Homotopy theory of bundles with fiber matrix algebra

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Abstract

In the present paper we consider a special class of locally trivial bundles with fiber a matrix algebra. On the set of such bundles over a finite CW-complex we define a relevant equivalence relation. The obtained stable theory gives us a geometric description of the H-space structure $\mathrm{BSU}_{\otimes}$ on BSU related to the tensor product of virtual SU-bundles of virtual dimension 1.

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Introduction

In the present paper we consider a special type of locally trivial bundles with fiber matrix algebra $M_k(\mathbb{C}),\ k>1$ that are equipped with an embedding into a trivial bundle with fiber

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$M_{kl}(\mathbb{C})$ for some l coprime with k. More precisely, using such bundles, we construct a homotopy functor $\widetilde{\mathrm{AB}}^1$ to the category of Abelian group. The construction can be viewed as a counterpart of the one for the usual topological K-functor by means of vector bundles. In particular, the role of classifying spaces in our case is played by matrix counterparts $\mathrm{Gr}_{k,l}$ of the usual Grassmann manifolds. Those $\mathrm{Gr}_{k,l}$ are homogeneous spaces parameterizing all subalgebras in $M_{kl}(\mathbb{C})$ that are isomorphic to $M_k(\mathbb{C})$ (for different pairs $\{k,\,l\}$, $(k,l)=1$). Furthermore, we define a stable equivalence relation on the set of considered bundles such that the classifying space for its equivalence classes is just the direct limit of the matrix Grassmannians $\varinjlim_{(k,l)=1} \mathrm{Gr}_{k,l}$. This equivalence relation can be treated as a counterpart of the usual stable equivalence relation on vector bundles considered in K-theory. Let us remark that the above condition $(k,l)=1$ allows us to avoid the localization of classifying spaces under the stabilization. Finally, we show that the obtained homotopy functor $\widetilde{\mathrm{AB}}^1$ is equivalent to the multiplicative group of the usual $\widetilde{\mathrm{KSU}}$-functor (i.e. to the group $\xi * \eta = \xi + \eta + \xi\eta,\ \xi,\,\eta \in \widetilde{\mathrm{KSU}}(X)$ for a finite CW-complex X). This gives us a geometric description of the H-space structure $\mathrm{BSU}_\otimes$ on BSU in terms of considered type of bundles.

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1 Main definitions

1.1 Category $\mathfrak{FAB}(X)$

Let X be a finite CW-complex. By $\widetilde{M}_n$ denote the trivial bundle (over X) with fiber $M_n(\mathbb{C})$, where $M_n(\mathbb{C})$ is the algebra of all $n\times n$ matrices over $\mathbb{C}$.

Definition 1. Let A_k $(k>1)$ be a locally trivial bundle over X with fiber $M_k(\mathbb{C})$. Suppose there is a bundle map μ

$$\begin{tikzcd} A_k \arrow[rr, "\mu"] \arrow[dr] & & \widetilde{M}_{kl} \arrow[dl] \\ & X & \end{tikzcd}$$

such that for any point $x\in X$ it embeds the fiber $(A_k)_x \cong M_k(\mathbb{C})$ into the fiber $(\widetilde{M}_{kl})_x \cong M_{kl}(\mathbb{C})$ as a central simple subalgebra. Then the triple $(A_k,\,\mu,\,\widetilde{M}_{kl})$ is called an *algebra bundle* (abbrev. *AB*) over X. Moreover, if the positive integers $k,\,l$ are coprime (i.e. their greatest common divisor $(k,l)=1$), then the triple $(A_k,\,\mu,\,\widetilde{M}_{kl})$ is called to be a *floating algebra bundle* (abbrev. *FAB*) over X.

Remark 2. Let A be a central simple algebra over a field $\mathbb{K}$, $B\subset A$ a central simple subalgebra in A. It is well known that the centralizer $\mathrm{Z}_A(B)$ of B in A is a central simple subalgebra in A again, moreover, the equality $A = B\underset{\mathbb{K}}{\otimes} \mathrm{Z}_A(B)$ holds. Taking centralizers for all fibers of the subbundle $A_k \subset \widetilde{M}_{kl}$ in the corresponding fibers of the trivial bundle $\widetilde{M}_{kl}$, we get the complementary subbundle B_l with fiber $M_l(\mathbb{C})$ together with its embedding $\nu\colon B_l \hookrightarrow M_{kl}(\mathbb{C})$. Thus, we have the complementary subbundle $\nu(B_l) \subset \widetilde{M}_{kl}$, where B_l is the locally trivial bundle with fiber $M_l(\mathbb{C})$, and ν is its evident embedding into $\widetilde{M}_{kl}$. Moreover, $A_k \otimes B_l = \widetilde{M}_{kl}$.

Conversely, to a given pair $(A_k,\, B_l)$ consisting of $M_k(\mathbb{C})$-bundle A_k and $M_l(\mathbb{C})$-bundle B_l over X such that $A_k \otimes B_l = \widetilde{M}_{kl}$ we can construct a unique triple $(A_k,\, \mu,\, \widetilde{M}_{kl})$, where μ is the embedding $A_k \hookrightarrow A_k \otimes B_l,\ a \mapsto a \otimes 1_{B_l}$. Note also that the operation of taking centralizer is involutive, i.e $\mathrm{Z}_A(\mathrm{Z}_A(B)) = B$.

Definition 3. A morphism from a (F)AB $(A_k,\, \mu,\, \widetilde{M}_{kl})$ to a (F)AB $(C_m,\, \nu,\, \widetilde{M}_{mn})$ over X is a pair $(f,\, g)$ of bundle maps $f\colon A_k \hookrightarrow C_m,\ g\colon \widetilde{M}_{kl} \hookrightarrow \widetilde{M}_{mn}$ such that

- they are fiberwise homomorphisms of algebras (i.e. they are embeddings in fact);
- the square diagram

$$\begin{tikzcd} \widetilde{M}_{kl} \arrow[r, "g"] & \widetilde{M}_{mn} \\ A_k \arrow[u, "\mu"] \arrow[r, "f"] & C_m \arrow[u, "\nu"'] \end{tikzcd}$$

  commutes;
- let $B_l \subset \widetilde{M}_{kl},\ D_n \subset \widetilde{M}_{mn}$ be the complementary subbundles (see the remark above) for $A_k,\ C_m$, respectively, then g maps B_l into D_n.

Note that a morphism $(f,\, g)\colon (A_k,\, \mu,\, \widetilde{M}_{kl}) \to (C_m,\, \nu,\, \widetilde{M}_{mn})$ exists only if $k|m,\ l|n$.

In particular, an *isomorphism between (F)ABs* $(A_k,\, \mu,\, \widetilde{M}_{kl})$ *and* $(C_k,\, \nu,\, \widetilde{M}_{kl})$ is a pair of bundle maps $f : A_k \to C_k,\ g : \widetilde{M}_{kl} \to \widetilde{M}_{kl}$ which are fiberwise isomorphisms of algebras such that the following diagram

$$\begin{tikzcd} \widetilde{M}_{kl} \arrow[r, "g"] & \widetilde{M}_{kl} \\ A_k \arrow[u, "\mu"] \arrow[r, "f"] & C_k \arrow[u, "\nu"'] \end{tikzcd}$$

commutes.

Clearly, ABs (resp. FABs) over X with morphisms just defined form a category which we will denote by $\mathfrak{AB}(X)$ (resp. $\mathfrak{FAB}(X)$).

For a continuous map $\varphi\colon X \to Y$ we have the functor $\varphi^*\colon (\mathfrak{F})\mathfrak{AB}(Y) \to (\mathfrak{F})\mathfrak{AB}(X)$.

Remark 4. Since $\mathrm{Aut}(M_n(\mathbb{C})) \cong \mathrm{Aut}(\mathbb{C}P^{n-1}) \cong \mathrm{PGL}_n(\mathbb{C})$, there is an equivalent theory of locally trivial bundles which can be developed by replacing matrix algebras $M_n(\mathbb{C})$ with the corresponding projective spaces $\mathbb{C}P^{n-1}$. For example, let us give a counterpart of Definition 1.

Let $\widetilde{\mathbb{C}P}^{kl-1}$ be the trivial bundle over X with fiber $\mathbb{C}P^{kl-1}$.

Definition 5. Let P^{k-1}, Q^{l-1} be locally trivial bundles over X with fibers $\mathbb{C}P^{k-1}$, $\mathbb{C}P^{l-1}$, respectively. Suppose there is a bundle map

$$\begin{tikzcd} P^{k-1}\underset{X}{\times}Q^{l-1} \arrow[rr, "\lambda"] \arrow[dr] & & \widetilde{\mathbb{C}P}^{kl-1} \arrow[dl] \\ & X & \end{tikzcd}$$

(here $P^{k-1}\underset{X}{\times}Q^{l-1}$ is the fibered product of bundles over X) such that for any point $x \in X$ its restriction $\lambda\mid_x \colon (P^{k-1}\underset{X}{\times}Q^{l-1})_x \cong \mathbb{C}P^{k-1} \times \mathbb{C}P^{l-1} \to (\widetilde{\mathbb{C}P}^{kl-1})_x \cong \mathbb{C}P^{kl-1}$ is Segre's embedding (in other words, under some appropriate choice of homogeneous coordinates on the projective spaces it is the map

$$\mathbb{C}P^{k-1} \times \mathbb{C}P^{l-1} \to \mathbb{C}P^{kl-1},$$

$$([x_0 : \ldots : x_{k-1}], [y_0 : \ldots : y_{l-1}]) \mapsto [x_0y_0 : \ldots : x_iy_j : \ldots : x_{k-1}y_{l-1}]).$$

Then $P^{k-1}\underset{X}{\times}Q^{l-1}$ is called a *bundle of Segre's product*, and in the case $(k,l) = 1$ a *floating bundle of Segre's product* (abbrev. *FBSP*).

For example, using FBSPs, we can construct a counterpart of Thom's spaces (see Appendix 2).

2 Classifying spaces

2.1 Canonical bundles over $\mathrm{Gr}_{k,l},\ k,l > 1$

For any pair $k,\ l > 1$ there is the space $\mathrm{Gr}_{k,l}$, parameterizing k-subalgebras (i.e. those isomorphic to $M_k(\mathbb{C})$) in the fixed matrix algebra $M_{kl}(\mathbb{C})$. As a homogeneous space it can be represented as follows:

$$\mathrm{Gr}_{k,l} = \mathrm{PGL}_{kl}(\mathbb{C})\big/\mathrm{PGL}_k(\mathbb{C}) \otimes \mathrm{PGL}_l(\mathbb{C}),$$

where by $\mathrm{PGL}_k(\mathbb{C})\otimes\mathrm{PGL}_l(\mathbb{C})$ we denote the image of the embedding $\mathrm{PGL}_k(\mathbb{C})\times\mathrm{PGL}_l(\mathbb{C}) \to \mathrm{PGL}_{kl}(\mathbb{C})$ induced by the Kronecker product of matrices.

By $\widetilde{\mathcal{M}}_{kl}$ denote the trivial bundle $\mathrm{Gr}_{k,l} \times M_{kl}(\mathbb{C})$. There is a *canonical (F)AB* $(\mathcal{A}_{k,l},\ \mu,\ \widetilde{\mathcal{M}}_{kl})$ over $\mathrm{Gr}_{k,l}$ which can be defined as follows: the fiber $(\mathcal{A}_{k,l})_x$ over $x \in \mathrm{Gr}_{k,l}$ is the k-subalgebra in $M_{kl}(\mathbb{C}) = (\widetilde{\mathcal{M}}_{kl})_x$ corresponding to this point.

By $\mathrm{Fr}_{k,l}$ denote the homogeneous space $\mathrm{PGL}_{kl}(\mathbb{C})\big/E_k \otimes \mathrm{PGL}_l(\mathbb{C})$. The space $\mathrm{Gr}_{k,l}$ is the base of the following principal $\mathrm{PGL}_k(\mathbb{C})$-bundle:

$$\begin{array}{ccc} \mathrm{PGL}_k(\mathbb{C}) & \overset{\ldots\otimes E_l}{\hookrightarrow} & \mathrm{Fr}_{k,l} \\ & & \downarrow \\ & & \mathrm{Gr}_{k,l}\,. \end{array} \tag{6}$$

Now we want to show that (6) can be treated as a frame fibration for the $M_k(\mathbb{C})$-bundle $\mathcal{A}_{k,l}$ over $\mathrm{Gr}_{k,l}$.

More precisely, an ordered collection of n^2 linearly independent matrices $\{\alpha_{i,j}\colon 1 \leq i, j \leq n\}$ in $M_n(\mathbb{C})$ such that $\alpha_{i,j}\alpha_{k,l} = \delta_{jk}\alpha_{i,l}$, $1 \leq i, j, k, l \leq n$ is called to be an *n-frame* in $M_n(\mathbb{C})$. In particular, the collection of matrices $\{e_{i,j}\colon 1 \leq i, j \leq n\}$, where $e_{i,j} = E_{ij}$ are the matrix units, is an n-frame. The map $e_{i,j} \mapsto \alpha_{i,j}$, $1 \leq i, j \leq n$ can be extended to the automorphism of $M_n(\mathbb{C})$ which is identity on the center $\mathbb{C}E_n \subset M_n(\mathbb{C})$. Now applying Noether-Scolem's theorem we see that there is an element $g \in \mathrm{GL}_n(\mathbb{C})$ such that $\alpha_{i,j} = ge_{i,j}g^{-1}$, $1 \leq i, j \leq n$. Therefore, the group $\mathrm{PGL}_n(\mathbb{C})$ acts transitive on the set of n-frames. Moreover, the stabilizer in $\mathrm{PGL}_n(\mathbb{C})$ of the n-frame $\{e_{i,j}\colon 1 \leq i, j \leq n\}$ is trivial. This implies that the set of n-frames in $M_n(\mathbb{C})$ can be identified with the group space of $\mathrm{PGL}_n(\mathbb{C})$. It follows easily that bundle (6) is the bundle of k-frames associated with the $M_k(\mathbb{C})$-bundle $\mathcal{A}_{k,l}$.

Similarly, we can define a k-frame in the matrix algebra $M_{kl}(\mathbb{C})$ as an ordered collection of k^2 linearly independent matrices $\{\alpha_{i,j}\colon 1 \leq i, j \leq k\}$ in $M_{kl}(\mathbb{C})$ such that $\alpha_{i,j}\alpha_{r,s} = \delta_{jr}\alpha_{i,s}$, $1 \leq i, j, r, s \leq k$. The space $\mathrm{Fr}_{k,l}$ can be treated as the space of k-frames in the matrix algebra $M_{kl}(\mathbb{C})$. Indeed, it follows from Noether-Scolem's theorem that the group $\mathrm{PGL}_{kl}(\mathbb{C}) = \mathrm{Aut}(M_{kl}(\mathbb{C}))$ acts transitive on the set of such frames and the stabilizer of the k-frame $\{e_{i,j} = E_{ij} \otimes E_l\colon 1 \leq i, j \leq k\}$ (here E_{ij} is a matrix unit in $M_k(\mathbb{C})$ and E_l is the unit $l \times l$ matrix) is just the subgroup $E_k \otimes \mathrm{PGL}_l(\mathbb{C}) \subset \mathrm{PGL}_{kl}(\mathbb{C})$.

Remark 6. The noncompact spaces $\mathrm{Gr}_{k,l}$ and $\mathrm{Fr}_{k,l}$ can be replaced by homotopy equivalent compact ones $\mathrm{Gr}^U_{k,l}$, $\mathrm{Fr}^U_{k,l}$, respectively. More precisely, let $M_k(\mathbb{C}) \otimes \mathbb{C}E_l \subset M_{kl}(\mathbb{C})$ be the "standard" k-subalgebra. A subalgebra $A \cong M_k(\mathbb{C})$ in $M_{kl}(\mathbb{C})$ is called *unitary* if there is an element $g \in \mathrm{U}(kl) \subset \mathrm{GL}_{kl}(\mathbb{C})$ such that $g(M_k(\mathbb{C}) \otimes \mathbb{C}E_l)g^{-1} = A$. In other words, A is unitary if it conjugates with the standard k-subalgebra by a unitary transformation. The set of unitary subalgebras in $M_{kl}(\mathbb{C})$ is parameterized by the subspace

$$\mathrm{Gr}^U_{k,l} := \mathrm{PU}(kl)/\,\mathrm{PU}(k) \otimes \mathrm{PU}(l) \subset \mathrm{Gr}_{k,l},$$

where $\mathrm{PU}(n)$ is the projective unitary group, i.e. the quotient group $\mathrm{U}(n)/\{\alpha E_n \mid \alpha \in \mathbb{C}^*,\ |\alpha| = 1\}$. It is clear that $\mathrm{Gr}^U_{k,l}$ is compact and homotopy equivalent to $\mathrm{Gr}_{k,l}$. The principle bundle

$$\begin{array}{ccll} \mathrm{PU}(k) & \overset{\ldots\otimes E_l}{\hookrightarrow} & \mathrm{Fr}^U_{k,l} & := \mathrm{PU}(kl)/E_k \otimes \mathrm{PU}(l) \\ & & \downarrow & \\ & & \mathrm{Gr}^U_{k,l} & \end{array} \tag{7}$$

(compare with bundle (6)) can be considered as the bundle of k-frames that are unitary with respect to the Hermite scalar product $\mathrm{tr}(X\overline{Y}^t)$ in $M_{kl}(\mathbb{C})$. There is a unitary counterpart $(\mathcal{A}^U_{k,l},\ \mu^U,\ \widetilde{\mathcal{M}}^U_{kl})$ over $\mathrm{Gr}^U_{k,l}$ of the canonical (F)AB over $\mathrm{Gr}_{k,l}$, where $\mathcal{A}^U_{k,l}$ is the $M_k(\mathbb{C})$-bundle associated with (7). Because of the homotopy-equivalences $\mathrm{Gr}^U_{k,l} \simeq \mathrm{Gr}_{k,l}$, $\mathrm{Fr}^U_{k,l} \simeq \mathrm{Fr}_{k,l}$, we will not make a difference between $\mathrm{Gr}^U_{k,l}$ and $\mathrm{Gr}_{k,l}$, $\mathrm{Fr}^U_{k,l}$ and $\mathrm{Fr}_{k,l}$ below.

2.2 Homotopy groups of $\mathrm{Gr}_{k,l}$

Let k, l, m, n be integers greater than 1 such that $k|m$, $l|n$. Clearly, a homomorphism of the matrix algebras $M_{kl}(\mathbb{C}) \to M_{mn}(\mathbb{C})$ induces the embedding

$$\mathrm{Gr}_{k,l} \hookrightarrow \mathrm{Gr}_{m,n}\,. \tag{8}$$

The aim of this subsection is to compute the homotopy groups of the spaces $\mathrm{Gr}_{k,l}$ and to study their behavior under the maps (8). We show that (8) induces an isomorphism $\pi_r(\mathrm{Gr}_{k,l}) \cong \pi_r(\mathrm{Gr}_{m,n})$, $r \leq 2\min\{k,\, l\}$ under the condition $(m,n)=1$ (see Corollary 10). This is the reason why the condition $(k,l)=1$ is imposed. In the general case the localization of the homotopy groups occurs.

Proposition 7. *The space $\mathrm{Gr}_{k,l}$ has the following homotopy groups in dimensions $\leq 2\min\{k,\, l\}$:*

$$\pi_2(\mathrm{Gr}_{k,l}) \cong \mathbb{Z}/(k,l)\mathbb{Z};\ \ \pi_{2r}(\mathrm{Gr}_{k,l}) \cong \mathbb{Z},\ \mathit{if}\, 2 \leq r \leq \min\{k,\, l\};$$

$$\pi_{2r-1}(\mathrm{Gr}_{k,l}) \cong \mathbb{Z}/(k,l)\mathbb{Z},\ \mathit{if}\, 1 \leq r \leq \min\{k,\, l\}.$$

Remark 8. The homotopy groups $\pi_r(\mathrm{Gr}_{k,l})$, $r \leq 2\min\{k,\, l\}$ will be called "stable" below.

Proof. Let us consider the homotopy sequence of the fibration:

$$\begin{array}{ccc} \mathrm{PU}(k)\otimes \mathrm{PU}(l) & \hookrightarrow & \mathrm{PU}(kl) \\ & & \downarrow \\ & & \mathrm{Gr}_{k,l}\,. \end{array}$$

Suppose $2 \leq r \leq \min\{k,\, l\}$, then we have:

$$0 \to \pi_{2r}(\mathrm{Gr}_{k,l}) \xrightarrow{1} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \pi_{2r-1}(\mathrm{Gr}_{k,l}) \to 0.$$

It is clear from the description of the embedding $\mathrm{PU}(k)\otimes\mathrm{PU}(l) \hookrightarrow \mathrm{PU}(kl)$ (recall that it is induced by the Kronecker product of matrices) that map 2 is following:

$$\mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z}, \quad (\alpha,\beta) \mapsto l\alpha + k\beta.$$

Now in the case $r \geq 2$ the required result follows from the exactness of the homotopy sequence. Moreover, we obtain the description of map 1:

$$\mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z}, \qquad \gamma \mapsto (\frac{k}{(l,k)}\gamma, -\frac{l}{(l,k)}\gamma).$$

In the case $r=1$ we have:

$$0 \to \pi_2(\mathrm{Gr}_{k,l}) \to \mathbb{Z}/k\mathbb{Z}\oplus\mathbb{Z}/l\mathbb{Z} \to \mathbb{Z}/kl\mathbb{Z} \to \pi_1(\mathrm{Gr}_{k,l}) \to 0.$$

The exactness of this sequence yields $\pi_2(\mathrm{Gr}_{k,l}) \cong \mathbb{Z}/(k,l)\mathbb{Z}$, $\pi_1(\mathrm{Gr}_{k,l}) \cong \mathbb{Z}/(k,l)\mathbb{Z}$. $\square$

Now we study the behavior of the stable homotopy groups under maps (8).

Proposition 9. *If $2 \le r \le \min\{k, l\}$, then the homomorphism of the stable homotopy groups $\pi_{2r}(\mathrm{Gr}_{k,l}) \to \pi_{2r}(\mathrm{Gr}_{m,n})$ is the following map:*

$$\mathbb{Z} \to \mathbb{Z}, \quad \gamma \mapsto \frac{(m,n)}{(k,l)}\gamma.$$

If $r = 1$, then the homomorphism $\pi_2(\mathrm{Gr}_{k,l}) \to \pi_2(\mathrm{Gr}_{m,n})$ is the monomorphism:

$$\mathbb{Z}/(k,l)\mathbb{Z} \hookrightarrow \mathbb{Z}/(m,n)\mathbb{Z}.$$

The image of $\pi_{2r-1}(\mathrm{Gr}_{k,l}) \cong \mathbb{Z}/(k,l)\mathbb{Z}$ in the group $\pi_{2r-1}(\mathrm{Gr}_{m,n}) \cong \mathbb{Z}/(m,n)\mathbb{Z}$ under the map $\pi_{2r-1}(\mathrm{Gr}_{k,l}) \to \pi_{2r-1}(\mathrm{Gr}_{m,n})$ is the subgroup generated by $\left[\frac{mn}{kl}\right] := \frac{mn}{kl} \bmod (m,n)$ (in particular, the order $\#\{\mathrm{im}(\pi_{2r-1}(\mathrm{Gr}_{k,l}))\}$ is equal to the order of $\left[\frac{mn}{kl}\right]$ in the group $\mathbb{Z}/(m,n)\mathbb{Z}$).

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
 & \mathrm{PU}(m)\otimes\mathrm{PU}(n) \arrow[r] & \mathrm{PU}(mn) \arrow[d] \\
\mathrm{PU}(k)\otimes\mathrm{PU}(l) \arrow[ur] \arrow[r] & \mathrm{PU}(kl) \arrow[ur] \arrow[d] & \mathrm{Gr}_{m,n} \\
 & \mathrm{Gr}_{k,l}\,. \arrow[ur] &
\end{tikzcd} \tag{10}$$

Let us describe all the maps in (10). The left-hand arrow: $(A, B) \mapsto (E_{m/k} \otimes A, B \otimes E_{n/l})$, where matrices $A \in \mathrm{U}(k)$, $B \in \mathrm{U}(l)$ represent elements of $\mathrm{PU}(k)$ and $\mathrm{PU}(l)$, respectively. The right-hand arrow: $C \mapsto E_{m/k} \otimes (C \otimes E_{n/l}) = (E_{m/k} \otimes C) \otimes E_{n/l}$, $C \in \mathrm{U}(kl)$. The two horizontal arrows are induced by the Kronecker product of matrices: $(A, B) \mapsto A \otimes B$. The commutativity of the diagram follows from the identity $(E_{m/k} \otimes A) \otimes (B \otimes E_{n/l}) = E_{m/k} \otimes ((A \otimes B) \otimes E_{n/l}) = (E_{m/k} \otimes (A \otimes B)) \otimes E_{n/l}$ for all $A \in \mathrm{U}(k)$, $B \in \mathrm{U}(l)$.

Diagram (10) induces the morphism of the homotopy sequences of fibrations. If $2 \le r \le \min\{k, l\}$, then we have the diagram:

$$\begin{tikzcd}
0 \arrow[r] & \pi_{2r}(\mathrm{Gr}_{m,n}) \arrow[r, "1'"] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "2'"] & \mathbb{Z} \arrow[r, "3'"] & \pi_{2r-1}(\mathrm{Gr}_{m,n}) \arrow[r] & 0 \\
0 \arrow[r] & \pi_{2r}(\mathrm{Gr}_{k,l}) \arrow[r, "1"'] \arrow[u, "4"] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "2"'] \arrow[u, "5"] & \mathbb{Z} \arrow[r, "3"'] \arrow[u, "6"] & \pi_{2r-1}(\mathrm{Gr}_{k,l}) \arrow[r] \arrow[u, "7"] & 0.
\end{tikzcd} \tag{11}$$

The description of monomorphisms $1, 1'$:

$$\mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z}, \qquad \gamma \mapsto \Bigl(\frac{k}{(l,k)}\gamma, -\frac{l}{(l,k)}\gamma\Bigr),$$

$$\mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z}, \qquad \nu \mapsto \Bigl(\frac{m}{(n,m)}\nu, -\frac{n}{(n,m)}\nu\Bigr),$$

and $2, 2'$:

$$(\alpha,\ \beta) \mapsto l\alpha + k\beta, \quad (\lambda,\ \mu) \mapsto n\lambda + m\mu$$

can be extracted from the proof of Proposition 7. The epimorphisms 3 and $3'$ are the reductions modulo (k, l) and modulo (m, n) respectively. The description of the diagonal arrows in (10) yields the following description of homomorphisms 5, 6 in (11):

$$(\alpha,\, \beta) \mapsto (\frac{m}{k}\alpha,\, \frac{n}{l}\beta), \quad \tau \mapsto \frac{mn}{kl}\tau,$$

respectively. Now in the case $r \geq 2$ the required result follows from the commutativity of diagram (11).

Now suppose $r = 1$. In this case diagram (10) yields us the diagram:

$$\begin{tikzcd}
0 \arrow[r] & \pi_2(\mathrm{Gr}_{m,n}) \arrow[r] & \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z} \arrow[r] & {} \\
0 \arrow[r] & \pi_2(\mathrm{Gr}_{k,l}) \arrow[r] \arrow[u, "4"] & \mathbb{Z}/k\mathbb{Z} \oplus \mathbb{Z}/l\mathbb{Z} \arrow[r] \arrow[u, "5"] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & \mathbb{Z}/mn\mathbb{Z} \arrow[r] & \pi_1(\mathrm{Gr}_{m,n}) \arrow[r] & 0 \\
{} \arrow[r] & \mathbb{Z}/kl\mathbb{Z} \arrow[r] \arrow[u, "6"] & \pi_1(\mathrm{Gr}_{k,l}) \arrow[r] \arrow[u] & 0.
\end{tikzcd} \tag{12}$$

The commutativity of (12) and the description of the diagonal arrows in (10) imply that arrows 4, 5, 6 are monomorphisms (here we use the evident fact that the map $\mathrm{PU}(k) \overset{\ldots\otimes E_l}{\to} \mathrm{PU}(kl)$ induces the *monomorphism* of the fundamental groups

$$\pi_1(\mathrm{PU}(k)) \hookrightarrow \pi_1(\mathrm{PU}(kl)), \quad \mathbb{Z}/k\mathbb{Z} \hookrightarrow \mathbb{Z}/kl\mathbb{Z}). \quad \square$$

Corollary 10. *Suppose $(k, l) = (m, n) = 1$. Then the map $\mathrm{Gr}_{k,l} \to \mathrm{Gr}_{m,n}$ induces isomorphisms $\pi_r(\mathrm{Gr}_{k,l}) \overset{\cong}{\to} \pi_r(\mathrm{Gr}_{m,n})$ of homotopy groups, where $r \leq 2\min\{k,\, l\}$. Thus, the spaces $\mathrm{Gr}_{k,l}$ and $\mathrm{Gr}_{m,n}$ are homotopy equivalent in the stable dimensions (i.e. up to dimension $r = 2\min\{k,\, l\}$).*

Now let us describe the (stable) homotopy groups of total spaces of frame fibrations (see (7)) and their behavior under maps.

The following Proposition follows easily from the exactness of the homotopy sequence of fibration (7).

Proposition 11.

$$\pi_{2r}(\mathrm{Fr}_{k,l}) = 0, \quad \pi_{2r-1}(\mathrm{Fr}_{k,l}) = \mathbb{Z}/k\mathbb{Z}, \quad 1 \leq r \leq l.$$

Remark 12. Suppose $(m, n) = 1$ ($\Rightarrow$ $(k, l) = 1$). Consider the following morphism of fiber bundles

$$\begin{tikzcd}
 & \mathrm{PU}(n) \arrow[r, "E_m\otimes\ldots"] & \mathrm{PU}(mn) \arrow[d] \\
\mathrm{PU}(l) \arrow[ur, "\ldots\otimes E_{\frac{n}{l}}"] \arrow[r, "E_k\otimes\ldots"] & \mathrm{PU}(kl) \arrow[ur, "E_{\frac{m}{k}}\otimes\ldots\otimes E_{\frac{n}{l}}"] \arrow[d] & \mathrm{Fr}_{m,n} \\
 & \mathrm{Fr}_{k,l} \arrow[ur, "\alpha"] &
\end{tikzcd}$$

and the corresponding morphism of homotopy sequences. Clearly, the order of element $\frac{mn}{kl} \bmod m \in \mathbb{Z}/m\mathbb{Z}$ equals k. It implies that α induces monomorphisms $\pi_{2r-1}(\mathrm{Fr}_{k,l}) \hookrightarrow \pi_{2r-1}(\mathrm{Fr}_{m,n})$, $\mathbb{Z}/k\mathbb{Z} \hookrightarrow \mathbb{Z}/m\mathbb{Z}$ for all $r \leq l$.

2.3 The universal property of $\mathrm{Gr}_{k,l}$

Let $\Psi_{k,l}(X)$ be the set of isomorphism classes of FABs of the form $(A_k,\, \mu,\, \widetilde{M}_{kl})$ over X ($\Rightarrow (k,l)=1$).

Proposition 13. *The assignment:*

$$[X,\, \mathrm{Gr}_{k,l}] \to \Psi_{k,l}(X), \quad \varphi \mapsto \varphi^*(\mathcal{A}_{k,l},\, \mu,\, \widetilde{\mathcal{M}}_{kl})$$

is a bijection for X, $\dim X \leq 2\min\{k,\, l\}$.

Proof. Let $\varphi_0,\, \varphi_1$ be two classifying maps $X \to \mathrm{Gr}_{k,l}$ for $(A_k,\, \mu,\, \widetilde{M}_{kl})$. We must prove that there is a map $\Phi\colon X \times I \to \mathrm{Gr}_{k,l}$ such that $\Phi|_{X\times\{0\}} = \varphi_0$, $\Phi|_{X\times\{1\}} = \varphi_1$. Suppose such a Φ exists, then $\Phi^*(\mathcal{A}_{k,l},\, \mu,\, \widetilde{\mathcal{M}}_{kl}) \cong \pi^*(A_k,\, \mu,\, \widetilde{M}_{kl})$, where $\pi\colon X \times I \to X$ is the projection.

We construct Φ by induction on dimension r of the skeleton of the relative CW-complex $(X \times I,\, X \times \{0\} \sqcup X \times \{1\})$.

Suppose we have already constructed a map $\Phi^{(r)}\colon (X \times I,\, X \times \{0\} \sqcup X \times \{1\})^{(r)} \to \mathrm{Gr}_{k,l}$ with the required properties ($\Phi^{(r)}|_{X\times\{0\}} = \varphi_0$, $\Phi^{(r)}|_{X\times\{1\}} = \varphi_1$) and with an isomorphism $\Phi^{(r)*}(\mathcal{A}_{k,l},\, \mu,\, \widetilde{\mathcal{M}}_{kl}) \cong \pi^*(A_k,\, \mu,\, \widetilde{M}_{kl})|_{(X\times I,\, X\times\{0\}\sqcup X\times\{1\})^{(r)}}$.

Let e_{r+1} be a relative cell in $X \times I$ (it has the form $e_r \times I$, where e_r is a cell in X).

If $r = 2s+1$, then using $\pi_{2s+1}(\mathrm{Gr}_{k,l}) = 0$ (see Proposition 7), we can extend $\Phi^{(r)}$ from the boundary $\partial e_{r+1} = S^{2s+1}$ to the whole cell e_{r+1} and therefore to the $r+1$-skeleton.

If $r = 2s$, then we construct $\Phi^{(r+1)}$ in the following way. The restriction of $\pi^*(A_k,\, \mu,\, \widetilde{M}_{kl})$ to a cell e_{r+1} is a trivial FAB (see Definition 14). Thus, we have the maps:

(i) $\pi^*(A_k,\, \mu,\, \widetilde{M}_{kl})|_{e_{r+1}} \overset{\cong}{\to} (e_{r+1} \times M_k(\mathbb{C}),\, \tau,\, e_{r+1} \times M_{kl}(\mathbb{C}))$;

(ii) $\Phi^{(r)}\colon \partial e_{r+1} \to \mathrm{Gr}_{k,l}$ satisfying the conditions $\Phi^{(r)}|_{e_r\times\{0\}} = \varphi_0$, $\Phi^{(r)}|_{e_r\times\{1\}} = \varphi_1$;

(iii) $\Phi^{(r)*}(\mathcal{A}_{k,l},\, \mu,\, \widetilde{\mathcal{M}}_{kl})|_{\partial e_{r+1}} \cong (\partial e_{r+1} \times M_k(\mathbb{C}),\, \tau,\, \partial e_{r+1} \times M_{kl}(\mathbb{C}))$.

$$\begin{tikzcd} \partial e_{r+1} \arrow[r, "\Psi^{(r)}"] \arrow[dr, "\Phi^{(r)}"'] & \mathrm{Fr}_{k,l} \arrow[d] \\ & \mathrm{Gr}_{k,l} \end{tikzcd} \tag{14}$$

Thus, we have a lifting $\Psi^{(r)}$ of $\Phi^{(r)}$ (see (14)) which to any point $x \in \partial e_{r+1}$ assigns a frame over this point (here we use isomorphism (iii)). Because of $r = 2s$ and $\pi_{2s}(\mathrm{Fr}_{k,l}) = 0$ (see Proposition 11), the lifting $\Psi^{(r)}$ can be extended to the whole $r+1$-cell e_{r+1} and therefore $\Phi^{(r)}$ can be extended to the $(r+1)$-skeleton $(X \times I,\, X \times \{0\} \sqcup X \times \{1\})^{(r+1)}$. $\square$

3 The stable theory

3.1 Stabilization

Suppose $(km, ln) = 1$. Define the product $\circ$ of two FABs $(A_k,\, \mu,\, \widetilde{M}_{kl})$, $(B_m,\, \nu,\, \widetilde{M}_{mn})$ over X as

$$(A_k,\, \mu,\, \widetilde{M}_{kl}) \circ (B_m,\, \nu,\, \widetilde{M}_{mn}) = (A_k \otimes B_m,\, \mu \otimes \nu,\, \widetilde{M}_{kl} \otimes \widetilde{M}_{mn})$$

(notice that $\widetilde{M}_{kl} \otimes \widetilde{M}_{mn} = \widetilde{M}_{klmn}$).

Definition 14. A FAB of the form $(\widetilde{M}_k,\, \tau,\, \widetilde{M}_{kl})$ is called to be *trivial* if $\tau\colon \widetilde{M}_k \to \widetilde{M}_{kl}$ is the following map:

$$X \times M_k(\mathbb{C}) \to X \times M_{kl}(\mathbb{C}), \qquad (x,\, T) \mapsto (x,\, T \otimes E_l)$$

(under some choice of trivializations on $\widetilde{M}_k$ and $\widetilde{M}_{kl}$) for any point $x \in X$, where E_l is the unit $l \times l$ matrix and $T \otimes E_l$ denotes the Kronecker product of matrices. In other words, the bundle $\widetilde{M}_k$ is embedded into $\widetilde{M}_{kl}$ as a fixed subalgebra.

Definition 15. Two FABs $(A_k,\, \mu,\, \widetilde{M}_{kl})$ and $(B_m,\, \nu,\, \widetilde{M}_{mn})$ over X are said to be *stable equivalent*, if there is a sequence of pairs $\{t_i,\, u_i\} \in \mathbb{N}^2$, $1 \leq i \leq s$ such that

- $\{t_1,\, u_1\} = \{k,\, l\}$, $\{t_s,\, u_s\} = \{m,\, n\}$;
- $(t_i t_{i+1},\, u_i u_{i+1}) = 1$ if $s > 1$, $1 \leq i \leq s-1$,

and a sequence of FABs $(A_{t_i},\, \mu_i,\, \widetilde{M}_{t_i u_i})$ over X such that

- $(A_{t_1},\, \mu_1,\, \widetilde{M}_{t_1 u_1}) = (A_k,\, \mu,\, \widetilde{M}_{kl})$, $(A_{t_s},\, \mu_s,\, \widetilde{M}_{t_s u_s}) = (B_m,\, \nu,\, \widetilde{M}_{mn})$;
- $(A_{t_i},\, \mu_i,\, \widetilde{M}_{t_i u_i}) \circ (\widetilde{M}_{t_{i+1}},\, \tau,\, \widetilde{M}_{t_{i+1} u_{i+1}}) \cong (A_{t_{i+1}},\, \mu_{i+1},\, \widetilde{M}_{t_{i+1} u_{i+1}}) \circ (\widetilde{M}_{t_i},\, \tau,\, \widetilde{M}_{t_i u_i})$, where $1 \leq i \leq s-1$ and $(\widetilde{M}_{t_i},\, \tau,\, \widetilde{M}_{t_i u_i})$ are trivial FABs.

By $\widetilde{\mathrm{AB}}^1(X)$ denote the set of stable equivalence classes of FABs over X.

The following theorem justifies the previous definition.

Theorem 16. *1) For all sequences of pairs of positive integers $\{k_j,\, l_j\}_{j \in \mathbb{N}}$ such that*

$$\text{(i)} \quad k_j,\, l_j \to \infty; \quad \text{(ii)}\ k_j | k_{j+1},\, l_j | l_{j+1}; \quad \text{(iii)}\ (k_j, l_j) = 1 \quad \forall j,$$

the corresponding direct limits $\varinjlim_j \mathrm{Gr}_{k_j, l_j}$ are homotopy equivalent. This unique homotopy type we denote by $\varinjlim_{(k,l)=1} \mathrm{Gr}_{k,l}$.

2) The space $\varinjlim_{(k,l)=1} \mathrm{Gr}_{k,l}$ is a classifying space for stable equivalence classes of FABs over a finite CW-complex X. In other words, the functor $X \mapsto \widetilde{\mathrm{AB}}^1(X)$ from the homotopy category of finite CW-complexes to the category $\mathfrak{Set}$ is represented by the space $\varinjlim_{(k,l)=1} \mathrm{Gr}_{k,l}$.

Proof. 1) First, suppose $(k, l) = 1 = (m, n)$, $(km, ln) = 1$, then the common stable parts of the spaces $\mathrm{Gr}_{k,l}$ and $\mathrm{Gr}_{m,n}$ are homotopy equivalent. Indeed, according to Corollary 10 the maps λ, κ :

$$\begin{tikzcd} & \mathrm{Gr}_{km,\,ln} & \\ \mathrm{Gr}_{k,\,l} \arrow[ur, "\lambda"] & & \mathrm{Gr}_{m,\,n} \arrow[ul, "\kappa"'] \end{tikzcd} \tag{15}$$

induce an isomorphism of homotopy groups.

Secondly, suppose $(km, ln) > 1$. Let us take sufficiently large t, u such that $(t, u) = 1$ and $(t, l) = (u, k) = (t, n) = (u, m) = 1$ (hence $(kt, lu) = 1 = (tm, nu)$). Now, using the diagram

$$\begin{tikzcd} & \mathrm{Gr}_{kt,\,lu} & & \mathrm{Gr}_{mt,\,nu} & \\ \mathrm{Gr}_{k,\,l} \arrow[ur] & & \mathrm{Gr}_{t,\,u} \arrow[ul] \arrow[ur] & & \mathrm{Gr}_{m,\,n}, \arrow[ul] \end{tikzcd} \tag{16}$$

we complete the proof of the first part of the theorem.

2) The proof of this part is based on Proposition 13. More precisely, suppose $\dim X \leq 2\min\{k,\, l,\, m,\, n\}$. Let $(A_k,\, \mu,\, \widetilde{M}_{kl})$, $(B_m,\, \nu,\, \widetilde{M}_{mn})$ be FABs over X, let $\varphi_A\colon X \to \mathrm{Gr}_{k,l}$, $\varphi_B\colon X \to \mathrm{Gr}_{m,n}$ be their classifying maps. Let us remark that according to Proposition 13 the classifying maps φ_A, φ_B exist and unique up to homotopy. Suppose the composite maps

$$X \overset{\varphi_A}{\to} \mathrm{Gr}_{k,l} \hookrightarrow \varinjlim_{(t,u)=1} \mathrm{Gr}_{t,\,u},$$

$$X \overset{\varphi_B}{\to} \mathrm{Gr}_{m,n} \hookrightarrow \varinjlim_{(t,u)=1} \mathrm{Gr}_{t,\,u}$$

are homotopic to each other. Under the condition $(km, ln) = 1$ we use diagram (15). Since $\dim X \leq 2\min\{k,\, l,\, m,\, n\}$, we see that already the maps $\lambda \circ \varphi_A$, $\kappa \circ \varphi_B$ are homotopic to each other. Note that

$$\lambda^*(\mathcal{A}_{km,\,ln},\, \mu,\, \widetilde{\mathcal{M}}_{klmn}) \cong (\mathcal{A}_{k,\,l},\, \mu,\, \widetilde{\mathcal{M}}_{kl}) \circ (\widetilde{\mathcal{M}}_m,\, \tau,\, \widetilde{\mathcal{M}}_{mn}),$$

$$\kappa^*(\mathcal{A}_{km,\,ln},\, \mu,\, \widetilde{\mathcal{M}}_{klmn}) \cong (\mathcal{A}_{m,\,n},\, \mu,\, \widetilde{\mathcal{M}}_{mn}) \circ (\widetilde{\mathcal{M}}_k,\, \tau,\, \widetilde{\mathcal{M}}_{kl}),$$

where $(\widetilde{\mathcal{M}}_k,\, \tau,\, \widetilde{\mathcal{M}}_{kl})$, $(\widetilde{\mathcal{M}}_m,\, \tau,\, \widetilde{\mathcal{M}}_{mn})$ are trivial FABs over $\mathrm{Gr}_{m,n}$ and $\mathrm{Gr}_{k,l}$ respectively. Therefore $\lambda \circ \varphi_A$ and $\kappa \circ \varphi_B$ are classifying maps for $(A_k,\, \mu,\, \widetilde{M}_{kl}) \circ (\widetilde{M}_m,\, \tau,\, \widetilde{M}_{mn})$ and $(B_m,\, \nu,\, \widetilde{M}_{mn}) \circ (\widetilde{M}_k,\, \tau,\, \widetilde{M}_{kl})$ respectively. Hence $(A_k,\, \mu,\, \widetilde{M}_{kl}) \circ (\widetilde{M}_m,\, \tau,\, \widetilde{M}_{mn}) \cong (B_m,\, \nu,\, \widetilde{M}_{mn}) \circ (\widetilde{M}_k,\, \tau,\, \widetilde{M}_{kl})$, i.e. $(A_k,\, \mu,\, \widetilde{M}_{kl}) \sim (B_m,\, \nu,\, \widetilde{M}_{mn})$.

Conversely, suppose $(A_k,\, \mu,\, \widetilde{M}_{kl}) \sim (B_m,\, \nu,\, \widetilde{M}_{mn})$. Since $\dim X \leq 2\min\{k,\, l,\, m,\, n\}$, we have $(A_k,\, \mu,\, \widetilde{M}_{kl}) \circ (\widetilde{M}_m,\, \tau,\, \widetilde{M}_{mn}) \cong (B_m,\, \nu,\, \widetilde{M}_{mn}) \circ (\widetilde{M}_k,\, \tau,\, \widetilde{M}_{kl})$ (recall that we have assumed that $(km, ln) = 1$). Then it follows from Proposition 13 that the compositions $\lambda \circ \varphi_A : X \to \mathrm{Gr}_{km,\,ln}$ and $\kappa \circ \varphi_B : X \to \mathrm{Gr}_{km,\,ln}$ are homotopic to each other.

The required assertion in the case $(kl, mn) > 1$ can be obtained similarly, but instead of (15) we should use diagram (16). $\square$

3.2 The group structure

For a FAB $(A_k,\, \mu,\, \widetilde{M}_{kl})$ over X by $[(A_k,\, \mu,\, \widetilde{M}_{kl})]$ we denote its stable equivalence class. Define the product $\diamond$ of two classes $[(A_k,\, \mu,\, \widetilde{M}_{kl})],\ [(B_m,\, \nu,\, \widetilde{M}_{mn})]$ as

$$[(A_k,\, \mu,\, \widetilde{M}_{kl})] \diamond [(B_m,\, \nu,\, \widetilde{M}_{mn})] = [(A_k,\, \mu,\, \widetilde{M}_{kl}) \circ (B_m,\, \nu,\, \widetilde{M}_{mn})].$$

Clearly, this product is well defined if $(km, ln) = 1$. The following lemma allows us to reject this restriction.

Lemma 17. For any pair $\{k,\, l\}$ such that (i) $(k, l) = 1$, (ii) $2\min\{k,\, l\} \geq \dim X$, any stable equivalence class of FABs over X has a representative of the form $(A_k,\, \mu,\, \widetilde{M}_{kl})$.

Proof easily follows from Theorem 16. $\square$

Clearly, the product $\diamond$ is associative, commutative, and has identity element $[(\widetilde{M}_k,\, \tau,\, \widetilde{M}_{kl})]$, where $(\widetilde{M}_k,\, \tau,\, \widetilde{M}_{kl})$ is an arbitrary trivial FAB. Moreover, for any class $[(A_k,\, \mu,\, \widetilde{M}_{kl})]$ there exists the inverse element. In order to find it, let us recall the following fact. The centralizer $\mathrm{Z}_P(Q)$ of a central simple subalgebra Q in a central simple algebra P (over some field $\mathbb{K}$) is a central simple subalgebra again, moreover, the equality $P = Q \underset{\mathbb{K}}{\otimes} \mathrm{Z}_P(Q)$ holds. Therefore, taking the centralizers for all fibers of the subbundle A_k in $\widetilde{M}_{kl}$, we obtain the *complementary* subbundle B_l with fiber $M_l(\mathbb{C})$ together with its embedding $\nu\colon B_l \hookrightarrow M_{kl}(\mathbb{C})$ into the trivial bundle. Moreover, $A_k \otimes B_l = \widetilde{M}_{kl}$. It is not hard to prove that $[(B_l,\, \nu,\, \widetilde{M}_{kl})]$ is the inverse element for $[(A_k,\, \mu,\, \widetilde{M}_{kl})]$. Thus, the functor $X \mapsto \widetilde{\mathrm{AB}}^1(X)$ takes values in the category of Abelian groups $\mathfrak{Ab}$.

The proof of the following proposition is clear.

Proposition 18. *There is the structure of H-space on $\varinjlim_{(k,l)=1} \mathrm{Gr}_{k,l}$ such that the H-space $\varinjlim_{(k,l)=1} \mathrm{Gr}_{k,l}$ represents the functor $X \mapsto \widetilde{\mathrm{AB}}^1(X)$ to the category of Abelian groups $\mathfrak{Ab}$. In other words, the functors $X \mapsto \widetilde{\mathrm{AB}}^1(X)$ and $[X, \varinjlim_{(k,l)=1} \mathrm{Gr}_{k,l}]$ are naturally equivalent as homotopy functors to the category $\mathfrak{Ab}$.*

Remark 19. Note that for any pairs $\{k,\, l\}$, $\{m,\, n\}$ such that $(km, ln) = 1$ we have the maps

$$\mathrm{Gr}_{k,l} \times \mathrm{Gr}_{m,n} \to \mathrm{Gr}_{km,\, ln}$$

induced by the tensor product of matrix algebras $M_{kl}(\mathbb{C}) \times M_{mn}(\mathbb{C}) \xrightarrow{\otimes} M_{kl}(\mathbb{C}) \underset{\mathbb{C}}{\otimes} M_{mn}(\mathbb{C}) \cong M_{klmn}(\mathbb{C})$. Clearly, the multiplication in the H-space $\varinjlim_{(k,l)=1} \mathrm{Gr}_{k,l}$ is determined by such maps on the finite subspaces $\mathrm{Gr}_{k,l} \subset \varinjlim_{(k,l)=1} \mathrm{Gr}_{k,l}$.

Furthermore, for any pair $\{k,\, l\}$ we have map $\mathrm{Gr}_{k,l} \to \mathrm{Gr}_{l,k}$ which assigns to any k-subalgebra in $M_{kl}(\mathbb{C})$ its centralizer and is a classifying map for the complementary bundle $(\mathcal{B}_{l,k},\, \nu,\, \widetilde{\mathcal{M}}_{kl})$ for the canonical bundle $(\mathcal{A}_{k,l},\, \mu,\, \widetilde{\mathcal{M}}_{kl})$ over $\mathrm{Gr}_{k,l}$. These maps induce the inversion map in the H-space $\varinjlim_{(k,l)=1} \mathrm{Gr}_{k,l}$.

Now we simplify our notation: the H-space $\varinjlim_{(k,l)=1} \mathrm{Gr}_{k,l}$ we denote simply by Gr.

3.3 One interesting property of a bundle A_k, which is a part of FAB $(A_k,\, \mu,\, \widetilde{M}_{kl})$

Definition 20. Let $(A_k,\, \mu,\, \widetilde{M}_{kl})$ be an AB over X. The locally trivial $\mathrm{Aut}(M_k(\mathbb{C})) \cong \mathrm{PGL}_k(\mathbb{C})$-bundle A_k is said to be a *core* of the AB $(A_k,\, \mu,\, \widetilde{M}_{kl})$.

Lemma 21. Suppose A_k is the core of a *FAB* $(A_k,\, \mu,\, \widetilde{M}_{kl})$, then the structural group of A_k can be reduced from $\mathrm{Aut}\, M_k(\mathbb{C}) \cong \mathrm{PGL}_k(\mathbb{C})$ to $\mathrm{SL}_k(\mathbb{C})$, i.e. actually to $\mathrm{SU}(k)$.

Proof. We have the covering

$$\begin{tikzcd}
\mu_n \arrow[r, hook] & \mathrm{SU}(n) \arrow[d] \\
& \mathrm{PU}(n),
\end{tikzcd}$$

where μ_n is the group of nth degree roots of unity. Suppose $(k,l)=1$, then the embedding $\mathrm{SU}(k)\times\mathrm{SU}(l) \overset{\otimes}{\hookrightarrow} \mathrm{SU}(kl)$ induces an isomorphism of the center of $\mathrm{SU}(k)\times\mathrm{SU}(l)$ (which is isomorphic to $\mu_k\times\mu_l=\mu_{kl}$) to the center of $\mathrm{SU}(kl)$. Therefore we have the diagram

$$\begin{tikzcd}
& \mathrm{PU}(k) \arrow[rr] & & \mathrm{PU}(kl)/E_k{\otimes}\mathrm{PU}(l) \arrow[d] \\
\mathrm{SU}(k) \arrow[ur] \arrow[r] & \mathrm{SU}(kl)/E_k{\otimes}\mathrm{SU}(l) \arrow[urr] \arrow[d] & & \mathrm{PU}(kl)/\,\mathrm{PU}(k){\otimes}\mathrm{PU}(l) \\
& \mathrm{SU}(kl)/\,\mathrm{SU}(k){\otimes}\mathrm{SU}(l), \arrow[urr, "\varphi"] & &
\end{tikzcd}$$

where φ is an isomorphism.

Another way to prove the lemma is based on the observation that $\pi_2(\mathrm{Gr}_{k,\,l}) = 0$ if $(k,l)=1$. Hence the $\mathrm{PU}(k)$-cocycle determining the $\mathrm{PU}(k)$-bundle $\mathcal{A}_{k,\,l}$ over $\mathrm{Gr}_{k,\,l}$ can be lifted to a $\mathrm{SU}(k)$-cocycle. $\square$

Corollary 22. *The core A_k of arbitrary FAB $(A_k,\, \mu,\, \widetilde{M}_{kl})$ has the form $\mathrm{End}(\xi_k)$ for some vector $\mathrm{SU}(k)$-bundle ξ_k. The bundle ξ_k is determined by A_k uniquely up to isomorphism.*

Proof. Let $\bar{\lambda}_k\colon\, \mathrm{Gr}_{k,\,l}\to \mathrm{BPU}(k)$ be a classifying map for $\mathcal{A}_{k,\,l}$ as a $\mathrm{PU}(k)$-bundle. We must prove the uniqueness of lifting λ_k of $\bar{\lambda}_k$:

$$\begin{tikzcd}
\mathrm{K}(\mu_k;\, 1) \arrow[r] & \mathrm{BSU}(k) \arrow[d] \\
\mathrm{Gr}_{k,\,l} \arrow[ur, "\lambda_k"] \arrow[r, "\bar{\lambda}_k"'] & \mathrm{BPU}(k).
\end{tikzcd} \tag{19}$$

It follows from (19) that the obstruction to a homotopy between two liftings λ_k, λ_k' of $\bar{\lambda}_k$ belongs to the group

$$H^1(\mathrm{Gr}_{k,\,l};\, \pi_1(\mathrm{K}(\mu_k;\, 1))) = H^1(\mathrm{Gr}_{k,\,l};\, \mu_k) = 0. \quad \square$$

It can be proved that any locally trivial $M_k(\mathbb{C})$-bundle A_k over X is the core of some AB. The following lemma shows that, in contrast to the general case, cores of FABs have some specific property.

Lemma 23. Let X be a finite CW-complex. Suppose $\dim X \leq 2\min\{k, m\}$; then the following conditions are equivalent:

(i) A_k is the core of some FAB over X;

(ii) for arbitrary m such that $2m \geq \dim X$ there is a bundle B_m with fiber $M_m(\mathbb{C})$ such that $A_k \otimes \widetilde{M}_m \cong B_m \otimes \widetilde{M}_k$;

(iii) $A_k \otimes \widetilde{M}_m \cong B_m \otimes \widetilde{M}_k$ for some locally trivial bundle B_m with fiber $M_m(\mathbb{C})$ such that $(k, m) = 1$.

Moreover, for any pair of bundles $(A_k,\, B_m)$ such that $(k, m) = 1$ and $A_k \otimes \widetilde{M}_m \cong B_m \otimes \widetilde{M}_k$ there exists a unique stable equivalence class of FABs over X which has (for sufficiently large n, $(km, n) = 1$) FABs of the form $(A_k,\, \mu,\, \widetilde{M}_{kn})$, $(B_m,\, \nu,\, \widetilde{M}_{mn})$ as representatives (for some embeddings $\mu,\, \nu$).

Proof. The implication (i)$\Rightarrow$(ii) follows from Lemma 17, the implication (ii)$\Rightarrow$(iii) is trivial. Thus, we have to prove the implication (iii)$\Rightarrow$(i).

Let $\bar{\lambda}_k \colon \mathrm{Gr}_{k,n} \to \mathrm{BPU}(k)$ be a classifying map for the core $\mathcal{A}_{k,n}$ of the canonical FAB over $\mathrm{Gr}_{k,n}$ as a $\mathrm{PU}(k)$-bundle. The map $\bar{\lambda}_k$ induces the morphism

$$\begin{tikzcd}
 & \mathrm{PU}(k) \arrow[r] & * \arrow[d] \\
\mathrm{PU}(k) \arrow[ur] \arrow[r, "\cdots\otimes E_n"] & \mathrm{Fr}_{k,n} \arrow[ur] \arrow[d] & \mathrm{BPU}(k) \\
 & \mathrm{Gr}_{k,n} \arrow[ur, "\bar{\lambda}_k"] &
\end{tikzcd}$$

from the k-frame bundle $\mathrm{Fr}_{k,n} \overset{\mathrm{PU}(k)}{\longrightarrow} \mathrm{Gr}_{k,n}$ to the universal principle $\mathrm{PU}(k)$-bundle over $\mathrm{BPU}(k)$. It follows from the corresponding morphism of the homotopy sequences that $\bar{\lambda}_{k*}$: $\pi_{2r}(\mathrm{Gr}_{k,n}) = \mathbb{Z} \to \pi_{2r}(\mathrm{BPU}(k)) = \mathbb{Z}$, $2 \leq r \leq \min\{k,\, n\}$ is the homomorphism which takes 1 to k (if $r = 1$, then $\pi_2(\mathrm{Gr}_{k,n}) = 0$ and $\pi_2(\mathrm{BPU}(k)) = \pi_1(\mathrm{PU}(k)) = \mu_k$ is the group of kth degree roots of unity).

The map $\bar{\lambda}_k$ can be considered as a fibration:

$$\begin{tikzcd}
\mathrm{Fr}_{k,n} \arrow[r] & \mathrm{Gr}_{k,n} \arrow[d, "\bar{\lambda}_k"] \\
X \arrow[ur, "\widetilde{\varphi}_{A_k}"'] \arrow[r, "\varphi_{A_k}"'] & \mathrm{BPU}(k).
\end{tikzcd} \tag{20}$$

Clearly, if $\varphi_{A_k} \colon X \to \mathrm{BPU}(k)$ is a classifying map for the bundle A_k (as a $\mathrm{PU}(k)$-bundle), then A_k is the core of some FAB iff φ_{A_k} has a lifting $\widetilde{\varphi}_{A_k} \colon X \to \mathrm{Gr}_{k,n}$ (see (20)). Furthermore, a fiber of $\bar{\lambda}_k$ is the space $\mathrm{Fr}_{k,n}$ of k-frames in $M_{kn}(\mathbb{C})$. Recall that

$$\pi_{2r}(\mathrm{Fr}_{k,n}) = 0,\ \pi_{2r-1}(\mathrm{Fr}_{k,n}) = \mathbb{Z}/k\mathbb{Z}, \quad 1 \leq r \leq n$$

(see Proposition 11).

Now consider the diagram

$$
\begin{tikzcd}
 & & \mathrm{BPU}(km) & & \\
\mathrm{BPU}(k) \arrow[urr, "\psi"] & \mathrm{Gr}_{k,n} \arrow[l, "\bar\lambda_k"'] \arrow[r, "\gamma_k"] & \mathrm{Gr}_{km,n^2} \arrow[u, "\bar\lambda_{km}"] & \mathrm{Gr}_{m,n} \arrow[l, "\gamma_m"'] \arrow[r, "\bar\lambda_m"] & \mathrm{BPU}(m) \arrow[ull, "\chi"'] \\
 & & X \arrow[ull, "\varphi_k"] \arrow[urr, "\varphi_m"'] & &
\end{tikzcd}
\tag{21}
$$

(where m is a positive integer such that $(km, n) = (k, m) = 1$ and $\dim X \leq 2\min\{k,\, m,\, n\}$). The maps $\gamma_k,\ \gamma_m$ in (21) are induced by matrix algebras homomorphisms (they are homotopy equivalences in the stable dimensions, see Corollary 10), ψ and χ are induced by the maps $\mathrm{PU}(k) \overset{\ldots\otimes E_m}{\longrightarrow} \mathrm{PU}(km)$, $\mathrm{PU}(m) \overset{\ldots\otimes E_k}{\longrightarrow} \mathrm{PU}(km)$, respectively, φ_k and φ_m are the classifying maps for the bundles $A_k,\ B_m$ as in (iii) of lemma's statement.

It follows from our assumption (iii) that $\psi \circ \varphi_k \simeq \chi \circ \varphi_m$. Moreover, it is clear that $\psi \circ \bar\lambda_k \simeq \bar\lambda_{km} \circ \gamma_k,\ \chi \circ \bar\lambda_m \simeq \bar\lambda_{km} \circ \gamma_m$. In order to verify the implication (iii) $\Rightarrow$ (i), we must construct a common lifting of the maps $\varphi_k,\ \varphi_m$ to Gr and prove its uniqueness up to homotopy.

We construct the common lifting of $\varphi_k,\ \varphi_m$ by induction on the dimension of the cellular skeleton of X. Suppose we have already constructed liftings $\widetilde\varphi_k^{2r}\colon X^{(2r)} \to \mathrm{Gr},\ \widetilde\varphi_m^{2r}\colon X^{(2r)} \to \mathrm{Gr}$ of $\varphi_k,\ \varphi_m$ to the $2r$-dimensional skeleton $X^{(2r)}$ which are homotopic, i.e. $\widetilde\varphi_k^{2r} \simeq \widetilde\varphi_m^{2r}$. Let $(\widetilde\varphi_k^{2r+1})',\ (\widetilde\varphi_m^{2r+1})'$ be arbitrary liftings of $\varphi_k,\ \varphi_m$ which extend $\widetilde\varphi_k^{2r},\ \widetilde\varphi_m^{2r}$ to the $(2r+1)$-skeleton $X^{(2r+1)}$. They can be considered as liftings of the map $\psi\varphi_k \simeq \chi\varphi_m$ to $X^{(2r+1)}$. Consider the distinguishing cochain $d((\widetilde\varphi_k^{2r+1})', (\widetilde\varphi_m^{2r+1})') \in \mathcal{C}^{2r+1}(X;\, \pi_{2r+1}(F_{km,n^2})) = \mathcal{C}^{2r+1}(X;\, \mathbb{Z}/km\mathbb{Z})$. The maps ψ, χ induce the monomorphisms of the cochain groups

$$\psi_*\colon \mathcal{C}^{2r+1}(X;\, \mathbb{Z}/k\mathbb{Z}) \to \mathcal{C}^{2r+1}(X;\, \mathbb{Z}/km\mathbb{Z}),$$

$$\chi_*\colon \mathcal{C}^{2r+1}(X;\, \mathbb{Z}/m\mathbb{Z}) \to \mathcal{C}^{2r+1}(X;\, \mathbb{Z}/km\mathbb{Z}).$$

Moreover, since $(k, m) = 1$ by our assumption, we have

$$\mathcal{C}^{2r+1}(X;\, \mathbb{Z}/km\mathbb{Z}) = \mathcal{C}^{2r+1}(X;\, \mathbb{Z}/k\mathbb{Z}) \oplus \mathcal{C}^{2r+1}(X;\, \mathbb{Z}/m\mathbb{Z}).$$

Note also that the distinguishing cochains for liftings of $\varphi_k,\ \varphi_m$ belong to $\mathcal{C}^{2r+1}(X;\, \mathbb{Z}/k\mathbb{Z})$ and $\mathcal{C}^{2r+1}(X;\, \mathbb{Z}/m\mathbb{Z})$, respectively. It is well-known from the obstruction theory that for an arbitrary cochain $d_1 \in \mathcal{C}^{2r+1}(X;\, \mathbb{Z}/k\mathbb{Z})$ and $(\widetilde\varphi_k^{2r+1})'$ as above there exists an extension $\widetilde\varphi_k^{2r+1}$ of the lifting $\widetilde\varphi_k^{2r}$ of φ_k such that $d(\widetilde\varphi_k^{2r+1}, (\widetilde\varphi_k^{2r+1})') = -d_1$; similarly, for $(\widetilde\varphi_m^{2r+1})'$ and $d_2 \in \mathcal{C}^{2r+1}(X;\, \mathbb{Z}/m\mathbb{Z})$.

Suppose $d((\widetilde\varphi_k^{2r+1})', (\widetilde\varphi_m^{2r+1})') = d_1 - d_2$, where $d_1 \in \mathcal{C}^{2r+1}(X; \mathbb{Z}/k\mathbb{Z})$, $d_2 \in \mathcal{C}^{2r+1}(X;\, \mathbb{Z}/m\mathbb{Z})$. Then

$$d(\widetilde\varphi_k^{2r+1}, (\widetilde\varphi_k^{2r+1})') + d((\widetilde\varphi_k^{2r+1})', (\widetilde\varphi_m^{2r+1})') + d((\widetilde\varphi_m^{2r+1})', \widetilde\varphi_m^{2r+1}) =$$

$$d(\widetilde\varphi_k^{2r+1}, \widetilde\varphi_m^{2r+1}) = -d_1 + d_1 - d_2 + d_2 = 0.$$

In other words, we can replace $(\widetilde{\varphi}_k^{2r+1})'$ by another lifting of φ_k to the $(2r+1)$-skeleton and $(\widetilde{\varphi}_m^{2r+1})'$ by another lifting of φ_m to the $(2r+1)$-skeleton such that the liftings extend the corresponding liftings φ_k^{2r} φ_m^{2r} and such that the obtained liftings are homotopic as maps from $X^{(2r+1)}$ to Gr. Clearly, the liftings $\widetilde{\varphi}_k^{2r+1}$ $\widetilde{\varphi}_m^{2r+1}$ which satisfy this condition are unique up to homotopy.

Now extend $\widetilde{\varphi}_k^{2r+1}, \widetilde{\varphi}_m^{2r+1}$ to the $(2r+2)$-skeleton $X^{(2r+2)}$. We have well-known formula $\delta d(f,g) = c(f) - c(g)$ in the obstruction theory, where $c(f)$ and $c(g)$ are the obstruction cochains for lifting of f and g, respectively, and δ is the coboundary map. Hence $0 = \delta d(\widetilde{\varphi}_k^{2r+1}, \widetilde{\varphi}_m^{2r+1}) = c(\widetilde{\varphi}_m^{2r+1}) - c(\widetilde{\varphi}_k^{2r+1})$. But $\widetilde{\varphi}_m^{2r+1}$ is a lifting of φ_m, therefore $c(\widetilde{\varphi}_k^{2r+1}) \in \mathcal{C}^{2r+1}(X;\mathbb{Z}/k\mathbb{Z})$; similarly, $c(\widetilde{\varphi}_k^{2r+1}) \in \mathcal{C}^{2r+1}(X;\mathbb{Z}/m\mathbb{Z})$. Thus, $c(\widetilde{\varphi}_m^{2r+1}) = c(\widetilde{\varphi}_k^{2r+1}) = 0$. $\square$

3.4 Localization

Let X be a finite CW-complex, $k \geq 2$ a fixed integer. The set of isomorphism classes of bundles of the form A_{k^m} (for arbitrary $m \in \mathbb{N}$) over X with fiber $M_{k^m}(\mathbb{C})$ is a monoid with respect to the operation $\otimes$ (with the identity element $M_{k^0}(\mathbb{C}) \cong \mathbb{C}$).

Let us consider the following equivalence relation

$$A_{k^m} \sim B_{k^n} \Longleftrightarrow \exists r, s \in \mathbb{N} \text{ such that } A_{k^m} \otimes \widetilde{M}_{k^r} \cong B_{k^n} \otimes \widetilde{M}_{k^s} \tag{22}$$

$(\Rightarrow m + r = n + s)$. The set of equivalence classes $[A_{k^m}]$ is a group with respect to the operation induced by $\otimes$. This group we denote by $\widetilde{\mathrm{AB}}^k(X)$.

Let us consider the direct limit $\varinjlim_n \mathrm{BPU}(k^n)$ with respect to the maps induced by the group monomorphisms

$$\begin{array}{ccc} \mathrm{PU}(k^n) & \hookrightarrow & \mathrm{PU}(k^{n+1}) \\ A & \mapsto & A \otimes E_k \end{array}$$

(here the symbol $\otimes$ denotes the Kronecker product of matrices). Clearly, the functor $X \mapsto \widetilde{\mathrm{AB}}^k(X)$ is represented by the space $\mathrm{BPU}(k^\infty) := \varinjlim_n \mathrm{BPU}(k^n)$.

According to Lemma 17, for any stable equivalence class of FABs over X there is a representative of the form $(A_{k^m},\, \mu,\, \widetilde{M}_{(kl)^m})$ (for some l coprime with k). Therefore, for any k we can define the natural transformation $\widetilde{\mathrm{AB}}^1(X) \to \widetilde{\mathrm{AB}}^k(X)$ by letting

$$[(A_{k^m},\, \mu,\, \widetilde{M}_{(kl)^m})] \mapsto [A_{k^m}].$$

More precisely, consider the diagram

$$\begin{array}{ccc} \uparrow & & \uparrow \\ \mathrm{Gr}_{k^2,\,l^2} & \xrightarrow{\bar{\lambda}_{k^2}} & \mathrm{BPU}(k^2) \\ \uparrow & & \uparrow \\ \mathrm{Gr}_{k,\,l} & \xrightarrow{\bar{\lambda}_k} & \mathrm{BPU}(k), \end{array} \tag{23}$$

where $\bar{\lambda}_k$ (resp. $\bar{\lambda}_{k^2}$) is a classifying map for the bundle $\mathcal{A}_{k,\,l}$ as a $\mathrm{PU}(k)$-bundle (resp. for $\mathcal{A}_{k^2,\,l^2}$ as a $\mathrm{PU}(k^2)$-bundle). The diagram determines the map of classifying spaces

$\bar{\lambda}_{k^\infty} := \varinjlim_n \bar{\lambda}_{k^n}\colon \mathrm{Gr} \to \mathrm{BPU}(k^\infty)$ such that $\bar{\lambda}_{k^\infty *}\colon [X;\,\mathrm{Gr}] \to [X;\,\mathrm{BPU}(k^\infty)]$ is the natural transformation $\widetilde{\mathrm{AB}}^1(X) \to \widetilde{\mathrm{AB}}^k(X)$ as above. Note that the kernel of the homomorphism $\bar{\lambda}_{k^\infty *}\colon \widetilde{\mathrm{AB}}^1(X) \to \widetilde{\mathrm{AB}}^k(X)$ is the k-torsion subgroup in $\widetilde{\mathrm{AB}}^1(X)$.

Remark 24. Consider the following equivalence relation on the set of bundles over X with fiber a matrix algebra:

$$A_k \sim B_l \Leftrightarrow \exists m,\, n \in \mathbb{N} \text{ such that } A_k \otimes \widetilde{M}_m \simeq B_l \otimes \widetilde{M}_n. \tag{24}$$

Let $\widetilde{\mathrm{AB}}(X)$ be the group of equivalence classes of the relation with respect to the operation induced by the tensor product of bundles. Then the functor $X \mapsto \widetilde{\mathrm{AB}}(X)$ is represented by the H-space $\varinjlim_k \mathrm{BPU}(k)$ (it is the direct limit over all $k \in \mathbb{N}$ with respect to the maps induced by $\mathrm{PU}(k) \hookrightarrow \mathrm{PU}(m),\ A \mapsto A \otimes E_{\frac{m}{k}}$ for any $k|m$). Note that $\pi_2(\varinjlim_k \mathrm{BPU}(k)) = \mathbb{Q}/\mathbb{Z}$ (more precisely, it is the group μ_∞ of finite degree roots of unity), $\pi_{2r}(\varinjlim_k \mathrm{BPU}(k)) = \mathbb{Q}$, if $r \geq 2$, and $\pi_{2r+1}(\varinjlim_k \mathrm{BPU}(k)) = 0 \quad \forall r \in \mathbb{N}$. We have the natural transformation of functors $\widetilde{\mathrm{AB}}^1 \to \widetilde{\mathrm{AB}},\ (A_m,\, \mu,\, \widetilde{M}_{mn}) \mapsto [A_m]$ which can be passed through $\widetilde{\mathrm{AB}}^k$ for any $k > 1$. Clearly, $\ker\{\widetilde{\mathrm{AB}}^1(X) \to \widetilde{\mathrm{AB}}(X)\}$ is just the torsion subgroup in $\widetilde{\mathrm{AB}}^1(X)$.

Remark 25. Note that the equivalence relations (22) and (24) do not preserve the dimension of bundles. Hence, $\widetilde{\mathrm{AB}}^k$ and $\widetilde{\mathrm{AB}}$ are counterparts of the reduced K-theory. In order to obtain the nonreduced theory, for example, instead of (22) we should consider the following equivalence relation:

$$A_{k^m} \sim B_{k^n} \Leftrightarrow m = n \text{ and there is } l \text{ such that } A_{k^m} \otimes \widetilde{M}_{k^l} \cong B_{k^m} \otimes \widetilde{M}_{k^l},$$

and repeat the above procedure for relation (24). Denote the corresponding functors by AB^k, AB, respectively. The group $\mathrm{AB}(\mathrm{pt})$ can easily be described: it is the symmetrization of the (multiplicative) monoid $\mathbb{N}^*$. Thus, $\mathrm{AB}(\mathrm{pt}) \cong \mathbb{Q}_+^*$, where $\mathbb{Q}_+^*$ is the group of positive rational numbers with respect to the multiplication. The group $\mathrm{AB}^k(\mathrm{pt})$ can be identified with the subgroup $\mathbb{Z} \hookrightarrow \mathbb{Q}_+^*,\ 1 \mapsto k$.

Theorem 26. *There is a homotopy equivalence* $\mathrm{Gr} \simeq \mathrm{BSU}$.

Proof. Let $p,\, q$ be two prime numbers, $p \neq q$. Clearly, $\mathrm{Gr}_{p^k,\, q^k} \simeq \mathrm{Gr}_{q^k,\, p^k}$. Consider the following homotopy commutative diagrams

$$\begin{tikzcd} {} & {} \\ \mathrm{Gr}_{p^2,\,q^2} \arrow[u] \arrow[r, "\lambda_{p^2}"'] & \mathrm{BSU}(p^2) \arrow[u, "\psi_{p^2}"'] \\ \mathrm{Gr}_{p,\,q} \arrow[u, "\varphi"] \arrow[r, "\lambda_p"'] & \mathrm{BSU}(p), \arrow[u, "\psi_p"'] \end{tikzcd} \qquad \begin{tikzcd} {} & {} \\ \mathrm{Gr}_{p^2,\,q^2} \arrow[u] \arrow[r, "\lambda_{q^2}"'] & \mathrm{BSU}(q^2) \arrow[u, "\psi_{q^2}"'] \\ \mathrm{Gr}_{p,\,q} \arrow[u, "\varphi"] \arrow[r, "\lambda_q"'] & \mathrm{BSU}(q), \arrow[u, "\psi_q"'] \end{tikzcd}$$

where $\lambda_{p^k},\, \lambda_{q^k}$ are classifying maps for the cores $\mathcal{A}_{p^k,\, q^k},\, \mathcal{A}_{q^k,\, p^k}$ of canonical FABs over $\mathrm{Gr}_{p^k,\, q^k}$ and $\mathrm{Gr}_{q^k,\, p^k}$, respectively (see Lemma 21; notice that $\bar{\lambda}_k$ in (23) is the composition of λ_k with the map $\mathrm{BSU}(k) \to \mathrm{BPU}(k)$ induced by the natural epimorphism $\mathrm{SU}(k) \to \mathrm{PU}(k)$ with the kernel μ_k), the map φ is induced by a homomorphism of matrix algebras $M_{pq}(\mathbb{C}) \hookrightarrow M_{(pq)^2}(\mathbb{C})$, and the map ψ_{p^k} (resp. ψ_{q^k}) is induced by the homomorphism

$$\mathrm{SU}(p^k) \overset{\ldots\otimes E_p}{\hookrightarrow} \mathrm{SU}(p^{k+1}) \qquad (\text{resp. } \mathrm{SU}(q^k) \overset{\ldots\otimes E_q}{\hookrightarrow} \mathrm{SU}(q^{k+1}))$$

of Lie groups.

Note that the map $\lambda_{p^k}\colon \mathrm{Gr}_{p^k,q^k} \to \mathrm{BSU}(p^k)$ is a fibration with fiber Fr_{p^k,q^k} (compare with diagram (20)). A simple computations with homotopy groups show that the map

$$\lambda_{p^\infty} := \varinjlim_k \lambda_{p^k}\colon\ \varinjlim_k \mathrm{Gr}_{p^k,\,q^k} \to \varinjlim_k(\mathrm{BSU}(p^k),\ \psi_{p^k})$$

is just the localization at p, similarly for q. In particular, $\mathrm{BSU}[\frac{1}{p}]$ (resp. $\mathrm{BSU}[\frac{1}{q}]$) is a $\mathbb{Z}[\frac{1}{p}]$-local space (resp. a $\mathbb{Z}[\frac{1}{q}]$-local space). Thus, we have the fibered square

$$\begin{tikzcd}
\varinjlim_k \mathrm{Gr}_{p^k,\,q^k} \arrow[r, "\lambda_{q^\infty}"] \arrow[d, "\lambda_{p^\infty}"'] & \mathrm{BSU}[\frac{1}{q}] \arrow[d, "g"] \\
\mathrm{BSU}[\frac{1}{p}] \arrow[r, "f"'] & \mathrm{BSU}[\frac{1}{pq}],
\end{tikzcd}$$

where the maps f, g correspond to the homomorphisms $\mathbb{Z}[\frac{1}{p}] \to \mathbb{Z}[\frac{1}{pq}]$ and $\mathbb{Z}[\frac{1}{q}] \to \mathbb{Z}[\frac{1}{pq}]$ respectively. It is known [1] that the space

$$\varinjlim_k \mathrm{Gr}_{p^k,\,q^k} = \mathrm{BSU}[\tfrac{1}{p}] \overset{h}{\underset{\mathrm{BSU}[\frac{1}{pq}]}{\times}} \mathrm{BSU}[\tfrac{1}{q}]$$

is determined by the maps f, g uniquely up to a homotopy equivalence. Now, applying Theorem 16, item 1), we complete the proof. $\square$

In the next section we construct a natural bijection between the sets $\widetilde{\mathrm{AB}}^1(X)$ and $\widetilde{\mathrm{KSU}}(X)$. This gives another way to prove the homotopy equivalence $\mathrm{Gr} \simeq \mathrm{BSU}$ (here $\widetilde{\mathrm{KSU}}$ is the reduced K-functor constructed by means of SU-bundles, recall that it is represented by the space BSU).

Now let us consider the following situation, in which FABs appear naturally. As usually, let X be a finite CW-complex, $\{k,\,l\}$ a pair of natural numbers such that $2\min\{k,\,l\} \geq \dim X$ and $(k,l)=1$. Let A_{kl} be a locally trivial bundle over X with fiber $M_{kl}(\mathbb{C})$. Suppose for simplicity that the structural group of A_{kl} is reducible to the special linear group $\mathrm{SL}_{kl}(\mathbb{C})$ (i.e. actually to $\mathrm{SU}(kl)$). Then it can be proved that there are bundles B_k, C_l over X with fibers $M_k(\mathbb{C})$, $M_l(\mathbb{C})$, respectively, such that their structural groups can be reduced to the corresponding special linear groups and $A_{kl} \cong B_k \otimes C_l$. Moreover, such pairs $(B_k,\,C_l)$ are in one-to-one correspondence with stable equivalence classes of FABs over X.

In order to prove the described results, consider the fibration

$$\begin{array}{ccc}
\mathrm{Gr}_{k,\,l} & \overset{\kappa}{\hookrightarrow} & \mathrm{BSU}(k)\times\mathrm{BSU}(l) \\
 & & \downarrow \vartheta \\
 & & \mathrm{BSU}(kl),
\end{array} \tag{25}$$

where ϑ is induced by the tensor product of the universal SU-bundles over classifying spaces. It follows from the obstruction theory that an arbitrary map $X \to \mathrm{BSU}(kl)$ can be lifted to the total space of the considered fibration. Thus, we have $A_{kl} = B_k \otimes C_l$ for some bundles B_k, C_l. Furthermore, the second part of the assertion follows from the exactness of the sequence

$$0 \to [X;\, \mathrm{Gr}_{k,l}] \stackrel{\kappa_*}{\to} [X;\, \mathrm{BSU}(k) \times \mathrm{BSU}(l)] \stackrel{\vartheta_*}{\to} [X;\, \mathrm{BSU}(kl)] \to 0$$

(let us remark that the injectivity of κ_* follows from the obvious fact that a FAB $(A_k,\, \mu,\, \widetilde{M}_{kl})$ is trivial iff its core A_k and its complementary subbundle B_l (i.e. the subbundle $B_l \subset \widetilde{M}_{kl}$ such that $A_k \otimes B_l = \widetilde{M}_{kl}$, see page 12) are both trivial as locally trivial bundles with fibers $M_k(\mathbb{C})$ and $M_l(\mathbb{C})$, respectively).

Remark 27. Consider also the sequence of fibrations (25):

$$\begin{array}{ccc} \mathrm{Gr}_{k^n,\, l^n} & \stackrel{\kappa_{k^n,\, l^n}}{\hookrightarrow} & \mathrm{BSU}(k^n) \times \mathrm{BSU}(l^n) \\ & & \downarrow \vartheta_{k^n,\, l^n} \\ & & \mathrm{BSU}((kl)^n) \end{array}$$

(clearly, $\kappa_{p,\,q} = \lambda_p \times \lambda_q$ in the notation of the previous theorem) and its direct limit as $n \to \infty$:

$$\mathrm{Gr} \stackrel{\kappa_{k^\infty,\, l^\infty}}{\hookrightarrow} \mathrm{BSU}(k^\infty) \times \mathrm{BSU}(l^\infty) \stackrel{\vartheta_{k^\infty,\, l^\infty}}{\longrightarrow} \mathrm{BSU}((kl)^\infty).$$

It can easily be checked that this fibration corresponds to the exact sequence of groups

$$0 \to \mathbb{Z} \to \mathbb{Z}[\tfrac{1}{k}] \times \mathbb{Z}[\tfrac{1}{l}] \to \mathbb{Z}[\tfrac{1}{kl}] \to 0.$$

4 Relation between $\widetilde{\mathrm{AB}}^1$ and $\widetilde{\mathrm{KSU}}$-theory

Let $\widetilde{\mathrm{KSU}}(X)$ be the reduced K-functor constructed by means of SU-bundles over X. Let k, m be a pair of positive integers such that $(k, m) = 1$. Without loss of generality we can assume that $m > k$. Suppose $\dim X \leq 2k$. Let ξ_k be a k-dimensional vector $\mathrm{SU}(k)$-bundle over X. Let us consider the pair

$$\xi_k \otimes [k] - [k(k-1)],\; (\xi_k \oplus [m-k]) \otimes [m] - [m(m-1)] \tag{27}$$

of virtual bundles of virtual dimensions k, m, respectively, where $[n]$ is the trivial n-dimensional bundle over X. By $\xi_k^\bullet$, $(\xi_k \oplus [m-k])^\bullet$ denote geometric representatives of stable equivalence classes (27) (since $\dim X \leq 2 \min\{k,\, m\}$, the geometric representatives exist and are unique up to isomorphism).

Proposition 28. $(\mathrm{End}\, \xi_k^\bullet) \otimes \widetilde{M}_m \cong (\mathrm{End}(\xi_k \oplus [m-k])^\bullet) \otimes \widetilde{M}_k$.

Proof. It can easily be checked that $\xi_k^\bullet \otimes [m] \cong (\xi_k \oplus [m-k])^\bullet \otimes [k]$. $\square$

Now we see that the pair $(A_k,\, B_m) = (\mathrm{End}(\xi_k^\bullet),\, \mathrm{End}(\xi_k \oplus [m-k])^\bullet)$ satisfies the condition of the second part of Lemma 23, and we conclude that every pair $\xi_k^\bullet, (\xi_k \oplus [m-k])^\bullet,\ (k, m) = 1$

determines a unique stable equivalence class of FABs over X. Moreover, one can easily verify that this assignment depends only on the stable equivalence class $[\xi_k] \in \widetilde{\mathrm{KSU}}(X)$ of the initial bundle ξ_k. Therefore, the corresponding natural transformation of functors

$$\phi_X \colon \widetilde{\mathrm{KSU}}(X) \to \widetilde{\mathrm{AB}}^1(X)$$

is well-defined. Moreover, it is a bijection:

Proposition 29. *For any finite CW-complex X the map $\phi_X \colon \widetilde{\mathrm{KSU}}(X) \to \widetilde{\mathrm{AB}}^1(X)$ is a natural bijection. Consequently, the spaces* Gr *and* BSU *are homotopy-equivalent.*

Proof. In order to prove the theorem, we produce the inverse map $\psi_X \colon \widetilde{\mathrm{AB}}^1(X) \to \widetilde{\mathrm{KSU}}(X)$ for ϕ_X. Let $(\xi_k,\, \xi_m)$ be a pair consisting of $\mathrm{SU}(k)$ and $\mathrm{SU}(m)$-bundle ($(k,m)=1$) such that $\xi_k \otimes [m] \cong \xi_m \otimes [k]$. Let us take l, n such that $kl+mn=1$. Then in the group $\mathrm{KSU}(X)$ the identity $\xi_k = kl\xi_k + mn\xi_k = kl\xi_k + kn\xi_m = k(l\xi_k + n\xi_m)$ holds. Similarly, $\xi_m = m(l\xi_k + n\xi_m)$. Thus, we have $\xi_k = k\eta$, $\xi_m = m\eta$ for the virtual SU-bundle η ($\eta = l\xi_k + n\xi_m$) of virtual dimension 1. Clearly, the assignment $(A_k,\, B_m) \mapsto \eta$ (where $A_k = \mathrm{End}\,\xi_k$, $B_m = \mathrm{End}\,\xi_m$) is the required map ψ_X satisfying $\psi_X \circ \phi_X = \mathrm{id}_{\widetilde{\mathrm{KSU}}(X)}$, $\phi_X \circ \psi_X = \mathrm{id}_{\widetilde{\mathrm{AB}}^1(X)}$. $\square$

Let us remember that $\mathrm{BSU}_\otimes$ is the space BSU with the structure of H-space related to the tensor product of virtual SU-bundles of virtual dimension 1.

Theorem 30. *The H-spaces* Gr *and* $\mathrm{BSU}_\otimes$ *are isomorphic to each other.*

Proof. Let ξ be a virtual SU-bundle over X of virtual dimension 1; ξ_k, ξ_m the geometric representatives of dimension k, m of the classes $k\xi$, $m\xi$, respectively (the numbers k, m are assumed to be sufficiently large). Given a virtual SU-bundle η of virtual dimension 1, the bundles η_l, η_n are defined in the same way. Clearly, $\xi_k \otimes \eta_l$ is a geometric representative of dimension kl of $kl\xi \otimes \eta \in \mathrm{KSU}(X)$; the same relation holds between $\xi_m \otimes \eta_n$ and $mn\xi \otimes \eta$. Clearly, $\mathrm{End}(\xi_k \otimes \eta_l) \otimes \widetilde{M}_{mn} \cong \mathrm{End}(\xi_m \otimes \eta_n) \otimes \widetilde{M}_{kl}$. Suppose $(kl, mn) = 1$; then, according to Lemma 23, the pair $(\mathrm{End}(\xi_k \otimes \eta_l),\, \mathrm{End}(\xi_m \otimes \eta_n))$ determines the stable equivalence class of FABs over X. We claim that this class is equal to the product of the classes corresponding to the pairs $(\mathrm{End}\,\xi_k,\, \mathrm{End}\,\xi_m)$ and $(\mathrm{End}\,\eta_l,\, \mathrm{End}\,\eta_n)$. Indeed, this also follows from Lemma 23. $\square$

We see that the group $\widetilde{\mathrm{AB}}^1(X)$ is isomorphic to the multiplicative group of the ring $\widetilde{\mathrm{KSU}}(X)$, i.e. to the group of elements of $\widetilde{\mathrm{KSU}}(X)$ with respect to the operation $\xi * \eta := \xi + \eta + \xi\eta$ ($\xi,\, \eta \in \widetilde{\mathrm{KSU}}(X)$).

Note that we have obtained a geometric description of the H-space structure on $\mathrm{BSU}_\otimes$. For example, the construction of the inverse stable equivalence class $[(B_m,\, \nu,\, \widetilde{M}_{mn})]$ for a given one $[(A_k,\, \mu,\, \widetilde{M}_{kl})]$ is closely connected with taking centralizer for a subalgebra in a fixed matrix algebra.

Let us remark that the H-space $\mathrm{BSU}_\otimes$ is an infinite loop space [2], so is Gr.

Now, using the described relation between $\widetilde{\mathrm{AB}}^1$ and $\widetilde{\mathrm{KSU}}$, we study (stable) characteristic classes of FABs. In particular, we obtain a formula which expresses characteristic classes of the product of stable equivalence classes of FABs in terms of the characteristic classes of factors (i.e. a counterpart of the Whitney formula).

Definition 31. Let $(A_k,\, \mu,\, \widetilde{M}_{kl})$ be a FAB over X (since we study the stable theory, without loss of generality we can assume that $k,\, l$ are sufficiently large). Let ξ_k be an SU-bundle over X such that $\phi_X[\xi_k] = [(A_k,\, \mu,\, \widetilde{M}_{kl})]$ (in particular, $A_k \cong \mathrm{End}\, \xi^\bullet_k$), where $[\xi_k] \in \widetilde{\mathrm{KSU}}(X)$ is the stable equivalence class of ξ_k and ϕ_X as in Proposition 29. Then, by definition, *Chern class* $\widetilde{c}_i((A_k,\, \mu,\, \widetilde{M}_{kl}))$ of $(A_k,\, \mu,\, \widetilde{M}_{kl})$ is the usual i-th Chern class $c_i(\xi_k)$ of ξ_k. In the same way we define *Newton classes* (corresponding to the Newton polynomials) $\widetilde{s}_i((A_k,\, \mu,\, \widetilde{M}_{kl}))$, $i \geq 1$. Furthermore, by definition, put $\widetilde{s}_0(A_k,\, \mu,\, \widetilde{M}_{kl}) = 1$ for any FAB (this convention is justified in Remark 32 below).

Below we will write simple $\widetilde{c}_i(A_k)$ instead of $\widetilde{c}_i((A_k,\, \mu,\, \widetilde{M}_{kl}))$, similarly for Newton classes.

It follows immediately from the definition that the introduced characteristic classes are well defined on the stable equivalence classes of FABs. Clearly, $H^*(\mathrm{Gr}\,;\, \mathbb{Z}) = \mathbb{Z}[\widetilde{c}_2,\, \widetilde{c}_3,\, \ldots]$.

Remark 32. Let $\xi^\bullet_k$ be an SU(k)-bundle such that $\mathrm{End}\, \xi^\bullet_k \cong A_k$. Then for integer $l > 1$ such that $(k, l) = 1$ and $2l \geq \dim X$ there exists an SU(l)-bundle $\eta^\bullet_l$ such that $\xi^\bullet_k \otimes [l] \cong \eta^\bullet_l \otimes [k]$ (compare with Lemma 23). Then for the *usual* Newton class s_i (recall that $s_0(\xi^\bullet_k) = \dim \xi^\bullet_k = k$) we have $s_i(\xi^\bullet_k \otimes [l]) = s_i(\eta^\bullet_l \otimes [k])$, i.e. $ls_i(\xi^\bullet_k) = ks_i(\eta^\bullet_l)$. Now we see that the assignment

$$\xi^\bullet_k \mapsto \frac{s_i(\xi^\bullet_k)}{k} = \widetilde{s}_i(\mathrm{End}\, \xi^\bullet_k) \in H^{2i}(X;\, \mathbb{Z}), \quad i \geq 0, \tag{28}$$

determines a well-defined characteristic class on the stable equivalence classes of FABs (which coincides with the Newton class $\widetilde{s}_i$ defined above). Note that the cohomology $H^*(\mathrm{Gr}\,;\, \mathbb{Z})$ are torsion-free, this allows us to divide $s_i(\xi^\bullet_k)$ in (28) by $k \in \mathbb{Z}$.

Remark 33. It follows from Definition 31 that the relation between the classes $\widetilde{c}_i$ and $\widetilde{s}_i$ is the same as the one between the usual Chern and Newton classes:

$$s_k - s_{k-1}c_1 + \cdots + (-1)^k kc_k = 0, \qquad k \geq 1. \tag{29}$$

Now we want to deduce the counterpart for FABs of Whitney's formula. Let $(A_k,\, \mu,\, \widetilde{M}_{kl})$ $(B_m,\, \nu,\, \widetilde{M}_{mn})$ be FABs over X, $\dim X \leq 2\min\{k,\, l,\, m,\, n\}$. Suppose $\xi^\bullet_k,\, \eta^\bullet_m$ are SU-bundles such that $A_k \cong \mathrm{End}\, \xi^\bullet_k$, $B_m \cong \mathrm{End}\, \eta^\bullet_m$. Then there are SU-bundles $\xi_k,\, \eta_m$ such that the bundles $\xi^\bullet_k,\, \eta^\bullet_m$ are geometric representatives of the equivalence classes $(\xi_k \otimes [k] - [k(k-1)])$, $(\eta_m \otimes [m] - [m(m-1)])$, respectively. By $(k, s_2, s_3, \ldots)$, $(m, s'_2, s'_3, \ldots)$ we denote the Newton classes of $\xi_k,\, \eta_m$, respectively. Then the bundles $\xi^\bullet_k,\, \eta^\bullet_m$ have the following Newton classes: $(k, ks_2, ks_3, \ldots)$ and $(m, ms'_2, ms'_3, \ldots)$ respectively. Recall that for any vector bundles $\xi,\, \eta$ there is the relation

$$s_r(\xi \otimes \eta) = \sum_{\substack{i+j=r\\ i,j\geq 0}} \frac{r!}{i!j!} s_i(\xi)s_j(\eta) \tag{30}$$

(recall that $s_0(\xi) = \dim \xi$) between Newton classes of $\xi,\, \eta$ and the ones of their tensor product. From the other hand, we have:

$$\xi^\bullet_k \otimes \eta^\bullet_m = \zeta^\bullet_{km} = \zeta_{km} \otimes [km] - [km(km-1)]$$

for some SU(km)-bundle ζ_{km}.

Set $s''_r := s_r(\zeta_{km})$. Using (30), we get:

$$s''_r = s_r + \sum_{\substack{i+j=r \\ i,j\geq 1}} \frac{r!}{i!j!} s_i s'_j + s'_r, \quad r \geq 1$$

(note that cohomology groups $H^*(\mathrm{Gr}\,; \mathbb{Z}) = H^*(\mathrm{BSU};\, \mathbb{Z})$ have no torsion). Hence,

$$\widetilde{s}_r(A_k \otimes B_m) = \widetilde{s}_r(A_k) + \sum_{\substack{i+j=r \\ i,j\geq 1}} \frac{r!}{i!j!} \widetilde{s}_i(A_k)\widetilde{s}_j(B_m) + \widetilde{s}_r(B_m)$$

$$= \sum_{\substack{i+j=r \\ i,j\geq 0}} \frac{r!}{i!j!} \widetilde{s}_i(A_k)\widetilde{s}_j(B_m), \quad r \geq 0,$$

because $\widetilde{s}_0(A_k) = 1 = \widetilde{s}_0(B_m)$ by our convention.

Now, using Newton's formulas (29) and Remark 33, one can obtain similar formulas for Chern classes. In particular,

$$\begin{aligned}
\widetilde{c}_2(A_k \otimes B_m) &= \widetilde{c}_2(A_k) + \widetilde{c}_2(B_m) \\
\widetilde{c}_3(A_k \otimes B_m) &= \widetilde{c}_3(A_k) + \widetilde{c}_3(B_m) \\
\widetilde{c}_4(A_k \otimes B_m) &= \widetilde{c}_4(A_k) - 5\widetilde{c}_2(A_k)\widetilde{c}_2(B_m) + \widetilde{c}_4(B_m) \\
\widetilde{c}_5(A_k \otimes B_m) &= \widetilde{c}_5(A_k) - 11\widetilde{c}_3(A_k)\widetilde{c}_2(B_m) - 11\widetilde{c}_2(A_k)\widetilde{c}_3(B_m) + \widetilde{c}_5(B_m),
\end{aligned}$$

etc. Using the previous formulas, one can also obtain the expressions for Chern classes of the inverse FAB for a given one.

5 Appendix 1: a GL-version

In this appendix we introduce the "GL-version" of the previous results which is related to the H-space structure $\mathrm{BU}_{\otimes}$.

Consider the canonical map $\mathrm{BU}(k) \to \mathrm{BPU}(k)$ induced by the group homomorphism $\mathrm{U}(k) \to \mathrm{PU}(k)$. By $\widehat{\mathrm{Gr}}_{k,\,l}$ denote the total space of the $\mathrm{Fr}_{k,\,l}$-fibration induced by the fibration $\mathrm{Gr}_{k,\,l} \xrightarrow{\mathrm{Fr}_{k,\,l}} \mathrm{BPU}(k)$ and the map $\mathrm{BU}(k) \to \mathrm{BPU}(k)$ (as ever, the integers k, l are assumed to be coprime).

It follows easily that there is a $\mathbb{C}P^\infty$-fibration $\widehat{\mathrm{Gr}}_{k,\,l} \to \mathrm{Gr}_{k,\,l}$. Moreover, there is the diagram of fibrations:

$$\begin{array}{ccc}
\widehat{\mathrm{Gr}}_{k,\,l} & \xrightarrow{\mathrm{Fr}_{k,\,l}} & \mathrm{BU}(k) \\
{\scriptstyle \mathbb{C}P^\infty}\downarrow & & \downarrow{\scriptstyle \mathbb{C}P^\infty} \\
\mathrm{Gr}_{k,\,l} & \xrightarrow{\mathrm{Fr}_{k,\,l}} & \mathrm{BPU}(k)
\end{array}$$

(the spaces over/next to the arrows are the corresponding fibers).

Consider the following morphism of $\mathrm{U}(k)$-fibrations:

$$\begin{tikzcd}
 & \mathrm{U}(k) \arrow[r] & * \arrow[d] \\
\mathrm{U}(k) \arrow[ur, "="] \arrow[r] & \mathrm{Fr}_{k,\,l} \arrow[ur] \arrow[d] & \mathrm{BU}(k) \\
 & \widehat{\mathrm{Gr}}_{k,\,l}, \arrow[ur, "\widehat{\lambda}_{k,\,l}"] &
\end{tikzcd}$$

where $\widehat{\lambda}_{k,\,l}$ is a classifying map for the canonical $\mathrm{U}(k)$-bundle over $\widehat{\mathrm{Gr}}_{k,\,l}$ and by $*$ we denote a contractible space. A simple computation with homotopy sequences of the fibrations shows that $\widehat{\lambda}_{k,\,l*}\colon \pi_{2r}(\widehat{\mathrm{Gr}}_{k,\,l}) \to \pi_{2r}(\mathrm{BU}(k))$, $r \leq \min\{k,\, l\}$ is just the monomorphism $\mathbb{Z} \to \mathbb{Z}$, $1 \mapsto k \cdot 1$ (note that the odd-dimensional stable homotopy groups of both spaces are equal to 0). This implies that the direct limit map $\widehat{\lambda}_{k^\infty}\colon \widehat{\mathrm{Gr}} \to \mathrm{BU}(k^\infty)$ is just the localization at k (in the sense that k is invertible; in particular, $\mathrm{BU}(k^\infty)$ is a $\mathbb{Z}[\frac{1}{k}]$-local space), where $\widehat{\mathrm{Gr}} := \varinjlim_{(k,l)=1} \widehat{\mathrm{Gr}}_{k,\,l}$.

The space $\widehat{\mathrm{Gr}}$ is an H-space with respect to the multiplication induced by the tensor product of bundles. It can be proved that $\widehat{\mathrm{Gr}} \cong \mathrm{BU}_\otimes$ as H-spaces. Let us also recall that there are isomorphisms of H-spaces

$$\mathrm{BU}_\otimes \cong \mathrm{BSU}_\otimes \times \mathbb{C}P^\infty$$

and $\mathrm{Gr} \cong \mathrm{BSU}_\otimes$, hence $\widehat{\mathrm{Gr}} \cong \mathrm{Gr} \times \mathbb{C}P^\infty$. In particular, the H-space $\widehat{\mathrm{Gr}}$ represents the functor of "multiplicative group" of the ring $\widetilde{\mathrm{K}}_{\mathbb{C}}$, i.e. the functor $X \mapsto \widetilde{\mathrm{K}}_{\mathbb{C}}(X)$, where $\widetilde{\mathrm{K}}_{\mathbb{C}}(X)$ is considered as a group with respect to the operation $\xi * \eta := \xi + \eta + \xi\eta,\ \xi,\, \eta \in \widetilde{\mathrm{K}}_{\mathbb{C}}(X)$ (here $\widetilde{\mathrm{K}}_{\mathbb{C}}$ is the reduced complex K-functor).

6 Appendix 2: A remark on Thom spaces

In algebraic topology for any stable type of bundles there is the corresponding bordism (and cobordism) theory. Therefore, it is natural to ask the following question: what is the bordism theory corresponding to FABs? In this appendix we define an analog of Thom spaces in this hypothetical theory.

In order to construct the analog of Thom spaces, we shall use the interpretation of floating bundles in term of bundles on projective spaces given in Remark 4.

First let us consider the embedding of pairs

$$\begin{gathered}
(\mathbb{C}P^k \times \mathbb{C}P^l,\, \mathbb{C}P^k \times \mathbb{C}P^{l-1} \cup \mathbb{C}P^{k-1} \times \mathbb{C}P^l) \hookrightarrow (\mathbb{C}P^{kl},\, \mathbb{C}P^{kl-1}),\\
([x_0 : \ldots : x_{k-1} : x_k],\, [y_0 : \ldots : y_{l-1} : y_l]) \mapsto [x_0y_0 : \ldots : x_iy_j : \ldots : x_{k-1}y_{l-1} : x_ky_l]
\end{gathered} \tag{37}$$

(note that the subspaces $\mathbb{C}P^{k-1} \subset \mathbb{C}P^k$ and $\mathbb{C}P^{l-1} \subset \mathbb{C}P^l$ are defined by the equations $x_k = 0$ and $y_l = 0$, respectively). Using the well-known formula $X \times Y/(X \times B \cup A \times Y) \cong (X/A) \wedge (Y/B)$, we get the map

$$S^{2k} \wedge S^{2l} \to S^{2kl}, \tag{38}$$

which corresponds (37).

Recall that the definition of a floating bundle of Segre's product (abbrev. FBSP) was given in Definition 5. Clearly, the space $\mathrm{Gr}_{k,l}$ parameterizes embeddings $\mathbb{C}P^{k-1}\times\mathbb{C}P^{l-1}\hookrightarrow\mathbb{C}P^{kl-1}$ that under some choice of homogeneous coordinates are equivalent to Segre's embedding $([x_0:\ldots:x_{k-1}],[y_0:\ldots:y_{l-1}])\mapsto[x_0y_0:\ldots:x_iy_j:\ldots:x_{k-1}y_{l-1}]$.

Clearly, there is the canonical FBSP $\mathcal{P}^{k-1}\underset{\mathrm{Gr}_{k,\,l}}{\times}\mathcal{Q}^{l-1}$ over $\mathrm{Gr}_{k,l}$ equipped with the canonical embedding $\lambda\colon\mathcal{P}^{k-1}\underset{\mathrm{Gr}_{k,\,l}}{\times}\mathcal{Q}^{l-1}\hookrightarrow\mathrm{Gr}_{k,l}\times\mathbb{C}P^{kl-1}$ into the trivial bundle (for any $x\in\mathrm{Gr}_{k,l}$ its restriction $\lambda\mid_x$ to the fiber over x coincides with the embedding $\mathbb{C}P^{k-1}\times\mathbb{C}P^{l-1}\hookrightarrow\mathbb{C}P^{kl-1}$ parameterized by x).

It follows from Lemma 21 that there are $\mathrm{SU}(k)$ and $\mathrm{SU}(l)$-bundles $\mathfrak{T}_k,\ \mathfrak{S}_l$ over $\mathrm{Gr}_{k,l}$ such that $\mathcal{P}^{k-1}=\mathbb{P}(\mathfrak{T}_k),\ \mathcal{Q}^{l-1}=\mathbb{P}(\mathfrak{S}_l)$, where $\mathbb{P}(\xi_m)$ is the projectivization of a vector bundle ξ_m. Clearly, $\mathfrak{T}_k\otimes\mathfrak{S}_l$ is a trivial $\mathbb{C}^{kl}$-bundle $[kl]$ over $\mathrm{Gr}_{k,l}$.

Now consider the following commutative diagram of the embeddings:

$$\begin{array}{ccc}\mathbb{P}(\mathfrak{T}_k\oplus[1])\underset{\mathrm{Gr}_{k,\,l}}{\times}\mathbb{P}(\mathfrak{S}_l)\cup\mathbb{P}(\mathfrak{T}_k)\underset{\mathrm{Gr}_{k,\,l}}{\times}\mathbb{P}(\mathfrak{S}_l\oplus[1]) & \longrightarrow & \mathbb{P}(\mathfrak{T}_k\oplus[1])\underset{\mathrm{Gr}_{k,\,l}}{\times}\mathbb{P}(\mathfrak{S}_l\oplus[1])\\ \downarrow & & \downarrow\\ \mathbb{P}(\mathfrak{T}_k\otimes\mathfrak{S}_l)=\mathbb{P}([kl]) & \longrightarrow & \mathbb{P}((\mathfrak{T}_k\otimes\mathfrak{S}_l)\oplus[1])\end{array}$$

and the corresponding map of the pairs

$$(\mathbb{P}(\mathfrak{T}_k\oplus[1])\underset{\mathrm{Gr}_{k,\,l}}{\times}\mathbb{P}(\mathfrak{S}_l\oplus[1]),\,\mathbb{P}(\mathfrak{T}_k\oplus[1])\underset{\mathrm{Gr}_{k,\,l}}{\times}\mathbb{P}(\mathfrak{S}_l)\cup\mathbb{P}(\mathfrak{T}_k)\underset{\mathrm{Gr}_{k,\,l}}{\times}\mathbb{P}(\mathfrak{S}_l\oplus[1]))$$

$$\hookrightarrow(\mathbb{P}((\mathfrak{T}_k\otimes\mathfrak{S}_l)\oplus[1]),\,\mathbb{P}(\mathfrak{T}_k\otimes\mathfrak{S}_l))$$

(compare with (37)), where by $[n]$ we denote a trivial $\mathbb{C}^n$-bundle. Thus, we have the canonical map of Thom spaces

$$\mathbf{T}(\mathfrak{T}_k)\underset{\mathrm{Gr}_{k,\,l}}{\wedge}\mathbf{T}(\mathfrak{S}_l)\hookrightarrow\mathbf{T}([kl]),$$

(compare with (38)), where $\mathbf{T}([kl])$ is the Thom space of the trivial bundle $[kl]=\mathfrak{T}_k\otimes\mathfrak{S}_l$ over $\mathrm{Gr}_{k,l}$, $\mathbf{T}(\mathfrak{T}_k)\underset{\mathrm{Gr}_{k,\,l}}{\wedge}\mathbf{T}(\mathfrak{S}_l)$ is the smash product over $\mathrm{Gr}_{k,l}$ of the Thom spaces $\mathbf{T}(\mathfrak{T}_k)$ and $\mathbf{T}(\mathfrak{S}_l)$.

Finally, let us remark that the natural stabilization on such Thom spaces does not coincide with the usual suspension. This is closely related to the stabilization of FABs: we stabilize our bundles by taking the tensor product with trivial bundles.

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