) \geq \pi/2 - \sin^{-1}(\limsup_{j\to\infty} (A_{ij} - B_{ij}) / C_j).$$

By a similar argument, we obtain

$$\begin{aligned}\limsup_{j\to\infty} (A_{ij} - B_{ij}) / C_j &= -\liminf_{j\to\infty} (B_{ij} - A_{ij}) / C_j \\ &\leq -\liminf_{j\to\infty} \sin(\pi/2 - \alpha(\zeta_{ij}, \beta_j)) \\ &= -\sin(\liminf_{j\to\infty} \alpha(\zeta_{*ij}, \beta_j) - \pi/2) \\ &\leq -\liminf_{j\to\infty} (A_{*ij} - B_{*ij}) / C_j).\end{aligned}$$

In particular, $\{(A_{ij} - B_{ij}) / C_j\}$ is a bounded. If $L = \lim_{j\to\infty} (A_{1j} - B_{1j}) / C_j$ exists, then $L' = \lim_{j\to\infty} (A_{2j} - B_{2j}) / C_j$ also exists, and $L' = -L$, with

$$\begin{aligned}\liminf_{j\to\infty} \alpha(\alpha_{1j}, \beta_j) &\geq \pi/2 - \sin^{-1}(L) \\ &= \pi/2 + \sin^{-1}(L') \\ &\geq \pi - \liminf_{j\to\infty} \alpha(\alpha_{2j}, \beta_j) \\ &= \limsup_{j\to\infty} (\pi - \alpha(\alpha_{2j}, \beta_j)) \\ &= \limsup_{j\to\infty} \alpha(\alpha_{1j}, \beta_j)).\end{aligned}$$

On the other hand, if $\varphi = \lim_{j\to\infty} \alpha(\alpha_{1j}, \beta_j)$ exists, and the above computation is applied to any convergent subsequences of $\{(A_{ij} - B_{ij}) / C_j\}$, $i = 1, 2$, then the limit is always $\sin(\pi/2 - \varphi)$. Since $\{(A_{ij} - B_{ij}) / C_j\}$ is bounded, it follows that L exists and has the required value.

Proof of Theorem 1. Suppose dim X = n, let $p \in X$ be arbitrary, and $\gamma_1, \ldots, \gamma_n \in T_p$ be a basis for $\overline{T}_p$. Choose $R > 0$ small enough that $\gamma_i|_{[-R,R]}$ is strictly minimal for all i, and $B(p, 3R)$ is a region of curvature $\geq k$. Define $u : X \to R^n$ by $u^i(x) = d(x, z_i)$, where $z_i = \gamma_i(R)$ (cf. [Be]). We will show first that u is injective, and hence a homeomorphism, near p. Suppose, to the contrary, there exist points $x_j, y_j \to p$ in B such that $u(x_j) = u(y_j)$. Let η_j be minimal from x_j to y_j, and γ_{ij} be minimal from x_j to z_i. By Lemma 17, $\lim_{j\to\infty} \alpha(\eta_j, \gamma_{ij}) = \pi/2$ for all i. By Lemma 10, if j is large, $\eta_j \notin sp\{\gamma_{1j}, \ldots, \gamma_{nj}\}$; but Lemma 10 and Corollary 16 also imply $sp\{\gamma_{1j}, \ldots, \gamma_{nj}\} = \overline{T}_{x_j}$, a contradiction.

Let $B = B(0, R) \subset T_p$ and define functions φ, $\psi : B \to R^n$ by

$\varphi(v) = u(\exp_p(v)) - u(p)$, and

$\psi(v) = (\|v - R\gamma_1\|^{1/2} - R, \ldots, \|v - R\gamma_n\|^{1/2} - R)$.

Since R^n can be identified with its own tangent space, Invariance of Domain and the above argument that u is a homeomorphism imply that, for small r, $\psi|_{B(0,r)}$ is a homeomorphism onto a neighborhood of 0 in R^n. Let $S(0, \epsilon)$ denote the sphere of radius $\epsilon > 0$ in R^n, and $p_\epsilon : R^n \setminus \{0\} \to S(0, \epsilon)$ be the radial projection. Then

$$\psi_\epsilon = p_\epsilon \circ \psi|_{S(0,\epsilon)} : S(0, \epsilon) \to S(0, \epsilon)$$

is defined and has degree ± 1 for small ϵ. On the other hand, Lemma 13 implies that, for small ϵ, $\varphi^{-1}(0) \cap B(0, 2\epsilon) = \{0\}$, and the map

$$\varphi_\epsilon = p_\epsilon \circ \varphi|_{S(0,\epsilon)} : S(0, \epsilon) \to S(0, \epsilon)$$

is defined. If we suppose $\varphi(B(0, 2\epsilon))$ contains no neighborhood of 0, 0 must be a topological boundary point of $\varphi(B(0, \epsilon))$. But then φ_ϵ has a continuous extension over $\overline{B}(0, \epsilon)$ (cf. [HW], p. 96), and so $\deg(\varphi_\epsilon) = 0$. Therefore, to obtain a contradiction we need only show that for small $\epsilon > 0$, φ_ϵ and ψ_ϵ are homotopic. Choose $\delta > 0$ such that for all $v \in \overline{T}_p$, there is some γ_i with $|\alpha(v, \gamma_i) - \pi/2| > \delta$. By Lemma 11 (with $\varphi = \pi/2 - \delta$) and T2, there exists a $\rho > 0$ such that for any $v \in S_p$, $\epsilon < \rho$, and i as above, $d(\exp_p(\epsilon v), \gamma_i(R))$ and $\|\epsilon v - (R)\gamma_i\|^{1/2}$ are either both < $R - \zeta$ or both > $R + \zeta$, where $\zeta = (\epsilon/2)\cdot\sin\delta$. Since $\varphi(\overline{B}(0, \epsilon))$ and $\psi(\overline{B}(0, \epsilon))$ are both bounded, we obtain that $\alpha(\varphi_\epsilon(\epsilon v), \psi_\epsilon(\epsilon v))$

$\leq \pi - \nu$, for some $\nu > 0$. Since S_p is dense in $\overline{S}_p$, the same inequality holds on $\overline{S}_p$, and φ_ϵ and ψ_ϵ are homotopic. This completes the proof that X is a manifold. The remainder of Theorem 1 follows from the discussion of the tangent space at the beginning of the paper and Proposition 7.

Remark 18. If an almost Riemannian space X is locally convex in the sense that each point is contained in a strictly convex ball ([P1]), then if we let $\overline{TX}$ be the set $X \times R^n$ and identify $p \times X$ with $\overline{T}_p$, $\overline{TX}$ can be given the structure of a C^0 vector bundle as follows: let $\pi : \overline{TX} \to X$ be the projection. For any $p \in X$, let $\overline{B}(p, R)$ be strictly convex and small enough that it is contained in a region of curvature $\geq k$ and homeomorphic to an open subset of R^n. Let $\gamma_1, \ldots, \gamma_n \in T_p$ be a basis for $\overline{T}_p$; then by strict convexity $\gamma_i|_{[0,R]}$ is strictly minimal for all i. By Lemma 10 we can choose $r > 0$ small enough that for all $q \in B(p, r)$, if γ_i^q is minimal from q to $\gamma_i(R)$ then $\gamma_1^q, \ldots, \gamma_n^q$ lie in B(p, R) and form a basis for $\overline{T}_q$. We define

$$\varphi : \pi^{-1}(B(p, r)) \to B(p, r) \times \overline{T}_p = B(p, r) \times R^n$$

by $\varphi(\sum c_i\gamma_i^q) = (q, \sum c_i\gamma_i)$, and obtain a vector bundle atlas for $\overline{TX}$.

Example 19. The "squashed sphere," Q, due to K. Grove and P. Petersen, is obtained as a limit of Riemannian manifolds of positive curvature by flattening the upper and lower hemispheres of S^2, while allowing curvature along the equator to go to infinity. Q may also be obtained by gluing together flat disks along their boundaries. Q is easily verified to be almost Riemannian. If $p \in Q$ lies on the interior of either disk, $T_p = R^2$ and $\exp_p$ is an isometry on $B(0, r)$ for small r. If p lies on the equator, T_p can be identified with $R^2 \setminus \{(t, 0) : t \neq 0\}$; i.e., S_p is S^1 minus two antipodal points. The missing points correspond to the two "directions" of the equator, which is not a geodesic (but is a limit of geodesics). Points along the equator are joined by pairs of minimal curves, Euclidean segments crossing each disk. The space $X = Q \times S^1$ can be given a natural "product" inner metric so that geodesics are "products" of geodesics in Q and S^1. If $p \in Q$ is on the equator, then at $x = (p, z) \in X$, T_p consists R^3 with two coplanar open half-planes removed. The cut locus map C is not continuous at the two points in S_p corresponding to the S^1-directions.

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References

[ABN] Aleksandrov, A. D., Berestovskii, V. N., and Nikolaev, I. G. Generalized riemannian spaces, *Russian Math. Sur.* 41 (1986) 1-54.

[Be] Berestovskii, V.N. Introduction of a Riemann structure into certain metric spaces, *Siberian Math. J.*, 16 (1975), 499-507.

[CE] Cheeger, J., and Ebin, D. G. *Comparison Theorems in Riemannian Geometry*, North Holland, Amsterdam, 1975.

[GP] Grove, K. and Petersen, P. Manifolds near the boundary of existence, preprint.

[GPW] Grove, K., Petersen, P., and Wu, J.Y. Geometric finiteness theorems via controlled topology, *Invent. math.* **99** (1990) 205-213.

[GW] Greene, R.E. and Wu, H. Lipschitz convergence of Riemannian manifolds, *Pacific Math. J.* 131 (1988), 119-141.

[HW] Hurewicz, W. and Wallman, H. *Dimension Theory*, Princeton University Press, Princeton, 1941.

[N] Nikolaev, I. G. Smoothness of the metric in spaces with bilaterally bounded curvature in the sense of A. P. Alexandrov, *Siberian Math J.* **24** (1983) 247-263.

[Na] Nagata, J. *Modern Dimension Theory*, Heldermann Verlag, Berlin, 1893.

[Pl] Plaut, C. A metric characterization of manifolds with boundary, Max-Planck-Institut preprint no. 89-60.

[Pe] Peters, S. Convergence of Riemannian manifolds, *Compositio Math.* **62** (1987), 3-16.

[R] Rinow, W. *Die Innere Geometrie der Metrischen Räume*, Springer-Verlag, Berlin, 1961.

Addendum

Proof of existence in Lemma 4. Since S_p is dense in $\overline{S}_p$, we can assume $\overline{\eta}_i = \eta_i \in S_p$ for all i. The case $\alpha_3 = \pi$ is simply geodesic completeness; assume now that $\alpha_3 < \pi$. Let $\eta_4 \in S_p$ be such that $\alpha(\eta_1, \eta_4) = \pi$; then $a = \alpha(\eta_3, \eta_4) > 0$. By taking successive approximations we can reduce to the case $\alpha_3 - \alpha_1 = a/2$; in other words, given any $\epsilon > 0$, we need only find some $\gamma \in S_p$ such that

$$|\alpha(\gamma, \eta_2) - a/2| < \epsilon \text{ and}$$
$$|\alpha(\gamma, \eta_4) - a/2| < \epsilon.$$

Let $\beta_i : [0, 1] \rightarrow B$ be minimal from $\eta_4(2^{-i})$ to $\eta_2(2^{-i})$, $q_i = \beta_i(1/2)$, and γ_i be minimal from p to q_i. Choose $T > 0$ small enough that $\eta_1|_{[0,T]}$ and $\eta_4|_{[0,T]}$ together form a minimal curve and let α_i be minimal from $q = \eta_1(T)$ to q_i. Let $\nu_1, \nu_2, \nu_3, \nu_4$ be unit minimal curves in S_k such that $\alpha(\nu_i, \nu_j) = \alpha(\eta_i, \eta_j)$ for all $i, j \neq 3$, and $\alpha(\nu_2, \nu_3) = \alpha(\nu_3, \nu_4) = a/2$. Finally, let $\zeta_i : [0, 1] \rightarrow S_k$ be minimal from $\nu_4(2^{-i})$ to $\nu_2(2^{-i})$. By definition of the angle, $\lim_{i\to\infty} 2^{-i}\cdot L(\beta_i) = \lim_{i\to\infty} 2^{-i}\cdot L(\zeta_i) = 2\cdot\sin(a/2)$, and T1 implies $\liminf_{i\to\infty} \alpha(\beta_i, \eta_4) \geq \lim_{i\to\infty} \alpha(\zeta_i, \nu_4)$. Therefore, Lemma 2 and T1 applied to the wedge $(\eta_4, \beta_i|_{[0,1/2]})$ implies $\limsup_{i\to\infty} L(\gamma_i) \leq \lim_{i\to\infty} d(\nu_4(0), \zeta_i(1/2))$. On the other hand, Lemma 2 and the definition of curvature $\geq k$ (applied to the wedge (η_4, β_i)) implies that $\liminf_{i\to\infty} L(\alpha_i) \geq$

$\lim_{i\to\infty} d(\nu_1(T), \zeta_i(1/2))$. These last inequalities imply (via the elementary geometry of S_k) that $\liminf_{i\to\infty} \alpha(\eta_1, \gamma_i) \geq \alpha(\nu_1, \nu_3) = \pi - a/2$. It now follows that $\limsup_{i\to\infty} \alpha(\eta_4, \gamma_i) \leq a/2$. By a similar argument one can show $\limsup_{i\to\infty} \alpha(\eta_2, \gamma_i) \leq a/2$. From the triangle inequality we get $\lim_{i\to\infty} \alpha(\eta_2, \gamma_i) = \lim_{i\to\infty} \alpha(\eta_4, \gamma_i) = a/2$, and the proof of existence π is complete.

Correction. The existence part of the proof of Lemma 5 is contained in the above proof, not in [P1].