Spectra related with localizations.

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Introduction

Categorical realization of non-affine varieties was started by J.-P. Serre in 1955, even before Grothendieck developed scheme theory, with a description of the category of quasi-coherent sheaves on a projective variety [Fac]. About thirty years later, Yu. I. Manin proposed to use Serre's description as a definition of the category of quasi-coherent sheaves on $Proj(R)$, where R is a noncommutative $\mathbb{Z}_+$-graded ring [M1]. The proposal triggered an active study of noncommutative $Proj$ (starting with [A], [AB], [AZ], [V1], [V2]) and largerly contributed to general philosophy according to which noncommutative spaces are represented by categories.

This philosophy was supported, at least in the case of noetherian schemes, by a spectral theory of abelian categories due to P. Gabriel [Gab]. Points of the Gabriel's spectrum are isomorphism classes of indecomposable injectives. If $\mathcal{A}$ is the category of modules over a commutative noetherian ring, then isomorphism classes of indecomposable injectives are in bijective correspondence with prime ideals of this ring. Injective objects are naturally related with localizations: there are localizations at points of Gabriel's spectrum and at sets of points. Gabriel used the injective spectrum for reconstruction of an arbitrary noetherian scheme (in particular, the $Proj$ of a noetherian commutative graded ring) from its category of quasi-coherent sheaves.

A different spectrum of an abelian category, which does not rely on noetherian hypothesis, was introduced in [R1] (see also [R, Ch.3]). This spectrum is, in general, much smaller than the Gabriel's spectrum, at the same time powerful enough to reconstruct the prime spectrum of an arbitrary commutative ring from the category of modules over this ring, or, more generally, a quasi-compact (not necessarily noetherian) scheme from its category of quasi-coherent sheaves [R4]. If an abelian category has enough objects of finite type (like the category of modules over an associative ring), then closed points of this spectrum are precisely isomorphism classes of simple objects. The spectrum can be realized as a certain family of Serre subcategories; so that there are localizations at points and at sets of points.

In this work, we introduce spectra directly related with localizations. Remarkably, they make sense for non-abelian categories and triangulated categories. A considerable part of the paper, however, is dedicated to the studying these spectra in the abelian case, where they are also new, and to the comparison with the already known spectra. In particular, we sketch briefly facts on associated points and prime decomposition showing that our spectral theory is capable to support a meaningful categorical version of local algebra.

The first part of this work can be regarded as an introduction to basic spectra associated with exact localizations of *general* 'spaces', i.e. 'spaces' represented by arbitrary categories regarded as categories of quasi-coherent sheaves.

Section 1 contains preliminaries on localizations and multiplicative systems, and the beginning of the dictionary of categorical pseudo-geometry: 'spaces' represented by categories and morphisms represented by (their inverse image) functors. This formal duality between 'spaces' and categories is reflected in notaions: a category C_X represents the 'space' X and a functor $C_Y \xrightarrow{f^*} C_X$ represents a morphism $X \xrightarrow{f} Y$.

In Section 2, we introduce the *spectrum of exact localizations*, or, shortly, the $\mathfrak{L}$-*spectrum*, of a 'space' and discuss its functorial properties.

In Section 3, we define $\mathfrak{L}$-local 'spaces' and show that the localization at a 'point' of the $\mathfrak{L}$-spectrum is an $\mathfrak{L}$-local 'space'.

In Section 4, we introduce the *complete* $\mathfrak{L}$*-spectrum* and show its functoriality with respect to exact localizations.

In Section 5, we define the *closed spectrum* (resp. the *complete closed spectrum*) and the *flat spectrum* (resp. the *complete flat spectrum*) of a 'space'.

In Section C1 (here 'C' stands for 'complemetary facts'), we introduce *cosubspaces* and *coimmertions*. These notions are mainly used in the case of 'spaces' represented by abelian categories.

The second part is dedicated to the spectra of 'spaces' represented by abelian categories. In Section 6.1, we translate the content of Sections 2, 3, and 4 to the language of thick subcategories (using a natural isomorphism between the preorder of thick subcategories of an abelian category and the preorder of its saturated multiplicative systems). In Section 6.2, we notice that this isomorphism induces an isomorphism between the preorder of left closed multiplicative systems and coreflective thick subcategories which leads to the description of the closed spectrum (introduced in Section 4) in terms of coreflective thick subcategories. In Section 6.3, we remind the notion of a *Serre subcategory* (as it is defined in [R1]) and introduce the corresponding spectra, $Spec^0_{\mathfrak{s}}(X)$ and $\mathbf{Spec}^1_{\mathfrak{s}}(X)$. If C_X is a Grothendieck category, then Serre subcategories are precisely coreflective thick subcategories and the spectrum $Spec^0_{\mathfrak{s}}(X)$ is isomorphic to the closed spectrum.

In Section 6.4, we establish, among other facts, the functoriality of spectra with respect to coimmersions.

In Section 6.5, we study $\mathfrak{L}$-local 'spaces' represented by abelian categories. We show that if X is $\mathfrak{L}$-local, then all simple objects of the category C_X (if any) are isomorphic to each other. In the case when C_X does have simple objects, this fact allows to associate with the local 'space' X a skew field, $k(X)$, defined uniquely up to isomorphism, which we call the *residue skew field* of X.

In Sections 7, we study spectra associated with the preorder of topologizing subcategories of an abelian category C_X and recover the spectrum, $\mathbf{Spec}(X)$, defined in [R1] (see also [R, Ch.3]) and mentioned above.

In Section 8, we define *local* 'spaces' as 'spaces' X such that the category C_X has the smallest non-trivial topologizing subcategory. This notion appeares in a different disguise in [R1]. Thick subcategories of a category C_X such that the corresponding quotient categories are local are points of the so called *complete spectrum* of X introduced in [R, Ch.6]. Taking the intersection of the complete spectrum with Serre subcategories, we obtain the spectrum $\mathbf{Spec}^-(X)$ which, also, made its first appearance in [R, Ch.6].

In Section 8.7, we show that if C_X is an abelian category with a Gabriel-Krull dimension, then the spectrum $\mathbf{Spec}^-(X)$ coincides with the Serre spectrum $\mathbf{Spec}^1_{\mathfrak{s}}(X)$ (defined in 6.3). In general, the latter is much larger.

In Section C2, the spectrum $\mathbf{Spec}^0_{\mathfrak{s}}(X)$ is extended to general 'spaces'.

The third part contains a fragment of noncommutative (categorical) local algebra. The results obtained here are not used in the rest of the work.

Section 9 treats the case of general 'spaces': we introduce and study supports and associated points of a class of arrows of an arbitrary category.

In Section 10, we apply and extend the notions and facts of Section 9 to studying supports and associated points of objects of an abelian category.

The main objective of the last part of the work is to define meaningful spectra of a ('space' represented by a) triangulated category.

In Section 11, we extend the notions of the spectra to the case of categories with an action of a monoidal category. This material, important by itself, is used futher only in the simplest case of so called $\mathbb{Z}$-categories, in order to give a background to spectral theory of triangulated categories.

Section 12 is dedicated to the spectra of triangulated categories. We follow the pattern of the corresponding constructions for abelian categories replacing thick subcategories by thick triangulated subcategories.

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I. General 'spaces' and their spectra.

1. Preliminaries on 'spaces' and localizations.

1.1. Categories and 'spaces'. As usual, Cat, or $Cat_{\mathfrak{U}}$ denotes the bicategory of categories which belong to a fixed universum $\mathfrak{U}$. Objects of the opposite category, Cat^{op}, are called 'spaces'. For any 'space' X, the corresponding category C_X is regarded as the category of quasi-coherent sheaves on X. For any 1-morphism $X \xrightarrow{f} Y$ of 'spaces', we denote by f^* the corresponding functor $C_Y \longrightarrow C_X$ and call it the *inverse image functor* of f. For any $\mathfrak{U}$-category $\mathcal{A}$, we denote by $|\mathcal{A}|$ the 'space' defined by $C_{|\mathcal{A}|} = \mathcal{A}$.

We denote by $|Cat|^o$ the category having same objects as Cat^{op}. Morphisms from X to Y are isomorphism classes of functors $C_Y \longrightarrow C_X$. For a morphism $X \xrightarrow{f} Y$, we denote by f^* any functor $C_Y \longrightarrow C_X$ representing f and call it an *inverse image functor of* f. We shall write $f = [F]$ to indicate that f is a morphism having an inverse image functor F. The composition of morphisms $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ is defined by $g \circ f = [f^* \circ g^*]$. The map which assigns to any functor $C_Y \xrightarrow{F} C_X$ the morphism $X \xrightarrow{[F]} Y$ is a functor $Cat^{op} \longrightarrow |Cat|^o$ which turns Cat^{op} into a fibered category over $|Cat|^o$.

1.2. Localizations and conservative morphisms. Let Y be an object of $|Cat|^o$ and Σ a class of arrows of the category C_Y. We denote by $\Sigma^{-1}Y$ the object of $|Cat|^o$ such that the corresponding category coincides with (the standard realization of) the quotient of the category C_Y by Σ (cf. [GZ, 1.1]): $C_{\Sigma^{-1}Y} = \Sigma^{-1}C_Y$. The canonical *localization functor* $C_Y \xrightarrow{p_\Sigma^*} \Sigma^{-1}C_Y$ is regarded as an inverse image functor of a morphism, $\Sigma^{-1}Y \xrightarrow{p_\Sigma} Y$.

For any morphism $X \xrightarrow{f} Y$ in $|Cat|^o$, we denote by Σ_f the family of all arrows s of the category C_Y such that $f^*(s)$ is invertible (notice that Σ_f does not depend on the choice of an inverse image functor f^*). Thanks to the universal property of localizations, f^* is represented as the composition of the localization functor $p_f^* = p_{\Sigma_f}^* : C_Y \longrightarrow \Sigma_f^{-1}C_Y$ and a uniquely determined functor $\Sigma^{-1}C_Y \xrightarrow{f_{\mathfrak{c}}^*} C_X$. In particular, $f = p_f \circ f_{\mathfrak{c}}$ for a uniquely determined morphism $X \xrightarrow{f_{\mathfrak{c}}} \Sigma_f^{-1}Y$.

A morphism $X \xrightarrow{f} Y$ is called *conservative* if Σ_f consists of isomorphisms only, or, equivalently, p_f is an isomorphism.

A morphism $X \xrightarrow{f} Y$ is called a *localization* if $f_{\mathfrak{c}}$ is an isomorphism, i.e. the functor $f_{\mathfrak{c}}^*$ is an equivalence of categories.

For a general morphism $X \xrightarrow{f} Y$, it follows from the universal property of localizations that $f = p_f \circ f_{\mathfrak{c}}$ is a decomposition of f into a localization and a conservative morphism.

1.3. Multiplicative systems. A family of arrows Σ of a category C_X is called a *left multiplicative system* if it has the following properties:

(S1) Σ is closed under composition and contains all identical arrows of C_X.

(SL2) Every diagram $M' \xleftarrow{s} M \xrightarrow{f} L$, where $s \in \Sigma$, can be completed to a commutative square

$$\begin{array}{ccc} M & \xrightarrow{f} & L \\ s \downarrow & & \downarrow s' \\ M' & \xrightarrow{f'} & L' \end{array}$$

where $s' \in \Sigma$.

(SL3) If $M \overset{f}{\underset{g}{\rightrightarrows}} N$ is a pair of arrows such that $f \circ s = g \circ s$ for some $s \in \Sigma$, then there exists a morphism $N \xrightarrow{t} N'$ of Σ such that $t \circ f = t \circ g$.

A family $\Sigma \subseteq HomC_X$ is a *right multiplicative system* if it has dual properties. Finally, Σ is called a *multiplicative system* if it is both right and left multiplicative.

We denote by $\mathcal{SM}_\ell(X)$ (resp. by $\mathcal{SM}_\mathfrak{r}(X)$) the family of all left (resp. right) multiplicative systems in C_X. We denote by $\mathcal{SM}(X)$ the family $\mathcal{SM}_\ell(X) \bigcap \mathcal{SM}_\mathfrak{r}(X)$ of all multiplicative systems in C_X.

We regard $\mathcal{SM}_\ell(X)$, $\mathcal{SM}_\mathfrak{r}(X)$, and $\mathcal{SM}(X)$ as preorders with respect to $\subseteq$.

1.3.1. Saturation. Let Σ be a family of morphisms of the category C_X. Let q_Σ be the localization morphism $\Sigma^{-1}X \longrightarrow X$ and $C_X \xrightarrow{q^*_\Sigma} C_{\Sigma^{-1}X} = \Sigma^{-1}C_X$ its canonical inverse image functor.

The family $\Sigma^\mathfrak{s} = \Sigma_{q_\Sigma}$ of all arrows of C_X which q^*_Σ transfers into isomorphisms (cf. 1.2) is called the *saturation* of Σ. A family of arrows Σ is called *saturated* if it coincides with its saturation.

1.3.2. Generalities on saturated families of arrows. It follows from the universal property of localizations, that for any morphism $Y \xrightarrow{f} X$, the family Σ_f of all arrows of C_X which f^* transforms to isomorphisms (see 1.2) is saturated. In particular, the saturation of any family of arrows is saturated.

Any set, $\{Y_i \xrightarrow{f_i} X \mid i \in J\}$, of morphisms of 'spaces' defines uniquely a morphism $\mathcal{Y} = \coprod_{i \in J} Y_i \xrightarrow{\mathbf{f}} X$ with an inverse image

$$C_X \xrightarrow{\mathbf{f}^*} C_\mathcal{Y} = \prod_{i \in J} C_{Y_i}$$

uniquely determined by a choice of inverse images, $C_X \xrightarrow{f_i^*} C_{Y_i}$, of morphisms f_i, $i \in J$. Evidently, $\Sigma_\mathbf{f} = \bigcap_{i \in J} \Sigma_{f_i}$. This shows that the intersection of any set of saturated families of morphisms is saturated.

1.3.3. Saturation of multiplicative systems. If Σ is a left multiplicative system, then its saturation, $\Sigma^\mathfrak{s}$, consists of all morphisms $L \xrightarrow{u} M$ which can be inserted in a commutative diagram of the form

$$\begin{tikzcd} L \arrow[r, "u"] \arrow[d, "s"'] & M \arrow[d, "t"] \arrow[dl, "v"] \\ V \arrow[r, "u_1"] & W \end{tikzcd}$$

where $s, t \in \Sigma$ (see [GZ, 1.1.3.5]).

It follows from this description that the saturation of a (left and right) multiplicative system Σ coincides with all arrows $s \in HomC_X$ such that there exist morphisms $\mathfrak{u}$ and $\mathfrak{v}$ such that $\mathfrak{u} \circ s \in \Sigma \ni s \circ \mathfrak{v}$.

1.3.3.1. Proposition. *The saturation of a (left and right) multiplicative system is a multiplicative system.*

Proof. Let Σ be a multiplicative system. It suffices to show that the saturation, $\Sigma^{\mathfrak{s}}$, of Σ has the properties (SL2) and (SL3).

Let $M \xrightarrow{s} M'$ be an element of $\Sigma^{\mathfrak{s}}$; i.e. there exist morphisms $M' \xrightarrow{\mathfrak{u}} M''$ and $N \xrightarrow{\mathfrak{v}} M$ such that $\mathfrak{u} \circ s \in \Sigma \ni s \circ \mathfrak{v}$. And let $M \xrightarrow{f} L$ be an arbitrary morphism.

By the property (SL3), the diagram $M'' \xleftarrow{\mathfrak{u} \circ s} M \xrightarrow{f} L$ can be inserted in a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & L \\ {\scriptstyle \mathfrak{u} \circ s}\downarrow & & \downarrow{\scriptstyle s'} \\ M'' & \xrightarrow{f'} & L' \end{array} \qquad (1)$$

where $s' \in \Sigma$. The diagram (1) can be rewritten as

$$\begin{array}{ccc} M & \xrightarrow{f} & L \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s'} \\ M' & \xrightarrow{f' \circ \mathfrak{u}} & L' \end{array}$$

which proves (SL2).

Let $M' \underset{g}{\overset{f}{\rightrightarrows}} L$ is a pair of arrows such that $f \circ s = g \circ s$. In particular, $f \circ (s \circ \mathfrak{v}) = g \circ (s \circ \mathfrak{v})$. Since $s \circ \mathfrak{v} \in \Sigma$, there exists (by the property (SL3)) a morphism $L \xrightarrow{t} L'$ of Σ such that $t \circ f = t \circ g$. ■

1.3.3.2. Note. The analogous assertion is not true, in general, for left (or right) multiplicative systems. It is true, however, if the category C_X has finite colimits (finite limits in the case of right multiplicative systems); see 1.4.1(b) and 1.4.2 below.

1.3.4. Notations. We denote by $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_\ell(X)$ (resp. by $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{r}}(X)$) the family of all saturated left (resp. right) multiplicative systems in C_X.

We denote by $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ the family of all saturated (left and right) multiplicative systems in C_X; that is $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) = \mathcal{S}^{\mathfrak{s}}\mathcal{M}_\ell(X) \bigcap \mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{r}}(X)$.

We regard $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_\ell(X)$, $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{r}}(X)$, and $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ as preorders with respect to $\subseteq$.

It follows from 1.3.3.1 that the saturation, $\Sigma \longmapsto \Sigma^{\mathfrak{s}}$ induces a functor

$$\mathcal{SM}(X) \longrightarrow \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$$

which is left adjoint to the inclusion functor $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) \longrightarrow \mathcal{SM}(X)$.

1.4. Left exact, right exact, and exact morphisms. A morphism $X \xrightarrow{f} Y$ is called *right exact* (resp. *left exact*, resp *exact*), if its inverse image functor preserves colimits (resp. limits, resp. both limits and colimits) of arbitrary finite diagrams.

The following assertion is a reformulation of Propositions 1.3.1 and 1.3.4 in [GZ].

1.4.1. Proposition. *(a) Let Σ be a left multiplicative system in C_X. Then the canonical morphism $\Sigma^{-1}X \xrightarrow{q_\Sigma} X$ is right exact.*

(b) Let $f = p_f \circ f_\mathfrak{c}$ be the canonical decomposition of a morphism $X \xrightarrow{f} Y$ into a conservative morphism $X \xrightarrow{f_\mathfrak{c}} \Sigma_f^{-1}Y$ and a localization $\Sigma_f^{-1}Y \xrightarrow{p_f} Y$. Suppose C_Y has finite limits (resp. finite colimits). Then f is left exact (resp. right exact) iff the family of arrows Σ_f is a left (resp. right) multiplicative system. In this case both the localization p_f and the conservative morphism $f_\mathfrak{c}$ are left (resp. right) exact.

In particular, if the category C_Y has limits and colimits of finite diagrams, then f is exact iff both the localization p_f and the conservative component $f_\mathfrak{c}$ are exact. The exactness of p_f is equivalent to that $\Sigma_f \in \mathcal{S}^\mathfrak{s}\mathcal{M}(X)$.

1.4.2. Corollary. *Suppose the category C_X has finite colimits. Then the saturation map, $\Sigma \longmapsto \Sigma^\mathfrak{s}$ induces a functor $\mathcal{S}\mathcal{M}_\ell(X) \longrightarrow \mathcal{S}^\mathfrak{s}\mathcal{M}_\ell(X)$ which is left adjoint to the corresponding inclusion functor $\mathcal{S}^\mathfrak{s}\mathcal{M}_\ell(X) \longrightarrow \mathcal{S}\mathcal{M}_\ell(X)$.*

1.4.3. Corollary. *Suppose C_X has finite colimits. Then the intersection of any set of saturated left multiplicative systems is a saturated left multiplicative system.*

1.5. Continuous morphisms and flat morphisms. A morphism f of $|Cat|^o$, or Cat^{op}, is called *continuous* if its inverse image functor has a right adjoint, f_*, which is called a *direct image functor* of f.

A morphism f is called *flat* if it is exact and continuous.

One can show that a morphism f is continuous iff both the localization p_f and the conservative component $f_\mathfrak{c}$ are continuous.

2. The $\mathfrak{L}$-spectrum.

Fix a 'space' X. Recall that $\mathcal{S}^\mathfrak{s}\mathcal{M}(X)$ denote the preorder (with resp. to $\subseteq$) of all saturated (left and right) multiplicative systems of the category C_X. The preorder $\mathcal{S}^\mathfrak{s}\mathcal{M}(X)$ has the initial object – the family $Iso(C_X)$ of all isomorphisms of C_X. Let $\mathcal{S}^\mathfrak{s}\mathcal{M}^\star(X)$ denote $\mathcal{S}^\mathfrak{s}\mathcal{M}(X) - \{Iso(C_X)\}$.

For any $\Sigma \subseteq HomC_X$, denote by $\widehat{\Sigma}$ the union of all saturated multiplicative systems of C_X which do not contain Σ. It follows that if $\Sigma_1 \subseteq \Sigma_2$, then $\widehat{\Sigma_2} \subseteq \widehat{\Sigma_1}$. Notice that if Σ_1 and Σ_2 are saturated multiplicative systems, then the inverse implication holds, i.e. $\Sigma_1 \subseteq \Sigma_2$ iff $\widehat{\Sigma_2} \subseteq \widehat{\Sigma_1}$.

2.1. Definition. The $\mathfrak{L}$-*spectrum*, $\mathbf{Spec}^0_\mathfrak{L}(X)$, of X consists of all saturated multiplicative systems Σ such that $\widehat{\Sigma}$ is a saturated multiplicative system.

In other words, elements of $\mathbf{Spec}^0_\mathfrak{L}(X)$, are saturated multiplicative systems Σ such that there exists the biggest saturated multiplicative system, $\widehat{\Sigma}$, which does not contain Σ. In particular, $\mathbf{Spec}^0_\mathfrak{L}(X) \subseteq \mathcal{S}^\mathfrak{s}\mathcal{M}^\star(X)$.

2.1.1. Note. If C_X is a groupoid, then $\mathcal{S}^{\mathfrak{s}}\mathcal{M}^\star(X)$ is empty, hence $\mathbf{Spec}^0_{\mathfrak{L}}(X) = \emptyset$.

2.1.2. Specialization preorder. We call the preorder, $\supseteq$, on $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ the *specialization preorder*: Σ is a *specialization* of Σ' if $\Sigma \subseteq \Sigma'$.

It follows that if Σ, Σ' are elements of $\mathbf{Spec}^0_{\mathfrak{L}}(X)$, then Σ is a specialication of Σ' iff the saturated multiplicative system $\widehat{\Sigma}$ is a specialization of $\widehat{\Sigma'}$.

2.2. Functorial properties of the $\mathfrak{L}$-spectrum. Let $\mathfrak{L}_{\mathfrak{e}}\mathfrak{Esp}$ denote the subcategory of $|Cat|^o$ formed by exact localizations (cf. 1.2). Since identical morphisms are exact localizations, $Ob\mathfrak{L}_{\mathfrak{e}}\mathfrak{Esp} = Ob|Cat|^o$. Let $\mathfrak{POrd}_\star$ denote the category of preorders with initial objects; its morphisms are morphisms of preorders mapping initial objects to initial objects.

2.2.1. Lemma. *The map $X \longmapsto \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ gives a rise to a contravariant functor*

$$\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{o}} : \mathfrak{L}_{\mathfrak{e}}\mathfrak{Esp}^{op} \longrightarrow \mathfrak{POrd}_\star$$

and to a covariant functor

$$\mathcal{S}^{\mathfrak{s}}\mathcal{M} : \mathfrak{L}_{\mathfrak{e}}\mathfrak{Esp} \longrightarrow \mathfrak{POrd}_\star.$$

Proof. Let $X \xrightarrow{u} Y$ be an exact localization and $C_Y \xrightarrow{u^*} C_X$ its inverse image functor. Set $\Sigma_u = \Sigma_{u^*} = \{s \in HomC_Y \mid u^*(s) \in IsoC_X\}$. The functor u^* induces a map

$$\mathcal{S}^{\mathfrak{s}}\mathcal{M}(Y) \xrightarrow{\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{o}}(u)} \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$$

which assigns to a family $\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(Y)$ the minimal saturated multiplicative system containing $u^*(\Sigma)$, and a map

$$\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) \xrightarrow{\mathcal{S}^{\mathfrak{s}}\mathcal{M}(u)} \mathcal{S}^{\mathfrak{s}}\mathcal{M}(Y)$$

which sends any saturated multiplicative system Σ' to its preimage, ${u^*}^{-1}(\Sigma')$. Notice that $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{o}}(u) \circ \mathcal{S}^{\mathfrak{s}}\mathcal{M}(u)$ is the identical map. This shows that $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{o}}(u)$ and $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(u)$ induce an isomorphism between $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ and the preorder $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\Sigma_u}(Y)$ of saturated multiplicative systems of C_Y containing Σ_u. Notice that the map $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{o}}(u)$ can be represented as the composition of the map

$$\mathcal{S}^{\mathfrak{s}}\mathcal{M}(Y) \longrightarrow \mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\Sigma_u}(Y), \quad \Sigma \longmapsto \Sigma \vee \Sigma_u,$$

and the restriction of $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{o}}(u)$ to $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\Sigma_u}(Y)$ (the inverse to $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(u)$). It is easy to see that both maps, $u \longmapsto \mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{o}}(u)$ and $u \longmapsto \mathcal{S}^{\mathfrak{s}}\mathcal{M}(u)$ are functorial. ■

2.2.2. Extended $\mathfrak{L}$-spectrum. For any 'space' X, set $\mathbf{Spec}^0_{\mathfrak{L}\star}(X) = \mathbf{Spec}^0_{\mathfrak{L}}(X) \cup \{\star_X\}$, where $\star_X = Iso(C_X)$. We call $\mathbf{Spec}^0_{\mathfrak{L}\star}(X)$ the *extended spectrum* of X. Notice that

$\widehat{Iso(C_X)} = \emptyset$. Thus, the added trivial multiplicative system $\star_X$ can be viewed as ∞ (with respect to the specialization preorder $\supseteq$).

2.2.3. Proposition. *Any exact localization $X \xrightarrow{u} Y$ induces a morphism of extended spectra $\mathbf{Spec}^0_{\mathfrak{L}\star}(Y) \longrightarrow \mathbf{Spec}^0_{\mathfrak{L}\star}(X)$. This correspondence defines a contravariant functor, $\mathbf{Spec}^0_{\mathfrak{L}\star}$, from the category $\mathfrak{L}_{\mathfrak{e}}\mathfrak{Esp}$ to the category $\mathfrak{POrd}_\star$ of preorders with initial objects.*

Proof. Fix an inverse image functor, $C_Y \xrightarrow{u^*} C_X$, of the morphism u. The map

$$\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{o}}(u) : \mathcal{S}^{\mathfrak{s}}\mathcal{M}(Y) \longrightarrow \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$$

(cf. 2.2.1) induces a morphism of spectra $\mathbf{Spec}^0_{\mathfrak{L}\star}(Y) \longrightarrow \mathbf{Spec}^0_{\mathfrak{L}\star}(X)$.

In fact, let $\Sigma_P \in \mathbf{Spec}^0_{\mathfrak{L}} Y$ and $\Sigma_P \not\subseteq \Sigma_u$. Then $\Sigma_u \subseteq \widehat{\Sigma}_P$. By (the argument of) 2.2.1, the map $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{o}}(u)$ induces an isomorphism between $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ and the preorder $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\Sigma_u}(Y)$ of saturated multiplicative systems of C_Y containing Σ_u. In particular, the image, $\widehat{\Sigma}'_P$, of $\widehat{\Sigma}_P$ is a saturated multiplicative system. It follows that $\widehat{\Sigma}'_P$ is the biggest saturated multiplicative system in C_X which does not contain the image, Σ'_P, of the saturated multiplicative system $\Sigma_u \vee \Sigma_P$. In fact, if $\Sigma'_P \not\subseteq \Sigma'$ for some saturated multiplicative system Σ', then Σ_P is not contained in the saturated multiplicative system $\Sigma = {u^*}^{-1}(\Sigma')$. Therefore $\Sigma \subseteq \widehat{\Sigma}_P$, hence the assertion. ∎

2.2.4. Remarks. (a) For any $S \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$, let $\mathcal{U}_{\mathfrak{L}}(S)$ denote $\{\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid \Sigma \not\subseteq S\}$ It follows that $\mathcal{U}_{\mathfrak{L}}(S) = \{\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid S \subseteq \widehat{\Sigma}\}$.

The argument of 2.2.3 proves that any exact localization, $X \xrightarrow{u} Y$, induces an injective map $\mathcal{U}_{\mathfrak{L}}(\Sigma_u) \longrightarrow \mathbf{Spec}^0_{\mathfrak{L}} Y$.

(b) In general, the map $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) \xrightarrow{\mathcal{S}^{\mathfrak{s}}\mathcal{M}(u)} \mathcal{S}^{\mathfrak{s}}\mathcal{M}(Y)$ corresponding to an exact localization $X \xrightarrow{u} Y$ does not induce a map $\mathbf{Spec}^0_{\mathfrak{L}\star}(X) \longrightarrow \mathbf{Spec}^0_{\mathfrak{L}\star}(Y)$.

For any exact localization $U \xrightarrow{u} X$, set $\mathbf{Spec}^0_{\mathfrak{L}}(U; X) = \{\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid \Sigma_u \subseteq \widehat{\Sigma}\}$.

2.2.5. Proposition. *Let $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ be a conservative set of exact localizations. Then* $\mathbf{Spec}^0_{\mathfrak{L}}(X) = \bigcup_{i \in J} \mathbf{Spec}^0_{\mathfrak{L}}(U_i; X)$.

Proof. By hypothesis, the family of localization functors, $\{C_X \xrightarrow{u_i^*} C_{U_i} \mid i \in J\}$, is conservative, i.e. $\bigcap_{i \in J} \Sigma_{u_i} = Iso(C_X)$. Therefore, for every $\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X)$, there exists $i \in J$ such that $\Sigma \not\subseteq \Sigma_{u_i}$ which means precisely that $\Sigma_{u_i} \subseteq \widehat{\Sigma}$ and the image of $\Sigma \vee \Sigma_{u_i}$ in $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(U_i)$ belongs to $\mathbf{Spec}^0_{\mathfrak{L}}(U_i)$ (see 2.2.3 and 2.2.4(a)), hence the assertion. ∎

3. $\mathfrak{L}$-Local 'spaces' and the spectrum $\mathbf{Spec}^0_{\mathfrak{L}}(X)$.

We call a 'space' X *$\mathfrak{L}$-local* (here $\mathfrak{L}$- stands for 'localization'), if $\mathcal{S}^{\mathfrak{s}}\mathcal{M}^\star(X)$ has the smallest element, or, equivalently, the intersection, Σ^X, of all non-trivial saturated multiplicative systems is a non-trivial saturated multiplicative system.

3.1. Proposition. *The following conditions on a 'space' X are equivalent:*

(a) The 'space' X is $\mathfrak{L}$-local.

(b) The family of arrows $\Sigma^X = \bigcap_{\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}^\star(X)} \Sigma$ *belongs to* $\mathbf{Spec}^0_{\mathfrak{L}}(X)$.

(c) The spectrum $\mathbf{Spec}^0_{\mathfrak{L}}(X)$ *has an element,* Σ', *such that* $\widehat{\Sigma'} = Iso(C_X)$.

Proof. $(a) \Rightarrow (b)$ & $(a) \Rightarrow (c)$: If X is $\mathfrak{L}$-local and Σ is a saturated multiplicative system, then $\Sigma^X \nsubseteq \Sigma$ iff $\Sigma \notin \mathcal{S}^{\mathfrak{s}}\mathcal{M}^\star(X)$, that is if $\Sigma = Iso(C_X)$.

$(b) \Rightarrow (a)$ follows from definitions: if $\Sigma^X \in \mathbf{Spec}^0_{\mathfrak{L}}(X)$, then $\Sigma^X \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}^\star(X)$, hence X is $\mathfrak{L}$-local.

$(c) \Rightarrow (a)$: Let $\Sigma' \in \mathbf{Spec}^0_{\mathfrak{L}}(X)$ be such that $\widehat{\Sigma'} = Iso(C_X)$. Then $\Sigma' \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}^\star(X)$ and Σ' is contained in any non-trivial saturated multiplicative system, i.e. $\Sigma' = \Sigma^X$. ∎

3.2. Proposition. *For any* $\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X)$, *the 'space'* $\widehat{\Sigma}^{-1}X$ *is $\mathfrak{L}$-local.*

Proof. The localization functor $C_X \xrightarrow{q_P^*} \widehat{\Sigma}^{-1}C_X$ induces an isomorphism between the preorder of (non-trivial) saturated multiplicative systems of $C_{\widehat{\Sigma}^{-1}X} = \widehat{\Sigma}^{-1}C_X$ and saturated multiplicative systems of C_X which contain $\widehat{\Sigma}$ properly. Since every saturated multiplicative system which contains $\widehat{\Sigma}$ properly contains Σ as well, the preorder of saturated multiplicative systems properly containing $\widehat{\Sigma}$ coincides with the preorder of saturated multiplicative systems containing $\widehat{\Sigma} \vee \Sigma$. Therefore the image of $\widehat{\Sigma} \vee \Sigma$ in $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(\widehat{\Sigma}^{-1}X)$ is the smallest element of $\mathcal{S}^{\mathfrak{s}}\mathcal{M}^\star(\widehat{\Sigma}^{-1}X)$. ∎

4. The complete $\mathfrak{L}$-spectrum.

For any 'space' X, we define its *complete $\mathfrak{L}$-spectrum*, $\mathbf{Spec}^1_{\mathfrak{L}}(X)$, as follows. Elements of $\mathbf{Spec}^1_{\mathfrak{L}}(X)$ are saturated multiplicative systems, Σ_x, of C_X such that the 'space' of fractions $\Sigma_x^{-1}X$ is $\mathfrak{L}$-local. In other words, elements of $\mathbf{Spec}^1_{\mathfrak{L}}(X)$ are saturated multiplicative systems, Σ_x, such that the intersection of all saturated multiplicative systems properly containing Σ_x is a saturated multiplicative system which contains Σ_x properly too. We consider $\mathbf{Spec}^1_{\mathfrak{L}}(X)$ together with the preorder $\subseteq$. By 3.2, there is a morphism $\mathbf{Spec}^0_{\mathfrak{L}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{L}}(X)$ defined by $\Sigma \longmapsto \widehat{\Sigma}$.

4.1. Proposition. *The map* $X \longmapsto \mathbf{Spec}^1_{\mathfrak{L}}(X)$ *extends to a functor,* $\mathbf{Spec}^1_{\mathfrak{L}}$, *from the category* $\mathfrak{L}_{\mathfrak{e}}\mathfrak{Esp}$ *to the category* $\mathfrak{POrd}$ *of preorders.*

Proof. The map $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) \xrightarrow{\mathcal{S}^{\mathfrak{s}}\mathcal{M}(u)} \mathcal{S}^{\mathfrak{s}}\mathcal{M}(Y)$, $\Sigma \longmapsto u^{*^{-1}}(\Sigma)$, corresponding to an exact localization $X \xrightarrow{u} Y$ induces a map $\mathbf{Spec}^1_{\mathfrak{L}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{L}}(Y)$.

Indeed, $\Sigma^{-1}X \simeq \mathcal{S}^{\mathfrak{s}}\mathcal{M}(u)(\Sigma)^{-1}Y$, so that $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(u)(\Sigma)^{-1}Y$ is $\mathfrak{L}$-local if $\Sigma^{-1}X$ is $\mathfrak{L}$-local, hence the assertion. ∎

4.2. Note. For any 'space' X,

$$\mathbf{Spec}^1_{\mathfrak{L}}(X) = \bigcup_{\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)} \mathbf{Spec}^0_{\mathfrak{L}}(\Sigma^{-1}X) = \bigcup_{\Sigma \in \mathbf{Spec}^1_{\mathfrak{L}}(X)} \mathbf{Spec}^0_{\mathfrak{L}}(\Sigma^{-1}X).$$

Here $\mathbf{Spec}^1_{\mathfrak{L}}(\Sigma^{-1}X)$ is identified with its image in $\mathbf{Spec}^1_{\mathfrak{L}}(X)$.

4.3. The extended complete $\mathfrak{L}$-spectrum. For any 'space' X, set $\mathbf{Spec}^1_{\mathfrak{L}\star}(X) = \mathbf{Spec}^1_{\mathfrak{L}}(X) \cup \{\star_X\}$, where $\star_X = Iso(C_X)$. We call $\mathbf{Spec}^1_{\mathfrak{L}\star}(X)$ the *extended complete $\mathfrak{L}$-spectrum* of X.

4.3.1. Proposition. *The map $X \longmapsto \mathbf{Spec}^1_{\mathfrak{L}}(X)$ gives rise to a contravariant functor*

$$^a\mathcal{S}\mathfrak{p} : \mathfrak{L}_\mathfrak{e}\mathfrak{Esp}^{op} \longrightarrow \mathfrak{POrd}_\star$$

and to a covariant functor

$$_a\mathcal{S}\mathfrak{p} : \mathfrak{L}_\mathfrak{e}\mathfrak{Esp} \longrightarrow \mathfrak{POrd}_\star$$

to the category $\mathfrak{POrd}_\star$ of preorders with initial objects.

Proof. The functor $_a\mathcal{S}\mathfrak{p} : \mathfrak{L}_\mathfrak{e}\mathfrak{Esp} \longrightarrow \mathfrak{POrd}_\star$ is the unique extension of the functor $\mathbf{Spec}^1_{\mathfrak{L}} : \mathfrak{L}_\mathfrak{e}\mathfrak{Esp} \longrightarrow \mathfrak{POrd}$ of 4.1.

Let $X \xrightarrow{u} Y$ be an exact localization. We define the map

$$^a\mathcal{S}\mathfrak{p}(u) : \mathbf{Spec}^1_{\mathfrak{L}}(Y) \longrightarrow \mathbf{Spec}^1_{\mathfrak{L}}(X)$$

as follows. Let $\Sigma_x \in \mathbf{Spec}^1_{\mathfrak{L}}(Y)$. If $\Sigma_u \subseteq \Sigma_x$, then $^a\mathcal{S}\mathfrak{p}(u)(\Sigma_x)$ is the minimal saturated multiplicative system containing $u^*(\Sigma_x)$. By transitivity of localizations, $^a\mathcal{S}\mathfrak{p}(u)(\Sigma_x) \in \mathbf{Spec}^1_{\mathfrak{L}}(X)$. If $\Sigma_u \not\subseteq \Sigma_x$, then $^a\mathcal{S}\mathfrak{p}(u)$ maps Σ_x to the trivial family, $\star_X = Iso(C_X)$. It is easy to check that the map $u \longmapsto ^a \mathcal{S}\mathfrak{p}(u)$ is functorial. ∎

4.4. Remark. The dualization functor, $X \longmapsto X^o$, establishes an isomorphism between the preorder of left (saturated) multiplicative systems on X and right (saturated) multiplicative systems on X^o. This isomorphism induces an isomorphism of preorders of (saturated) multiplicative systems:

$$\mathcal{SM}(X) \xrightarrow{\sim} \mathcal{SM}(X^o) \quad \text{and} \quad \mathcal{S}^\mathfrak{s}\mathcal{M}(X) \xrightarrow{\sim} \mathcal{S}^\mathfrak{s}\mathcal{M}(X^o). \tag{1}$$

In particular, a 'space' X is $\mathfrak{L}$-local iff its dual, X^o is $\mathfrak{L}$-local.

Thus, the isomorphism $\mathcal{S}^\mathfrak{s}\mathcal{M}(X) \xrightarrow{\sim} \mathcal{S}^\mathfrak{s}\mathcal{M}(X^o)$ induces isomorphisms of spectra

$$\mathbf{Spec}^1_{\mathfrak{L}}(X) \xrightarrow{\sim} \mathbf{Spec}^1_{\mathfrak{L}}(X^o) \quad \text{and} \quad \mathbf{Spec}^0_{\mathfrak{L}}(X) \xrightarrow{\sim} \mathbf{Spec}^0_{\mathfrak{L}}(X^o). \tag{2}$$

as well as the extended versions of these spectra.

The spectra $\mathbf{Spec}^1_{\mathfrak{L}}(X)$ and $\mathbf{Spec}^0_{\mathfrak{L}}(X)$ are too large, which is one of the reasons why the duality (2) takes place. In the next section, we single out smaller spectra inside of $\mathbf{Spec}^1_{\mathfrak{L}}(X)$ and $\mathbf{Spec}^0_{\mathfrak{L}}(X)$.

5. Closed spectra and flat spectra.

5.1. Σ-Torsion free objects. Let $\Sigma \subseteq HomC_X$. We say that an object, M, of the category C_X is *Σ-torsion free* if every morphism $M \xrightarrow{t} N$ which belongs to Σ is a monomorphism. We denote by C_{X_Σ} the full subcategory of the category C_X whose objects are Σ-torsion free.

5.1.1. Lemma. *Let $\Sigma \subseteq HomC_X$ be such that for every diagram $\widetilde{L} \xleftarrow{s} L \xrightarrow{g} M$, where $s \in \Sigma$ and g is a monomorphism, there exists a commutative diagram*

$$\begin{array}{ccc} L & \xrightarrow{g} & M \\ s \downarrow & & \downarrow t \\ \widetilde{L} & \xrightarrow{\widetilde{g}} & \widetilde{M} \end{array} \qquad (1)$$

where $t \in \Sigma$ (e.g. Σ is a left multiplicative system).

Then any subobject of a Σ-free object is Σ-free.

Proof. Let $L \xrightarrow{g} M$ be a monomorphism, and $L \xrightarrow{s} \widetilde{L}$ a morphism of Σ. Then there exists a commutative diagram (1) in which $t \in \Sigma$. If M is Σ-torsion free, then $M \xrightarrow{t} \widetilde{M}$ is a monomorphism. Thus $t \circ g$ is a monomorphism. It follows from the equality $t \circ g = \widetilde{g} \circ s$ that s is a monomorphism, hence the assertion. ■

5.2. Closed families of morphisms and closed spectra. Let $\Sigma \subseteq HomC_X$. We say that Σ is *closed*, or *right closed*, if for every $M \in ObC_X$, there exists a morphism $M \longrightarrow \widetilde{M}$ of Σ such that $\widetilde{M} \in ObC_{X_\Sigma}$.

5.2.1. Proposition. *Let $\Sigma \subseteq HomC_X$ be a left saturated multiplicative system such that the canonical morphism $\Sigma^{-1}X \xrightarrow{q_\Sigma} X$ is continuous; and let q_Σ^* and $q_{\Sigma*}$ be resp. its inverse and direct image functors. Then the following conditions on an object M of C_X are equivalent:*

(i) M is Σ-torsion free.

(ii) An adjunction morphism $M \xrightarrow{\eta_\Sigma(M)} q_{\Sigma*}q_\Sigma^*(M)$ is a monomorphism.

Proof. Let Σ be a saturated multiplicative system and $C_X \xrightarrow{q_\Sigma^*} C_{\Sigma^{-1}X} = \Sigma^{-1}C_X$ a localization functor at Σ. If the family Σ is flat, the functor q_Σ^* has a right adjoint, $q_{\Sigma*}$. For every $M \in ObC_X$, the adjunction arrow, $M \xrightarrow{\eta_\Sigma(M)} q_{\Sigma*}q_\Sigma^*(M)$, belongs to Σ. In particular, if M is a Σ-torsion free object, then the adjunction morphism $\eta_\Sigma(M)$ is a monomorphism.

Let $M \xrightarrow{s} N$ be a morphism from Σ. Then the upper horizontal arrow in the commutatative diagram

$$\begin{array}{ccc} q_{\Sigma*}q_\Sigma^*(M) & \xrightarrow{q_{\Sigma*}q_\Sigma^*(s)} & q_{\Sigma*}q_\Sigma^*(N) \\ \eta_\Sigma(M) \uparrow & & \uparrow \eta_\Sigma(N) \\ M & \xrightarrow{s} & N \end{array}$$

is an isomorphism. If $\eta_\Sigma(M)$ is a monomorphism, then $\eta_\Sigma(N) \circ s = q_{\Sigma*}q_\Sigma^*(s) \circ \eta_\Sigma(M)$ is a monomorphism. Therefore s is a monomorphism. This shows that the object M is Σ-torsion free. ■

5.2.2. Corollary. *Let Σ be a left saturated multiplicative system such that the canonical morphism $\Sigma^{-1}X \xrightarrow{q_\Sigma} X$ is continuous. Then Σ is closed.*

Proof. For every $M \in ObC_X$, the adjunction arrow, $M \xrightarrow{\eta_\Sigma(M)} q_{\Sigma *}q^*_\Sigma(M)$ belongs to Σ. If $\widetilde{M} = q_{\Sigma *}q^*_\Sigma(M)$, then the adjunction arrow $\eta_\Sigma(\widetilde{M})$ is an isomorphism, in particular it is a monomorphism. ■

5.2.3. Closed spectra. We denote by $\mathfrak{C}\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ the preorder of all closed saturated multiplicative systems on X. The *complete closed spectrum*, $\mathbf{Spec}^1_{\mathfrak{C}}(X)$, is defined by

$$\mathbf{Spec}^1_{\mathfrak{C}}(X) = \mathfrak{C}\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) \bigcap \mathbf{Spec}^1_{\mathfrak{L}}(X);$$

that is elements of $\mathbf{Spec}^1_{\mathfrak{C}}(X)$ are closed saturated multiplicative systems, Σ, such that $\Sigma^{-1}X$ is $\mathfrak{L}$-local.

We call $\mathbf{Spec}^0_{\mathfrak{C}}(X) = \{\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid \widehat{\Sigma} \in \mathbf{Spec}^1_{\mathfrak{C}}(X)\}$ the *closed spectrum* of X.

5.3. Continuous localizations and flat spectra. Let $\Sigma \subseteq HomC_X$. Recall that an object M of C_X is called *left closed for* Σ if $C_X(s, M)$ is a bijection for each morphism s of Σ [GZ, I.4].

5.3.1. Lemma. *(a) Let $\Sigma \subseteq HomC_X$, and let M be an object of C_X such that $C_X(s, M)$ is a surjection for each morphism s of Σ. Then every morphism $M \xrightarrow{t} N$ which belongs to Σ is a retraction (i.e. $u \circ t = id_M$ for some morphism u). In particular, M is Σ-torsion free.*

(b) Suppose for any diagram $\widetilde{L} \xleftarrow{s} L \xrightarrow{g} M$ such that $s \in \Sigma$, there exists a commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{f} & M \\ {\scriptstyle s}\big\downarrow & & \big\downarrow{\scriptstyle t} \\ \widetilde{L} & \xrightarrow{\widetilde{f}} & \widetilde{M} \end{array} \qquad (1)$$

where $t \in \Sigma$ (e.g. Σ is a left multiplicative system). Then $C_X(s, M)$ is a surjection for each morphism s of Σ iff every morphism $M \xrightarrow{t} N$ which belongs to Σ is a retraction.

Proof. (a) If $M \xrightarrow{t} N$ is a morphism of Σ, then the map

$$C_X(N, M) \longrightarrow C_X(M, M), \quad f \longmapsto f \circ t,$$

is surjective. In particular, there exists a morphism $N \xrightarrow{u} M$ such that $u \circ t = id_M$.

(b) Suppose that the object M is such that every morphism $M \longrightarrow N$ which belongs to Σ is a retraction. Then $C_X(s, M)$ is surjective for any morphism $L \xrightarrow{s} \widetilde{L}$ of Σ.

In fact, let $L \xrightarrow{f} M$ be an arbitrary morphism. By hypothesis, there is a commutative diagram (1), where $t \in \Sigma$. By condition, t is a retraction, i.e. there exists a morphism $\widetilde{M} \xrightarrow{u} M$ such that $u \circ t = id_M$. Then $(u \circ \widetilde{f}) \circ s = u \circ (t \circ f) = f$. This shows the surjectivity of $C_X(s, M)$. ■

5.3.2. Proposition. *Suppose that $\Sigma \subseteq HomC_X$ is a left multiplicative system. Then the following conditions on an object M of C_X are equivalent:*

(a) M is left closed for Σ;
(b) $C_X(s, M)$ is surjective for any $s \in \Sigma$;
(c) any morphism $M \longrightarrow N$ which belongs to Σ is a retraction.

Proof. The implications $(b) \Leftrightarrow (c)$ follow from 5.3.1(b). The implication $(a) \Rightarrow (b)$ holds by definition. The implication $(b) \Rightarrow (a)$ is proven in [GZ, 1.4.1.1]. ■

5.3.3. Localizations and continuous localizations. Let $X \xrightarrow{f} Y$ be a morphism with an inverse image functor $C_Y \xrightarrow{f^*} C_X$. An object N of C_Y is called f*-free* over an object M of C_X, if there exists a morphism $f^*(N) \xrightarrow{u} M$ such that for any morphism $f^*(L) \xrightarrow{v} M$ there exists a unique morphism $L \xrightarrow{\widetilde{v}} N$ satisfying $v = u \circ f^*(\widetilde{v})$. In other words, $(N, f^*(N) \xrightarrow{u} M)$ is a final object of the category f^*/M, or, what is the same, the object N represents the functor $C_X(f^*(-), M) : C_Y^{op} \longrightarrow Sets$. We denote by $C_{\mathcal{L}(f)}$ the full subcategory of the category C_Y generated by f-free objects.

Let $C_{\mathfrak{D}(f_*)}$ denote the full subcategory of C_X generated by all $M \in ObC_X$ such that the functor $C_X(f^*(-), M) : C_Y^{op} \longrightarrow Sets$ is representable. A choice for each $M \in ObC_{\mathfrak{D}(f_*)}$ of an object, $f_*(M)$, of the subcategory $C_{\mathcal{L}(f)}$ representing the functor $C_X(f^*(-), M)$ extends uniquely to a functor $\mathfrak{D}(f_*) \longrightarrow C_Y$ taking values in $C_{\mathcal{L}(f)}$. Let $C_{\mathfrak{R}(f_*)}$ denote ${f^*}^{-1}(\mathfrak{D}(f_*))$. The morphism f is continuous iff $\mathfrak{D}(f_*) = X$ and, therefore, $\mathfrak{R}(f_*) = Y$.

5.3.3.1. Proposition. *Suppose that $\Sigma \subseteq HomC_X$ is a left multiplicative system. And let $\Sigma^{-1}X \xrightarrow{q_\Sigma} X$ be the localization morphism. Then*

(a) $C_{\mathcal{L}(q_\Sigma)}$ is the full subcategory of C_X generatated by all objects which are left closed for Σ.

(b) The subcategory $C_{\mathfrak{R}(q_\Sigma)}$ is generated by all $M \in ObC_X$ such that there exists a morphism $M \xrightarrow{s} N$, where N is left closed for Σ and $q_\Sigma(s)$ is invertible.

(c) The composition of the inclusion $C_{\mathcal{L}(q_\Sigma)} \longrightarrow C_X$ and the canonical localization functor $C_X \xrightarrow{q_\Sigma^*} \Sigma^{-1}C_X$ is a fully faithful functor injective on objects. This functor induces an isomorphism $\mathfrak{D}(q_\Sigma) \xrightarrow{\sim} \mathcal{L}(q_\Sigma)$.

(d) The inclusion functor $C_{\mathcal{L}(q_\Sigma)} \xrightarrow{\widetilde{q}_{\Sigma*}} C_{\mathfrak{R}(q_\Sigma)}$ has a left adjoint, $\widetilde{q}_\Sigma^*$.

Proof. (a) Let $Y \xrightarrow{f} X$ be a morphism, M an object of C_Y such that the functor $C_(f^*(-), M)$ is representable. Then any object, N, representing $C_(f^*(-), M)$ is, obviously, left closed for $\Sigma_f = \{s \in HomC_X \mid f^*(s) \in Iso(C_Y)\}$.

If $f = q_\Sigma$, then the converse is true: if $N \in ObC_X$ is left closed for Σ, then it follows from the universal property of the localization at Σ that the object N represents the functor $C_{\Sigma^{-1}X}(q_\Sigma^*(-), q_\Sigma^*(N))$.

(b) By definition, the subcategory $C_{\mathfrak{R}(q_\Sigma)}$ is generated by all $M \in ObC_X$ such that the functor $C_{\Sigma^{-1}X}(q_\Sigma^*(-), q_\Sigma^*(M))$ is representable by some object, N, of the category C_X. In particular, there exists a canonical morphism $M \xrightarrow{t} N$ corresponding to the identical arrow $q_\Sigma^*(M) \longrightarrow q_\Sigma^*(M)$. It follows that $q_\Sigma^*(t)$ is an isomorphism.

(c) The canonical localization functor $C_X \xrightarrow{q^*_\Sigma} C_{\Sigma^{-1}X}$ is identical on objects, hence the composition of q^*_Σ with the inclusion functor $C_{\mathcal{L}(q_\Sigma)} \longrightarrow C_X$ is injective on objects. For any $M \in ObC_{\mathcal{L}(q_\Sigma)}$ and any $L \in ObC_X$, we have a functorial isomorphism $C_{\Sigma^{-1}X}(q^*_\Sigma(L), q^*_\Sigma(M)) \simeq C_X(L, M)$. In particular, the composition of the embedding $C_{\mathcal{L}(q_\Sigma)} \longrightarrow C_X$ with q^*_Σ is a fully faithful functor.

(d) By (b), for any $M \in ObC_{\mathfrak{R}(q_\Sigma)}$, there exists a morphism $M \xrightarrow{t} N$, where N is left closed for Σ. It follows from 5.3.2 that the object N here is defined uniquely up to isomorphism. A choice of N for every $M \in ObC_{\mathfrak{R}(q_\Sigma)}$ defines a functor, $\widetilde{q}^*_\Sigma$, from $C_{\mathfrak{R}(q_\Sigma)}$ to $C_{\mathcal{L}(q_\Sigma)}$. This functor is a left adjoint to the inclusion functor $C_{\mathcal{L}(q_\Sigma)} \longrightarrow C_{\mathfrak{R}(q_\Sigma)}$. ■

One of the corollaries of 5.3.3.1 is the following fact:

5.3.3.2. Proposition [GZ, 1.4.1]. *Suppose that $\Sigma \subseteq HomC_X$ is a left multiplicative system. Then the following conditions are equivalent:*

(a) The canonical morphism $\Sigma^{-1}X \xrightarrow{q_\Sigma} X$ is continuous.

(b) For every object M of the category C_X, there exists an object $\widetilde{M}$ left closed for Σ and a morphism $M \xrightarrow{s} \widetilde{M}$ such that $q_\Sigma(s)$ is invertible.

5.3.4. Continuous and flat multiplicative systems. We call $\Sigma \subseteq HomC_X$ *continuous* if it is a left multiplicative system and the canonical morphism $\Sigma^{-1}X \xrightarrow{q_\Sigma} X$ is continuous. It follows from [GZ, 1.1.3] that if Σ is continuous, then the saturation of Σ is a left multiplicative system, hence it is continuous. We denote by $\mathfrak{Lc}_\ell(X)$ the preorder of all continuous left saturated multiplicative systems and by $\mathfrak{Lc}(X)$ the preorder of all continuous saturated multiplicative systems, i.e. $\mathfrak{Lc}(X) = \mathfrak{Lc}_\ell(X) \cap \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$.

We will call continuous saturated multiplicative systems *flat*.

5.3.5. The flat spectra $\mathbf{Spec}^0_{\mathfrak{fL}}(X)$ and $\mathbf{Spec}^1_{\mathfrak{fL}}(X)$. The elements of the *flat complete $\mathfrak{L}$-spectrum* $\mathbf{Spec}^1_{\mathfrak{fL}}(X)$ are flat multiplicative systems Σ such that the 'space' of fractions $\Sigma^{-1}X$ is $\mathfrak{L}$-local.

We call $\mathbf{Spec}^1_{\mathfrak{fL}}(X)$ the *complete flat $\mathfrak{L}$-spectrum* of X.

The *flat $\mathfrak{L}$-spectrum*, $\mathbf{Spec}^0_{\mathfrak{fL}}(X)$, of the 'space' X is defined by setting

$$\mathbf{Spec}^0_{\mathfrak{fL}}(X) = \{\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid \widehat{\Sigma} \in \mathbf{Spec}^1_{\mathfrak{fL}}(X)\}.$$

It follows that

$$\mathbf{Spec}^0_{\mathfrak{fL}}(X) \subseteq \mathbf{Spec}^0_{\mathfrak{C}}(X) \quad \text{and} \quad \mathbf{Spec}^1_{\mathfrak{fL}}(X) \subseteq \mathbf{Spec}^1_{\mathfrak{C}}(X).$$

We leave to the reader the definition of the extended versions of these spectra.

C1. Cosubspaces and coimmersions.

Let X be a 'space'. We call Y a *cosubspace* of X if C_Y is a full subcategory of the category C_X which is closed under finite limits and colimits taken in C_X and, in addition, the following condition holds:

If $M \rightrightarrows N$ is a pair of arrows such that $M \in ObC_Y$ (resp. $N \in ObC_Y$) and the kernel (resp. cokernel) of the pair $M \rightrightarrows N$ exists in C_X, then this kernel (resp. cokernel) belongs to the subcategory C_Y. In particular, C_Y is strictly full (i.e. it contains with every object all objects of C_X isomorphic to this object).

We call a morphism $X \xrightarrow{u} V$ a *coimmersion* if it induces an equivalence between V and a cosubspace of X.

C1.1. Proposition. *Let X be a 'space' such that the category C_X has finite limits and colimits. Then any coimmersion $X \xrightarrow{\mathfrak{v}} Y$ induces a morphism $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) \xrightarrow{\mathfrak{v}_{\mathfrak{L}}} \mathcal{S}^{\mathfrak{s}}\mathcal{M}(Y)$ which maps every saturated multiplicative system Σ to its preimage ${\mathfrak{v}^*}^{-1}(\Sigma)$ by an inverse image functor $\mathfrak{v}^*$ of the coimmersion $\mathfrak{v}$.*

The functor $\mathfrak{v}^*$ determines a functor $\mathfrak{v}^*_\Sigma : \mathfrak{v}_{\mathfrak{L}}(\Sigma)^{-1}C_Y \longrightarrow \Sigma^{-1}C_X$ which is an inverse image functor of a coimmersion.

Proof. We can and will assume that Y is a cosubspace of X and $\mathfrak{v}^*$ is the inclusion functor $C_Y \longrightarrow C_X$.

(a) The morphism $X \xrightarrow{\mathfrak{v}} Y$ induces a morphism $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) \xrightarrow{\mathfrak{v}_{\mathfrak{L}}} \mathcal{S}^{\mathfrak{s}}\mathcal{M}(Y)$ which maps every saturated multiplicative system Σ of C_X to its restriction, $\Sigma|_Y$, to C_Y.

We need to verify that $\Sigma|_Y$ belongs to $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(Y)$. Let q^*_Σ be the localization functor $C_X \longrightarrow \Sigma^{-1}C_X = C_{\Sigma^{-1}X}$. Since Σ is saturated, it coincides with the family $\{t \in HomC_X \mid q^*_\Sigma(t) \text{ is invertible}\}$. In particular, $\Sigma|_Y = \{t \in HomC_Y \mid q^*_\Sigma(t) \text{ is invertible}\}$. Therefore $\Sigma|_Y$ is saturated too. Because C_X has finite limits and colimits, it follows from [GZ, 1.3.2 and 1.3.4] that Σ is a saturated multiplicative system iff the localization functor q^*_Σ is exact, i.e. it preserves finite limits and colimits. Since the category C_Y is closed under finite limits and colimits taken in C_X, that is the inclusion functor $C_Y \xrightarrow{\mathfrak{v}^*} C_X$ is exact, the restriction of q^*_Σ to C_Y (i.e. the composition $q^*_\Sigma \circ \mathfrak{v}^*$) is exact. Therefore $\Sigma|_Y$ is a multiplicative system.

(b) For any $\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$, there is a commutative diagram

$$\begin{array}{ccc} C_Y & \xrightarrow{\mathfrak{v}^*} & C_X \\ {\scriptstyle q^*_{\Sigma|_Y}}\downarrow & & \downarrow{\scriptstyle q^*_\Sigma} \\ \Sigma|_Y^{-1}C_Y & \xrightarrow{\mathfrak{v}^*_\Sigma} & \Sigma^{-1}C_X \end{array} \qquad (1)$$

in which the functor $\mathfrak{v}^*_\Sigma$ is uniquely determined by the other three. Notice that $\mathfrak{v}^*_\Sigma$ is an inverse image functor of a coimmersion.

In fact, by [GZ, 1.3.4], the functor $\mathfrak{v}^*_\Sigma$ is exact, because the functors $q^*_{\Sigma|_Y}$ and $q^*_\Sigma \circ \mathfrak{v}^*$ are exact (see the diagram (1)). It follows from the description of the categories of fractions $\Sigma|_Y^{-1}C_Y$ and $\Sigma^{-1}C_X$ (cf. [GZ, 1.3]) that the functor $\mathfrak{v}^*_\Sigma$ is fully faithful.

Moreover, if Y is a cosubspace of X and $C_Y \xrightarrow{\mathfrak{v}^*} C_X$ is the inclusion functor, then $\Sigma|_Y^{-1}Y$ is a cosubspace of $\Sigma^{-1}X$ and $C_{\Sigma|_Y^{-1}Y} = \Sigma|_Y^{-1}C_Y \xrightarrow{\mathfrak{v}^*_\Sigma} \Sigma^{-1}C_X = C_{\Sigma^{-1}X}$ is the inclusion functor of a cosubspace. Here we take the standard realization of categories of fractions (see [GZ, 1.3]). ∎

II. Spectra of 'spaces' represented by abelian categories.

6. Spectra in terms of thick and Serre subcategories.

Fix a 'space' X such that C_X is an abelian category. Let $\mathfrak{Th}(X)$ denote the preorder (with respect to $\subseteq$) of all thick subcategories of the category C_X. Recall that a full subcategory, $\mathbb{T}$, of the category C_X is *thick* if it is closed under subquotients and extensions. In other words, an object M in the exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ belongs to $\mathbb{T}$ iff M' and M'' are objects of $\mathbb{T}$.

6.1. Thick subcategories, saturated multiplicative systems, and $\mathbf{Spec}^0_{\mathfrak{L}}$. There is a bijective map from $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ to the preorder $\mathfrak{Th}(X)$ which assigns to each saturated multiplicative system Σ the kernel of the localization functor $C_X \longrightarrow \Sigma^{-1}C_X$. The inverse map assigns to each thick subcategory $\mathbb{T}$ of the category C_X the family, $\Sigma_{\mathbb{T}}$, of all morphisms s of C_X whose kernel and cokernel are objects of $\mathbb{T}$. One can see that the map $\mathfrak{Th}(X) \longrightarrow \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$, $\mathbb{T} \longmapsto \Sigma_{\mathbb{T}}$, (hence its inverse) is an isomorphism of preorders.

The trivial saturated multiplicative system, $Iso(C_X)$, corresponds to the zero subcategory of the category C_X. Thus, the isomorphism $\mathfrak{Th}(X) \xrightarrow{\sim} \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ induces an isomorphism between the preorder $\mathfrak{Th}_\star(X)$ of all nonzero thick subcategories of C_X and the preorder $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_\star(X)$ of all non-trivial saturated multiplicative systems of C_X.

Let $\mathbf{Spec}^0_{\mathfrak{Th}}(X)$ denote the preorder (with respect to $\subseteq$) of all thick subcategories, P, of the category C_X such that there exists the biggest thick subcategory, $\widehat{\mathcal{P}}$, of C_X which does not contain $\mathcal{P}$. The map $\mathbb{T} \longmapsto \Sigma_{\mathbb{T}}$ induces an isomorphism of preorders

$$\mathbf{Spec}^0_{\mathfrak{Th}}(X) \xrightarrow{\sim} \mathbf{Spec}^0_{\mathfrak{L}}(X).$$

For any object M of the category C_X, let $[M]_\bullet$ denote the *thick envelope of* M which is, by definition, the smallest thick subcategory of C_X containing M.

6.1.1. Lemma. *Let $\mathcal{P} \in \mathbf{Spec}^0_{\mathfrak{Th}}(X)$. Then $\mathcal{P} = [M]_\bullet$ for any $M \in Ob\mathcal{P} - Ob\widehat{\mathcal{P}}$.*

Proof. (a) Let $\mathcal{P}, \mathcal{P}' \in \mathbf{Spec}^0_{\mathfrak{Th}}(X)$. It follows from 2.1.2 that $\mathcal{P}' \subseteq \mathcal{P}$ iff $\widehat{\mathcal{P}}' \subseteq \widehat{\mathcal{P}}$.

(b) For any $M \in Ob\mathcal{P} - Ob\widehat{\mathcal{P}}$, the category $\widehat{\mathcal{P}}$ is the biggest thick subcategory which does not contain M, that is $[M]_\bullet \in \mathbf{Spec}^0_{\mathfrak{Th}}(X)$ and $\widehat{[M]_\bullet} = \widehat{\mathcal{P}}$. By (a) above, the latter equality implies that $\mathcal{P} = [M]_\bullet$. ∎

6.1.2. Representatives of elements of $\mathbf{Spec}^0_{\mathfrak{Th}}(X)$. Let $Spec^0_{\mathfrak{Th}}(X)$ denote the family of all objects M of the category C_X such that there exists the biggest thick subcategory, $\widehat{[M]_\bullet}$, which does not contain M. It follows from 6.1.1 that

$$\mathbf{Spec}^0_{\mathfrak{Th}}(X) = \{[M]_\bullet \mid M \in Spec^0_{\mathfrak{Th}}(X)\}.$$

6.1.3. The categories $\langle M\rangle_\bullet$ and another description of $Spec^0_{\mathfrak{Th}}(X)$. For any object M of the category C_X, denote by $\langle M\rangle_\bullet$ the full subcategory of C_X whose objects are $N \in ObC_X$ such that $M \notin [N]_\bullet$.

We define a relation $\succcurlyeq$ on ObC_X by $M \succcurlyeq L$ iff $L \in [M]_\bullet$.

6.1.3.1. Lemma. *The following conditions on objects L, M of C_X are equivalent:*
(i) $M \succcurlyeq L$*;*
(ii) $[L]_\bullet \subseteq [M]_\bullet$*;*
(iii) $\langle L\rangle_\bullet \subseteq \langle M\rangle_\bullet$.

Proof. The implications (i)⇔(ii) are evident. The implications (ii)⇒(iii) and (iii)⇒(i) are left to the reader. ■

The equivalence of (i) and (ii) in 6.1.3.1 shows, in particular, that $\succcurlyeq$ is a preorder.

6.1.3.2. Note. It follows from 6.1.3.1 that for any object M of the category C_X, the subcategory $\langle M\rangle_\bullet$ is closed with respect $\succcurlyeq$; i.e. if $N \in Ob\langle M\rangle_\bullet$ and $N \succcurlyeq L$, then $L \in Ob\langle M\rangle_\bullet$. In particular, the subcategory $\langle M\rangle_\bullet$ contains all subquotients of its objects. But, in general, $\langle M\rangle_\bullet$ is not closed under extensions.

6.1.3.3. Proposition. *(a) For any object M of the category C_X, the subcategory $\langle M\rangle_\bullet$ is the union of all thick subcategories of C_X which do not contain the object M.*

(b) The following conditions on an object M of the category C_X are equivalent:

(i) $M \in Spec^0_{\mathfrak{Th}}(X)$*;*

(ii) the subcategory $\langle M\rangle_\bullet$ is thick.

For every $M \in Spec^0_{\mathfrak{Th}}(X)$, the subcategory $\widehat{[M]_\bullet}$ coincides with $\langle M\rangle_\bullet$.

Proof. (a) If $\mathbb{T}$ is a thick subcategory which does not contain the object M, then, obviously, $M \notin Ob[N]_\bullet$ for any $N \in Ob\mathbb{T}$, that is $\mathbb{T} \subseteq \langle M\rangle_\bullet$. This observation shows also that $\langle M\rangle_\bullet$ is the union of its thick subcategories $[N]_\bullet$, $N \in Ob\langle M\rangle_\bullet$.

(b) (i)⇒(ii). Let $M \in Spec^0_{\mathfrak{Th}}(X)$. If $N \in Ob\langle M\rangle_\bullet$, that is M does not belong to the thick subcategory $[N]_\bullet$ spanned by N, then $[N]_\bullet \subseteq \widehat{[M]_\bullet}$. Therefore $\langle M\rangle_\bullet$ is a subcategory of $\widehat{[M]_\bullet}$. The inverse inclusion, $\widehat{[M]_\bullet} \subseteq \langle M\rangle_\bullet$, is obvious.

(ii)⇒(i). Notice that any thick subcategory $\mathbb{T}$ of the category C_X which does not contain the object M is a subcategory of $\langle M\rangle_\bullet$. This shows that if $\langle M\rangle_\bullet$ is a thick subcategory itself, the it is the biggest thick subcategory of C_X which does not contain M, hence the assertion. ■

6.2. Coreflective thick subcategories and the closed spectrum. Recall that a subcategory $\mathbb{T}$ of the category C_X is called *coreflective*, or (*right*) *closed*, if the inclusion functor $\mathbb{T} \hookrightarrow C_X$ has a right adjoint. In other words, the subcategory $\mathbb{T}$ is coreflective iff every object of C_X has the biggest subobject which belongs to $\mathbb{T}$.

6.2.1. Proposition. *(a) Let $\mathbb{T}$ be a full subcategory of the category C_X closed under coproducts and taking quotients (e.g. $\mathbb{T}$ is topologizing). If the family $\Sigma_{\mathbb{T}} = \{s \in HomC_X \mid Ker(s) \in Ob\mathbb{T} \ni Cok(s)\}$ is closed, then the subcategory $\mathbb{T}$ is coreflective.*

(b) Suppose the $\mathbb{T}$ is a coreflective subcategory of C_X closed under extensions. Then the family $\Sigma_{\mathbb{T}}$ is closed.

(c) Suppose $\mathbb{T}$ is closed under taking quotients and extensions (e.g. $\mathbb{T}$ is a thick subcategory). Then the family $\Sigma_{\mathbb{T}}$ is closed iff the subcategory $\mathbb{T}$ is coreflective.

Proof. Notice that $\Sigma_{\mathbb{T}}$-torsion free objects are precisely $\mathbb{T}$-torsion free objects.

(a) Suppose, $\Sigma_{\mathbb{T}}$ is closed; i.e. for every object M of the category C_X, there exists a morphism $M \xrightarrow{s} N$ of $\Sigma_{\mathbb{T}}$ (that is $Ker(s) \in Ob\mathbb{T} \ni Cok(s)$) such that N is $\mathbb{T}$-torsion free. Then $Ker(s)$ is the $\mathbb{T}$-torsion of M. In fact, if $L \hookrightarrow M$ is a subobject of M such that $L \in Ob\mathbb{T}$, then the image, L_s, of $L \oplus Ker(s) \longrightarrow M$ is a subobject of M which belongs to $\mathbb{T}$ and $Ker(s) \hookrightarrow L_s$. The image of the composition $L_s \hookrightarrow M \xrightarrow{s} N$ is a subobject of N which belongs to $\mathbb{T}$. Since N is $\mathbb{T}$-torsion free, this image equals to zero. Therefore $Ker(s) \hookrightarrow L_s$ is an isomorphism which means that L is a subobject of $Ker(s)$.

(b) Let $\mathbb{T}$ be a coreflective subcategory closed under extensions. For an arbitrary object M, let $\mathfrak{t}_{\mathbb{T}}M$ denote the $\mathbb{T}$-torsion of M and $M_{\mathbb{T}}$ the quotient $M/\mathfrak{t}_{\mathbb{T}}M$. By definition, the canonical epimorphism $M \longrightarrow M_{\mathbb{T}}$ belongs to $\Sigma_{\mathbb{T}}$. Notice that the object $M_{\mathbb{T}}$ is $\mathbb{T}$-torsion free.

In fact, let $L \hookrightarrow M_{\mathbb{T}}$ be a subobject of $M_{\mathbb{T}}$ such that $L \in Ob\mathbb{T}$. Then we have a short exact sequence

$$0 \longrightarrow \mathfrak{t}_{\mathbb{T}}M \longrightarrow \widetilde{L} \longrightarrow L \longrightarrow 0,$$

where $\widetilde{L} = L \times_{M_{\mathbb{T}}} M$. Since $\mathbb{T}$ is closed under extensions, $\widetilde{L} \in Ob\mathbb{T}$. Because $\widetilde{L}$ is a subobject of M, it factors through $\mathfrak{t}_{\mathbb{T}}M \hookrightarrow M$, hence $\mathfrak{t}_{\mathbb{T}}M \longrightarrow \widetilde{L}$ is an isomorphism, or, equivalently, $L = 0$.

(c) The assertion follows from (a) and (b). ■

6.2.2. The closed spectrum. Let $\mathfrak{CTh}(X)$ denote the preorder of coreflective thick subcategories. We set

$$\mathbf{Spec}^0_{\mathfrak{C}}(X) = \{\mathcal{P} \in \mathbf{Spec}^0_{\mathfrak{Th}}X \mid \widehat{\mathcal{P}} \in \mathfrak{CTh}(X)\}.$$

The map $\mathbb{T} \longmapsto \Sigma_{\mathbb{T}}$ induces an isomorphism of the preorder $\mathbf{Spec}^0_{\mathfrak{C}}(X)$ onto the *closed spectrum* $\mathbf{Spec}^0_{\mathfrak{C}}(X)$ of X.

Let $Spec^0_{\mathfrak{C}}(X)$ denote the family of all objects M such that $\langle M\rangle_{\bullet}$ is a coreflective thick subcategory of C_X and M is $\langle M\rangle_{\bullet}$-torsion free. It follows from this definition that $Spec^0_{\mathfrak{C}}(X) \subseteq Spec^0_{\mathfrak{Th}}X$.

6.2.3. Proposition. $\mathbf{Spec}^0_{\mathfrak{C}}(X) = \{[M]_{\bullet} \mid M \in Spec^0_{\mathfrak{C}}(X)\}$.

Proof. Let $\mathcal{P} \in \mathbf{Spec}^0_{\mathfrak{C}}(X)$, and let $M \in Ob\mathcal{P} - Ob\widehat{\mathcal{P}}$. By 6.1.1, $\mathcal{P} = [M]_{\bullet}$ and, therefore, $\widehat{\mathcal{P}} = \langle M\rangle_{\bullet}$ (cf. 6.1.3). Since the category $\mathcal{P}$ is coreflective, the object M has the $\widehat{\mathcal{P}}$-torsion, $\mathfrak{t}_{\widehat{\mathcal{P}}}M \hookrightarrow M$, and (by 6.2.1(c)) the quotient object, $M_{\widehat{\mathcal{P}}} = M/\mathfrak{t}_{\widehat{\mathcal{P}}}M$, is $\widehat{\mathcal{P}}$-torsion free. Since $M \in Ob\mathcal{P} - Ob\widehat{\mathcal{P}}$, the object $M_{\widehat{\mathcal{P}}}$ is nonzero and belongs to the subcategory $\mathcal{P}$. By 6.1.1 and 6.1.3.3, $\mathcal{P} = [M_{\widehat{\mathcal{P}}}]_{\bullet}$ and $\widehat{\mathcal{P}} = \langle M_{\widehat{\mathcal{P}}}\rangle_{\bullet}$. ■

6.3. A spectrum associated with Serre subcategories. We remind the notion of a Serre subcategory of an abelian category as it is defined in [R, III.2.3.2].

Let $\mathbb{T}$ be a subcategory of C_X. We denote by $\mathbb{T}^-$ the full subcategory of C_X generated by all objects L of C_X such that any nonzero subquotient of L has a nonzero subobject which belongs to $\mathbb{T}$.

6.3.1. Proposition. *Let $\mathbb{T}$ be a subcategory of C_X. Then* *(a) The subcategory $\mathbb{T}^-$ is thick (i.e. it is closed under taking subquotients and extensions).*

(b) $(\mathbb{T}^-)^- = \mathbb{T}^-$.

(c) $\mathbb{T} \subseteq \mathbb{T}^-$ iff any nonzero subquotient of an object of $\mathbb{T}$ has a nonzero subobject which is isomorphic to an object of $\mathbb{T}$.

Proof. See [R, III.2.3.2.1]. ■

6.3.2. Corollary. *For any subcategory, $\mathbb{T}$, of the category C_X, $Ob\mathbb{T}^-$ consists of all $N \in ObC_X$ such that any nonzero object of $[N]_\bullet$ contains a nonzero subobject which belongs to $\mathbb{T}$.*

Proof. By 6.3.1, $\mathbb{T}^-$ is a thick subcategory of C_X, hence $N \in Ob\mathbb{T}^-$ iff $[N]_\bullet \subseteq \mathbb{T}^-$. ■

6.3.3. Definition. A subcategory $\mathbb{T}$ of C_X is called a *Serre subcategory* if $\mathbb{T}^- = \mathbb{T}$.

6.3.4. The spectrum $\mathbf{Spec}^0_{\mathfrak{s}}(X)$. For any object L of the category C_X, denote by ${}^\perp L$ the full subcategory of C_X whose objects are all $N \in ObC_X$ such that $C_X(N, L) = 0$. Let $Spec^0_{\mathfrak{s}}(X)$ denote the family of all nonzero objects M of the category C_X such that $M \in Ob[N]_\bullet$ for any $N \in ObC_X - Ob^\perp M$. In other words, if N is an object of the category C_X such that if there exists a nonzero morphism $N \longrightarrow M$, then $M \in Ob[N]_\bullet$.

Thus, if $M \in Spec^0_{\mathfrak{s}}(X)$, then $Ob\langle M\rangle_\bullet = \{N \in ObC_X \mid [N]_\bullet \subseteq^\perp M\}$.

6.3.5. Proposition. *For every $M \in Spec^0_{\mathfrak{s}}(X)$, the subcategory $\langle M\rangle_\bullet$ is a Serre subcategory. In particular, $Spec^0_{\mathfrak{s}}(X) \subseteq Spec^0_{\mathfrak{T}\mathfrak{h}}(X)$.*

Proof. By 6.1.3.2, $\langle M\rangle_\bullet$ is closed under taking subquotients (in C_X). Therefore, by 6.3.1(c), $\langle M\rangle_\bullet$ is a subcategory of its "Serre closure" $\langle M\rangle_\bullet^-$. Notice that $\langle M\rangle_\bullet$ is properly contained in $\langle M\rangle_\bullet^-$ iff $M \in Ob\langle M\rangle_\bullet^-$.

In fact, if $N \in Ob\langle M\rangle_\bullet^- - Ob\langle M\rangle_\bullet$, then, by the definition of $\langle M\rangle_\bullet$, the object M belongs to the minimal thick subcategory, $[N]_\bullet$, spanned by N. By 6.3.1(a), the subcategory $\langle M\rangle_\bullet^-$ is thick, hence it contains the subcategory $[N]_\bullet$.

If M is a nonzero object and $M \in \langle M\rangle_\bullet^-$, there exists a nonzero morphism $L \longrightarrow M$ with $L \in Ob\langle M\rangle_\bullet$. But, if $M \in Spec^0_{\mathfrak{s}}(X)$, then $C_X(L, M) = 0$ for all $L \in Ob\langle M\rangle_\bullet$. Thus, $\langle M\rangle_\bullet^- = \langle M\rangle_\bullet$. ■

6.3.6. Proposition. *The following conditions on a nonzero object M of the category C_X are equivalent:*

(i) $M \in Spec^0_{\mathfrak{s}}(X)$;

(ii) M is $\langle M\rangle_\bullet$-torsion free;

(iii) $\langle M\rangle_\bullet$ is a Serre subcategory of C_X and M is $\langle M\rangle_\bullet$-torsion free.

Proof. Clearly (iii)⇒(ii).

If $M \in Spec^0_{\mathfrak{s}}(X)$, then M is $\langle M\rangle_\bullet$-torsion free, because if there is a nonzero morphism $N \longrightarrow M$, then $M \in [N]_\bullet$, i.e. $N \notin Ob\langle M\rangle_\bullet$. Together with 6.3.5, this proves the implication (i)⇒(iii).

(ii)⇒(i). By definition of $Spec^0_{\mathfrak{s}}(X)$, a nonzero object, M, belongs to $Spec^0_{\mathfrak{s}}(X)$ iff for any nonzero morphism $N \longrightarrow M$, the object M belongs to the thick subcategory $[N]_\bullet$ spanned by N.

Suppose the conditions of (ii) hold, and let $N \longrightarrow M$ be a nonzero morphism. Since M is $\langle M\rangle_\bullet$-torsion free, and $\langle M\rangle_\bullet$ is closed under taking subquotients in C_X (see 6.1.3.2), the object N does not belong to $\langle M\rangle_\bullet$ which means that $M \in [N]_\bullet$. ■

We denote by $\mathbf{Spec}^0_{\mathfrak{s}}(X)$ the preordered subset of $\mathbf{Spec}^0_{\mathfrak{Th}}(X)$ formed by all $[M]_\bullet$ such that $M \in Spec^0_{\mathfrak{s}}(X)$.

6.3.7. The property (sup). Recall that X (or the corresponding category C_X) has the property (sup) if for any ascending chain, Ω, of subobjects of an object M, the supremum of Ω exists, and for any subobject L of M, the natural morphism

$$sup(N \cap L \mid N \in \Omega) \longrightarrow (sup\Omega) \cap L$$

is an isomorphism.

6.3.8. Lemma. *(a) Any coreflective thick subcategory of an abelian category C_X is a Serre subcategory.*

(b) If X has the property (sup), then any Serre subcategory of C_X is coreflective.

Proof. See [R, III.2.4.4]. ■

6.3.9. Note. If C_X is a category with small coproducts, then a thick subcategory of C_X is coreflective iff it is closed under small coproducts (taken in C_X).

6.3.10. Proposition. *Suppose that X has the property (sup). Then*

$$\begin{aligned}\mathbf{Spec}^0_{\mathfrak{s}}(X) &= \{\mathcal{P} \in \mathbf{Spec}^0_{\mathfrak{Th}}(X) \mid \widehat{\mathcal{P}} \textit{ is a Serre subcategory of } C_X\} \\ &= \{[M]_\bullet \mid M \neq 0 \textit{ and } \langle M\rangle_\bullet \textit{ is a Serre subcategory of } C_X\} = \mathbf{Spec}^0_{\mathfrak{C}}(X).\end{aligned}$$

Proof. It follows from 6.3.6 that

$$Spec^0_{\mathfrak{C}}(X) = \{M \in Spec^0_{\mathfrak{s}}(X) \mid \langle M\rangle_\bullet \text{ is coreflective}\}.$$

If X has the property (sup), then, by 6.3.8, every Serre subcategory is coreflective. The assertion follows now from 6.2.3. ■

6.4. Cosubspaces and coimmersions. Let C_X be an abelian category. Then Y is a cosubspace of X iff C_Y is a *topologizing* subcategory of C_X. Here 'topologizing' means that C_Y is a full subcategory of C_X closed under finite coproducts and subquotients taken in C_X. In particular, C_Y is an abelian category.

For any cosubspace Y of X, we define cosubspaces Y^- and Y^∞ by setting $C_{Y^-} := C_Y^-$ and taking as C_{Y^∞} the thick envelope of C_Y in C_X.

We call a cosubspace Y *thick* if C_Y is a thick subcategory of C_X, i.e. $Y = Y^\infty$. We call Y a *Serre cosubspace* if $Y = Y^-$.

A morphism $X \xrightarrow{\mathfrak{v}} Y$ is a coimmersion if it induces an isomorphism between a cosubspace of X and Y.

6.4.1. Gabriel multiplication. Recall that the Gabriel product, $\mathbb{T} \bullet_X \mathbb{S}$, of subcategories $\mathbb{T}$ and $\mathbb{S}$ of the category C_X is the full subcategory of C_X generated by all $M \in ObC_X$ for which there exists an exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

with $M' \in Ob\mathbb{S}$ and $M'' \in Ob\mathbb{T}$.

We define *infinitesimal neighborhoods* of the subcategory $\mathbb{T}$ by $\mathbb{T}^{\bullet 2} = \mathbb{T} \bullet_X \mathbb{T}$ and $\mathbb{T}^{\bullet n} = \mathbb{T}^{\bullet n-1} \bullet_X \mathbb{T}$ for $n \geq 3$. Finally, we set $\mathbb{T}^{\infty} = \bigcup_{n\geq 2} \mathbb{T}^{\bullet n}$.

6.4.2. Gabriel product of topologizing subcategories. Let C_X be an abelian category and $\mathbb{S}$, $\mathbb{T}$ a pair of subcategories of C_X.

(i) If the subcategories $\mathbb{S}$ and $\mathbb{T}$ are topologizing, (i.e. they are closed under subquotients and coproducts taken in C_X), then $\mathbb{T} \bullet_X \mathbb{S}$ is topologizing too.

(ii) Gabriel product of topologizing subcategories is associative and has a unit element: $\mathbb{T} \bullet_X \mathbb{O} = \mathbb{O} \bullet_X \mathbb{T} = \mathbb{T}$ for any topologizing subcategory $\mathbb{T}$. Here $\mathbb{O}$ denotes the zero subcategory of C_X.

(iii) If $\mathbb{S}$ and $\mathbb{T}$ are coreflective (resp. reflective) topologizing subcategories, then $\mathbb{T} \bullet_X \mathbb{S}$ is coreflective (resp. reflective).

(iv) A topologizing subcategory $\mathbb{T}$ of C_X is thick iff $\mathbb{T} \bullet_X \mathbb{T} = \mathbb{T}$.

(v) Let $\mathbb{T}$ be a topologizing subcategory. Then $\mathbb{T}^{\infty} = \bigcup_{n\geq 2} \mathbb{T}^{\bullet n}$ is the thick envelope of $\mathbb{T}$.

(vi) In particular, if $\mathbb{T}$ and $\mathbb{S}$ are topologizing subcategories of C_X, then $(\mathbb{T} \bullet_X \mathbb{S})^{\infty}$ is the thick envelope of $\mathbb{T} \cup \mathbb{S}$.

6.4.3. Gabriel product of coimmersions. If $V \xleftarrow{\mathfrak{v}} X \xrightarrow{\mathfrak{w}} W$ are coimmersions, then we define their Gabriel coproduct, $V \bullet_X W$ by setting $C_{V \bullet_X W} = \mathfrak{v}^*(C_V) \bullet_X \mathfrak{w}^*(C_W)$. Since $\mathfrak{v}^*(C_V)$ and $\mathfrak{w}^*(C_X)$ are equivalent to topologizing subcategories (resp. $\mathfrak{v}^*(C_V) \bullet_X 0$ and $\mathfrak{w}^*(C_X) \bullet_X 0$) of C_X, the subcategory $C_{V \bullet_X W}$ is topologizing, i.e. $V \bullet_X W$ is a cosubspace of X.

6.4.4. Proposition. *(a) Let $X \xrightarrow{\mathfrak{v}} Y$ be a morphism having an exact inverse image functor, $C_Y \xrightarrow{\mathfrak{v}^*} C_X$. The the map $\mathbb{T} \longmapsto \mathfrak{v}^{*^{-1}}(\mathbb{T})$ sends topologizing (resp. thick) subcategories of the category C_X to topologizing (resp. thick) subcategories of C_Y.*

(b) If $X \xrightarrow{\mathfrak{v}} Y$ is a coimmersion, then the map $\mathbb{T} \longmapsto \mathfrak{v}^{*^{-1}}(\mathbb{T})$ is compatible with Gabriel multiplication:

$$\mathfrak{v}^{*^{-1}}(\mathbb{T} \bullet_X \mathbb{S}) = \mathfrak{v}^{*^{-1}}(\mathbb{T}) \bullet_Y \mathfrak{v}^{*^{-1}}(\mathbb{S}) \tag{1}$$

for any pair $\mathbb{S}$ and $\mathbb{T}$ of topologizing subcategories of the category C_X.

Proof. The argument is left to the reader. ■

6.4.4.1. Note. If Y is a cosubspace of X and $\mathfrak{v}^*$ the inclusion functor, then we write $\mathbb{T}|_Y$ instead of $\mathfrak{v}^{*^{-1}}(\mathbb{T})$. The equality (1) turns into $(\mathbb{T} \bullet_X \mathbb{S})|_Y = \mathbb{T}|_Y \bullet_Y \mathbb{S}|_Y$.

6.4.5. Proposition. *Every coimmersion, $X \xrightarrow{\mathfrak{v}} Y$, induces morphisms of spectra which are vertical arrows of the commutative diagram*

$$\begin{array}{ccccccc}
\mathbf{Spec}^0_{\mathfrak{s}}(Y) & \longleftarrow & Spec^0_{\mathfrak{s}}(Y) & \longrightarrow & Spec^0_{\mathfrak{T}\mathfrak{h}}(Y) & \longrightarrow & \mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}}(Y) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbf{Spec}^0_{\mathfrak{s}}(X) & \longleftarrow & Spec^0_{\mathfrak{s}}(X) & \longrightarrow & Spec^0_{\mathfrak{T}\mathfrak{h}}(X) & \longrightarrow & \mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}}(X)
\end{array} \qquad (2)$$

Proof. It suffices to show that an inverse image functor, $\mathfrak{v}^*$, of the coimmersion $\mathfrak{v}$ induces embeddings $Spec^0_{\mathfrak{s}}(Y) \longrightarrow Spec^0_{\mathfrak{s}}(X)$ and $Spec^0_{\mathfrak{T}\mathfrak{h}}(Y) \longrightarrow Spec^0_{\mathfrak{T}\mathfrak{h}}(X)$. These embeddings determine the remaining two vertical arrows of the diagram (2).

For the rest of the argument, we can and will assume that Y is a cosubspace of X.

(a) Let $M \in Spec^0_{\mathfrak{s}}(Y)$. We claim that $M \in Spec^0_{\mathfrak{s}}(X)$, i.e. for any nonzero subobject, L, of the object M the thick subcategory, $[L]_\bullet$, of C_X spanned by L contains M.

By hypothesis, the subcategory C_Y is closed under taking subquotients (in particular, subobjects) and coproducts in C_X, i.e. inclusion functor $\mathfrak{v}^*$ is exact. This means that every subobject, L, of M in C_X belongs to C_Y. Since $M \in Spec^0_{\mathfrak{s}}(Y)$, the thick subcategory $[L]^Y_\bullet$ of C_Y spanned by L contains M. But, $[L]^Y_\bullet$ is the subcategory of $[L]_\bullet$, hence $M \in Ob[L]_\bullet$.

(b) Let now $M \in Spec^0_{\mathfrak{T}\mathfrak{h}}(Y)$. By 6.1.3.3, this means that the full subcategory $\langle M\rangle^Y_\bullet$ of C_Y whose objects are $N \in ObC_Y$ such that $M \notin [N]^Y_\bullet$ is a thick subcategory of C_Y. We claim that $\langle M\rangle_\bullet$ is a thick subcategory of the category C_X.

Let $\mathbb{T}$ and $\mathbb{S}$ be thick subcategories of the category C_X which do not contain the object M. Then their Gabriel product, $\mathbb{T} \bullet \mathbb{S}$, does not contain M.

Indeed, $\mathbb{T}|_Y$ and $\mathbb{S}|_Y$ do not contain M which means precisely that $\mathbb{T}|_Y$ and $\mathbb{S}|_Y$ are subcategories of the category $\langle M\rangle_\bullet$. Since the subcategory $\langle M\rangle^Y_\bullet$ is thick, that is $\langle M\rangle^Y_\bullet \bullet_Y \langle M\rangle^Y_\bullet = \langle M\rangle^Y_\bullet$, the Gabriel product $\mathbb{T}|_Y \bullet_Y \mathbb{S}|_Y$ is a subcategory of $\langle M\rangle_\bullet$, hence does not contain the object M. Since $\mathbb{T}|_Y \bullet_Y \mathbb{S}|_Y = (\mathbb{T} \bullet_X \mathbb{S})|_Y$ (see 6.4.3 and 6.4.3.1), M does not belong to $\mathbb{T} \bullet_X \mathbb{S}$. Same argument shows that the Gabriel product of any number of thick subcategories of C_X which do not contain M does not contain M. In particular, $M \notin \bigcup_{n\geq 1}(\mathbb{T} \bullet_X \mathbb{S})^{\bullet n} = (\mathbb{T} \bullet_X \mathbb{S})^\infty$. But, $(\mathbb{T} \bullet_X \mathbb{S})^\infty$ is the thick envelope of $\mathbb{S}$ and $\mathbb{T}$ (see 6.4.2(vi)).

This shows that $\langle M\rangle_\bullet \bullet_X \langle M\rangle_\bullet = \langle M\rangle_\bullet$, i.e. $\langle M\rangle_\bullet$ is a thick subcategory of C_X. ■

6.4.6. Corollary. *(a) For any pair of coimmersions $W \xleftarrow{\mathfrak{w}} X \xrightarrow{\mathfrak{v}} V$, there are inclusions*

$$Spec^0_{\mathfrak{s}}(V \bullet_X W) \supseteq Spec^0_{\mathfrak{s}}(V) \bigcup Spec^0_{\mathfrak{s}}(W)$$

$$Spec^0_{\mathfrak{T}\mathfrak{h}}(V \bullet_X W) \supseteq Spec^0_{\mathfrak{T}\mathfrak{h}}(V) \bigcup Spec^0_{\mathfrak{T}\mathfrak{h}}(W) \qquad (3)$$

(b) If the functors $\mathfrak{w}^*$ and $\mathfrak{v}^*$ induce equivalences between the categories C_V and C_W and thick subcategories of the category C_X, then the inclusions in (3) can be replaced by equalities. In particular, in this case

$$\mathbf{Spec}^0_{\mathfrak{s}}(V \bullet_X W) = \mathbf{Spec}^0_{\mathfrak{s}}(V) \bigcup \mathbf{Spec}^0_{\mathfrak{s}}(W)$$

$$\mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}}(V \bullet_X W) = \mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}}(V) \bigcup \mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}}(W) \tag{4}$$

Proof. (a) The inclusions

$$Spec^0_{\mathfrak{s}}(V \bullet_X W) \supseteq Spec^0_{\mathfrak{s}}(V) \bigcup Spec^0_{\mathfrak{s}}(W)$$

and

$$Spec^0_{\mathfrak{T}\mathfrak{h}}(V \bullet_X W) \supseteq Spec^0_{\mathfrak{T}\mathfrak{h}}(V) \bigcup Spec^0_{\mathfrak{T}\mathfrak{h}}(W)$$

follow from 6.4.5.

(b) For the rest of the argument, we can and will assume that V and W are cosubspaces of X, such that C_V and C_W are thick subcategories of C_X.

Let $M \in Spec^0_{\mathfrak{s}}(V \bullet_X W)$, i.e. there exists an exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

such that $M' \in ObC_W$ and $M'' \in ObC_V$. If M' is nonzero, then $M \in Ob[M']_\bullet$. Since C_W is a thick subcategory, $[M']_\bullet \subseteq C_W$; in particular, M is an object of the subcategory C_W. Because C_W is a thick subcategory of C_X, the intersection $\langle M\rangle_\bullet \bigcap C_W$ coincides with $\langle M\rangle^W_\bullet$. Therefore $M \in Spec^0_{\mathfrak{s}}(W)$.

If the object M' is zero, then $M \in ObC_V$ which implies that $M \in Spec^0_{\mathfrak{s}}(V)$.

The argument for $Spec^0_{\mathfrak{T}\mathfrak{h}}$ is similar and is left to the reader. ■

6.4.7. Proposition. *Let Y be a cosubspace of X such that C_Y is a thick subcategory of C_X. Then there are natural inclusions:*

$$\mathbf{Spec}^0_{\mathfrak{s}}(X) \longrightarrow \mathbf{Spec}^0_{\mathfrak{s}}(Y) \coprod \mathbf{Spec}^0_{\mathfrak{s}}(X/Y) \tag{5}$$

and

$$\mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}}(X) \longrightarrow \mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}}(Y) \coprod \mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}}(X/Y). \tag{6}$$

Here X/Y is defined by $C_{X/Y} = C_X/C_Y$.

Proof. (i) It follows from 2.2.3 that the exact localization $X/Y \xrightarrow{q_Y} X$ induces an embedding

$$\mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}}(X) - \mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}}(Y) \longrightarrow \mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}}(X/Y).$$

This map, together the identical map $\mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}}(Y)$ onto itself, determines a morphism (6).

(ii) The morphism (6) can be also described in terms of the map

$$Spec^0_{\mathfrak{T}\mathfrak{h}}(X) - ObC_Y \longrightarrow Spec^0_{\mathfrak{T}\mathfrak{h}}(X/Y)$$

induced by an inverse image (localization) functor $C_X \xrightarrow{q^*_Y} C_X/C_Y$.

In fact, if $M \in ObC_X - ObC_Y$, then, $C_Y = Ker(q^*_Y) \subseteq \langle M\rangle_\bullet$. Therefore $q^*_Y(\langle M\rangle_\bullet) = \langle q^*_Y(M)\rangle_\bullet$. Because if $\mathbb{T}$ is a thick subcategory of C_X/C_Y which does not contain $q^*_Y(M)$, then ${q_Y^*}^{-1}(\mathbb{T})$ is a thick subcategory of C_X which does not contain M, hence ${q_Y^*}^{-1}(\mathbb{T}) \subseteq \langle M \rangle_\bullet$ which implies that $\mathbb{T} \subseteq q_Y^*(\langle M \rangle_\bullet)$. If $M \in Spec^0_{\mathfrak{Th}}(X)$, i.e. the subcategory $\langle M \rangle_\bullet$ is thick, then $q_Y^*(\langle M \rangle_\bullet) = \langle q_Y^*(M) \rangle_\bullet$ is a thick subcategory of the category C_X/C_Y, that is $q_Y^*(M) \in Spec^0_{\mathfrak{Th}}(X/Y)$.

(iii) Let $M \in Spec^0_{\mathfrak{s}}(X) - ObC_Y$, i.e. M is $\langle M \rangle_\bullet$-torsion free. Since $Ker(q_Y^*) \subseteq \langle M \rangle_\bullet$, this implies that $q_Y^*(M)$ is $\langle q_Y^*(M) \rangle_\bullet$-torsion free, hence $q_Y^*(M)$ belongs to $Spec^0_{\mathfrak{s}}(X/Y)$. ■

6.5. $\mathfrak{L}$-local 'spaces' and categories. Let C_X be an abelian category. We define the *residue thick cosubspace*, $X_\star$, of the 'space' X by setting $C_{X_\star}$ equal to the intersection of all nonzero thick subcategories of X. Then X is *$\mathfrak{L}$-local* iff the residue thick cosubspace of X is non-trivial, i.e. $C_{X_\star}$ is a nonzero subcategory of C_X. It follows that $C_{X_\star} = [M]_\bullet$ and $\langle M \rangle_\bullet = 0$ for any nonzero object M of $C_{X_\star}$. In particular, every nonzero object of $C_{X_\star}$ belongs to $Spec^0_{\mathfrak{s}}(X)$.

6.5.1. Proposition. *Suppose X is $\mathfrak{L}$-local and the category C_X has simple objects. Then all simple objects of C_X are isomorphic to each other, and $C_{X_\star} = [M]_\bullet$, where M is a simple object.*

Proof. Let $M \in ObC_X$ be a simple object. In particular, M is nonzero, hence $C_{X_\star} \subseteq [M]_\bullet$. But, if L is a nonzero object of the minimal thick subcategory, $[M]_\bullet$, spanned by M, then M is a subobject of L. Therefore, $M \in [L] \subseteq [L]_\bullet$. This shows the inverse inclusion, $[M]_\bullet \subseteq C_{X_\star}$, hence $C_{X_\star} = [M]_\bullet$.

If L is another simple object, then $[M]_\bullet = [L]_\bullet$, in particular, $L \in Ob[M]_\bullet$ which implies that M is a subobject of L (see above). Since the object L is simple, $M \simeq L$. ■

6.5.2. Residue (skew) field of an $\mathfrak{L}$-local category with simple objects. Suppose X is a $\mathfrak{L}$-local category such that the category C_X has a simpe object, M. We denote by $k(X)$ the ring $C_X(M,M)^o$ opposite to the ring of endomorphims of the object M. Since M is simple, $k(X)$ is a skew field which we call the *residue skew field* of the $\mathfrak{L}$-local 'space' X. By 6.5.1, the residue field of X is defined uniquely up to isomorphism.

6.5.3. The thick spectrum and the Serre spectrum. For any thick subcategory, $\mathbb{T}$ of the category C_X. let $\mathbb{T}^\star$ denote the *residue category* of $\mathbb{T}$ defined as the intersection of all thick subcategories of C_X properly containing $\mathbb{T}$. We denote by $\mathbf{Spec}^1_{\mathfrak{Th}}(X)$ the preorder of all thick subcategories $\mathbb{T}$ such that $\mathbb{T} \neq \mathbb{T}^\star$.

One can see that $\mathbf{Spec}^1_{\mathfrak{Th}}(X)$ can be identified with the family of all thick cosubspaces, $\mathcal{P}$, of X (that is, $C_\mathcal{P}$ is a thick subcategory of C_X) such that the quotient 'space' $X/\mathcal{P}$ (defined by $C_{X/\mathcal{P}} = C_X/C_\mathcal{P}$) is $\mathfrak{L}$-local. The canonical functor

$$\mathfrak{Th}(X) \longrightarrow \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X), \quad \mathbb{T} \longmapsto \Sigma_\mathbb{T},$$

(cf. 6.1) induces an isomorphism from $\mathbf{Spec}^1_{\mathfrak{Th}}(X)$ to the complete spectrum, $\mathbf{Spec}^1_{\mathfrak{L}}(X)$, of X (defined in Section 4).

Let $\mathbf{Spec}^1_{\mathfrak{s}}(X)$ denote the family of all Serre cosubspaces, $\mathcal{P}$, of X (i.e. $C_\mathcal{P}$ is a Serre subcategory of C_X) such that $X/\mathcal{P}$ is $\mathfrak{L}$-local. By 3.2 and 6.3.6, we have inclusions

$$\mathbf{Spec}^0_{\mathfrak{Th}}(X) \hookrightarrow \mathbf{Spec}^1_{\mathfrak{Th}}(X) \quad \text{and} \quad \mathbf{Spec}^0_{\mathfrak{s}}(X) \hookrightarrow \mathbf{Spec}^1_{\mathfrak{s}}(X).$$

It follows from 4.2 that

$$\mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}}(X) = \bigcup_{C_Y \in \mathfrak{T}\mathfrak{h}(X)} \mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}}(X/Y) = \bigcup_{\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}}(X)} \mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}}(X/\mathcal{P}).$$

Similarly,

$$\mathbf{Spec}^1_{\mathfrak{s}}(X) = \bigcup_{C_Y \in \mathfrak{S}\mathfrak{e}(X)} \mathbf{Spec}^0_{\mathfrak{s}}(X/Y) = \bigcup_{\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{s}}(X)} \mathbf{Spec}^0_{\mathfrak{s}}(X/\mathcal{P}).$$

Here $\mathfrak{S}\mathfrak{e}(X)$ is the family of all Serre subcategories of the category C_X.

7. Spectra associated with the category of cosubspaces.

Let C_X be an abelian category. We denote by $\mathfrak{T}(X)$ the preorder (with respect to $\supseteq$) of all topologizing subcategories of the category C_X. The Gabriel multiplication, $\bullet_X$, is a structure of a monoidal category on $\mathfrak{T}(X)$ with the unit object $\mathbb{O} = \{0\}$.

7.1. The subcategories $\langle \mathcal{P} \rangle$. Let $\mathfrak{T}_\star(X)$ denote $\mathfrak{T}(X) - \{\mathbb{O}\}$. For any $\mathcal{P} \in \mathfrak{T}_\star(X)$, let $\mathcal{P}^\vdash$ denote the family of all $\mathbb{T} \in \mathfrak{T}(X)$ such that $\mathcal{P} \not\subseteq \mathbb{T}$. Let $\langle \mathcal{P} \rangle$ denote the union of all $\mathbb{T} \in \mathcal{P}^\vdash$.

If $\mathcal{P},\ \mathcal{P}'$ are nonzero topologizing subcategories of C_X, then $\mathcal{P} \subseteq \mathcal{P}'$ iff $\langle \mathcal{P} \rangle \subseteq \langle \mathcal{P}' \rangle$.

7.2. Subcategories $[M]$ and the preorder $\succ$. For any $M \in ObC_X$, let $[M]$ denote the topologizing subcategory spanned by M; i.e. $[M]$ is the intersection of all topologizing subcategories containing M.

For any nonzero object M of the category C_X, let $\langle M \rangle$ denote the union of all topologizing subcategories of C_X which do not contain the object M. It follows that $\langle M \rangle = \langle [M] \rangle$.

For two objects, M and N, of the category C_X, we write $M \succ N$ if N is a subquotient of a coproduct of finite number of copies of M.

7.2.1. Lemma. *For any two objects $M,\ N$ of the category C_X, the following conditions are equivalent:*

(i) $M \succ N$,

(ii) $[N] \subseteq [M]$,

(iii) $\langle N \rangle \subseteq \langle M \rangle$.

Proof. (i)⇒(ii). If $M \succ N$, then $N \in Ob[M]$, hence $[N] \subseteq [M]$.

(ii)⇒(i). The relation $\succ$ is transitive and for any object, N, a finite coproduct, N', of copies of N $N \succ N'$. This implies that the full subcategory of C_X whose objects are $L \in ObC_X$ such that $M \succ L$ is topologizing, hence it coincides with $[M]$. Thus, we have the implications: $\big([N] \subseteq [M]\big) \Rightarrow \big(N \in [M]\big) \Rightarrow \big(M \succ N\big)$.

The equivalence of (ii) and (iii) is observed in 7.1. ■

7.3. Proposition. *The following conditions on a nonzero topologizing subcategory $\mathcal{P}$ of C_X are equivalent:*

(i) $\langle \mathcal{P} \rangle$ is a thick subcategory of the category C_X;

(ii) $\langle\mathcal{P}\rangle$ is a topologizing subcategory of C_X and $\mathcal{P}^{\vdash}$ is a monoidal subcategory of the monoidal category $(\mathfrak{T}(X), \bullet_X)$;

(iii) there exists an object $M \in \mathcal{P}$ such that $\mathcal{P} = [M]$ and $\langle M\rangle$ is a thick subcategory of the category C_X.

If the equivalent conditions above hold, then $\mathcal{P} = [M]$ for every $M \in Ob\mathcal{P} - Ob\langle\mathcal{P}\rangle$.

Proof. (i)⇒(ii). Suppose $\langle\mathcal{P}\rangle$ is a thick subcategory of the category C_X. This means precisely the following:

(a) $\langle\mathcal{P}\rangle$ is topologizing, hence it is a final object of the category $\mathcal{P}^{\vdash}$;

(b) $\langle\mathcal{P}\rangle = \langle\mathcal{P}\rangle \bullet_X \langle\mathcal{P}\rangle$.

If $\mathbb{T}$ and $\mathbb{S}$ are objects to $\mathcal{P}^{\vdash}$, then $\mathbb{S} \subseteq \langle\mathcal{P}\rangle \supseteq \mathbb{T}$, whence $\mathbb{T} \bullet_X \mathbb{S} \subseteq \langle\mathcal{P}\rangle \bullet_X \langle\mathcal{P}\rangle = \langle\mathcal{P}\rangle$.

(ii)⇒(iii). Suppose that $\mathcal{P} \in \mathfrak{T}_\star(X)$ is such that $\langle\mathcal{P}\rangle$ is a topologizing subcategory of C_X. Then $\mathcal{P} = [M]$ for every $M \in Ob\mathcal{P} - Ob\langle\mathcal{P}\rangle$.

In fact, $[M] \subseteq \mathcal{P}$, or, equivalently, $\langle M\rangle \subseteq \langle\mathcal{P}\rangle$, for every $M \in Ob\mathcal{P}$. If $M \in Ob\mathcal{P} - Ob\langle\mathcal{P}\rangle$, then, since $\langle\mathcal{P}\rangle \in \mathfrak{T}(X)$, the inverse inclusion holds: $\langle\mathcal{P}\rangle \subseteq \langle M\rangle$. Therefore $\langle M\rangle = \langle\mathcal{P}\rangle$, or, equivalently, $\mathcal{P} = [M]$.

The implication (iii)⇒(i) is obvious. ∎

7.4. The spectrum $\mathbf{Spec}^0_{\mathfrak{t}}(X)$. The *spectrum*, $\mathbf{Spec}^0_{\mathfrak{t}}(X)$, of X consists of all $\mathcal{P} \in \mathfrak{T}_\star(X)$ such that $\langle\mathcal{P}\rangle$ is a thick subcategory of C_X. Here $_{\mathfrak{t}}$ stands for 'topologizing'.

Let $Spec^0_{\mathfrak{t}}(X)$ the family of all objects M of C_X such that $[M] \in \mathbf{Spec}^0_{\mathfrak{t}}(X)$. In other words, $M \in Spec^0_{\mathfrak{t}}(X)$ iff $\langle M\rangle$ is a thick subcategory of C_X.

7.5. Proposition. *$Spec^0_{\mathfrak{t}}(X) \subseteq Spec^0_{\mathfrak{T}\mathfrak{h}}X$.*

Proof. For every $M \in ObC_X$, the subcategory $\langle M\rangle_\bullet$ is contained in $\langle M\rangle$. In fact, $Ob\langle M\rangle_\bullet = \{N \in ObC_X \mid M \notin [N]_\bullet\} \subseteq \{N \in ObC_X \mid M \notin [N]\} = Ob\langle M\rangle$.

If $M \in Spec^0_{\mathfrak{t}}(X)$, i.e. $\langle M\rangle$ is a thick subcategory of the category C_X, then we have the inverse inclusion: $\langle M\rangle \subseteq \langle M\rangle_\bullet$. Therefore $\langle M\rangle = \langle M\rangle_\bullet$ and $\langle M\rangle_\bullet$ is a thick subcategory of the category C_X, i.e. $M \in Spec^0_{\mathfrak{T}\mathfrak{h}}X$. ∎

7.6. The spectra $Spec(X)$ and $\mathbf{Spec}(X)$. Let $Spec(X)$ denote the family of all nonzero objects M such that $\langle M\rangle$ is contained in ${}^\perp M$. Recall that ${}^\perp M$ is the full subcategory of C_X whose objects are left orthogonal to M; i.e. $Ob^\perp M = \{L \in ObC_X \mid C_X(L, M) = 0\}$. In other words, $M \in Spec(X)$ iff $M \in [N]$ for every object N such that there exists a nonzero morphism $N \longrightarrow M$.

We set $\mathbf{Spec}(X) = \{[M] \mid M \in Spec(X)\}$.

7.6.1. Proposition. *(a) If $M \in Spec(X)$, then $\langle M\rangle$ is a Serre subcategory of the category C_X. In particular, $\mathbf{Spec}(X)$ is contained in $\mathbf{Spec}^0_{\mathfrak{t}}(X)$.*

(b) $Spec(X) = Spec^0_{\mathfrak{t}}(X) \bigcap Spec^0_{\mathfrak{s}}(X)$.

(c) If X has the property (sup), then $\mathbf{Spec}(X) = \{\mathcal{P} \in \mathfrak{T}_\star(X) \mid \langle\mathcal{P}\rangle$ is a Serre subcategory of $C_X\}$.

Proof. (a) Since $\langle M\rangle$ is closed under taking subquotients, $\langle M\rangle \subseteq \langle M\rangle^-$ for any nonzero object M of the category C_X. Since $\langle M\rangle^-$ is a Serre (in particular topologizing) category, $\langle M\rangle \neq \langle M\rangle^-$ iff $M \in Ob\langle M\rangle^-$. The latter implies the existence of a nonzero morphism $N \longrightarrow M$ with $N \in \langle M\rangle$. If M were an object of $Spec(X)$, then $M \in [N] \subseteq \langle M\rangle$ which is impossible by the definition of $\langle M\rangle$. Therefore $\langle M\rangle = \langle M\rangle^-$.

(b) If $M \in Spec(X)$, then $\langle M\rangle$ is a Serre (in particular, thick) subcategory of the category C_X. By (the argument of) 7.5, $\langle M\rangle_\bullet = \langle M\rangle$. In particular, $\langle M\rangle_\bullet$ is a Serre subcategory and M is $\langle M\rangle_\bullet$-torsion free, i.e. $M \in Spec^0_{\mathfrak{s}}(X)$. Together with (a) above, this gives the inclusion $Spec(X) \subseteq Spec^0_{\mathfrak{t}}(X) \bigcap Spec^0_{\mathfrak{s}}(X)$.

Conversely, let $M \in Spec^0_{\mathfrak{t}}(X) \bigcap Spec^0_{\mathfrak{s}}(X)$. This means that $\langle M\rangle$ is a thick subcategory and $\langle M\rangle_\bullet$ is a Serre subcategory of the category C_X, and M is $\langle M\rangle_\bullet$-torsion free. But, by (the argument of) 7.5, the subcategories $\langle M\rangle$ and $\langle M\rangle_\bullet$ coincide. Therefore $M \in Spec(X)$.

(c) If X has a property (sup), then by 6.3.10, $\mathbf{Spec}^0_{\mathfrak{s}}(X) = \{\mathcal{P} \in \mathfrak{T}_\star(X) \mid \langle\mathcal{P}\rangle_\bullet$ is a Serre subcategory of $C_X\}$.

It follows from (the argument of) (b) above that $\mathbf{Spec}(X) = \mathbf{Spec}^0_{\mathfrak{t}}(X) \bigcap \mathbf{Spec}^0_{\mathfrak{s}}(X) = \{\mathcal{P} \in \mathfrak{T}_\star(X) \mid \langle\mathcal{P}\rangle_\bullet = \langle\mathcal{P}\rangle$ is a Serre subcategory of $C_X\}$. ∎

7.6.2. Proposition. *The following conditions are equivalent:*

(a) Every object of $Spec^0_{\mathfrak{s}}(X)$ contains a subobject which belongs to $Spec(X)$.

(b) The map which assigns to every topologizing subcategory, $\mathbb{T}$, of the category C_X the minimal thick subcategory, $\mathbb{T}_\bullet$, containing $\mathbb{T}$ induces an isomorphism

$$\mathbf{Spec}(X) \xrightarrow{\sim} \mathbf{Spec}^0_{\mathfrak{s}}(X).$$

Proof. (a)⇒(b). It follows from 7.6.1 that the map $\mathbb{T} \longmapsto \mathbb{T}_\bullet$ induces an embedding $\mathbf{Spec}(X) \hookrightarrow \mathbf{Spec}^0_{\mathfrak{s}}(X)$. By definition, $\mathbf{Spec}^0_{\mathfrak{s}}(X) = \{[P]_\bullet \mid P \in Spec^0_{\mathfrak{s}}(X)\}$. If P' is a nonzero subobject of an object $P \in Spec^0_{\mathfrak{s}}(X)$, then $[P']_\bullet = [P]_\bullet$. If $P' \in Spec(X)$, then $[P]_\bullet$ is the image of an object of $\mathbf{Spec}(X)$.

(b)⇒(a). By (b), for every $P' \in Spec^0_{\mathfrak{s}}(X)$, there exists a $P \in Spec(X)$ such that $[P]_\bullet = [P']_\bullet$. In particular, $P' \in Ob[P]_\bullet$. Since $[P]_\bullet \subseteq [P]^-$ and P' is a nonzero object, P' has a nonzero subobject, P'', which belongs to the subcategory $[P]$. Because $P' \in Spec^0_{\mathfrak{s}}(X)$, and P'' is a nonzero subobject of P', the objects P' and P'' are equivalent, i.e. $[P'']_\bullet = [P']_\bullet$, whence $[P'']_\bullet = [P]_\bullet$. The equality $[P'']_\bullet = [P]_\bullet$ is equivalent to the equality $\langle P''\rangle_\bullet = \langle P\rangle_\bullet$, and $\langle P''\rangle_\bullet = \langle P\rangle$. In particular, $P'' \notin \langle P\rangle$. Since $P \in Spec(X)$, the latter means that $P'' \succ P$, or, what is the same, $P \in Ob[P'']$. Thus, $P'' \in Ob[P]$ and $P \in Ob[P'']$ which means that $[P''] = [P] \in \mathbf{Spec}(X)$. ∎

8. Local 'spaces' and the related spectra.

Let C_X be an abelian category. We define the *residue cosubspace*, $X_{\mathfrak{t}}$, of X by setting $C_{X_{\mathfrak{t}}}$ equal to the intersection of all nonzero topologizing subcategories of C_X. We call X and the category C_X *local* if the *residue subcategory* $C_{X_{\mathfrak{t}}}$ is nonzero.

Thus, if X is local, then $[M] = C_{X_{\mathfrak{t}}}$ and $\langle M\rangle = \mathbb{O}$ for every nonzero object of the residue subcategory $C_{X_{\mathfrak{t}}}$. In particular, every nonzero object of $C_{X_{\mathfrak{t}}}$ belongs to $Spec(X)$, and all nonzero objects of $C_{X_{\mathfrak{t}}}$ are *equivalent* to each other, i.e. they define a unique smallest element of $\mathbf{Spec}(X)$.

8.1. Note. Local categories are defined in [R, Ch.3] as categories with *quasi-final* objects. An object P is called *quasi-final* if $N \succ P$ for every nonzero object N. One can see that quasi-final objects are precisely nonzero objects of the residue subcategory $C_{X_{\mathfrak{t}}}$.

8.2. Proposition. *(a) Every local 'space' is $\mathfrak{L}$-local.*

(b) Let X be an $\mathfrak{L}$-local 'space' with the residue thick cosubspace $X_\star$ such that $Spec^0_{\mathfrak{t}}(X)_\star$ is non-empty. Then X is local.

Proof. (a) If X is local, then the residue thick subcategory (the intersection of all nonzero thick subcategories of C_X) coincides with the thick envelope of the residue subcategory $C_{X_{\mathfrak{t}}}$.

(b) Let X be a $\mathfrak{L}$-local space with non-empty $Spec^0_{\mathfrak{t}}(X)_\star$. Let $P \in Spec^0_{\mathfrak{t}}(X)_\star$, that is P is a nonzero object of the residue thick subcategory $C_{X_\star}$ and $\langle P\rangle$ is a thick subcategory. Since $P \notin Ob\langle P\rangle$ and $C_{X_\star} = [P]_\bullet$ is the smallest nonzero thick subcategory of the category C_X, the subcategory $\langle P\rangle$ is zero. This means that P is a quasi-final object (see 8.1), or, equivalently, P is contained in any nonzero topologizing subcategory of the category C_X. Therefore X is local with the residue subcategory $[P]$. ■

8.3. Corollary. *If C_X has simple objects, then X is local iff it is $\mathfrak{L}$-local.*

Proof. Suppose X is $\mathfrak{L}$-local and it has a simple object, M. By 6.5.1, the residue thick subcategory of C_X coincides with the minimal thick subcategory, $[M]_\bullet$, containing M. In particular, $M \in Spec(X)$. The assertion now follows from 8.2(b). ■

8.4. Proposition. *For every $P \in Spec(X)$, the category $C_X/\langle P\rangle$ is local.*

Proof. The localization functor $C_X \xrightarrow{q^*_{\langle P\rangle}} C_X/\langle P\rangle$ is exact and its kernel coincides with the Serre subcategory $\langle P\rangle$. Every nonzero object of the quotient category $C_X/\langle P\rangle$ is isomorphic to an object $q^*_{\langle P\rangle}(N)$, where $N \notin Ob\langle P\rangle$, i.e. $N \succ P$. Being exact, the functor $q^*_{\langle P\rangle}$ respects the relation $\succ$; in particular, $q^*_{\langle P\rangle}(N) \succ q^*_{\langle P\rangle}(P)$. Since $q^*_{\langle P\rangle}(P) \neq 0$, this shows that $q^*_{\langle P\rangle}(P)$ is a quasi-final object of the category $C_X/\langle P\rangle$, that is $[q^*_{\langle P\rangle}(P)]$ is the residue subcategory of $C_X/\langle P\rangle$. ■

8.5. The complete spectrum and the S-spectrum. The elements of the complete spectrum, $\mathbf{Spec}^1_{\mathfrak{t}}(X)$, of the 'space' X is the family of all thick subcategories C_Y such that the quotient 'space' X/Y (defined by $C_{X/Y} = C_X/C_Y$) is local.

The *S-spectrum*, $\mathbf{Spec}^-(X)$, is formed by Serre subcategories, C_V of C_X such that X/V is local. Thus, $\mathbf{Spec}^-(X) \subseteq \mathbf{Spec}^1_{\mathfrak{t}}(X)$.

The notion of the complete spectrum is introduced in [R, Ch.6], details on the S-spectrum can be found in [R, Ch.3 and Ch.6], where it is called the *flat spectrum.*

The map $\mathcal{P} \longmapsto \langle\mathcal{P}\rangle$ induces injective morphisms

$$\mathbf{Spec}^0_{\mathfrak{t}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{t}}(X) \quad \text{and} \quad \mathbf{Spec}(X) \longrightarrow \mathbf{Spec}^-(X). \tag{1}$$

Since 'local' implies '$\mathfrak{L}$-local', we have inclusions

$$\mathbf{Spec}^1_{\mathfrak{t}}(X) \subseteq \mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}}(X) \quad \text{and} \quad \mathbf{Spec}^-(X) \subseteq \mathbf{Spec}^1_{\mathfrak{s}}(X). \tag{2}$$

The image of the map $\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}^-(X)$ is described as follows.

8.5.1. Proposition. *An element $\mathcal{P}$ of $\mathbf{Spec}^-(X)$ belongs to the image of $\mathbf{Spec}(X)$ iff there exists an element, M, of $Spec(X)$ which belongs to $Ob\mathcal{P}^\star - Ob\mathcal{P}$.*

Proof. The image of $\mathbf{Spec}(X)$ in $\mathbf{Spec}^-(X)$ consists of Serre subcategories of the form $\langle M\rangle$, where $M \in Spec(X)$. By definition of $\langle M\rangle$, the object M does not belong to $\langle M\rangle$ and, obviously, does belong to $\langle M\rangle^\star = ([M] \cup \langle M\rangle)^-$.

Conversely, let $\mathcal{P} \in \mathbf{Spec}^-(X)$ and let $M \in Spec(X)$ does not belong to the subcategory $\mathcal{P}$. The claim is that $\mathcal{P} = \langle M\rangle$.

Since $M \notin Ob\mathcal{P}$, we have the inclusion $\mathcal{P} \subseteq \langle M\rangle$. Let q^* denote a localization functor $C_X \longrightarrow C_X/\mathcal{P}$. By [R, 3.2.2], since M belongs to $Spec(X)$ and does not belong to $\mathcal{P}$, the object $q^*(M)$ belongs to $Spec(X/|\mathcal{P}|)$. Here $C_{X/|\mathcal{P}|} = C_X/\mathcal{P}$. Because $M \in Ob\mathcal{P}^\star$, the object $q^*(M)$ is *quasi-final.* The latter means that $q^*(N) \succ q^*(M)$ for every $N \in ObC_X - Ob\mathcal{P}$; i.e., there exists a diagram

$$\bigoplus_{m\ times} q^*(N) \xleftarrow{\widetilde{\mathfrak{j}}} q^*(\widetilde{K}) \xrightarrow{\widetilde{\mathfrak{e}}} q^*(M), \tag{3}$$

where $\widetilde{\mathfrak{j}}$ is a monomorphism and $\widetilde{\mathfrak{e}}$ is an epimorphism. To the diagram (3), there corresponds a diagram

$$\bigoplus_{m\ times} N \xleftarrow{\mathfrak{j}'} K' \xrightarrow{s} \widetilde{K} \xleftarrow{t} K'' \xrightarrow{\mathfrak{e}''} M \tag{4}$$

such that $q^*(\mathfrak{j}')$ is a monomorphism, $q^*(\mathfrak{e}'')$ is an epimorphism, and $q^*(s)$, $q^*(t)$ are isomorphisms. The diagram (4) can be replaced by the diagram

$$\bigoplus_{m\ times} N \xleftarrow{\mathfrak{j}_1} K_1 \xrightarrow{\mathfrak{e}_1} M \tag{5}$$

where $K_1 = K' \times_{\widetilde{K}} K''$ and $\mathfrak{j}_1$ (resp. $\mathfrak{e}_1$) is the composition of $\mathfrak{j}'$ (resp. $\mathfrak{e}''$) with the projection $K_1 \longrightarrow K'$ (resp. $K_1 \longrightarrow K''$). Since $q^*(\mathfrak{j}_1)$ is a monomorphism, $Ker(\mathfrak{j}_1)$ is an object of $\mathcal{P}$. Notice that the composition of $K_1 \xrightarrow{\mathfrak{e}_1} M$ with the canonical monomorphism $Ker(\mathfrak{j}_1) \hookrightarrow K_1$ is zero, because M is $\mathcal{P}$-torsion free. Therefore $\mathfrak{e}_1$ factors through a morphism $K = K_1/Ker(\mathfrak{j}_1) \xrightarrow{\mathfrak{e}} M$. Thus, we have obtained a diagram

$$\bigoplus_{m\ times} N \xleftarrow{\mathfrak{j}} K \xrightarrow{\mathfrak{e}} M \tag{4}$$

in which $\mathfrak{j}$ is a monomorphism and $\mathfrak{e}$ is nonzero. Let M_1 denote the image of $\mathfrak{e}$. The diagram (4) means, by definition, that $N \succ M_1$. Since $M \in Spec(X)$ and M_1 is a nonzero subobject of M, $M_1 \succ M$. Therefore, $N \succ M$ for every $N \in ObC_X - Ob\mathcal{P}$, or, equivalently, $\langle M\rangle \subseteq \mathcal{P}$. ∎

8.6. The flat spectra. Recall that the *flat complete $\mathfrak{L}$-spectrum*, $\mathbf{Spec}^1_{\mathfrak{f}\mathfrak{L}}(X)$ is formed by all saturated multiplicative systems Σ such that the 'space' of fractions $\Sigma^{-1}X$

is $\mathfrak{L}$-local and the localization, $\Sigma^{-1}X \xrightarrow{q_\Sigma} X$, is continuous (i.e. it has a direct image functor) (see 5.3.5).

The *flat $\mathfrak{L}$-spectrum*, $\mathbf{Spec}^0_{\mathfrak{f}\mathfrak{L}}(X)$ is defined as $\{\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}^\star(X) \mid \widehat{\Sigma} \in \mathbf{Spec}^1_{\mathfrak{f}\mathfrak{L}}(X)\}$.

The map $\mathbb{T} \longmapsto \Sigma_{\mathbb{T}}$ identifies $\mathbf{Spec}^1_{\mathfrak{f}\mathfrak{L}}(X)$ and $\mathbf{Spec}^0_{\mathfrak{f}\mathfrak{L}}(X)$ with corresponding preorders of coreflective thick (hence Serre) subcategories.

8.6.1. Proposition. *(a) There are inclusions:*

$$\mathbf{Spec}^1_{\mathfrak{f}\mathfrak{L}}(X) \subseteq \mathbf{Spec}^1_{\mathfrak{C}}(X) \subseteq \mathbf{Spec}^1_{\mathfrak{s}}(X) \quad \textit{and} \quad \mathbf{Spec}^0_{\mathfrak{f}\mathfrak{L}}(X) \subseteq \mathbf{Spec}^0_{\mathfrak{C}}(X) \subseteq \mathbf{Spec}^0_{\mathfrak{s}}(X).$$

(b) If C_X is a Grothendieck category, then

$$\mathbf{Spec}^1_{\mathfrak{f}\mathfrak{L}}(X) = \mathbf{Spec}^1_{\mathfrak{C}}(X) = \mathbf{Spec}^1_{\mathfrak{s}}(X) \quad \textit{and} \quad \mathbf{Spec}^0_{\mathfrak{f}\mathfrak{L}}(X) = \mathbf{Spec}^0_{\mathfrak{C}}(X) = \mathbf{Spec}^0_{\mathfrak{s}}(X).$$

Proof. (a) Let q^* denote the localization functor $C_X \longrightarrow C_X/\mathcal{P}$, where $\mathcal{P}$ is a thick subcategory. If q^* has a right adjoint, q_*, then the subcategory $\mathcal{P}$ is *coreflective*, i.e. every object of C_X has a $\mathcal{P}$-torsion. This follows from that the $\mathcal{P}$-torsion of an object M is the kernel of the canonical (adjunction) morphism $M \longrightarrow q_*q^*(M)$. The assertion (a) follows now from the fact that every coreflective topologizing (in particular, thick) subcategory is a Serre subcategory of the category C_X.

(b) If C_X is a Grothendieck category, then every Serre subcategory of the category C_X is coreflective, and any exact localization whose kernel is a Serre subcategory has a right adjoint. ■

8.7. The spectra of abelian categories with Gabriel-Krull dimension. We recall the notion of the *Gabriel filtration* of an abelian category as it is presented in [R, 6.6]. Let C_X be an abelian category. The *Gabriel filtration of X* assigns to every ordinal α a Serre subcategory C_{X_α} of C_X which is constructed as folows:

Set $C_{X_0} = \mathbb{O}$.

If α is not a limit ordinal, then C_{X_α} is the smallest Serre subcategory of C_X containing all objects M such that the localization $q^*_{\alpha-1}(M)$ of M at $C_{X_{\alpha-1}}$ has a finite length.

If β is a limit ordinal, then C_{X_β} is the smallest Serre subcategory containing all subcategories C_{X_α} for $\alpha < \beta$.

Let C_{X_ω} denote the smallest Serre subcategory containing all the subcategories C_{X_α}. Clearly the quotient category C_X/C_{X_ω} has no simple objects.

An object M is said to have the *Gabriel-Krull dimension β*, if β is the smallest ordinal such that M belongs to C_{X_β}.

The 'space' X has a Gabriel-Krull dimension if $X = X_\omega$.

Every locally noetherian abelian category (e.g. the category of quasi-coherent sheaves on a noetherian scheme, or the category of left modules over a left noetherian associative algebra) has a Gabriel-Krull dimension.

8.7.1. Proposition. *Suppose C_X is an abelian category with Gabriel-Krull dimension. Then* $\mathbf{Spec}^-(X) = \mathbf{Spec}^1_{\mathfrak{s}}(X)$.

Proof. By [R, 6.6.0.3], if $\mathbb{S}$ is a proper Serre subcategory of C_X, then the quotient category, $C_X/\mathbb{S}$, has simple objects. In particular, for any $\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{s}}(X)$, the $\mathfrak{L}$-local quotient category $C_X/\mathcal{P}$ has simple objects. By 8.3, any $\mathfrak{L}$-local category with simple objects is local, that is $\mathcal{P} \in \mathbf{Spec}^-(X)$. This shows the inclusion $\mathbf{Spec}^1_{\mathfrak{s}}(X) \subseteq \mathbf{Spec}^-(X)$ which together with 8.5(2) above implies the equality $\mathbf{Spec}^-(X) = \mathbf{Spec}^1_{\mathfrak{s}}(X)$. ■

8.7.2. Proposition. *Suppose C_X is an abelian category with Gabriel-Krull dimension and the property (sup). Then the following conditions are equivalent:*

(a) Any nonzero object of C_X has a subobject which belongs to Spec(X).

(b) $\mathbf{Spec}^-(X) = \mathbf{Spec}^1_{\mathfrak{s}}(X) = \mathbf{Spec}^0_{\mathfrak{s}}(X) = \mathbf{Spec}(X)$.

Proof. (a)⇒(b). The equality $\mathbf{Spec}^-(X) = \mathbf{Spec}^1_{\mathfrak{s}}(X)$ holds by 8.7.1. Since there are inclusions $\mathbf{Spec}(X) \subseteq \mathbf{Spec}^0_{\mathfrak{s}}(X) \subseteq \mathbf{Spec}^1_{\mathfrak{s}}(X)$, and $\mathbf{Spec}(X) \subseteq \mathbf{Spec}^-(X)$, it suffices to show that $\mathbf{Spec}^-(X) \subseteq \mathbf{Spec}(X)$.

Let $\mathcal{P} \in \mathbf{Spec}^-(X)$, and let M be an object of C_X such that the image of M in $C_X/\mathcal{P}$ is a simple object. Since C_X has the property (sup), every Serre subcategory of C_X in particular, $\mathcal{P}$, is coreflective. Therefore, every object of C_X has a $\mathcal{P}$-torsion. Replacing the object M by its quotient by the $\mathcal{P}$-torsion, we will assume that M is $\mathcal{P}$-torsion free. By hypothesis (a), M has a subobject, M' which belongs to *Spec*(X). Since M' is $\mathcal{P}$-torsion free, $\mathcal{P} \subseteq \langle M' \rangle$. On the other hand, the Serre subcategory $\langle M' \rangle/\mathcal{P}$ of the category $C_X/\mathcal{P}$ does not contain the image of M, hence $\langle M' \rangle/\mathcal{P} = \mathbb{O}$, that is $\langle M' \rangle = \mathcal{P}$.

(b)⇒(a). Let N be a nonzero object of the category C_X, and let the Gabriel-Krull dimension of N be $\alpha + 1$. Then N is C_{X_α}-torsion free and the image of N in $C_X/C_{X_\alpha} = C_{X/X_\alpha}$ has a simple subobject, M. Let q^*_α denote the localization functor $C_X \longrightarrow C_{X/X_\alpha}$. We can and will assume that M is the image of a subobject, M', of N. We claim that $M' \in$ *Spec*(X).

In fact, the preimage, $\mathcal{P}$, of the Serre subcategory $\langle M \rangle \subseteq C_{X/X_\alpha}$ in C_X belongs to $\mathbf{Spec}^-(X)$. Since, by (b), $\mathbf{Spec}^-(X) = \mathbf{Spec}(X)$, i.e. $\mathcal{P} = \langle P \rangle$ for some $P \in$ *Spec*(X). The object P can be chosen in such a way that $q^*_\alpha(P)$ is a simple object of the category C_{X/X_α}. This implies that $q^*_\alpha(P)$ is isomorphic to the object M. In particular, there exists a monomorphism $q^*_\alpha(P) \longrightarrow q^*_\alpha(N)$, or, equivalently, there exists a diagram $P \xleftarrow{s} P' \xrightarrow{h} N$ such that $q^*_\alpha(s)$ is an isomorphism and $q^*_\alpha(h)$ is a monomorphism. Taking quotient with respect to C_{X_α}-torsion, we can assume that both s and h are monomorphisms. Since $P \in$ *Spec*(X), every nonzero subobject of P is equivalent to P. Thus $P' \in$ *Spec*(X) and $[P'] = [P]$. ■

8.7.3. Corollary. *Let C_X be the category $R - mod$ of left modules over a left noetherian PI ring R. Then*

$$\mathbf{Spec}^-(X) = \mathbf{Spec}^1_{\mathfrak{s}}(X) = \mathbf{Spec}^0_{\mathfrak{s}}(X) = \mathbf{Spec}(X) \simeq SpecR.$$

Here SpecR denotes the prime spectrum of the ring R.

Proof. (a) If $C_X = R - mod$ for an arbitrary associative ring, then every object of *Spec*(X) is equivalent to any of its cyclic submodules, i.e. modules of the form R/p for some left ideal p. The set $Spec_\ell R = \{p \in I_\ell R \mid R/p \in Spec(X)\}$ is called in [R] the *left spectrum* of the ring R. For any pair of left ideals, $\mathfrak{m}$ and $\mathfrak{n}$, in R, the relation $R/\mathfrak{m} \succ R/\mathfrak{n}$ (same as $R/\mathfrak{m} \in [R/\mathfrak{n}]$) means exactly that there exists a finite set, x, of elements of R such that $(\mathfrak{m} : x) \subseteq \mathfrak{n}$ (cf. [R, 3.4]), where $(\mathfrak{m} : x) = \{r \in R \mid rx \subset \mathfrak{m}\}$. If $\mathfrak{m} \in Spec_\ell R$, and a finite set $x \subset R$ is not contained in $\mathfrak{m}$, then $(\mathfrak{m} : x)$ is equivalent to $\mathfrak{m}$, that is $[R/\mathfrak{m}] = [R/(\mathfrak{m} : x)$ (cf. [R, Ch.1]. For any $\mathfrak{m} \in Spec_\ell R$, the set $(\mathfrak{m} : R) = \{a \in R \mid aR \subseteq \mathfrak{m}\}$ (which is the biggest two-sided ideal contained in $\mathfrak{m}$) is a prime ideal in R. If R is a PI ring, then for any left ideal, $\mathfrak{m}$, there exists a finite subset, x, of elements of the ring R such that $(\mathfrak{m} : R) = (\mathfrak{m} : x)$. In particular, every object of $Spec(X)$ is equivalent to the quotient module R/p, where p is a prime ideal. It remains to observe that the restriction of the map $\mathfrak{m} \longmapsto [R/\mathfrak{m}]$ to two-sided ideals is injective (see [R, Ch.1]). All together shows that the map $p \longmapsto [R/p]$ is an isomorphism of $SpecR$ onto $\mathbf{Spec}(X)$.

(b) Let A be an associative unital ring, m a left ideal in A. Consider the set of ideals $(m : x)$, where x runs through $A - m$. Suppose this set has a maximal element, $(m : y)$, with respect to $\leq$. Then $(m : y) \in Spec_\ell(A)$.

In fact, for any $r \in A - (m : y)$, we have $(m : y) \leq ((m : y) : r) = (m : ry) \neq A$. Thanks to the maximality of $(m : y)$, $((m : y) : r) = (m : ry) \leq (m : y)$, hence the assertion.

(c) Every nonzero R-module has a submodule which belongs to $Spec(X)$.

It suffices to prove this fact for an arbitrary quotient module, R/m. For every $x \in R$, let $\bar{x}$ denote the image of x in R/m. Then $Ann(\bar{x}) = (m : x)$. Thus, $\{(m : x) \mid x \in R - m\}$ is the set of annihilators of nonzero elements of the module R/m. By (a), each ideal $(m : x)$ is equivalent to the maximal two-sided ideal contained in $(m : x)$ (the annihilator of the module $R/(m : x)$, which is $((m : x) : R) = (m : Rx)$

Since the ring R is noetherian, the set of two-sided ideals $(m : Rx) \mid x \in R - m\}$ has a maximal element, $(m : Rz)$. Therefore, $(m : z)$ is a maximal element of the set $\{(m : x) \mid x \in R - m\}$. By (b), $(m : z)$ belongs to $Spec_\ell(R)$, hence the cyclic submodule $R\bar{z} \simeq R/(m : z)$ belongs to $Spec(X)$.

The assertion follows now from 8.7.2. ∎

8.7.4. Corollary. *Let C_X be the category $Qcoh_{\mathbf{X}}$ of quasi-coherent sheaves on a scheme $\mathbf{X} = (\mathcal{X}, \mathcal{O})$. If the scheme $\mathbf{X}$ is noetherian, then*

$$\mathbf{Spec}^-(X) = \mathbf{Spec}^1_{\mathfrak{s}}(X) = \mathbf{Spec}^0_{\mathfrak{s}}(X) = \mathbf{Spec}(X) \simeq \mathcal{X}.$$

Proof. Since the category of quasi-coherent sheaves on a noetherian scheme is locally noetherian, the equalities $\mathbf{Spec}^-(X) = \mathbf{Spec}^1_{\mathfrak{s}}(X) = \mathbf{Spec}^0_{\mathfrak{s}}(X) = \mathbf{Spec}(X)$ follow from 8.7.2. By [R, 6.1.1 and 6.1.2], the spectrum $\mathbf{Spec}^-(X)$ is naturally isomorphic to the Gabriel's spectrum of C_X formed by isomorphism classes of indecomposable injective objects. Gabriel proved (in [Gab]) that the underlying space of a noetherian scheme is isomorphic to the Gabriel's spectrum of the category of quasi-coherent sheaves on that scheme. ∎

C2. The spectrum $\mathbf{Spec}^0_{\mathfrak{S}}(X)$.

Fix a triple $(X;\mathfrak{M},\mathcal{E})$, where X is a 'space', $\mathfrak{M}$ (resp. $\mathcal{E}$) is a family of monomorphisms (resp. epimorphisms) closed under the composition and such that $\mathfrak{M}\cap\mathcal{E}$ contains the class $Iso(C_X)$ of all isomorphisms of the category C_X.

A choice of $\mathfrak{M}$ to keep in mind is the class $\mathfrak{M}_s(X)$ of all *strict* monomorphisms of the category C_X. Recall that an arrow $L \xrightarrow{g} M$ is called a *strict monomorphism* if it is determined uniquely up to isomorphism by the family of all pairs of arrows $M \rightrightarrows N$ equalizing g. If fibred coproducts of pairs of monomorphisms $M \xleftarrow{g} L \xrightarrow{g} M$ exist in C_X, then $\mathfrak{M}_s(X)$ consists of all arrows $L \longrightarrow M$ such that the canonical diagram $L \longrightarrow M \rightrightarrows M\coprod_L M$ is exact.

Dually, a standard choice of $\mathcal{E}$ is the family $\mathcal{E}_s(X)$ of all strict epimorphisms of C_X. If C_X has fibred products of pairs of epimorphisms $M \xrightarrow{h} N \xleftarrow{h} M$, then $\mathcal{E}_s(X)$ consists of all arrows $L \longrightarrow M$ such that the canonical diagram $M\prod_N M \rightrightarrows M \longrightarrow N$ is exact.

Thus, if C_X is an abelian category, then every monomorphism (resp. epimorphism) of C_X is strict.

C2.1. Definition of $\mathbf{Spec}^0_{\mathfrak{S}}(X)$. Let $\mathrm{Spec}^0_{\mathfrak{S}}(X)$ denote the preorder (with respect to $\subseteq$) of all $\Sigma \subset Hom(C_X)$ satisfying the following conditions:

(i) The family Σ contains all identical arrows, and $\Sigma \not\subseteq Iso(C_X)$.

(ii) If $ts \in \Sigma$, and either $s \in \mathcal{E}$, or $t \in \mathfrak{M}$, then $s \in \Sigma$.

(iii) If $ts \in \Sigma$ and $s \in Iso(C_X)$, then $t \in \Sigma$.

(iv) $\Sigma \cap \widehat{\Sigma} = Iso(C_X)$, and $\widehat{\Sigma} \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$.

Note that already the properties (i) and (iii) imply that, $Iso(C_X) \subsetneq \Sigma$.

For any family $\Sigma \subseteq Hom(C_X)$, we denote by $\Sigma_\bullet$ the smallest saturated multiplicative system containing Σ. It follows that $\widehat{\Sigma_\bullet} = \widehat{\Sigma}$. We denote by $\mathbf{Spec}^0_{\mathfrak{S}}(X)$ the preorder $(\{\Sigma_\bullet \mid \Sigma \in \mathrm{Spec}^0_{\mathfrak{S}}(X)\}, \subseteq)$. It follows from (iv) above that $\mathbf{Spec}^0_{\mathfrak{S}}(X) \subseteq \mathbf{Spec}^0_{\mathfrak{L}}(X)$.

C2.2. Another realization of $\mathbf{Spec}^0_{\mathfrak{S}}(X)$. Consider the following equivalence relation on $\mathrm{Spec}^0_{\mathfrak{S}}(X)$: $\Sigma \sim \Sigma'$ iff $\Sigma_\bullet = \Sigma'_\bullet$; that is Σ and Σ' are equivalent iff they have the same image in $\mathbf{Spec}^0_{\mathfrak{S}}(X)$. Every equivalence class has the biggest element which is the union of all $\Sigma' \in \mathrm{Spec}^0_{\mathfrak{S}}(X)$ such that $\Sigma'_\bullet = \Sigma_\bullet$, or, what is the same, $\widehat{\Sigma'} = \widehat{\Sigma}$.

In fact, the properties (i), (ii), and (iii) are preserved by the union of every family of classes having these properties. The property (iv) is preserved by the union of a family of equivalent classes.

We denote by $\mathfrak{Spec}^0_{\mathfrak{S}}(X)$ the preorder (with respect to $\subseteq$) of the biggest elements of $\mathrm{Spec}^0_{\mathfrak{S}}(X)$. The map $\Sigma \longmapsto \Sigma_\bullet$ induces an isomorphism of preorders

$$\mathfrak{Spec}^0_{\mathfrak{S}}(X) \xrightarrow{\sim} \mathbf{Spec}^0_{\mathfrak{S}}(X).$$

C2.3. Subcategories $\mathcal{S}_P$ and associated classes of morphisms. Let C_X be an abelian category; and let $\mathfrak{M}$ and $\mathcal{E}$ be the families of resp. all monomorphisms and all epimorphisms of C_X. For any subcategory $\mathbb{T}$ of the category C_X, we set $\Sigma_{\mathbb{T}} = \{s \in HomC_X \mid Ker(s) \in Ob\mathbb{T} \ni Cok(s)\}$.

For any object $P \in C_X$, we denote by $\mathcal{S}_P$ the full subcategory of the category C_X whose objects are all subobjects of P.

C2.3.1. Proposition. *For any nonzero object P, the family of arrows $\Sigma_{\mathcal{S}_P}$ has properties (i), (ii), (iii) of C2.1.*

The family $\Sigma_{\mathcal{S}_P}$ belongs to $\mathrm{Spec}^0_{\mathfrak{S}}(X)$ (i.e. it satisfies the property C2.1(iv)) iff P is an object of $Spec^0_{\mathfrak{s}}(X)$.

Proof. (i) For any $P \in ObC_X$ the family of arrows $\Sigma_{\mathcal{S}_P}$ contains all isomorphisms of the category C_X. If $P = 0$, then $\Sigma_{\mathcal{S}_P} = Iso(C_X)$. If $P \neq 0$, then $Iso(C_X) \subsetneq \Sigma_{\mathcal{S}_P}$, because, for example, the morphism $0 \longrightarrow P$ belongs to $\Sigma_{\mathcal{S}_P}$.

(ii) Suppose that the composition of morphisms $L \xrightarrow{s} M \xrightarrow{t} N$ belongs to $\Sigma_{\mathcal{S}_P}$; i.e. $Ker(ts) \in Ob\mathcal{S}_P \ni Cok(ts)$. Since $Ker(s)$ is a subobject of $Ker(ts)$ the subcategory $\mathcal{S}_P$ is closed under taking subobjects, $Ker(s) \in Ob\mathcal{S}_P$. If $s \in \mathcal{E}$, this means that $s \in \Sigma_{\mathcal{S}_P}$.

If $s \notin \mathcal{E}$, but $t \in \mathfrak{M}$, then the sequence

$$0 \longrightarrow Cok(s) \longrightarrow Cok(ts) \longrightarrow Cok(t) \longrightarrow 0$$

is exact. In particular, $Cok(s) \in Ob\mathcal{S}_P$, because $Cok(ts) \in Ob\mathcal{S}_P$. Therefore $s \in \Sigma_{\mathcal{S}_P}$.

(iii) Evidently, if s is an isomorphism, then $ts \in \Sigma_{\mathcal{S}_P}$ iff $t \in \Sigma_{\mathcal{S}_P}$.

(iv) Let $\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$, and let $\Sigma_{\mathcal{S}_P} \nsubseteq \Sigma$. If $P \in Spec^0_{\mathfrak{s}}(X)$, then $\Sigma_{\mathcal{S}_P} \cap \Sigma = Iso(C_X)$.

In fact, let $\mathbb{T}$ be the kernel of the localization $C_X \longrightarrow \Sigma^{-1}C_X$. Clearly $\Sigma_{\mathcal{S}_P} \nsubseteq \Sigma$ iff $P \notin Ob\mathbb{T}$; and $\Sigma_{\mathcal{S}_P} \cap \Sigma = Iso(C_X)$ iff $\mathcal{S}_P \cap \mathbb{T} = \mathbb{O}$. Thus, the property (iv) is equivalent to the following property:

(iv') For any thick subcategory $\mathbb{T}$ of the category C_X, $P \notin Ob\mathbb{T}$ iff $\mathcal{S}_P \cap \mathbb{T} = \mathbb{O}$.

Notice that $\mathcal{S}_P \cap \mathbb{T} = \mathbb{O}$ iff $\mathbb{T} \in^{\perp} P$. Thus, the condition (iv') means precisely that P is an object of $Spec^0_{\mathfrak{s}}(X)$. ■

C2.3.2. Proposition. *Let C_X be an abelian category. The map $P \longmapsto \Sigma_{\mathcal{S}_P}$ induces an isomorphism $\mathbf{Spec}^0_{\mathfrak{s}}(X) \longrightarrow \mathbf{Spec}^0_{\mathfrak{S}}(X)$.*

Proof. By C2.3.1, the map $P \longmapsto \Sigma_{\mathcal{S}_P}$ induces a morphism $Spec^0_{\mathfrak{s}}(X) \longrightarrow \mathrm{Spec}^0_{\mathfrak{S}}(X)$. The claim is that this morphism induces an isomorphism $\mathbf{Spec}^0_{\mathfrak{s}}(X) \longrightarrow \mathbf{Spec}^0_{\mathfrak{S}}(X)$.

Let $\Sigma \in \mathrm{Spec}^0_{\mathfrak{S}}(X)$, i.e. Σ is a family of arrows of the category C_X satisfying the conditions (i)–(iv) in C2.1. In particular, $\Sigma \cap \widehat{\Sigma} = Iso(C_X)$, or, what is the same, $\Sigma - \widehat{\Sigma} = \Sigma - Iso(C_X)$. Therefore, $s \in \Sigma - \widehat{\Sigma}$ iff $s \in \Sigma$ and either $Ker(s) \neq 0$, or $Cok(s) \neq 0$.

(a) Suppose $Ker(s) \neq 0$. We claim that $Ker(s)$ is an object of $Spec^0_{\mathfrak{T}\mathfrak{h}}X$.

The morphism $M \xrightarrow{s} N$ can be represented as a composition, ts', of an epimorphism $M \xrightarrow{s'} N'$ and a monomorphism $N' \xrightarrow{t} N$. By the property C2.1(ii), $s' \in \Sigma$.

Let $\mathbb{T}$ be a thick subcategory of the category C_X. Suppose $Ker(s)$ has a nonzero subobject, M', which belongs to $\mathbb{T}$. Then s' can be presented as a composition of two epimorphisms, $M \xrightarrow{s''} N'' \xrightarrow{t'} N'$ such that $Ker(s'') = M'$. By C2.1(ii), $s'' \in \Sigma$, hence $s'' \in \Sigma \cap \Sigma_{\mathbb{T}}$, where $\Sigma_{\mathbb{T}} = \{h \in HomC_X \mid Ker(h) \in Ob\mathbb{T} \ni Cok(h)\}$. Since s'' is not an isomorphism, this implies that $\Sigma \subseteq \Sigma_{\mathbb{T}}$, in particular, $Ker(s) \in Ob\mathbb{T}$.

(b) Suppose now that $s \in \Sigma$ is such that $Ker(s) = 0$ and $Cok(s) \neq 0$. We claim that $Cok(s) \in Spec^0_{\mathfrak{T}\mathfrak{h}}X$.

Suppose $Cok(s)$ has a nonzero subobject, M', which belongs to a thick subcategory $\mathbb{T}$ of the category C_X. Then we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{s'} & N' & \longrightarrow & M' & \longrightarrow & 0 \\
 & & {\scriptstyle id_M}\downarrow & & {\scriptstyle t}\downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \xrightarrow{s} & N & \longrightarrow & Cok(s) & \longrightarrow & 0
\end{array}$$

whose vertical arrows are monomorphisms and rows are exact sequences. Here $N' = M' \times_{Cok(s)} N$. Since $s = ts' \in \Sigma$, $s' \in \Sigma \cap \Sigma_{\mathbb{T}}$. Because s' is not an isomorphism, this implies that $\Sigma \subseteq \Sigma_{\mathbb{T}}$, in particular, $Cok(s) \in Ob\mathbb{T}$.

(c) It follows from (a) and (b) that every $\Sigma \in \mathrm{Spec}^0_{\mathfrak{S}}(X)$ contains either an epimorphism, s, such that $Ker(s) \in Spec^0_{\mathfrak{s}}(X)$, or a monomorphism, t, such that $Cok(t) \in Spec^0_{\mathfrak{s}}(X)$.

(c') Consider the first case, i.e. suppose there exists an $s \in \Sigma$ such that $P_s = Ker(s) \in Spec^0_{\mathfrak{s}}(X)$. It follows from the property C2.1(ii) that $\Sigma_{[P_s)} \subseteq \Sigma$, hence $\widehat{\Sigma_{[P_s)}} \subseteq \widehat{\Sigma}$. On the other hand, $\Sigma_{[P_s)}$ contains the morphism s which does not belong to $\widehat{\Sigma}$. Therefore, the inverse inclusion, $\widehat{\Sigma} \subseteq \widehat{\Sigma_{[P_s)}}$, holds, i.e. $\widehat{\Sigma} = \widehat{\Sigma_{[P_s)}} = \Sigma_{\langle P_s\rangle}$. The latter equalities are equivalent to the equalities $\Sigma_{\bullet} = \Sigma_{[P_s)} = \Sigma_{[P_s]_{\bullet}}$.

(c") Suppose now that $s \in \Sigma$ is a monomorphism such that $Cok(s) = P^s$ belongs to $Spec^0_{\mathfrak{s}}(X)$. It follows from the argument in (b) above that $\Sigma_{[P^s)} \subseteq \Sigma$, hence $\widehat{\Sigma_{[P^s)}} \subseteq \widehat{\Sigma}$. Since $s \in \Sigma_{[P^s)} - \widehat{\Sigma}$, the inverse inclusion holds, i.e. $\Sigma_{\bullet} = \Sigma_{[P^s)} = \Sigma_{[P^s]_{\bullet}}$, or, equivalently, $\Sigma_{\bullet} = \Sigma_{[P^s)} = \Sigma_{[P^s]_{\bullet}}$. This concludes the argument. ∎

C2.4. A description of $\mathfrak{Spec}^0_{\mathfrak{S}}(X)$. Let C_X be an abelian category. For any $M \in ObC_X$, we denote by $[M)$ the smallest full subcategory of the category C_X which contains all objects equivalent to M with respect to the preorder $\succ$ and is closed under taking subobjects. Recall that $N \succ L$ iff L is a subquotient of a finite coproduct of copies of N. Objects N and L are equivalent with respect to $\succ$ iff $[N] = [L]$.

One can show that for any object M, the subcategory $[M)$ is closed under finite coproducts. In particular, $[M)$ is an additive (but, in general, not abelian) subcategory of the category C_X.

C2.4.1. Proposition. *Let C_X be an abelian category. Then*

$$\mathfrak{Spec}^0_{\mathfrak{S}}(X) = \{\Sigma_{[P)} \mid P \in Spec^0_{\mathfrak{s}}(X)\}.$$

Proof. If P is an object of $Spec^0_{\mathfrak{s}}(X)$, then every nonzero object of the subcategory $[P)$ is equivalent to P with respect to $\succ$. Thus, $[P)$ is the union of the full subcategories $\mathcal{S}_M$ (whose objects are subobjects of M), where M runs through the class of objects equivalent to P, i.e. such that $M \succ P \succ M$, or, equivalently, $[M] = [P]$. The assertion follows now from (the arguments of) C2.3.1 and C2.3.2. Details are left to the reader. ∎

III. Supports, associated points, and primary decomposition.

9. Supports and weakly associated points.

Fix a 'space' X. For any $S \subseteq HomC_X$, let $Supp_{\mathfrak{L}}(S)$ denote $\{\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid S \not\subseteq \widehat{\Sigma}\}$. We call $Supp_{\mathfrak{L}}(S)$ the *support* of S in $\mathbf{Spec}^0_{\mathfrak{L}}(X)$, or simply the support of S.

It follows that $Supp_{\mathfrak{L}}(S)$ is closed with respect to the specialization preorder: if $\Sigma \in Supp_{\mathfrak{L}}(S)$ and $\Sigma' \in \mathbf{Spec}^0_{\mathfrak{L}}(X)$ is contained in Σ, then $\Sigma' \in Supp_{\mathfrak{L}}(S)$.

9.1. Proposition. *For any set* $\{S_i \subseteq HomC_X \mid i \in J\}$,

$$Supp_{\mathfrak{L}}(\bigcup_{i \in J} S_i) = \bigcup_{i \in J} Supp_{\mathfrak{L}}(S_i).$$

Proof. In fact, by definition, $Supp_{\mathfrak{L}}(\bigcup_{i\in J} S_i) = \{\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid \bigcup_{i\in J} S_i \not\subseteq \widehat{\Sigma}\}$, and $\{\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid \bigcup_{i\in J} S_i \not\subseteq \widehat{\Sigma}\} = \bigcup_{i\in J}\{\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid S_i \not\subseteq \widehat{\Sigma}\} = \bigcup_{i\in J} Supp_{\mathfrak{L}}(S_i)$. ∎

9.2. Corollary. *Let* $S_1 \subseteq HomC_X \supseteq S_2$. *Suppose that* $Iso(C_X)$ *is contained either in* S_1, *or in* S_2. *Then*

$$Supp_{\mathfrak{L}}(S_1 S_2) = Supp_{\mathfrak{L}}(S_1) \bigcup Supp_{\mathfrak{L}}(S_2).$$

Proof. It follows from the hypothesis that $S_1 \cup S_2 \subseteq S_1 S_2$, hence, by 9.1,

$$Supp_{\mathfrak{L}}(S_1) \bigcup Supp_{\mathfrak{L}}(S_2) \subseteq Supp_{\mathfrak{L}}(S_1 \cup S_2) \subseteq Supp_{\mathfrak{L}}(S_1 S_2).$$

Let $[S]_\bullet$ denote the intersection of all saturated multiplicative systems containing S. Then $Supp_{\mathfrak{L}}(S) = Supp_{\mathfrak{L}}([S]_\bullet)$. Therefore, since $S_1 S_2 \subseteq [S_1 \cup S_2]_\bullet$,

$$Supp_{\mathfrak{L}}(S_1 S_2) \subseteq Supp_{\mathfrak{L}}([S_1 \cup S_2]_\bullet) = Supp_{\mathfrak{L}}(S_1 \cup S_2) = Supp_{\mathfrak{L}}(S_1) \bigcup Supp_{\mathfrak{L}}(S_2),$$

whence the assertion. ∎

9.3. Weakly associated points in $\mathbf{Spec}^0_{\mathfrak{L}}(X)$. For any $S \subseteq HomC_X$, set

$$Ass_{\mathfrak{L}}(S) = \{\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid (\Sigma \vee \widehat{\Sigma}) \cap S \not\subseteq \widehat{\Sigma}\}.$$

We call the elements of $Ass_{\mathfrak{L}}(S)$ *weakly associated points* of S in $\mathbf{Spec}^0_{\mathfrak{L}}(X)$, or simply *weakly associated points* of S.

9.3.1. Proposition. *(i) For any* $S \subseteq HomC_X$, $Ass_{\mathfrak{L}}(S) \subseteq Supp_{\mathfrak{L}}(S)$.

(ii) If S *is a saturated multiplicative system, then* $Ass_{\mathfrak{L}}(S) = Supp_{\mathfrak{L}}(S) = \{\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid \Sigma \subseteq S\}$.

(iii) For any set $\{S_i \subseteq HomC_X \mid i \in J\}$,

$$Ass_{\mathfrak{L}}(\bigcup_{i \in J} S_i) = \bigcup_{i \in J} Ass_{\mathfrak{L}}(S_i).$$

Proof. (i) The inclusion $Ass_{\mathfrak{L}}(S) \subseteq Supp_{\mathfrak{L}}(S)$ follows from definitions.

(ii) Suppose $S \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$. Then

$$Supp_{\mathfrak{L}}(S) = \{\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid S \not\subseteq \widehat{\Sigma}\} = \{\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid \widehat{\Sigma} \subseteq \widehat{S}\}$$

and

$$\{\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid \widehat{\Sigma} \subseteq \widehat{S}\} = \{\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid \Sigma \subseteq S\},$$

because $\widehat{\Sigma} \subseteq \widehat{S}$ iff $\Sigma \subseteq S$ (see 2). Obviously, $\{\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid \Sigma \subseteq S\} \subseteq Ass_{\mathfrak{L}}(S)$. By (i), $Ass_{\mathfrak{L}}(S) \subseteq Supp_{\mathfrak{L}}(S)$, hence the assertion.

(iii) We have:

$$\begin{aligned} Ass_{\mathfrak{L}}\Big(\bigcup_{i\in J} S_i\Big) &= \{\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid (\Sigma \vee \widehat{\Sigma}) \bigcap (\bigcup_{i\in J} S_i) \not\subseteq \widehat{\Sigma}\} \\ &= \{\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid \bigcup_{i\in J}((\Sigma \vee \widehat{\Sigma}) \bigcap S_i) \not\subseteq \widehat{\Sigma}\} \\ &= \bigcup_{i\in J}\{\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid (\Sigma \vee \widehat{\Sigma}) \bigcap S_i \not\subseteq \widehat{\Sigma}\} = \bigcup_{i\in J} Ass_{\mathfrak{L}}(S_i) \end{aligned}$$

for any set $\{S_i \subseteq HomC_X \mid i \in J\}$. ■

9.3.2. Proposition. *(i) Let $\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X)$, and let $\Sigma \supseteq S \not\subseteq \Sigma \cap \widehat{\Sigma}$. Then $Supp_{\mathfrak{L}}(S) = \{\Sigma' \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid \Sigma' \subseteq \Sigma\}$.*

(ii) Suppose that the intersection, $S_\bullet$, of all saturated multiplicative systems containing S belongs to $\mathbf{Spec}^0_{\mathfrak{L}}(X)$ and $S \cap \widehat{S} = Iso(C_X)$ (e.g. $S \in \mathrm{Spec}^0_{\mathfrak{S}}(X)$; cf. C2.1). Then $Ass_{\mathfrak{L}}(S) = \{S_\bullet\}$.

Proof. (i) Clearly $\Sigma \in Supp_{\mathfrak{L}}(S)$. Since $Supp_{\mathfrak{L}}(S)$ is closed under specializations, $\{\Sigma' \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid \Sigma' \subseteq \Sigma\} \subseteq Supp_{\mathfrak{L}}(S)$. Suppose $\Sigma' \in Supp_{\mathfrak{L}}(S)$, that is $S \not\subseteq \widehat{\Sigma'}$. Since $S \subseteq \Sigma$, this implies that $\Sigma \not\subseteq \widehat{\Sigma'}$, that is $\widehat{\Sigma'} \subseteq \widehat{\Sigma}$, or, equivalently, $\Sigma' \subseteq \Sigma$.

(ii) Suppose that the conditions of (ii) hold. Let $\Sigma \in Ass_{\mathfrak{L}}(S)$. Set $S' = S \cap (\Sigma \vee \widehat{\Sigma})$. By definition, $\Sigma \in Ass_{\mathfrak{L}}(S)$, iff $S' \not\subseteq \widehat{\Sigma}$; in particular, $S' \not\subseteq Iso(C_X)$. Because $S' \cap \widehat{S} \subseteq Iso(C_X)$, the latter implies that $S' \not\subseteq \widehat{S}$. Since $\Sigma \in Supp_{\mathfrak{L}}(S)$, it follows from (i) that $\Sigma \subseteq S_\bullet$ and, equivalently, $\widehat{\Sigma} \subseteq \widehat{S}$.

Suppose $\widehat{\Sigma} \neq \widehat{S}$, i.e. $\widehat{S}$ contains $\widehat{\Sigma}$ properly. Then, since $\Sigma \vee \widehat{\Sigma}$ is the smallest saturated multiplicative system containing properly $\widehat{\Sigma}$, we have the inclusion $\Sigma \vee \widehat{\Sigma} \subseteq \widehat{S}$. In particular, $S' \subseteq \widehat{S}$, which contradicts to the fact that $S' \not\subseteq \widehat{S}$. ■

9.4. Support and weakly associated points in the complete $\mathfrak{L}$-spectrum. Fix a 'space' X. Let $S \subseteq HomC_X$. We define the *support of S in $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$* by

$$Supp_{\mathfrak{L}}(S) = \{\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) \mid S \not\subseteq \Sigma\}$$

and the *support*, $Supp^1_{\mathfrak{L}}(S)$, of S in $\mathbf{Spec}^1_{\mathfrak{L}}(X)$ by

$$Supp^1_{\mathfrak{L}}(S) = Supp_{\mathfrak{L}}(S) \bigcap \mathbf{Spec}^1_{\mathfrak{L}}(X) = \{\Sigma \in \mathbf{Spec}^1_{\mathfrak{L}}(X) \mid S \not\subseteq \Sigma\}.$$

These supports have the same general properties as the support in $\mathbf{Spec}^0_{\mathfrak{L}}(X)$, as the following proposition shows.

9.4.1. Proposition. *Let $Supp(S)$ denote either $Supp_{\mathfrak{L}}(S)$, or $Supp^1_{\mathfrak{L}}(S)$.*

(a) Let $S \subseteq HomC_X$, and let $[S]_\bullet$ be the intersection of all saturated multiplicative systems containing S. Then $Supp(S) = Supp([S]_\bullet)$.

(b) For any set $\{S_i \subseteq HomC_X \mid i \in J\}$,

$$Supp\big(\bigcup_{i\in J} S_i\big) = \bigcup_{i\in J} Supp(S_i).$$

(c) Let $S_1 \subseteq HomC_X \supseteq S_2$. Suppose that $Iso(C_X)$ is contained either in S_1, or in S_2. Then

$$Supp(S_1S_2) = Supp(S_1)\bigcup Supp(S_2).$$

Proof. It suffices to verify these assertions for the support in $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$.

(a) The equality $Supp_{\mathfrak{L}}(S) = Supp_{\mathfrak{L}}([S]_\bullet)$ is obvious.

(b) For any set $\{S_i \subseteq HomC_X \mid i \in J\}$, we have:

$$\begin{aligned} Supp_{\mathfrak{L}}\big(\bigcup_{i\in J} S_i\big) &= \{\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) \mid \bigcup_{i\in J} S_i \not\subseteq \Sigma\} \\ &= \bigcup_{i\in J}\{\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) \mid S_i \not\subseteq \Sigma\} = \bigcup_{i\in J} Supp_{\mathfrak{L}}(S_i). \end{aligned}$$

(c) The argument is similar to that of 9.2. Details are left to the reader. ∎

9.4.2. Weakly associated points in $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$. For any $\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$, let $\Sigma^\star$ denote the intersection of all saturated multiplicative systems which contain Σ *properly.* For any $S \subseteq HomC_X$, we set

$$Ass^1_{\mathfrak{L}}(S) = \{\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) \mid S \cap \Sigma^\star \not\subseteq \Sigma\}. \tag{1}$$

The elements of $Ass^1_{\mathfrak{L}}(S)$ will be called *weakly associated points* of S in $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$.

9.4.3. Proposition. *(i) For any $S \subseteq HomC_X$, $Ass^1_{\mathfrak{L}}(S) \subseteq Supp^1_{\mathfrak{L}}(S)$. In particular, $Ass^1_{\mathfrak{L}}(S) \subseteq \mathbf{Spec}^1_{\mathfrak{L}}(X)$.*

(ii) For any set $\{S_i \subseteq HomC_X \mid i \in J\}$,

$$Ass^1_{\mathfrak{L}}\big(\bigcup_{i\in J} S_i\big) = \bigcup_{i\in J} Ass^1_{\mathfrak{L}}(S_i).$$

Proof. (i) In fact, $\mathbf{Spec}^1_{\mathfrak{L}}(X)$ consists of all saturated multiplicative systems Σ such that $\Sigma \neq \Sigma^\star$. Therefore $Ass^1_{\mathfrak{L}}(S) \subseteq \mathbf{Spec}^1_{\mathfrak{L}}(X)$ which implies the inclusion $Ass^1_{\mathfrak{L}}(S) \subseteq Supp^1_{\mathfrak{L}}(S)$.

(ii) We have:

$$\begin{aligned} Ass^1_{\mathfrak{L}}\Big(\bigcup_{i\in J} S_i\Big) &= \{\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) \mid \Sigma^\star \bigcap \Big(\bigcup_{i\in J} S_i\Big) \neq \Sigma\} \\ &= \{\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) \mid \bigcup_{i\in J}(\Sigma^\star \cap S_i) \not\subseteq \widehat{\Sigma}\} \\ &= \bigcup_{i\in J}\{\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) \mid \Sigma^\star \cap S_i \neq \Sigma\} = \bigcup_{i\in J} Ass^1_{\mathfrak{L}}(S_i) \end{aligned}$$

for any set $\{S_i \subseteq HomC_X \mid i \in J\}$. ■

9.4.4. Proposition. *For any $S \subseteq HomC_X$,*

$$Supp_{\mathfrak{L}}(S) = \{\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid \widehat{\Sigma} \in Supp^1_{\mathfrak{L}}(S)\} \tag{2}$$

and

$$Ass_{\mathfrak{L}}(S) = \{\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid \widehat{\Sigma} \in Ass^1_{\mathfrak{L}}(S)\}. \tag{3}$$

Proof. The equality (2) follows from the definitions of $Supp_{\mathfrak{L}}(S)$ and $Supp^1_{\mathfrak{L}}(S)$.

By definition, $Ass_{\mathfrak{L}}(S) = \{\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid (\Sigma \vee \widehat{\Sigma}) \cap S \not\subseteq \widehat{\Sigma}\}$ (cf. 9.3). On the other hand, $\{\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid \widehat{\Sigma} \in Ass^1_{\mathfrak{L}}(S)\} = \{\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid S \cap (\widehat{\Sigma})^\star \not\subseteq \widehat{\Sigma}\}$. Notice that, for any $\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X)$, $(\widehat{\Sigma})^\star$ coincides with the smallest saturated multiplicative system, $\widehat{\Sigma} \vee \Sigma$, containing $\Sigma \cup \widehat{\Sigma}$, hence the equality (3). ■

For any $S \subseteq HomC_X$, let $\mathcal{U}_{\mathfrak{L}}(S)$ denote $\{\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) \mid S \subseteq \Sigma\}$.

9.4.5. Proposition. *Let $X \xrightarrow{u} Y$ be an exact localization with an inverse image functor u^*. Then for any $S \subseteq HomC_Y$, the localization u induces*

(i) a bijection of $Ass^1_{\mathfrak{L}}(S) \cap \mathcal{U}_{\mathfrak{L}}(\Sigma_{u^*})$ onto $Ass^1_{\mathfrak{L}}(u^*(S))$;

(ii) a bijection of $Supp^1_{\mathfrak{L}}(S) \cap \mathcal{U}_{\mathfrak{L}}(\Sigma_{u^*})$ onto $Supp^1_{\mathfrak{L}}(u^*(S))$.

Here $\Sigma_{u^*} = \{t \in HomC_Y \mid u^*(t) \in Iso(C_X)\}$.

Proof. By (the argument of) 4.1, the localization functor $C_Y \xrightarrow{u^*} C_X$ induces a map

$$\mathbf{Spec}^1_{\mathfrak{L}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{L}}(Y), \quad \Sigma \longmapsto {u^*}^{-1}(\Sigma). \tag{4}$$

This map induces an isomorphism

$$\mathbf{Spec}^1_{\mathfrak{L}}(X) \longrightarrow \mathcal{U}_{\mathfrak{L}}(\Sigma_{u^*}).$$

(i) By definition 9.4.2,

$$\begin{aligned} Ass^1_{\mathfrak{L}}(u^*(S)) &= \{\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) \mid u^*(S) \cap \Sigma^\star \not\subseteq \Sigma\} \\ &= \{\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) \mid S \cap {u^*}^{-1}(\Sigma^\star) \not\subseteq {u^*}^{-1}(\Sigma)\}. \end{aligned} \tag{5}$$

Since $u^{*^{-1}}(\Sigma^\star) = (u^{*^{-1}}(\Sigma))^\star$, it follows from (5) that

$$Ass^1_{\mathfrak{L}}(u^*(S)) = \{\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) \mid S \cap (u^{*^{-1}}(\Sigma))^\star \nsubseteq u^{*^{-1}}(\Sigma)\}. \tag{6}$$

The map $\Sigma \longmapsto u^{*^{-1}}(\Sigma)$ (see (4) above) induces a bijection of the right hand side of the equality (6) onto $Ass^1_{\mathfrak{L}}(S) \cap \mathcal{U}_{\mathfrak{L}}(\Sigma_{u^*})$.

(ii) Similarly,

$$\begin{aligned} Supp^1_{\mathfrak{L}}(u^*(S)) &= \{\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) \mid u^*(S) \nsubseteq \Sigma\} \\ &= \{\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) \mid S \nsubseteq u^{*^{-1}}(\Sigma)\}. \end{aligned} \tag{7}$$

The map (4) induces a bijection of the right hand side of (7) onto $Supp^1_{\mathfrak{L}}(S) \cap \mathcal{U}_{\mathfrak{L}}(\Sigma_{u^*})$. ■

9.5. Supports and weakly associated points in other spectra. The other spectra considered in this work (and in [R]) are contained (or embedded) in $\mathbf{Spec}^0_{\mathfrak{L}}(X)$, or in $\mathbf{Spec}^1_{\mathfrak{L}}(X)$. If $\mathbf{Spec}_?(X)$ is one of those spectra, we define, for any $S \subseteq HomC_X$, the *support*, $Supp_?(S)$, and *weakly associated points*, $Ass_?(S)$, of S in $\mathbf{Spec}_?(X)$ as the intersection (pull-back) of resp. $Supp^1_{\mathfrak{L}}(S)$ and $Ass^1_{\mathfrak{L}}(S)$ with $\mathbf{Spec}_?(X)$. Thus, we have the *flat support*, $Supp^1_{\mathfrak{fL}}(S) = Supp^1_{\mathfrak{L}}(S) \cap \mathbf{Spec}^1_{\mathfrak{fL}}(X)$, and *flat weakly associated points*, $Ass^1_{\mathfrak{fL}}(S) = Ass^1_{\mathfrak{L}}(S) \cap \mathbf{Spec}^1_{\mathfrak{fL}}(X)$. Similarly, we have support and weakly associated points of S in $\mathbf{Spec}^0_{\mathfrak{S}}(X)$: $Supp_{\mathfrak{S}}(S) = Supp_{\mathfrak{L}}(S) \cap \mathbf{Spec}^0_{\mathfrak{S}}(X)$ and $Ass_{\mathfrak{S}}(S) = Ass_{\mathfrak{L}}(S) \cap \mathbf{Spec}^0_{\mathfrak{S}}(X)$.

9.6. Supports and weakly associated points of objects. For any object M of the category C_X, let $\mathcal{E}_M$ denote the family of all quotient objects $M \longrightarrow L$. We define the *support of* M *in* $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ as the support, $Supp_{\mathfrak{L}}(\mathcal{E}_M)$, of $\mathcal{E}_M$ in $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$, and the *support of* M *in* $\mathbf{Spec}_?(X)$ as the support of $\mathcal{E}_M$ in $\mathbf{Spec}_?(X)$. Here $\mathbf{Spec}_?(X)$ is one of the spectra considered in this work. The notation remains the same: $Supp_?(M)$.

Similarly, we define *weakly associated points* of an object M, as associated points, $Ass_?(\mathcal{E}_M)$, of $\mathcal{E}_M$, and denote it by $Ass_?(M)$.

10. Application to abelian categories.

Let C_X be an abelian category. Then saturated multiplicative systems are represented by thick subcategories of the category C_X, and $\mathbf{Spec}^0_{\mathfrak{L}}(X)$ (resp. $\mathbf{Spec}^1_{\mathfrak{L}}(X)$, resp. $\mathbf{Spec}^0_{\mathfrak{S}}(X)$) is identified with $\mathbf{Spec}^0_{\mathfrak{Th}}(X)$ (resp. $\mathbf{Spec}^1_{\mathfrak{Th}}(X)$, resp. $\mathbf{Spec}^0_{\mathfrak{s}}(X)$).

For any $S \subseteq HomC_X$, the isomorphism $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) \xrightarrow{\sim} \mathfrak{Th}(X)$ (cf. 6.1) identifies $Supp_{\mathfrak{L}}(S)$ with $Supp_{\mathfrak{T}}(S) = \{\mathbb{T} \in \mathfrak{Th}(X) \mid S \nsubseteq \Sigma_{\mathbb{T}}\}$, where

$$\Sigma_{\mathbb{T}} = \{s \in HomC_X \mid Ker(s) \in Ob\mathbb{T} \ni Cok(s)\}.$$

In other words, $\mathbb{T} \in Supp_{\mathfrak{T}}(S)$ iff there is an $s \in S$ such that either the kernel of s, or the cokernel of s does not belong to $\mathbb{T}$.

Recall that, for any thick subcategory, $\mathbb{T}$ of the category C_X, we denote by $\mathbb{T}^\star$ the *residue category* of $\mathbb{T}$ defined as the intersection of all thick subcategories of C_X properly containing $\mathbb{T}$. The *complete spectrum*, $\mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}}(X)$, consists of all thick subcategories $\mathbb{T}$ of the category C_X such that $\mathbb{T}^\star \neq \mathbb{T}$.

10.1. Proposition. *(a) For any $M \in ObC_X$,*

$$Supp_{\mathfrak{T}}(M) = \{\mathbb{T} \in \mathfrak{T}\mathfrak{h}(X) \mid M \notin \mathbb{T}\}.$$

In particular,

$$Supp^1_{\mathfrak{T}\mathfrak{h}}(M) = Supp_{\mathfrak{T}}(M) \bigcap \mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}}(X) = \{\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}}(X) \mid M \notin Ob\mathcal{P}\}.$$

(b) $Ass^1_{\mathfrak{T}\mathfrak{h}}(M)$ consists of all $\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}}(X)$ such that M has a subobject which belongs to $Ob\mathcal{P}^\star - Ob\mathcal{P}$.

(c) Suppose X has the property (sup) (cf. 6.5). Then $Ass_{\mathfrak{s}}(M)$ consists of all $[P]_\bullet$, where $P \in Spec^0_{\mathfrak{s}}(X)$ is a quotient of a subobject of M by its $\langle P\rangle_\bullet$-torsion.

Proof. (a) If an object M of C_X does not belong to a thick subcategory $\mathbb{T}$, then the morphism $M \to 0$ does not belong to the saturated multiplicative system $\Sigma_{\mathbb{T}}$ associated with $\mathbb{T}$, hence $\mathcal{E}_M \nsubseteq \Sigma_{\mathbb{T}}$.

Conversely, $M \in Ob\mathbb{T}$ iff $\mathcal{E}_M$ is a subcategory of $\mathbb{T}$. Thus, $\{\mathbb{T} \in \mathfrak{T}\mathfrak{h}(X) \mid M \notin Ob\mathbb{T}\} = \{\mathbb{T} \in \mathfrak{T}\mathfrak{h}(X) \mid \mathcal{E}_M \nsubseteq \Sigma_{\mathbb{T}}\}$, hence the assertion.

(b) By definition (9.4.2),

$$Ass^1_{\mathfrak{T}\mathfrak{h}}(M) = Ass^1_{\mathfrak{T}\mathfrak{h}}(\mathcal{E}_M) = \{\mathcal{P} \in \mathfrak{T}\mathfrak{h}(X) \mid \mathcal{E}_M \cap (\Sigma_{\mathcal{P}})^\star \nsubseteq \Sigma_{\mathcal{P}}\}.$$

Here $(\Sigma_{\mathcal{P}})^\star$ is the intersection of all saturated multiplicative systems properly containing $\Sigma_{\mathcal{P}}$. Due to the isomorphism between the preorder $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ of saturated multiplicative systems in C_X and the preorder $\mathfrak{T}\mathfrak{h}(X)$ of thick subcategories of C_X, the equality $(\Sigma_{\mathcal{P}})^\star = \Sigma_{\mathcal{P}^\star}$ holds. Therefore, $\mathcal{E}_M \cap (\Sigma_{\mathcal{P}})^\star \nsubseteq \Sigma_{\mathcal{P}}$ iff there is an epimorphism $M \xrightarrow{f} L$ which belongs to $\Sigma_{\mathcal{P}^\star} - \Sigma_{\mathcal{P}}$. This means precisely that $Ker(f) \in Ob\mathcal{P}^\star - Ob\mathcal{P}$.

(c) By definition, $Ass_{\mathfrak{s}}(M) = \{\mathcal{P} \in \mathbf{Spec}^0_{\mathfrak{s}}(X) \mid \mathcal{P} \in Ass^1_{\mathfrak{s}}(M)\}$. Therefore, it follows from (b) above that $Ass_{\mathfrak{s}}(M)$ consists of all $\mathcal{P} \in \mathbf{Spec}^0_{\mathfrak{s}}(X)$ such that M has a subobject, L, which belongs to $Ob\widehat{\mathcal{P}}^\star - Ob\widehat{\mathcal{P}}$. Notice that $\widehat{\mathcal{P}}^\star$ is the thick envelope of $\mathcal{P} \cup \widehat{\mathcal{P}}$; i.e. $\widehat{\mathcal{P}}^\star = \mathcal{P} \vee \widehat{\mathcal{P}}$. Since X has the property (sup), the Serre subcategory $\widehat{\mathcal{P}}$ is coreflective; in particular, the object L has a $\widehat{\mathcal{P}}$-torsion, i.e. a subobject, L' which belong to $\widehat{\mathcal{P}}$ and such that the quotient object L/L' is $\widehat{\mathcal{P}}$-torsion free. Since L/L' is a nonzero object of the smallest Serre subcategory, $(\mathcal{P} \cup \widehat{\mathcal{P}})^-$, containing $\mathcal{P} \cup \widehat{\mathcal{P}}$. Therefore, L/L' has a nonzero subobject, L'', which belongs to $\mathcal{P} \cup \widehat{\mathcal{P}}$. Since L/L' is $\widehat{\mathcal{P}}$-torsion free, the subobject L'' belongs to $\mathcal{P}$. Let $\widetilde{L}$ be the pull-back of L'' in L, that is $\widetilde{L} = L'' \times_{L/L'} L$. Thus $\widetilde{L}$ is a subobject of L which contains L' and such that $\widetilde{L}/L' = L''$ is a $\widehat{\mathcal{P}}$-free nonzero object of the subcategory $\mathcal{P}$; in particular, $L'' \in Ob\mathcal{P} - Ob\widehat{\mathcal{P}}$. Therefore, $\widehat{\mathcal{P}} = \langle L''\rangle_\bullet$, or, equivalently, $[L]_\bullet = \mathcal{P}$. ■

10.2. Proposition. *(i) Let $\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}}X$, and let $M \in Ob\mathcal{P}^\star - Ob\mathcal{P}$. Then $Supp^1_{\mathfrak{T}\mathfrak{h}}(M) = \{\mathcal{P}' \in \mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}} \mid \mathcal{P}' \subseteq \mathcal{P}\}$.*

(ii) If M is a $\mathcal{P}$-torsion free nonzero object of $\mathcal{P}^\star$, then $Ass^1_{\mathfrak{T}\mathfrak{h}}(M) = \{\mathcal{P}\}$.

Proof. (i) Let $\mathcal{P}' \in Supp^1_{\mathfrak{T}\mathfrak{h}}(M)$. The quotient category $\mathcal{P}'/(\mathcal{P}' \cap \mathcal{P})$ is (identified with) a thick subcategory of $C_X/\mathcal{P}$. If $\mathcal{P}'$ were not contained in $\mathcal{P}$, then $\mathcal{P}'/(\mathcal{P}' \cap \mathcal{P})$ would be a nonzero thick subcategory of $C_X/\mathcal{P}$, hence it should contain the subcategory $\mathcal{P}^\star/\mathcal{P}$ which contradicts to the fact that the image of the object M in $C_X/\mathcal{P}$ does not belong to $\mathcal{P}'/(\mathcal{P}' \cap \mathcal{P})$. Thus, $\mathcal{P}'/(\mathcal{P}' \cap \mathcal{P}) = \mathbb{O}$, i.e. $\mathcal{P}' \subseteq \mathcal{P}$.

(ii) Let $\widetilde{\mathcal{P}} \in Ass^1_{\mathfrak{T}\mathfrak{h}}(M)$. By 10.1(b), this means that M has a subobject, L, which belongs to $Ob\widetilde{\mathcal{P}}^\star - Ob\widetilde{\mathcal{P}}$. By hypothesis, M is a $\mathcal{P}$-torsion free object of $\mathcal{P}^\star$, hence every nonzero subobject of M, in particular L, is a $\mathcal{P}$-torsion free object of $\mathcal{P}^\star$.

Since $\widetilde{\mathcal{P}} \in Supp^1_{\mathfrak{T}\mathfrak{h}}(M)$, by (i) above, $\widetilde{\mathcal{P}} \subseteq \mathcal{P}$. Thus, L is a $\widetilde{\mathcal{P}}$-torsion free object of $\widetilde{\mathcal{P}}^\star$. Since $\mathcal{P} \in Supp^1_{\mathfrak{T}\mathfrak{h}}(L)$, by the argument above, $\mathcal{P} \subseteq \widetilde{\mathcal{P}}$, hence $\widetilde{\mathcal{P}} = \mathcal{P}$. ∎

10.3. Proposition. *For any exact sequence*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

$$Supp^1_{\mathfrak{T}\mathfrak{h}}(M) = Supp^1_{\mathfrak{T}\mathfrak{h}}(M') \bigcup Supp^1_{\mathfrak{T}\mathfrak{h}}(M'') \tag{1}$$

and

$$Ass^1_{\mathfrak{T}\mathfrak{h}}(M') \subseteq Ass^1_{\mathfrak{T}\mathfrak{h}}(M) \subseteq Ass^1_{\mathfrak{T}\mathfrak{h}}(M') \bigcup Ass^1_{\mathfrak{T}\mathfrak{h}}(M''). \tag{2}$$

Proof. (a) If $\mathbb{T}$ is a thick subcategory, then $M \in Ob\mathbb{T}$ iff M' and M'' are objects of $\mathbb{T}$. Equivalently, $M \notin Ob\mathbb{T}$ iff either $M' \notin Ob\mathbb{T}$, or $M'' \notin Ob\mathbb{T}$.

(b) Clearly, $Ass^1_{\mathfrak{T}\mathfrak{h}}(M') \subseteq Ass^1_{\mathfrak{T}\mathfrak{h}}(M)$.

Let $\mathcal{P} \in Ass^1_{\mathfrak{T}\mathfrak{h}}(M)$, i.e. M has a subobject, L, which belongs to $Ob\mathcal{P}^\star - Ob\mathcal{P}$. Consider the short exact sequence

$$0 \longrightarrow L' = L \cap M' \longrightarrow L \longrightarrow L'' \longrightarrow 0.$$

Since $L \in Ob\mathcal{P}^\star$, both L' and L'' belong to the subcategory $\mathcal{P}^\star$. Since L does not belong to $\mathcal{P}$, either $L' \notin Ob\mathcal{P}$, and then $\mathcal{P} \in Ass^1_{\mathfrak{T}\mathfrak{h}}(L')$, or $L'' \notin Ob\mathcal{P}$, and then $\mathcal{P} \in Ass^1_{\mathfrak{T}\mathfrak{h}}(L'')$. Notice that L' is a subobject of M' and L'' is a subobject of the object M''. In particular, $Ass^1_{\mathfrak{T}\mathfrak{h}}(L') \subseteq Ass^1_{\mathfrak{T}\mathfrak{h}}(M')$ and $Ass^1_{\mathfrak{T}\mathfrak{h}}(L'') \subseteq Ass^1_{\mathfrak{T}\mathfrak{h}}(M'')$. This proves the assertion. ∎

10.4. Corollary. *For any finite set, $\{M_i \mid i \in J\}$, of objects of C_X,*

$$Ass^1_{\mathfrak{T}\mathfrak{h}}(\oplus_{i\in J} M_i) = \bigcup_{i \in J} Ass^1_{\mathfrak{T}\mathfrak{h}}(M_i). \tag{3}$$

Proof. It suffices to verify the equality (3) for a set consisting of two objects, M_1 and M_2. By 10.3, we have the inclusions

$$Ass^1_{\mathfrak{T}\mathfrak{h}}(M_1) \bigcup Ass^1_{\mathfrak{T}\mathfrak{h}}(M_2) \subseteq Ass^1_{\mathfrak{T}\mathfrak{h}}(M_1 \oplus M_2) \subseteq Ass^1_{\mathfrak{T}\mathfrak{h}}(M_1) \bigcup Ass^1_{\mathfrak{T}\mathfrak{h}}(M_2),$$

hence the assertion. ■

10.5. Corollary. *Let $\{L_i \mid i \in J\}$ be a finite set of subobjects of an object M such that $\bigcap_{i\in J} L_i = 0$. Then*

$$Ass^1_{\mathfrak{Th}}(M) \subseteq \bigcup_{i\in J} Ass^1_{\mathfrak{Th}}(M/L_i). \tag{4}$$

Proof. Since $\bigcap_{i\in J} L_i = 0$, the natural morphism $M \longrightarrow \oplus_{i\in J} M/L_i$ is a monomorphism. Therefore, by 10.3, we have the inclusion $Ass^1_{\mathfrak{Th}}(M) \subseteq Ass^1_{\mathfrak{Th}}(\oplus_{i\in J} M/L_i)$. By 10.4, $Ass^1_{\mathfrak{Th}}(\oplus_{i\in J} M/L_i) = \bigcup_{i\in J} Ass^1_{\mathfrak{Th}}(M/L_i)$. ■

For every subcategory $\mathbb{T}$ of the category C_X, let $\mathcal{U}_{\mathfrak{Th}}(\mathbb{T})$ denote the family of all $\mathcal{P} \in \mathfrak{Th}(X)$ which contain $\mathbb{T}$.

10.6. Proposition. *Let C_Y be an abelian category, and let $X \xrightarrow{u} Y$ be an exact localization with an inverse image functor u^*. Then for any object M of the category C_Y, the localization u induces*

(i) a bijection of $Ass^1_{\mathfrak{Th}}(M) \cap \mathcal{U}_{\mathfrak{Th}}(Ker(u^*))$ onto $Ass^1_{\mathfrak{Th}}(u^*(M))$;

(ii) a bijection of $Supp^1_{\mathfrak{Th}}(M) \cap \mathcal{U}_{\mathfrak{Th}}(Ker(u^*))$ onto $Supp^1_{\mathfrak{Th}}(u^*(M))$.

Proof. The assertion can be deduced from 9.4.5. For the reader's convenience, we give below a direct proof.

The localization functor $C_Y \xrightarrow{u^*} C_X$ induces an isomorphism

$$\mathcal{U}_{\mathfrak{Th}}(Ker(u^*)) \xrightarrow{\widetilde{u}^*} \mathbf{Spec}^1_{\mathfrak{Th}} X$$

with the inverse isomorphism given by $\mathcal{P} \longmapsto u^{*^{-1}}(\mathcal{P})$.

(i) Let $\mathcal{P} \in Ass^1_{\mathfrak{Th}}(M) \cap \mathcal{U}_{\mathfrak{Th}}(Ker(u^*))$, i.e. $Ker(u^*) \subseteq \mathcal{P}$ and the object M has a subobject, L, which belongs to $Ob\mathcal{P}^\star - Ob\mathcal{P}$. It follows that $u^*(L) \in Ob\widetilde{u}^*(\mathcal{P}^\star) - Ob\widetilde{u}^*(\mathcal{P}) = Ob(\widetilde{u}^*\mathcal{P})^\star - Ob\widetilde{u}^*(\mathcal{P})$. Since the functor u^* is left exact, $u^*(L)$ is a subobject of the object $u^*(M)$. This shows that $\widetilde{u}^*(\mathcal{P}) \in Ass^1_{\mathfrak{Th}}(u^*(M))$; i.e. the isomorphism $\widetilde{u}^*$ induces an injective map from $Ass^1_{\mathfrak{Th}}(M) \cap \mathcal{U}_{\mathfrak{Th}}(Ker(u^*))$ to $Ass^1_{\mathfrak{Th}}(u^*(M))$. This map is bijective.

In fact, let $\mathcal{P}_x$ be an element of $Ass^1_{\mathfrak{Th}}(u^*(M))$, i.e. there exists a monomorphism, $L' \longrightarrow u^*(M)$, such that L' belongs to $Ob\mathcal{P}_x^\star - Ob\mathcal{P}_x$. Since u^* is an exact localization (it suffices for u^* to be left exact) and the category C_Y has finite limits, the monomorphism $L' \longrightarrow u^*(M)$ is the composition of an isomorphism $L' \xrightarrow{\sim} u^*(L)$ and a morphism $u^*(L) \xrightarrow{u^*(\mathfrak{j})} u^*(M)$, where $L \xrightarrow{\mathfrak{j}} M$ is a monomorphism. Thus, $u^*(L) \in Ob\mathcal{P}_x^\star - Ob\mathcal{P}_x$, or, equivalently, $L \in Ob(u^{*^{-1}}(\mathcal{P}_x^\star)) - Ob(u^{*^{-1}}(\mathcal{P}_x)) = Ob(u^{*^{-1}}(\mathcal{P}_x))^\star - Ob(u^{*^{-1}}(\mathcal{P}_x))$. This means that, $u^{*^{-1}}(\mathcal{P}_x) \in Ass^1_{\mathfrak{Th}}(M)$.

(ii) The assertion (ii) follows directly from the definition of the support. Details are left to the reader. ■

10.7. The support and weakly associated points in Serre subcategories. Let $\mathfrak{Se}(X)$ denote the preorder of Serre subcategories of the category C_X, that is

$$\mathfrak{Se}(X) = \big(\{\mathbb{T} \in \mathfrak{Th}(X) \mid \mathbb{T}^- = \mathbb{T}\}, \subseteq\big)$$

(cf. 6.3). For any $M \in ObC_X$, we define the *support* of M in $\mathfrak{Se}(X)$ and the *weakly associated points* of M in $\mathfrak{Se}(X)$ by respectively

$$Supp_{\mathfrak{Se}}(M) = Supp_{\mathfrak{T}} \bigcap \mathfrak{Se}(X) \quad \text{and} \quad Ass_{\mathfrak{Se}}(M) = Ass^1_{\mathfrak{Th}} \bigcap \mathfrak{Se}(X).$$

10.7.1. Proposition. *(a) For any exact sequence*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

$$Supp_{\mathfrak{Se}}(M) = Supp_{\mathfrak{Se}}(M') \bigcup Supp_{\mathfrak{Se}}(M'') \tag{5}$$

and

$$Ass_{\mathfrak{Se}}(M') \subseteq Ass_{\mathfrak{Se}}(M) \subseteq Ass_{\mathfrak{Se}}(M') \bigcup Ass_{\mathfrak{Se}}(M''). \tag{6}$$

(b) Suppose X has the property (sup) (cf. 6.5). And let an object M of C_X is a supremum of an ascending family, Ξ, of its subobjects. Then

$$Supp_{\mathfrak{Se}}(M) = \bigcup_{M' \in \Xi} Supp_{\mathfrak{Se}}(M') \quad \text{and} \quad Ass_{\mathfrak{Se}}(M) = \bigcup_{M' \in \Xi} Ass_{\mathfrak{Se}}(M').$$

Proof. The assertion (a) follows from 10.3.

(b) The inclusions

$$\bigcup_{M' \in \Xi} Supp_{\mathfrak{Se}}(M') \subseteq Supp_{\mathfrak{Se}}(M) \quad \text{and} \quad \bigcup_{M' \in \Xi} Ass_{\mathfrak{Se}}(M') \subseteq Ass_{\mathfrak{Se}}(M)$$

follow from (a). Suppose $\mathbb{T} \in Supp_{\mathfrak{Se}}(M)$, i.e. $M \notin Ob\mathbb{T}$ and $\mathbb{T} = \mathbb{T}^-$. Since X has the property (sup), any Serre subcategory of C_X is closed under taking supremums of subobject. In particular, since $M = sup\{M' | M' \in \Xi\}$, $M \notin Ob\mathbb{T}$ iff $M' \notin Ob\mathbb{T}$ for some $M' \in \Xi$. This proves the inverse inclusion, $Supp_{\mathfrak{Se}}(M) \subseteq \bigcup_{M' \in \Xi} Supp_{\mathfrak{Se}}(M')$.

Let $\mathcal{P} \in Ass_{\mathfrak{Se}}(M)$, i.e. M has a subobject L which belongs to $\mathcal{P}^\star - \mathcal{P}$. Thanks to the property (sup), $L = sup\{M' \cap L | M' \in \Xi\}$. Since $\mathcal{P}$ is a Serre subcategory, it is closed under taking supremums of subobjects. Therefore, $L \in Ob\mathcal{P}^\star - Ob\mathcal{P}$ iff $L \cap M' \in Ob\mathcal{P}^\star - Ob\mathcal{P}$ for some $M' \in \Xi$. This proves the inclusion $Ass_{\mathfrak{Se}}(M) \subseteq \bigcup_{M' \in \Xi} Ass_{\mathfrak{Se}}(M')$, hence the assertion. ∎

10.7.2. Corollary. *Let X have the property (sup). Then for any set, $\{M_i \mid i \in J\}$, of objects of C_X,*

$$Ass_{\mathfrak{Se}}(\oplus_{i \in J} M_i) = \bigcup_{i \in J} Ass_{\mathfrak{Se}}(M_i). \tag{7}$$

Proof. If the set $\{M_i \mid i \in J\}$ is finite, then the assertion is a consequence of 10.4. Since $\oplus_{i \in J} M_i$ is the supremum of coproducts $\oplus_{i \in J'} M_i$, where J' runs through finite subsets of J, the fact follows from 10.7.1(b). ∎

10.7.3. Corollary. *Let $\{L_i \mid i \in J\}$ be a finite set of subobjects of an object M such that $\bigcap_{i\in J} L_i = 0$. Then*

$$Ass_{\mathfrak{S}\mathfrak{e}}(M) \subseteq \bigcup_{i\in J} Ass_{\mathfrak{S}\mathfrak{e}}(M/L_i). \tag{8}$$

Proof. The assertion follows from 10.5. ∎

10.7.4. Proposition. *Let C_Y be an abelian category, and let $X \xrightarrow{u} Y$ be an exact localization with an inverse image functor u^*. Then for any object M of the category C_Y, the localization u induces*

(i) a bijection of $Ass_{\mathfrak{S}\mathfrak{e}}(M) \cap \mathcal{U}_{\mathfrak{T}\mathfrak{h}}(Ker(u^*))$ onto $Ass_{\mathfrak{S}\mathfrak{e}}(u^*(M))$;

(ii) a bijection of $Supp_{\mathfrak{S}\mathfrak{e}}(M) \cap \mathcal{U}_{\mathfrak{T}\mathfrak{h}}(Ker(u^*))$ onto $Supp_{\mathfrak{S}\mathfrak{e}}(u^*(M))$.

Proof. The assertion follows from 10.6. ∎

10.7.5. Proposition. *Let $Y \xrightarrow{u} X$ be an exact localization such that $\mathbb{S} = Ker(u^*)$ is a coreflective (in particular, Serre) subcategory of the category C_X. Let $M \in ObC_X$, and let $\mathsf{t}_{\mathbb{S}}M$ denote the $\mathbb{S}$-torsion of M and $M_{\mathbb{S}}$ the quotient object $M/\mathsf{t}_{\mathbb{S}}M$. Then*

$$Ass_{\mathfrak{S}\mathfrak{e}}(\mathsf{t}_{\mathbb{S}}M) = Ass_{\mathfrak{S}\mathfrak{e}}(M) - \mathcal{U}_{\mathfrak{T}\mathfrak{h}}(\mathbb{S}) = Ass_{\mathfrak{S}\mathfrak{e}}(M) - Ass_{\mathfrak{S}\mathfrak{e}}(M_{\mathbb{S}})$$

and

$$\begin{aligned} Ass_{\mathfrak{S}\mathfrak{e}}(M) \cap \mathcal{U}_{\mathfrak{T}\mathfrak{h}}(\mathbb{S}) &= Ass_{\mathfrak{S}\mathfrak{e}}(M_{\mathbb{S}}) \cap \mathcal{U}_{\mathfrak{T}\mathfrak{h}}(\mathbb{S}) \\ &= \{\mathcal{P} \in Ass_{\mathfrak{S}\mathfrak{e}}(M_{\mathbb{S}}) \mid \mathcal{P} \supseteq \mathbb{S} = Ker(u^*)\} \subseteq Ass_{\mathfrak{S}\mathfrak{e}}(M_{\mathbb{S}}). \end{aligned}$$

Proof. By 10.7.4, an inverse image (localization) functor $C_X \xrightarrow{u^*} C_Y$ of the morphism u induces an isomorphism between $Ass_{\mathfrak{S}\mathfrak{e}}(u^*(M))$ and $Ass_{\mathfrak{S}\mathfrak{e}}(M) \cap \mathcal{U}_{\mathfrak{T}\mathfrak{h}}(\mathbb{S})$. Notice that the functor u^* maps the natural epimorphism $M \longrightarrow M_{\mathbb{S}}$ to an isomorphism, i.e.

$$Ass_{\mathfrak{S}\mathfrak{e}}(M) \cap \mathcal{U}_{\mathfrak{T}\mathfrak{h}}(\mathbb{S}) \simeq Ass_{\mathfrak{S}\mathfrak{e}}(u^*(M)) \simeq Ass_{\mathfrak{S}\mathfrak{e}}(u^*(M_{\mathbb{S}})) \simeq Ass_{\mathfrak{S}\mathfrak{e}}(M_{\mathbb{S}}) \cap \mathcal{U}_{\mathfrak{T}\mathfrak{h}}(\mathbb{S}), \tag{9}$$

and $Ass_{\mathfrak{S}\mathfrak{e}}(\mathsf{t}_{\mathbb{S}}M) \subseteq Ass_{\mathfrak{S}\mathfrak{e}}(M) \cap \mathcal{V}^1_{\mathfrak{s}}(\mathbb{S})$, where $\mathcal{V}^1_{\mathfrak{s}}(\mathbb{S}) = \{\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{s}}(X) \mid \mathbb{S} \nsubseteq \mathcal{P}\}$. ∎

It follows from the short exact sequence

$$0 \longrightarrow \mathsf{t}_{\mathbb{S}}M \longrightarrow M \longrightarrow M_{\mathbb{S}} \longrightarrow 0,$$

(9), and 10.7.1 that

$$\begin{aligned} Ass_{\mathfrak{S}\mathfrak{e}}(\mathsf{t}_{\mathbb{S}}M) \subseteq Ass_{\mathfrak{S}\mathfrak{e}}(M) &= \big(Ass_{\mathfrak{S}\mathfrak{e}}(M) \cap \mathcal{V}^1_{\mathfrak{s}}(\mathbb{S})\big) \bigcup \big(Ass_{\mathfrak{S}\mathfrak{e}}(M) \cap \mathcal{U}_{\mathfrak{T}\mathfrak{h}}(\mathbb{S})\big) \\ &= \big(Ass_{\mathfrak{S}\mathfrak{e}}(M) \cap \mathcal{V}^1_{\mathfrak{s}}(\mathbb{S})\big) \bigcup \big(Ass_{\mathfrak{S}\mathfrak{e}}(M_{\mathbb{S}}) \cap \mathcal{U}_{\mathfrak{T}\mathfrak{h}}(\mathbb{S})\big) \\ &\subseteq Ass_{\mathfrak{S}\mathfrak{e}}(\mathsf{t}_{\mathbb{S}}M) \bigcup Ass_{\mathfrak{S}\mathfrak{e}}(M_{\mathbb{S}}). \end{aligned} \tag{10}$$

Since all unions in (10) are disjoint, we obtain that

$$Ass_{\mathfrak{S}\mathfrak{e}}(\mathsf{t}_{\mathbb{S}}M) = Ass_{\mathfrak{S}\mathfrak{e}}(M) \cap \mathcal{V}^1_{\mathfrak{s}}(\mathbb{S}) = Ass_{\mathfrak{S}\mathfrak{e}}(M) - \mathcal{U}_{\mathfrak{T}\mathfrak{h}}(\mathbb{S}) = Ass_{\mathfrak{S}\mathfrak{e}}(M) - Ass_{\mathfrak{S}\mathfrak{e}}(M_{\mathbb{S}}). \quad ∎$$

10.8. Associated points in a proper sense. For every $M \in ObC_X$, we set $\mathfrak{Ass}_{\mathfrak{L}}(M) = \{\mathcal{P} \in \mathfrak{T}(X) \mid$ there exists a nonzero subobject, L, of M which belongs to $\mathcal{P}^\star$ and is $\mathcal{P}$-torsion free$\}$. We call elements of $\mathfrak{Ass}_{\mathfrak{L}}(M)$ *associated points* of M.

10.8.1. Lemma. $\mathfrak{Ass}_{\mathfrak{L}}(M) \subseteq Ass_{\mathfrak{S}\mathfrak{e}}(M)$.

Proof. Let $\mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(M)$, i.e. there exists a nonzero subobject L of M which $\mathcal{P}$-torsion free and belongs to $\mathcal{P}^\star$. But, if L is $\mathcal{P}$-torsion free, it is $\mathcal{P}^-$-torsion free.

Indeed, if an object has a nonzero subobject which belongs to $\mathcal{P}^-$, then the latter should have a nonzero subobject which belongs to $\mathcal{P}$.

If $\mathcal{P}^- \neq \mathcal{P}$, then $\mathcal{P}^\star \subseteq \mathcal{P}^-$ which contradicts to that L is a nonzero $\mathcal{P}^-$-torsion free object of $\mathcal{P}^\star$. ■

General properties of $\mathfrak{Ass}_{\mathfrak{L}}(-)$ are similar to those of $Ass^1_{\mathfrak{T}\mathfrak{h}}(-)$, except for a weaker version of functoriality (see 10.8.2(c) below).

10.8.2. Proposition. *(a) For any exact sequence*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

$$\mathfrak{Ass}_{\mathfrak{L}}(M') \subseteq \mathfrak{Ass}_{\mathfrak{L}}(M) \subseteq \mathfrak{Ass}_{\mathfrak{L}}(M') \bigcup \mathfrak{Ass}_{\mathfrak{L}}(M'').$$

(b) Suppose X has the property (sup) (cf. 6.5). Let an object M of C_X be a supremum of an ascending family, Ξ, of its subobjects. Then

$$\mathfrak{Ass}_{\mathfrak{L}}(M) = \bigcup_{M' \in \Xi} \mathfrak{Ass}_{\mathfrak{L}}(M').$$

(c) For every object M of C_X, any exact localization, $Y \xrightarrow{u} X$, induces an injective map $\mathfrak{Ass}_{\mathfrak{L}}(M) \cap \mathcal{U}_{\mathfrak{T}\mathfrak{h}}(Ker(u^*)) \longrightarrow \mathfrak{Ass}_{\mathfrak{L}}(u^*(M))$.

(d) Let $Y \xrightarrow{u} X$ be an exact localization such that $\mathbb{S} = Ker(u^*)$ is a coreflective (in particular, Serre) subcategory of the category C_X. Let $M \in ObC_X$, and let $\mathfrak{t}_{\mathbb{S}}M$ be the $\mathbb{S}$-torsion of M and $M_{\mathbb{S}}$ the quotient object $M/\mathfrak{t}_{\mathbb{S}}M$. Then the bijective map

$$\mathcal{U}_{\mathfrak{T}\mathfrak{h}}(\mathbb{S}) \longrightarrow \mathfrak{T}\mathfrak{h}(Y), \quad \mathcal{P} \longmapsto \mathcal{P}/\mathbb{S},$$

induces an isomorphism

$$\mathfrak{Ass}_{\mathfrak{L}}(M_{\mathbb{S}}) \xrightarrow{\sim} \mathfrak{Ass}_{\mathfrak{L}}(u^*(M))$$

and

$$\mathfrak{Ass}_{\mathfrak{L}}(\mathfrak{t}_{\mathbb{S}}M) = \mathfrak{Ass}_{\mathfrak{L}}(M) - \mathcal{U}_{\mathfrak{T}\mathfrak{h}}(\mathbb{S}) = \mathfrak{Ass}_{\mathfrak{L}}(M) - \mathfrak{Ass}_{\mathfrak{L}}(M_{\mathbb{S}}) \tag{11}$$

Proof. (a) The inclusion $\mathfrak{Ass}_{\mathfrak{L}}(M') \subseteq \mathfrak{Ass}_{\mathfrak{L}}(M)$ follows from definitions.

Let $\mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(M)$, i.e. there exists a nonzero subobject, L, of M which belongs to $\mathcal{P}^\star$ and is $\mathcal{P}$-torsion free. Suppose $L' = L \cap M' \neq 0$. Then L' is a nonzero subobject of M' which belongs to $\mathcal{P}^\star$ and is $\mathcal{P}$-torsion free, i.e. $\mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(M')$. If $L' = 0$, then the composition of $L \hookrightarrow M$ and the canonical epimorphism $M \longrightarrow M''$ is a monomorphism, hence $\mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(M'')$. This proves the inclusion $\mathfrak{Ass}_{\mathfrak{L}}(M) \subseteq \mathfrak{Ass}_{\mathfrak{L}}(M') \bigcup \mathfrak{Ass}_{\mathfrak{L}}(M'')$.

(b) It follows from (a) that the inclusion $\mathfrak{Ass}_{\mathfrak{L}}(M) \supseteq \bigcup_{M' \in \Xi} \mathfrak{Ass}_{\mathfrak{L}}(M')$ holds without any additional conditions on X.

Let $\mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(M)$, i.e. M has a nonzero subobject, L, which belongs to $\mathcal{P}^\star$ and is $\mathcal{P}$-torsion free. Since X has the property (sup), $L \cap M' \neq 0$ for some $M' \in \Xi$. Then $\mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(M')$ (see the argument in (a) above). This verifies the inverse inclusion, $\mathfrak{Ass}_{\mathfrak{L}}(M) \subseteq \bigcup_{M' \in \Xi} \mathfrak{Ass}_{\mathfrak{L}}(M')$.

(c) Let u^* be an inverse image functor of $Y \xrightarrow{u} X$. Set $Ker(u^*) = \mathbb{S}$. The claim is that the bijective map $\mathcal{U}_{\mathfrak{Th}}(X) \longrightarrow \mathfrak{Th}(Y)$, $\mathcal{P} \longrightarrow \mathcal{P}/\mathbb{S}$, induces a (forcibly injective) map $\mathfrak{Ass}_{\mathfrak{L}}(M) \cap \mathcal{U}_{\mathfrak{Th}}(\mathbb{S}) \longrightarrow \mathfrak{Ass}_{\mathfrak{L}}(u^*(M))$.

Let $\mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(M) \cap \mathcal{U}_{\mathfrak{Th}}(\mathbb{S})$, that is $\mathbb{S} \subseteq \mathcal{P}$, and there exists a nonzero subobject, L, of M which is $\mathcal{P}$-torsion free and belongs to $\mathcal{P}^\star$. Since $\mathbb{S} \subseteq \mathcal{P}$, the object L is $\mathbb{S}$-torsion free. Therefore, $u^*(L)$ is a nonzero subobject of $u^*(M)$ which belongs to $\mathcal{P}^\star/\mathbb{S} = (\mathcal{P}/\mathbb{S})^\star$ and is $\mathcal{P}/\mathbb{S}$-torsion free; that is $\mathcal{P}/\mathbb{S} \in \mathfrak{Ass}_{\mathfrak{L}}(u^*(M))$.

(d) (i) Notice that $\mathfrak{Ass}_{\mathfrak{L}}(M_{\mathbb{S}}) = \mathfrak{Ass}_{\mathfrak{L}}(M_{\mathbb{S}}) \cap \mathcal{U}_{\mathfrak{L}}(\mathbb{S})$.

In fact, let $\mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(M_{\mathbb{S}})$, i.e. $M_{\mathbb{S}}$ has a nonzero subobject, L, such that $L \in \mathcal{P}^\star$ and L is $\mathcal{P}$-torsion free. If $\mathbb{S} \nsubseteq \mathcal{P}$, then $\mathcal{P}^\star \subseteq \mathcal{P} \vee \mathbb{S}$. Therefore, $L \in Ob\mathcal{P} \vee \mathbb{S}$. Since $\mathcal{P} \vee \mathbb{S} \subseteq (\mathcal{P} \cup \mathbb{S})^-$, L contains a nonzero subobject, L', which belongs to $\mathcal{P} \cup \mathbb{S}$. Since L is $\mathcal{P}$-torsion free, the object L' belongs to $\mathbb{S}$. But, L is $\mathbb{S}$-torsion free as well, hence $L' = 0$ which contradicts the choice of L'. Thus, $\mathbb{S} \subseteq \mathcal{P}$.

This shows that $\mathfrak{Ass}_{\mathfrak{L}}(M_{\mathbb{S}}) = \mathfrak{Ass}_{\mathfrak{L}}(M_{\mathbb{S}}) \cap \mathcal{U}_{\mathfrak{Th}}(\mathbb{S})$. Therefore, by (c), the bijective map

$$\mathcal{U}_{\mathfrak{Th}}(\mathbb{S}) \xrightarrow{\sim} \mathfrak{Th}(Y), \quad \mathcal{P} \longmapsto \mathcal{P}/\mathbb{S},$$

induces an embedding $\mathfrak{Ass}_{\mathfrak{L}}(M_{\mathbb{S}}) \longrightarrow \mathfrak{Ass}_{\mathfrak{L}}(u^*(M_{\mathbb{S}}))$. The claim is that this embedding is surjective (hence it is bijective).

Indeed, let $\mathcal{P} \in \mathcal{U}_{\mathfrak{Th}}(\mathbb{S})$ be such that $\mathcal{P}/\mathbb{S} \in \mathfrak{Ass}_{\mathfrak{L}}(u^*(M_{\mathbb{S}}))$, i.e. there exists a nonzero subobject, L', of $u^*(M_{\mathbb{S}}))$ which belongs to the subcategory $(\mathcal{P}/\mathbb{S})^\star = \mathcal{P}^\star/\mathbb{S}$ and is $\mathcal{P}/\mathbb{S}$-torsion free. Replacing L' by an isomorphic object, we can assume that $L' = u^*(L)$. A monomorphism $u^*(L') \longrightarrow u^*(M_{\mathbb{S}}))$ is represented by a diagram $L' \xleftarrow{t} L'' \xrightarrow{f} M_{\mathbb{S}}$ such that $u^*(t)$ is an isomorphism and $u^*(f)$ is a monomorphism. The morphism f can be represented as a composition of an epimorphism, $L'' \xrightarrow{\mathfrak{e}_f} L$, and a monomorphism, $L \xrightarrow{\mathfrak{j}_f} M_{\mathbb{S}}$. Since the functor u^* is exact, in particular right exact, $u^*(\mathfrak{e}_f)$ is an epimorphism. On the other hand, $u^*(\mathfrak{j}_f) \circ u^*(\mathfrak{e}_f) = u^*(f)$ is a monomorphism, hence $u^*(\mathfrak{e}_f)$ is a monomorphism. Thus, $u^*(\mathfrak{e}_f)$ is an isomorphism which implies that $u^*(L) \xrightarrow{u^*(\mathfrak{j}_f)} u^*(M_{\mathbb{S}})$ is a nonzero monomorphism.

Since $\mathbb{S} \subseteq \mathcal{P}^\star$ and $u^*(L) \in \mathcal{P}^\star/\mathbb{S}$, the object L belongs to $\mathcal{P}^\star$. Notice that L is $\mathcal{P}$-torsion free, because $u^*(L)$ is $\mathcal{P}/\mathbb{S}$-torsion free and L, being a subobject of an $\mathbb{S}$-torsion free object $M_{\mathbb{S}}$, is $\mathbb{S}$-torsion free. This means that $\mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(M_{\mathbb{S}})$.

The functor u^* transforms the epimorphism $M \longrightarrow M_{\mathbb{S}}$ into an isomorphism. Therefore, we have isomorphisms

$$\mathfrak{Ass}_{\mathfrak{L}}(M_{\mathbb{S}}) \xrightarrow{\sim} \mathfrak{Ass}_{\mathfrak{L}}(u^*(M_{\mathbb{S}})) \xleftarrow{\sim} \mathfrak{Ass}_{\mathfrak{L}}(u^*(M)).$$

(ii) If $N \in Ob\mathbb{S}$, then $Supp_{\mathfrak{Th}}(N) \cap \mathcal{U}_{\mathfrak{Th}}(\mathbb{S}) = \emptyset$; in particular, $\mathfrak{Ass}_{\mathfrak{L}}(N) \cap \mathcal{U}_{\mathfrak{Th}}(\mathbb{S}) = \emptyset$.

Therefore, we have the inclusion $\mathfrak{Ass}_{\mathfrak{L}}(\mathfrak{t}_{\mathbb{S}}M) \subseteq \mathfrak{Ass}_{\mathfrak{L}}(M) - \mathcal{U}_{\mathfrak{Th}}(\mathbb{S})$. Applying the assertion (a) to the exact sequence

$$0 \longrightarrow \mathfrak{t}_{\mathbb{S}}M \longrightarrow M \longrightarrow M_{\mathbb{S}} \longrightarrow 0,$$

we obtain the inclusions

$$\mathfrak{Ass}_{\mathfrak{L}}(\mathfrak{t}_{\mathbb{S}}M) \subseteq \mathfrak{Ass}_{\mathfrak{L}}(M) - \mathcal{U}_{\mathfrak{Th}}(\mathbb{S}) \subseteq M \subseteq \mathfrak{Ass}_{\mathfrak{L}}(\mathfrak{t}_{\mathbb{S}}M) \bigcup \mathfrak{Ass}_{\mathfrak{L}}(M_{\mathbb{S}}).$$

Since, by (i), $\mathfrak{Ass}_{\mathfrak{L}}(M_{\mathbb{S}}) \subseteq \mathcal{U}_{\mathfrak{Th}}(\mathbb{S})$, this implies the equalities (11). ■

10.8.3. Corollary. *(i) For any finite set, $\{M_i \mid i \in J\}$, of objects of C_X,*

$$\mathfrak{Ass}_{\mathfrak{L}}(\oplus_{i \in J} M_i) = \bigcup_{i \in J} \mathfrak{Ass}_{\mathfrak{L}}(M_i).$$

If X has the property (sup), then the finiteness condition can be dropped.

(ii) Let $\{L_i \mid i \in J\}$ be a finite set of subobjects of an object M such that $\bigcap_{i \in J} L_i = 0$. Then

$$\mathfrak{Ass}_{\mathfrak{L}}(M/(\cap_{i \in J} L_i)) \subseteq \bigcup_{i \in J} \mathfrak{Ass}_{\mathfrak{L}}(M/L_i).$$

Proof. (i) For a finite set $\{M_i \mid i \in J\}$, the assertion follows from 10.8.2(a). The infinite case is a consequence of 10.8.2(b) (see the argument of 10.7.2).

(ii) The assertion follows from (i) and 10.8.2(a) applied to the canonical monomorphism $M/(\cap_{i \in J} L_i) \longrightarrow \oplus_{i \in J} M/L_i$. ■

10.8.4. Corollary. *The full subcategory, $C_{X^{\emptyset}_{\mathfrak{Ass}}}$, of the category C_X whose objects, M, have no associated points, $\mathfrak{Ass}_{\mathfrak{L}}(M) = \emptyset$, is closed under extensions, taking subobjects, and colimits of filtered diagrams of monoarrows.*

Proof. The assertion is a consequence of 10.8.2(a) and (b). ■

10.8.5. Corollary. *Let $Y \xrightarrow{u} X$ be a continuous localization, i.e. u has a direct image functor, u_*. Then for any object, $\mathcal{M}$, of the category C_Y, the map*

$$\mathcal{U}_{\mathfrak{Th}}(Ker(u^*)) \longrightarrow \mathfrak{Th}(Y), \quad \mathcal{P} \longmapsto \mathcal{P}/Ker(u^*),$$

induces an isomorphism $\mathfrak{Ass}_{\mathfrak{L}}(\mathcal{M}) \xrightarrow{\sim} \mathfrak{Ass}_{\mathfrak{L}}(u_*(\mathcal{M}))$.

Proof. If u is continuous than $\mathbb{S} = Ker(u^*)$ is a coreflective thick subcategory of the category C_X: the $\mathbb{S}$-torsion of an object L is the kernel of an adjunction morphism $L \longrightarrow u_*u^*(L)$. Because u is a localization, for any object $\mathcal{M}$ of C_Y, an adjunction arrow $u^*u_*(\mathcal{M}) \longrightarrow \mathcal{M}$ is an isomorphism, in particular, the object $u_*(\mathcal{M})$ is $\mathbb{S}$-torsion free. By 10.8.2(d), the map $\mathcal{P} \longmapsto \mathcal{P}/\mathbb{S}$ induces a bijection $\mathfrak{Ass}_{\mathfrak{L}}(u_*(\mathcal{M})) \xrightarrow{\sim} \mathfrak{Ass}_{\mathfrak{L}}(u^*u_*(\mathcal{M}))$, and $\mathfrak{Ass}_{\mathfrak{L}}(u^*u_*(\mathcal{M})) \simeq \mathfrak{Ass}_{\mathfrak{L}}(\mathcal{M})$. ■

10.8.6. Proposition. *Let $Y \xrightarrow{u} X$ be an exact localization such that $\mathbb{S} = Ker(u^*)$ is a coreflective subcategory of the category C_X. Let $\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{s}}(X)$ and $\mathbb{S} \subseteq \mathcal{P}$. Let M be an object of C_X such that $\mathfrak{Ass}_{\mathfrak{L}}(L) \neq \emptyset$ for any nonzero subobject, L, of M. Then the following conditions are equivalent:*

(a) $\mathfrak{Ass}_{\mathfrak{L}}(M) = \{\mathcal{P}\}$;

(b) $\mathfrak{Ass}_{\mathfrak{L}}(u^*(M)) = \{\mathcal{P}/\mathbb{S}\}$ and M is $\mathbb{S}$-torsion free.

Proof. (a)⇒(b). Let $\mathfrak{t}_{\mathbb{S}}M$ denote the $\mathbb{S}$-torsion of M. If $\mathfrak{t}_{\mathbb{S}}M \neq 0$, then, by hypothesis, $\mathfrak{Ass}_{\mathfrak{L}}(\mathfrak{t}_{\mathbb{S}}M) \neq \emptyset$, i.e. $\mathfrak{Ass}_{\mathfrak{L}}(\mathfrak{t}_{\mathbb{S}}M) = \{\mathcal{P}\}$. The latter means that $\mathfrak{t}_{\mathbb{S}}M$ has a nonzero subobject, L, which belongs to $\mathcal{P}^\star$ and is $\mathcal{P}$-torsion free; in particular, it is $\mathbb{S}$-torsion free which contradicts to that L is a nonzero object of the subcategory $\mathbb{S}$.

Since M is $\mathbb{S}$-torsion free, it follows from 10.8.2(d) that $\mathfrak{Ass}_{\mathfrak{L}}(u^*(M)) = \{\mathcal{P}/\mathbb{S}\}$.

(b)⇒(a). This implication follows from 10.8.2(d). Details are left to the reader. ■

10.8.7. Proposition. *Suppose X has the property (sup). Let $M \in ObC_X$, and let Φ be a subset of $\mathfrak{Ass}_{\mathfrak{L}}(M)$. Then there exists a subobject $L \longrightarrow M$ such that*

$$\mathfrak{Ass}_{\mathfrak{L}}(M/L) = \mathfrak{Ass}_{\mathfrak{L}}(M) - \Phi \quad \textit{and} \quad \mathfrak{Ass}_{\mathfrak{L}}(L) = \Phi. \tag{12}$$

Proof. (a) Let $\mathfrak{D}_\Phi$ be the set of subobjects, M', of M such that $\mathfrak{Ass}_{\mathfrak{L}}(M') \subseteq \Phi$. The set $\mathfrak{D}_\Phi$ is not empty, because it contains the zero subobject. It follows from 10.8.2(b) that $sup\Xi \in \mathfrak{D}_\Phi$ for every filtered subset Ξ of $\mathfrak{D}_\Phi$. Therefore, by Zorn's lemma, there exists a maximal element (subobject), L, in $\mathfrak{D}_\Phi$. We claim that the subobject L satisfies the conditions (12). Thanks to 10.8.2(a), it suffices to show that $\mathfrak{Ass}_{\mathfrak{L}}(M/L) \subseteq \mathfrak{Ass}_{\mathfrak{L}}(M) - \Phi$.

(b) Let $\mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(M/L)$, i.e. M/L has a subobject, $N \longrightarrow M/L$, which belongs to $\mathcal{P}^\star$ and is $\mathcal{P}$-torsion free. Consider the associated with $N \longrightarrow M/L$ short exact sequence

$$0 \longrightarrow L \longrightarrow \widetilde{N} = M \times_{M/L} N \longrightarrow N \longrightarrow 0. \tag{10}$$

By 10.8.2(a), $\mathfrak{Ass}_{\mathfrak{L}}(\widetilde{N}) \subseteq \mathfrak{Ass}_{\mathfrak{L}}(L) \bigcup \mathfrak{Ass}_{\mathfrak{L}}(N)$. By 10.2(ii), $\mathfrak{Ass}_{\mathfrak{L}}(N) = \{\mathcal{P}\}$. Since L is a maximal element of $\mathfrak{D}_\Phi$ and a proper subobject of $\widetilde{N}$, the latter does not belong to $\mathfrak{D}_\Phi$. Therefore $\mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(\widetilde{N}) - \Phi$. ■

10.9. Primary decomposition.

10.9.1. Definition. Let M be an object of an abelian category C_X. We call a subobject N of M *primary*, or *$\mathcal{P}$-primary*, if $\mathfrak{Ass}_{\mathfrak{L}}(M/N)$ consists of one element, $\mathcal{P}$.

10.9.2. Proposition. *Let $\{N_i \mid i \in J\}$ be a finite set of $\mathcal{P}$-primary subobjects of an object M of an abelian category C_X. Then $\bigcap_{i\in J} N_i$ is a $\mathcal{P}$-primary subobject of M.*

Proof. The fact follows from 10.8.3(ii). ■

10.9.3. Definition. Let N be a subobject of an object M of the category C_X. A *primary decomposition of $N \hookrightarrow M$* is a finite set, $\{N_i \mid i \in J\}$, of primary subobjects of M such that N is a subobject of $\bigcap_{i\in J} N_i$ and $\mathfrak{Ass}_{\mathfrak{L}}(\bigcap_{i\in J} N_i/N) = \emptyset$.

10.9.3.1. Note. It follows from this definition and 10.8.3(ii) that if a subobject N of M has a primary decomposition, then $\mathfrak{Ass}_{\mathfrak{L}}(M/N)$ is a subset of $\{\mathcal{P}_i \mid i \in J\}$, in particular, $\mathfrak{Ass}_{\mathfrak{L}}(M/N$ is finite. Here $\mathfrak{Ass}_{\mathfrak{L}}(M/N_i) = \{\mathcal{P}_i\}$.

10.9.4. Proposition. *Let N be a subobject of an object M such that $\mathfrak{Ass}_{\mathfrak{L}}(M/N)$ is finite. Then there exists a primary decomposition, $\{N_{\mathcal{P}} \mid \mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(M/N)\}$, such that $N_{\mathcal{P}}$ is $\mathcal{P}$-primary for every $\mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(M/N)$.*

Proof. Replacing M by M/N, we can and will assume that $N = 0$. By 10.8.7, for every $\mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(M)$, there exists a subobject $N_{\mathcal{P}}$ of M such that $\mathfrak{Ass}_{\mathfrak{L}}(M/N_{\mathcal{P}}) = \{\mathcal{P}\}$ and $\mathfrak{Ass}_{\mathfrak{L}}(N_{\mathcal{P}}) = \mathfrak{Ass}_{\mathfrak{L}}(M) - \{\mathcal{P}\}$. Set $M_0 = \bigcap_{\mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(M)} N_{\mathcal{P}}$. For each $\mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(M)$, $\mathfrak{Ass}_{\mathfrak{L}}(M_0) \subseteq \mathfrak{Ass}_{\mathfrak{L}}(N_{\mathcal{P}})$, hence $\mathfrak{Ass}_{\mathfrak{L}}(M_0) = \emptyset$. ■

10.9.5. Definition. Let N be a subobject of an object M such that $\mathfrak{Ass}_{\mathfrak{L}}(M/N)$ is finite. Let $\{N_i \mid i \in J\}$ be a primary decomposition of N in M with $\mathfrak{Ass}_{\mathfrak{L}}(M/N_i = \{\mathcal{P}_i\}$. The primary decompsition $\{N_i \mid i \in J\}$ is called *reduced* if

(a) for any $i \in J$, $\mathfrak{Ass}_{\mathfrak{L}}(\bigcap_{J \ni j \neq i} N_j / \bigcap_{j \in J} N_j) \neq \emptyset$; in particular, the intersection $\bigcap\limits_{J \ni j \neq i} N_j$ is not a subobject of N_i;

(b) if $i \neq j$, then $\mathcal{P}_i \neq \mathcal{P}_j$.

10.9.5.1. Note. Starting with an arbitrary primary decomposition, one can obtain a reduced primary decomposition as follows. Let $\{N_i \mid i \in J\}$ be any primary decomposition of $N \hookrightarrow M$ with $\mathfrak{Ass}_{\mathfrak{L}}(M/N_i) = \{\mathcal{P}_i\}$, $i \in J$. Set $\Phi = \{\mathcal{P}_i \mid i \in J\}$. Let J_0 is a minimal element of the set of subsets, I, of J such that $\{N_i \mid i \in I\}$ is a primary decomposition. Clearly, $\{N_i \mid i \in J_0\}$ satisfies the condition (a). For each $\mathcal{P} \in \Phi$, let $N_{\mathcal{P}} = \bigcap_{\mathcal{P}_i = \mathcal{P}} N_i$. By 10.9.2, $N_{\mathcal{P}} \hookrightarrow M$ is $\mathcal{P}$-primary. Since $\bigcap_{\mathcal{P} \in \Phi} N_{\mathcal{P}} = \bigcap_{i \in J} N_i$, the set of subobjects $\{N_{\mathcal{P}} \mid \mathcal{P} \in \Phi\}$ is a reduced primary decomposition of $N \hookrightarrow M$.

10.9.6. Proposition. *Let N be a subobject of an object M such that $\mathfrak{Ass}_{\mathfrak{L}}(M/N)$ is finite. Let $\{N_i \mid i \in J\}$ be a primary decomposition of N in M with $\mathfrak{Ass}_{\mathfrak{L}}(M/N_i) = \{\mathcal{P}_i\}$.*

(i) The following conditions are equivalent:

(a) The decomposition $\{N_i \mid i \in J\}$ is reduced.

(b) All $\mathcal{P}_i$ belong to $\mathfrak{Ass}_{\mathfrak{L}}(M/N)$ and $\mathcal{P}_i \neq \mathcal{P}_j$ if $i \neq j$.

(ii) If the equivalent conditions (a), (b) are fulfilled, then

$$\mathfrak{Ass}_{\mathfrak{L}}(M/N) = \{\mathcal{P}_i \mid i \in J\} \quad \textit{and}$$
$$\mathfrak{Ass}_{\mathfrak{L}}(N_i/N) = \{\mathcal{P}_j \mid j \in J,\ j \neq i\} \quad \textit{for all} \ \ i \in J.$$

Proof. (a)⇒(b). Let $\{N_i \mid i \in J\}$ be a reduced primary decomposition. By 10.9.3.1, $\mathfrak{Ass}_{\mathfrak{L}}(M/N)$ is a subset of $\{\mathcal{P}_i \mid i \in J\}$.

Set $N_i^{\vee} = \bigcap_{J \ni j \neq i} N_j$. We can and will assume that $N = \bigcap_{j \in J} N_j = N_i^{\vee} \cap N_i$. Since the decomposition $\{N_i \mid i \in J\}$ is reduced, $\mathfrak{Ass}_{\mathfrak{L}}(N_i^{\vee}/N) \neq \emptyset$. Because $N_i^{\vee}/N$ is isomorphic to the subobject $sup(N_i^{\vee}, N_i)/N_i$ of M/N_i, this implies that $\mathfrak{Ass}_{\mathfrak{L}}(N_i^{\vee}/N) = \{\mathcal{P}_i\}$. This shows the inverse inclusion: $\{\mathcal{P}_i \mid i \in J\} \subseteq \mathfrak{Ass}_{\mathfrak{L}}(M/N)$.

(b)$\Rightarrow$(a). If the condition (b) holds, $\{N_j \mid j \in J - \{i\}\}$ cannot be a primary decomposition, because this would imply that $\mathcal{P}_i \notin \mathfrak{Ass}_{\mathfrak{L}}(M/N)$. Therefore the primary decomposition $\{N_i \mid i \in J\}$ of $N \hookrightarrow M$ is reduced.

The equality $\mathfrak{Ass}_{\mathfrak{L}}(M/N) = \{\mathcal{P}_i \mid i \in J\}$ is already established. It remains to show that for any $i \in J$, $\mathfrak{Ass}_{\mathfrak{L}}(N_i/N) = \{\mathcal{P}_j \mid j \in J,\ j \neq i\}$. Applying 10.8.2(a) to the exact sequence

$$0 \longrightarrow N_i/N \longrightarrow M/N \longrightarrow M/N_i \longrightarrow 0,$$

we obtain inclusions

$$\mathfrak{Ass}_{\mathfrak{L}}(N_i/N) \subseteq \mathfrak{Ass}_{\mathfrak{L}}(M/N) \subseteq \mathfrak{Ass}_{\mathfrak{L}}(N_i/N) \bigcup \mathfrak{Ass}_{\mathfrak{L}}(M/N_i) = \mathfrak{Ass}_{\mathfrak{L}}(N_i/N) \bigcup \{\mathcal{P}_i\}.$$

This and the equality $\mathfrak{Ass}_{\mathfrak{L}}(M/N) = \{\mathcal{P}_j \mid j \in J\}$ imply that

$$\{\mathcal{P}_j \mid j \in J - \{i\}\} \subseteq \mathfrak{Ass}_{\mathfrak{L}}(N_i/N) \subseteq \{\mathcal{P}_j \mid j \in J\}.$$

On the other hand, since $N = \bigcap_{j \in J - \{i\}} (N_i \cap N_j)$, we have an inclusion

$$\mathfrak{Ass}_{\mathfrak{L}}(N_i/N) \subseteq \bigcup_{j \in J - \{i\}} \mathfrak{Ass}_{\mathfrak{L}}(N_i/(N_i \cap N_j)).$$

But, $N_i/(N_i \cap N_j)$ is isomorphic to the subobject $sup(N_i, N_j)/N_j$ of the object M/N_j, hence $\mathfrak{Ass}_{\mathfrak{L}}(N_i/(N_i \cap N_j)) \subseteq \mathfrak{Ass}_{\mathfrak{L}}(M/N_j) = \{\mathcal{P}_j\}$. This gives the inverse inclusion: $\mathfrak{Ass}_{\mathfrak{L}}(N_i/N) \subseteq \{\mathcal{P}_j \mid j \in J - \{i\}\}$. ■

10.9.7. Corollary. *Let $\{N_i \mid i \in J\}$ be a primary decomposition of a subobject N of an object M. Then $Card(\mathfrak{Ass}_{\mathfrak{L}}(M/N)) \leq Card(J)$. The decomposition $\{N_i \mid i \in J\}$ is reduced iff $Card(\mathfrak{Ass}_{\mathfrak{L}}(M/N)) = Card(J)$.*

Proof. Following the procedure of 10.9.5.1, one can obtain, starting from $\{N_i \mid i \in J\}$, a reduced primary decomposition, $\{\widetilde{N}_j \mid j \in I\}$ such that $Card(I) \leq Card(J)$. The rest follows from 10.9.6. ■

For any object M of the category C_X, let $\mathfrak{D}_{\wp}(M)$ denote the set of reduced primary decompositions of $0 \hookrightarrow M$. By 10.9.6, each element of $\mathfrak{D}_{\wp}(M)$ is a set, $\{N_{\mathcal{P}} \mid \mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(M)\}$ of subobjects of M such that $\mathfrak{Ass}_{\mathfrak{L}}(M/N_{\mathcal{P}}) = \{\mathcal{P}\}$ and $\mathfrak{Ass}_{\mathfrak{L}}\big(\bigcap_{\mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(M)} N_{\mathcal{P}}\big) = \emptyset$.

10.9.8. Proposition. *Let $\{N_{\mathcal{P}} \mid \mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(M)\}$ and $\{\widetilde{N}_{\mathcal{P}} \mid \mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(M)\}$ be two elements of $\mathfrak{D}_{\wp}(M)$, and let Φ be a subset of $\mathfrak{Ass}_{\mathfrak{L}}(M)$. Then $\{N_{\mathcal{P}} \mid \mathcal{P} \in \Phi\} \bigcup \{\widetilde{N}_{\mathcal{P}} \mid \mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(M) - \Phi\}$ is an element of $\mathfrak{D}_{\wp}(M)$.*

Proof. Set $N_{\Phi} = \bigcap_{\mathcal{P} \in \Phi} N_{\mathcal{P}}$ and $\widetilde{N}_{\Phi}^{\vee} = \bigcap_{\mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(M) - \Phi} \widetilde{N}_{\mathcal{P}}$. Since $\mathfrak{Ass}_{\mathfrak{L}}(M/N_{\mathcal{P}}) = \{\mathcal{P}\}$ and $\mathfrak{Ass}_{\mathfrak{L}}(M/\widetilde{N}_{\mathcal{P}}) = \{\mathcal{P}\}$ for all $\mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(M)$, it suffices to verify (thanks to 10.9.6) that $\mathfrak{Ass}_{\mathfrak{L}}(N_{\Phi} \bigcap \widetilde{N}_{\Phi}^{\vee}) = \emptyset$.

By 10.9.6(ii), $\mathfrak{Ass}_{\mathfrak{L}}(N_{\mathcal{P}}) = \mathfrak{Ass}_{\mathfrak{L}}(M) - \{\mathcal{P}\}$, in particular, $\mathcal{P} \notin \mathfrak{Ass}_{\mathfrak{L}}(N_{\mathcal{P}})$. Therefore, every element of Φ does not belong to $\mathfrak{Ass}_{\mathfrak{L}}(N_{\Phi})$, i.e. $\Phi \bigcap \mathfrak{Ass}_{\mathfrak{L}}(N_{\Phi}) = \emptyset$. Similarly $(\mathfrak{Ass}_{\mathfrak{L}}(M) - \Phi) \bigcap \mathfrak{Ass}_{\mathfrak{L}}(\widetilde{N}^{\vee}_{\Phi}) = \emptyset$. Thus, $\mathfrak{Ass}_{\mathfrak{L}}(N_{\Phi} \bigcap \widetilde{N}^{\vee}_{\Phi}) \subseteq \Phi \bigcap (\mathfrak{Ass}_{\mathfrak{L}}(M) - \Phi) = \emptyset$. ■

Let $Y \xrightarrow{u} X$ be an exact localization such that $\mathbb{S} = Ker(u^*)$ is a coreflective subcategory of the category C_X. Let $M \in ObC_X$. For every subobject N of the object M, the preimage in M, $N^{\mathbb{S}} = M \times_{M/N} \mathfrak{t}_{\mathbb{S}}(M/N)$, of the $\mathbb{S}$-torsion of M/N is called $\mathbb{S}$-*saturation* of N in M.

10.9.9. Proposition. *(a) Let $Y \xrightarrow{u} X$ be an exact localization such that $\mathbb{S} = Ker(u^*)$ is a coreflective subcategory of the category C_X.*

(a) Let M be an object of the category C_X, and let $\{N_{\mathcal{P}} \mid \mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(M/N)\}$ be a reduced primary decomposition of a subobject N of M. Then

(i) $\{N^{\mathbb{S}}_{\mathcal{P}} \mid \mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(M/N),\ \mathbb{S} \subseteq \mathcal{P}\}$ is a reduced primary decomposition of the $\mathbb{S}$-saturation, $N^{\mathbb{S}}$, of N in M;

(ii) $\{u^*(N_{\mathcal{P}}) \mid \mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(M/N),\ Ker(u^*) \subseteq \mathcal{P}\}$ is a reduced primary decomposition of $u^*(N) \hookrightarrow u^*(M)$.

(b) Suppose $Y \xrightarrow{u} X$ is a continuous localization, i.e. it has a direct image functor, $C_Y \xrightarrow{u_*} C_X$. Let $\mathcal{M}$ be an object of C_Y, and let $\{\mathcal{N}_{\mathcal{P}} \mid \mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(\mathcal{M}/\mathcal{N})\}$ be a reduced primary decomposition of a subobject $\mathcal{N}$ of $\mathcal{M}$. Then $\{u_*(\mathcal{N}_{\mathcal{P}}) \mid \mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(\mathcal{M}/\mathcal{N})\}$ is a reduced primary decomposition of $u_*(\mathcal{N}) \hookrightarrow u_*(\mathcal{M})$.

Proof. (a) Because $M/N^{\mathbb{S}}$ is $\mathbb{S}$-torsion free, it follows from 10.8.2(d) that

$$\mathfrak{Ass}_{\mathfrak{L}}(M/N^{\mathbb{S}}) = \{\mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(M) \mid \mathbb{S} \subseteq \mathcal{P}\}$$

and the map $\mathcal{P} \longmapsto \mathcal{P}/\mathbb{S}$ induces a bijective map $\mathfrak{Ass}_{\mathfrak{L}}(M/N^{\mathbb{S}}) \xrightarrow{\sim} \mathfrak{Ass}_{\mathfrak{L}}(u^*(M/N^{\mathbb{S}}))$. In particular, $\mathfrak{Ass}_{\mathfrak{L}}(M/N^{\mathbb{S}}_{\mathcal{P}}) = \{\mathcal{P}\}$ and $\mathfrak{Ass}_{\mathfrak{L}}(u^*(M/N^{\mathbb{S}}_{\mathcal{P}})) = \{\mathcal{P}/\mathbb{S}\}$. By 10.9.6(ii),

$$\mathfrak{Ass}_{\mathfrak{L}}(N_{\mathcal{P}}/N) = \mathfrak{Ass}_{\mathfrak{L}}(M/N) - \{\mathcal{P}\}.$$

This implies that if $\mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(M/N)$ is such that $\mathbb{S} \subseteq \mathcal{P}$, then $\mathfrak{Ass}_{\mathfrak{L}}(u^*(N_{\mathcal{P}}/N)) = \mathfrak{Ass}_{\mathfrak{L}}(u^*(M/N)) - \{\mathcal{P}/\mathbb{S}\}$. By 10.8.2(d), this equality is equivalent to the equality

$$\mathfrak{Ass}_{\mathfrak{L}}(N^{\mathbb{S}}_{\mathcal{P}}/N^{\mathbb{S}}) = \mathfrak{Ass}_{\mathfrak{L}}(M/N^{\mathbb{S}}) - \{\mathcal{P}\}.$$

Since the functor u^* is left exact,

$$u^*\big(\bigcap_{\mathcal{P}\in\mathfrak{Ass}_{\mathfrak{L}}(M/N^{\mathbb{S}})} N^{\mathbb{S}}_{\mathcal{P}}/N^{\mathbb{S}}\big) \simeq u^*\big(\bigcap_{\mathcal{P}\in\mathfrak{Ass}_{\mathfrak{L}}(M/N^{\mathbb{S}})} N_{\mathcal{P}}/N\big) \simeq \bigcap_{\mathcal{P}\in\mathfrak{Ass}_{\mathfrak{L}}(M/N^{\mathbb{S}})} u^*(N_{\mathcal{P}}/N).$$

It follows from the argument above that $\mathfrak{Ass}_{\mathfrak{L}}\big(\bigcap_{\mathcal{P}\in\mathfrak{Ass}_{\mathfrak{L}}(M/N^{\mathbb{S}})} u^*(N_{\mathcal{P}}/N)\big) = \emptyset$. Therefore, $\{u^*(N_{\mathcal{P}}) \mid \mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(M/N^{\mathbb{S}})$ is a reduced primary decomposition of $u^*(N) \hookrightarrow u^*(M)$. This implies, by 10.8.2(d), that $\{N^{\mathbb{S}}_{\mathcal{P}} \mid \mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(M/N^{\mathbb{S}})$ is a reduced primary decomposition of $N^{\mathbb{S}} \hookrightarrow M$.

(b) By 10.8.5, the bijective map $\mathcal{U}_{\mathfrak{Th}}(\mathbb{S}) \xrightarrow{\sim} \mathfrak{Th}(Y)$, $\mathcal{P} \longmapsto \mathcal{P}/\mathbb{S}$, induces a bijective map $\mathfrak{Ass}_{\mathfrak{L}}(u_*(\mathcal{M})) \xrightarrow{\sim} \mathfrak{Ass}_{\mathfrak{L}}(\mathcal{M})$.

Let $\{\mathcal{N}_{\mathcal{P}} \mid \mathcal{P}/\mathbb{S} \in \mathfrak{Ass}_{\mathfrak{L}}(\mathcal{M})\}$ be a reduced primary decomposition of $\mathcal{N} \hookrightarrow \mathcal{M}$. Since the functor u^* is left exact, for every $\mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(u_*(\mathcal{M}))$ (i.e. such $\mathcal{P}$ that $\mathcal{P}/\mathbb{S} \in \mathfrak{Ass}_{\mathfrak{L}}(\mathcal{M})$) $u_*(\mathcal{N}_{\mathcal{P}})/u_*(\mathcal{N})$ is a subobject of $u_*(\mathcal{N}_{\mathcal{P}}/\mathcal{N})$, and the functor u^* maps the canonical morphism $u_*(\mathcal{N}_{\mathcal{P}})/u_*(\mathcal{N}) \longrightarrow u_*(\mathcal{N}_{\mathcal{P}}/\mathcal{N})$ to an isomorphism. The latter implies (by 10.8.5) that $\mathfrak{Ass}_{\mathfrak{L}}(u_*(\mathcal{N}_{\mathcal{P}})/u_*(\mathcal{N})) = \mathfrak{Ass}_{\mathfrak{L}}(u_*(\mathcal{N}_{\mathcal{P}})/\mathcal{N})) = \{\mathcal{P}/\mathbb{S}\}$. Since u_* is left exact,

$$u_*\Big(\bigcap_{\mathcal{P}\in\mathfrak{Ass}_{\mathfrak{L}}(u_*(\mathcal{M}))} \mathcal{N}_{\mathcal{P}}/\mathcal{N}\Big) \simeq \bigcap_{\mathcal{P}\in\mathfrak{Ass}_{\mathfrak{L}}(u_*(\mathcal{M}))} u_*(\mathcal{N}_{\mathcal{P}}/\mathcal{N}).$$

Therefore,

$$\mathfrak{Ass}_{\mathfrak{L}}\Big(\bigcap_{\mathcal{P}\in\mathfrak{Ass}_{\mathfrak{L}}(u_*(\mathcal{M}))} u_*(\mathcal{N}_{\mathcal{P}})/u_*(\mathcal{N})\Big) \subseteq \mathfrak{Ass}_{\mathfrak{L}}\Big(\bigcap_{\mathcal{P}\in\mathfrak{Ass}_{\mathfrak{L}}(u_*(\mathcal{M}))} u^*(\mathcal{N}_{\mathcal{P}}/\mathcal{N})\Big) = \emptyset.$$

Thanks to 10.9.6, this proves that $\{u_*(\mathcal{N}_{\mathcal{P}}) \mid \mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(\mathcal{M}/\mathcal{N})\}$ is a reduced primary decomposition of $u_*(\mathcal{N}) \hookrightarrow u_*(\mathcal{M})$. ■

10.9.10. Proposition. *Suppose X has the property (sup). Let N be a subobject of an object M such that $\mathfrak{Ass}_{\mathfrak{L}}(L) \neq \emptyset$ for any nonzero subquotient of M/N. Suppose that $\mathfrak{Ass}_{\mathfrak{L}}(M/N)$ is finite, and let $\{N_i \mid i \in J\}$ be a reduced primary decomposition of $N \hookrightarrow M$ with $\mathfrak{Ass}_{\mathfrak{L}}(M/N_i) = \{\mathcal{P}_i\}$. If $\mathcal{P}_i$ is a maximal element of $\mathfrak{Ass}_{\mathfrak{L}}(M/N)$, then N_i equals to $\mathcal{P}_i$-saturation of N, i.e. N_i/N coincides with $\mathcal{P}_i$-torsion of M/N.*

Proof. Replacing M by M/N, we assume that $N = 0$. let $\{N_i \mid i \in J\}$ be a reduced primary decomposition of $0 \hookrightarrow M$ with $\mathfrak{Ass}_{\mathfrak{L}}(M/N_i) = \{\mathcal{P}_i\}$.

Since X has the property (sup), any Serre subcategory of C_X is coreflective. In particular, every $\mathcal{P}_i$, $i \in J$, is coreflective.

By 10.9.9(a)(i), $\{N_j^{\mathcal{P}_i} \mid \mathcal{P}_i \subseteq \mathcal{P}_j\}$ is the rediced primary decomposition of the $\mathcal{P}$-torsion, $\mathfrak{t}_{\mathcal{P}_i}M$, of M. In particular, $\mathfrak{Ass}_{\mathfrak{L}}(M^{\mathcal{P}_i}) = \{\mathcal{P}_j \mid \mathcal{P}_i \subseteq \mathcal{P}_j\} = \{\mathcal{P}_i\}$, because $\mathcal{P}_i$ is maximal. Here $M^{\mathcal{P}_i} = M/\mathfrak{t}_{\mathcal{P}_i}M$. Thus, $\mathfrak{t}_{\mathcal{P}_i}M \hookrightarrow M$ is a $\mathcal{P}_i$-primary subobject. By 10.9.2, the intersection $\widetilde{N}_i = N_i \cap \mathfrak{t}_{\mathcal{P}_i}M$ is a $\mathcal{P}_i$-primary subobject. By 10.8.6, the object $M/\widetilde{N}_i$ is $\mathcal{P}_i$-torsion free for every $i \in J$. Since $\widetilde{N}_i$ is an object of $\mathcal{P}_i$, this implies that $\widetilde{N}_i = \mathfrak{t}_{\mathcal{P}_i}M$, i.e. $\mathfrak{t}_{\mathcal{P}_i}M \hookrightarrow N_i$. Thus, $\{N_i\}$ is a reduced primary decomposition of $\mathfrak{t}_{\mathcal{P}_i}M \hookrightarrow N_i$. Applying to this primary decomposition 10.9.6(ii), we obtain that $\mathfrak{Ass}_{\mathfrak{L}}(N_i/\mathfrak{t}_{\mathcal{P}_i}M) = \emptyset$. Therefore, by hypothesis, $N_i/\mathfrak{t}_{\mathcal{P}_i}M = 0$. ■

10.10. Supports and associated points of objects with a Gabriel-Krull dimension. We fix a 'space' X such that C_X is an abelian category. Let $\{C_{X_\alpha}\}$ be the Gabriel filtration of X (see 8.7).

10.10.1. Proposition. *Suppose $M \in ObC_X$ has a Gabriel-Krull dimension. Then*
(i) $\mathfrak{Ass}_{\mathfrak{L}}(M) \neq \emptyset$. In particular, $Ass_{\mathfrak{Se}}(M)$ is not empty.
(ii) Every element of $Supp_{\mathfrak{Se}}(M)$ is contained in some element of $Ass_{\mathfrak{Se}}(M)$.

Proof. (i) Let β be the biggest ordinal such that the object M is C_{X_β}-torsion free. Then the image, M_β, of M in $C_{X/X_\beta} = C_X/C_{X_\beta}$ has a simple subobject, L. Let $\mathcal{P}$ be the preimage of $\langle L\rangle$ in C_X. This is a Serre subcategory. We claim that $\mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(M)$.

In fact, there exists a subobject, $\widetilde{L}$, of M such that the image of $\widetilde{L}$ in C_X/C_{X_β} is isomorphic to L. The object $\widetilde{L}$ is $\mathcal{P}$-torsion free: if L' is a nonzero subobject of $\widetilde{L}$, then its image in C_X/C_{X_β} is a nonzero subobject of L, and L is $\langle L\rangle$-torsion free.

On the other hand, $\widetilde{L} \in Ob\mathcal{P}^\star$, because $\mathcal{P}^\star$ is the preimage of $\langle L\rangle^\star = [L]_\bullet \vee \langle L\rangle$, and the image of $\widetilde{L}$ belongs to $[L]_\bullet$.

(ii) Let $\mathbb{S} \in Supp_{\mathfrak{Se}}(M)$; i.e. $\mathbb{S}$ is a Serre subcategory of the category C_X such that $M \notin Ob\mathbb{S}$. The latter means that the image, $\widetilde{M}$, of M in the quotient category $C_X/\mathbb{S}$ is nonzero. Since M has a Gabriel-Krull dimension, the object $\widetilde{M}$ has a Gabriel-Krull dimension too. By (i), $\mathfrak{Ass}_{\mathfrak{L}}(\widetilde{M}) \neq \emptyset$. Let $\widetilde{\mathcal{P}} \in \mathfrak{Ass}_{\mathfrak{L}}(\widetilde{M})$, and let $\mathcal{P}$ be the preimage of $\widetilde{\mathcal{P}}$ in C_X. By construction, $\mathbb{S} \subseteq \mathcal{P}$. We claim that $\mathcal{P} \in Ass_{\mathfrak{Se}}(M)$.

In fact, there exists a nonzero subobject, $\widetilde{L}$, of the object $\widetilde{M}$ which is $\widetilde{\mathcal{P}}$-torsion free and belongs to $\widetilde{\mathcal{P}}^\star$. There exists a nonzero subobject, L, of the object M such that $\widetilde{L}$ is isomorphic to the image of L in $C_X/\mathbb{S}$. Since $\mathcal{P}^\star$ is the preimage of $\widetilde{\mathcal{P}}^\star$, the object L belongs to $\mathcal{P}^\star$ and does not belong to $\mathcal{P}$. ■

10.10.2. Canonical filtrations. Let X have the property (sup). Then all Serre subcategories of C_X, in particular the Serre subcategories $\{C_{X_\alpha}\}$ of the Gabriel filtration, are coreflective.

Let $\mathbb{T}$ be a topologizing subcategory of the category C_X. To any ordinal, α, we assign a full subcategory, $\mathbb{T}_\alpha$, of C_X defined as follows:

$\mathbb{T}_0 = \mathbb{O}$, and $Ob\mathbb{T}_1$ consists of all objects of C_X which are supremums of their subobjects from $\mathbb{T}$; i.e. $\mathbb{T}_1$ is the smallest coreflective subcategory of C_X containing $\mathbb{T}$.

If α is not a limit ordinal, then $\mathbb{T}_\alpha = \mathbb{T}_{\alpha-1} \bullet_X \mathbb{T}_1$.

If β is a limit ordinal, then $\mathbb{T}_\beta$ is the smallest coreflective subcategory containing $\bigcup_{\alpha<\beta} \mathbb{T}_\alpha$, that is $\mathbb{T}_\beta = \big(\bigcup_{\alpha<\beta} \mathbb{T}_\alpha\big)_1$.

Since the category $\mathbb{T}_1$ is coreflective and the Gabriel product of coreflective categories is coreflective, it follows that all subcategories $\mathbb{T}_\alpha$ are coreflective. Their union coincides with the Serre subcategory, $\mathbb{T}^-$, spanned by $\mathbb{T}$.

We use this construction to obtain a refinement of the Gabriel filtration, $\{C_{X_\alpha}\}$.

We set $X^0_\alpha = X_\alpha$, that is $C_{X^0_\alpha} = C_{X_\alpha}$.

Let $C_{X^1_\alpha}$ denote the preimage in C_X of the *socle* of the quotient category C_X/C_{X_α} which is, by definition, the full subcategory, $(C_X/C_{X_\alpha})_1$, of the category C_X/C_{X_α} spanned by semisimple objects. Since $(C_X/C_{X_\alpha})_1$ is a coreflective topologizing subcategory of C_X/C_{X_α}, its preimage is a coreflective topologizing subcategory of the category C_X. For $\beta \geq 1$, we define the subcategories $C_{X^\beta_\alpha}$ by setting $C_{X^\beta_\alpha} = (C_{X^1_\alpha})_\beta$.

It follows that every $C_{X^\beta_\alpha}$ is a coreflective subcategory of the Serre subcategory $C_{X_{\alpha+1}}$, and the union $\bigcup_\beta C_{X^\beta_\alpha}$ coincides with $C_{X_{\alpha+1}}$.

Thus, we have obtained an increasing filtration, $\{C_{X^\beta_\alpha} \mid \alpha, \beta \geq 0\}$ of the category C_X by coreflective topologizing subcategories such that $\bigcup_\beta C_{X^\beta_\alpha} = C_{X_\alpha}$ for every ordinal α, and $\bigcup_\alpha \bigcup_\beta C_{X^\beta_\alpha} = C_{X_\omega}$, where C_{X_ω} is the full subcategory of the category C_X spanned by all objects having a Gabriel-Krull dimension.

10.10.2.1. Proposition. *Suppose X has the property (sup). Then*

$$\mathbf{Spec}^1_{\mathfrak{s}}(X_\omega) = \mathbf{Spec}^-(X_\omega) \simeq \bigcup_\alpha \mathbf{Spec}^-(X^1_\alpha) \simeq \coprod_\alpha \mathbf{Spec}(X^1_\alpha/X_\alpha). \qquad (1)$$

Here $C_{X^1_\alpha/X_\alpha} = C_{X^1_\alpha}/C_{X_\alpha}$.

Proof. The equality $\mathbf{Spec}^1_{\mathfrak{s}}(X_\omega) = \mathbf{Spec}^-(X_\omega)$ is a consequence of 8.7.1.

Let $\mathcal{P} \in \mathbf{Spec}^-(X_\omega)$. Since $C_{X_\omega} = \bigcup_\alpha C_{X_\alpha}$ and $\mathcal{P}$ is a proper Serre subcategory of C_{X_ω}, there exists an ordinal α such that $C_{X_\alpha} \subseteq \mathcal{P}$ and $C_{X_{\alpha+1}} \nsubseteq \mathcal{P}$. Since $\mathcal{P}$ is a Serre subcategory, and $C_{X_{\alpha+1}} = C^-_{X^1_\alpha}$, $C_{X_{\alpha+1}} \nsubseteq \mathcal{P}$ iff $C_{X^1_\alpha} \nsubseteq \mathcal{P}$. The latter means that $C_{X^1_\alpha}/C_{X_\alpha} \nsubseteq \mathcal{P}/C_{X_\alpha}$. Since all objects of the category $C_{X^1_\alpha/X_\alpha} = C_{X^1_\alpha}/C_{X_\alpha}$ are semisimple, there exists a simple object, L, of the category $C_{X_\omega}/C_{X_\alpha}$ which does not belong to the Serre subcategory $\mathcal{P}/C_{X_\alpha}$. By 8.5.1, this implies that $\mathcal{P}/C_{X_\alpha} = \langle L \rangle$; i.e. $\mathcal{P}/C_{X_\alpha}$ is a closed point of $\mathbf{Spec}(X_\omega/X_\alpha)$. Since every simple object of the category $C_{X_\omega}/C_{X_\alpha} = C_{X_\omega/X_\alpha}$ belongs to the subcategory $C_{X^1_\alpha/X_\alpha}$, the family of closed points of $\mathbf{Spec}(X_\omega/X_\alpha)$ is in one-to-one correspondence with $\mathbf{Spec}(X^1_\alpha/X_\alpha)$. ■

10.10.2.2. The canonical filtration of objects. For any $M \in ObC_X$, the filtration $\{C_{X^\beta_\alpha}\}$ induces a canonical increasing filtration, $\{M^\beta_\alpha\}$, of M. Here the subobject M^β_α is the $C_{X^\beta_\alpha}$-torsion of M. The supremum of the subobjects M^β_α coincides with the C_{X_ω}-torsion of the object M. Let q^*_α denote a localization functor $C_X \longrightarrow C_{X/X_\alpha} = C_X/C_{X_\alpha}$. Then for any α and β, the objects $q^*_\alpha(M^{\beta+1}_\alpha/M^\beta_\alpha)$ are either zero, or semisimple.

10.10.2.3. Proposition. *Suppose X has a Gabriel-Krull dimension (i.e. $X = X_\omega$) and the property (sup). Then for any $M \in ObC_X$,*

$$\mathfrak{Ass}_{\mathfrak{L}}(M) = \bigcup_{\alpha,\beta} \mathfrak{Ass}_{\mathfrak{L}}(M^\beta_\alpha) \subseteq \bigcup_\alpha \bigcup_\beta \mathfrak{Ass}_{\mathfrak{L}}(M^{\beta+1}_\alpha/M^\beta_\alpha).$$

$$Ass_{\mathfrak{Se}}(M) = \bigcup_{\alpha,\beta} Ass_{\mathfrak{Se}}(M^\beta_\alpha) \subseteq \bigcup_\alpha \bigcup_\beta Ass_{\mathfrak{Se}}(M^{\beta+1}_\alpha/M^\beta_\alpha) = \bigcup_\alpha \bigcup_\beta \mathfrak{Ass}_{\mathfrak{L}}(M^{\beta+1}_\alpha/M^\beta_\alpha)$$

$$Supp_{\mathfrak{Se}}(M) = \bigcup_{\alpha,\beta} Supp_{\mathfrak{Se}}(M^\beta_\alpha) = \bigcup_\alpha \bigcup_\beta Supp_{\mathfrak{Se}}(M^{\beta+1}_\alpha/M^\beta_\alpha)$$

Proof. The equality and the inclusion

$$\mathfrak{Ass}_{\mathfrak{L}}(M) = \bigcup_{\alpha,\beta} \mathfrak{Ass}_{\mathfrak{L}}(M^\beta_\alpha) \subseteq \bigcup_\alpha \bigcup_\beta \mathfrak{Ass}_{\mathfrak{L}}(M^{\beta+1}_\alpha/M^\beta_\alpha)$$

follow from 10.8.2. Similarly, the equality and the inclusion

$$Ass_{\mathfrak{Se}}(M) = \bigcup_{\alpha,\beta} Ass_{\mathfrak{Se}}(M^\beta_\alpha) \subseteq \bigcup_\alpha \bigcup_\beta Ass_{\mathfrak{Se}}(M^{\beta+1}_\alpha/M^\beta_\alpha)$$

and the equalities

$$Supp_{\mathfrak{Se}}(M) = \bigcup_{\alpha,\beta} Supp_{\mathfrak{Se}}(M^\beta_\alpha) = \bigcup_\alpha \bigcup_\beta Supp_{\mathfrak{Se}}(M^{\beta+1}_\alpha/M^\beta_\alpha)$$

follow from 10.7.1. It remains to show that

$$Ass_{\mathfrak{S}\mathfrak{e}}(M_\alpha^{\beta+1}/M_\alpha^\beta) = \mathfrak{Ass}_{\mathfrak{L}}(M_\alpha^{\beta+1}/M_\alpha^\beta) \tag{2}$$

for all α and β. If $M_\alpha^{\beta+1}/M_\alpha^\beta = 0$, the equality (2) holds by a trivial reason. Suppose $M_\alpha^{\beta+1}/M_\alpha^\beta \neq 0$; and let $\mathcal{P} \in Ass_{\mathfrak{S}\mathfrak{e}}(M_\alpha^{\beta+1}/M_\alpha^\beta)$; i.e. there exists a subobject, L, of $M_\alpha^{\beta+1}/M_\alpha^\beta$ which belongs to $Ob\mathcal{P}^\star - Ob\mathcal{P}$. By (the argument of) 10.10.2.1, there exists an ordinal γ such that $\mathcal{P}$ is the preimage in C_X of the Serre subcategory $\langle\mathcal{N}\rangle$ for some simple object, $\mathcal{N}$, of the quotient category $C_{X/X_\gamma} = C_X/C_{X_\gamma}$. It follows from this description that $\gamma = \alpha$. In fact, it cannot be that $\gamma > \alpha$, because in this case the object L belongs to the subcategory C_{X_γ} which contradicts to the condition that $L \notin Ob\mathcal{P}$.

The condition $L \in Ob\mathcal{P}^\star - Ob\mathcal{P}$ implies that the image, $q_\gamma^*(L)$, of L in C_X/C_{X_γ} belongs to the Serre subcategory, $(\{\mathcal{N}\} \cup \langle\mathcal{N}\rangle)^-$, containing the subcategory $\langle\mathcal{N}\rangle$ and the object $\mathcal{N}$, but does not belong to $\langle\mathcal{N}\rangle$. This implies that $\mathcal{N}$ is a subobject of the object $q_\gamma^*(L)$. By a standard argument, there exists a subobject, L_1, of L such that $q_\gamma^*(L_1) \simeq \mathcal{N}$. In particular, $L_1 \in ObC_{X_{\gamma+1}}$. If $\gamma < \alpha$, then L_1 is $C_{X_{\gamma+1}}$-torsion free, hence it is equal to zero which is not the case. Thus, $\gamma = \alpha$ and L_1 is a nonzero object of the category $\mathcal{P}^\star$ which is $\mathcal{P}$-torsion free. ■

10.10.2.4. Note. Suppose X has the property (sup), and let $M \in ObC_X$. Consider the canonical filtration, $\{M_\alpha^\beta\}$ of the object M. If $M_\alpha^{\beta+1}/M_\alpha^\beta \neq 0$, then the object $q_\alpha^*(M_\alpha^{\beta+1}/M_\alpha^\beta)$ of the quotient category C_X/C_{X_α} is semisimple (cf. 10.10.2.2). By a standard argument, this means that there exist monomorphisms $L_i \longrightarrow M_\alpha^{\beta+1}/M_\alpha^\beta,\ i \in J$, such that all objects $q_\alpha^*(L_i)$ are simple and the natural morphism

$$\bigoplus_{i\in J} q_\alpha^*(L_i) \longrightarrow q_\alpha^*(M_\alpha^{\beta+1}/M_\alpha^\beta)$$

is an isomorphism. In particular, $q_\alpha^*(L_i) \cap q_\alpha^*(L_j) = 0$ if $i \neq j$. Since the localization q_α^* is exact, in particular left exact, $q_\alpha^*(L_i) \cap q_\alpha^*(L_j) \simeq q_\alpha^*(L_i \cap L_j)$, hence $L_i \cap L_j$ belongs to the subcategory C_{X_α}. Since the object $M_\alpha^{\beta+1}/M_\alpha^\beta$ is C_{X_α}-torsion free, this means that $L_i \cap L_j = 0$ whenever $i \neq j$. Therefore the monomorphisms $L_i \longrightarrow M_\alpha^{\beta+1}/M_\alpha^\beta,\ i \in J$, induce an *essential* monomorphism

$$\bigoplus_{i\in J} L_i \longrightarrow M_\alpha^{\beta+1}/M_\alpha^\beta \tag{3}$$

whose cokernel belongs to the subcategory C_{X_α}. Recall that *essential* means that every nonzero subobject of $M_\alpha^{\beta+1}/M_\alpha^\beta$ has a nonzero intersection with $\bigoplus_{i\in J} L_i$.

The set $Ass_{\mathfrak{S}\mathfrak{e}}(M_\alpha^{\beta+1}/M_\alpha^\beta) = \mathfrak{Ass}_{\mathfrak{L}}(M_\alpha^{\beta+1}/M_\alpha^\beta)$ consists of preimages in C_X of the Serre subcategories $\langle q_\alpha^*(L_i)\rangle,\ i \in J$.

Notice that if $\alpha = 1$, then all objects $L_i,\ i \in J$, are simple, and the morphism (3) is an isomorphism.

10.10.3. Proposition. *Suppose $M \in ObC_X$ has a Gabriel-Krull dimension. The following properties are equivalent:*

(a) *The dimension of M is zero.*
(b) *Every element of $Ass_{\mathfrak{S}\mathfrak{e}}(M)$ is of the form $\langle P\rangle$ for some simple object P.*
(c) *Every element of $Supp_{\mathfrak{S}\mathfrak{e}}(M)$ is of the form $\langle P\rangle$ for some simple object P.*

Proof. The implication $(c) \Rightarrow (b)$ follows from the inclusion $Ass_{\mathfrak{S}\mathfrak{e}}(M) \subseteq Supp_{\mathfrak{S}\mathfrak{e}}(M)$.

$(b) \Rightarrow (c)$. By 10.10.1, every $\mathcal{P} \in Supp_{\mathfrak{S}\mathfrak{e}}(M)$ is contained in an element of $Ass_{\mathfrak{S}\mathfrak{e}}(M)$. By hypothesis, the latter is of the form $\langle L\rangle$ for some simple object L. The inclusion $\mathcal{P} \subseteq \langle L\rangle$ means precisely that $L \notin Ob\mathcal{P}$. Let q^* be a localization functor $C_X \longrightarrow C_X/\mathcal{P}$. Since the simple object L does not belong to $\mathcal{P}$ its image, $q^*(L)$, in the quotient category $C_X/\mathcal{P}$ is a simple object. Let $N \in ObC_X - Ob\mathcal{P}$. Since the category $C_X/\mathcal{P}$ is local and $q^*(N) \neq 0$, of the object $q^*(N)$ is 'greater' than $q^*(L)$, i.e. $q^*(N) \succ q^*(L)$. This means that $L \in Ob\mathcal{P}^\star$. By 8.5.1, $\mathcal{P} = \langle L\rangle$.

$(a) \Rightarrow (c)$. Let C_{X_0} be the full subcategory of C_X spanned by objects of zero dimension. And let $C_{X_0^1}$ be the full subcategory of C_X spanned by semisimple objects. It follows from 10.10.2.1 that $\mathbf{Spec}^0_{\mathfrak{s}}(X)_0 = \mathbf{Spec}_{\mathfrak{s}}(X)^1_0 = \mathbf{Spec}X^1_0 = \{\langle L\rangle \mid L \in SimpleC_X\}$.

$(c) \Rightarrow (a)$. If the dimension of M is greater than zero, then M has a quotient which belongs to C_{X_α} for $\alpha \geq 1$. This implies that $Supp_{\mathfrak{S}\mathfrak{e}}(M)$ contains an element which is not of the form $\langle L\rangle$ for a simple object L. ∎

10.10.4. Corollary. *For any object, M, of zero dimension $Supp_{\mathfrak{S}\mathfrak{e}}(M) = Ass_{\mathfrak{S}\mathfrak{e}}(M)$.*

Proof. The fact follows from the argument $(b) \Rightarrow (c)$ in the proof of 10.10.3. ∎

10.10.5. Proposition. *Let M be an object with Gabriel-Krull dimension. Let $\mathcal{P} \in Supp_{\mathfrak{S}\mathfrak{e}}(M)$ and let q^* denote the localization functor $C_X \longrightarrow C_X/\mathcal{P}$. The object $q^*(M)$ is of zero dimension iff $\mathcal{P}$ is a maximal element of $Ass_{\mathfrak{S}\mathfrak{e}}(M)$.*

Proof. By 10.7.4, the map $\mathcal{P}' \longrightarrow \mathcal{P}'/\mathcal{P}$ induces a bijection $\{\mathcal{P}' \in Ass_{\mathfrak{S}\mathfrak{e}}(M) \mid \mathcal{P} \subseteq \mathcal{P}'\}$ and $Ass_{\mathfrak{S}\mathfrak{e}}(q^*(M))$. Since the category $C_X/\mathcal{P}$ is local, it follows from 10.10.3 that the object $q^*(M)$ is of dimension zero iff $Ass_{\mathfrak{S}\mathfrak{e}}(q^*(M)) = \{\mathbb{O}\}$. This happens iff $\mathcal{P}$ is a maximal element of $Ass_{\mathfrak{S}\mathfrak{e}}(M)$. ∎

10.10.6. Proposition. *Let M be an object of finite length. There exists only one (necessarily reduced) primary decomposition of $0 \hookrightarrow M$.*

Proof. By (the argument of) 10.10.3, all elements of $Ass_{\mathfrak{S}\mathfrak{e}}(M)$ are maximal. In particular, all elements of $\mathfrak{Ass}_{\mathfrak{L}}(M)$ are maximal. The uniqueness of the primary decomposition follows now from 10.9.10. For every $\mathcal{P} \in \mathfrak{Ass}_{\mathfrak{L}}(M)$, the $\mathcal{P}$-primary subobject of M coincides with the $\mathcal{P}$-torsion of M. ∎

10.11. Supports and associated points in other spectra.

The other spectra introduced in this work (and in [R]) are contained (or embedded) in one of the spectra considered above: $\mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}}(X)$, or in $\mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}}(X)$, or in $\mathbf{Spec}^1_{\mathfrak{s}}(X)$. If $\mathbf{Spec}_?(X)$ is one of those spectra, we define, for any $M \in ObC_X$, the *support*, $Supp_?(M)$, and *weakly associated points*, $Ass_?(M)$, of M in $\mathbf{Spec}_?(X)$ as the intersection (a pull-back) of resp. $Supp^1_{\mathfrak{T}\mathfrak{h}}(M)$ and $Ass^1_{\mathfrak{T}\mathfrak{h}}(M)$ with $\mathbf{Spec}_?(X)$. Thus, we have the *flat support*,

$$Supp^1_{\mathfrak{f}\mathfrak{L}}(M) = Supp^1_{\mathfrak{T}\mathfrak{h}}(M) \bigcap \mathbf{Spec}^1_{\mathfrak{f}\mathfrak{L}}(X)$$

and *flat weakly associated points*,

$$Ass^1_{\mathfrak{fL}}(M) = Ass^1_{\mathfrak{Th}}(M) \bigcap \mathbf{Spec}^1_{\mathfrak{fL}}(X).$$

Similarly, we have support and weakly associated points of M in $\mathbf{Spec}^0_{\mathfrak{s}}(X)$:

$$Supp_{\mathfrak{s}}(M) = Supp_{\mathfrak{Th}}(M) \bigcap \mathbf{Spec}^0_{\mathfrak{s}}(X) \quad \text{and} \quad Ass_{\mathfrak{s}}(M) = Ass_{\mathfrak{Th}}(M) \bigcap \mathbf{Spec}^0_{\mathfrak{s}}(X).$$

Finally, we define $Supp(M)$ and $Ass(M)$ by taking the intersection of $\mathbf{Spec}(X)$ (cf. 7.6) with respectively $Supp_{\mathfrak{Th}}(M)$ and $Ass_{\mathfrak{Th}}(M)$.

The general properties described in 10.2, 10.3, 10.4, and 10.5 are inherited by $Supp_?$ and $Ass_?$ for all ?. If $Supp_?$ and $Ass_?$ are contained in $\mathbf{Spec}^1_{\mathfrak{s}}(X)$, the assertion 10.7.1(b) and its corollary 10.7.2 are valid for them.

One should be careful with the functorial properties described by 10.6 and 10.7.4: the functoriality breaks for $Supp_{\mathfrak{s}}(-)$ and $Ass_{\mathfrak{s}}(-)$ and for $Supp(-)$ and $Ass(-)$.

IV. Spectra of 'spaces' represented by triangulated categories.

11. Actions of monoidal categories and their spectra.

We fix a monoidal category $\widetilde{C_T} = (C_T, \odot, \mathfrak{a}; 1, \phi_{\mathfrak{l}}, \phi_{\mathfrak{r}})$. Here $C_T \times C_T \xrightarrow{\odot} C_T$ is a monoidal structure ('tensor product'); $\mathfrak{a}$ is an associativity constraint, i.e. a functor isomorphism

$$\odot \circ (\odot \times Id_{C_T}) \longrightarrow \odot \circ (Id_{C_T} \times \odot)$$

satisfying certain natural compatibility conditions; 1 denotes the unit object,

$$1 \odot - \xrightarrow{\phi_{\mathfrak{l}}} Id_{C_T} \xleftarrow{\phi_{\mathfrak{r}}} - \odot 1$$

are canonical isomorphisms.

11.1. Actions. Let X be a 'space'. Fix an action, $C_T \times C_X \xrightarrow{\gamma^*} C_X$, of the monoidal category $\widetilde{C_T} = (C_T, \odot, 1)$ on the category C_X. The functor γ^* induces a functor

$$C_T \xrightarrow{\Gamma} \mathcal{E}nd(C_X), \quad a \longmapsto \Gamma_a,$$

where $\mathcal{E}nd(C_X)$ denote the category of functors $C_X \longrightarrow C_X$, and $\Gamma_a(M) = \gamma^*(a, M)$. The functor γ^* being an 'action' means precisely that Γ is a monoidal functor, i.e. for any $a, b \in ObC_T$, there are natural morphisms

$$\Gamma_a \circ \Gamma_b \xrightarrow{\phi_{a,b}} \Gamma_{a \odot b} \quad \text{and} \quad \Gamma_1 \xrightarrow{\sim} Id_{C_X}$$

related in a natural way between themselves and with associativity constraint on $\widetilde{C_T}$.

A pair (C_X, γ^*), where γ^* is a $\widetilde{C_T}$-action, is called a $\widetilde{C_T}$*-category.* We call a pair (X, γ^*) a $\widetilde{C_T}$*-'space'.* A morphism (more precisely, a 1-morphism) between two $\widetilde{C_T}$-'spaces', $(X, \gamma^*) \longrightarrow (Y, \widetilde{\gamma}^*)$, is given by a pair (F, ϕ), where F is a functor $C_Y \xrightarrow{F} C_X$ such that the diagram

$$\begin{array}{ccc} C_T \times C_Y & \xrightarrow{Id \times F} & C_T \times C_X \\ \gamma^* \downarrow & & \downarrow \widetilde{\gamma}^* \\ C_Y & \xrightarrow{F} & C_X \end{array}$$

quasi-commutes, and ϕ is a functor isomorphism $\gamma^* \circ (Id \times F) \xrightarrow{\sim} F \circ \gamma^*$ satisfying a standard cocycle condition. The composition of morphisms is defined naturally.

11.1.1. Note. In the language of 'spaces', the monoidal structure, $C_T \times C_T \xrightarrow{\odot} C_T$, can be regarded as an inverse image of a morphism (a coaction) $T \longrightarrow T \coprod T$, and the action γ^* as an inverse image functor of a morphism $X \xrightarrow{\gamma} T \coprod X$.

11.1.2. Actions of a monoid, $\mathbb{Z}$-categories. Any monoid, $\mathcal{G}$, might be regarded as a discrete category with the monoidal structure given by multiplication. This defines an isomorphism between the category of monoids and the category of discrete 'small' monoidal categories which allows to define actions of monoids on categories. Thus, a $\mathcal{G}$-*category* is a pair (C_X, γ^*), where γ^* is a monoidal functor from $\mathcal{G}$ to the monoidal category $\mathfrak{End}(C_X)$ of functors $C_X \longrightarrow C_X$. If $\mathcal{G}$ is a group, the functor γ takes values in the monoidal subcategory $\mathbf{Pic}(X)$ of $\mathfrak{End}(C_X)$ formed by invertible functors and isomorphisms between them. The group $\mathbb{Z}$ is of particular interest because triangulated categories, categories of graded modules, and the category of quasi-coherent sheaves on (commutative, or noncommutative) Proj are $\mathbb{Z}$-categories.

11.2. A graded category associated with an action. Suppose the category C_T is 'small'. For any pair of objects, L, M, of the category C_X and an object a of the category C_T, set $C_X^a(L,M) = C_X(L, \gamma^*(a,M)) = C_X(L, \Gamma_a(M))$. If $N \in ObC_X$ and $b \in ObC_T$, then we define a map

$$C_X^a(L,M) \times C_X^b(M,N) \longrightarrow C_X^{a\odot b}(L,N) \tag{1}$$

as the composition of the maps

$$\begin{array}{ccc} C_X(L,\Gamma_a(M)) \times C_X(M,\Gamma_b(N)) & \longrightarrow & C_X(L,\Gamma_a(M)) \times C_X(\Gamma_a(M), \Gamma_a \circ \Gamma_b(N)) \\ & & \downarrow \\ & & C_X(L,\Gamma_a(M)) \times C_X(\Gamma_a(M), \Gamma_{a\odot b}(N)), \end{array}$$

where the vertical arrow is induced by the functor morphism $\Gamma_a \circ \Gamma_b \xrightarrow{\phi_{a,b}} \Gamma_{a\odot b}$, and the composition map

$$C_X(L,\Gamma_a(M)) \times C_X(\Gamma_a(M), \Gamma_{a\odot b}(N)) \longrightarrow C_X(L, \Gamma_{a\odot b}(N)) = C_X^{a\odot b}(L,N)$$

This defines an *enriched* category, $C_{(X,\gamma^*)}$, with the same objects as C_X; morphisms between objects are $\widetilde{C_T}$-graded sets. To every morphism, $(X,\gamma^*) \longrightarrow (Y,\widetilde{\gamma}^*)$, of $\widetilde{C_T}$-actions, there corresponds a morphism, $C_{(X,\gamma^*)} \longrightarrow C_{(Y,\widetilde{\gamma}^*)}$, of enriched categories.

11.2.1. The case of 'spaces' over a monoid. Let $\mathcal{G}$ be a monoid and (X,γ^*) a $\mathcal{G}$-'space' such that C_X is a k-linear category for for some commutative ring k. Then $C_{(X,\gamma^*)}$ gives rise to an enriched category over the monoidal category of $\mathcal{G}$-graded k-modules. In particular, a $\mathbb{Z}$-'space' defines an enriched category over the monoidal category of $\mathbb{Z}$-graded k-modules.

11.3. Stable saturated multiplicative systems. Let $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X,\gamma^*)$ denote the family of saturated multiplicative systems in C_X which are invariant with respect to the action of $\widetilde{C_T}$. It follows from the universal property of localizations that for every $\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X,\gamma^*)^\star$, the 'space' of fractions, $\Sigma^{-1}X$, inherits a $\widetilde{C_T}$-action uniquely defined by the condition that the canonical morphism $\Sigma^{-1}X \xrightarrow{q_\Sigma} X$ is a morphism of actions.

11.3.1. Proposition. *(a) If Σ is a $\widetilde{C_T}$-stable multiplicative system, then its saturation, $\Sigma^{\mathfrak{s}}$ is a $\widetilde{C_T}$-stable multiplicative system too.*

(b) If Σ_1 and Σ_2 are $\widetilde{C_T}$-stable saturated multiplicative systems, then the smallest saturated multiplicative system, $\Sigma_1 \vee \Sigma_2$, spanned by Σ_1 and Σ_2 is $\widetilde{C_T}$-stable.

(c) Suppose C_X has finite colimits. Then the intersection of any set of $\widetilde{C_T}$-stable saturated left multiplicative systems is a $\widetilde{C_T}$-stable saturated left multiplicative system.

Proof. (a) The saturation, $\Sigma^{\mathfrak{s}}$, of a multiplicative system Σ consists of all morphisms s of C_X such that $u \circ s \in \Sigma \ni s \circ v$ for some morphisms u and v (see 1.3.3). If Σ is $\widetilde{C_T}$-stable, then

$$\Gamma_a(u) \circ \Gamma_a(s) = \Gamma_a(u \circ s) \in \Sigma \ni \Gamma_a(s \circ v) = \Gamma_a(s) \circ \Gamma_a(v)$$

for all $a \in ObC_T$ (cf. 11.1 for notations); hence $\Gamma_a(s) \in \Sigma$ for all $a \in ObC_T$.

(b) We need several steps.

(i) Let Σ_1 and Σ_2 be two left multiplicative systems, and let $\Sigma_1 \sqcup \Sigma_2$ denote the smallest family of arrows closed under composition and containing Σ_1 and Σ_2. We claim that $\Sigma_1 \sqcup \Sigma_2$ is a left multiplicative system.

The family $\Sigma_1 \sqcup \Sigma_2$ consists of all possible compositions of arrows of $\Sigma_1 \bigcup \Sigma_2$. Since Σ_1 and Σ_2 are closed under composition and contain all identical morphisms, a generic element, s, of $\Sigma_1 \sqcup \Sigma_2$ can be represented as a composition

$$M_0 \xrightarrow{s_1} L_1 \xrightarrow{t_1} M_1 \xrightarrow{s_2} \ \ldots \ \xrightarrow{s_n} L_n \xrightarrow{t_n} M_n, \tag{2}$$

where $s_i \in \Sigma_1$ and $t_i \in \Sigma_2$, $i = 1, ..., n$.

Let s be an element of $\Sigma_1 \sqcup \Sigma_2$ given by the composition (2), and let $M_0 \xrightarrow{f} M_0'$ be an arbitrary morphism. By the property (SL2) (see 1.3), there exists a commutative square

$$\begin{array}{ccc}
M_0 & \xrightarrow{s_1} & L_1 \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g_1} \\
M_0' & \xrightarrow{s_1'} & L_1'
\end{array}$$

where $s_1' \in \Sigma_1$. Applying the property (SL2) to the pair of morphisms $L_1' \xleftarrow{g_1} L_1 \xrightarrow{t_1} M_1$, we complete it to a commutative diagram

$$\begin{array}{ccc}
L_1 & \xrightarrow{t_1} & M_1 \\
{\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle f_1} \\
L_1' & \xrightarrow{t_1'} & M_1'
\end{array}$$

where $t_1' \in \Sigma_2$. Continuing this process, we obtain a commutative diagram

$$\begin{array}{ccccccccccc}
M_0 & \xrightarrow{s_1} & L_1 & \xrightarrow{t_1} & M_1 & \xrightarrow{s_2} & \ldots & \xrightarrow{s_n} & L_n & \xrightarrow{t_n} & M_n \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle f_1} & & \ldots & & {\scriptstyle g_n}\downarrow & & \downarrow{\scriptstyle f_n} \\
M_0' & \xrightarrow{s_1'} & L_1' & \xrightarrow{t_1'} & M_1' & \xrightarrow{s_2'} & \ldots & \xrightarrow{s_n'} & L_n' & \xrightarrow{t_n'} & M_n'
\end{array}$$

where $s_i' \in \Sigma_1$ and $t_i' \in \Sigma_2$, $i = 1, ..., n$. Thus, $\Sigma_1 \sqcup \Sigma_2$ has the property (SL2). It remains to verify the property (SL3) (see 1.3).

Let s be an element of $\Sigma_1 \sqcup \Sigma_2$ presented as the composition (2). And let $M_n \overset{f}{\underset{g}{\rightrightarrows}} N$ be a pair of arrows such that $f \circ s = g \circ s$. This equality can be presented as

$$(f \circ t_n \circ s_n \circ ... \circ t_1) \circ s_1 = (g \circ t_n \circ s_n \circ ... \circ t_1) \circ s_1.$$

By the property (SL3), there exists an element $s'_1 \in \Sigma_1$ such that

$$s'_1 \circ (f \circ t_n \circ s_n \circ ... \circ t_1) = s'_1 \circ (g \circ t_n \circ s_n \circ ... \circ t_1). \tag{3}$$

Applying (SL3) to the equality (3) presented in the form

$$(s'_1 \circ f \circ t_n \circ s_n \circ ... \circ s_2) \circ t_1 = (s'_1 \circ g \circ t_n \circ s_n \circ ... \circ s_2) \circ t_1,$$

we find an element $t'_1 \in \Sigma_2$ such that

$$t'_1 \circ (s'_1 \circ f \circ t_n \circ s_n \circ ... \circ s_2) = t'_1 \circ (s'_1 \circ g \circ t_n \circ s_n \circ ... \circ s_2).$$

By an induction argument, we obtain the equality

$$(t'_n \circ s'_n \circ ... \circ t'_1 \circ s'_1) \circ f = (t'_n \circ s'_n \circ ... \circ t'_1 \circ s'_1) \circ g,$$

where $s'_i \in \Sigma_1$ and $t'_i \in \Sigma_2$, $i = 1, ..., n$.

(ii) It follows from the description of $\Sigma_1 \sqcup \Sigma_2$ in (i) that if Σ_1 and Σ_2 are $\widetilde{C_T}$-stable, then $\Sigma_1 \sqcup \Sigma_2$ is $\widetilde{C_T}$-stable too.

(iii) The smallest saturated multiplicative system, $\Sigma_1 \vee \Sigma_2$, spanned by Σ_1 and Σ_2 is, evidently, the saturation of $\Sigma_1 \sqcup \Sigma_2$. By (ii), the left multiplicative system $\Sigma_1 \sqcup \Sigma_2$ is $\widetilde{C_T}$-stable. Therefore, by (a), its saturation, $\Sigma_1 \vee \Sigma_2$, is $\widetilde{C_T}$-stable.

(c) The assertion follows from 1.4.3. ■

11.4. Spectra. Clearly, the trivial multiplicative system, $Iso(C_X)$, is $\widetilde{C_T}$-invariant, i.e. $Iso(C_X) \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X, \gamma^*)$. Let $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X, \gamma^*)^\star$ denote the family $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X, \gamma^*) - \{Iso(C_X)\}$ of non-trivial $\widetilde{C_T}$-invariant saturated multiplicative systems.

All notions and facts considered so far in this work are extended to 'spaces' with an action of a monoidal category $\widetilde{C_T} = (C_T, \odot, 1)$ by simply replacing $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)^\star$ by $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X, \gamma^*)^\star$ and inserting "$\widetilde{C_T}$-invariant" whenever it is required.

Thus, $\mathfrak{L}$-*local* $\widetilde{C_T}$-'spaces' are those $\widetilde{C_T}$-'spaces' (X, γ^*) for which the intersection of all $\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X, \gamma^*)^\star$ belongs to $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X, \gamma^*)^\star$.

The *complete* $\mathfrak{L}$-*spectrum*, $\mathbf{Spec}^1_{\mathfrak{L}}(X, \gamma^*)$, of a $\widetilde{C_T}$-'space' (X, γ^*) consists of all $\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X, \gamma^*)^\star$ such that the $\widetilde{C_T}$-'space' of fractions $(\Sigma^{-1}X, \widetilde{\gamma}^*)$ is $\mathfrak{L}$-local. In other words, there exists the smallest $\widetilde{C_T}$-invariant saturated multiplicative system, $\Sigma^\star$, properly containing Σ.

The $\mathfrak{L}$-*spectrum*, $\mathbf{Spec}^0_{\mathfrak{L}}(X, \gamma^*)$, of a $\widetilde{C_T}$-'space' (X, γ^*) is formed by $\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X, \gamma^*)$ such that there exists the biggest $\widetilde{C_T}$-invariant saturated multiplicative system, $\widehat{\Sigma}$, which does not contain Σ.

11.4.1. Proposition. *The map* $\Sigma \longmapsto \widehat{\Sigma}$ *is an injective morphism from* $\mathbf{Spec}^0_{\mathfrak{L}}(X, \gamma^*)$ *to* $\mathbf{Spec}^1_{\mathfrak{L}}(X, \gamma^*)$.

Proof. Let $\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X, \gamma^*)$. By 11.3.1(b), $\Sigma \vee \widehat{\Sigma}$ is the smallest $\widetilde{C_T}$-invariant saturated multiplicative system which contains Σ and $\widehat{\Sigma}$. If Σ_1 is a $\widetilde{C_T}$-invariant saturated multiplicative system which properly contains $\widehat{\Sigma}$, then it contains Σ too, hence it contains $\Sigma \vee \widehat{\Sigma}$. Therefore, $\Sigma \vee \widehat{\Sigma}$ is the smallest $\widetilde{C_T}$-invariant saturated multiplicative system which properly contains $\widehat{\Sigma}$; in particular, $\widehat{\Sigma} \in \mathbf{Spec}^1_{\mathfrak{L}}(X, \gamma^*)$.

Injectivity of the map $\Sigma \longmapsto \widehat{\Sigma}$ follows from that $\Sigma_1 \subseteq \Sigma_2$ iff $\widehat{\Sigma}_1 \subseteq \widehat{\Sigma}_2$. ∎

The definitions of the remaining spectra are even more straightforward.

11.4.2. Closed spectra and flat spectra. Elements of the *closed complete* $\mathfrak{L}$-*spectrum*, $\mathbf{Spec}^1_{\mathfrak{C}}(X, \gamma^*)$, are those $\Sigma \in \mathbf{Spec}^1_{\mathfrak{L}}(X, \gamma^*)$ which are closed in the sense of 5.2. The *closed* $\mathfrak{L}$-*spectrum*, $\mathbf{Spec}^0_{\mathfrak{C}}(X, \gamma^*)$, consists of all $\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X, \gamma^*)$ such that $\widehat{\Sigma}$ is closed.

The *flat complete* $\mathfrak{L}$-*spectrum*, $\mathbf{Spec}^1_{\mathfrak{f}\mathfrak{L}}(X, \gamma^*)$, is formed by $\Sigma \in \mathbf{Spec}^1_{\mathfrak{L}}(X, \gamma^*)$ such that the localization $\Sigma^{-1}X \longrightarrow X$ is continuous (i.e. it has a direct image functor).

The *flat* $\mathfrak{L}$-*spectrum*, $\mathbf{Spec}^0_{\mathfrak{f}\mathfrak{L}}(X, \gamma^*)$, is formed by $\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X, \gamma^*)$ such that $\widehat{\Sigma}$ belongs to $\mathbf{Spec}^1_{\mathfrak{f}\mathfrak{L}}(X, \gamma^*)$.

11.4.2.1. Note. For any $\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X, \gamma^*)$, the full subcategory of the category C_X generated by objects which are left closed for Σ (cf. 5.3) is $\widetilde{C_T}$-stable. But, the full subcategory of C_X whose objects are Σ-torsion free objects of C_X is not, in general, $\widetilde{C_T}$-stable.

11.5. Locally trivial actions and spectra.

11.5.1. Proposition. *Let* (X, γ^*) *be a* $\widetilde{C}_T$*-'space' such that there exists a family* $\{\Sigma_i \mid i \in J\}$ *of saturated stable multiplicative systems with the following properties:*

(a) $\bigcap_{i \in J} \Sigma_i = Iso(C_X)$;

(b) every $\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ *containing some of* Σ_i *is* $\widetilde{C}_T$*-stable.*

Then the canonical map $\mathbf{Spec}^0_{\mathfrak{L}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{L}}(X)$, $\Sigma \mapsto \widehat{\Sigma}$, *takes values in* $\mathbf{Spec}^1_{\mathfrak{L}}(X, \gamma^*)$.

Proof. Let U_i denote the 'space' $\Sigma_i^{-1}X$ and u_i the canonical morphism $U_i \longrightarrow X$, $i \in J$. Since each Σ_i is stable, the action γ^* induces a $\widetilde{C}_T$-action, γ_i^* on U_i. The condition (a) means that the family of localizations $\{U_i \xrightarrow{q_i} X \mid i \in J\}$ is conservative. By 2.2.5, $\mathbf{Spec}^0_{\mathfrak{L}}(X) = \bigcup_{i \in J} \mathbf{Spec}^0_{\mathfrak{L}}(U_i; X)$, where $\mathbf{Spec}^0_{\mathfrak{L}}(U_i; X) = \{\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid \Sigma_{u_i} \subseteq \widehat{\Sigma}\}$. The condition (b) means that for every $i \in J$, all $\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(U_i)$ are $\widetilde{C}_T$-stable. In particular, $\mathbf{Spec}^0_{\mathfrak{L}}(U_i, \gamma_i^*) = \mathbf{Spec}^0_{\mathfrak{L}}(U_i)$ for every $i \in J$. This implies that for every $\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X)$, the multiplicative system $\widehat{\Sigma}$ is $\widetilde{C}_T$-stable. Thus, the map $\Sigma \longmapsto \widehat{\Sigma}$ is an embedding $\mathbf{Spec}^0_{\mathfrak{L}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{L}}(X, \gamma^*)$. ∎

12. Spectra of triangulated categories.

12.1. 'Spaces' represented by triangulated categories. Triangulated categories are triples $(C_X, \gamma; \mathfrak{D})$, where (C_X, γ) is an additive $\mathbb{Z}$-category, and $\mathfrak{D}$ a family of 'distinguished' triangles. We shall denote a triangulated category $(C_X, \gamma; \mathfrak{D})$ by $\mathcal{CT}_{\mathfrak{X}}$ and regard it as a triangulated category representing a 'space' $\mathfrak{X}$.

12.2. Preliminaries: thick subcategories and saturated multiplicative systems of a triangulated category.

12.2.1. Multiplicative systems in triangulated categories. Fix a triangulated category $\mathcal{CT}_{\mathfrak{X}} = (C_X, \gamma; \mathfrak{D})$. A multiplicative system Σ of (X, γ) is said to be *compatible with triangulation* if for any pair of distinguished triangles (L, M, N, u, v, w) and (L', M', N', u', v', w') and any commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{u} & M \\ s \Big\downarrow & & \Big\downarrow s' \\ L' & \xrightarrow{u'} & M' \end{array}$$

where s and s' are elements of Σ, there exists a morphism $N \xrightarrow{t} N'$ which belongs to Σ and such that (s, s', t) is a morphism of triangles.

We shall use same notations: $\mathcal{SM}(\mathfrak{X})$ (resp. $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(\mathfrak{X})$) for the preorder of multiplicative (resp. saturated multiplicative) systems of the triangulated 'space' $\mathfrak{X}$. The dualization functor $\mathfrak{X} \longmapsto \mathfrak{X}^o$ induces an isomorphism of preorders

$$\mathcal{SM}(\mathfrak{X}) \xrightarrow{\sim} \mathcal{SM}(\mathfrak{X}^o) \quad \text{and} \quad \mathcal{S}^{\mathfrak{s}}\mathcal{M}(\mathfrak{X}) \xrightarrow{\sim} \mathcal{S}^{\mathfrak{s}}\mathcal{M}(\mathfrak{X}^o).$$

12.2.2. Triangulated subcategories. Recall that a full subcategory, $\mathbb{T}$, of the category C_X is called a *triangulated subcategory* if it is stable by translations, and has a triangulated structure such that the inclusion functor $\mathbb{T} \hookrightarrow C_X$ is exact.

Let $\mathbb{T}$ be a full subcategory of C_X stable by translations. The subcategory $\mathbb{T}$ admits a triangulated structure which makes it a triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$ iff for any morphism $L \xrightarrow{f} M$ of $\mathbb{T}$, there exists a distinguished triangle (L, M, N, f, g, h) such that $N \in Ob\mathbb{T}$.

12.2.3. Definitions. (1) A full triangulated subcategory, $\mathbb{T}$, of $\mathcal{CT}_{\mathfrak{X}}$ is called *saturated* (in [Ve2]), if every direct summand (in C_X) of an object of $\mathbb{T}$ belongs to $\mathbb{T}$.

(2) A full triangulated subcategory, $\mathbb{T}$, of $\mathcal{CT}_{\mathfrak{X}}$ is called *thick* (in [Ve1] and everywhere else), if every distinguished triangle (L, M, N, u, v, w) such that $N \in Ob\mathbb{T}$ and $L \xrightarrow{u} M$ factors through an object of $\mathbb{T}$, belongs to $\mathbb{T}$ (that is L and M are objects of $\mathbb{T}$).

These two notions are equivalent: *A full triangulated subcategory of a triangulated category is thick iff it is saturated.*

12.2.4. Triangulated subcategories and multiplicative systems. For any full triangulated subcategory, $\mathbb{T}$, of the triangulated category $\mathcal{CT}_{\mathfrak{X}}$, let $\Sigma_{\mathbb{T}}$ denote the family of all morphisms $L \xrightarrow{u} M$ of $\mathcal{CT}_{\mathfrak{X}}$ such that there exists a distinguished triangle (L, M, N, u, v, w), where N is an object of $\mathbb{T}$.

12.2.5. Proposition [Ve2, 2.1.8]. *For any full triangulated subcategory $\mathbb{T}$ of a triangulated category $\mathcal{CT}_{\mathfrak{X}}$, the family $\Sigma_{\mathbb{T}}$ is a multiplicative system. The system $\Sigma_{\mathbb{T}}$ is saturated iff the subcategory $\mathbb{T}$ is thick.*

For any multiplicative system Σ in the triangulated category $\mathcal{CT}_{\mathfrak{X}}$, let $\mathbb{T}_{\Sigma}$ denote the full subcategory of $\mathcal{CT}_{\mathfrak{X}}$ generated by objects N contained in a distinguished triangle (L, M, N, u, v, w) such that $L \xrightarrow{u} M$ belongs to Σ.

12.2.6. Proposition [Ve1, 2.1]. *The map $\Sigma \longmapsto \mathbb{T}_{\Sigma}$ is an isomorphism of the preorder $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(\mathfrak{X})$ of saturated multiplicative systems of a triangulated category $\mathcal{CT}_{\mathfrak{X}}$ onto the preorder $\mathfrak{Tht}(\mathfrak{X})$ of thick triangulated subcategories of $\mathcal{CT}_{\mathfrak{X}}$. The inverse isomorphism is given by the map $\mathbb{T} \longmapsto \Sigma_{\mathbb{T}}$.*

12.2.6.1. Corollary. *The intersection of any set of saturated multiplicative systems of a triangulated category is a saturated multiplicative system.*

Proof. The assertion follows from an easily checked fact that the intersection of any set of thick triangulated subcategories of a triangulated category is a thick triangulated subcategory. ∎

12.3. The spectrum $\mathbf{Spec}^0_{\mathfrak{L}}$. Fix a triangulated category $\mathcal{CT}_{\mathfrak{X}}$. We define $\mathbf{Spec}^0_{\mathfrak{L}}(\mathfrak{X})$ the same way as $\mathbf{Spec}^0_{\mathfrak{L}}(X)$ is defined in 2.1: elements of $\mathbf{Spec}^0_{\mathfrak{L}}(\mathfrak{X})$ are saturated multiplicative systems, Σ, of $\mathfrak{X}$ such that the union, $\widehat{\Sigma}$, of all $\Sigma' \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(\mathfrak{X})$ which do not contain Σ is a saturated multiplicative system.

Thanks to the isomorphism between the preorder $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(\mathfrak{X})$ of saturated multiplicative systems and the preorder $\mathfrak{Tht}(\mathfrak{X})$ of thick triangulated subcategories of $\mathcal{CT}_{\mathfrak{X}}$, (see 12.2.6.), one can, like in the abelian case, define spectra in terms of thick triangulated subcategories.

We denote by $\mathbf{Spec}^0_{\mathfrak{Tht}}(\mathfrak{X})$ the preorder (with respect to the inclusion) of all thick triangulated subcategories $\mathcal{P}$ of $\mathcal{CT}_{\mathfrak{X}}$ such that there exists the biggest thick triangulated subcategory, $\widehat{\mathcal{P}}$ of $\mathcal{CT}_{\mathfrak{X}}$ which does not contain $\mathcal{P}$. The map $\mathcal{P} \longmapsto \Sigma_{\mathcal{P}}$ (cf. 12.2.4) induces an isomorphism of preorders $\mathbf{Spec}^0_{\mathfrak{Tht}}(\mathfrak{X}) \xrightarrow{\sim} \mathbf{Spec}^0_{\mathfrak{L}}(\mathfrak{X})$.

12.3.1. Representatives of elements of $\mathbf{Spec}^0_{\mathfrak{Tht}}(\mathfrak{X})$. For any object M of the category $\mathcal{CT}_{\mathfrak{X}}$, let $[M]_{\mathfrak{t}}$ denote the *thick envelope of* M which is the smallest thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$ containing M. And let $\langle M\rangle_{\mathfrak{t}}$ denote the full subcategory of $\mathcal{CT}_{\mathfrak{X}}$ whose objects are $N \in Ob\mathcal{CT}_{\mathfrak{X}}$ such that $M \notin Ob[N]_{\mathfrak{t}}$.

The following assertion is a triangulated version of 6.1.3.3.

12.3.2. Proposition. *(a) For any object M of the category $\mathcal{CT}_{\mathfrak{X}}$, the subcategory $\langle M\rangle_{\mathfrak{t}}$ is the union of all thick triangulated subcategories which do not contain M.*

(b) The following conditions are equivalent:

(i) $[M]_{\mathfrak{t}} \in \mathbf{Spec}^0_{\mathfrak{Tht}}(\mathfrak{X})$;

(ii) $\langle M\rangle_{\mathfrak{t}}$ is a thick triangulated subcategory.

Proof. (a) If $\mathbb{T} \in \mathfrak{Tht}(\mathfrak{X})$ and $M \notin Ob\mathbb{T}$, then $M \notin [N]_{\mathfrak{t}}$ for every $N \in Ob\mathbb{T}$, hence the assertion.

(b) $(i) \Rightarrow (ii)$. Clearly, $[M]_{\mathfrak{t}} \in \mathbf{Spec}^0_{\mathfrak{Tht}}(\mathfrak{X})$ iff there exists the biggest thick triangulated subcategory $\mathcal{P}'$ such that $M \notin Ob\mathcal{P}'$. Therefore $\langle M\rangle_{\mathfrak{t}} \subseteq \mathcal{P}'$. The inverse inclusion, $\mathcal{P}' \subseteq \langle M\rangle_{\mathfrak{t}}$, follows from the definition of the subcategory $\langle M\rangle_{\mathfrak{t}}$.

$(ii) \Rightarrow (i)$. Every thick triangulated subcategory which does not contain M is a subcategory of $\langle M\rangle_{\mathfrak{t}}$. Therefore, if $\langle M\rangle_{\mathfrak{t}} \in \mathfrak{Tht}(\mathfrak{X})$, then it is the biggest thick triangulated subcategory which does not contain M. ∎

We denote by $Spec^0_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$ the family of all objects M of the category $\mathcal{CT}_{\mathfrak{X}}$ satisfying the equivalent conditions of 12.3.1(b).

12.4. $\mathfrak{L}$-Local triangulated categories and the associated spectrum.

12.4.1. $\mathfrak{L}$-Local triangulated categories. A triangulated category $\mathcal{CT}_{\mathfrak{X}}$ is $\mathfrak{L}$-*local* iff the intersection of all nonzero thick triangulated subcategories of $\mathcal{CT}_{\mathfrak{X}}$ is nonzero. In other words, the preorder $\mathfrak{T}\mathfrak{h}\mathfrak{t}_\star(\mathfrak{X})$ of all nonzero thick triangulated subcategories of $\mathcal{CT}_{\mathfrak{X}}$ has the smallest element.

12.4.2. The spectrum $\mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}$. The elements of $\mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$ are thick triangulated subcategories, $\mathcal{P}$, of $\mathcal{CT}_{\mathfrak{X}}$ such that the intersection $\mathcal{P}^\star$ of all thick triangulated subcategories of $\mathcal{CT}_{\mathfrak{X}}$ properly containing $\mathcal{P}$ is not equal to $\mathcal{P}$.

The map $\mathcal{P} \longmapsto \Sigma_{\mathcal{P}}$ (cf. 12.2.4) induces an isomorphism of preorders

$$\mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X}) \xrightarrow{\sim} \mathbf{Spec}^1_{\mathfrak{L}}(\mathfrak{X}).$$

12.4.3. Proposition. *(a) A thick triangulated subcategory $\mathcal{P}$ belongs to $\mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$ iff the quotient triangulated category $\mathcal{CT}_{\mathfrak{X}}/\mathcal{P}$ is $\mathfrak{L}$-local.*

(b) There is an embedding $\mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X}) \longrightarrow \mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$ defined by $\mathcal{P} \longmapsto \widehat{\mathcal{P}}$.

Proof. The assertion (a) is obvious. An argument similar to the argument of 3.2 proves (b). ■

12.5. Functorial properties. The following proposition (which is a part of [Ve2, 2.3.1]) is a convenient reference for the rest of this section.

12.5.1. Proposition. *Let $\mathcal{B}$ and $\mathcal{A}$ be full triangulated subcategories of a triangulated category $\mathcal{CT}_{\mathfrak{X}}$ such that $\mathcal{B} \subseteq \mathcal{A}$.*

(a) The canonical functor $\mathcal{A}/\mathcal{B} \longrightarrow \mathcal{CT}_{\mathfrak{X}}/\mathcal{B}$ is fully faithful and injective on objects. The image of this functor is $q^*_{\mathcal{B}}(\mathcal{A})$, where $q^*_{\mathcal{B}}$ is the canonical functor $\mathcal{CT}_{\mathfrak{X}} \longrightarrow \mathcal{CT}_{\mathfrak{X}}/\mathcal{B}$.

The subcategory $\mathcal{A}$ is thick iff the subcategory $q^*_{\mathcal{B}}(\mathcal{A})$ is thick.

(b) The map $\mathcal{A} \longmapsto q^*_{\mathcal{B}}(\mathcal{A})$ is an isomorphism of the preorder of strictly full triangulated subcategories of $\mathcal{CT}_{\mathfrak{X}}$ containing the kernel, $\mathcal{B}^{\mathfrak{t}}$ of the functor $q^*_{\mathcal{B}}$ onto the preorder of strictly full triangular subcategories of $\mathcal{CT}_{\mathfrak{X}}/\mathcal{B}$.

(c) The canonical functor $\mathcal{CT}_{\mathfrak{X}}/\mathcal{A} \longrightarrow (\mathcal{CT}_{\mathfrak{X}}/\mathcal{B})/(\mathcal{A}/\mathcal{B})$ is an isomorphism of triangulated categories.

12.5.2. Corollary. *Let $\mathcal{B}$ and $\mathcal{A}$ be full triangulated subcategories of a triangulated category $\mathcal{CT}_{\mathfrak{X}}$ such that $\mathcal{B} \subseteq \mathcal{A}$. Let $\mathcal{B}^{\mathfrak{t}}$ be the thick envelope of $\mathcal{B}$ in $\mathcal{CT}_{\mathfrak{X}}$. Then $\mathcal{B}^{\mathfrak{t}} \cap \mathcal{A}$ is the thick envelope of $\mathcal{B}$ in $\mathcal{A}$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccc}
\mathcal{B}^{\mathfrak{t}} & \longrightarrow & \mathcal{CT}_{\mathfrak{X}} & \longrightarrow & \mathcal{CT}_{\mathfrak{X}}/\mathcal{B} \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{B}^{\mathfrak{t}}_{\mathcal{A}} & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{A}/\mathcal{B}
\end{array}$$

with exact rows. By 12.5.1(a), the functor $\mathcal{A}/\mathcal{B}\longrightarrow\mathcal{CT}_{\mathfrak{X}}/\mathcal{B}$ is faithful. Therefore, the kernel, $\mathcal{B}^{\mathfrak{t}}_{\mathcal{A}}$, of the localization functor $\mathcal{A}\longrightarrow\mathcal{A}/\mathcal{B}$, which is the thick envelope of $\mathcal{B}$ in $\mathcal{A}$, coincides with $\mathcal{B}^{\mathfrak{t}}\cap\mathcal{A}$. ∎

12.5.3. Proposition. *Let $\mathcal{CT}_{\mathfrak{Y}}$ be a full triangulated subcategory of a triangulated category $\mathcal{CT}_{\mathfrak{X}}$.*

(a) The localization $\mathcal{CT}_{\mathfrak{X}}\longrightarrow\mathcal{CT}_{\mathfrak{X}}/\mathcal{CT}_{\mathfrak{Y}}=\mathcal{CT}_{\mathfrak{X}/\mathfrak{Y}}$ induces an isomorphism

$$\mathcal{U}_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(C_{\mathfrak{Y}})\bigcap\mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})\xrightarrow{\sim}\mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X}/\mathfrak{Y})$$

and an injective morphism

$$\mathcal{U}_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(C_{\mathfrak{Y}})\bigcap\mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})\longrightarrow\mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X}/\mathfrak{Y}).$$

Here $\mathcal{U}_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(C_{\mathfrak{Y}})=\{\mathbb{T}\in\mathfrak{T}\mathfrak{h}\mathfrak{t}(\mathfrak{X})\mid C_{\mathfrak{Y}}\subseteq\mathbb{T}\}$.

(b) $Spec^0_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})\bigcap ObC_{\mathfrak{Y}}\subseteq Spec^0_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{Y})$.

(c) Suppose the triangulated subcategory $\mathcal{CT}_{\mathfrak{Y}}$ is thick. Then the map

$$\mathfrak{T}\mathfrak{h}\mathfrak{t}(\mathfrak{X})\longrightarrow\mathfrak{T}\mathfrak{h}\mathfrak{t}(\mathfrak{Y}),\ \mathbb{T}\longmapsto\mathbb{T}\cap\mathcal{CT}_{\mathfrak{Y}},$$

induces an injective morphism

$$\mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})-\mathcal{U}_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(C_{\mathfrak{Y}})\longrightarrow\mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{Y}).\tag{1}$$

Proof. (a) The assertion follows from 12.5.1 (or 4.3.1).

(b) Let $M\in Ob\mathcal{CT}_{\mathfrak{Y}}$; and let $\langle M\rangle_{\mathfrak{t}}$ be the union of all thick triangulated subcategories of the triangulated category $\mathcal{CT}_{\mathfrak{X}}$ which do not contain the object M. Then $\langle M\rangle_{\mathfrak{t}}\bigcap\mathcal{CT}_{\mathfrak{Y}}$ is the union, $\langle M\rangle^{\mathfrak{Y}}_{\mathfrak{t}}$, of all thick triangulated subcategories of $\mathcal{CT}_{\mathfrak{Y}}$ which do not contain M.

In fact, let $\mathbb{T}$ be a thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{Y}}$ which does not contain M. And let $\mathbb{T}^{\mathfrak{t}}$ be the thick envelope of $\mathbb{T}$ in $\mathcal{CT}_{\mathfrak{X}}$. By 12.5.2, $\mathbb{T}^{\mathfrak{t}}\bigcap\mathcal{CT}_{\mathfrak{Y}}=\mathbb{T}$; in particular, $M\notin Ob\mathbb{T}^{\mathfrak{t}}$. The latter means precisely that $\mathbb{T}^{\mathfrak{t}}\subseteq\langle M\rangle_{\mathfrak{t}}$, hence $\mathbb{T}=\mathbb{T}^{\mathfrak{t}}\bigcap\mathcal{CT}_{\mathfrak{Y}}\subseteq\langle M\rangle_{\mathfrak{t}}\bigcap\mathcal{CT}_{\mathfrak{Y}}$. This shows that $\langle M\rangle_{\mathfrak{t}}\bigcap\mathcal{CT}_{\mathfrak{Y}}$ is the union of all thick triangulated subcategories of $\mathcal{CT}_{\mathfrak{Y}}$ which do not contain M.

If $M\in Spec^0_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$, i.e. $\langle M\rangle_{\mathfrak{t}}$ is a thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$, then the intersection $\langle M\rangle_{\mathfrak{t}}\bigcap\mathcal{CT}_{\mathfrak{Y}}$ is a thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{Y}}$. Therefore $M\in Spec^0_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{Y})$.

(c) Let $M\in Spec^0_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$, i.e. $[M]_{\mathfrak{t}}\in\mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$. Then

$$\big[M\notin ObC^{\mathfrak{t}}_{\mathfrak{Y}}\big]\Leftrightarrow\big[C^{\mathfrak{t}}_{\mathfrak{Y}}\subseteq\langle M\rangle_{\mathfrak{t}}\big]\Leftrightarrow\big[C_{\mathfrak{Y}}\subseteq\langle M\rangle_{\mathfrak{t}}\big].$$

Here $C^{\mathfrak{t}}_{\mathfrak{Y}}$ is the thick envelope of $\mathcal{CT}_{\mathfrak{Y}}$ in $C_{\mathfrak{Y}}$, and $\langle M\rangle_{\mathfrak{t}}$ is the biggest thick subcategory which does not contain M (cf. 12.3.1). Thus, if the triangulated subcategory $C_{\mathfrak{Y}}$ is thick, then $\big[M\in ObC_{\mathfrak{Y}}\big]\Leftrightarrow\big[[M]_{\mathfrak{t}}\subseteq ObC_{\mathfrak{Y}}\big]\Leftrightarrow\big[C_{\mathfrak{Y}}\nsubseteq\langle M\rangle_{\mathfrak{t}}\big]$. The assertion follows now from (a). ∎

12.6. Preliminaries on orthogonality. For any subcategory $\mathcal{B}$ of $\mathcal{CT}_{\mathfrak{X}}$, the *left orthogonal*, ${}^{\perp}\mathcal{B}$, of $\mathcal{B}$ is the full subcategory of $\mathcal{CT}_{\mathfrak{X}}$ generated by all objects N such that $\mathcal{CT}_{\mathfrak{X}}(N, M) = 0$ for all $M \in Ob\mathcal{B}$. The right orthogonal of $\mathcal{B}$ is defined dually and is denoted by $\mathcal{B}^{\perp}$. Its objects are $\mathcal{B}$-torsion free objects of $\mathcal{CT}_{\mathfrak{X}}$. If $\mathcal{B}$ is a triangulated subcategory, then $\mathcal{B}^{\perp}$ and ${}^{\perp}\mathcal{B}$ are thick triangulated subcategories.

12.6.1. Proposition [Ve2, 2.3.3]. *Let $\mathcal{B}$ be a full triangulated subcategory of a triangulated category $\mathcal{CT}_{\mathfrak{X}}$ and $\mathcal{CT}_{\mathfrak{X}} \xrightarrow{q^*_{\mathcal{B}}} \mathcal{CT}_{\mathfrak{X}}/\mathcal{B}$ the canonical localization functor.*

(a) For an object M of $\mathcal{CT}_{\mathfrak{X}}$, the following conditions are equivalent:

(i) The object M is $q^*_{\mathcal{B}}$-free.

(ii) The object M is left closed for $\Sigma_{\mathcal{B}}$, i.e. $\mathcal{CT}_{\mathfrak{X}}(s, M)$ is an isomorphism for every $s \in \Sigma_{\mathcal{B}}$. Here $\Sigma_{\mathcal{B}}$ is the multiplicative system corresponding to $\mathcal{B}$ (cf. 12.2.4).

(iii) Every morphism $M \xrightarrow{s} N$ with $s \in \Sigma_{\mathcal{B}}$ admits a retraction.

(iv) The object M is $\mathcal{B}$-torsion free, that is for every $L \in Ob\mathcal{B}$, $\mathcal{CT}_{\mathfrak{X}}(L, M) = 0$.

(v) For every $N \in Ob\mathcal{CT}_{\mathfrak{X}}$, the map

$$\mathcal{CT}_{\mathfrak{X}}(N, M) \longrightarrow \mathcal{CT}_{\mathfrak{X}}/\mathcal{B}(q^*_{\mathcal{B}}(N), q^*_{\mathcal{B}}(M))$$

is an isomorphism.

(b) The full subcategory $\mathcal{L}(q_{\mathcal{B}})$ of $\mathcal{CT}_{\mathfrak{X}}$ generated by $q^*_{\mathcal{B}}$-free objects is a thick triangulated subcategory.

(c) The composition of the inclusion functor $\mathcal{CT}_{\mathcal{L}(q_{\mathcal{B}})} \hookrightarrow \mathcal{CT}_{\mathfrak{X}}$ and the localizaiton functor $\mathcal{CT}_{\mathfrak{X}} \xrightarrow{q^*_{\mathcal{B}}} \mathcal{CT}_{\mathfrak{X}}/\mathcal{B}$ is a fully faithful functor injective on objects.

(d) Let $\mathcal{CT}_{\mathfrak{D}(q_{\mathcal{B}*})}$ be the full subcategory of the quotient triangulated category $\mathcal{CT}_{\mathfrak{X}}/\mathcal{B}$ generated by all objects M such that the functor $\mathcal{CT}_{\mathfrak{X}}/\mathcal{B}(q^*_{\mathcal{B}}(-), M)$ is representable. The subcategory $\mathcal{CT}_{\mathfrak{D}(q_{\mathcal{B}*})}$ is triangulated and strictly full. If infinite coproducts or products exist in $\mathcal{CT}_{\mathfrak{X}}$, then $\mathcal{CT}_{\mathfrak{D}(q_{\mathcal{B}*})}$ is thick.

(e) The localization functor $q^*_{\mathcal{B}}$ induces an equivalence of categories

$$\mathcal{CT}_{\mathcal{L}(q_{\mathcal{B}})} \longrightarrow \mathcal{CT}_{\mathfrak{D}(q_{\mathcal{B}*})}.$$

(f) An object N of $\mathcal{CT}_{\mathfrak{X}}$ belongs to the preimage, $\mathcal{CT}_{\mathfrak{R}(q_{\mathcal{B}})} = {q^*_{\mathcal{B}}}^{-1}(\mathcal{CT}_{\mathfrak{D}(q_{\mathcal{B}*})})$, of the subcategory $\mathcal{CT}_{\mathfrak{D}(q_{\mathcal{B}*})}$ iff there exists a morphism $N \xrightarrow{s} M$ such that M is $q^*_{\mathcal{B}}$-free and $q^*_{\mathcal{B}}(s)$ is invertible.

(g) The inclusion functor $\mathcal{CT}_{\mathcal{L}(q_{\mathcal{B}})} \longrightarrow \mathcal{CT}_{\mathfrak{R}(q_{\mathcal{B}})}$ has a left adjoint.

Proof. The equivalence of (i), (ii), (iii), and (v) follow from 5.3.3.2, as well as the assertions (c), (e), (f), and (g). The proof of the remaining statements can be found in [Ve2, 2.3.3]. ∎

12.6.2. Corollary [Ve1, 6–3]. *Let $\mathbb{T}$ be a thick triangulated subcategory of the triangulated category $\mathcal{CT}_{\mathfrak{X}}$. The full subcategory of $\mathcal{CT}_{\mathfrak{X}}$ generated by objects which are left closed for $\Sigma_{\mathbb{T}}$, is the right orthogonal, $\mathbb{T}^{\perp}$, of the subcategory $\mathbb{T}$.*

Proof. The fact follows from the equivalence of (ii) and (iv) in 12.6.1. ∎

12.6.3. Proposition [Ve1, 6–5]. *Let $\mathbb{T}$ be a thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$, and let*

$$\mathbb{T} \xrightarrow{\iota_{\mathbb{T}}^*} \mathcal{CT}_{\mathfrak{X}} \xrightarrow{q_{\mathbb{T}}^*} \mathcal{CT}_{\mathfrak{X}}/\mathbb{T}$$

be the inclusion and localization functors. The following properties are equivalent:

(a) The functor $\iota_{\mathbb{T}}^*$ has a right adjoint.

(b) The functor $q_{\mathbb{T}}^*$ has a right adjoint.

12.7. The structure of $\mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$.

12.7.1. Proposition. *Let $\mathcal{P}$ be a thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$ such that the intersection $\mathcal{P}_0 = \mathcal{P}^{\perp} \cap \mathcal{P}^{\star}$ is nonzero. Then*

(a) The subcategory $\mathcal{P}$ is an object of $\mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$ and $\mathcal{P} = {}^{\perp}\mathcal{P}_0$.

(b) The triangulated category $\mathcal{P}^{\perp}$ is $\mathfrak{L}$-local, and $\mathcal{P}_0$ is its smallest nonzero thick triangulated subcategory.

Proof. (a) The condition $\mathcal{P}_0 \neq 0$ implies, obviously, that $\mathcal{P}^{\star}$ contains $\mathcal{P}$ properly, i.e. $\mathcal{P}$ is an object of $\mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$

The inclusion $\mathcal{P}_0 \subseteq \mathcal{P}^{\perp}$ is equivalent to the inclusion $\mathcal{P} \subseteq^{\perp} \mathcal{P}_0$. If the (thick triangulated) subcategory ${}^{\perp}\mathcal{P}_0$ contains $\mathcal{P}$ properly, then ${}^{\perp}\mathcal{P}_0/\mathcal{P}$ is a nonzero thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}/\mathcal{P}$, hence it contains the image, $\widetilde{\mathcal{P}}_0$, of the subcategory $\mathcal{P}_0$. This means that for every $L \in Ob\mathcal{P}_0$, there exists an object, M, of ${}^{\perp}\mathcal{P}_0$ and an isomorphism $q_{\mathcal{P}}^*(M) \xrightarrow{\sim} q_{\mathcal{P}}^*(L)$. The latter is determined by a diagram $M \xrightarrow{s'} K \xleftarrow{s} L$ whose both arrows belong to $\Sigma_{\mathcal{P}}$. Since L is an object of $\mathcal{P}^{\perp}$, it follows from the equivalence of (iii) and (iv) in 12.6.1 that the morphism $K \xleftarrow{s} L$ admits a retraction, $K \xrightarrow{s''} L$. Let $M \xrightarrow{t} L$ be the composition $s'' \circ s'$. Since the morphism t belongs to $\Sigma_{\mathcal{P}}$, there exists a distinguished triangle $M \xrightarrow{t} L \longrightarrow N$ such that $N \in Ob\mathcal{P}$. In particular, $N \in Ob{}^{\perp}\mathcal{P}_0$. Thus, we have a distinguished triangle, $M \xrightarrow{t} L \longrightarrow N$, such that M and N are objects of the thick subcategory ${}^{\perp}\mathcal{P}_0$. Therefore, L is an object of ${}^{\perp}\mathcal{P}_0$, which cannot happen, unless $L = 0$. Thus, ${}^{\perp}\mathcal{P}_0$ cannot contain $\mathcal{P}$ properly, i.e. $\mathcal{P} = {}^{\perp}\mathcal{P}_0$.

(b) Let $\mathbb{T}$ be a nonzero thick triangulated subcategory of $\mathcal{P}^{\perp}$. Then the image, $q_{\mathcal{P}}^*(\mathbb{T})$, in the quotient category $\mathcal{CT}_{\mathfrak{X}}/\mathcal{P}$ is nonzero, hence its thick envelope contains the subcategory $\mathcal{P}^{\star}/\mathcal{P}$. In particular, it contains the image of the subcategory $\mathcal{P}_0$. Since objects of the thick envelope of $q_{\mathcal{P}}^*(\mathbb{T})$ are direct summands of objects of $q_{\mathcal{P}}^*(\mathbb{T})$, this means that for every object L of $\mathcal{P}_0$, there exists an object M of $\mathbb{T}$ such that $q_{\mathcal{P}}^*(L)$ is a direct summand of $q_{\mathcal{P}}^*(M)$. Since both objects, L and M, belong to the subcategory $\mathcal{P}^{\perp}$ and, by 12.6.1(c) (and 12.6.1(i) $\Leftrightarrow$ (iv)), the restriction of the localization functor $q_{\mathcal{P}}^*$ to the subcategory $\mathcal{P}^{\perp}$ is a fully faithful functor, it follows that L is a direct summand of M. Since $\mathbb{T}$ is a thick subcategory of $\mathcal{CT}_{\mathfrak{X}}$, it contains all direct summands of its objects. Thus $\mathcal{P}_0 \subseteq \mathbb{T}$. ∎

12.7.2. Corollary. *Let $\mathcal{P}$ be a thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$ such that the intersection $\mathcal{P}_0 = \mathcal{P}^{\perp} \cap \mathcal{P}^{\star}$ is nonzero. Then $\mathcal{P}$ is closed under all colimits which exist in $\mathcal{CT}_{\mathfrak{X}}$, in particular, $\mathcal{P}$ is closed under all coproducts which exist in $\mathcal{CT}_{\mathfrak{X}}$.*

Proof. In fact, by 12.7.1, $\mathcal{P} = {}^{\perp}\mathcal{P}_0$; and the left orthogonal to any subcategory is closed under arbitrary colimits which exist in $\mathcal{CT}_{\mathfrak{X}}$. ∎

12.7.3. Proposition. *Suppose that infinite coproducts or products exist in $C\mathcal{T}_{\mathfrak{X}}$. Let $\mathcal{P}$ be a thick triangulated subcategory of $C\mathcal{T}_{\mathfrak{X}}$. Then the following properties of are equivalent:*

(i) $\mathcal{P}_0 = \mathcal{P}^{\perp} \cap \mathcal{P}^{\star}$ is nonzero;

(ii) $\mathcal{P}$ belongs to $\mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$ and the composition of the inclusion $\mathcal{P}_0 \hookrightarrow C\mathcal{T}_{\mathfrak{X}}$ and the localization functor $C\mathcal{T}_{\mathfrak{X}} \xrightarrow{q^*_{\mathcal{P}}} C\mathcal{T}_{\mathfrak{X}}/\mathcal{P}$ induces an equivalence of triangulated categories $\mathcal{P}_0 \xrightarrow{\sim} \mathcal{P}^{\star}/\mathcal{P}$.

(iii) $\mathcal{P}$ belongs to $\mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$ and the inclusion functor $\mathcal{P} \hookrightarrow \mathcal{P}^{\star}$ has a right adjoint.

(iv) $\mathcal{P}$ belongs to $\mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$ and $\mathcal{P}^{\perp}$ is nonzero.

Proof. The implications $(iii) \Leftarrow (ii) \Rightarrow (i) \Rightarrow (iv) \Leftarrow (iii)$ hold by obvious reasons (see 12.7.1(a)). The implication $(iii) \Rightarrow (ii)$ follows from 12.6.3 (see also 12.6.2). Thus, $(iii) \Leftrightarrow (ii) \Rightarrow (i) \Rightarrow (iv)$ without any additional hypothesis on $C\mathcal{T}_{\mathfrak{X}}$. The existence of infinite coproducts or products is needed for the implication

$(iv) \Rightarrow (ii)$. Fix an object $\mathcal{P}$ of $\mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$. By 12.6.1(e) (see also 12.6.1$(i) \Leftrightarrow (iv)$), the composition of the localization functor $q^*_{\mathcal{P}}$ with the inclusion functor $\mathcal{P}^{\perp} \hookrightarrow C\mathcal{T}_{\mathfrak{X}}$ induces an equivalence of the triangulated categories $\mathcal{P}^{\perp} \longrightarrow C\mathcal{T}_{\mathfrak{D}(q_{\mathcal{P}*})}$. Here $C\mathcal{T}_{\mathfrak{D}(q_{\mathcal{P}*})}$ is the full subcategory of the quotient category $C\mathcal{T}_{\mathfrak{X}}/\mathcal{P}$ generated by all objects M such that the functor $C\mathcal{T}_{\mathfrak{X}}/\mathcal{P}(q^*_{\mathcal{P}}(-), M)$ is representable. By 12.6.1(d), if infinite coproducts, or infinite products exist in $C\mathcal{T}_{\mathfrak{X}}$, then $C\mathcal{T}_{\mathfrak{D}(q_{\mathcal{P}*})}$ is a thick triangulated subcategory of the $\mathfrak{L}$-local triangulated category $C\mathcal{T}_{\mathfrak{X}}/\mathcal{P}$. If $\mathcal{P}^{\perp} \neq 0$, then (and only then) the subcategory $C\mathcal{T}_{\mathfrak{D}(q_{\mathcal{P}*})}$ is nonzero, hence it contains the (smallest non-trivial thick) subcategory $\mathcal{P}^{\star}/\mathcal{P}$ which implies that $C\mathcal{T}_{\mathfrak{D}(q_{\mathcal{P}*})}$ is an $\mathfrak{L}$-local triangulated category having $\mathcal{P}^{\star}/\mathcal{P}$ as the smallest nonzero thick subcategory. This, in turn, implies that $\mathcal{P}_0 = \mathcal{P}^{\perp} \cap \mathcal{P}^{\star}$ is nonzero and, moreover, the localization $q^*_{\mathcal{P}}$ induces an equivalence between $\mathcal{P}_0$ and $\mathcal{P}^{\star}/\mathcal{P}$. ∎

12.7.4. Decomposition of $\mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$. Fix a triangulated category $C\mathcal{T}_{\mathfrak{X}}$. Let $\mathbf{Spec}^{1,0}_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$ (resp. $\mathbf{Spec}^{1,1}_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$) denote the full subcategory of $\mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$ generated by $\mathcal{P}$ such that $\mathcal{P}_0 = \mathcal{P}^{\star} \cap \mathcal{P}^{\perp} = 0$ (resp. $\mathcal{P}_0 \neq 0$). Thus, we have the decomposition

$$\mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X}) = \mathbf{Spec}^{1,0}_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X}) \bigsqcup \mathbf{Spec}^{1,1}_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X}).$$

If 'small' coproducts or products exist in $C\mathcal{T}_{\mathfrak{X}}$, then, by 12.7.3(iv), the subcategory $\mathbf{Spec}^{1,0}_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$ is generated by all thick triangulated subcategories $\mathcal{P}$ such that $\mathcal{P} \neq \mathcal{P}^{\star}$ and $\mathcal{P}^{\perp} = 0$. The latter equality means, by definition, that $\mathcal{P}$ *generates* the category $C\mathcal{T}_{\mathfrak{X}}$. Thus, $\mathbf{Spec}^{1,1}_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$ might be regarded as the 'proper' part of $\mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$.

In the case of an arbitrary triangulated category $C\mathcal{T}_{\mathfrak{X}}$, objects of $\mathbf{Spec}^{1,0}_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$ are $\mathcal{P}$ such that $\mathcal{P}$ generates $\mathcal{P}^{\star}$ (and $\mathcal{P} \neq \mathcal{P}^{\star}$).

12.8. The intermediate spectrum $\mathbf{Spec}^{1/2}_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$. Let $\mathbf{Spec}^{1/2}_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$ denote the full subpreorder of $\mathfrak{T}\mathfrak{h}\mathfrak{t}(\mathfrak{X})$ whose objects are thick triangulated subcategories $\mathcal{Q}$ such that ${}^{\perp}\mathcal{Q}$ belongs to $\mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$ and every thick triangulated subcategory of $C\mathcal{T}_{\mathfrak{X}}$ properly containing ${}^{\perp}\mathcal{Q}$ contains $\mathcal{Q}$; i.e. ${}^{\perp}\mathcal{Q} \vee \mathcal{Q}$ is the smallest thick triangulated subcategory of $C\mathcal{T}_{\mathfrak{X}}$ properly containing ${}^{\perp}\mathcal{Q}$.

12.8.1. Proposition. *(a) The map $\mathcal{Q} \longmapsto {}^{\perp}\mathcal{Q}$ induces an isomorphism of preorders $\mathbf{Spec}^{1/2}_{\mathfrak{Tht}}(\mathfrak{X}) \xrightarrow{\sim} \mathbf{Spec}^{1,1}_{\mathfrak{Tht}}(\mathfrak{X})$.*

(b) If $\mathcal{Q}$ is an object of $\mathbf{Spec}^{1/2}_{\mathfrak{Tht}}(\mathfrak{X})$, then $\mathcal{Q}$ is a minimal nonzero thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$.

(c) Suppose that $\mathcal{CT}_{\mathfrak{X}}$ has infinite coproducts or products. Then the following properties of a thick triangulated subcategory $\mathcal{Q}$ are equivalent:

(i) $\mathcal{Q}$ belongs to $\mathbf{Spec}^{1/2}_{\mathfrak{Tht}}(\mathfrak{X})$;

(ii) $\mathcal{Q}$ is a minimal nonzero thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$ such that ${}^{\perp}\mathcal{Q}$ belongs to $\mathbf{Spec}^{1}_{\mathfrak{Tht}}(\mathfrak{X})$.

Proof. (a) Let $\mathcal{Q}$ be an object of $\mathbf{Spec}^{1/2}_{\mathfrak{Tht}}(\mathfrak{X})$. This means that ${}^{\perp}\mathcal{Q}$ belongs to $\mathbf{Spec}^{1}_{\mathfrak{Tht}}(\mathfrak{X})$ and $({}^{\perp}\mathcal{Q})^{\star} = {}^{\perp}\mathcal{Q} \vee \mathcal{Q}$. Since $\mathcal{Q}$ is contained in the intersection

$$\mathcal{Q}_1 = ({}^{\perp}\mathcal{Q})^{\perp} \cap ({}^{\perp}\mathcal{Q})^{\star} = ({}^{\perp}\mathcal{Q})^{\perp} \cap ({}^{\perp}\mathcal{Q} \vee \mathcal{Q})$$

and $\mathcal{Q} \neq 0$, the subcategory ${}^{\perp}\mathcal{Q}$ belongs to $\mathbf{Spec}^{1,1}_{\mathfrak{Tht}}(\mathfrak{X})$.

By 12.7.1(b), the triangulated category $({}^{\perp}\mathcal{Q})^{\perp}$ is $\mathfrak{L}$-local and $\mathcal{Q}_1$ is its smallest nonzero thick triangulated subcategory. Therefore, $\mathcal{Q}_1 = \mathcal{Q}$. Thus, the composition of the map

$$\mathbf{Spec}^{1/2}_{\mathfrak{Tht}}(\mathfrak{X}) \longrightarrow \mathbf{Spec}^{1,1}_{\mathfrak{Tht}}(\mathfrak{X}), \qquad \mathcal{Q} \longmapsto {}^{\perp}\mathcal{Q}, \tag{1}$$

with the map $\mathcal{P} \longmapsto \mathcal{P}_0 = \mathcal{P}^{\perp} \cap \mathcal{P}^{\star}$ is identical. It follows from 12.7.1 that the latter map is a morphism of preorders

$$\mathbf{Spec}^{1,1}_{\mathfrak{Tht}}(\mathfrak{X}) \longrightarrow \mathbf{Spec}^{1/2}_{\mathfrak{Tht}}(\mathfrak{X}). \tag{2}$$

The argument above shows that the map (2) is inverse to the map (1).

(b) If $\mathcal{Q}$ is an object of $\mathbf{Spec}^{1/2}_{\mathfrak{Tht}}(\mathfrak{X})$, then, by (a), $\mathcal{Q}$ is the smallest thick triangulated subcategory of the $\mathfrak{L}$-local category $({}^{\perp}\mathcal{Q})^{\perp}$; in particular, $\mathcal{Q}$ is a minimal nonzero thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$.

(c) Suppose that $\mathcal{CT}_{\mathfrak{X}}$ has infinite coproducts or products. Let $\mathcal{Q}$ be a thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$ such that ${}^{\perp}\mathcal{Q}$ belongs to $\mathbf{Spec}^{1}_{\mathfrak{Tht}}(\mathfrak{X})$. Then $({}^{\perp}\mathcal{Q})^{\perp}$ contains a nonzero subcategory $\mathcal{Q}$, hence it is nonzero. By 12.7.3(iv), this is equivalent to that $\mathcal{Q}_1 = ({}^{\perp}\mathcal{Q})^{\perp} \cap ({}^{\perp}\mathcal{Q})^{\star}$ is nonzero. By 12.7.1(b), $\mathcal{Q}_1$ is the smallest thick triangulated subcategory of the $\mathfrak{L}$-local triangulated category $({}^{\perp}\mathcal{Q})^{\perp}$. In particular, $\mathcal{Q}_1 \subseteq \mathcal{Q}$. If $\mathcal{Q}$ is a minimal nonzero thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$, then the inclusion $\mathcal{Q}_1 \subseteq \mathcal{Q}$ implies that $\mathcal{Q}$ coincides with $\mathcal{Q}_1$. The assertion follows now from (a). ■

12.8.2. Corollary. *(a) If $\mathcal{Q}$ is an object of $\mathbf{Spec}^{1/2}_{\mathfrak{Tht}}(\mathfrak{X})$, then $\mathcal{Q} = [M]_{\mathfrak{t}}$ for any nonzero object M of $\mathcal{Q}$.*

(b) The following properties of an object M of the category $\mathcal{CT}_{\mathfrak{X}}$ are equivalent:

(i) The thick envelope, $[M]_{\mathfrak{t}}$, of M belongs to $\mathbf{Spec}^{1/2}_{\mathfrak{Tht}}(\mathfrak{X})$.

(ii) ${}^{\perp}M$ belongs to $\mathbf{Spec}^{1}_{\mathfrak{Tht}}(\mathfrak{X})$, and if $\mathbb{T}$ is a thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$ properly containing ${}^{\perp}M$, then $M \in Ob\mathbb{T}$.

(iii) $[M]_{\mathfrak{t}}$ *is a minimal nonzero thick subcategory, and* $^\perp M$ *belongs to* $\mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$.

(c) The equivalent conditions (i), (ii), or (iii) imply the following property:

(iv) $^\perp M$ *belongs to* $\mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$*, and every nonzero thick triangulated subcategory of* $(^\perp M)^\perp$ *contains* M.

(d) If infinite coproducts or products exist in $\mathcal{C}\mathcal{T}_{\mathfrak{X}}$*, then* (iv) *is equivalent to the properties (i), (ii), and (iii).*

Proof. (a) The assertion follows from the minimality of $\mathcal{Q}$ (see 12.8.1(b)).

(b) $(i) \Rightarrow (ii)$. Let $\mathcal{Q}$ be an object of $\mathbf{Spec}^{1/2}_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$; and let M be a nonzero object of $\mathcal{Q}$. Since $\mathcal{Q} = [M]_{\mathfrak{t}}$ and $^\perp M = {}^\perp[M]_{\mathfrak{t}}$, the subcategory $^\perp M$ coincides with $^\perp\mathcal{Q}$. Since $\mathcal{Q}$ belongs to $\mathbf{Spec}^{1/2}_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$, $^\perp\mathcal{Q}\vee\mathcal{Q}$ is the smallest thick subcategory of $\mathcal{C}\mathcal{T}_{\mathfrak{X}}$ properly containing $^\perp\mathcal{Q}$. It contains the object M.

$(ii) \Rightarrow (i)$. The conditions (ii) mean that $^\perp M \vee [M]_{\mathfrak{t}}$ is the smallest thick triangulated subcategory of $\mathcal{C}\mathcal{T}_{\mathfrak{X}}$ properly containing $^\perp M = {}^\perp[M]_{\mathfrak{t}}$.

The implications $(ii) \Leftrightarrow (iii)$ follow from 12.5.1.

(c) $(iii) \Rightarrow (iv)$. Let $\mathcal{Q} = [M]_{\mathfrak{t}}$. It follows from 12.5.1(ii) that $(^\perp M)^\perp = (^\perp\mathcal{Q})^\perp$ is an $\mathfrak{L}$-local triangulated category and $\mathcal{Q} = [M]_{\mathfrak{t}}$ is its smallest nonzero thick subcategory. Clearly $M \in Ob\mathcal{Q}$.

(d) The implication $(iv) \Rightarrow (iii)$ follows from 12.5.1(c). ■

12.9. The structure of $\mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$. The decomposition

$$\mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X}) = \mathbf{Spec}^{1,0}_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})\bigsqcup\mathbf{Spec}^{1,1}_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X}). \tag{1}$$

induces a decomposition

$$\mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X}) = \mathbf{Spec}^{0,0}_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})\bigsqcup\mathbf{Spec}^{0,1}_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X}). \tag{2}$$

Here $\mathbf{Spec}^{0,i}_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$ is the full subcategory of $\mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$ whose objects are all thick triangulated subcategories $\mathcal{P}$ such that $\widehat{\mathcal{P}}$ belongs to $\mathbf{Spec}^{1,i}_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$, $i = 0, 1$.

12.9.1. Proposition. *Let* $\mathcal{P}$ *be an object of* $\mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$*. The following conditions are equivalent:*

(i) $\mathcal{P}$ *belongs to* $\mathbf{Spec}^{0,1}_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$.

(ii) $\mathcal{P} = \widehat{\mathcal{P}}^\perp \cap \widehat{\mathcal{P}}^\star = \widehat{\mathcal{P}}^\perp \cap (\widehat{\mathcal{P}} \vee \mathcal{P})$.

(iii) $\widehat{\mathcal{P}} = {}^\perp\mathcal{P}$*, i.e. if* $\mathbb{T}$ *is a thick triangulated subcategory of* $\mathcal{C}\mathcal{T}_{\mathfrak{X}}$ *such that* $\mathcal{P} \not\subseteq \mathbb{T}$*, then* $\mathbb{T} \subseteq {}^\perp\mathcal{P}$.

Proof. $(i) \Rightarrow (ii)$. The condition (i) means that $\widehat{\mathcal{P}}$ is a thick triangulated subcategory and $\widehat{\mathcal{P}}_0 = \widehat{\mathcal{P}}^\perp \cap \widehat{\mathcal{P}}^\star$ is nonzero. In particular, $\widehat{\mathcal{P}}_0 \not\subseteq \widehat{\mathcal{P}}$. By the definition of $\widehat{\mathcal{P}}$ this implies the inclusion $\mathcal{P} \subseteq \widehat{\mathcal{P}}_0$. By 12.7.1(b), $\widehat{\mathcal{P}}_0$ is the smallest nonzero thick triangulated subcategory of $\widehat{\mathcal{P}}^\perp$. Therefore, $\mathcal{P} = \widehat{\mathcal{P}}_0$.

$(ii) \Rightarrow (iii)$. By 12.7.1(a), $\widehat{\mathcal{P}} = {}^\perp\widehat{\mathcal{P}}_0$. Therefore, since by (ii) $\mathcal{P} = \widehat{\mathcal{P}}_0$, we obtain the equality $\widehat{\mathcal{P}} = {}^\perp\mathcal{P}$.

$(iii) \Rightarrow (i)$. Suppose $\widehat{\mathcal{P}} = {}^\perp\mathcal{P}$. Then

$$\widehat{\mathcal{P}}_0 = \widehat{\mathcal{P}}^\perp \cap \widehat{\mathcal{P}}^\star = ({}^\perp\mathcal{P})^\perp \cap (\widehat{\mathcal{P}} \vee \mathcal{P}) = ({}^\perp\mathcal{P})^\perp \cap ({}^\perp\mathcal{P} \vee \mathcal{P}).$$

Notice that $({}^\perp\mathcal{P})^\perp \supseteq \mathcal{P} \subseteq ({}^\perp\mathcal{P} \vee \mathcal{P})$, hence $\mathcal{P} \subseteq ({}^\perp\mathcal{P})^\perp \cap ({}^\perp\mathcal{P} \vee \mathcal{P}) \subseteq \widehat{\mathcal{P}}_0$. In particular, $\widehat{\mathcal{P}}_0 \neq 0$ which means, by definition, that $\mathcal{P}$ belongs to $\mathbf{Spec}^{0,1}_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$. ■

12.9.2. Note. It follows from 12.9.1 that

$$\mathbf{Spec}^{0,1}_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X}) = \mathbf{Spec}^{0}_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X}) \bigcap \mathbf{Spec}^{1/2}_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X}).$$

12.10. Flat spectra. Let $\mathfrak{Se}(\mathfrak{X})$ denote the family of all thick triangulated subcategories of the triangulated category $\mathcal{CT}_{\mathfrak{X}}$ which satisfy equivalent conditions of 12.6.3. We define the *complete flat spectrum* of $\mathfrak{X}$, $\mathbf{Spec}^1_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X})$, by setting

$$\mathbf{Spec}^1_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X}) = \mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X}) \bigcap \mathfrak{Se}(\mathfrak{X}). \tag{1}$$

We define the *flat spectrum of* $\mathfrak{X}$ as a full subpreorder, $\mathbf{Spec}^0_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X})$, of $\mathfrak{T}\mathfrak{h}\mathfrak{t}(\mathfrak{X})$ whose objects are all $\mathcal{P}$ such that $\widehat{\mathcal{P}} \in \mathbf{Spec}^1_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X})\}$.

It follows from these definitions that the map $\mathcal{P} \longmapsto \widehat{\mathcal{P}}$ defines an injective morphism

$$\mathbf{Spec}^0_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X}) \longrightarrow \mathbf{Spec}^1_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X}). \tag{2}$$

Let $\mathbf{Spec}^{1/2}_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X})$ denote the full subpreorder of $\mathbf{Spec}^{1/2}_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$ whose objects are all $\mathcal{Q}$ such that ${}^\perp\mathcal{Q}$ belongs to $\mathfrak{Se}(\mathfrak{X})$.

12.10.1. Proposition. *(a) The map*

$$\mathfrak{T}\mathfrak{h}\mathfrak{t}(\mathfrak{X}) \longrightarrow \mathfrak{T}\mathfrak{h}\mathfrak{t}(\mathfrak{X}), \quad \mathcal{Q} \longmapsto {}^\perp\mathcal{Q},$$

induces an isomorphism

$$\mathbf{Spec}^{1/2}_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X}) \xrightarrow{\sim} \mathbf{Spec}^1_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X}). \tag{3}$$

(b) $\mathbf{Spec}^0_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X}) = \mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X}) \bigcap \mathbf{Spec}^{1/2}_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X})$. *The canonical morphism (2) is the composition of the inclusion* $\mathbf{Spec}^0_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X}) \hookrightarrow \mathbf{Spec}^{1/2}_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X})$ *and the isomorphism (3).*

Proof. Notice that $\mathbf{Spec}^1_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X}) \subseteq \mathbf{Spec}^{1,1}_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$, hence $\mathbf{Spec}^0_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X}) \subseteq \mathbf{Spec}^{0,1}_{\mathfrak{T}\mathfrak{h}\mathfrak{t}}(\mathfrak{X})$. This follows from 12.6.3 and the definitions of these spectra. Now the assertion (a) becomes a consequence of 12.8.1, and (b) follows from 12.9.1 and 12.9.2. ■

12.10.2. Proposition. *(a) If $\mathcal{Q}$ is an object of* $\mathbf{Spec}^{1/2}_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X})$*, then $\mathcal{Q} = [M]_{\mathfrak{t}}$ (hence ${}^\perp\mathcal{Q} = {}^\perp M$) for any nonzero object M of $\mathcal{Q}$.*

(b) The following properties of an object M of the category $\mathcal{CT}_{\mathfrak{X}}$ are equivalent:

(i) The thick envelope, $[M]_{\mathfrak{t}}$, of M belongs to $\mathbf{Spec}^{1/2}_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X})$.

(ii) $^{\perp}M$ *belongs to* $\mathbf{Spec}^1_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X})$, *and if* $\mathbb{T}$ *is a thick triangulated subcategory of* $\mathcal{CT}_{\mathfrak{X}}$ *properly containing* $^{\perp}M$, *then* $M \in Ob\mathbb{T}$.

(iii) $[M]_{\mathfrak{t}}$ *is a minimal nonzero thick subcategory, and* $^{\perp}M$ *belongs to* $\mathbf{Spec}^1_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X})$.

(iv) M *is a nonzero object which belongs to every nonzero thick triangulated subcategory of* $(^{\perp}M)^{\perp}$ *and such that the inclusion functor* $^{\perp}M \longrightarrow \mathcal{CT}_{\mathfrak{X}}$ *has a right adjoint.*

Proof. (a) The assertion is a consequence of the minimality of $\mathcal{Q}$ (see 12.8.1(b)).

(b) The implications $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iv)$ follow from the corresponding implications of 12.8.2.

$(iv) \Rightarrow (iii)$. By 2.6.3, the inclusion functor $^{\perp}M \longrightarrow \mathcal{CT}_{\mathfrak{X}}$ has a right adjoint iff the localization functor $\mathcal{CT}_{\mathfrak{X}} \longrightarrow \mathcal{CT}_{\mathfrak{X}}/^{\perp}M$ has a right adjoint. The latter implies that the quotient category $\mathcal{CT}_{\mathfrak{X}}/^{\perp}M$ is equivalent to the triangulated category $(^{\perp}M)^{\perp}$. The condition that M is contained in every nonzero thick triangulated subcategory of $(^{\perp}M)^{\perp}$ means that $(^{\perp}M)^{\perp}$ is $\mathfrak{L}$-local and $[M]_{\mathfrak{t}}$ is its smallest thick triangulated subcategory. Therefore, $^{\perp}M$ belongs to $\mathbf{Spec}^1_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X})$, and $[M]_{\mathfrak{t}}$ is a minimal nonzero thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$. ■

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