Quasiperiodic solutions of nearly integrable infinite–dimensional Hamiltonian systems

by

S.B. Kuksin

Max–Planck–Institut
für Mathematik
Gottfried–Claren–Straße 26
D–5300 Bonn 3

Federal Republic of Germany

Institute for Information
Transmission Problems
Ermolovoy St., 19
101447, MOSCOW

USSR

MPI/91-62

Contents

Introduction

The book is devoted to nonlinear Hamiltonian perturbations of stable linear Hamiltonian systems of large and infinite dimension. Such systems arise in physics in many different ways. As a working hypothesis for theirs study it was postulated in the physical literature after the works of Boltzmann that in a "typical situation" their solutions are stochastic. This postulate ("ergodic hypothesis") was succesfully used to explain many properties of matter. On the other hand, a lot of numerical experiments starting from the ones of Fermi–Pasta–Ulam (see [FPU], [U]) have shown quite regular recurrent behavior of many solutions of the systems under consideration (see e.g. [ZIS]). This effect cannot be explained by means of the Poincare recurrence theorem [A4] because the Poincare recurrence time is much larger than the one obtained in the experiments. It seems that the investigated systems have a lots of quasiperiodic trajectories or trajectories abnormally close to the quasiperiodic ones (see [LL], [DEGM],[Mo]). These trajectories correspond to low-frequency oscillations of the underlying physical object. In these oscillations the energy is frozen in low frequencies for a very long time. So the recurrence effect causes a low rate of stochasticity (the ergodic hypothesis works now in a slow way). This effect seemed rather strange to the physicists who observed it.

Our goal is to obtain some general theorem to prove the existence many of quasiperiodic solutions in perturbed linear infinite-dimensional Hamiltonian systems corresponding to conservative physical systems with one spatial dimension. The theorem gives some explanation to the recurrence effect in spatially one-dimensional systems. It proves that in some strict sense *the one-dimensional world "is not very ergodic"*.

The introduction is devoted to a rather expanded discussion of the theorem and its applications. Sometimes the discussion supplements the results from the main text. We preface the survey of our results with a survey of the finite dimensional situation.

1 Finite dimensional situation

"Regular" (periodic and quasiperiodic) solutions of $2n$-dimensional Hamiltonian systems are important for classical and celestial mechanics. Some quite general existence theorems for this class of solutions have been obtained. Here we are interested in perturbation-type results only.

1.1 Lyapunov's theorem

The first classical result in this direction was Lyapunov's thorem (see e.g. [AbM], [AKN], [SM]). It states that "nonresonant" periodic solutions of a Hamiltonian system survive under Hamiltonian perturbations. In particular if the unperturbed system is a stable linear system with the spectrum

$$\{\pm i\lambda_j \mid j = 1, \ldots, n\} \tag{1}$$

and

$$n\lambda_j \neq \lambda_k \ \forall j \neq k, \forall n \in \mathbf{Z},$$

then the perturbed system has n two-dimensional invariant manifolds and the manifold number j (j=1,...,n) is filled with periodic solutions of periods close to $2\pi/\lambda_j$.

1.2 Kolmogorov's theorem

The second classical result concerning the subject is Kolmogorov's theorem [Kol] which inspired Arnold and Moser to create a powerful technique to handle nonlinear problems, well known nowadays as KAM (Kolmogorov–Arnold–Moser) theory; see [A2], [A3], [AA], [Mo], [SM] and bibliographies of the last three books. Kolmogorov's theorem states that most of the quasiperiodic n-frequency solutions of a nondegenerate integrable analytical system with n degrees of freedom survive under analytical Hamiltonian perturbations or, equivalently, Hamiltonian perturbations preserve most of invariant n-tori of a nondegenerate integrable system. Here integrability means that in a phase space $\mathbf{T}^n \times P$ (P is an n-dimensional domain) the system has the form:

$$\dot{q} = \nabla h(p),\ \dot{p} = 0, \tag{2}$$

(i.e. it has a hamiltonian h depending on $p \in P$ only) and the nondegeneracy means that

$$\text{Hess}\, h(p) := \det\{\partial^2 h(p)/\partial p_i \partial p_j\} \not\equiv 0. \tag{3}$$

Invariant tori of the system (2) are of the form

$$T^n(p) = \mathbf{T}^n \times \{p\}, p \in P, \tag{4}$$

and most of them survive in the perturbed system with the hamiltonian $h(p) + \varepsilon H(q,p)$,

$$\dot{q} = \nabla_p(h(p) + \varepsilon H(q,p)), \dot{p} = -\varepsilon \nabla_q H(q,p), \tag{5}$$

if positive ε is small enough. That means that there exists a subset $P_\varepsilon \subset P$ such that $\text{mes}(P \setminus P_\varepsilon) \to 0\, (\varepsilon \to 0)$, and for $p \in P_\varepsilon$ there exists a map $\Sigma_p : \mathbf{T}^n \to \mathbf{T}^n \times P$ such that for all $q \in \mathbf{T}^n$ $\text{dist}(\Sigma_p(q), (q,p)) < C\varepsilon$ and the curve

$$t \mapsto \Sigma_p(q + t\nabla h(p)) \tag{6}$$

is a solution of (5).

For other versions and important improvements of the theorem see [AKN], [Bru1] , [Bru2], [Laz], [Mo], [P3], [Ru], [Z1].

1.3 Melnikov's theorem

Lyapunov's theorem states the preservation of nondegenerate one-dimensional invariant tori (=periodic solutions) under Hamiltonian perturbations, and Kolmogorov's theorem states the preservation of most of the invariant n-tori of integrable system with n degrees of freedom. The natural question is if most of invariant tori of an intermediate dimension k, $1 < k < n$, survive under perturbations. For perturbations of a linear Hamiltonian system the question means the following. In the phase space

$$\mathbf{T}^n \times P \times \mathbf{R}^{2m} = \{(q,p,z)\}, z = (z_+, z_-) \in \mathbf{R}^{2m}, \tag{7}$$

the Hamiltonian equations

$$\dot{q} = \lambda + \varepsilon \nabla_p H, \dot{p} = -\varepsilon \nabla_q H, \dot{z} = J(Az + \varepsilon \nabla_z H) \tag{8}$$

are considered. Here $J(z_+, z_-) = (-z_-, z_+)$, A is a symmetric linear operator in $\mathbf{R}^{2m}$, $\varepsilon H = \varepsilon H(q,p,z)$ is an analytical perturbation and $\lambda \in \Lambda \subset\subset \mathbf{R}^n$ is a parameter. For $\varepsilon = 0$ the system (8) has invariant n-tori $T^{n,m}(p) = \mathbf{T}^n \times \{p\} \times \{0\}$, $p \in P$. The question is if these tori survive in the system (8) for $\varepsilon > 0$.

Let us denote the spectrum of the operator JA by $M = \{\mu_1, \ldots, \mu_{2m}\}$. We have to consider three cases:

a) (nondegenerate hyperbolic tori) $M \subset \mathbf{C} \setminus i\mathbf{R}$, $\mu_j \neq \mu_k \forall j \neq k$. In this situation a hyperbolic torus $T^{n,m}(p)$ survives for most λ. That is, for positive ε small enough and for $\lambda \in \Lambda(\varepsilon, p)$, mes $\Lambda \setminus \Lambda(\varepsilon, p) \to 0$ $(\varepsilon \to 0)$, the equation (8) has an invariant torus at a distance $< C\varepsilon$ from $T^{n,m}(p)$. See [Bi], [Gr], [Mol], [Z1].

b_0) (nondegenerate elliptic tori) $M \subset i\mathbf{R} \setminus \{0\}$, $\mu_j \neq \mu_k \forall j \neq k$. This situation is more complicated. The preservation theorem for the elliptic torus $T^{n,m}(p)$ for most λ was formulated by Melnikov [Me1], [Me2]. The complete proof of the theorem was published only 15 years later by Eliasson [El], Pöschel [P1] and the author [K1], [K2] (the infinite-dimensional theorems of the last two works are applicable to equations (8) as well). The proofs given in the papers just mentioned are also valid in the more general situation:

b) (nondegenerate tori) $0 \notin M$, $\mu_j \neq \mu_k \forall j \neq k$.

In the degenerate case

c) $0 \in M$ or $\mu_j = \mu_k$ for some $j \neq k$

no preservation theorem for tori $T^{n,m}(p)$, formulated in terms of the unperturbed equation (8) with $\varepsilon = 0$ only, is known yet.

Remark. Melnikov's theorem (case b_0)) remains true for $m = 0$, too. In such a case $Y = \{0\}$ and the theorem then asserts the preservation of the n-dimensional invariant torus $\mathbf{T}^n \times \{0\}$ of the system with the linear hamiltonian $h(p) = \lambda \cdot p$,

$$\dot{q} = \lambda, \dot{p} = 0,$$

under small analytical Hamiltonian perturbations for most parameters $\lambda \in \Lambda$. This result implies Kolmogorov's theorem as it was formulated above via some simple substitution; see [Mo], p.171 (and [K5], Part 1 with $Y = \{0\}$). Conversely, one can easily extract this version of Melnikov's theorem from Kolmogorov's theorem. So these two statement are equivalent. This equivalence (we had found it in the paper [Mo]) was important for our insigt into infinite-dimensional problems.

Remark. The case c) is important for a better understanding of the situation studied in Kolmogorov's theorem. Indeed, if some torus $T^n(p)$ (see (4)) is resonant and

$$\dim_{\mathbf{Q}} E(p) = n - 1, E(p) = \mathbf{Q}\partial h/\partial p_1 + \ldots + \mathbf{Q}\partial h/\partial p_n$$

(we treat $E(p) \subset \mathbf{R}$ as a linear space over the field $\mathbf{Q}$ here), then the torus $T^n(p)$ is a union of invariant $(n-1)$-tori. Near each $(n-1)$-torus the perturbed system (5) may be reduced to a system (8) of the type c) with $m = 1$ and JA equal to Jordan cell with zero eigenvalue.

2 Infinite dimensional systems

2.1 The problem

In a Hilbert space Z with inner product $< \cdot, \cdot >$ we consider the equation

$$\dot{u}(t) = J\nabla\mathcal{K}(u(t)), \ u(t) \in Z. \tag{9}$$

Here J is an antiselfadjoint operator in Z and $\nabla\mathcal{K}$ is the gradient of a functional $\mathcal{K}$ relative to the inner product $< \cdot, \cdot >$. In the most interesting situations the linear operator J, or the nonlinear operator $\nabla\mathcal{K}$, or both of them are unbounded. So one has to be careful with the equation and its solutions. For the exact definition of solutions of (9) and for some their properties see Part 1 of the main text. Equation (9) is Hamiltonian if the phase space Z is provided with a symplectic structure by means of 2-form $- < J^{-1}du, du >$ (by definition, $- < J^{-1}du, du > [\xi, \eta] = - < J^{-1}\xi, \eta > \forall \xi, \eta \in Z$).

In this book we are most interested in equations of the form

$$\dot{u}(t) = J(Au(t) + \varepsilon\nabla H(u(t))). \tag{10}$$

This equation is Hamiltonian with the hamiltonian

$$\mathcal{K}_\varepsilon = \tfrac{1}{2} < Au, u > + \varepsilon H(u).$$

Here A is a selfadjoint linear operator in Z and H is an analytic functional. The linear operators J,A and the nonlinear operator ∇H are assumed to be characterized by their orders d^J, d^A and d^H. We suppose that

$$d^J \geq 0, d^A \geq 0, d^J + d^A \geq 1, d^J + d^H \leq 0. \tag{11}$$

In the most important examples Z is the L_2-space of square-summable functions on a segment, and J and A are differential operators. In such a case d^J, d^A are the orders of the differential operators and $\nabla H(u)$ is a variational derivative $\delta H/\delta u(x)$. In particular, if

$$H(u) = \int h(x, u(x))\, dx$$

then $\delta H/\delta u(x) = h_u(x, u(x))$ and $d^H = 0$; if the density h depends on integral of $u(x)$ instead of $u(x)$ itself, then $d^H < 0$. To define the orders d^J, d^A, d^H in a general case, we must include the space Z into a scale of Hilbert spaces. See Part 1 below.

The assumption (11) implies that equation (10) is quasilinear. This assumption is rather natural for the study of long-time behavior of solutions because for some strongly nonlinear Hamiltonian equations (i.e. ones of the form (10) with $d^H = d^A$) it is known that the equations have no nontrivial solutions existing for all time; see [Lax].

We suppose that J and A commute, and that Z admits an ortonormal basis $\{\varphi_j^\pm | j \geq 1\}$ such that

$$A\varphi_j^\pm = \lambda_j^A \varphi_j^\pm, J\varphi_j^\pm = \mp\lambda_j^J \varphi_j^\mp, \ \forall j \geq 1. \tag{12}$$

So, in particular, the spectrum of JA is equal to

$$\{\pm i\lambda_j | j \geq 1, \lambda_j = \lambda_j^A \lambda_j^J\}.$$

Let us fix some $n \geq 1$. The $2n$-dimensional linear space

$$Z^0 = \operatorname{span}\{\varphi_j^\pm | 1 \leq j \leq n\}$$

is invariant for the flow of equation (10) and is foliated into invariant n-tori

$$T^n = T^n(I) = \{\sum_{j=1}^n x_j^\pm \varphi_j^\pm \,|x_j^{+^2} + x_j^{-^2} = 2I_j \,\forall j\},$$

$I = (I_1, \ldots, I_n) \in \mathbf{R}_+^n$. Every torus T^n is filled with quasiperiodic solutions of equation (10) with $\varepsilon = 0$. One can treat (10) with $\varepsilon = 0$ as an infinite chain of free harmonic oscillators with frequencies $\lambda_1, \lambda_2, \ldots$. The solutions lying on the tori $T^n(I)$ correspond to oscillations with only the first n oscillators being excited. One can treat these solutions as low-frequency oscillations.

We study the question: *under what assumptions do the tori $T^n(I)$ and the corresponding low-frequency quasiperiodic solutions survive in equation (10) for $\varepsilon > 0$?*

It is convenient to introduce the angle-action variables $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ in the space Z^0,

$$x_j^+ + ix_j^- = \sqrt{2p_j}\exp(iq_j), \; j = 1, \ldots, n$$

($x_j^\pm$ are coordinates with respect to the basis $\{\varphi_j^\pm | 1 \leq j \leq n\}$); to denote $Y = Z \ominus Z^0$ (i.e. Y is equal to the closure of $\operatorname{span}\{\varphi_j^\pm | j \geq n+1\}$) and to pass to the variables (q, p, y),

$$q = (q_1, \ldots, q_n) \in \mathbf{T}^n, p = (p_1, \ldots, p_n) \in \mathbf{R}_+^n, y \in Y. \tag{13}$$

Let us denote by Σ^0 the imbedding

$$\Sigma^0 : \mathbf{T}^n \times \mathbf{R}_+^n \to Z, (q, p) \mapsto (q, p, 0).$$

(we use coordinates (13) in Z). The invariant space Z^0 is the image of this map.

In the new variables (13) equation (10) takes the form:

$$\dot{q} = \nabla_p \mathcal{H}, \dot{p} = -\nabla_q \mathcal{H}, \dot{y} = J^Y \nabla_y \mathcal{H} \tag{14}$$

with

$$\mathcal{H} = \mathcal{H}_\varepsilon = \omega \cdot p + \tfrac{1}{2} < A^Y y, y > + \varepsilon H_1(q, p, y).$$

Here $\omega = (\lambda_1, \ldots, \lambda_n)$, $J^Y = J_{|Y}, A^Y = A_{|Y}$. So the operator $J^Y A^Y$ has pure imaginary spectrum $\{\pm i\lambda_j | j \geq n+1\}$ and one can easily recognize in the last equations an infinite-dimensional analogy to the elliptic case of the system (8). The form of Melnikov's theorem we gave above in Section 1.3 has a natural infinite-dimensional reformulation. It is remarkable that this reformulation becomes a true statement after adding essentially just one infinite-dimensional condition.

2.2 The result

Keeping in mind the applications, we generalise the situation and suppose that equation (10) analytically depends on n outer parameters $(a_1, \ldots, a_n) = a \in \mathcal{A}$, $\mathcal{A}$ is a bounded open domain in $\mathbf{R}^n$. So $A = A_a$, $H = H_a$ and $\lambda_j = \lambda_j(a)$. Let us assume that

$$\det\{\partial\lambda_j(a)/\partial a_k | 1 \leq j, k \leq n\} \not\equiv 0 \tag{15}$$

and consider a torus $T^n(I_1, \ldots, I_n)$ such that $I_k > 0 \, \forall k$.

Theorem 1. Let us suppose that the assumptions (12), (15) take place together with

1) (quasilinearity)

$$d^J \geq 0, d^A \geq 0, d_1 := d^J + d^A \geq 1, d^J + d^H \leq 0, d^J + d^H < d_1 - 1;$$

2) (spectral asymptotics)

$$\lambda_j(a) = K_1 j^{d_1} + K_2 + \mu_j(a)$$
$$|\mu_j(a)| + |\nabla\mu_j(a)| \leq K_3 j^{d_1 - \kappa}$$

for some $\kappa > 1$;

3) for some $N \geq n$ and $M \geq 1$ depending on the problem (10) nonresonance relations

$$s_1\lambda_1(a) + s_2\lambda_2(a) + \ldots + s_N\lambda_N(a) \not\equiv 0 \tag{16}$$

hold for all $s \in \mathbf{Z}^N$ such that $1 \leq |s| \leq M$ and $|s_{n+1}| + \ldots + |s_N| \leq 2$.

Then for positive ε small enough there exist a Borel subset $\mathcal{A}_\varepsilon(I) \subset \mathcal{A}$ and analytical embeddings

$$\Sigma^\varepsilon_{a,I} : \mathbf{T}^n \to Z, a \in \mathcal{A}_\varepsilon(I), \tag{17}$$

such that

a) $\operatorname{mes}(\mathcal{A} \setminus \mathcal{A}_\varepsilon(I)) \to 0$ $(\varepsilon \to 0)$;

b) the map $(q, I, a) \mapsto \Sigma^\varepsilon_{a,I}(q)$ is Lipschitz and is $C\varepsilon$-close to the map $(q, I, a) \mapsto \Sigma^0(q, I)$;

c) for $a \in \mathcal{A}_\varepsilon$ the torus $\Sigma^\varepsilon_{a,I}(\mathbf{T}^n)$ is invariant for the equation (10) and is filled with quasiperiodic solutions of the form $u_\varepsilon(t) = \Sigma^\varepsilon_{a,I}(q + \omega_\varepsilon t)$ with a vector $\omega_\varepsilon \in \mathbf{R}^n$ which is $C\varepsilon$-close to $\omega = (\lambda_1, \ldots, \lambda_n)$. All Lyapunov exponents of these solutions are equal to zero.

Refinement 1. If $K^{-1} \leq I_j \leq K$ for all $j = 1, \ldots, n$, then the numbers M, N in the assumption 3) and the rate of convergence in the statement a) of the theorem depend on $n, K, K_1 - K_3, d_1, \kappa$, the radius of analyticity of H and its norm only.

Refinement 2 (see Part 3, Theorem 1.1). In the variables (13) the unperturbed hamiltonian is equal to $\omega \cdot p + \frac{1}{2} < A^Y y, y >$ and the perturbation is $\varepsilon H_1(q, p, y)$. The statememt of the theorem remains true for perturbations of the more general form

$$\varepsilon H_1(q, p, y) + H^3(q, p, y), H^3 = O(|p - I|^2 + \| y \|^3 + \| y \| \, |p - I|).$$

This form of the result is suitable for applications to perturbations of nonlinear problems (see below).

Remark. The formulations of our results given above are "almost exact". For the exact statements see the main text.

Remark. If the first n frequencies $(\lambda_1, \dots, \lambda_n) = \omega$ are taken for the parameters $a_1, \dots, a_n$ and if λ_j does not depend on ω for $j \geq n+1$, then the assumption 3) is fulfilled trivially. If in addition $\dim Z < \infty$, then the assumptions 1), 2) hold trivially, too. So for finite-dimensional systems (written in the form (14)) Theorem 1 coincides with Melnikov's theorem.

As another infinite-dimensional version of Melnikov-type theorem we want to mention the result of Wayne's paper [W1], devoted to the nonlinear-string equation with a random potential. We discuss the approach the work [W1] is based on, below.

Remark. As the map (17) is $C\varepsilon$-close to the map $q \mapsto \Sigma^0(q, I)$, then the solutions $u_\varepsilon(t)$ are $C\varepsilon$-close to the curves $t \mapsto \Sigma^0(q + \omega_\varepsilon t, I)$ for all t. The vector ω_ε is equal to $\omega + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \dots$ and the vector ω_1 may be obtained via some natural averaging (see [K4]). So Theorem 1 gives an averaging procedure for solutions of equation (10) as a simple consequence ([K4], [K8]).

Under the assumptions of the theorem an unperturbed torus $T^n(I)$ with

$$I \in \mathcal{I} = \{x \in \mathbb{R}^n | K^{-1} \leq x_j \leq K \;\forall j\}$$

survives in the equation (10) if $\varepsilon \leq \varepsilon_0$ and a belongs to a set $\mathcal{A}_\varepsilon(I)$ such that

$$\mathrm{mes}(\mathcal{A} \setminus \mathcal{A}_\varepsilon(I)) \leq \nu(\varepsilon)\mathrm{mes}\mathcal{A},$$

$$\nu(\varepsilon) \to 0 \; (\varepsilon \to 0)$$

By Refinement 1 the number ε_0 and the function $\nu(\varepsilon)$ don't depend on I (but depend on K). Let us denote

$$\mathcal{I}_\varepsilon(a) = \{I \in \mathcal{I} | a \in \mathcal{A}_\varepsilon(I)\}.$$

The torus $T^n(I)$ survives if $I \in \mathcal{I}_\varepsilon(a)$. By Fatou theorem,

$$(\mathrm{mes}\mathcal{A})^{-1} \int_{\mathcal{A}} \mathrm{mes}(\mathcal{I} \setminus \mathcal{I}_\varepsilon(a))\, da =$$

$$= (\mathrm{mes}\mathcal{A})^{-1} \int_{\mathcal{I}} \mathrm{mes}(\mathcal{A} \setminus \mathcal{A}_\varepsilon(I))\, dI \leq \mathrm{mes}\mathcal{I}\; \nu(\varepsilon) \tag{18}$$

(the set $\{(a, I) | a \in \mathcal{A}_\varepsilon(I)\}$ is measurable, see the main part of the text). Let us consider the sets

$$Z_K^0 = \{(q, I) \in Z^0 | I \in \mathcal{I}\}$$

and

$$Z_K^\varepsilon = \{(q, I) | I \in \mathcal{I}_\varepsilon(a)\}.$$

By (18) for a typical a the relative measure of Z_K^ε in Z_K^0 is no less then $1 - \nu(\varepsilon)$. The image of the set Z_K^ε under the map

$$(q, I) \mapsto \Sigma_{a,I}^\varepsilon(q) \tag{19}$$

is invariant for the flow of equation (10) and is filled with quasiperiodic solutions. The mapping (19) is Lipschitz and $C\varepsilon$-close to the embedding Σ^0. So the Hausdorf measure $\mathcal{H}^{2n}$ (see [Fe]) of the invariant set as above is no less then

$$(1-\nu_1(\varepsilon))\,\mathrm{mes}_{2n} Z_K^0, \tag{20}$$

with some $\nu_1(\varepsilon) \to 0$ as $\varepsilon \to 0$. Taking K large enough and ε sufficiently small one can make (20) as large as desired. So we have seen that under the assumptions of Theorem 1 for typical a and for ε small enough the equation (10) has invariant sets of the Hausdorf measure $\mathcal{H}^{2n}$ as large as desired. These sets are filled with quasiperiodic trajectories with zero Lyapunov exponents. They form obstacles to the fast stochastisation of solutions of a typical system of form (10). Our guess is that the recurrence effect "of FPU type" is caused by such sets. See Part 2.3 of the main text for some more results concerning this insight.

Our results leave without any answer the natural question: do the infinite-dimemsional invariant tori of the system (10) with $\varepsilon = 0$ survive under Hamiltonian perturbations? The answer is affirmative if the following three assumptions are satisfied:

a) the perturbation H has short range interactions, i.e for $u(t)$ written as $\sum x_k^{\pm}(t)\varphi_k^{\pm}$, and for some finite N the equation for $x_k^{\pm}$ does not depend on $x_m^{\pm}$ with $|k-m| \geq N$, or depends on $x_m^{\pm}$ in an exponentially small (with respect to $|k-m|$) way;

b) $|H(u)| = O(\| u \|^d)$ for some $d > 2$;

c) the coefficients $x_k^{\pm}$ decrease for example exponentially when k is growing.

The assumtions a), b) are broken for nonlinear partial differential equations (but they are fulfiled for some equations from the physics of crystals). For the exact statements see [FSW], [VB] and [P2], [W2], [AlFS]. We want to mention that the works [FSW], [VB] were the first ones where KAM theory was applied to infinite-dimensional Hamiltonian systems.

Without the assumptions a)-b) the maximal magnitute of the perturbation which allows one to prove Kolmogorov's theorem (=to prove preservation most of half-dimensional tori) exponentially decrease with the dimension of the phase-space (see e.g. [P2, p.364]). We suppose that the exponential estimate is the best possible one. In particular, infinite-dimensional tori "in general" do not survive under perturbations of infinite-dimensional systems.

We end this part with the remark that some results concerning the preservation of infinite-dimensional tori in equation (10) with the spectrum $\{\pm i\lambda_j\}$ of a special type may be obtained via infinite-dimensional versions of Siegel's theorem. See [War], [Z2] and especially [Nik].

3 Applications

3.1 Perturbations of linear differential equations

As a rule, the assumption 1) of Theorem 1 is fulfiled if JA is a differential operator on a segment of the line with some self-adjoint boundary conditions. So the theorem is applicable to spatially one-dimensional quasilinear Hamiltonian partial differential equations depending on a vector parameter.

Example 1 (see [K1] and Part 2.4). Let us consider nonlinear Schrödinger equation with a bounded real potential $V(x;a)$ depending on an n-dimensional parameter a:

$$\dot{u} = i(-u'' + V(x;a)u + \varepsilon\varphi'(x,|u|^2;a)u),$$

$$u = u(t,x), t \in \mathbf{R}, x \in (0,\pi), \tag{21}$$
$$u(t,0) \equiv u(t,\pi) \equiv 0.$$

Here φ is a real function analytic in $|u|^2$ and $\varphi' = \partial\varphi/\partial|u|^2$. To apply the theorem one has to set Z equal to the space of square-summable complex-valued functions on $(0,\pi)$ (and consider it as a real Hilbert space), to set A_a equal to the differential operator $-\partial^2/\partial x^2 + V(x;a)$ under the Dirichlet boundary condition, to set $Ju(x) = iu(x)$ and

$$H_a(u(x)) = \tfrac{1}{2}\int_0^\pi \varphi(x, |u(x)|^2; a)\, dx.$$

Let us denote by $\{\varphi_j(x;a)\}$, $\{\lambda_j(a)\}$ a complete system of real eigenfunctions and eigenvalues of the operator A_a. The invariant n-tori of the unperturbed problem are of the form

$$T(I) = \{\sum_{j=1}^{n}(\alpha_j^+ + i\alpha_j^-)\varphi_j(x;a) | \alpha_j^{+^2} + \alpha_j^{-^2} = 2I_j > 0\, \forall j\}.$$

By the well-known asymptotics of the spectrum of the Sturm–Liouville problem ([Ma], [PT]), $\lambda_j(a) = j^2 + O(1)$ and the assumption 1) of Theorem 1 is fulfilled with $d_1 = 2$, $\kappa = 3/2$. The theorem is applicable to the problem (21); therefore the torus $T(I)$ survives in the problem (21) for most of a and ε small enough if the potential V depends on a in a nondegenerate way. So for nondegenerate families of potentials $\{V(\cdot;a)\}$ and for typical parameters a equation (21) has a lot of quasiperiodic on t solutions, localized in the phase-space Z in a $C\varepsilon$-neighborhood of the low-frequency tori $T(I)$. One can compare this frequency-localization result with the spatial Anderson localization for random linear and non-linear Schrödinger equations (see [CFKS], [FSW], [PasF]).

Example 2 (see [K2] and Part 2.5). Let us consider the equation of oscillations of a string with fixed ends in nonlinear-elastic media depending on n-dimensional parameter:

$$\ddot{w} = (\partial^2/\partial x^2 - V(x;a))w - \varepsilon\varphi_w(x,w;a),$$
$$w = w(t,x), 0 \le x \le \pi, t \in \mathbf{R}, \tag{22}$$
$$w(t,0) \equiv w(t,\pi) \equiv 0.$$

After some reduction (see Part 2.5) Theorem 1 is applicable to this problem with the choice $d_1 = 1, \kappa = 3/2, d_H = -1$. So in a nondegenerate case quasiperiodic in t solutions of the unperturbed problem (22) with $\varepsilon = 0$ survive in the problem (22) for most of a and ε small enough.

3.2 Perturbations of nonlinear systems

A) *Perturbations of Birkhoff-integrable systems* (see [K5], Example 1).

We call a Hamiltonian system Birkhoff integrable if it may be analytically reduced to an infinite sequence of Hamiltonian equations of the form

$$\dot{x}_j^+ = \partial H_0/\partial x_j^-, \dot{x}_j^- = -\partial H_0/\partial x_j^+, j = 1,2,\ldots$$

with

$$H_0 = H_0(p_1, p_2, \ldots), p_j = \tfrac{1}{2}(x_j^{+^2} + x_j^{-^2})$$

(i.e. it may be reduced to the Birkhoff normal form, see [Mo], [SM]). The n-tori

$$T(I) = \{x | x_j^{+^2} + x_j^{-^2} = 2p_j, j = 1, \ldots, n; 0 = x_{n+1}^{\pm} = x_{n+2}^{\pm} = \ldots\}$$

are invariant for the system. It is convenient to pass to the variables (q, p, y) as in (13) with $p = (p_1, \ldots p_n)$, $y = (y_1^+, y_1^-, y_2^+, \ldots)$, $y_j^{\pm} = x_{n+j}^{\pm}$ $(j = 1, 2, \ldots)$. In these variables the equations have the form (14) with

$$\mathcal{H}(q, p, y) = h(p) + \tfrac{1}{2} < A(p)y, y > + O(|y|^3)$$

and

$$h(p) = H_0(p_1, \ldots, p_n, 0, \ldots),$$

$$< A(p)y, y > = \sum_{j=1}^{\infty} (y_j^{+^2} + y_j^{-^2}) \frac{\partial}{\partial p_{n+j}} h(p_1, \ldots, p_n, 0, \ldots).$$

So the ε-perturbed hamiltonian in the new variables is equal to

$$\mathcal{H}_\varepsilon = \mathcal{H} + \varepsilon H_1 = h(p) + \tfrac{1}{2} < A(p)y, y > + O(\| y \|^3) + \varepsilon H_1. \tag{23}$$

Let us fix for a moment some $a \in \mathbf{R}_+^n$ and rewrite $\mathcal{H}_\varepsilon$ as follows:

$$\mathcal{H}_\varepsilon = [h(a) - \omega(a) \cdot a] + \omega(a) \cdot p + \tfrac{1}{2} < A(a)y, y > +$$

$$+ \varepsilon H_1 + O(\| y \|^3 + |p - a|^2 + |p - a| \, \| y \|^2)$$

with $\omega(a) = \nabla h(a)$. The term in the square brackets does not affect the dynamics and may be neglected. Let us suppose that the system possesses nondegenerate amplitude-frequency modulation:

$$\text{Hess}\, h(a) = \det\{\partial \omega_j(a) / \partial a_k\} \not\equiv 0. \tag{24}$$

Then one can treat the vector a as a parameter of the problem and apply to the perturbed problem Theorem 1, taking into account Refinement 2. So if spectral asymptotics and nondegeneracy assumptions are fulfilled, then most of invariant tori $T(a)$ survive under perturbations.

The trick we have just discussed is well suited to study perturbations of finite-dimensional integrable systems but not perturbations of integrable partial differential equations of Hamiltonian form. The reason is that in the last case the transition to the Birkhoff coordinates (or to the action-angle ones) is not regular. To handle infinite-dimensional systems one needs more sophisticated approach; see item C) below.

B) *Use of the partial Birkhoff normal form.*

One can treat the unperturbed linear Hamiltonian system (10) as a Birkhoff integrable system with the linear hamiltonian $H_0 = \frac{1}{2} \sum \lambda_j (x_j^{+^2} + x_j^{-^2})$ and $h(p) = \lambda_1 p_1 + \ldots +$

$\lambda_n p_n$, $\omega(p) = (\lambda_1, \ldots, \lambda_n)$. Now the condition (24) is broken and one can not use an amplitude-frequency modulation to avoid outer parameters a. Nevertheless sometimes one can extract the modulation from the perturbation. This trick was successfully used in a number of works, starting (as far as we know) with Arnold's paper [A3] devoted to Hamiltonian systems with proper degeneration (see also [AKN]); Pöschel [P1] used the trick in his investigations of lower-dimensional tori, Wayne [W1] used similar approach to prove the existence of quasiperiodic on time solutions of nonlinear string equation. Now we turn to its discussion.

For the sake of symplicity we restrict ourselves to the perturbations of the form $H = H^3 + H^4$ with homogeneos of order j functions $H^j, j = 3, 4$. Let us pass to the variables (13). Then the perturbed hamiltonian is $\mathcal{H}_\varepsilon = \mathcal{H}_0 + \varepsilon H_1$ with $\mathcal{H}_0 = h(p) + \frac{1}{2} < Ay, y >$ and

$$H_1 = H^0(q,p) + < H^1(q,p), y > + \tfrac{1}{2} < H^2(q,p)y, y > + O(\| y \|^3).$$

Here H^1 is a vector in Y and H^2 is a selfadjoint operator. So

$$\mathcal{H}_\varepsilon = h(p) + \tfrac{1}{2} < Ay, y > +$$

$$+\varepsilon[H^0(q,p) + < H^1(q,p), y > + \tfrac{1}{2} < H^2(q,p)y, y > + O(\| y \|^3)].$$

It is known since Birkhoff that with the help of a formally-analytic symplectic change of variables $\mathcal{H}_\varepsilon$ may be put into a partial normal form as follows:

$$\mathcal{H}_\varepsilon = h^1(p) + \tfrac{1}{2} < A(p)y, y > + \varepsilon^2 H_\Delta(q,p,y) + \varepsilon O(\| y \|^3). \tag{25}$$

Here $h^1(p) = h(p) + \varepsilon \overline{H}^0(p)$ (the bar means the averaging over $q \in \mathbf{T}^n$) and $A^1 = A + \varepsilon A_\Delta(p)$ with some operator $A_\Delta(p)$ constructed in terms of the operator $\overline{H}^2(p)$. The function (25) is of the same form as (23) and in general the assumption (24) is fulfilled for function $h^1(p)$. Now Hess h^1 is of order ε and the norm of the inverse map $\omega \mapsto p$ is of order ε^{-1}. Following the scheme of item A) one has to pass to the parameter ω. After this the perturbation is of order $\varepsilon^{-1} \times \varepsilon^2 = \varepsilon$ and Theorem 1 is applicable provided the change of variables is analytic. The exact formulaes (see e.g. [P1] and Part 3 below) show that the normal-form transformation is defined as a series with some regular numerators and with denominators of the form $D(s) = s_1\lambda_1 + \ldots + s_N\lambda_N$. Here N is an arbitrary natural number and

$$3 \leq |s_1| + \ldots + |s_N| \leq 4, s_N \neq 0, |s_{n+1}| + \ldots + |s_N| \leq 2$$

So if

$$|D(s)| \geq C^{-1} \tag{26}$$

for all s as above, then the scheme works and one can use it to prove the existence of an invariant torus of the perturbed equation at a distance of order ε from the unperturbed torus.

The condition (26) is not very restrictive because it holds for typical sequences $\{\lambda_j\}$ satisfying assumption 2) of Theorem 1. Now we prove this statement for $d_1 > 1$. To do it let us suppose for a moment that $N \geq n+1$. Then

$$|D(s)| \geq |\lambda_{n+1}s_{n+1} + \ldots + \lambda_N s_N| - |\lambda_1 s_1 + \ldots + \lambda_n s_n| \geq$$

$$\geq |\lambda_N - \lambda_{N-1}| - 3\max\{|\lambda_j||1 \leq j \leq n\} \geq C_1 N^{d_1-1} - C_2.$$

So (26) holds with $C = 1$ if N is greater than some N_0. Therefore the inequality (26) holds if $D(s) \neq 0$ for the finite set of resonance relations consisting of all admissible relations with $N \leq N_0$ (one can choose C^{-1} equal to $\min\{1, \min\{|D(s)| \, |N \leq N_0\}\}$).

This scheme was mostly proposed by Pöschel in his work [P1] devoted to finite-dimensional systems, where he also conjectured that it may be used to study infinite-dimensional systems. We have a few doubts that described above infinite-dimensional realization of the scheme via Theorem 1 may be done without too much efforts, although this work still has not been done.

C) *On the integrable equations of mathematical physics*

One of the main achievements of mathematical physics during the last decades was the discovery of theta-integrable nonlinear partial differential equations (see e.g. [DEGN], [Nov]). Such equations are quasilinear Hamiltonian equations of the form (9). They possess invariant symplectic $2n$-dimensional manifolds $\mathcal{T}^{2n}$ such that the restriction of the system (9) on $\mathcal{T}^{2n}$ is integrable. So $\mathcal{T}^{2n}$ is symplectomorphic to $\mathbf{T}^n_q \times P_p$, $P \subset \mathbf{R}^n$, and in coordinates (q,p) the restriction of the system onto $\mathcal{T}^{2n}$ has the form

$$\dot{q} = \nabla h(p), \dot{p} = 0.$$

Therefore $\mathcal{T}^{2n}$ is foliated into invariant n-tori $T^n(p) = \{(q,p)|p = \text{const}\}$ filled with quasiperiodic solutions $u_0(t) = (q + t\nabla h(p), p)$. The question is if the tori $T^n(p)$ survive under Hamiltonian perturbations of the equation. To formulate the corresponding result we have to consider variational equations about the solutions $u_0(t)$:

$$\dot{v} = J(\nabla \mathcal{K}(u_0(t)))_* v$$

and to suppose that all these equations are reducible to constant coefficient linear equations by means of a quasi-periodic substitutions $v = B(t,p)V$ (B is a linear operator in Z quasiperiodically depending on t). It is proved (see [K5] and [K7], [K8]) that under the reducibility assumption the quasilinear equation (9) near the manifold $\mathcal{T}^{2n}$ may be written in the form (14) with

$$\mathcal{H} = h(p) + \tfrac{1}{2} < A(p)y, y > + O(\| y \|^3).$$

A perturbed equation under this reduction takes exactly the form (23). So as in item A) one can prove that in a nondegenerate situation most of the tori $T^n(p)$ survive under perturbations.

For an exact realization of this scheme for a perturbed Korteweg–de Vries equation see [K5] and for a perturbed Sine-Gordon equation see the forthcoming paper of R.Bikbaev and the author. The finite-dimensional version of this result is of interest also. One can treat it as a version of Kolmogorov's theorem for lower-dimensional invariant tori, see [K7].

3.3 Last remark

Above we have proposed as an explanation of the recurrence effect of FPU-type in infinite-dimensional Hamiltonian systems, the theorem on the preservation of most of quasi-periodic solutions under Hamiltonian perturbations. It is well-understood however that long-time regular behavior of solutions may be explained by means of an averaging theorem as well. In a finite-dimensional situation Nekhoroshev's theorem (see [N], [BGG], [Lo]) suites this purpose very well. For infinite-dimensional systems with discrete spectrum versions of this result are known only for systems with short range interactions ([W3],[BFG]). We are rather sceptical that there exists a version of Nekhoroshev's theorem applicable to *all* solutions of a nearly-integrable nonlinear partial differential equation of Hamiltonian form.

Acknowledgements

I am grateful to V.I.Arnold, V.F.Lazutkin and M.I.Vishik for useful discussions concerning the results obtained. I am indebted to Jürgen Pöschel for helpful remarks and friendly discussions during the preparation of the final version of this work; to M.B.Sevrjuk and Gene Wayne for pointing out a lot of misprints and mistakes; to Ms. Manuela Sarlette (MPI) for her quality typing of the text. I am much obliged to my wife Julija for encouragement and for typeing the original Russian versions of *all* my papers.

This text was written while I was visiting the Max-Planck-Institut für Mathematik in Bonn and was originally printed as preprints of the MPI [K6]. I want to thank the MPI for its warm hospitality and the Max-Planck-Gesellschaft for financial support.

Part 1.

Symplectic structures and Hamiltonian systems in the scales of Hilbert spaces

The following notations are used: for Hilbert spaces X, Y, Z the norms are denoted by $|\cdot|_X$, $|\cdot|_Y$, $|\cdot|_Z$ and inner products by $\langle\cdot,\cdot\rangle_X$, $\langle\cdot,\cdot\rangle_Y$, $\langle\cdot,\cdot\rangle_Z$; dist_X – distance in the space X; for domains $O_X \subset X$, $O_Y \subset Y$ the space of k–times Fréchet differentiable mappings $O_X \longrightarrow O_Y$ is denoted by $C^k(O_X;O_Y)$ and $C(O_X;O_Y) = C^0(O_X;O_Y)$, $C^k(O_X;\mathbb{R}) = C^k(O_X)$ $\forall k \geq 0$; for $\phi \in C^1(O_X;O_Y)$ the tangent (cotangent) mapping is denoted by $\phi_*(\phi^*)$ (tangent spaces T_xO_X, T_yO_Y are identified with X and Y, cotangent spaces $T_x^*O_X$, $T_y^*O_Y$ are identified with X and Y through Riesz's isomorphism). For a mapping $G: O_X \longrightarrow O_Y$ we denote by $\mathrm{Lip}(G) = \mathrm{Lip}(G: O_X \longrightarrow O_X)$ its Lipschitz constant,

$$\mathrm{Lip}(G) = \sup_{x_1 \neq x_2} \frac{|G(x_1)-G(x_2)|_Y}{|x_1-x_2|_X}.$$

1. Symplectic Hilbert scales and Hamitonian equations

Let Z be a Hilbert space with inner product $\langle\cdot,\cdot\rangle_Z$ and $\{Z_s | s \in \mathbb{R}\}$ a scale of Hilbert spaces (see [RS1], [RS2] with following properties:

a) the Hilbert space Z_{s_1} is densely inclosed in Z_{s_2} if $s_1 \geq s_2$ and the linear space $Z_\infty = \cap Z_s$ is dense in Z_s $\forall s$;

b) $Z_0 = Z$;

c) the spaces Z_s and Z_{-s} are dual with respect to inner product $\langle\cdot,\cdot\rangle_Z$.

The norm (inner product) in Z_s will be denoted by $\|\cdot\|_s$ ($\langle\cdot,\cdot\rangle_s$). In particular $\|\cdot\|_0 = |\cdot|_Z$ and $\langle\cdot,\cdot\rangle_0 = \langle\cdot,\cdot\rangle_Z$. The pairing between Z_s and Z_{-s} will be denoted $\langle\cdot,\cdot\rangle_0$ or $\langle\cdot,\cdot\rangle_Z$.

Let $J: Z_\infty \longrightarrow Z_\infty$ be a linear operator such that $J(Z_\infty) = Z_\infty$ and

d) J determines an isomorphism of the scale $\{Z_s\}$ of order $d_J \geq 0$, i.e. for every

$s \in \mathbb{R}$ J may be extended to a continuous linear isomorphism $J : Z_s \xrightarrow[\sim]{} Z_{s-d_J}$;

e) the operator J with domain of definition Z_∞ is antisymmetric in Z , i.e. $\langle Jz_1,z_2\rangle_Z = -\langle z_1 , Jz_2\rangle_Z \quad \forall\, z_1,z_2 \in Z_\infty$.

Let us denote by $\bar{J}$ the isomorphism of order $-d_J$ of scale $\{Z_s\}$:

$$\bar{J} = -(J)^{-1} : Z_s \xrightarrow[\sim]{} Z_{s+d_J} \quad \forall s \in \mathbb{R} \tag{1.1}$$

Lemma 1.1. The operator $\bar{J} : Z \longrightarrow Z_{d_J} \subset Z$ is anti selfadjoint in Z .

Proof. Let $x,y \in Z_\infty$ and $Jx = x_1$, $Jy = y_1$. Then $\bar{J}x_1 = -x$, $\bar{J}y_1 = -y$ and

$$\langle x_1,\bar{J}y_1\rangle_Z = -\langle Jx,y\rangle_Z = \langle x,Jy\rangle_Z = -\langle \bar{J}x_1,y_1\rangle_Z \ .$$

The operator $\bar{J} : Z \longrightarrow Z$ is continuous, and the space Z_∞ is dense in Z , so the lemma is proved. ■

Let us introduce in every space Z_s with $s \geq 0$ a 2–form $\alpha = \langle \bar{J}\, dz,dz\rangle_Z$. Here by definition

$$\langle \bar{J}\, dz,dz\rangle_Z\ [z_1,z_2] = \langle \bar{J}\, z_1,z_2\rangle_Z \ \forall\, z_1,z_2 \in Z_s \tag{1.2}$$

The form α is closed and nondegenerate [A,Ch–B].

Definition. The triple $\{Z,\{Z_s \mid s \in \mathbb{R}\},\ \alpha = \langle \bar{J}\, dz,dz\rangle\}$ is called symplectic Hilbert scale (or SHS for brevity).

Example 1.1. Let $Z = \mathbb{R}^n_p \times \mathbb{R}^n_q$, $Z_s = Z\ \forall s$ and $J : Z \longrightarrow Z$, $(p,q) \longmapsto (-q,p)$. In this case $J^2 = -E$ so $\bar{J} = -J^{-1} = J$, $d_J = 0$ and

$$\alpha = \langle \bar{J}\, dz,dz\rangle_Z = \langle J\, dz,dz\rangle_Z = dp \wedge dq \ .$$

Properties a)–e) are obvious and we obtain the classical symplectic structure for even–dimensional spaces [A].

Example 1.2. Let $Z = L_2(S^1) \times L_2(S^1)$, $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, be a space of pairs of square–summable periodic functions $(p(x), q(x))$. Let $Z_s = H^s(S^1) \times H^s(S^1)$. Here $H^s(S^1)$ is the Sobolev space of periodic functions, $s \in \mathbb{R}$ [Ch–B,RS2]. Let us take

$$J : Z_s \longrightarrow Z_s\,,\ (p(x),q(x)) \longmapsto (-q(x),p(x))\ .$$

Then $\bar{J} = J$ is an isomorphism of scale $\{Z_s\}$ of order zero. Properties a)–e) are evident.

Example 1.3. Let

$$Z_s = H_0^s(S^1) = \{u(x) \in H^s(S^1) \mid \int_0^{2\pi} u(x)dx = 0\}\ .$$

Let us take $J = \partial/\partial x$. Then J is an isomorphism of the scale of order one and $\bar{J} = -(J)^{-1}$ is an isomorphism of order -1. Properties a)–e) are evident again and we have got SHS corresponding to symplectic structure of KdV–equation (see below and [A, Appendix 13; N]).

For $f \in C^1(O_s)$ let $\nabla f \in Z_{-s}$ be the gradient of f with respect to the inner product $\langle\cdot,\cdot\rangle_Z$:

$$\langle \nabla f(u),v\rangle_Z = Df(u)(v) = \frac{\partial}{\partial\epsilon} f(u+\epsilon v)\,|_{\,\epsilon=0}\ \forall v \in O_s\ .$$

The mapping $O_s \longrightarrow Z_{-s}$, $u \longmapsto \nabla f(u)$, is continuous.

For $H \in C^1(O_s)$ the Hamiltonian vector–field V_H is the mapping $V_H : V_s \longrightarrow Z_{-\infty} = \bigcup_s Z_s$ defined by the following relation [A, Ch–B]:

$$\alpha(\xi,V_H(u)) = \langle \xi,\nabla H(u)\rangle_Z\ \forall \xi \in Z_\infty$$

or

$$\langle J\xi, V_H(u)\rangle_Z = \langle \xi, \nabla H(u)\rangle_Z \ \forall \xi \in Z_\infty \ .$$

So $V_H(u) = J\nabla H(u)$ and

$$\dot{u} = J\nabla H(u) \tag{1.3}$$

is the Hamiltonian equation corresponding to the hamiltonian H . Let us denote

$$D_s(V_H) = \{u \in O_s \mid V_H(u) = J\nabla H(u) \in Z_s\} \ .$$

Definition (cf. [B]). A curve $u(t)$, $0 \le t \le T$, is called a strong solution in the space Z_s of the equation (1.3) iff $u \in C^1([0,T];Z_s)$, $u(t) \in D_s(V_H)\ \forall t \in [0,T]$ and $\forall t$ equation (1.3) is satisfied. A curve $u \in C([0,T];Z_s)$ is called a weak solution of (1.3) iff it is the limit in the $C([0,T];Z_s)$–norm of some sequence of strong solutions.

Definition. Let $O_s^1 \subset O_s$ be a domain such that for every $u_0 \in O_s^1$ there exists a unique weak solution $u(t) = S^t(u_0)$ $(0 \le t \le T)$ of equation (1.3) with initial condition $u(0) = u_0$. The set of mappings

$$S^t : O_s^1 \longrightarrow O_s \ , \ u_0 \longmapsto S^t(u_0) \quad (0 \le t \le T)$$

is called the "local semiflow of equation (1.3)" or the "flow of equation (1.3)" for short.

Weak solutions of equations (1.3) are generalized ones in the sense of distributions (see [L] for systematic use of this type of solutions):

Proposition 1.4. Let us suppose that for some $s_1 \in \mathbb{R}$, $\mathrm{Lip}(\nabla H : O_s \longrightarrow Z_{s_1}) < \infty$. Then a weak solution $u(t) \in O_s$ $(0 \le t \le T)$ of equation (1.3) is a generalized solution and after substitution of $u(t)$ into (1.3) the left and right hand sides of the equation coincide as elements of the space $D'((0,T);Z_{s_2})$ of distributions on $(0,T)$ with values in Z_{s_2} , $s_2 = \min\{s, s_1 - d_J\}$.

Proof. By definition of weak solution there exist a sequence of strong solutions $u_n(t)$ such that $u_n(\cdot) \longrightarrow u(\cdot)$ in $C([0,T];X_s)$. For this sequence

$$\dot{u}_n \longrightarrow \dot{u} \text{ in } D'((0,T);Z_s) ,$$

$J\nabla H(u_n) \longrightarrow J\nabla H(u)$ in $C([0,T];Z_{s_1-d_J})$. After transition to the limit in equation (1.3) one obtains the result. ∎

Example 1.1, again. Let $H \in C^1(\mathbb{R}^n_p \times \mathbb{R}^n_q)$. The Hamiltonian equation takes the classical form:

$$\dot{p} = -\nabla_q H(p,q) , \quad \dot{q} = \nabla_p H(p,q) .$$

If $H \in C^2(\mathbb{R}^{2n})$ then a weak solution is a strong one and it exists for some $T > 0$, $T = T(p(0),q(0))$.

Example 1.2, again. Let us consider the hamiltonian

$$H = \frac{1}{2}\int_0^{2\pi} (p_x(x)^2 + q_x(x)^2 + V(x)(p(x)^2 + q(x)^2) + \chi(p(x)^2 + q(x)^2))dx$$

with analytical function χ and smooth function V. Then $H \in C^1(Z_s)$ for $s \geq 1$ and

$$\nabla H(p,q) = (-p_{xx} + V(x)p + \chi'(p^2+q^2)p , -q_{xx} + V(x)q(x) + \chi'(p^2 + q^2)q) .$$

The equation (1.3) takes now the following form:

$$\begin{aligned} \dot{p} &= q_{xx} - V(x)q - \chi'(p^2+q^2)q , \\ \dot{q} &= -p_{xx} + V(x)p + \chi'(p^2+q^2)p . \end{aligned}$$

Let us denote $u(t,x) = p(t,x) + iq(t,x)$. The last equations are equivalent to nonlinear

Schrödinger equation with real potential $V(x)$ for complex functions $u(t,x)$:

$$\begin{aligned} &\dot{u} = i(-u_{xx} + V(x)u + \chi'(|u(x)|^2)u) \ , \\ &u(t,x) \equiv u(t,x+2\pi) \ . \end{aligned} \tag{1.4}$$

The problem (1.4) has an unique strong solution $u(t,x)$, $u(t,\cdot) \in Z_s$, $0 \le t \le T = T(u(0,x))$, if $s \ge 1$ and $u(0,x) \in Z_{s+2}$ (we interpret here Z_s as the Sobolev space of periodic complex–valued functions), and (1.4) has an unique weak solution for $0 \le t \le T$ if $u(0,x) \in Z_s$. For the simple proof see part 3 below.

Example 1.3, again. In the situation of example 1.3 let us consider the hamiltonian

$$H = \int_0^{2\pi} (\tfrac{1}{2} u_x{}^2 + u^3) dx \ .$$

Then $H \in C^1(Z_s)$ for $s \ge 1$ and $\nabla H(u(x)) = -u_{xx} + 3u^2$. So now equation (1.3) is the KdV equation

$$\dot{u}(t,x) = -u_{xxx} + 6uu_x \tag{1.5}$$

for periodic in x functions with zero mean value:

$$u(t,x) \equiv u(t,x+2\pi) \ , \ \int_0^{2\pi} u(t,x)dx \equiv 0 \tag{1.5'}$$

It is well known [K] that for $s \ge 3$ the problem (1.5), (1.5′) has an unique strong solution $u(t,x)$, $u(t,\cdot) \in Z_s \forall t$, for every initial condition $u(0,x) = u_0(x) \in Z_{s+3}$ and has an unique weak solution for every $u_0(x) \in Z_s$. The flow of problem (1.5), (1.5′) defines a homeomorphisms of phase space

$$S^t : Z_s \xrightarrow[\sim]{} Z_s \ \forall\, t \ge 0 \quad \forall\, s \ge 3 \ .$$

It is worth mentioning that any Hamiltonian equation (including (1.4) and (1.5), (1.5′)) may be written down in a form (1.3) in many different ways. For this statement see below Corollary 2.3.

2. Canonical transformations

Let $\{X,\{X_s\},\alpha^X\}$ and $\{Y,\{Y_s\},\alpha^Y\}$ be two SHS with 2–forms $\alpha^X = \langle J^X dx,dx\rangle_X$ and $\alpha^Y = \langle J^Y dy,dy\rangle_Y$ respectively; let J^X (J^Y) be an isomorphism of scale $\{X_s\}$ $(\{Y_s\})$ of order $-d_{J^X}$ $(-d_{J^Y})$; $d_{J^X}, d_{J^Y} \geq 0$. A mapping $\phi : O^X_{s_X} \longrightarrow O^Y_{s_Y}$ is a C^1–diffeomorphism of domains $O^X_{s_X} \subset X_{s_X}$ and $O^Y_{s_Y} \subset Y_{s_Y}$ $(s_X \geq 0, s_Y \geq 0)$, if ϕ is one–to–one onto $O^Y_{s_Y}$ and

$$\phi \in C^1(O^X_{s_X};O^Y_{s_Y}) , \quad \phi^{-1} \in C^1(O^Y_{s_Y};O^X_{s_X}) \tag{2.1}$$

Definition. A C^1–diffeomorphism $\phi : O^X_{s_X} \longrightarrow O^Y_{s_Y}$ is a canonical transformation iff it transforms the 2–form α^Y into the 2–form α^X :

$$\phi^* \alpha^Y = \alpha^X \ . \tag{2.2}$$

Proposition 2.1. A C^1–diffeomorphism $\phi : O^X_{s_X} \longrightarrow O^Y_{s_Y}$ is canonical iff

$$\phi^* J^Y \phi_* \equiv J^X \tag{2.3}$$

(the identity takes place in the space $L(X_{s_X};X_{-s_X})$) .

Proof. From (2.2) one has for $v \in O^X_{s_X}$ and $\xi_1,\xi_2 \in X_{s_X}$

$$\langle J^Y \phi_*(v)\xi_1, \phi_*(v)\xi_2\rangle_Y = \langle J^X \xi_1,\xi_2\rangle_X \ . \tag{2.4}$$

Therefore

$$\langle \phi^*(v) J^Y \phi_*(v) \xi_1, \xi_2 \rangle_X = \langle J^X \xi_1, \xi_2 \rangle_X$$

for all $\xi_1, \xi_2 \in X_{s_X}$. This identity implies the stated assertion.

■

As in the finite–dimensional case [A] a canonical transformation transforms solutions of Hamiltonian equation into solutions of the equation with transformed hamiltonian:

Theorem 2.2. Let $\phi : O^X_{s_X} \longrightarrow O^Y_{s_Y}$ be a canonical transformation and let $y : [0,T] \longrightarrow O^Y_{s_Y}$ be a strong solution of the Hamiltonian equation

$$\dot{y} = V_{H^Y}(y) = J^Y \nabla H^Y(y) , \quad H^Y \in C^1(O^Y_{s_Y}; \mathbb{R}) . \tag{2.5}$$

Then $x(t) = \phi^{-1}(y(t))$ is a strong solution in $O^X_{s_X}$ of equation

$$\dot{x} = V_{H^X}(x) = J^X \nabla H^X(x) , \quad H^X = H^Y \circ \phi . \tag{2.6}$$

If the mapping $\phi^{-1} : O^Y_{s_Y} \longrightarrow O^X_{s_X}$ is Lipschitz and y is a weak solution of (2.5) then x is a weak solution of (2.6).

Proof. For $H^X = H^Y \circ \phi$ and $x = \phi^{-1} \circ y$ $\quad \nabla H^X = \phi^* \nabla H^Y$. Then $x : [0,T] \longrightarrow O^X_{s_X}$ is C^1 and for $y = \phi \circ x$

$$\phi_* \dot{x} = \dot{y} = J^Y \nabla H^Y(y) = J^Y (\phi^*)^{-1} \nabla H^X(x) \tag{2.7}$$

or

$$\dot{x} = (\phi_*)^{-1} J^Y (\phi^*)^{-1} \nabla H^X(x) \tag{2.8}$$

(the right–hand side is well defined because $J^Y(\phi^*)^{-1} \nabla H^X(x) \in C([0,T]; O^Y_{s_Y})$ for (2.1)). By (2.3), $J^X = (\phi_*)^{-1} J^Y (\phi^*)^{-1}$, hence

$$\dot{x} = J^X \nabla H^X(x)$$

as stated.

The second statement of the theorem follows from the first one and the definition of a weak solution because the mapping ϕ^{-1} is Lipshitz. ■

Let $\{Y,\{Y_s\},\alpha^Y\}$ be a SHS, let L be an isomorphism of scale $\{Y_s\}$ of order $\Delta \leq \frac{1}{2} d_{J^Y}$, $L : Y_s \xrightarrow[\sim]{} Y_{s-\Delta}$ $\forall s$. Let us define a second SHS $\{X,\{X_s\},\alpha^X\}$ where $X = Y$, $X_s = Y_s$ and $\alpha^X = \langle \bar{J}^X dx,dx \rangle_X$, $J^X = L^* J^Y L$. The operator J^X is antisymmetric in X and it defines an isomorphism of the scale $\{X_s\}$ of the order $-d_{J^Y} + 2\Delta \leq 0$, so the new triple is a SHS indeed. Let $O^X_{s_X}$ be a domain in X_{s_Y} and $O^Y_{s_Y} = L(O^X_{s_X}) \subset Y_{s_Y}$, $s^Y = s^X - \Delta$. The mapping $L : O^X_{s_X} \longrightarrow O^Y_{s_Y}$ is canonical due to Proposition 2.1. So we have the trivial

<u>Corollary 2.3</u> (change of symplectic structure). Let $H^Y \in C^1(O^Y_{s_Y})$ and let $y(t) \in O^Y_{s_Y}$ $(0 \leq t \leq T)$ be a solution of equation (2.5) (strong or weak). Then $x(t) = L^{-1} y(t)$ is a solution of Hamiltonian equation

$$\dot{x} = J^X \nabla H^X(x), \ J^X = L^{-1} J^Y (L^*)^{-1},$$

with a hamiltonian $H^X = H^Y \circ L \in C^1(O^X_{s_X})$.

Let $\{X,\{X_s\}, \alpha = \langle J\,dx,dx\rangle_X\}$ be a SHS, O_s^1 and O_s be domains in X_s, $O_s^1 \subset O_s$ and

$$\mathrm{dist}_{X_s}(O_s^1; X_s \backslash O_s) > \delta > 0 \tag{2.10}$$

Let $H \in C^2(O_s)$ and

$$\nabla H \in C^1(O_s; X_{s+d_J}),\ \|J\nabla H(x)\|_s \le K,\ \mathrm{Lip}(J\nabla H : O_s \longrightarrow X_s) \le K, \tag{2.11}$$

Let us consider the Hamiltonian equation

$$\dot{x} = J\nabla H(x) \tag{2.12}$$

From (2.10), (2.11) one can easily obtain that the flow of equation (2.12) defines mappings $S^t \in C^1(O_s^1;O_s)$ $\forall t \in [0,T]$, $T = \delta/K$, and every S^t is a C^1–diffeomorphism onto its image.

Theorem 2.4. For every $0 \le t \le T$ the mapping S^t is a canonical transformation.

Proof. One has to prove that

$$(S^t)^*\alpha(x)[\eta_1,\eta_2] = \alpha[\eta_1,\eta_2]\ \forall x \in O_s^1\ \forall \eta_1,\eta_2 \in X_s$$

Since $S^0 = \mathrm{Id}$ it is sufficient to prove that

$$(S^{\cdot})^*\alpha(x)[\eta_1,\eta_2] = \mathrm{const} \tag{2.13}$$

Let $x(\tau)$ be the solution of equation (2.12) for $x(0) = x$, and $\eta^j(t)$ $(j = 1,2)$ be the solution of the Cauchy problem for linearized about $x(\cdot)$ equation:

$$\dot{\eta}^j(\tau) = J(\nabla H)_*(x(\tau))\eta^j(\tau),\ \eta^j(0) = \eta_j\,. \tag{2.14}$$

Then $(S^\tau)_*(z)\eta_j = \eta^j(\tau)$, $j = 1,2$ and

$$(S^\tau)^* \alpha(x)[\eta_1,\eta_2] = \alpha[\eta^1(\tau),\eta^2(\tau)] = \tag{2.15}$$
$$= \langle J\eta^1(\tau),\eta^2(\tau)\rangle_X \equiv \ell(\tau)$$

The function $\ell(\tau)$ is continuously differentable. So (2.13) is equivalent to the relation $d/d\tau\ \ell(\tau) \equiv 0$. One has

$$\frac{d}{d\tau}\ell(\tau) = \langle J\,\dot\eta^1,\eta^2\rangle_X + \langle J\,\eta^1,\dot\eta^2\rangle_X =$$

$$= \langle JJ(\nabla H)_*(x)\eta^1,\eta^2\rangle_X + \langle J\,\eta^1,J(\nabla H)_*(x)\eta^2\rangle_X =$$

$$= -\langle(\nabla H)_*(x)\eta^1,\eta^2\rangle_X + \langle\eta^1,(\nabla H)_*(x)\eta^2\rangle_X = 0$$

because operator J is anti selfadjoint (Lemma 1.1) and operator $(\nabla H)_*$ is selfadjoint. The theorem is proved. ∎

Let $H_j \in C^1(O_s)$, $\nabla H_j \in C(O_s;X_{s_j})$ $(j = 1,2)$.

<u>Definition</u>. Let $s_1 + s_2 \geq d_J$. The Poisson bracket of the functions H_1, H_2 is the function $\{H_1,H_2\} \in C(O_s)$ defined by

$$\{H_1,H_2\} = \langle J\nabla H_1,\nabla H_2\rangle_X \ .$$

Let $0 < \epsilon \leq 1$ and $H \in C^2(O_s)$, let conditions (2.10), (2.11) be satisfied and $S^t \in C^1(O_s^1;O_s)$, $0 \leq t \leq T = \delta/K$, be the flow of the equation

$$\dot x = \epsilon\ J\nabla H(x) \ .$$

<u>Theorem 2.5</u>. For every $G \in C^1(O_s)$ $G(S^t(x)) = G(x) + t\epsilon\{H,G\}(x) + O((\epsilon t)^2)$

$\forall x \in O_s^1$, $\forall\, 0 \leq t \leq T$.

Proof. From the conditions on H it is easy to see that

$$S^t(x) = x + t\epsilon J\nabla H(x) + O(t\epsilon)^2 \text{ in } X_s \ .$$

So

$$G(S^t(x)) - G(x) = \langle \nabla G(x), S^t(x) - x \rangle_X + O\|S^t x - x\|_s^2 =$$

$$= t\epsilon \langle \nabla G(x), J\nabla H(x) \rangle_X + O(\epsilon t)^2$$

and the theorem is proved. ■

3. Local solvability of Hamiltonian equations

Let $\{Y, \{Y_s\}, \alpha\}$ be SHS, let O_s be a domain in Y_s and let

$$H \in C^2(O_s) \ , \ \ H(y) = \tfrac{1}{2} \langle Ay, y \rangle_Y + H_0(y) \ .$$

Here A is an isomorphism of scale $\{Y_s\}$ of order $d_A \geq 0$;

$$A : Y_s \xrightarrow[\sim]{} Y_{s-d_A} \quad \forall\, s \in \mathbb{R} \ , \tag{3.1}$$

and the operator

$$A : D(A) \subset Y \longrightarrow Y \ , \ \ D(A) = Y_{d_A}$$

is selfadjoint. So $\nabla(\frac{1}{2} \langle Ay, y \rangle_Y)(y) = Ay$, and the Hamiltonian equation corresponding to H has the form

$$\dot{y} = J(Ay + \nabla H_0(y)) \tag{3.2}$$

We shall prove a simple theorem on the local solvability of equation (3.2) which will suit well to our aims. To formulate the theorem let us suppose that

$$\mathrm{Lip}(J\nabla H_0 : O_s \longrightarrow Y_s) \le K \tag{3.3}$$

for some $s \ge 0$ and let $O^2, O^1 \subset Y_s$ be domains with the following properties:

$$O^2 \subset O^1 \subset O_s \,, \quad \mathrm{dist}_{Y_s}(O^1, Y_s \backslash O_s) \ge \delta > 0 \ . \tag{3.4}$$

Theorem 3.1. Let

$$AJy = JAy \quad \forall\, y \in Y_\infty \tag{3.5}$$

$$\langle Ay_1, y_2\rangle_s = \langle y_1, Ay_2\rangle_s \,, \quad \langle Jy_1, y_2\rangle_s = -\langle y_1, Jy_2\rangle_s \quad \forall\, y_1, y_2 \in Y_\infty \ . \tag{3.6}$$

Suppose that every strong solutions $y(t)$ of equation (3.2) with initial condition $y(0) = y_0 \in O^2$ stays inside O^1 for $0 \le t \le T$. Then for $y_0 \in O^2 \cap Y_{s+d_1}$, $d_1 = d_A + d_J$, there exists a unique strong solution $y(t)$ for $0 \le t \le T$, and for $y_0 \in O^2$ there exists a unique weak solution $y(t)$ for $0 \le t \le T$.

Proof. Let us continue the mapping $J\nabla H_0 : O^1 \longrightarrow Y_s$ to a Lipschitz one $V : Y_s \longrightarrow Y_s$. One may take for example

$$V(y) = \begin{cases} \chi(y) J\nabla H_0(y) \,, & y \in O_s \\ 0, y \notin O_s \,, & \end{cases}$$

where $\chi(y) = \delta^{-1} \max\,(0, \delta - \mathrm{dist}_{Y_s}(y; O^1))$ (see (3.4)). The function χ is Lipschitz, it is equal to 1 in O^1 and to 0 out of O_s. So $\mathrm{Lip}(V) \le K^1$ and $V|_{O^1} = J\nabla H_0$.

Let us consider the equation

$$\dot{y} = JAy + V(y) \tag{3.7}$$

Its solution $y(t)$ is a solution of equation (3.2) as long as $y(t) \in O^1$. Let us consider the linear equation

$$\dot{y} = JAy \ , \tag{3.8}$$

too. From (3.5), (3.6) it follows that

$$\langle AJy_1,y_2\rangle_s = -\langle y_1,AJy_2\rangle_s \quad \forall y_1,y_2 \in Y_\infty$$

so by repeating the proof of Lemma 1.1 one can obtain that operator $(AJ)^{-1} : Y_s \longrightarrow Y_s$ is anti selfadjoint. So the operator

$$AJ : D(AJ) = Y_{s+d_1} \subset Y_s \longrightarrow Y_s$$

is anti selfadjoint, too. Due to Stone's theorem [RS1] for $y(0) = y_0 \in Y_{s+d_1}$ equation (3.8) has a unique strong solution and the mapping

$$S^T : Y_{s+d_1} \longrightarrow Y_{s+d_1} \ , \ y(0) \longmapsto y(T) \ , \ T > 0 \ ,$$

is isometric with respect to the Y_s–norm. Equation (3.7) is a Lipschitz perturbation of (3.8). So it has the unique strong solution $y(t)$, $t \geq 0$, for every $y(0) \in Y_{s+d_1}$ and the unique weak solution for every $y(0) \in Y_s$ (see [B]). If $y(0) = y_0 \in O^2$ then due to the theorem's hypotheses such a solution does not leave domain O^1 for $0 \leq t \leq T$ and for such a " t " it is the unique solution of equation (3.7).

■

The theorem above reduces the problem of solving equation (3.2) to the problem of finding a *priori* estimate for its solutions.

4. Toroidal phase space

Let us consider a toroidal phase space of the form $\mathcal{Y} = \mathbf{T}^n \times \mathbb{R}^n \times Y$. Here $\mathbf{T}^n = \mathbb{R}^n / 2\pi \mathbb{Z}^n$ is the n–dimensional torus, $Y = Y_0$, $\{Y_s \mid s \in \mathbb{R}\}$ is a scale of Hilbert spaces which satisfies properties a)–c) (see above). Let us denote $\mathcal{Y}_s = \mathbf{T}^n \times \mathbb{R}^n \times Y_s$. Every space $\mathcal{Y}_s$ has a natural metric dist_s and a natural structure of a Hilbert manifold with local charts

$$K(q^0) \times \mathbb{R}^n \times Y_s\,, \quad K(q^0) = \{q \in \mathbb{R}^n \,\big|\, |q_j - q_j^0| < \pi \ \forall j\}$$

(see [Ch–B]). So

$$T_u\, \mathcal{Y}_s \cong \mathbb{R}^n \times \mathbb{R}^n \times Y_s \equiv Z_s \quad \forall u \in \mathcal{Y}_s\ ,$$

Let J^Y be an isomorphism of the scale $\{Y_s\}$ with properties d), e) and

$$J^T : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^n\,, \quad (q,p) \longmapsto (-p,q)\ .$$

Let us denote by $J^{\mathcal{Y}}$ the operator

$$J^{\mathcal{Y}} = J^T \times J^Y : Z_s = (\mathbb{R}^n \times \mathbb{R}^n) \times Y_s \longrightarrow Z_{s-d_J} = (\mathbb{R}^n \times \mathbb{R}^n) \times Y_{s-d_J}$$

and introduce in $\mathcal{Y}_s$, $s \geq 0$, a 2–form

$$\alpha^{\mathcal{Y}} = \langle \bar{J}^{\mathcal{Y}}\, du, du \rangle_Z\,, \quad \bar{J}^{\mathcal{Y}} = -(J^{\mathcal{Y}})^{-1}\,, \quad T_u \mathcal{Y}_s \cong Z_s\ .$$

<u>Definition</u>. The triple $\{\mathcal{Y}, \{\mathcal{Y}_s\}, \alpha^{\mathcal{Y}}\}$ is called toroidal symplectic Hilbert scale (TSHS).

Let O_s be a domain in $\mathcal{Y}_s$ and $H \in C^1(O_s)$. Then the Hamiltonian equations corresponding to H have the form

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j} \quad (1 \le j \le n), \quad \dot{y} = J^Y \nabla_y H \tag{4.1}$$

The definitions of strong and weak solutions for equations (4.1) are analogous to those for equation (1.3).

The Poisson bracket of two functions H_1, H_2 with $H_j \in C^1(O_s)$, $\nabla_y H_j \in C(O_s; Y_{s_j})$ $(j = 1,2)$, $s_1 + s_2 \ge d_J$, takes the form

$$\{H_1,H_2\}(q,p,y) = \sum_{j=1}^{n} \left[-\frac{\partial H_1}{\partial q_j}\frac{\partial H_2}{\partial p_j} + \frac{\partial H_1}{\partial p_j}\frac{\partial H_2}{\partial q_j} \right] + \langle J^Y \nabla_y H_1, \nabla_y H_2 \rangle_Y .$$

The results of section 1–3 readily extend to canonical transformations and Hamiltonian equations in TSHS. We'll formulate analogs of Theorems 2.2, 2.4, 2.5 and 3.1 only.

Proposition 4.1. The statements of Theorem 2.2 remain true if anyone of the spaces X, Y is replaced by a toroidal symplectic Hilbert space (with equations of motion replaced accordingly).

Let O_s^1, O_s be domains in $\mathcal{Y}_s$, $O_s^1 \subset O_s$ and

$$\operatorname{dist}_{\mathcal{Y}_s}(O_s^1; \mathcal{Y}_s \backslash O_s) > \delta > 0 . \tag{4.2}$$

Let $H \in C^2(O_s)$ and $V_H = (\nabla_p H, -\nabla_q H, J^Y \nabla_y H)$ be corresponding Hamitonian vector–field. Let us suppose that $V_H \in C^1(O_s; Z_s)$ and

$$|V_H(q,p,y)| \le K \ \ \forall (q,p,y), \ \operatorname{Lip}(V_H : O_s \longrightarrow Z_s) \le K \tag{4.3}$$

Then the flow mappings $S^t : O_s^1 \longrightarrow O_s$ exist for $0 \le t \le T = \delta/K$ and every S^t is C^1–diffeomorphism on its image.

Proposition 4.2. For every $0 \le t \le \delta/K$ the mapping S^t is a canonical transformation.

Let conditions (4.2), (4.3) be fulfilled and $S^t \in C^1(O^1_S;O_S)$ be the flow of equation $\frac{d}{dt}(q,p,y) = \epsilon V_H(q,p,y)$.

Proposition 4.3. For every $G \in C^1(O_s)$ $G(S^t(\mathfrak{h})) = G(\mathfrak{h}) + t\epsilon\{H,G\}(\mathfrak{h}) + O(\epsilon t)^2$ $\forall \mathfrak{h} = (q,p,y) \in O^1_s$, $\forall\, 0 \leq t \leq T = \delta/K$.

Let in (4.1) $H = \frac{1}{2}\langle Ay,y\rangle_Y + H_0(p,q,y)$ and let the linear operator A be the same as in part 3. Let O^1_s, O^2_s, O_s be domains in $\mathcal{Y}_s$, $O^2_s \subset O^1_s \subset O_s$ and suppose inequality (4.2) is fulfilled. Let us suppose that $\mathrm{Lip}(V_{H_0} : O_s \longrightarrow Z_s) \leq K$.

Proposition 4.4. Let us suppose that relations (3.5), (3.6) are fulfilled and that every strong solution of (4.1) with initial point $\mathfrak{h}_0 = (q_0,p_0,y_0) \in O^2_s$ stays in domain O^1_s for $0 \leq t \leq T$. Then for $\mathfrak{h}_0 \in O^2_s \cap \mathcal{Y}_{s+d_1}$, $d_1 = d_A + d_J$, and for $0 \leq t \leq T$ there exists a unique strong solution of (4.1); for $\mathfrak{h}_0 \in O^2_s$, $0 \leq t \leq T$, there exists a unique weak solution of (4.1).

The proofs of Propositions 4.1–4.3 are the same as the proofs of the corresponding theorems.

5. A version of the former constructions

All construction of the sections 1–4 have natural analogs for the scales of Hilbert spaces depending on an integer index, i.e. for the scales $\{Z_s \mid s \in \mathbb{Z}\}$. SHS and TSHS with discrete scales $\{Z_s\}$ are sometimes more convenient to study Hamiltonian equations of form (3.2) with integer d_A, d_J. For example, KdV equation (1.5), (1.5′) $(d_J = 1,\ d_A = 2)$ and nonlinear Schrödinger equation (1.4) $(d_J = 0,\ d_A = 2)$.

All the statements of sections 1–4 have natural analogs for discrete scales. The proofs are the same.

Part 2.

Statement of the main theorem and its consequences

The following notations are used: for Hilbert spaces Y and Z the norms are denoted by $|\cdot|_Y$, $|\cdot|_Z$ and inner products by $<\cdot,\cdot>_Y$, $<\cdot,\cdot>_Z$; dist_Z – distance in the space Z.

The usual norms in $\mathbb{R}^n$ and $\mathbb{C}^n$ $(n \geq 1)$ are denoted $|\cdot|$. For metric spaces B_1, B_2, for a subset $Q_1 \subset B_1$ and a mapping $h : Q_1 \longrightarrow B_2$ we denote

$$\mathrm{Lip}\, h = \mathrm{Lip}(h : Q_1 \longrightarrow B_2) = \sup_{b_1 \neq b_2} \frac{\mathrm{dist}_{B_2}(h(b_1); h(b_2))}{\mathrm{dist}_{B_1}(b_1; b_2)}$$

If the space B_2 is a Banach one with a norm $|\cdot|_{B_2}$, we denote

$$|h|_{B_2}^{Q_1,\mathrm{Lip}} = \max\{\sup_{b \in Q_1} |h(b)|_{B_2}, \mathrm{Lip}\, h\}. \qquad (0.1)$$

Let B_1, B_2 be Banach spaces with norms $|\cdot|_{B_1}$, $|\cdot|_{B_2}$, let B_1^c, B_2^c be their complexifications, let V_j^c be an (open) domain in B_j^c $j = 1,2$. We denote by $\mathcal{A}^R(V_1^c; V_2^c)$ the set of Fréchet complex–analytical mappings from V_1^c to V_2^c which map $V_1^c \cap B_1$ into $V_2^c \cap B_2$. Let M be some metric space. We denote by $\mathcal{A}^R_M(V_1^c; V_2^c)$ a class of mappings $G : V_1^c \times M \longrightarrow V_2^c$ with the following properties:

i) $G(\cdot\, ; m) \in \mathcal{A}^R(V_1^c; V_2^c)$ $\forall m \in M$,

ii) the map $G(b\, ; \cdot) : M \longrightarrow V_2^c$ is Lipschitz $\forall b \in V_1^c$ and

$$|G|_{B_2}^{V_1^c\,;\,M} \equiv \sup_{b\in V_1^c} |G(b\,;\,\cdot)|_{B_2}^{M,\mathrm{Lip}} < \infty \tag{0.2}$$

(the norm in B_2^c is denoted by $|\cdot|_{B_2}$).

For domains $V_Y \subset Y$, $V_Z \subset Z$ we use standard notations $C^k(V_Y\,;\,V_Z)$ $(k \in \mathbb{Z}, k \geq 0)$ for the spaces of Fréchet–differentiable mappings $\varphi : V_Y \longrightarrow V_Z$ and notation $\varphi_*(\varphi^*)$ for tangent (cotangent) map.

For abstract sets $\mathfrak{A}$, $\mathcal{J}$, for a subset Θ of their product $\mathfrak{A} \times \mathcal{J}$ and for $I \in \mathcal{J}$ we denote by $\Theta[I]$ a subset of $\mathfrak{A}$ of the form

$$\Theta[I] = \{a \in \mathfrak{A} \,|\, (a,I) \in \Theta\} \tag{0.3}$$

In the notations of functions and mappings we sometimes omit a part of arguments; we denote by C, C_1, C_2 etc. different positive constants which arrive at estimates and denote by K, K_1 etc. constants in the assumptions of theorems.

1. Statement of the main theorem

Let $\{Z, \{Z_s \,|\, s \in \mathbb{R}\}, \alpha = \langle \bar{J}^Z dz, dz\rangle_Z\}$ be a symplectic Hilbert scale as it was defined in Part 1. It means that Z is a Hilbert space, $\{Z_s\}$ is a scale of Hilbert spaces with norms $\|\cdot\|_s$ and inner products $\langle\cdot,\cdot\rangle_s$, $Z_{s_1} \subset Z_{s_2}$ if $s_1 \geq s_2$, Z_s^* is dual to Z_s with respect to scalar product $\langle\cdot,\cdot\rangle_0$ and $Z_0 = Z$. $\bar{J}^Z$ is an isomorphism of scale $\{Z_s\}$ of order $-d_J \leq 0$, i.e. $\bar{J}^Z : Z_s \xrightarrow[\sim]{} Z_{s+d_J}$ $\forall s \in \mathbb{R}$. The operator $\bar{J}^Z : Z \longrightarrow Z_{d_J} \subset Z$ is supposed to be antisymmetric in Z. The operator $J^Z = -(\bar{J}^Z)^{-1}$ is an isomorphism of the scale $\{Z_s\}$ of order d_J, its restriction on Z is aniselfadjoint (and possibly unbounded). The 2–form $\alpha = \langle \bar{J}^Z dz, dz\rangle_Z$,

$$\langle \bar{J}^Z dz, dz\rangle_Z[z_1,z_2] \equiv \langle \bar{J}^Z z_1, z_2\rangle_Z,$$

is continous, antisymmetric and nondegenerate in any space Z_s, $s \geq 0$. Now every $Z_s (s \geq 0)$ is a linear symplectic space. See Part 1 for more details.

Let us suppose that the operator J^Z depends on a vector–parameter $a \in \mathfrak{A}$, $\mathfrak{A}$ is a bounded open domain in $\mathbf{R}^n$. So the symplectic form α depends on the parameter a , too. Let $A^Z(a)$ be a self–adjoint operator in Z depending on $a \in \mathfrak{A}$ and let $\forall a \in \mathfrak{A}$ $A^Z(a)$ defines an isomorphism of the scale $\{Z_s\}$ of order $d_A \geq 0$,

$$A^Z(a) : Z_s \xrightarrow[\sim]{} Z_{s-d_A} \qquad \forall s \in \mathbf{R} \tag{1.1}$$

Let us suppose that there exists a basis $\{\varphi_j^{\pm} \mid j \geq 1\}$ of the space Z with the following properties:

i) there exist positive numbers $\lambda_j^{(s)}$, $s \in \mathbf{R}$, $j \in \mathbf{N}$, such that $\lambda_j^{(-s)} = (\lambda_j^{(s)})^{-1}$ $\forall j,s$,

$$K^{-1} j^s \leq \lambda_j^{(s)} \leq K j^s \quad \forall j \geq 1, \forall s \in \mathbf{R}, \tag{1.2}$$

and $\{\varphi_j^{\pm} \lambda_j^{(-s)} \mid j \geq 1\}$ is a Hilbert basis of the space Z_s $\forall s \in \mathbf{R}$, i.e. $< \varphi_j^{\sigma_1} \lambda_j^{(-s)} , \varphi_k^{\sigma_2} \lambda_k^{(-s)} >_{Z_s} = \delta_{j,k} \, \delta_{\sigma_1,\sigma_2}$ $\forall j,k \in \mathbf{N}$, $\forall \sigma_1,\sigma_2 = \pm$;

$$\text{ii)} \quad J^Z(a) \, \varphi_j^{\pm} = \mp \lambda_j^J(a) \, \varphi_j^{\mp} \quad \forall j \geq 1, \forall a \tag{1.3}$$

$$A^Z(a) \, \varphi_j^{\pm} = \lambda_j^A(a) \, \varphi_j^{\pm} \; \forall j , \forall a , \tag{1.3$'$}$$

Here real numbers λ_j^J , λ_j^A are positive for j large enough:

$$\lambda_j^A(a) > 0 , \lambda_j^J(a) > 0 \; \forall a , \; \forall j \geq j_0 \tag{1.4}$$

Let us consider a hamiltonian

$$\mathscr{H}(z;a,\varepsilon) = \frac{1}{2} < A^Z(a)\, z, z >_Z + \varepsilon\, H(z;a,\varepsilon)$$

depending on a parameter $a \in \mathfrak{A}$ and a small parameter $\varepsilon \in [0,1]$. The corresponding Hamiltonian equation (with respect to 2–form $\alpha(a)$) has the form

$$\dot{z} = J^Z(a)\,(A^Z(a)z + \varepsilon\, \nabla\, H(z;a,\varepsilon))\,. \tag{1.5}$$

Here and in what follows, ∇ is the gradient in $z \in Z$ with respect to the scalar product $<\cdot,\cdot>_Z$. Equation (1.5) is a perturbation of linear Hamiltonian equation

$$\dot{z} = J^Z(a)A^Z(a)z \tag{1.6}$$

In view of conditions (1.3), (1.3′) the spectrum of operator $J^Z(a)\, A^Z(a)$ is purely imaginary,

$$\sigma(J^Z(a)A^Z(a)) = \{\pm\, i\, \lambda_j(a) \,|\, j \geq 1\}\ ,\ \lambda_j(a) = \lambda_j^{\,J}(a)\ \lambda_j^{\,A}(a)\,.$$

It is supposed that the functions

$$a \longmapsto \lambda_j^J(a)\ ,\ \ a \longmapsto \lambda_j^A(a)\ ,\ \ j \leq n\ \ ,$$

are C^2–smooth and for $j \leq n$, $\alpha \in \mathbb{Z}^n$, $|\alpha| \leq 2$

$$|\,\partial_a^\alpha\, \lambda_j^J(a)| + |\,\partial_a^\alpha\, \lambda_j^A(a)| \leq K_1 \tag{1.7}$$

and the mapping $a \longmapsto \omega = (\lambda_1,\dots,\lambda_n) \in \mathbb{R}^n$ is nondegenerate at some point $a_0 \in \mathfrak{A}$,

$$|\det(\partial\omega_j/\partial a_k)(a_0)| \geq K_0 > 0\ (j,k = 1,\dots,n)\,.$$

(1.8)

Let us denote

$$\lambda^A_{j0} = \lambda^A_j(a_0)\,,\ \lambda^J_{j0} = \lambda^J_j(a_0)\,,\ \lambda_{j0} = \lambda_j(a_0),\ \omega_0 = \omega(a_0) \tag{1.9}$$

Let us set $Z^0 \subset Z$ be a 2n–dimensional linear span of the vectors $\{\varphi^\pm_j \,|\, j \le n\}$. The space Z^0 is foliated into tori $T(I)$ which are invariant for linear equation (1.6),

$$T(I) = \{\sum_{j=1}^{n} \alpha^+_j \varphi^+_j + \alpha^-_j \varphi^-_j \,|\, {\alpha^+_j}^2 + {\alpha^-_j}^2 = 2\ I_j \ge 0\,,\ 1 \le j \le n\}\,. \tag{1.9′}$$

A torus $T(I)$ with $I_j > 0\ \ \forall j$ is n–dimensional, $T(I) \simeq \mathbf{T}^n$ and it is filled with quasiperiodic solutions of the form

$$\dot q = \omega(a)\,. \tag{1.10}$$

Here q is a coordinate on $T(I)$,

$$q_j = \mathrm{Arg}(\alpha^+_j + i\,\alpha^-_j)\,,\ j = 1,\dots,n\,.$$

Let

$$\textstyle\sum^0_I : \mathbf{T}^n \longrightarrow Z^0 \subset Z$$

be an imbedding identifying a point of $\mathbf{T}^n$ with a point of $T(I)$ having the same coordinates.

Let us consider a family of tori $\{T(I) \,|\, I \in \mathcal{J}\}$ where

$$\mathcal{J} \subset \{I \in \mathbb{R}^n \,|\, K_1^{-1} \le I_j \le K_1\,,\ j = 1,\dots,n\} \tag{1.11}$$

is some Borel set (possibly $\mathcal{J}$ consists of the only point, $\mathcal{J} = \{I_0\}$). Let us denote

$$\mathcal{T} = \cup\{T(I) \,|\, I \in \mathcal{J}\}\,.$$

Let us fix some number d ,

$$d_A/2 \le d \quad , \tag{1.12}$$

and let choose a domain O_d^c in the complexification of the space Z_d, $O_d^c \subset Z_d^c = Z_d \otimes_{\mathbb{R}} \mathbb{C}$, such that $\mathcal{T} \subset O_d^c$ and

$$\mathrm{dist}_{Z_d}(\mathcal{T};Z_d^c \backslash O_d^c) \ge K_1^{-1} . \tag{1.13}$$

We suppose that the function H may be extended to a function $H : O^c \times \mathfrak{A} \times [0,1] \longrightarrow \mathbb{C}$ which is complex–analytical on $z \in O_d^c$ and Lipschitz on $a \in \mathfrak{A}$, i.e. $H \in \mathcal{A}_{\mathfrak{A}}^{R}(O_d^c;\mathbb{C})\ \forall \epsilon$.

Theorem 1.1. Let the conditions mentioned above hold together with

1) (analyticity and quasilinearity): for some $d_H \in \mathbb{R}$ such that

$$d_H < d_A - 1 \, , \ d_J + d_H \le 0 \quad , \tag{1.14}$$

and for all $\epsilon \in [0,1]$

$$|H(\cdot;\cdot,\epsilon)|^{O_d^c;\mathfrak{A}} \le K_1, \ \ |\nabla H(\cdot;\cdot,\epsilon)|_{Z_{d-d_H}}^{O_d^c;\mathfrak{A}} \le K_1 \tag{1.15}$$

(see (0.2));

2) (spectral asymptotics):

$$d_1 \equiv d_A + d_J \ge 1$$

and there exists an asymptotic expansion for the frequencies λ_{j0} , $j \longrightarrow \infty$:

$$|\lambda_{j0} - K_2 j^{d_1} - K_2^1 j^{d_{1,1}} - \ldots - K_2^{r-1} j^{d_{1,r-1}}| \le K_1 j^{d_{1,r}} \quad , \tag{1.16}$$

here $K_2 > 0$, $r \ge 1$ and $d_1 > d_{1,1} > \ldots > d_{1,r}$, $d_1 - 1 > d_{1,r}$;

$$K_1^{-1} j^{d_A} \le |\lambda_j^A(a)| \le K_1 j^{d_A} \quad \forall j \ge 1\,, \tag{1.17}$$

$$K_1^{-1} j^{d_J} \le |\lambda_j^J(a)| \le K_1 j^{d_J} \quad \forall j \ge 1\,; \tag{1.18}$$

$$\mathrm{Lip}\,\lambda_j^A \le K_1 j^{d_A}\,,\ \mathrm{Lip}\,\lambda_j^J \le K_1 j^{d_J}\,,\ \mathrm{Lip}\,\lambda_j \le K_1 j^{d_{1,r}}\,; \tag{1.19}$$

Then there exist integers j_1, M_1 such that if condition

3) (nonresonance):

$$\begin{gathered} |\ell_1\lambda_{10}+\ell_2\lambda_{20}+\dots+\ell_{j_1}\lambda_{j_1 0}| \ge K_3 > 0 \\ \forall \ell \in \mathbb{Z}^{j_1} \quad |\ell| \le M_1\,,\ 1 \le |\ell_{n+1}| + \dots + |\ell_{j_1}| \le 2 \end{gathered} \tag{1.20}$$

is satisfied, then for sufficiently small $\epsilon > 0$ there exist $\delta_* > 0$ (sufficiently small and independent from ϵ), a Borel set $\Theta_\epsilon^{a_0}$ of vectors (a,I) ,

$$\Theta_\epsilon^{a_0} \subset \Theta^{a_0} \equiv \mathfrak{A}(a_0,\delta_*) \times \mathcal{J}\,\mathfrak{A}(a_0,\delta_*) \equiv \{a \in \mathfrak{A}\,|\ |a-a_0| < \delta_*\}, \tag{1.21}$$

and analytical embeddings

$$\Sigma^{\epsilon}_{(a,I)} : \mathbf{T}^n \longrightarrow Z_{d_c},\ (a,I) \in \Theta_\epsilon^{a_0},\ d_c = d+d_A-d_H-1 \tag{1.22}$$

with the following properties a)–d):

$$\text{a) mes}\ \Theta_\epsilon^{a_0}[I] \xrightarrow[\epsilon \to 0]{} \text{mes}\ \mathfrak{A}(a_0,\delta_*) \tag{1.23}$$

uniformly with respect to $I \in \mathcal{J}$ (see (0.3));

b) the mapping

$$\sum^{\epsilon} : \mathbb{T}^n \times \Theta_{\epsilon}^{a_0} \longrightarrow Z_{d_c} , \quad (q,a,I) \longmapsto \sum^{\epsilon}_{(a,I)}(q)$$

is Lipschitz and is close to the mapping $\sum^0 : (q;a,I) \longmapsto \sum^0_I(q)$

$$|\sum^{\epsilon} - \sum^{0}|^{\mathbb{T}^n \times \Theta_{\epsilon}^{a_0}, \mathrm{Lip}}_{Z_{d_c}} \leq C\,\epsilon ;$$

c) every torus $\sum^{\epsilon}_{(a,I)}(\mathbb{T}^n)$, $(a,I) \in \Theta_{\epsilon}^{a_0}$, is invariant for the equation (1.5) and is filled with weak in Z_d solutions of (1.5) of the form $z^{\epsilon}(t) = \sum^{\epsilon}_{(a,I)}(q+\omega' t)$, here $q \in \mathbb{T}^n$, $\omega' = \omega'(a,I,\epsilon) \in \mathbb{R}^n$ and $|\omega - \omega'| \leq C\epsilon$;

d) all Liapunov exponents of solutions $z^{\epsilon}(t)$ are equal to zero.

The theorem will be proved in a part 3 of the text.

An immediate consequence of the stated result is a strong averaging principle for nonresonant systems of the form (1.5):

Corollary 1.2. Under the assumptions of Theorem 1.1 for every $(a,I) \in \Theta_{\epsilon}^{a_0}$, $q \in \mathbb{T}^n$, and for all t a curve $t \longmapsto \sum^0_I(q+\omega' t)$ for ϵ small enough is $C\epsilon$–close to some weak solution of (1.5). Here ω' is an averaged frequency vector, $|\omega' - \omega| \leq C\epsilon$.

Remarks. 1) From the second estimate in (1.15) one can see that the order of nonlinear operator in equation (1.5) is equal to $d_J + d_H$. The order of the linear one is equal to $d_J + d_A$. So the condition (1.14) of theorem 1.1 indeed means the quasilinearity of equation (1.5) because the order of the linear term exceeds the order of the nonlinear one at least by one.

2) If $d_a \le d_c - d_1 = d - d_J - d_H - 1$ then the r.h.s. in (1.5) with $z(t) = z^\epsilon(t)$ belongs to $C([0,T];Z_{d_a})$. So $z^\epsilon \in C^1([0,T];Z_{d_a})$ is a strong in Z_{d_a} solution of (1.5).

3) The numbers j_1, M_1 in the assumption 3) of Theorem 1.1 depends on $K, K_0 - K_2, K_2^j, d_1, d_{1,j}, d_A, d_J, d_H, d, n$ and j_0 only. The maximal possible values of ϵ, δ_* and the rate of convergence in (1.23) depends on the same quantities and on K_3.

4) All the tori $\sum^{\epsilon}_{(a,I)}(\mathbf{T}^n)$ are isotropic, i.e. $\left[\sum^{\epsilon}_{(a,I)}\right]^* \alpha = 0 \ \forall (a,I) \in \Theta_\epsilon^{a_0}$.

5) The frequencies $\{\lambda_{j0}\}$ are ordered asymptotically only (see (1.16)). So for a space Z^0 one can choose any 2n–dimensional invariant subspace of the operator $J(a)A(a)$.

6) If instead of the condition $d_1 \ge 1$ a weaker condition $d_1 > 0$ takes place then the statements of Theorem 1.1 seems to be wrong in a general case. But the statement of Corollary 1.2 remains true for $0 \le t \le \epsilon^{-1}$ and $C\,\epsilon^\rho$ instead of C_ϵ with some $\rho > 0$ (see [K4]).

7) The form (1.16) of a spectral condition is not the most general one we need for our proof. For example for $d_1 > 1$ it is sufficient to demand that

$$C^{-1} j^{d_1} \le |\lambda_j| \le C\, j^{d_1}, \quad |\lambda_{j+1} - \lambda_j| \ge C_1 j^{d_1 - 1} \ \forall j. \tag{1.24}$$

See [K1] for (1.24) and [K2] for a possible form of a spectral condition with $d_1 = 1$. For the profound investigation of this problem see [DPRV].

8) The necessity of the quasilinearity condition $d_H \le d_A - 1$ results from (1.16) (or (1.24)). Indeed for arbitrary $d_H^1 > d_A - 1$ one can easily find perturbation H of the form

$$H = \tfrac{1}{2} < A^p(a,\epsilon)z,z >_Z, \quad A^p A^Z = A^Z A^p ,$$

such that the condition (1.15) is satisfied with $d_H = d_H^1$ and for the operator $A^Z(a) + A^p(a,\epsilon)$ condition $|\lambda_{j+1} - \lambda_j| \geq C_1 j^{d_1-1}$ is broken for some j large enough.

9) The analyticity of the tori $\sum^{\epsilon}_{(a,I)}(T^n)$ was observed by J. Pöschel [P1]. In the author's works [K1–K3] only smoothness of the tori was stated.

10) If all the numbers d, d_H, d_A, d_J are the integers then Theorem 1.1 may be stated in the framework of discrete symplectic Hilbert scales $\{Z,\{Z_s \mid s \in \mathbb{Z}\},\alpha\}$ (see Part 1, part 5).

2. Reformulation of Theorem 1.1

Let us suppose that the boundary $\partial\mathfrak{A}$ is smooth, the domain $\mathfrak{A}$ is connected, all eigen–values λ_j^J, λ_j^A are analytical functions of $a \in \mathfrak{A}$ and

$$\det\{\partial\omega_j/\partial a_k(a) \mid 1 \leq j,k \leq n\} \not\equiv 0 \ . \tag{2.1}$$

For some fixed point $a_0 \in \mathfrak{A}$ we define numbers λ^A_{j0}, λ^J_{j0}, λ_{j0} and a vector ω_0 as in (1.9).

Let us consider some resonance identical relation of a form

$$s \cdot \omega(a) + \Lambda(a) \equiv 0 \ , \ \Lambda = \ell_1 \lambda_{n+1}(a) + \ldots + \ell_p \lambda_{n+p}(a) \ , \tag{2.2}$$

$$s \in \mathbb{Z}^n \ , \ 1 \leq |\ell|_1 \equiv |\ell_1| + |\ell_2| + \ldots + |\ell_p| \leq 2 \ . \tag{2.3}$$

<u>Lemma 2.1</u>. Let all the functions λ^J_j, λ^A_j be analytical in $\mathfrak{A}$, $d_1 \geq 1$ and asymptotic (1.16) together with assumptions (1.19) and (2.1) take place. Then there exist numbers M_2, j_2 with the following property: if some identical relation of form (2.2), (2.3) holds and in (2.2) $\ell_p \neq 0$, then $|s| \leq M_2$, $p \leq j_2$.

Proof. By the assumption (2.1) there exist a point $a' \in \mathfrak{A}$ such that

$$C^{-1} \le |\omega_*(a')|_{\mathbb{R}^n,\mathbb{R}^n} \le C \tag{2.4}$$

for some C. Let ∇_a be a gradient with respect to the usual scalar product in $\mathbb{R}^n$. Then $\nabla_a(s \cdot \omega(a)) = \omega^*(a)s$ and by (2.4)

$$|\nabla_a(s \cdot \omega(a'))| \ge C^{-1}|s| \tag{2.5}$$

Till the end of the proof let us denote by $\lambda_n(p)$ zero function: $\lambda_n(p) \equiv 0$. Then every relation of the form (2.2), (2.3) with $l_p \ne 0$ may be rewritten in the following way:

$$s^1 \cdot w(a) \pm \lambda_m(a) + \lambda_{n+p}(a) \equiv 0 \tag{2.6}$$

Here $s^1 0 s$ or $s^1 = s/2$ and $n \ge m < n+p$. It follows from (2.6) that $|\lambda_{n+p}(a) \pm \lambda_m(a)| \le C_1|s|$ and by the assumption (1.16)

$$C_2|s| \ge (n+p)^{d_1} - (n+p)^{d_{1,r}} - m^{d_1} . \tag{2.7}$$

It follows from (2.5), (2.6) that $C_3^{-1}|s| \le |\nabla\lambda_m(a)|$. So by the assumption (1.19)

$$|s| \le C_3((n+p)^{d_{1,r}} + m^{d_{1,r}}) . \tag{2.8}$$

By (2.7), (2.8)

$$(n+p)^{d_1} - m^{d_1} \le C((n+p)^{d_{1,r}} + m^{d_{1,r}}) . \tag{2.9}$$

Since $n + p \ge m + 1$ and the function t^{d_1} is convex it follows that

$$\tfrac{1}{2}((n+p)^{d_1} - m^{d_1}) \ge C_1^{-1}(n+p)^{d_1-1}$$

and

$$\tfrac{1}{2}\left((n+p)^{d_1}-m^{d_1}\right)\geq C_1^{-1}(n+p-m)m^{d_1-1}\ .$$

By the three last estimates $(n+p)^{d_1-1}+(n+p-m)m^{d_1-1}\leq C_2((n+p)^{d_{1,r}}+m^{d_{1,r}})$. So $n+p\leq C^1$ (to prove the estimate one has to use the inequality $d_{1,r}<d_1-1$ and to consider the cases $d_{1,r}<0$ and $d_{1,r}\geq 0$ separately). Now by (2.8) $|s|\leq C^{11}$ and the lemma is proved with $j_2=C^1-n$, $M_2=c^{11}$. ■

<u>Theorem 2.2</u>. Let all eigen–values λ_j^A, λ_j^J be analytical functions of a parameter $a\in\mathfrak{A}$ and conditions (1.2), (1.3), (1.3′), (1.7), (1.11)–(1.13), (2.1) hold together with assumptions 1), 2) of Theorem 1.1. Then there exist integers j_1, M_1 such that if an assumption

$$\begin{gathered}s_1\lambda_1(a)+s_2\lambda_2(a)+\ldots+s_{j_1}\lambda_{j_1}(a)\not\equiv 0\\ \forall s\in\mathbb{Z}^{j_1},\ |s|\leq M_1,\ 1\leq|s_{n+1}|+\ldots+|s_{j_1}|\leq 2\end{gathered}\tag{2.10}$$

is satisfied then for every $\delta>0$ and for sufficiently small $\epsilon>0$ there exists a Borel subset $\mathfrak{A}_\epsilon^\delta\subset\mathfrak{A}$ and analytical embeddings

$$\sum\nolimits_a^\epsilon:\mathbf{T}^n\longrightarrow Z_{d_c},\ a\in\mathfrak{A}_\epsilon^\delta,\ d_c=d+d_A-d_H-1\ ,$$

with the following properties a)–c):

a) mes $\mathfrak{A}\setminus\mathfrak{A}_\epsilon^\delta<\delta$,
b) the mapping

$$\sum^{\epsilon} : \mathbb{T}^n \times \mathfrak{A}^{\delta}_{\epsilon} \longrightarrow Z_{d_c}, \ (q,a) \longmapsto \sum^{\epsilon}_{a}(q)$$

is Lipschitz and

$$|\sum^{\epsilon} - \sum^{0}|^{\mathbb{T}^n \times \mathfrak{A}^{\delta}_{\epsilon}, \mathrm{Lip}}_{Z_{d_c}} \leq C_{\delta}\epsilon \ , \tag{2.11}$$

c) every torus $\sum^{\epsilon}_{a}(\mathbb{T}^n)$, $a \in \mathfrak{A}^{\delta}_{\epsilon}$, is invariant for the equation (1.5) and is filled with weak in Z_d quasiperiodidc solutions of the form $\sum^{\epsilon}_{a}(q+\omega' t)$ and

$$|\omega - \omega'| \leq C^1_{\delta}\epsilon \ . \tag{2.12}$$

All Liapunov exponents of these solutions are equal to zero.

Proof. By the analyticity of functions λ^J_j, λ^A_j and by the assumption (2.1) the set $\{a \in \mathfrak{A} \mid |\det \partial\omega_j/\partial a_k| > 0\}$ is open and of full Lebesque measure in $\mathfrak{A}$ (i.e. a measure of its complement is equal to zero). Let define

$$\mathfrak{A}_t = \{a \in \mathfrak{A} \mid |\det \partial\omega_j/\partial a_k| \geq t \ , \ \mathrm{dist}(a, \partial\mathfrak{A}) \geq t\} \ .$$

Then $\mathfrak{A}_t$, $t \longrightarrow 0$, is an increasing sequence of compact sets and $\cup\{\mathfrak{A}_t \mid t > 0\}$ is of full measure. So there exists $K_0 = K_0(\delta) > 0$ such that

$$\mathrm{mes}\, \mathfrak{A} \backslash \mathfrak{A}_{K_0} < \gamma = \delta/4 \ . \tag{2.13}$$

Let us choose $j_1 \geq j_2 + n$, $M_1 \geq M_2 + 2$ with j_2, M_2 as in Lemma 2.1. Then by the assumption (2.10) and Lemma 2.1 there is no identical resonance relation of the form (2.2), (2.3). So every set

$$\{a \in \mathfrak{A} \mid s_1\lambda_1(a) + \ldots + s_p\lambda_p(a) \neq 0\} \tag{2.14}$$

with $1 \leq |s_{n+1}|+...+|s_p| \leq 2$, is of full measure in $\mathfrak{A}$.

Let us take a point $a_0 \in \mathfrak{A}_{K_0}$. For the remark 3, Theorem 1.1 is applicable with this choice of a_0 if condition (1.20) is fulfilled with some $j_1 = j_{1,0}$, $M_1 = M_{1,0}$ which does not depend on a_0. Let us consider a set

$$\overline{\mathfrak{A}}_t = \{a \in \overline{\mathfrak{A}} \mid |s_1\lambda_1(a) +...+ s_{j_{1,0}} \lambda_{j_{1,0}}(a)| \geq t$$

$$\forall\ |s| \leq M_{1,0},\ 1 \leq |s_{n+1}|+...+|s_{j_{1,0}}| \leq 2\} .$$

As the sets (2.14) are of full measure, then for some $t_0 > 0$

$$\text{mes}\ \mathfrak{A} \backslash \overline{\mathfrak{A}}_{t_0} < \gamma . \tag{2.15}$$

Theorem 1.1 is applicable with arbitrary $a_0 \in \mathfrak{A}_{K_0} \cap \mathfrak{A}_{t_0}$, $\mathcal{J} = \{I_0\}$ and a constant K_0 in the assumption (1.8) as in (2.13). In this situation for remark 3 δ_* does not depend on a_0 and the set $\Theta_\epsilon^{a_0}$ is of the form

$$\Theta_\epsilon^{a_0} = \mathfrak{A}_\epsilon^{a_0} \times \{I_0\},\ \mathfrak{A}_\epsilon^{a_0} \subset \mathfrak{A}(a_0,\delta_*) \tag{2.16}$$

The open balls $\mathfrak{A}(a_0,\delta_*)$, $a_0 \in \mathfrak{A}_{K_0} \cap \overline{\mathfrak{A}}_{t_0}$, form a covering of the compact set $\mathfrak{A}_{K_0} \cap \overline{\mathfrak{A}}_{t_0}$. Let us fix some finite subcovering, $\mathfrak{A}_{K_0} \cap \overline{\mathfrak{A}}_{t_0} \subset \bigcup_{j=1}^{M} D_j$, $D_j = \mathfrak{A}(a_{0j},\delta_*)$. By the statement a) of Theorem 1.1

$$\text{mes}\ D_j \backslash \mathfrak{A}_\epsilon^{a_{0j}} < \gamma/M \quad \forall\ j=1,...,M \ \text{ if }\ \epsilon < \epsilon(\delta) \tag{2.17}$$

For every $j = 1,...,M$ let us choose a closed subset $D_j^0 \subset D_j$ such that

$$\text{dist}(D_j^0,D_k^0) \geq \delta' > 0 \quad \forall\ j \neq k \tag{2.18}$$

and

$$\operatorname{mes}(\cup D_j \setminus \cup D_j^0) < \gamma \; . \tag{2.19}$$

Let us set

$$\mathfrak{A}_\epsilon^\delta = \bigcup_{j=1}^{M} (\mathfrak{A}_\epsilon^{a_{0j}} \cap D_j^0)$$

and define a map $\sum_a^\epsilon$, $a \in \mathfrak{A}_\epsilon^{a_{0j}} \cap D_j^0$, being equal to the map $\sum_{(a,I_0)}^\epsilon$ constructed by means of Theorem 1.1 for $a \in \mathfrak{A}_\epsilon^{a_{0j}}$. This definition is correct because every point $a \in \mathfrak{A}_\epsilon^\delta$ belongs to only one set $\mathfrak{A}_\epsilon^{a_{0j}} \cap D_j^0$.

The statements a)–c) of the theorem are true with this choice of $\mathfrak{A}_\epsilon^\delta$ and $\sum_a^\epsilon$. Indeed, the assertion a) results from the estimates (2.13), (2.15), (2.17), (2.19). The assertion c) is local with respect to the parameter a and it results from Theorem 1.1.

For to prove the assertion b) let us mention that by Theorem 1.1 for $\Delta\Sigma = \Sigma^\epsilon - \Sigma^0$ we have

$$|\Delta\Sigma|_{Z_{d_c}}^{T^n \times \mathfrak{A}_\epsilon^{a_{0j}}, Lip} \le C'_\delta \epsilon$$

(2.20)

If $b_j \in \mathfrak{A}_\epsilon^{a_{0j}} \cap D_j^0$ then for $j_1 \neq j_2$ by (2.18) $|b_{j_1} - b_{j_2}| \ge \delta'$. So by (2.20)

$$\|\Delta\Sigma(q;b_1) - \Delta\Sigma(q;b_2)\|_{d_c} \le 2\, C'_\delta\, \delta'^{-1} |b_1 - b_2|\, \epsilon \quad \forall b_1, b_2 \in \mathfrak{A}_\epsilon^\delta . \tag{2.21}$$

By (2.20), (2.21) we get an estimate (2.11) with $C_\delta = C'_\delta(1 + 2\delta'^{-1})$. ■

Corollary 2.3. If under the assumptions of Theorem 2.2 condition (2.10) is satisfied then for arbitrary $\rho \in (0,1)$ and for $0 < \epsilon << 1$ there exists a Borel subset $\mathfrak{A}_\epsilon \subset \mathfrak{A}$ and analytical embeddings $\sum_a^\epsilon : \mathbf{T}^n \longrightarrow Z_{d_c}$, $a \in \mathfrak{A}_\epsilon$, $d_c = d + d_A - d_H - 1$, with the following properties:

a) $\operatorname{mes} \mathfrak{A}\backslash\mathfrak{A}_\epsilon \longrightarrow 0 \ (\epsilon \longrightarrow 0)$,

b) $\left|\sum^\epsilon - \sum^0\right|_{Z_{d_c}}^{\mathbb{T}_n \times \mathfrak{A}_\epsilon, \mathrm{Lip}} \leq \epsilon^\rho$,

c) every torus $\sum_a^\epsilon(\mathbf{T}^n)$, $a \in \mathfrak{A}_\epsilon$, is invariant for the equation (1.5) and is filled with weak in Z_d solutions of the form $\sum_a^\epsilon(q+\omega' t)$, $|\omega'-\omega| \leq \epsilon^\rho$. All Liapunov exponents of these solutions are equal to zero.

Proof. By Theorem 2.2 with $\delta = 1/n$, $n = 1,2,\ldots$ for $\epsilon \leq \epsilon_n$, $\epsilon_n > 0$, we have the sets $\mathfrak{A}_\epsilon^{1/n}$ and maps $\sum_a^\epsilon$ satisfying the assertions a)–c) of the theorem. If $\epsilon_n << 1$ then

$$C_\delta \epsilon \leq \epsilon^\rho , \ C_\delta^1 \epsilon \leq \epsilon^\rho \quad \forall\, \epsilon < \epsilon_n \ . \tag{2.22}$$

We may assume that $\epsilon_n \searrow 0$ $(n \longrightarrow \infty)$ and set $\delta(\epsilon) = 1/n$ if $\epsilon \in (\epsilon(n+1), \epsilon(n)]$. Now the assertions of the corollary result from Theorem 2.2 and (2.22). ■

3. On systems with random spectrum

Theorems 1.1 and 2.2 may be applied to the Hamiltonian perturbations of random linear system for proving that quasiperiodic solutions of the unperturbed linear system survive in perturbed system with probability 1 (w.pr.1). Here we prove a simple theorem of this sort which deals with perturbations of a linear system equivalent to a countable set of free harmonic oscillators with random frequencies $\omega_1, \omega_2, \dots$.

The perturbations of a countable system of random oscillators by means of a short range interacting hamiltonians have been studied in a number of works (see [FSW], [P2] and bibliographies of these papers). For applications of our theorems we don't need short range interaction assumption. Instead of the last we use the assumption of linear or super–linear growth of frequencies $(\omega_j \sim Cj^d\ ,\ d \geq 1)$.

In the work [W1] non–linear perturbations of the string equation with a random potential were studied. The theorems of [W1] are similar to our results of this section.

Let Z be a Hilbert space with an orthobasis $\{\varphi_j^\pm \mid j \geq 1\}$; Z_s , $s \in \mathbb{R}$, be Hilbert spaces with the orthobasis $\{j^{-s}\varphi_j^\pm \mid j \geq 1\}$ and

$$J : Z \longrightarrow Z\ ,\ J(\varphi_j^\pm) = \mp\, \varphi_j^\mp \quad \forall j\ . \tag{3.1}$$

Then $J = (-J)^{-1} = J$ and the triple $\{Z, \{Z_s\}\ ,\ < Jdz, dz >_Z\}$ is a symplectic Hilbert scale with properties (1.3) being fulfilled with $\lambda_j^J \equiv 1$.

Let $(\mathcal{U}, \mathcal{F}, \mathcal{P})$ be some probability space and $A = A(\mu)$, $\mu \in \mathcal{U}$, be a random selfadjoint operator in Z such that $\forall\, j \in \mathbb{N}$

$$A(\varphi_j^\pm) = \lambda_j^A(\mu)\varphi_j^\pm,\ \lambda_j^A(\mu) = Kj^{d_A} + \Lambda_j(\mu)\ . \tag{3.2}$$

Here $K > 0$ and $\{\Lambda_j \mid j \geq 1\}$ are independent random variables (r.v.) such that every Λ_j is uniformly distributed on a segment

$$\Delta_j = [-\tfrac{1}{2}\, j^p, \tfrac{1}{2}\, j^p]\ . \tag{3.3}$$

Let O_d^c be a neighborhood of Z_d in $Z_d^c = Z_d \otimes_{\mathbb{R}} \mathbb{C}$ and $H \in \mathcal{A}^R(O_d^c;\mathbb{C})$. Let us consider a Hamiltonian equation with a hamiltonian $\mathcal{H} = \frac{1}{2} < Az, z >_Z + \,\epsilon\, H(z)$, i.e. the equation

$$\dot{z} = J(Az + \epsilon\, \nabla H(z)) \ . \tag{3.4}$$

<u>Theorem 3.1</u>. Let $d_A \geq 1$, $d \geq \frac{1}{2} d_A$, $H \in \mathscr{A}^R(O_d^c;\mathbb{C})$, $\nabla H \in \mathscr{A}^R(O_d^c;Z_{d-d_H})$ with some $d_H \leq 0$, $d_H < d_A - 1$ and H, ∇H are bounded on bounded subsets of O_d^c. Let in (3.3) $p < d_A - 1$. Let Q_d be an arbitrary open domain in Z_d. Then $\forall\, \epsilon > 0$ there exists a set $\mathscr{U}_\epsilon \in \mathscr{F}$ such that

a) $\mathscr{P}(\mathscr{U}_\epsilon) \longrightarrow 0 \ (\epsilon \longrightarrow 0)$,

b) if $\mu \notin \mathscr{U}_\epsilon$ then the equation (3.4) has a quasiperiodic solution passing through Q_d. All Liapunov exponents of this solution are equal to zero.

<u>Remark</u>. For a "not so small ϵ" one has $\mathscr{U}_\epsilon = \mathscr{U}$ and the statement of the theorem is empty.

<u>Corollary 3.2</u>. Let $\{\epsilon_j\}$ be a sequence such that $\epsilon_j \searrow 0$ for $j \longrightarrow \infty$. Then under the assumptions of Theorem 3.1 w.pr.1 equation (3.4) has a quasiperiodic solution through Q_d for ϵ equal to some ϵ_j.

<u>Proof</u>. Let us set $\mathscr{U}_0 = \cap\, \mathscr{U}_{\epsilon_j}$. For Theorem 3.1 $\mathscr{P}(\mathscr{U}_0) = 0$. If $\mu \notin \mathscr{U}_0$ then μ lies out of some $\mathscr{U}_{\epsilon_j}$ and equation (3.4) with $\epsilon = \epsilon_j$ has a quasiperiodic solution though Q_d. ∎

<u>Corollary 3.3</u>. Let $\epsilon_j \searrow 0$ $(j \longrightarrow \infty)$ and $QP(\epsilon_j)$ be the union of all quasiperiodic trajectories of equation (3.4) with $\epsilon = \epsilon_j$. Then w.pr.1 $\bigcup_j QP(\epsilon_j)$ is dense in Z_d.

<u>Proof</u>. As the Hilbert space Z_d is separable there exists a countable system $\{B_j \,|\, j \in \mathbb{N}\}$ of balls $B_j \subset Z_d$ such that any open set B_* contains some ball B_{j_*}. Now the statement results from Corollary 3.2 being applied to the balls B_j $(j = 1,2,\ldots)$, because the intersection of a countable system of sets of full measure is of full measure, again. ∎

<u>Proof of the theorem</u>. Let us take some point $z_0 \in Q_d$ of the form

$$z_0 = \sum_{j=1}^{n} z_{0j}^{\pm} \varphi_j^{\pm} , \quad n = n(z_0) < \infty ,$$

and denote

$$\mathrm{dist}_{Z_d}(z_0, Z_d \backslash Q_d) = \delta_0, \quad \delta_0 > 0 . \tag{3.5}$$

After the rearrangement of n first pairs of basis vectors $\{\varphi_j^{\pm} \mid j = 1,...,n\}$ and decreasing the number n (if there is need in it) one may suppose that $z_{0j}^{+2} + z_{0j}^{-2} > 0 \quad \forall\, j = 1,...,n$. So the point z_0 belongs to some torus $T(I_0) \simeq \mathbb{T}^n$, $I_0 \in \mathbb{R}_+^n$.

Let us denote $\omega_j = \lambda_j^A(\mu)$, $j \in \mathbb{N}$. By Corollary 2.3 for every fixed $\omega_\infty = (\omega_{n+1}, \omega_{n+2}, ...)$ there exists a set $\Omega_\epsilon = \Omega_\epsilon(\omega_\infty)$ of vectors $\omega = (\omega_1, ..., \omega_n)$, $\Omega_\epsilon \subset \Delta_1 \times \Delta_2 \times ... \times \Delta_n$, such that

$$\mathrm{mes}\, \Omega_\epsilon \le m(\epsilon) , \quad m(\epsilon) \xrightarrow[\epsilon \to 0]{} 0 , \tag{3.6}$$

(here mes is the normalized Lebesque measure) and for $\omega \notin \Omega_\epsilon$ the equation (3.4) has an invariant torus $T_\epsilon \simeq \mathbb{T}^n$ at a distance $< \epsilon^{1/2}$ from the torus $T(I)$. The torus T_ϵ is filled with the quasiperiodic solutions. So if $\epsilon < \delta_0^2$ then equation (3.4) has a quasiperiodic solution passing through Q_d provided ω lies out of Ω_ϵ .

In the present situation a n–dimensional parameter of the problem (3.4) is the frequency vector ω itself. So condition (1.8) is fulfilled with $K_0 = 1$. All the constants mentioned in the remark 3 (see part 1) are uniform with respect to ω_∞ . So the remark and an analysis of the proof of Theorem 2.2 (we omit the routine) show that in (3.6) the function $m(\epsilon)$ does not depend on ω_∞ . Let us set $\mathcal{U}_\epsilon = \{\mu \in \mathcal{U} \mid \omega \in \Omega_\epsilon(\omega_\infty)\}$. As the r.v. ω and ω_∞ are independent, then $\mathcal{P}(\mathcal{U}_\epsilon) \le m(\epsilon)$. So the theorem is proved because for $\mu \notin \mathcal{U}_\epsilon$ equation (3.4) has a quasiperiodic solution through Q_d . ∎

4. Nonlinear Schrödinger equation

A nonlinear Schrödinger equation

$$\dot{u} = i(-u_{xx} + V(x;a)u + \epsilon \frac{\partial}{\partial |u|^2} \chi(x,|u|^2;a)u)$$

(V and χ depend on a parameter $a \in \mathfrak{A}$) will be considered under the Dirichlet boundary condition

$$0 \leq x \leq \pi\ ,\ \ u(t,0) \equiv u(t,\pi) \equiv 0\ \ .$$

Let $Z = L_2(0,\pi;\mathbb{C})$ which is regarded as a real Hilbert space with inner product

$$< u,v >_Z = \mathrm{Re} \int u(x)\ \overline{v(x)}\ dx\ \ .$$

A differential operator $-\partial^2/\partial x^2$ with the Dirichlet boundary conditions defines a positive selfadjoint operator $\mathcal{A}_0$ in Z with the domain of definition $D(\mathcal{A}_0) = (\overset{\circ}{H}{}^1 \cap H^2)(0,\pi;\mathbb{C})$. For $s \geq 0$ let Z_s be the domain of definition of the operator $\mathcal{A}_0^{s/2}$. Every space Z_s is a closed subspace of $H^s(0,\pi;\mathbb{C})$ and the norm in Z_s is equivalent to the norm induced from $H^s(0,\pi;\mathbb{C})$. In particular

$$Z_1 = \overset{\circ}{H}{}^1(0,\pi;\mathbb{C})\ ,\ \ Z_2 = (\overset{\circ}{H}{}^1 \cap H^2)(0,\pi;\mathbb{C})\ \ . \tag{4.1}$$

Let Z_{-s} be the space adjoint to Z_s with respect to the scalar product in Z .

Let us consider the antiselfadjoint operator J ,

$$J : Z \longrightarrow Z\ ,\ \ u(x) \longmapsto i\ u(x)\ \ .$$

Then $J^2 = -E$, so $\bar{J} \equiv -(J^{-1}) = J$ and the triple $\{Z,\{Z_s\}\ ,\ < Jdz,dz >_Z\}$ is a symplectic Hilbert scale Part 1.

Let $\mathfrak{A}$ be a bounded domain in $\mathbb{R}^n$ and $V : [0,\pi] \times \mathfrak{A} \longrightarrow \mathbb{R}$ be a C^2–function. The differential operator $-\partial^2/\partial x^2 + V(x;a)$ defines a selfadjoint operator $\mathscr{A}(a)$ in Z with the domain of definition Z_2. $\mathscr{A}(a)$ depends on a parameter $a \in \mathfrak{A}$. For a full system of eigen–vectors of $\mathscr{A}(a)$ let us take $\{\varphi_j^{\pm}(a)\}$. Here $\varphi_j^{+}(a) = \varphi_j(x;a)$, $\varphi_j^{-}(a) = i\,\varphi_j(x;a)$ and $\{\varphi_j(x;a)\}$ is the full in $L_2(0,\pi;\mathbb{R})$ system of real eigen–functions of the operator $-\partial^2/\partial x^2 + V(x;a)$ under Dirichlet boundary conditions. So

$$\mathscr{A}(a)\,\varphi_j^{\pm}(a) = \lambda_j^A(a)\,\varphi_j^{\pm}(a) \quad \forall\, j \geq 1 .$$

Let us suppose that the numbers $\{\lambda_j^A(a)\}$ are asymptotically ordered, i.e. $\lambda_j^A(a) > \lambda_k^A(a)$ if $j > k$ and k is large enough.

Let $O^c \subset \mathbb{C}$ be a complex neighborhood of $\mathbb{R}$ and let χ may be extended to a function $\chi:\ [0,\pi] \times O^c \times \mathfrak{A} \longrightarrow \mathbb{C}$ such that

$$\begin{gathered} \chi(\cdot,\cdot;a) \in C^2([0,\pi]\times O^c;\mathbb{C}) \ \ \forall a \in \mathfrak{A}\ , \\ \frac{\partial^s}{\partial x^s}\,\chi(x,\cdot;\cdot) \in \mathscr{A}^R_{\mathfrak{A}}(O^c;\mathbb{C}) \ \ \forall s \leq 2,\ \forall x \in [0,\pi]\ . \end{gathered} \tag{4.2}$$

Let us set

$$H_0(u;a) = \frac{1}{2}\int_0^{\pi} \chi(|u(x)|^2,x;a)dx\ . \tag{4.3}$$

<u>Lemma 4.1</u>. For any $R > 0$ there exists a complex δ–neighbourhood $B_R^c \subset Z_2^c$ of a ball $\{u \in Z_2 \mid \|u\|_2 \leq R\}$ such that $\delta = \delta(R) > 0$ and $H_0 \in \mathscr{A}^R_{\mathfrak{A}}(B_R^c;\mathbb{C})$,

$$\nabla H_0(u;a) = \frac{\partial}{\partial |u|^2}\,\chi(x,|u|^2;a)u \tag{4.4}$$

and $\nabla H_0 \in \mathscr{A}^R_{\mathfrak{A}}(B_R^c;Z_2^c)$.

Proof. The existence of the set B_R^c and analyticity of H_0 result from Corollary A2 from the Appendix. Relation (4.4) results from the identities

$$< v(x), \nabla H_0(u(x);a) >_Z = d\, H_0(u;a)(v) =$$

$$= \frac{1}{2}\int_0^\pi \frac{\partial}{\partial |u|^2} \chi(x,|u|^2;a)(u\,\bar{v} + \bar{u}\,v)dx =$$

$$= < v(x), \frac{\partial}{\partial |u|^2} \chi(x,|u(x)|^2;a)u >_Z .$$

By Corollary A2 the map ∇H_0 belongs to $A^R_{\mathfrak{A}}(B_R^c; H^2(0,\pi;\mathbb{C}))$. But for $u(x) \in B_R^c$ the function $\nabla H_0(u(x);a)$ is equal to zero for $x = 0$, $x = \pi$. Therefore $\nabla H_0(u(x);a) \in B_R^c$ for $u \in B_R^c$ and the last statement is proved. ■

So the Hamiltonian equation with a hamiltonian $\frac{1}{2} < \mathcal{A}(a)u,u >_Z + H_0(u;a)$ has the form

$$\dot{u} = i(-u_{xx} + V(x;a)u + \epsilon \frac{\partial}{\partial |u|^2} \chi(x,|u|^2;a)u) \tag{4.5}$$

This equation is of the form (1.5) but operators $\mathcal{A}(a)$ don't commute one with another and the condition (1.3′) is not satisfied. For applying the theorem we at first must do linear transformations U_a of the phase space depending on a parameter a,

$$U_a : Z \longrightarrow Z, \quad z\varphi_j(x) \longmapsto z\varphi_j(x;a) \quad \forall z \in \mathbb{C} \quad \forall j .$$

Here $\varphi_j(x) = (2/\pi)^{1/2} \sin jx$.

Lemma 4.2. For every $a \in \mathfrak{A}$ the transformation U_a is canonical and orthogonal with respect to scalar product $< \cdot,\cdot >_Z$. For every $a, a_1, a_2 \in \mathfrak{A}$ and every $s \in [0,2]$

$$|\lambda_j(a_1) - \lambda_j(a_2)| \leq C|a_1 - a_2| \quad , \tag{4.6}$$

$$\|U_{a_1} - U_{a_2}\|_{s,s} \leq C_s |a_1 - a_2| \quad , \tag{4.7}$$

$$\|U_a\|_{s,s} \leq C'_s \quad . \tag{4.8}$$

Here $\|\cdot\|_{s,s} = |\cdot|_{Z_s,Z_s}$.

Proof. The orthogonality of U_a results from the fact that it maps one Hilbert basis of the space Z into another. The canonicity results from identities

$$< i\, U_a u, U_a v >_Z = < U_a iu, U_a v >_Z = < iu, v >_Z$$

(we use the orthogonality of U_a).

The estimate (4.6) for the spectrum of Sturm–Liouville problem is well–known [PT,Ma].

To prove (4.7) let us mention that for the eigen–functions $\varphi_j(x;a)$ one has the estimate

$$\|\varphi_j(a_1) - \varphi_j(a_2)\|_0 \leq C \sup_x |V(x;a_1) - V(x;a_2)|/j \leq C_1 |a_1 - a_2|/j \tag{4.9}$$

(see [PT,Ma]). As

$$\frac{\partial^2}{\partial x^2} \varphi_j(x;a) = (V(x;a) - \lambda_j(a))\varphi_j(x;a)$$

then we get from estimates (4.6), (4.9) that $\|\varphi_j(a_1) - \varphi_j(a_2)\|_2 \leq C_2 |a_1 - a_2|\, j$. From (4.9), the last inequalities and interpolation inequality [RS2] it follows that for all $s \in [0,2]$

$$\|\varphi_j(a_1) - \varphi_j(a_2)\|_s \leq C_2\, j^{s-1} |a_1 - a_2| \tag{4.10}$$

Let $u \in Z_s$ and

$$u = \sum (u_k^+ + iu_k^-)\varphi_k(x)\ ,\ \ \|u\|_s^2 = \sum |u_k^\pm|^2 k^{2s} < \infty$$

(one has to mention that $\|\varphi_j\|_s = j^s$) . Then

$$\|U_{a_1}u - U_{a_2}u\|_s = \|\sum_k (u_k^+ + iu_k^-)(U_{a_1} - U_{a_2})\varphi_k(x)\|_s \le$$

$$\le \sum_k |u_k^+ + iu_k^-|\ \|\varphi_k(x;a_1) - \varphi_k(x;a_2)\|_s \le$$

$$\le C_s'\Big(\sum_k |u_k^+ + iu_k^-|^2 k^{2s}\Big)^{1/2} |a_1 - a_2| \Big(\sum k^{-2}\Big)^{1/2} \le$$

$$\le C_s'' \ |a_1 - a_2|\ \|u\|_s$$

and we get the estimate (4.7). The estimate (4.8) results from the inequality $\|\varphi_j(x;a) - \varphi_j(x)\|_s \le C_j^1 j^{s-1}$ in the same way as (4.7) results from (4.10). ∎

For Lemma 4.2 and Theorem 2.2 from Part 1 the substitution

$$u = U_a\, v \tag{4.11}$$

transforms solutions of equation (4.5) to solutions of equation

$$\dot v = J(A(a)v + \epsilon\, \nabla H(v;a)) \tag{4.12}$$

with

$$A(a) = U_a^* \mathcal{A}_a U_a\ ,\ \ H = H_0(U_a v;a)\ .$$

So

$$\nabla\, H(v;a) = U_a^* \nabla H_0(U_a v;a)\ \ ,$$

$$A(a)\,\varphi_j^{\pm}(a) = \lambda_j^A(a)\,\varphi_j^{\pm} \quad \forall j\ .$$

Equation (4.5) with $\epsilon = 0$ is a linear Schrödinger equation

$$\dot u = i(-u_{xx}+V(x;a)u)\ ,\quad u(t) \in (\overset{\circ}{H}{}^1 \cap H^2)(0,\pi;\mathbb{C})\ \forall t$$

and it has invariant n–tori

$$T_a^n(I) = \left\{ \sum_{j=1}^{n} (\alpha_j^+ + i\alpha_j^-)\varphi_j(x;a) \mid {\alpha_j^+}^2 + {\alpha_j^-}^2 = 2\,I_j > 0 \right\}$$

Let a Borel set $\mathcal{J} \subset \mathbb{R}_+^n$ be as in (1.11) and $\mathcal{T}_a = \cup\,\{T_a^n(I) \mid I \in \mathcal{J}\}$. For every $a \in \mathfrak{A}$ $U_a^{-1}(T_a^n(I))$ is an invariant torus $T(I)$ of equation (4.12) with $\epsilon = 0$. It is of the form (1.9′), does not depend on a and

$$U_a^{-1}\,\mathcal{T}_a = \mathcal{T} = \cup\,\{T(I) \mid I \in \mathcal{J}\}\ .$$

Moreover, if R is large enough then one can choose a domain O_2^c

$$O_2^c \subset \bigcap_{a\in\mathfrak{A}} U_a^{-1}\,B_R^c \tag{4.13}$$

which satisfies relation (1.13) with $s = 2$.

Let us check that Theorem 1.1 with

$$\lambda_j^J \equiv 1\ ,\ d_J = 0\ ,\ d_A = 2\ ,\ d_H = 0\ ,\ d = 2\ ,\ d_c = 3$$

may be applied to the equation (4.12). Indeed, the validity of assumption 1) with $O_d^c = O_2^c$ (see (4.13)) results from (4.3), (4.7), (4.8); assumption 2) with $d_1 = d_A = 2$ results from (4.6) and from the well–known asymptotic $\lambda_j = j^2 + O(1)$ (see [PT],

[Ma]). So we get the following statement.

Theorem 4.3. Let a_0 be a point in $\mathfrak{A}$ such that

$$|\det(\partial \lambda_j^A(a_0)/\partial a_k \,|\, 1 \le j, k \le n)| \ge K_0 > 0 . \tag{4.14}$$

Then there exist integers j_1, M_1 such that if

$$\begin{gathered} \lambda_1^A(a_0)s_1 + \lambda_2^A(a_0)s_2 + \ldots + \lambda_{j_1}^A(a_0)s_{j_1} \neq 0 \\ \forall\, s \in \mathbb{Z}^{j_1},\ \ |s| \le M_1,\ 1 \le |s_{n+1}| + \ldots + |s_{j_1}| \le 2 , \end{gathered} \tag{4.15}$$

then for sufficiently small $\epsilon > 0$ there exists $\delta_* > 0$ (sufficiently small and independent of ϵ), a Borel subset

$$\Theta_\epsilon^{a_0} \subset \Theta^{a_0} = \mathfrak{A}(a_0,\delta_*) \times \mathcal{J}$$

and analytic embeddings

$$\sum\nolimits_{(a,I)}^{\epsilon} : \mathbf{T}^n \longrightarrow (\overset{\circ}{H}{}^1 \cap H^3)(0,\pi;\mathbb{C}),\ \ (a,I) \in \Theta_\epsilon^{a_0} ,$$

with the following properties:

a) mes $\Theta_\epsilon^{a_0}[I] \longrightarrow$ mes $\mathfrak{A}(a_0,\delta_*)$ $(\epsilon \longrightarrow 0)$ uniformly with respect to I ;

b) every torus $\sum_{(a,I)}^{\epsilon}(\mathbf{T}^n)$ is invariant for the equation (4.5) and is filled with weak in $(\overset{\circ}{H}{}^1 \cap H^2)$ solutions of (4.5) of the form $\sum_{(a,I)}^{\epsilon}(q_0 + \omega' t)$ (q_0 is an arbitrary point from $\mathbf{T}^n$, $\omega' = \omega'(a,I,\epsilon) \in \mathbb{R}^n$);

c) $\mathrm{dist}_{H^2}(\sum_{(a,I)}^{\epsilon}(\mathbf{T}^n) , T_a^n(I)) \le C\epsilon$ and $|\omega - \omega'| \le C\,\epsilon$;

d) the numbers j_1, M_1 depend on K_0, n and the C^2–norm of $V(x;a)$ only.

Let us discuss assumptions (4.14), (4.15) of the theorem. For this purpose let us consider a mapping $\mathcal{U}$ from the set $\mathfrak{A}$ into the space $C[0,\pi]$ of potentials $V(x)$,

$$\mathcal{U} : \mathfrak{A} \longrightarrow C[0,\pi] , \quad a \longmapsto V(\cdot;a) .$$

Every λ_j^A is an analytical function of potential $V(x)$. So condition (4.1) means that the point $\mathcal{U}(a_0)$ lies in the space $C[0,\pi]$ out of the zero set of some nontrivial analytical function. To discuss assumption (4.14) let us mention that

$$\frac{\partial \lambda_j}{\partial a_k}(a_0) = \int_0^\pi \varphi_j^2(x;a_0) \frac{\partial V(x;a_0)}{\partial a_k} dx$$

(see [PT, Ma]). It is proved in [PT] that the system of the functions $\{\varphi_1^2(\cdot;a),\ldots,\varphi_n^2(\cdot;a)\}$ is linearly independent for all a. So the function

$$(\xi_1(x),\ldots,\xi_n(x)) \longmapsto \det(\int \varphi_j^2(x;a)\xi_\ell(x)dx \mid 1\le j,\ \ell\le n)$$

turns out to be a non–trivial n–form on the space $C[0,\pi]$ and the condition (4.14) means that the restriction of this n–form on the image of the tangent mapping

$$\mathcal{U}_*(a_0) : \mathbb{R}^n \longrightarrow T_{\mathcal{U}(a_0)} C[0,\pi] \simeq C[0,\pi]$$

is nondegenerate, too.

So the assumption (4.14)+(4.15) is an non–degeneracy condition on the 1–jet of the map $\mathcal{U}$ at the point a_0.

Remark. Theorem 1.1 is applicable to study equation (4.5) under Neumann boundary conditions, or in the space of even periodic with respect to x functions,

$$x\in\mathbb{R},\ \ u(t,x) \equiv u(t,x+2\pi),\ \ u(t,x) \equiv u(t,-x) \ , \tag{4.16}$$

if the functions V and χ are even periodic and smooth on x. In the last situation one has to take for spaces $\{Z_s\}$ the spaces of even periodic Sobolev functions. In such a case relation (4.4) defines an analytical mapping from the space Z_s into itself for every $s \geq 1$. So Theorem 1.1 is applicable with arbitrary $d \geq 1$ and in the case of the problem (4.5), (4.16) one may prove the existence of arbitrary smooth invariant tori (i.e. being in the space $H^k(0,\pi;\mathbb{C})$ with k arbitrarily large) at a distance of order ϵ from $\mathcal{T}_a$.

5. Nonlinear string equation

The next application of our theorem will be to the equation of oscillation of a string with fixed ends in a nonlinear–elastic medium depending on a parameter $a \in \mathfrak{A}$:

$$\frac{\partial^2}{\partial t^2} w = (\partial^2/\partial x^2 - V(x;a))w - \epsilon \frac{\partial}{\partial w} \chi(x,w;a) \ ; \tag{5.1}$$

$$w = w(t,x), \ \ 0 \leq t \leq \pi; \ \ w(t,0) \equiv w(t,\pi) \equiv 0 \tag{5.2}$$

For writing down this non–linear boundary value problem in a form (1.5) we need some preliminary work. Let $V : [0,\pi] \times \mathfrak{A} \longrightarrow \mathbb{R}_+$ be a smooth function. The differential operator $-\partial^2/\partial x^2 + V(x;a)$ defines a positive selfadjoint operator in the space $L_2(0,\pi;\mathbb{R})$ with the domain of definition $(\overset{\circ}{H}{}^1 \cap H^2)(0,\pi;\mathbb{R})$. The space $\mathcal{Z} = D(\sqrt{\mathcal{A}_a})$ is the Sobolev space $\overset{\circ}{H}{}^1(0,\pi;\mathbb{R})$ with the scalar product

$$< u,v >^{(a)} = \int_0^\pi (u_x v_x + V(x;a)uv)dx$$

For $t \geq 0$ let $\mathcal{Z}_t$ be the space $\mathcal{Z}_t = D(\mathcal{A}_a^{(t+1)/2})$ with the norm $\|u\|_t^{(a)} = \| \mathcal{A}_a^{t/2} u \|_0^{(a)}$. In particular $\|u\|_0^{(a)} = (< u,u >^{(a)})^{1/2}$. For $-t \leq 0$ let $\mathcal{Z}_{-t}$ be a space dual to $\mathcal{Z}_t$ with respect to scalar product $< \cdot ; \cdot >^{(a)}$. Let us set $Z_t^{(a)} = \mathcal{Z}_t \times \mathcal{Z}_t$ with the natural norm and scalar product which will be denoted as

$<\cdot,\cdot>^{(a)}$, too. In the scale $\{Z_t^{(a)}\}$ let us consider an operator J_a of order $d_J = 1$,

$$J_a : Z_t^{(a)} \longrightarrow Z_{t-1}^{(a)},\ w = (w_1, w_2) \longmapsto (\mathcal{A}_a^{1/2} w_2, -\mathcal{A}_a^{1/2} w_1)$$

This operator is anti–selfadjoint in $Z^{(a)} = Z_0^{(a)}$ with the domain of definition $D(J_a) = Z_1^{(a)}$. The triple

$$\{Z^{(a)}, \{Z_s^{(a)} \mid s \in \mathbb{R}\}, < \bar{J}_a dw, dw >^{(a)}\}, \bar{J}_a = -(J_a)^{-1}$$

is a symplectic Hilbert scale Part 1 depending on a parameter a.

Let $\{\varphi_j^{(a)} \mid j \geq 1\}$ be a full in $L_2(0,\pi;\mathbb{R})$ system of eigen–functions of operator $-\partial^2/\partial x^2 + V(x;a)$,

$$(-\partial^2/\partial x^2 + V(x;a))\varphi_j^{(a)} = \lambda_j^{(a)}\varphi_j^{(a)},\quad |\varphi_j^{(a)}|_{L_2} = 1 ,$$

and $\lambda_j^{(a)} > \lambda_k^{(a)}$ for $j > k$ and k large enough. Let us set

$$\varphi_j^{+(a)} = (\varphi_j^{(a)}(x), 0)(\lambda_j^{(a)})^{-1/2},\ \varphi_j^{-(a)} = (0, \varphi_j^{(a)}(x))(\lambda_j^{(a)})^{-1/2} .$$

Then the set of functions $\{(\lambda_j^{(a)})^{-s/2}\varphi_j^{\pm(a)} \mid j \geq 1\}$ is a Hilbert basis of $Z_s^{(a)}\ \forall s \in \mathbb{R}$ and

$$J_a \varphi_j^{\pm(a)} = \mp(\lambda_j^{(a)})^{1/2}\varphi_j^{\mp(a)}\ \forall j \tag{5.3}$$

Let the function $\chi(x,w;a)$ and domain $O^c \subset \mathbb{C}$ be the same as in § 4 with the additional property

$$\chi(0,0;a) \equiv \chi(\pi,0;a) \equiv 0 ,$$

and

$$H^0(w_1,w_2) = \int_0^{\pi} \chi(x,w_1(x);a)dx$$

Lemma 5.1. For any $R > 0$ there exists a complex δ–neighborhood $B_R^c \subset Z_1^{(a)c} = Z_1^{(a)} \otimes \mathbb{C}$ of a ball $\{u \in Z_1^{(a)} \mid \|u\|_1^{(a)} \leq R\}$ such that $\delta = \delta(R) > 0$ and $H^0 \in \mathscr{A}_{\mathfrak{A}}^R(B_R^c;\mathbb{C})$,

$$\nabla^a H^0(u;a) = (\mathscr{A}_a^{-1} \frac{\partial}{\partial w_1} \chi(x,w_1(x);a),0) \ , \tag{5.4}$$

$\nabla^a H^0 \in \mathscr{A}_{\mathfrak{A}}^R (B_R^c;Z_3^{(a)c})$ (here ∇^a is the gradient with respect to scalar product $<\cdot;\cdot>^{(a)}$).

Proof of analyticity of H^0 and $\nabla^a H^0$ is the same as in Lemma 4.1. The formula for $\nabla^a H^0$ results from identities

$$< (v_1,v_2), \nabla^a H^0(w) >^{(a)} = dH^0(w)(v_1,v_2) =$$

$$= \int (\frac{\partial}{\partial w_1} \chi(x,w_1(x);a)v_1(x))dx =$$

$$= \int (\mathscr{A}_a^{-1} \frac{\partial}{\partial w_1} \chi(x,w_1(x);a) \ \mathscr{A}_a v_1(x))dx =$$

$$= < (v_1,v_2), (\mathscr{A}_a^{-1} \frac{\partial}{\partial w_1} \chi(x,w_1(x);a), 0) >^{(a)} \ . \qquad \blacksquare$$

The Hamiltonian equation corresponding to a hamiltonian $\mathscr{H}_a(w) = \frac{1}{2} \|w\|_0^2 + \epsilon H^0(w)$ in a symplectic structure with the 2–form $< J_a dw,dw >^{(a)}$ is the following:

$$(\dot{w}_1,\dot{w}_2) = \dot{w} = J_a \nabla \mathscr{H}_a =$$

$$= (\mathcal{A}_a^{1/2}\, w_2, - \mathcal{A}_a^{1/2}(w_1 + \mathcal{A}_a^{-1}\, \epsilon \frac{\partial}{\partial w_1} \chi(x,w_1(x);a)))$$

or

$$\dot{w}_1 = \mathcal{A}_a^{1/2} w_2$$
$$\dot{w}_2 = - \mathcal{A}_a^{1/2}(w_1 + \mathcal{A}_a^{-1}\, \epsilon \frac{\partial}{\partial w_1} \chi(x,w_1(x);a)) \ . \tag{5.5}$$

After elimination w_2 from these equations one gets an equation on w_1,

$$\frac{\partial^2}{\partial t^2} w_1 = (\frac{\partial^2}{\partial x^2} - V(x;a))w_1 - \epsilon \frac{\partial}{\partial w_1} \chi(x,w_1(x);a) \ . \tag{5.6}$$

So equation (5.5) is equivalent to equation (5.1). In what follows we shall discuss equation (5.1) in the form (5.5).

As in § 4 we have to do some linear transformation before to apply the theorem. So let $\{Z_s\}$ be the scale of spaces of the form $\{Z_s^{(a)}\}$ with $V(x;a) \equiv 0$, i.e. defined by operator $-\partial^2/\partial x^2$ instead of $-\partial^2/\partial x^2 + V(x;a)$. Let us set

$$\varphi_j^+(x) = (\sin jx,0)(2/\pi j)^{1/2}, \ \ \varphi_j^- = (0,\sin jx)(2/\pi j)^{1/2}$$

and denote an antiselfadjoint operator $J(a)$ of order 1 in the scale $\{Z_s\}$,

$$J(a)\, \varphi_j^{\pm} = \mp (\lambda_j^{(a)})^{1/2}\, \varphi_j^{\mp} \tag{5.7}$$

The triple $\{Z = Z_0, \{Z_s\}, \ < \bar{J}(a)dw,dw >_0\}$ is a symplectic Hilbert scale depending on a parameter a of the same sort as in § 1, i.e. with condition (1.3) being fulfilled.

For the relations (5.3), (5.7) the mapping

$$U_a : Z \longrightarrow Z^{(a)}, \ \ \varphi_j^{\pm} \longmapsto \varphi_j^{\pm(a)}$$

$$A^Z(a)\varphi_j^{\pm} = \mp \lambda_j^A(a)\varphi_j^{\pm},\ J^Z(a)\varphi_j^{\pm} = \mp \lambda_j^J(a)\varphi_j^{\mp},\ j = n+1,\dots,n+p \tag{6.1}$$

(see [K3]). In such a case

$$\begin{gathered}\sigma(J^Z(a)A^Z(a)) = \{\pm i\lambda_j(a) \mid j=1,\dots,n,\ \ n+p+1,\ \ n+p+2,\dots\} \cup \\ \cup \{\pm\lambda_j(a) \mid j=n+1,\dots,n+p\},\ \ \lambda_j(a) = \lambda_j^J(a)\lambda_j^A(a)\ .\end{gathered} \tag{6.2}$$

So the spectrum contains p pairs of real eigenvalues.

<u>Example</u>. Let us consider the problem (5.6), (5.2) without the limitation $V(x;a) \geq 0$ (and, so, with the possibility of negative points in the spectrum $\sigma(\mathscr{A}_a)$ of the operator $\mathscr{A}_a = -\partial^2/\partial x^2 + V(x;a))$. Let us suppose that $0 \notin \sigma(\mathscr{A}_a)$ and denote by $\mathscr{Z}_t$, $t \geq 0$, a space $\mathscr{Z}_t = D(|\mathscr{A}_a|^{(t+1)/2})$. Let us define spaces $\{Z_s^{(a)}\}$, $\{Z_s\}$ and operators J_a, $J(a)$ and function H^0 in the same way as in § 5 but with the operator $|\mathscr{A}_a|$ instead of $\mathscr{A}_a$ and $|\lambda_j^{(a)}|$ instead of $\lambda_j^{(a)}$, $j = 1,2,\dots$ (by definition, $|\mathscr{A}_a|\varphi_j^{(a)} = |\lambda_j^{(a)}|\varphi_j^{(a)}\ \forall\, j$).

Let us consider a hamiltonian

$$\mathscr{H}(w;a) = \frac{1}{2}\,\|w\|^2_{Z_0^{(a)}} + \frac{1}{2} < \operatorname{sgn} \mathscr{A}_a w^1, w^1 >_{Z_0^{(a)}} + \epsilon\, H^0(w;a); \tag{6.3}$$

$$\operatorname{sgn} \mathscr{A}_a\, \varphi_j^{(a)} = \operatorname{sgn} \lambda_j^{(a)}\, \varphi_j^{(a)}\ \forall\, j\ .$$

Corresponding Hamiltonian equations have the form

$$\begin{aligned}\dot w^1 &= |\mathscr{A}_a|^{1/2} w^2\ ,\\ \dot w^2 &= -|\mathscr{A}_a|^{1/2}((\operatorname{sgn} \mathscr{A}_a)w^1 + \epsilon\,|\mathscr{A}_a|^{-1}\frac{\partial}{\partial w^1}\,\chi)\end{aligned} \tag{6.4}$$

then for sufficiently small $\epsilon > 0$ there exists $\delta_* > 0$, Borel subset $\Theta_\epsilon^{a_0} \subset \Theta^{a_0} = \mathfrak{A}(a_0, \delta_*) \times \mathcal{J}$ and smooth embeddings $\sum_{(a,I)}^{\epsilon} : \mathbf{T}^n \longrightarrow Z_2$, $(a,I) \in \Theta_\epsilon^{a_0}$, with the following properties:

a) $\operatorname{mes} \Theta_\epsilon^{a_0}[I] \longrightarrow \operatorname{mes} \mathfrak{A}(a_0, \delta_*)$ $(\epsilon \longrightarrow 0)$ uniformly with respect to I;

b) every torus $\sum_{(a,I)}^{\epsilon}(\mathbf{T}^n)$ is invariant for the equation (5.5) and is filled with weak in Z_1 solutions.

The conditions (5.9), (5.10) are ones of non–degeneracy in the same sense as in § 4.

Example. Let $n=1$ and $V(x;a) \equiv a$ for $a \in \mathfrak{A} = (0,1)$. Then $\lambda_m^{(a)} = m^2 + a$ and assumption (5.9) is trivially fulfilled. Let us consider some resonance relation of the form (5.10):

$$\sum_{m=1}^{j\,1} (m^2+a)^{1/2}\, s_m \equiv 0 \ . \tag{5.11}$$

The l.h.s. in (5.11) defines an analytical function of argument $a \in [0,1]$. This function is not identically zero because after the analytical extension into complex plain with the cut along $(-\infty,0]$ the function has an essential singularity at the point $a = -m^2$ if $s_m \neq 0$. So the function has only a finite number of zeroes in $\mathfrak{A}$. Let a countable set $\mathfrak{A}_0$ be equal to the union of zeroes of all the functions corresponding to the relations of the form (5.11) with $j_1 \geq n+1$ and $1 \leq |s_{n+1}| + \dots + |s_{j_1}| \leq 2$. Then all the assumptions of Theorem 5.2 are fulfilled for $a_0 \in \mathfrak{A} \backslash \mathfrak{A}_0$.

6. On real points in the spectrum

The proof of the statements a)–c) of Theorem 1.1 is valid if some finite number of eigenvalues of the operator $J^Z(a)\, A^Z(a)$ are real, i.e. if for some finite number of indexes j instead of the conditions (1.3′), (1.3) one has defines a canonical transformation from Z_s to $Z_s^{(a)}$ for every $s \geq 0$. So U_a transforms solutions of equation

$$\dot{v} = J(a)(v + \epsilon \nabla H(v;a)), \quad H(v;a) = H^0(U_a(v);a) \tag{5.8}$$

into solutions of (5.5). As in § 4 one can prove the following statement.

Lemma 5.2. For any $R > 0$ there exists a complex δ–neighborhood $O_1^c \subset Z_1^c$ of a ball $\{u \in Z_1 \mid \|u\|_1 \leq R\}$ such that $\delta = \delta(R) > 0$ and $H \in \mathscr{A}_{\mathfrak{A}}^R(O_1^c;\mathbb{C})$, $\nabla H \in \mathscr{A}_{\mathfrak{A}}^R(O_1^c;Z_3^c)$.

Let us check that Theorem 1.1 with

$$\lambda_j^A = 1, \ \lambda_j^J = (\lambda_j^{(a)})^{1/2}, \ d_J = 1, \ d_A = 0, \ d_H = -2, \ d = 1, \ d_c = 2$$

is applicable to equation (5.8). Indeed, assumption 1) results from Lemma 5.2, assumption 2) with $r = 1$, $K_2 = 1$, is satisfied because $\lambda_j(a) = \lambda_j^J(a) = (\lambda_j^{(a)})^{1/2}$, where $\{\lambda_j^{(a)} = j^2 + C(a) + O(1)\}$ is a spectrum of the Sturm–Liouville problem. So we get the following statement on equation (5.5) (or (5.6)).

Theorem 5.2. Let a Borel set $\mathscr{J}$ be as in (1.11) and a_0 be a point in $\mathfrak{A}$ such that

$$\det(\partial \lambda_j^{(a_0)} / \partial a_k \mid 1 \leq j, \ k \leq n) \neq 0 \ . \tag{5.9}$$

Then there exist integers j_1, M_1 such that if

$$\begin{gathered}(\lambda_1^{(a_0)})^{1/2} s_1 + (\lambda_2^{(a_0)})^{1/2} s_2 + \ldots + (\lambda_{j_1}^{(a_0)})^{1/2} s_{j_1} \neq 0 \\ \forall s \in \mathbb{Z}^{j_1}, \ |s| \leq M_1, \ 1 \leq |s_{n+1}| + \ldots + |s_{j_1}| \leq 2\end{gathered} \tag{5.10}$$

and we get the equation (5.6) for the function $w^1(t,x)$, again. One can repeat the proofs of § 5 and to write down the equations (6.4) in a form (5.8) which satisfies the conditions of Theorem 1.1 with the condition (6.1) instead of (1.3′), (1.3). So the statements of Theorem 5.2 are true without the assumption $V(x;a) \geq 0$.

Appendix. On superposition operator in Sobolev spaces.

Let $O^c \subset \mathbb{C}$ be a complex neighborhood of the real line and $\chi : O^c \times [0,\pi] \longrightarrow \mathbb{C}^p$ be a C^k–function which is real for real arguments. Let $H^k(0,\pi;\mathbb{C}^p)$ $(H^k(0,\pi;\mathbb{R}^p))$ be the usual Sobolev space of $\mathbb{C}^p(\mathbb{R}^p)$–valued functions on $[0,\pi]$; B_R be a ball in $H^k(0,\pi;\mathbb{R}^p)$ of radius R centered at zero and $B_R^c(\delta)$ be a δ–neighborhood of B_R in $H^k(0,\pi;\mathbb{C}^p)$. As $H^k(0,\pi;\mathbb{R}^p) \subset C(0,\pi;\mathbb{R}^p)$ for $k \geq 1$ then for such a k $B_R^c(\delta) \subset C(0,\pi;O^c)$ if $\delta = \delta(R) << 1$. So the superposition operator

$$\phi : B_R^c(\delta) \longrightarrow C(0,\pi;\mathbb{C}^p), \quad u(x) \longmapsto \chi(u(x),x)$$

is well–defined.

Theorem A1. Let $k \in \mathbb{N}$, $\chi \in C^k(O^c \times [0,\pi])$ and $\forall s \leq k$

$$\frac{\partial^s}{\partial x^s}\chi(\cdot,x) \in \mathscr{A}^R(O^c;\mathbb{C}) \;\; \forall x \in [0,\pi], \;\; \left|\frac{\partial^s}{\partial x^s}\chi(u,x)\right| \leq K_* \quad \forall\, u \in O^c, \; x \in [0,\pi] .$$

Then $\phi \in \mathscr{A}^R(B_R^c(\delta); H^k(0,\pi;\mathbb{C}^p))$ and

$$|\phi(u)|_{H^k(0,\pi;\mathbb{C}^p)} \leq C(R)K_* \quad \forall\, u \in B_R^c(\delta) \tag{A1}$$

Proof. By taking a derivative of order $\ell \leq k$ from the function $\chi(u(x),x)$, $u \in B_R^c(\delta)$, one gets the estimate (A1). If $u \in B_R^c(\delta)$ and $v,w \in H^k(0,\pi;\mathbb{C}^p)$ then the function

$$\lambda \longmapsto \langle \phi(u+\lambda v),w \rangle_{H^k(0,\pi;\mathbb{C}^p)}$$

is complex–analytic in some neighborhood of the origin in $\mathbb{C}$; so the map ϕ is weakly analytic on $B^c_R(\delta)$.

As ϕ is bounded and weakly analytic then it is Fréchet–analytic (see [PT], Appendix A). ■

Let the function $\chi = \chi(u,x;a)$ depends on a parameter $a \in \mathfrak{A}$ in a Lipschitzian way, i.e. $\chi(\cdot,\cdot;a) \in C^k(O^c \times [0,\pi])$ $\forall a \in \mathfrak{A}$ and

$$\frac{\partial^s}{\partial x^s}\chi(\cdot,x;\cdot) \in \mathscr{A}^R(O^c;\mathbb{C}^p) \;\; \forall s \le k, \;\; \forall x \in [0,\pi] . \tag{A2}$$

Then by applying Theorem A1 to functions $\chi(u(x),x,a)$ and $\chi(u(x),x,a_1) - \chi(u(x),x,a_2)$ $(a,a_1,a_2 \in \mathfrak{A})$ we get

Corollary A2. If assumption (A2) holds for some $k \in \mathbb{N}$, then $\phi \in \mathscr{A}^R_{\mathfrak{A}}(B^c_R(\delta);\ H^k(0,\pi;\mathbb{C}^p))$. In particular, the function $u(x) \longmapsto \int \phi(u)(x)dx$ belongs to $\mathscr{A}^R_{\mathfrak{A}}(B^c_R(\delta);\mathbb{C}^p)$.

Part 3.

Proof of the main theorem

We use the notations from Part 1, 2 and some new ones. A list of them is given at the end of the paper. Sometimes we refer the reader to the formulas from Part 2. We write (2.2.3) for the formula (2.3) from Part 2 and so on. We use the abbreviations r.h.s. (l.h.s) for "right–hand–side" ("left–hand–side") and write ε_0 instead of ε . By " $\varepsilon_0 << 1$" ("$K >> 1$") we mean "positive ε_0 is small enough" (" K is large enough").

1. Preliminary transformations.

In a symplectic Hilbert scale $\{Z,\{Z_s \mid s \in \mathbb{R}\}\ ,\ \alpha(a) = \langle \bar{J}^Z(a)\, dz, dz \rangle_Z\}$ (see Part 1) we study a Hamiltonian equation with the hamiltonian

$$\mathscr{H}(z;a,\varepsilon_0) = \tfrac{1}{2} \langle A^Z(a)\, z,z \rangle_Z + \varepsilon_0\, H(z;a,\varepsilon_0)\ ;$$

i. e. the equation

$$\dot{z} = J^Z(a)(A^Z(a)z + \varepsilon_0 \nabla H(z;a,\varepsilon_0))\ ,\ \ J^Z(a) = -(\bar{J}^Z(a))^{-1}. \tag{1.1}$$

Here $a \in \mathfrak{A} \subset\subset \mathbb{R}^n$ is a n–dimensional parameter, $\varepsilon_0 \in [0,1]$ is a small parameter, H is an analytical function, $J^Z(a)$, $A^Z(a)$ are linear operators and for some Hilbert basis $\{\varphi_j^{\pm} \mid j \geq 1\}$ of the space Z the following relations take place:

$$J^Z(a)\, \varphi_j^{\pm} = \mp\, \lambda_j^J(a)\, \varphi_j^{\mp} \qquad \forall j\, ,\, \forall a\, , \tag{1.2}$$

$$A^Z(a)\, \varphi_j^{\pm} = \lambda_j^A(a)\, \varphi_j^{\pm} \qquad \forall j\, ,\, \forall a\, . \tag{1.3}$$

For the exact assumptions on equation (1.1) see Part 2.

1.1. Change of the symplectic structure.

The numbers $\{\lambda_j^J(a)\}$ are nonzero $\forall j,a$ and are positive for all j large enough (see (2.1.4), (2.1.18)). So after unessential exchange $\varphi_j^{\pm}$ on $\varphi_j^{\mp}$ for some finite number of indexes j we may suppose that $\lambda_j^J(a) > 0$ $\forall j,a$. Let us consider a linear operator L_a which maps $\varphi_j^{\pm}$ into $(\lambda_j^J(a))^{1/2}\varphi_j^{\pm}$, $j = 1, 2, \dots$. By assumption (2.1.18) this operator defines an isomorphism of the scale $\{Z_s\}$ of order $d_J/2$, $L_a : Z_s \xrightarrow[\sim]{} Z_{s-d_J/2}$ $\forall s$. It is selfadjoint in Z with the domain of definition $Z_{d_J/2}$. By Corollary 2.3 from Part 1 the mapping L_a^{-1} transforms solutions of the Hamiltonian equation (1.1) in the symplectic Hilbert scale $\{Z, \{Z_s\}, \alpha(a)\}$ into solutions of a Hamiltonian equation with the hamiltonian $\mathscr{H}_1(z;a,\varepsilon_0) = \frac{1}{2} < A_1(a)\, z,z >_Z + \varepsilon_0\, H_1(z;a,\varepsilon_0)$ in a symplectic Hilbert scale $\{Z, \{Z_s\}, \alpha_1(a) = < \bar{J}_1(a)\, dz, dz >_Z\}$. Here

$$\bar{J}_1(a) = L_a \bar{J}^Z(a)\, L_a,\ \ A_1(a) = L_a A^Z(a)\, L_a,\ \ H_1 = H(L_a\, z;a,\varepsilon_0)\,.$$

By the definition of the operator L_a and by (1.2) one has

$$\bar{J}_1(a)\,\varphi_j^{\pm} = \mp\,\varphi_j^{\mp} \qquad \forall j, \forall a\,. \tag{1.4}$$

So operator $\bar{J}_1(a)$ does not depend on the parameter a, $J_1 = -(\bar{J}_1)^{-1} = \bar{J}_1$, and a Hamiltonian equation with the hamiltonian $\mathscr{H}_1$ has the form

$$\dot{z} = J_1(A_1(a)\, z + \varepsilon_0 \nabla H_1(z;a,\varepsilon_0)) \tag{1.5}$$

and

$$\nabla H_1 = L_a \nabla H_1(L_a\, z;a,\varepsilon_0) \tag{1.6}$$

Let us denote by $\mathscr{L}(Z_s; Z_{s_1})$ the space of linear continuous operators from Z_s to Z_{s_1}

with the operator norm $\|\cdot\|_{s,\,s_1}$, and by $L:\mathfrak{A}\longrightarrow\mathscr{L}(Z_s;\,Z_{s-d_J/2})$ the mapping $a\mapsto L_a$.

Lemma 1.1. For every s

$$\operatorname{Lip}(L:\mathfrak{A}\longrightarrow\mathscr{L}(Z_s;\,Z_{s-d_J/2}))\le C\,. \tag{1.7}$$

For every s and every a

$$\|L_a\|_{s,s-d_J/2}+\|L_a^{-1}\|_{s,s+d_J/2}\le C_1\,. \tag{1.8}$$

Proof. An operator $L_{a_1}-L_{a_2}$ is diagonal in the basis $\{\varphi_j^{\pm}\}$ with eigenvalues $\Delta\ell_j^{\pm}=(\lambda_j^J(a_1))^{1/2}-(\lambda_j^J(a_2))^{1/2}$. By the assumptions (2.1.18), (2.2.19)

$$|\Delta\ell_j^{\pm}|\le\frac{K_1|a_1-a_2|\;j^{d_J}}{2\,\min\,(\lambda_j^J(a_1)^{1/2},\;\lambda_j^J(a_2)^{1/2})}\le\frac{K_1^{3/2}}{2}\,|a_1-a_2|\;j^{d_J/2}$$

and inequality (1.7) results from (2.1.2). Inequaltiy (1.8) results from (2.1.2) and (2.1.18). ■

Let us denote $d'=d+\frac{1}{2}d_J$ and $T_a(I)=L_a^{-1}T(I)$, $\mathscr{T}_a=\cup\,\{T_a(I)\,|\,I\in\mathscr{I}\}$, $O^c_{d',a}=L_a^{-1}\,O_d^c$. Then

$$T_a(I)=\{\sum_{j=1}^{n}\alpha_j^{\pm}\,\varphi_j^{\pm}\,|\,{\alpha_j^{+}}^2+{\alpha_j^{-}}^2=2\,I_j^a,\ j=1,\dots,n\},\qquad I_j^a=\frac{I_j}{\lambda_j^J(a)}$$

and by the assumption (2.1.13) and estimate (1.8)

$$\operatorname{dist}_Z(\mathscr{T}_a\,;\,Z^c_{d'}\setminus O^c_{d',a})>\gamma>0\qquad\qquad\forall a\in\mathfrak{A}\,. \tag{1.9}$$

By the analyticity assumption (2.1.15), Lemma 1.1 and identity (1.6) one can see that the mappings

$$H_1 : O^c_{d',a} \times \mathfrak{A} \times [0,1] \longrightarrow \mathbb{C}$$
$$\nabla H_1 : O^c_{d',a} \times \mathfrak{A} \times [0,1] \longrightarrow Z^c_{d'-d_H-d_J} \tag{1.10}$$

are complex–analytical with respect to the first variable and Lipschitz with respect to the second one unifomly with respect to $\varepsilon_0 \in [0,1]$.
The operator $A_1(a)$ is an isomorphism of the scale $\{Z_s\}$ of the order $d_1 = d_A + d_J$ and

$$A_1(a)\,\varphi_j^{\pm} = \lambda_j(a)\,\varphi_j^{\pm} \qquad \forall j, \forall a . \tag{1.11}$$

Equation (1.5) satisfies conditions 2) of the theorem with $d_A' = d_A + d_J$, $d_J' = 0$, $d_H' = d_H + d_J$. So it is sufficient to prove the theorem in a case $d_J = 0$.

1.2. A change of parameter.

The statements of the theorem are local with respect to the parameter a . So one may replace the set $\mathfrak{A}$ of parameters a by arbitrary δ_a–neighbourhood $\mathfrak{A}(a_0,\delta_a)$ of the point a_0 in $\mathfrak{A}$. If positive δ_a is sufficiently small, then by the assumptions (2.1.7), (2.1.8) the mapping

$$\omega : \mathfrak{A}(a_0,\delta_a) \longrightarrow \mathbb{R}^n, \qquad a \mapsto \omega(a) = (\lambda_1(a),\dots,\lambda_n(a))$$

is a C^1–differomorphism on some neighbourhood Ω_0 of the point $\omega_0 = \omega_0(a_0)$ and

$$\mathrm{Lip}\,\omega + \mathrm{Lip}\,\omega^{-1} \le K^1, \tag{1.12}$$

$$\mathrm{diam}\,\Omega_0 \le K^1 \delta_{a,} \tag{1.13}$$

$$K^{-1}\delta_a^n \le \text{mes}\, \Omega_0 \le K\, \delta_a{}^n . \tag{1.14}$$

So Lipschitz dependence on the parameter $a \in \mathfrak{A}\,(a_0,\delta_a)$ is equivalent to Lipschitz dependence on the parameter $\omega \in \Omega_0$.

1.3. A transition to angle variables

In what follows we use the notation $O(Q,\delta,B)$ for the δ–neighbourhood of a subset Q of a metric space B ; for a Banach space Z we write $O(\delta,Z)$ instead of $O(0,\delta,Z)$.

Let us set $Z^0 \subset Z$ be equal to the 2n–dimensional linear span of the vectors $\{\varphi_j^{\pm} \mid j \le n\}$ and $Y_s \subset Z_s$, $s \in \mathbb{R}$, be equal to the closure in Z_s of the linear span of the vectors $\{\varphi_j^{\pm} \mid j \ge n+1\}$ and $Y = Y_0$. For a vector from Z^0 let $\{\chi_j^{\pm} \mid 1 \le j \le n\}$ be its coefficients for the basis $\{\varphi_j^{\pm} \mid j \le n\}$. In some small enough neighbourhood of a torus $T_a(I)$ let us change coordinates $\{\chi_j^{\pm}\}$ to (q,ξ), $q \in \mathbf{T}^n$, $\xi \in O(3\delta_0,\mathbb{R}^n)$ $(\delta_0 << 1)$:

$$q_j = \operatorname{Arg}\,(\chi_j^- + i\chi_j^+), \qquad\qquad \xi_j = \tfrac{1}{2}\left[{\chi_j^+}^2 + {\chi_j^-}^2\right] - I_j^a . \tag{1.15}$$

Let us consider toroidal spaces $\mathcal{Y}_s = \mathbf{T}^n \times \mathbb{R}^n \times Y_s$, $s \in \mathbb{R}$, with a natural metric dist_s and tangent spaces $T_u \mathcal{Y}_s \cong \mathbb{R}^n \times \mathbb{R}^n \times Y_s = E_s$, $u \in \mathcal{Y}_s$. Let J be a restriction on Y_s of the operator J_1, i.e. $J\,\varphi_j^{\pm} = \mp\,\varphi_j^{\mp}$ $\forall j \ge n+1$ (see (1.4)); let

$$J^T : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^n,\ (\delta q,\delta\xi) \mapsto (\delta\xi, -\delta q)\ ,$$

and

$$J^{\mathcal{Y}} = J^T \times J^Y : E_s = (\mathbb{R}^n \times \mathbb{R}^n) \times Y_s \longrightarrow E_s\ .$$

Let us introduce in $\mathcal{Y}_s$, $s \ge 0$, a symplectic structure with the help of the 2–form $\alpha^{\mathcal{Y}} = \langle J^{\mathcal{Y}} d\,\eta, d\,\eta \rangle_E$. The triple $\{\mathcal{Y}_0, \{\mathcal{Y}_s\}, \alpha^{\mathcal{Y}}\}$ is a toroidal symplectic Hilbert scale. See Part 1, § 4, for details.

For the fixed $s \in \mathbb{R}$, $I \in \mathcal{J}$, $\omega \in \Omega_0$ and $\delta_0 << 1$ let us consider a map

$$L : \mathbb{T}^n \times O(3\delta_0, \mathbb{R}^n) \times Y_s \longrightarrow Z_s, \quad (q,\xi,y) \mapsto \sum_{j=1}^{n} \chi_j^{\pm}\varphi_j^{\pm} + y$$

(see (1.15)). It defines a complex–analytical diffeomorphism of the domain

$$Q^c(s) = O(\mathbb{T}^n \times \{0\} \times \{0\}, 3\delta_0, \mathcal{Y}_s^{\ c}) \subset \mathcal{Y}_s^{\ c} = (\mathbb{C}^n/\, 2\pi\,\mathbb{Z}^n) \times \mathbb{C}^n \times Y_s^{\ c} \tag{1.16}$$

on a complex neighbourhood of $T_a(I)$ in Z_s. This diffeomorphism is Lipschitz in I and in ω (via the dependence $a = a(\omega)$), i.e.

$$L \in \mathcal{A}^R_{\Omega_0} \times \mathcal{J}(Q^c(s); Z_s^{\ c}) \tag{1.17}$$

for all s.

The subspaces $Z^0 \subset Z$, $Y \subset Z$ are skew–ortogonal with respect to the 2–form $\alpha_1 = < \bar{J}_1\, dz, dz >_Z$. A restriction of α_1 on Z^0 is of the form $d\chi^- \wedge d\chi^+$ and, so, it is equal to $d\xi \wedge dq$ (see [A2]). A restriction of the form $\alpha^{\mathcal{Y}}$ on Z^0 is $d\xi \wedge dq$, too. Hence

$$L^*\alpha_1 = L^*(d\chi^- \wedge d\chi^+ + < J^1 dy, dy >_Z) = d\xi \wedge dq + < J\, dy, dy >_Y = \alpha^{\mathcal{Y}}$$

and the map L is canonical. So the equation (1.1) in the coordinates (q,ξ,y) is Hamiltonian with the hamiltonian

$$\mathcal{H}_0(q,\xi,y;\omega,I,\varepsilon_0) = \text{const} + \sum_{j=1}^{n} \xi_j\omega_j + \tfrac{1}{2} < A(\omega)\, y,y > + \varepsilon_0 H^0(q,\xi,y;\omega,I,\varepsilon_0) \tag{1.18}$$

(see Part 1, Proposition 4.1). Here we use the identity

$$\tfrac{1}{2} < A_1(\omega)\, z_0, z_0 >_Z = \tfrac{1}{2}\sum_{j=1}^{n} \lambda_j(\omega)\left[\chi_j^{+2} + \chi_j^{-2}\right] = \sum \omega_j \xi_j$$

$$\forall z_0 = \sum_{j=1}^{n} \chi_j^+ \varphi_j^+ + \chi_j^- \varphi_j^- \in Z^0 ,$$

denote by $A(\omega)$ a restriction of the operator $A_1(\omega)$ on the space Y and denote by $< \cdot , \cdot > = < \cdot , \cdot >_Y$ the scalar product in Y induced from Z. The Hamiltonian equations have the form

$$\dot q_j = \omega_j + \varepsilon_0 \frac{\partial}{\partial \xi_j} H^0, \qquad \dot\xi_j = -\varepsilon_0 \frac{\partial}{\partial q_j} H^0$$
$$\dot y = J(A(\omega)y + \varepsilon_0 \nabla_y H^0) . \tag{1.29}$$

Let us set $\Theta_0 = \Omega_0 \times \mathcal{J}$. A Borel set $\mathcal{J}$ is the same as in (2.1.11), i.e.

$$\mathcal{J} \subset \{I \in \mathbb{R}^n \mid K^{-1} \leq I_j \leq K \qquad \forall j = 1,\dots,n\} . \tag{1.20}$$

It results from (1.9), (1.17) and from the analyticity of the mappings (1.10) that $\forall \varepsilon_0 \in [0,1]$

$$| H^0(\cdot ; \cdot , \varepsilon_0) |^{Q^c(d'),\Theta_0} \leq K_1' , \; \| \nabla_y H^0(\cdot ; \cdot , \varepsilon_0)\|^{Q^c(d'),\Theta_0}_{d^{\gamma - d_H - d_J}} \leq K_1'$$

if in (1.16) $\delta_0 << 1$.

The operator $A(\omega)$ has the double spectrum $\{\lambda_j(\omega) \mid j = n+1, n+2,\dots\}$ and the operator $JA(\omega)$ has the spectrum $\{\pm i\lambda_j(\omega) \mid j \geq n+1\}$. Let us shift the numeration:

$$\lambda_j(\omega) := \lambda_{j+n}(\omega), \qquad \varphi_j^{\pm} := \varphi_{j+n}^{\pm}, \qquad \lambda_j^{(s)} := \lambda_{j+n}^{(s)}$$

and redenote

$$d_H := d_H + d_J, \qquad d := d' = d + \tfrac{1}{2} d_J .$$

Then by the condition (2.1.2) the set of vectors $\{\varphi_j^{\pm}\, \lambda_j^{(-s)} \mid j \geq 1\}$ is a Hilbert basis of Y_s and for some new K

$$K^{-1} j^s \leq \lambda_j^{(s)} \leq K j^s, \quad \lambda_j^{(-s)} = (\lambda_j^{(s)})^{-1} \quad \forall j \geq 1, \forall s \in \mathbb{R} \tag{1.21}$$

By this condition the scale $\{Y_s\}$ is interpolational. See below appendix A.

For the shifted sequence $\{\lambda_{j0} = \lambda_j(\omega_0)\}$ relation (2.1.16) takes place with the same d_1, some new r, $K_2^{\,1},\ldots,K_2^{\,r-1}$, $d_{1,1},\ldots,d_{1,r-1}$ and some new K^1. For all $j \geq 1$, $\omega \in \Omega_0$

$$A(\omega)\, \varphi_j^{\pm} = \lambda_j(\omega)\, \varphi_j^{\pm}, \ J\, \varphi_j^{\pm} = \mp\, \varphi_j^{\mp} \tag{1.22}$$

and

$$\lambda_j(\omega) > 0 \qquad \forall j \geq j_0 . \tag{1.24}$$

Theorem 1.1 from Part 2 may be reformulated for equations (1.19). Here we formulate some more general result. To do it, we suppose that the operator $A(\omega)$ depends on ε_0, $A = A(\omega,\varepsilon_0)$; so $\lambda_j = \lambda_j(\omega,\varepsilon_0)$, and $\lambda_{j0} = \lambda_{j0}(\varepsilon_0)$. We suppose that

$$\varepsilon_0\, H^0 = \varepsilon_0 H_0(q,\xi,y;\theta,\varepsilon_0) + H^3(q,\xi,y;\theta,\varepsilon_0) ,$$

the functions H_0, H^3 may be continued to complex–analytic functions on a domain $Q^c(d)$, $d \geq \frac{1}{2} d_1$. It is supposed that $\forall \varepsilon_0 \in [0,1]$

$$| H_0(\,\cdot\,;\,\cdot\,;\varepsilon_0) |^{Q^c(d),\Theta_0} + \| \nabla_y H_0(\,\cdot\,;\,\cdot\,;\varepsilon_0) \|^{Q^c(d),\Theta_0}_{d-d_H} \leq K_1 \tag{1.24}$$

and $\forall \mathfrak{h} = (q,\xi,y) \in Q^c(d)$

$$|H^3(\mathfrak{h};\cdot,\varepsilon_0)|^{\Theta_0,\mathrm{Lip}} \le K_1(|\xi|^2 + |\xi|\ \|y\|_d + \|y\|_d{}^3)\,, \tag{1.25}$$

$$|\nabla_\xi H^3(\mathfrak{h};\cdot,\varepsilon_0)|^{\Theta_0,\mathrm{Lip}} \le K_1(|\xi| + \|y\|_d)\,, \tag{1.26}$$

$$\|\nabla_y H^3(\mathfrak{h};\cdot,\varepsilon_0)\|^{\Theta_0,\mathrm{Lip}}_{d-d_H} \le K_1(|\xi| + \|y\|_d{}^2)\,. \tag{1.27}$$

Here

$$d_H \le 0\ ,\ d_H < d_1 - 1\,. \tag{1.28}$$

In the terms of the decomposition $\varepsilon_0 H^0 = \varepsilon_0 H_0 + H^3$ the results may be formulated in more exact way, important for some applications. Now the equations (1.19) take the form

$$\dot q_j = w_j + \epsilon_0 \frac{\partial}{\partial \xi_j} H_0 + \frac{\partial}{\partial \xi_j} H^3\ ,$$

$$\dot \xi_j = \epsilon_0 \frac{\partial}{\partial q_j} H_0 - \frac{\partial}{\partial q_j} H^3\ , \tag{1.29}$$

$$\dot y = J(A(w)y + \epsilon_0 \nabla_y H_0 + \nabla_y H^3)\ .$$

<u>Theorem 1.1</u>. Let the conditions (1.20) – (1.25) hold together with

1) $d_1 \ge 1$ and

$$|\lambda_{j0} - K_2 j^{d_1} - K_2{}^1 j^{d_{1,1}} - \ldots - K_2{}^{r-1} j^{d_{1,r-1}}| \le K_1 j^{d_{1,r}} \tag{1.30}$$

for some $K_2 = K_2(\varepsilon_0) > 0$, $r \ge 1$, $K_2{}^j = K_2{}^j(\varepsilon_0) \in \mathbb{R}$ $(j = 1, \ldots, r-1)$ and for $d_1 > d_{1,1} > \ldots > d_{1,r-1} > d_{1,r}$ such that

$$d_1 - 1 > d_{1,r}, \qquad K_3^{-1} \le K_2(\varepsilon_0) \le K_3, \qquad |K_2{}^j(\varepsilon_0)| \le K_3 \qquad \forall j, \forall \varepsilon_0\ ;$$

moreover

$$\mathrm{Lip}(\lambda_j : \Omega_0 \longrightarrow \mathbf{R}) \leq K_1 j^{d_{1,r}} \qquad \forall j, \forall \varepsilon_0 ; \tag{1.31}$$

Then there exist integers j_1, M_1 such that if a condition

$$\begin{gathered} 2) \quad |s \cdot \omega_0 + \ell_1 \lambda_{10} + \ell_2 \lambda_{20} + \dots + \ell_{j_1} \lambda_{j_1 0}| \geq K_5 > 0 \\ \forall s \in \mathbf{Z}^n, \quad |s| \leq M_1, \quad \forall \ell \in \mathbf{Z}^{j_1}, \qquad 1 \leq |\ell_1| + \dots + |\ell_{j_1}| \leq 2 , \end{gathered} \tag{1.32}$$

is satisfied, then for sufficiently small $\varepsilon_0 > 0$ there exist $\delta_a > 0$ sufficiently small and independent on ε_0 (see (1.13), (1.14)), a Borel subset $\Theta_{\varepsilon_0} \subset \Theta_0$ and analytic embeddings

$$\Sigma^{\varepsilon_0}_{(\omega,I)} : \mathbf{T}^n \longrightarrow \mathscr{Y}_{d_c}, \quad (\omega,I) \in \Theta_{\varepsilon_0}, \quad d_c = d + d_1 - d_H - 1 ,$$

with the following properties:

a) $\quad \mathrm{mes}\, \Theta_{\varepsilon_0}[I] \longrightarrow \mathrm{mes}\, \Omega_0 \quad (\varepsilon_0 \longrightarrow 0) \hfill (1.33)$

uniformly with respect to $I \in \mathscr{J}$;

b) the mapping

$$\Sigma^{\varepsilon_0} : \mathbf{T}^n \times \Theta_{\varepsilon_0} \longrightarrow \mathscr{Y}_{d_c}, \qquad (q,\omega,I) \mapsto \Sigma^{\varepsilon_0}_{(\omega,I)}(q) , \tag{1.34}$$

is Lipschitz and is close to the mapping

$$\Sigma^{0} : \mathbf{T}^n \times \Theta_{\varepsilon_0} \longrightarrow \mathscr{Y}_{d_c}, \qquad (q,\omega,I) \mapsto (q,0,0) \in \mathscr{Y}_{d_c}.$$

That is

$$\operatorname{dist}_{d_c}\Big[\sum^{0}(q;\omega,I),\sum^{\varepsilon_0}(q;\omega,I)\Big]\le C_\varrho\varepsilon_0{}^{\varrho}\qquad\forall\varrho<1/3\,,$$

$$\operatorname{Lip}\Big[\sum^{0}-\sum^{\varepsilon_0}:\mathbb{T}^n\times\Theta_{\varepsilon_0}\longrightarrow\mathscr{Y}_{d_c}\Big]\le C_\varrho\varepsilon_0{}^{\varrho}\qquad\forall\rho<1/3\,;\tag{1.35}$$

c) every torus $\sum^{\varepsilon_0}_{(\omega,I)}(\mathbb{T}^n)$, $(\omega,I)\in\Theta_{\varepsilon_0}$, is invariant for the equations (1.29) and is filled with weak in $\mathscr{Y}_d$ solutions of the form $z^{\varepsilon_0}(t)=\sum^{\varepsilon_0}_{(\omega,I)}(q+\omega' t)$, $q\in\mathbb{T}^n$, $\omega'=\omega'(\omega,I,\varepsilon_0)\in\mathbb{R}^n$ and

$$|\,\omega-\omega'\,|\le C\,\varepsilon_0{}^{1/3}\,;\tag{1.36}$$

d) all Lyapunow exponents of the solutions $z^{\varepsilon_0}(t)$ are equal to zero.

<u>Statement 1.2</u>. Under the assumptions of Theorem 1.1 a sharper form of estimates (1.31), (1.32) is true:

$$\operatorname{dist}_{d_c}\Big[\sum^{0}(q;\omega,I),\sum^{\varepsilon_0}(q;\omega,I)\Big]\le C\,\varepsilon_0\,,\tag{1.37}$$

$$|\,\omega-\omega'\,|\le C_1\,\varepsilon_0\,.\tag{1.38}$$

2. Proof of Theorem 1.1

We extend the scalar product $<\cdot,\cdot>$ to a bilinear over $\mathbb{C}$ map $Y^c\times Y^c\longrightarrow\mathbb{C}$ and denote by $\mathscr{L}^s(Y_{s_1}{}^c;Y_{s_2}{}^c)$ a subspace of operators $L\in\mathscr{L}(Y^c_{s_1};Y_{s_2}{}^c)$ symmetric with respect to $<\cdot,\cdot>$, i.e. $<Ly_1,y_2>=<y_1,Ly_2>$ $\forall y_1,y_2\in Y^c_\infty$. We denote

$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We shall use the following domains in $\mathbb{C}^n/2\pi\mathbb{Z}^n$, $\mathcal{Y}_s$ and $\mathcal{Y}_s{}^c$:

$$U(\delta) = \{\xi \in \mathbb{C}^n/2\pi\mathbb{Z}^n \mid |\operatorname{Im}\xi| < \delta\},$$

$$O^c(\xi_0,\xi_1,\xi_2;\mathcal{Y}_s{}^c) = U(\xi_0) \times O(\xi_1, \mathbb{C}^n) \times O(\xi_2, Y_s{}^c)\,.$$

Let us fix some

$$\gamma_* \in (0,1], \qquad \rho \in (0, \tfrac{1}{3}) \tag{2.1}$$

and set $\gamma_0 = 2(1^{-2} + 2^{-2} + \dots)$,

$$e_m = \begin{cases} 0, & m = 0 \\ (1^{-2} + \dots + m^{-2})\gamma_0{}^{-1}, & m \geq 1\,, \end{cases} \tag{2.2}$$

$$\varepsilon_m = \varepsilon_0{}^{(1+\varrho)^m}, \quad \delta_m = \delta_0(1 - e(m)), \qquad U_m = U(\delta_m)\,, \tag{2.3}$$

$$O_m{}^c = O^c(\delta_m, \varepsilon_m{}^{2/3}, \varepsilon_m{}^{1/3}; \mathcal{Y}_d^c)\,, \quad O_m = O_m^c \cap \mathcal{Y}_d\,.$$

We shall need some subdomains of U_m and $O_m{}^c$. For this end let us set

$$\delta_m^j = \frac{6-j}{6}\,\delta_m + \frac{j}{6}\,\delta_{m+1}\,, \; 0 \leq j \leq 5 \tag{2.4}$$

(so $\delta_m = \delta_m{}^0 > \delta_m{}^1 > \dots > \delta_m{}^5$) , and denote

$$O_m{}^{jc} = O^c(\delta_m{}^j, (2^{-j}\varepsilon_m)^{2/3}, (2^{-j}\varepsilon_m)^{1/3}; \mathcal{Y}_d{}^c),\; U_m{}^j = U(\delta_m{}^j)\,.$$

If $\varepsilon_0 << 1$ then $2^{-j}\varepsilon_m > \varepsilon_{m+1}$, $j = 1, \dots, 5$, and so the domains $O_m{}^{jc}$ are neighbourhoods of O_{m+1}^c, $O_m{}^c \supset O_m{}^{1c} \supset \dots \supset O_m{}^{5c} \supset O_{m+1}^c$.

We denote by $C, C_1, C_2, \dots$ different positive constants independent of ε_0 and m ; by

$C(m)$, $C_1(m)$, ... different functions of m of the form $C(m) = C_1 m^{C_2}$; by $C^e(m)$, $C_1{}^e(m)$, ... different functions of the form $\exp C(m)$. By C_*, C_{*1}, ..., $C_*(m)$, $C_{*1}(m)$,... we denote fixed constants and functions of the form $C(m)$. Let us mention that $\forall\, C(m)$, $\forall\, C^e(m)$ and $\forall\, \sigma < 0$

$$C(m) \le \varepsilon_m^\sigma \quad \forall m, \qquad C^e(m) \le \varepsilon_m^\sigma \quad \forall m \qquad \text{if } \varepsilon_0 << 1 .$$

Let $m \in \mathbb{N}_0$ and Θ_m be a Borel subset of $\Theta_0 = \Omega_0 \times \mathcal{J}$ such that

$$\operatorname{mes} \Theta_m[I] \ge K_6(1 - \gamma_* e(m)) \qquad \forall I \in \mathcal{J}. \tag{2.5}$$

Here $K_6 = \operatorname{mes} \Omega_0$ and γ_* as in (2.1).

We shall denote a pair $(\omega, I) \in \Theta_m$ by θ and shall omit dependence of functions and sets on the parameter ε_0. All estimates will be uniform with respect to $\varepsilon_0 \in [0,1]$.

At domain $O_m{}^c$ let us consider a hamiltonian depending on the parameter $\theta \in \Theta_m$;

$$\mathscr{H}_m = H_{0m}(q,y;\theta) + \varepsilon_m H_m(q,\xi,y;\theta) + H^3(q,\xi,y;\theta), \tag{2.6}$$

$$H_{0m} = \xi \cdot \Lambda_m(\theta) + \tfrac{1}{2} < A_m(q;\theta)y,y > . \tag{2.7}$$

Here the function H^3 is the same as in (1.24) and

$$\Lambda_m : \Theta_m \longrightarrow \mathbb{R}^n, \ |\Lambda_m(\omega,I) - \omega|^{\Theta_m,\,\mathrm{Lip}} \le \varepsilon_0{}^\rho e(m); \tag{2.8}$$

the operator $A_m(q;\theta)$ is equal to $A(\theta) + A_m^1(q;\theta)$ and

$$A_m^1(q;\theta)\, \varphi_j^{\pm} = \beta_{jm}(q;\theta)\, \varphi_j^{\pm} \qquad \forall j, \tag{2.9}$$

$$\beta_{jm} \in \mathscr{A}^{R}_{\Theta_m}(U_m;\mathbb{C}), \qquad |\beta_{jm}|^{U_m,\Theta_m} \le \varepsilon_0{}^{\rho}e(m)\, j^{d_H}. \tag{2.10}$$

We suppose that $H_m \in \mathscr{A}^{R}_{\Theta_m}(O^c_m;\mathbb{C})$ and

$$|H_m|^{O^c_m;\Theta_m} \le C_*(m) \equiv K_7{}^{m+1}, \tag{2.11}$$

$$\|\nabla_y H_m\|^{O_m{}^c,\Theta_m}_{d\ -d_H} \le \varepsilon_m{}^{-1/3}C_*(m). \tag{2.12}$$

For $m = 0$ the hamiltonian $\mathscr{H}_0$ in (1.18) has a form (2.6) with $\Lambda_0(\omega,I) \equiv \omega$, $A^1_0 \equiv 0$ and the assumptions (2.11), (2.12) are fulfilled by the theorem's assumptions.

Hamiltonian equations with the hamiltonian $\mathscr{H}_m$ have the form

$$\dot q = \Lambda_m(\theta) + \nabla_\xi(\varepsilon_m H_m + H^3)\,(q,\xi,y;\theta), \tag{2.13}$$

$$\dot\xi = -\nabla_q(\tfrac{1}{2} < A_m(q;\theta)y,y > + (\varepsilon_m H_m + H^3)\,(q,\xi,y;\theta)), \tag{2.14}$$

$$\dot y = J(A_m(q;\theta)y + \nabla_y(\varepsilon_m H_m + H^3)\,(q,\xi,y;\theta)). \tag{2.15}$$

For $m = 0$ these equations coincide with the equations (1.19).

The theorem will be proved via KAM–procedure. For $m = 0,1,2,\ldots$ we shall construct canonical transformation $S_m : O_{m+1} \longrightarrow O_m$ which is well–defined for $\theta \in \Theta_{m+1}$ and transforms the equations (2.13) – (2.15) into Hamiltonian equations in O_{m+1} with a hamiltonian of form (2.6) with $m := m + 1$. For $\theta \in \Theta_{\varepsilon_0} = \cap\Theta_m$ the limit transformation $\sum^{\varepsilon_0} : S_0 \circ S_1 \circ \ldots$ transforms equations (1.19) into an equation in a set $\cap O_m = \mathbb{T}^n \times \{0\} \times \{0\}$. The last one has solutions $(q + t\Lambda_\infty(\theta),0,0)$, $\Lambda_\infty = \lim \Lambda_m$, $q \in \mathbb{T}^n$. So for $\theta \in \Theta_{\varepsilon_0}$ equation (1.19) has desired quasiperiodic solutions of a form

$\sum^{\varepsilon_0}(q_0 + t\Lambda_\infty , 0 , 0)$.

Let us extract from H_m a linear on ξ and quadratic on y part:

$$H_m(q,\xi,y;\theta) = h^q(q;\theta) + \xi \cdot h^{1\xi}(q;\theta) + < y,h^y(q;\theta) > +$$

$$+ < y,h^{yy}(q;\theta)y > + H_{3m}(q,\xi,y;\theta) , \tag{2.16}$$

$$H_{3m} = O(|\xi|^2 + \|y\|_d^3 + |\xi| \|y\|_d) . \tag{2.17}$$

Here $h^q \in \mathbb{C}$, $h^{1\xi} \in \mathbb{C}^n$, $h^y \in Y^c$ and h^{yy} is an operator in the scale $\{Y_s\}$. We may vary H_m on a constant depending on θ and so may suppose that

$$\int h^q(q;\theta)\, dq/(2\pi)^n = 0 . \tag{2.18}$$

Here and in what follows

$$\int f(q)\, dq/(2\pi)^n = (2\pi)^{-n} \int_{\mathbb{T}^n} f(q)\, dq$$

for an arbitrary vector–valued function integrable on $\mathbb{T}^n$. Let us define a function $h^{0\xi}(\theta) = \int h^{1\xi}(q;\theta)dq/(2\pi)^n$ and set

$$h^\xi(q;\theta) = h^{1\xi} - h^{0\xi}, \qquad\qquad \Lambda_{m+1} = \Lambda_m + \varepsilon_m h^{0\xi} \tag{2.19}$$

and rearrange the terms of $\mathscr{H}_m$ in the following way:

$$\mathscr{H}_m = H'_{0m}(q,\xi,y;\theta) + \varepsilon_m(H_{2m} + H_{3m}) + H^3 \tag{2.20}$$

Here

$$H'_{0m} = \xi \cdot \Lambda_{m+1}(\theta) + \tfrac{1}{2} < A_m(q;\theta)y,y > , H_{3m} = H_m - H_{2m},$$

$$H_{2m} = h^q + \xi \cdot h^\xi + < y, h^y > + < y, h^{yy} y > .$$

Lemma 2.1. If $\varepsilon_0 << 1$ then

a) $$|h^q|^{U_m, \Theta_m} \le C_*(m), \tag{2.21}$$

(2.22) $$|h^\xi|^{U_m, \Theta_m} \le 2C_*(m)\varepsilon_m^{-2/3}, \quad |h^{0\xi}|^{\Theta_m, Lip} \le C_*(m)\varepsilon_m^{-2/3},$$

$$\|h^y\|_{d-d_H}^{U_m, \Theta_m} \le C_*(m)\, \varepsilon_m^{-1/3}; \tag{2.23}$$

b) $$h^{yy}(q,\theta) \in \mathscr{L}^s(Y_d, Y_{d-d_H}) \qquad \forall q \in \mathbf{T}^n, \forall \theta$$

$$\|h^{yy}\|_{d, d-d_H}^{U_m, \Theta_m} \le C_*(m)\, \varepsilon_m^{-2/3}; \tag{2.24}$$

c) if in (2.11) $K_7 >> 1$ then

$$|H_{3m}|^{O_{m+1}, \Theta_m} + \varepsilon_m^{1/3} \|\nabla_y H_{3m}\|_{d-d_H}^{O^c_{m+1}, \Theta_m} \le \tfrac{1}{8} C_*(m+1)\varepsilon_m^\rho; \tag{2.25}$$

d) $$H'_{0m}, H_{2m}, H_{3m} \in \mathscr{A}^R_{\Theta_m}(O_m^c; \mathbb{C});$$

e) $$|\Lambda_{m+1}(\theta) - \omega|^{\Theta_m, Lip} \le \varepsilon_0^\rho e(m+1). \tag{2.26}$$

Proof.

a) The estimate (2.21) results from (2.11) because $h^q(q;\theta) = H_m(q,0,0;\theta)$. To estimate the mapping $h^{1\xi}$ let us define a function of an argument $z \in \mathbb{C}$, $|z| < \varepsilon_m^{2/3}$:

$$z \longrightarrow H_m(q,z\xi,0;\theta),\ \xi \in \mathbb{C}^n,\ |\xi| \le 1\,.$$

By (2.11) its module is no greater than $C_*(m)$ and by Cauchy estimate its derivative at zero is no greater than $\varepsilon_m^{-2/3}\,C_*(m)$. So $|\xi\cdot h^{1\xi}(q;\theta)| \le \varepsilon_m^{-2/3}\,C_*(m)\ \ \forall|\xi| \le 1$ and $|h^{1\xi}| \le \varepsilon_m^{-2/3}\,C_*(m)$. By considering a function $z \longrightarrow H_m(q,z\xi,0;\theta_1) - H_m(q,z\xi,0;\theta_2)$, one can get an analogous estimate for the Lipschitz constant on θ. So $|h^{1\xi}|^{U_m,\Theta_m} \le \varepsilon_m^{-2/3}\,C_*(m)$. From this estimates results (2.22).

The estimate (2.23) results from (2.12) with $y = 0$.

b) Let us consider a map

$$\{|z| < \varepsilon_m^{\,1/3}\} \longrightarrow Y^{\,c}_{d-d_H}\,, \qquad z \mapsto \nabla_y H_m(q,0,zy;\theta)\,, \tag{2.27}$$

($\|y\|_d \le 1$). Its derivative at zero is equal to $h^{yy}(q;\theta)y$. So by (2.12) and Cauchy estimate

$$\|h^{yy}y\|^{U_m,\Theta_m}_{d-d_H} \le \varepsilon_m^{-2/3}\,C_*(m) \qquad \forall\,\|y\|_d \le 1\,.$$

The last estimate implies (2.24). The inclusion $h^{yy} \in \mathscr{L}^s(Y_d;Y_{d-d_H})$ results from the general fact that Hessian of a function is a symmetric linear operator.

c) Let $\mathfrak{h} = (q,\xi,y) \in O^{\,c}_{m+1}$ and $\nu = \varepsilon_m^{\,\rho/3}$. Then $(q,(z/\nu)^2\xi,(z/\nu)y) \in O_m^{\,c}$ for $z \in \mathbb{C}$, $|z| \le 1$. Let us consider a function $z \longrightarrow H_m(q,(z/\nu)^2\xi,\,(z/\nu)y;\theta)$ and its Taylor series at zero:

$$H_m(q,(\tfrac{z}{\nu})^2\xi,\,(\tfrac{z}{\nu})y;\theta) = h_0 + h_1 z + h_2 z^2 + \dots\,.$$

By (2.11), $|h_k| \le C_*(m)$ $\forall k$. Since $H_{3m}(\mathfrak{h};\theta) = h_3\nu^3 + h_4\nu^4 + \dots$, then

$$|H_{3m}(\mathfrak{h};\theta)| = |h_3\nu^3 + h_4\nu^4 + \dots| \le \frac{C_*(m)\ \varepsilon_m{}^{\rho}}{1-\varepsilon_m{}^{\rho/3}} \le \frac{C_*(m+1)\,\varepsilon_m{}^{\rho}}{8}$$

if $K_7 >> 1$. In a similar way one can estimate a Lipschitz constant of H_{3m}.

To estimate $\nabla_y H_{3m}$ let us consider a map

$$z \longrightarrow \nabla_y H_m(q,(\tfrac{z}{\nu})^2\xi,(\tfrac{z}{\nu})y;\theta) = h_0' + h_1' z + \dots \in Y^c_{d-d_H} .$$

By (2.12) $\|h_k'\|_{d-d_H} \le \varepsilon_m^{-1/3}\, C_*(m)$ $\forall k$. So

$$\begin{aligned}\|\nabla_y H_{3m}(\mathfrak{h};\theta)\|_{d-d_H} &= \|h_2'\nu^2 + h_3'\nu^3 + \dots\|_{d-d_H} \le \\ &\le \frac{\nu^2}{1-\nu}\varepsilon_m^{-1/3}\, C_*(m) \le \tfrac{1}{8}\,\varepsilon_{m+1}^{-1/3}\,\varepsilon_m^{\rho}\, C_*(m+1) .\end{aligned}$$

A similar estimate is true for the Lipschitz constant, so (2.25) is proved.

d) The analyticity of the functions is evident. Their real–valuedness for real (q,ξ,y) results from the real–valuedness of $\mathcal{H}_m$.

e) The estimate results from (2.8), (2.19), (2.22). ■

Let us consider an auxiliary hamiltonian $\varepsilon_m F$,

$$F = f^q(q;\theta) + \xi \cdot f^{\xi}(q;\theta) + \langle y, f^y(q;\theta)\rangle + \langle y, f^{yy}(q;\theta)y\rangle ,$$

and the corresponding Hamiltonian equations

$$\dot q = \varepsilon_m \nabla_\xi F, \qquad \dot\xi = -\varepsilon_m \nabla_q F, \qquad \dot y = \varepsilon_m J\nabla_y F . \tag{2.28}$$

A flow of these equations consists of canonical transformations $\{S^t\}$ of the phase space (see Part 1, Theorem 2.4). Let us set $S_m = S^1$ and denote $(q,\xi,y) = \mathfrak{h}$. Then

$$\mathscr{H}_m(S_m(\mathfrak{h};\theta);\theta) = \mathscr{H}_m(\mathfrak{h};\theta) + \varepsilon_m\{F,\mathscr{H}_m\} + O(\varepsilon_m{}^2) .$$

Here $\{\cdot,\cdot\}$ is a Poisson bracket; see Part 1, Proposition 4.3. So if $\mathfrak{h},S_m(\mathfrak{h}) \in O_m$, then by (1.25), (2.20) and (2.25)

$$\mathscr{H}_m(S_m(\mathfrak{h})) = H_{0m}'(\mathfrak{h}) + \varepsilon_m(H_{2m}(\mathfrak{h}) + \{F(\mathfrak{h}), H_{0m}'(\mathfrak{h})\}) +$$

$$+ O(\varepsilon_m{}^{1+\rho}) = H_{0m}'(\mathfrak{h}) + \varepsilon_m(H_{2m}(\mathfrak{h}) - \nabla_q F(\mathfrak{h}) \cdot \nabla_\xi H_{0m}'(\mathfrak{h}) +$$

$$+ \nabla_\xi F \cdot \nabla_q H_{0m}' + < J\nabla_y F(\mathfrak{h}), \nabla_y H_{0m}'(\mathfrak{h}) >) + O(\varepsilon_m{}^{1+\rho})$$

(we omit the parmeter θ) . As

$$\nabla_\xi H_{0m}' = \Lambda_{m+1}, \quad \nabla_y H_{0m}' = A_m y, \quad \nabla_q H_{0m}' = \tfrac{1}{2} < \nabla_q A_m y,y > ,$$

then we may denote

$$\omega' = \Lambda_{m+1}(\omega;\theta), \qquad \frac{\partial}{\partial\omega'} = \sum \omega_j' \frac{\partial}{\partial q_j} \tag{2.29}$$

and rewrite $\mathscr{H}_m \circ S_m$ as follows:

$$\mathscr{H}_m(S_m(\mathfrak{h};\theta);\theta) = H_{0m}' + \varepsilon_m[\tfrac{1}{2} f^\xi \cdot < \nabla_q A_m y,y > - \partial f^q/\partial\omega' -$$

$$- \xi \cdot \partial f^\xi/\partial\omega' - < y,\partial f^y/\partial\omega' > - < y,(\partial f^{yy}/\partial\omega')y > +$$

$$+ < A_m y,Jf^y > + 2 < A_m y,Jf^{yy}y > + h^q + \xi \cdot h^\xi + < y,h^y > +$$

$$+ < y,h^{yy}y >] + O(\varepsilon_m^{1+\rho}) . \tag{2.30}$$

We try to find a transformation S_m such that the contents of the square brackets is $O(\varepsilon_m^{\rho})$. For this end we have to find $f^q, f^{\xi}, f^y, f^{yy}$ solving homological equations:

$$\partial f^q/\partial\omega' = h^q(q;\theta), \qquad \partial f^{\xi}/\partial\omega' = h^{\xi}(q;\theta), \tag{2.31}$$

$$\partial f^y/\partial\omega' - A_m(\theta)Jf^y = h^y(q;\theta), \tag{2.32}$$

$$\partial f^{yy}/\partial\omega' + f^{yy}J\,A_m - A_mJ\,f^{yy} = h^{yy}(q;\theta) -$$

$$- \Delta h^{yy}(q;\theta) + \tfrac{1}{2}f^{\xi}\cdot\nabla_q A_m(q;\theta). \tag{2.33}$$

• Here Δh^{yy} is an admissible disparity.

Lemma 2.2. If $\varepsilon_0 << 1$ then there exists a Borel subset $\Theta_{m+1} \subset \Theta_m$ such that

$$\text{mes}\,(\Theta_m \setminus \Theta_{m+1})[I] \le \gamma_* K_6(m+1)^{-2}/\gamma_0 \qquad \forall I \tag{2.34}$$

and for all $\theta \in \Theta_{m+1}$

a) equations (2.31) have solutions $f^q \in \mathcal{A}^R_{\Theta_{m+1}}(U_m{}^1;\mathbb{C})$, $f^{\xi} \in \mathcal{A}^R_{\Theta_{m+1}}(U_m{}^1;\mathbb{C}^n)$ and

$$|f^q|^{U_m{}^1,\Theta_{m+1}} \le C(m), \quad |f^{\xi}|^{U_m{}^1,\Theta_{m+1}} \le \varepsilon_m^{-2/3}\,C(m)\,; \tag{2.35}$$

b) equation (2.32) has an analytical solution $f^y \in \mathcal{A}^R_{\Theta_{m+1}}(U_m{}^2;\,Y^c_{d-d_H} + d_1)$ and

$$\|f^y\|^{U_m{}^2,\,\Theta_{m+1}}_{d-d_H\,+\,d_1} \le C^e(m)\,\varepsilon_m^{-1/3}\,; \tag{2.36}$$

c) there exist $\Delta h^{yy} \in \mathcal{A}^R_{\Theta_{m+1}}(U_m{}^2;\,\mathcal{L}^s(Y_d, Y_{d-d_H}))$ such that

$$\Delta h^{yy}\varphi_j^{\pm} = b_j(q;\theta)\varphi_j^{\pm} \qquad \forall j, \forall q, \tag{2.37}$$

$$|b_j|^{U_m^2,\Theta_{m+1}} \le C(m)\varepsilon_m^{-2/3} \qquad \forall j, \tag{2.38}$$

equation (2.33) has a solution f^{yy} belonging to the same class as Δh^{yy},

$$\|f^{yy}\|^{U_m^2,\Theta_{m+1}}_{a\,,\,a+\Delta d} \le C^e(m)\,\varepsilon_m^{-2/3} \qquad \forall a \in [-d-\Delta d, d]\,, \tag{2.39}$$

(here $\Delta d = d_1 - d_H - 1$) and

$$\|A_m J f^{yy} - f^{yy} J\, A_m\|^{U_m^2,\Theta_{m+1}}_{d\,,\,d\text{-}d_H} \le C^e(m)\,\varepsilon_m^{-2/3}\,. \tag{2.40}$$

A proof of the lemma is given below in § 3.

Let us denote

$$\Pi_{\mathscr{Y}} : \mathscr{Y}^c \times \Theta_0 \longrightarrow \mathscr{Y}^c, \qquad (\mathfrak{h},\theta) \mapsto \mathfrak{h}\,, \tag{2.41}$$

$$\Pi_\Theta : \mathscr{Y}^c \times \Theta_0 \longrightarrow \Theta_0, \qquad (\mathfrak{h},\theta) \mapsto \theta\,, \tag{2.42}$$

let Π_q, Π_ξ, Π_y be projectors of $\mathscr{Y}^c = (\mathbb{C}^n/\,2\pi\,\mathbb{Z}^n) \times \mathbb{C}^n \times Y^c$ on the first, second and third term respectively and let S_m be a time one shift along the trajectories of the system (2.28).

Let $d_c = d + d_1 - d_H - 1$ and $O^c_{m,d_c} = O_m{}^c \cap \mathscr{Y}^c_{d_c}$ with the norm dist d_c. The set O^c_{m,d_c} is dense in O^c_m and is unbounded in $\mathscr{Y}^c_{d_c}$.

We may identify the torus $\mathbb{T}^n$ with a measurable subset $T^{(n)} \subset \mathbb{R}^n$,

$$\{q \in \mathbb{R}^n \mid |q_j| < \pi \ \forall j\} \subset T^{(n)} \subset \{q \in \mathbb{R}^n \mid |q_j| \le \pi \ \forall j\}$$

(the map $\mathbf{T}^n \longrightarrow T^{(n)}$ is one–to–one, measurable and discontinuous), and may identify $\mathscr{Y}$ with a subset $T^{(n)} \times \mathbb{R}^n \times Y$ of $E = \mathbb{R}^{2n} \times Y$. The identifications depend on a choice of $T^{(n)}$, but if $\mathrm{dist}_{\mathscr{Y}}(\mathfrak{h}_1, \mathfrak{h}_2) < \pi$ then the point in E corresponding to $\mathfrak{h}_1 - \mathfrak{h}_2$ does not depend on $T^{(n)}$. We shall use these identifications and treat a difference of two close $\mathscr{Y}$–valued (or $\mathbb{T}^n$–valued) maps as a E–valued ($\mathbb{R}^n$–valued) map.

<u>Lemma 2.3</u>. If $\varepsilon_0 << 1$ then

a)
$$S_m \in \mathscr{A}^{R}_{\Theta_{m+1}}(O_m^{4c}; O_m{}^c) \tag{2.43}$$

and

$$|S_m - \Pi_{\mathscr{Y}}|_{E_{d_c}}^{O_m^{5c} \times \Theta_{m+1}, \mathrm{Lip}} \le \varepsilon_m^{\rho} . \tag{2.44}$$

More precisely,

$$|\Pi_q \circ (S_m - \Pi_{\mathscr{Y}})|^{O_m^{5c} \times \Theta_{m+1}, \mathrm{Lip}} \le C(m)\, \varepsilon_m^{1/3} , \tag{2.45}$$

$$|\Pi_\xi \circ (S_m - \Pi_{\mathscr{Y}})|^{O_m^{5c} \times \Theta_{m+1}, \mathrm{Lip}} \le C^e(m)\, \varepsilon_m , \tag{2.46}$$

$$\|\Pi_y \circ (S_m - \Pi_{\mathscr{Y}})\|_{d_c}^{O_m^{5c} \times \Theta_{m+1}, \mathrm{Lip}} \le C^e(m)\, \varepsilon_m^{2/3} . \tag{2.47}$$

b) A restriction of S_m on O_{m+1} is a canonical transformation which transforms equations (2.13) – (2.15) on the domain O_m into Hamiltonian equations with a hamiltonian $\mathscr{H}_{m+1}$ of the form (2.6) with $m := m+1$ on the domain O_{m+1}.

The lemma is proved in § 5.

Let us set $\Theta_{\varepsilon_0} = \bigcap \Theta_m$. Then Θ_{ε_0} is a Borel set. For the definition of γ_0 and e_m (see (2.2)) and for (2.5)

$$\operatorname{mes} \Theta_{\varepsilon_0}[I] \geq (1 - \tfrac{1}{2}\gamma_*) \operatorname{mes} \Omega_0 \qquad \forall I \in \mathcal{J}. \tag{2.48}$$

For $\theta \in \Theta_{\varepsilon_0}$ and $r, N \in \mathbb{N}_0$ let us set

$$\sum\nolimits^{r}_{r+N+1}(\cdot\,;\theta) = S_r(\cdot\,;\theta) \circ \ldots \circ S_{r+N}(\cdot\,;\theta) : O^c_{r+N+1} \longrightarrow O_r{}^c \tag{2.49}$$

and let us set $\sum^r_r$ be equal to the identical map of $O_r{}^c$.

Lemma 2.4. For all $r, m \geq 0$

$$|\sum\nolimits^{r}_{r+m} - \Pi_{\mathcal{Y}}|^{O^c_{r+m,d_c} \times \Theta_{m+1},\mathrm{Lip}}_{E_{d_c}} \leq 3\,\varepsilon_r^{\rho}, \tag{2.50}$$

Proof. Let us denote the l.h.s. in (2.50) by D^r_{r+m}. One may rewrite the identity $\sum^r_{r+m}(\mathfrak{h};\theta) = S_r(\sum^{r+1}_{r+m}(\mathfrak{h};\theta);\theta)$ in the form

$$\sum\nolimits^{r}_{r+m} - \Pi_{\mathcal{Y}} = (S_r - \Pi_{\mathcal{Y}}) \circ (\sum\nolimits^{r+1}_{r+m} \times \Pi_{\Theta}) + \sum\nolimits^{r+1}_{r+m} - \Pi_{\mathcal{Y}}.$$

So by (2.44) we get the estimate

$$D^r_{r+m} \leq \varepsilon_r^{\rho}(D^{r+1}_{r+m} + 2) + D^{r+1}_{r+m}. \tag{2.51}$$

As $D^{r+m}_{r+m} = 0$, then the lemma's assertion results by the induction. ∎

Let us denote $T_0{}^n = \mathbf{T}^n \times \{0\} \times \{0\}$ and $O^c_\infty = U(\delta_0/2) \times \{0\} \times \{0\} \subset \mathscr{Y}^c_\infty$. Then $T_0{}^n \subset O^c_\infty$ and O^c_∞ lies in O^c_m for every $m \geq 1$ as $\delta_m > \frac{1}{2}\delta_0$ $\forall m$.

Lemma 2.5. If $\varepsilon_0 << 1$ then $\forall m \in \mathbf{N}_0$ the maps $\sum^m_{m+N} : O^c_\infty \times \Theta_{\varepsilon_0} \longrightarrow \mathscr{Y}^c_{d_c}$ $(N \longrightarrow \infty)$ converge to a map $\sum^m_\infty : O^c_\infty \times \Theta_{\varepsilon_0} \longrightarrow \mathscr{Y}^c_{d_c}$ such that

a) for every θ the map $\sum^m_\infty(\cdot\,;\theta) : O^c_\infty \longrightarrow \mathscr{Y}^c_{d_c}$ is complex–analytical;

b)

$$\sum^m_p(\cdot\,;\theta) \circ \sum^p_\infty(\cdot\,;\theta) = \sum^m_\infty(\cdot\,;\theta)\,;\ \ \forall 0 \leq m \leq p,\ \forall \theta \in \Theta_{\varepsilon_0} \tag{2.52}$$

c) $$|\sum^m_\infty - \Pi_{\mathscr{Y}}|^{O^c_\infty \times \Theta_{\varepsilon_0},\mathrm{Lip}}_{E_{d_c}} \leq 3\,\varepsilon_m^\rho\,; \tag{2.53}$$

d) $$\|\Pi_y \circ \sum^m_\infty(\mathfrak{h};\theta)\|_{d_c} \leq \varepsilon_m{}^{1/3+\rho} \qquad \forall(\mathfrak{h},\theta) \in T^n_0 \times \Theta_{\varepsilon_0}\,, \tag{2.54}$$

$$|\Pi_\xi \circ \sum^m_\infty(\mathfrak{h};\theta)| \leq \tfrac{1}{4}\varepsilon_m{}^{2/3} \qquad \forall(\mathfrak{h},\theta) \in T^n_0 \times \Theta_{\varepsilon_0}\,. \tag{2.55}$$

Proof. Let $\mathfrak{h}_0 \in O^c_\infty$ and for $j \geq 1$ let $\mathfrak{h}_j = \sum^m_{m+j}(\mathfrak{h}_0;\theta)$. Then by (2.44), (2.50)

$$\mathrm{dist}_{d_c}(\mathfrak{h}_{N+1}, \mathfrak{h}_N) = \mathrm{dist}_{d_c}(\sum^m_{m+N}(S_{m+N}(\mathfrak{h}_0;\theta)),$$

$$\sum^m_{m+N}(\mathfrak{h}_0;\theta)) \leq (1 + 3\,\varepsilon^\rho_m)\,\varepsilon^\rho_{m+N} \leq 2\,\varepsilon^\rho_{m+N}\,.$$

So the sequence $\{\mathfrak{h}_j\}$ is fundamental and converges to a point $\mathfrak{h}_\infty \in \mathscr{Y}^c_{d_c}$. The r.h.s. of the last estimate does not depend on $\mathfrak{h}_0$. So the sequence $\{\sum^m_{m+N}(\cdot\,;\theta)\}$ converges uniformly in O^c_∞ to an analytical map $\sum^m_\infty(\cdot\,;\theta) : O^c_\infty \longrightarrow \mathscr{Y}^c_{d_c}, \sum^m_\infty(\mathfrak{h}_0;\theta) = \mathfrak{h}_\infty$. The relations (2.52) take place and the items a), b) are proved.

The estimate (2.53) results from (2.50) by going to a limit.

To prove (2.54), (2.55) let us take $\mathfrak{h} \in O^c_\infty$ and set $\mathfrak{h}^{m+N+1} = \mathfrak{h}$,

$$\mathfrak{h}^j = \sum^j_{m+N+1}(\mathfrak{h}^{m+N+1};\theta) \in O^c_j \ \forall j \in [m, m+N] \cap \mathbb{N}_0 .$$

Then $\mathfrak{h}^j = S_j(\mathfrak{h}^{j+1};\theta)$ and by (2.47)

$$\|\Pi_y \mathfrak{h}^j\|_{d_c} \le \|\Pi_y \mathfrak{h}^{j+1}\|_{d_c} + \tfrac{1}{2}\,\varepsilon_j^{\rho+1/3}, \qquad m \le j \le m+N .$$

As $\Pi_y \mathfrak{h}^{m+N+1} = 0$, then $\|\Pi_y \mathfrak{h}^m\|_{d_c} \le \varepsilon_m^{\rho+1/3}$. So $\|\Pi_y \circ \sum^m_{m+N+1}(\mathfrak{h};\theta)\|_{d_c} \le \varepsilon_m^{\rho+1/3}$ and by going to the limit when $m \longrightarrow \infty$ one gets (2.54).

Estimate (2.55) results from (2.46) because for the last $|\Pi_\xi \mathfrak{h}^j| \le |\Pi_\xi \mathfrak{h}^{j+1}| + C^e(j)\,\varepsilon_j$. ∎

As $\Lambda_0(\omega,I) \equiv \omega$ then by (2.19) with $m = 0, 1, \dots, r-1$

$$\Lambda_r(\omega,I) = \omega + \varepsilon_0 h_0^{0\xi} + \varepsilon_1 h_1^{0\xi} + \dots + \varepsilon_{r-1} h_{r-1}^{0\xi} . \tag{2.56}$$

Here the vector–function $h_j^{0\xi}$ corresponds to the hamiltonian $\mathscr{H}_m$ with $m = j$. So

$$|\varepsilon_j h_j^{0\xi}|^{\Theta_{\varepsilon_0},\mathrm{Lip}} \leq C(j)\,\varepsilon_j^{1/3}\,, \tag{2.57}$$

the maps $\Lambda_r : \Theta_{\varepsilon_0} \longrightarrow \mathbb{R}^n$ $(r \longrightarrow \infty)$ converge to a Lipschitz one

$$\Lambda_\infty : \Theta_{\varepsilon_0} \longrightarrow \mathbb{R}^n,\ \Lambda_\infty = \omega + \varepsilon_0 h_0^{0\xi} + \varepsilon_1 h_1^{0\xi} + \dots \tag{2.58}$$

and by (2.57)

$$|\Lambda_\infty(\omega,I) - \omega|^{\Theta_{\varepsilon_0},\mathrm{Lip}} \leq C\,\varepsilon_0^{1/3}\,. \tag{2.59}$$

Let us fix $\theta_0 \in \Theta_{\varepsilon_0}$ and denote $\omega_m = \Lambda_m(\theta_0)$, $m \leq \infty$. Then by (2.56), (2.57)

$$|\omega_m - \omega_{m+p}| \leq C(m)\,\varepsilon_m^{1/3} \qquad \forall m, \forall p \geq 1\,. \tag{2.60}$$

Let us consider a curve $t \mapsto \mathfrak{h}_\infty(t) = (q_0 + t\omega_\infty',0,0)$, $0 \leq t \leq 1$, on the torus $T_0^n = \mathbf{T}^n \times \{0\} \times \{0\}$. The map $\sum_\infty^m(\cdot\,;\theta_0)$, $m \geq 0$, transforms it into a curve $\mathfrak{h}_m(t) = (q_m(t),\xi_m(t),y_m(t)) \in O_m$. By the estimates (2.53) – (2.55)

$$\mathrm{dist}\,(q_m(t), q_m(0) + t\omega_\infty') \leq C\,\varepsilon_m^\rho\,, \tag{2.61}$$

$$\|y_m(t)\|_{d_c} \leq \varepsilon_m^{1/3+\rho}\,, \tag{2.62}$$

$$|\xi_m(t)| \leq \tfrac{1}{4}\,\varepsilon_m^{2/3}\,. \tag{2.63}$$

Let $\mathfrak{h}(t)$ be some strong solution of the system with hamiltonian $\mathscr{H}_m$, staying inside $O_m \cap O_m^{1c}$ for $0 \leq t \leq T$. Taking the inner product in Y_d of equation (2.15) by $y(t)$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|y(t)\|_d^2 = \varepsilon_m < J\,\nabla_y H_m, y >_d + < J\,\nabla_y H^3, y >_d \le$$

$$\le \varepsilon\|y\|_d\|\nabla_y H_m\|_d + \|y\|_d\, K_1\,(\|y\|_d^2 + |\xi|)\,,$$

and

$$\|y(t)\|_d \le \|y(0)\|_d + \tfrac{1}{3}\,t\,\varepsilon_m{}^{1/3} \quad \text{for } 0 \le t \le T\,. \tag{2.64}$$

By equations (2.13), (2.14) we have

$$|\xi(t)| \le |\xi(0)| + \tfrac{1}{3}\,t\,\varepsilon_m{}^{2/3}\,,\ \text{dist}\,(q(t), q(0) + t\omega_m) \le Ct\,\varepsilon_m^{\delta} \quad (0 \le t \le T)\,. \tag{2.65}$$

So if

$$\|y(0)\|_d \le \tfrac{1}{3}\,\varepsilon_m{}^{1/3},\ |\xi(0)| \le \tfrac{1}{3}\,\varepsilon_m{}^{2/3},\ q(0) \in \mathbf{T}^n\,, \tag{2.66}$$

then the solution $\mathfrak{h}(t)$ stays inside $O_m \cap O_m{}^{1c}$ for $0 \le t \le 1$. If $\mathfrak{h}^m(t)$ is a weak solution of the equations with hamiltonian $\mathscr{H}_m$ and $\mathfrak{h}^m(0) = \mathfrak{h}_m(0)$, then (2.66) is true by (2.61) – (2.63) with $t = 0$. So by Theorem 3.1 from Part 1 solution $\mathfrak{h}^m(t)$ exists for $0 \le t \le 1$ and for this solution estimates (2.64), (2.65) take place. By the inequalities (2.61)–(2.63), (2.64), (2.65) and (2.60), $\text{dist}_d(f^m(t), f_m(t)) \le C\,\epsilon_m^{\rho}\ \forall\ 0 \le t \le 1$. The mapping $\sum_m^0(\cdot;f) : \mathscr{Y}_d \longrightarrow \mathscr{Y}_d$ is Lipschitz by Lemma 2.4. So $\text{dist}_d(f^0(t), f_0(t)) \le C'\,\epsilon_m^{\rho}\ \forall\ 0 \le t \le 1$ for arbitrary m . Hence $f^0 = f_0$ and $\sum_\infty^0(f_\infty(z);\theta)$ is a weak in $\mathscr{Y}_d$ solution of the initial Hamiltonian system.

Now the assertions b) – c) of the theorem are proved by setting
$\sum^{\varepsilon_0}(q;\theta) = \sum_\infty^0(q,0,0;\theta)$, because estimate (1.31) results from (2.53) and (1.32) results from (2.59).

In order to prove the assertion a), we set in (2.1) $\gamma_* = \gamma_*(M) \searrow 0$, where M is a natural parameter tending to infinity. Assertions b) – c) are valid for $\varepsilon_0 = \varepsilon_0(M) > 0$, and we may assume that $\varepsilon_0(M) \searrow 0$. Then by (2.48) for $\varepsilon_0 \in (\varepsilon_0(M+1), \varepsilon_0(M)]$ mes Ω_0 – mes $\Theta_{\varepsilon_0}[I] \leq \gamma_*(M) \searrow 0$ and the assertion is proved.

To prove assertion d) let us mention that Liapunov exponents are stable under a change of phase variable. So the exponents of a solution $\mathfrak{h}_0(t)$ of equations (2.13) – (2.15) with $m = 0$ are equal to ones of the solution $\mathfrak{h}_m(t) = (\sum_m^0)^{-1} \mathfrak{h}_0(t)$ of the equations with $m = m$. Let $\delta\mathfrak{h} = (\delta q, \delta\xi, \delta y)(t)$ be a strong solution of the variational equations for (2.13) – (2.15) along $\mathfrak{h}_m(t)$:

$$\delta\dot{q} = \nabla_\xi(\varepsilon_m H_m + H^3)(\mathfrak{h}_m(t))_*(\delta q, \delta\xi, \delta y) ,$$

$$\delta\dot{\xi} = -\nabla_q(H_{0m} + \varepsilon_m H_m + H^3)(\mathfrak{h}_m(t))_*(\delta q, \delta\xi, \delta y) ,$$

$$\begin{aligned}\delta\dot{y} = J\,[A_m(q_m(t))\delta y + (\delta q \cdot \nabla q\, A_m(q_m(t)))y\, + \\ + \nabla_y(\varepsilon_m H_m + H^3)(\mathfrak{h}_m(t))_*(\delta\mathfrak{h})] \,.\end{aligned}$$

Taking the inner product in E_d of these equations with $\delta\mathfrak{h}(t)$ we get an inequality:

$$\frac{d}{dt}\|\mathfrak{h}(t)\|_d \leq \varepsilon_m{}^\rho \|\mathfrak{h}(t)\|_d \,.$$

The same is true after the change $t \longrightarrow -t$. So modules of the exponents of the variational equations do not exceed $\varepsilon_m{}^\rho$. As m is arbitrary, they are equal to zero.

3. Proof of Lemma 2.2 (solving of homological equations)

In § 3 – 5 we write ε, δ instead of ε_m, δ_m and sometimes we omit the argument θ for functions and maps. In the deductions of estimates, we use systematically the conditions $\varepsilon_0 << 1$, $\delta_a << 1$. We denote $\mathbb{Z}_0^s = \mathbb{Z}^s \setminus \{0\}$, $\mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}$.

The assertions of the lemma will be proved for $\Theta_{m+1} = \Theta_m \setminus (\Theta^1 \cup \Theta^2 \cup \Theta^3)$, where Θ^p are Borel sets, and for $p = 1, 2, 3$

$$\operatorname{mes} \Theta^p[I] \le \gamma_* K_6 (m+1)^{-2} / (3\gamma_0) \qquad \forall I. \tag{3.1}$$

By (2.8) the map

$$\Theta_m[I] \ni \omega \mapsto \omega' = \Lambda_{m+1}(\omega, I) \tag{3.2}$$

for all I is a Lipschitz homeomorphism , changing the Lebesgue measure by a factor no greater than two. I.e. for every Borel subset $\Omega \subset \Theta_m[I]$

$$\tfrac{1}{2} \operatorname{mes} \Omega \le \operatorname{mes} \Lambda_{m+1}(\Omega, I) \le 2 \operatorname{mes} \Omega \tag{3.3'}$$

(see Appendix C, Treorem C 1). Besides,

$$|\Lambda_{m+1} - \omega|^{\Theta_m[I], \mathrm{Lip}} \le C \varepsilon_0^{\rho}, \ |\Lambda_{m+1}(\omega, I) - \omega_0| \le C(\varepsilon_0^{\rho} + \delta_a) \qquad \forall \omega \in \Theta_m[I]. \tag{3.3}$$

Therefore, if

$$\Theta^1 = \cup \{\Theta_s^1 \mid s \in \mathbb{Z}_0^n\}, \ \Theta_s^1 = \{\theta \in \Theta_m \mid \ |\omega'(\theta) \cdot s| \le \le [(m+1)^2 |s|^n C]^{-1}\}, \tag{3.4}$$

then

$$\operatorname{mes} \Theta^1[I] \le \sum_{s \in \mathbb{Z}_0^n} \operatorname{mes} \Theta_s^1[I] \le$$

$$\leq 2 \sum_{s \in \mathbb{Z}_0^n} \text{mes}\,\{\omega' \mid |\omega' \cdot s| \leq (m+1)^{-2} |s|^{-n} C^{-1}\} \leq$$

$$\leq \frac{2}{C(m+1)^2} \sum_{s \in \mathbb{Z}_0^n} C_1 |s|^{-n-1} \leq 2\, C_2 C^{-1} (m+1)^{-2}$$

and condition (3.1) is satisfied if $C >> 1$. For $\theta \in \Theta_m \setminus \Theta^1$, $q \in U_m{}^1$, the solutions of equations (2.31) are given by convergent trigonometric series and satisfy the estimates (2.35) (see [A, Sec. 4.2] and Lemmas B1, B2 in Appendix B below).

We turn to the equation (2.33) (a proof of the assertion b) on the equation (2.32) is much simpler, a sketch of it is given at the end of the section). For $j \in \mathbb{Z}_0$ we set

$$w_j = (\varphi^+_{|j|} + (\text{sgn } j) i\, \varphi^-_{|j|}) / \sqrt{2}\,.$$

Then $\{w_j \lambda_{|j|}^{(-s)} \mid j \in \mathbb{Z}_0\}$ is a Hilbert basis of a space $Y_s{}^c$, $s \in \mathbb{R}$. For complex numbers χ_j, $j \in \mathbb{Z}_0$, we denote by diag (χ_j) an operator in Y^c which maps w_j to $\chi_j w_j$ $\forall j \in \mathbb{Z}_0$. In particular by (2.9)

$$J\, A_m(q;\theta) = \text{diag}\,(i \lambda_j^1(q;\theta)) \tag{3.5}$$

with $\lambda_j^1(q;\theta) = \lambda_j(\omega) + \beta_{jm}(q;\theta)$ $\forall j \in \mathbb{Z}_0$. Here for $j \in \mathbb{N}$ $\lambda_{-j}(\omega) = -\lambda_j(\omega)$, $\beta_{-jm}(q;\theta) = -\beta_{jm}(q;\theta)$. By (2.10) and (1.27) $\forall j \in \mathbb{Z}_0$

$$\begin{gathered} |\lambda_j^1(\cdot\,;\cdot) - \lambda_j(\cdot)|^{U_m;\Theta_m} \leq \varepsilon_0{}^\rho |j|^{d_H}, \\ |\lambda_j^1(q;\theta) - \lambda_{j0}| \leq C(\delta_a |j|^{d_{1,r}} + \varepsilon_0{}^\rho |j|^{d_H}) \quad \forall q, \theta \end{gathered} \tag{3.6}$$

(here $\lambda_{j0} = \lambda_j(\omega_0)$). Let us choose functions $b_j(q;\theta)$, $j \in \mathbb{N}$ (see (2.37), (2.38)), as follows:

$$b_j(q;\theta) = \frac{1}{2} \sum_{\sigma=\pm} < (\tfrac{1}{2} f^{\xi} \cdot \nabla_q A_{m+1} + h^{yy}) \varphi_j^{\sigma} , \varphi_j^{\sigma} >$$

and define the operator Δh^{yy}, $\Delta h^{yy} \varphi_j^{\pm} = b_j(q) \varphi_j^{\pm}$ $\forall j \in \mathbb{N}$. By (2.24) and (2.10), (2.35)

$$| b_j |^{U_m^1; \Theta_{m+1}} \le C(m) \, \varepsilon_m^{-2/3} \, j^{d_H} \qquad \forall j \in \mathbb{N} . \qquad (3.7)$$

So operator Δh^{yy} satisfies (2.37), (2.38).

Let us denote $h^{1yy}(q;\theta) = h^{yy} + \frac{1}{2} f^{\xi} \cdot \nabla_q A_m - \Delta h^{yy}$. Then by (2.24), (2.35) and (3.7)

$$\|h^{1yy}\|^{U_m^1, \Theta_{m+1}}_{d\,,\, d-d_H} \le C(m) \, \varepsilon_m^{-2/3} \qquad (3.8)$$

As the operators J and A_m commute we may write equation (2.33) as follows:

$$\frac{\partial}{\partial \omega} f^{yy} + [f^{yy}, J A_m] = h^{1yy} \qquad (3.9)$$

Let us fix for a moment some functions $W_j(q;\theta)$, $j \in \mathbb{Z}_0$, such that $W_j = - W_{-j}$, and

$$W_j \in \mathscr{A}^R_{\Theta_{m+1}}(U_m^{\,1} ; \mathbb{C}) , \; | W_j |^{U_m^1, \Theta_{m+1}} \le C(m) \qquad \forall j \qquad (3.10)$$

(they will be chosen later) and denote $W(q;\theta) = \mathrm{diag}(\exp i \, W_j(q;\theta))$. Then

$$\frac{\partial}{\partial \omega} W^{\pm 1}(q;\theta) = \pm \, \mathrm{diag} \, (i \, \frac{\partial}{\partial \omega} W_j(q,\theta)) \, W^{\pm 1}(q;\theta) .$$

So if we substitute into (3.9)

$$f^{yy} = W F^{yy} W^{-1} , \;\; h^{1yy} = W H^{yy} W^{-1} , \qquad (3.11)$$

then by (3.5) we get for F^{yy} an equation

$$\frac{\partial}{\partial\omega'} F^{yy} + [F^{yy}, \operatorname{diag}(i(\lambda^1_j - \frac{\partial}{\partial\omega'} W_j))] = H^{yy}. \tag{3.12}$$

Let us take functions W_k be solutions of equations

$$\frac{\partial}{\partial\omega'} W_k(q;\theta) = \lambda^1_k(q;\theta) - \lambda'_k(\theta), \; \lambda'_k = \int \lambda^1_k(q;\theta)\, dq/(2\pi)^n. \tag{3.13}$$

If $\theta \in \Theta_m \setminus \Theta^1$ then the equations (3.13) may be solved just as equations (2.31) and by (3.6) estimates (3.10) take place for the solutions $W_j, j \in \mathbb{Z}_0$. By (3.10), (3.11) $\|\cdot\|^{U^1_m, \Theta_{m+1}}_{a\,,\,b}$–norms of operators h^{1yy} and H^{yy}, f^{yy} and F^{yy} differ by a factor no greater than $C^e(m)$. Thus to get estimate (2.39) for a solution of (2.33) is equivalent to get it for one of (3.12).

Let us mention that elements of the matrix $\{F_{jk}\}$ of the operator F^{yy} in the basis $\{W_j | j \in \mathbb{Z}_0\}$ are given by the formula $F_{jk} = \langle F^{yy} W_k, W_{-j} \rangle$ and the same is true for a matrix $\{H_{jk}\}$ of the operator H^{yy}. So we may apply quadratic forms corresponding to the operators in l.h.s. and r.h.s. of (3.12) to vectors W_k, W_{-j} and get equations on the matrix elements $F_{jk}(q;\theta)$:

$$\frac{\partial}{\partial\omega'} F_{jk}(q;\theta) + (\lambda'_k(\theta) - \lambda'_j(\theta))\, F_{jk} = H_{jk}(q;\theta). \tag{3.14}$$

For a vector–function $f(q)$, $q \in \mathbf{T}^n$, we denote by $\widehat{f}(s)$, $s \in \mathbb{Z}^n$, its Fourier coefficients: $f(q) = \sum \widehat{f}(s)\, e^{iq\cdot s}$. By (3.8) and Lemma B1

$$\|\widehat{H}^{yy}(s)\|^{\Theta_{m+1}, \mathrm{Lip}}_{d\,,\,d-d_H} \le C^e(m)\, \varepsilon_m^{-2/3} e^{-5/6\, \delta |s|}. \tag{3.15}$$

For the diagonal elements $\{h^1_{jj}\}$ of the matrix of operator h^{1yy} we have:

$$h^1_{jj}(q) = \tfrac{1}{2} < h^{1yy}(q)(\varphi^+_{|j|} + i(\mathrm{sgn}\ j)\,\varphi^-_{|j|}),\ \varphi^+_{|j|} - i(\mathrm{sgn}\ j)\,\varphi^-_{|j|} >$$
$$= \tfrac{1}{2}\,(< h^{1yy}(q)\,\varphi^+_{|j|},\varphi^+_{|j|} > + < h^{1yy}(q)\,\varphi^-_{|j|},\varphi^-_{|j|} >)\,. \tag{3.16}$$

So by the definitions of the functions b_j and operator h^{1yy}, $h^1_{jj}(q) \equiv 0\ \ \forall j$ and the same is true for the operator H^{yy}:

$$H_{jj}(q;\theta) \equiv 0 \qquad \forall j \tag{3.17}$$

By (3.17) equations (3.14) are equivalent to the following relations on Fourier coefficients:

$$i(\omega' \cdot s - \lambda'_j + \lambda'_k)\,\widehat{F}_{kj}(s) = \begin{cases} 0 & \text{if } k = j\,, \\ \widehat{H}_{kj} & \text{if } k \neq j\,. \end{cases}$$

Let us choose $\widehat{F}_{kk}(s) \equiv 0\ \ \forall k \in \mathbb{Z}_0$ and denote

$$D(k,j,s;\theta) = \begin{cases} i(\omega' \cdot s - \lambda'_j + \lambda'_k)\,, & j \neq k\,, \\ i\,, & j = k\,. \end{cases}$$

Then

$$\widehat{F}_{kj}(s) = \widehat{H}_{kj}(s;\theta)\,D^{-1}(k,j,s;\theta)\,. \tag{3.18}$$

<u>Lemma 3.1</u>. There exists a Borel subset $\Theta^2 \subset \Theta_m$ with the property (3.1) and a constant $c > 0$ such that if $\varepsilon_0 << 1$ and $\delta_a << 1$ then for all $\theta \in \Theta_m \setminus (\Theta^1 \cup \Theta^2)$ and for all $j,k \in \mathbb{Z}_0$, $j \neq k$, $s \in \mathbb{Z}^n$ the following estimate takes place:

$$|D^{-1}(k,j,s;\cdot)|^{\Theta_m \setminus \Theta^2,\mathrm{Lip}} \leq$$
$$\leq C(m)(1 + |s|)^{2c+1}(1 + |\lambda_{k0} - \lambda_{j0}|)^{-1}\,. \tag{3.19}$$

The proof is given in § 4 below.

For a map $g(k,p)$, $g : \mathbb{Z}_0 \times P \longrightarrow \mathbb{C}$, where P is an abstract set, we denote

$$|g(k,p)|_{\ell^r(k)} = \Big(\sum_{k\in\mathbb{Z}_0} |g(k,p)|^r \Big)^{1/r}$$

and treat g as a map from P to $\ell^r(\mathbb{Z}_0)$.

We have to estimate the norm of the operator $F^{yy}(q;\theta)$. By Lemmas B 1, B 2 this is equivalent to estimate the operator norms of Fourier coefficients $\widehat{F}^{yy}(s)$. For this end we:

(1) estimate matrix coefficients $\widehat{H}_{kj}(s)$ of operator $\widehat{H}^{yy}(s)$,

(2) estimate coefficients $\widehat{F}_{kj}(s)$ via the relation (3.18);

(3) estimate the norm of a matrix $\widehat{F}^{yy}(s)$ via coefficients $\widehat{F}_{kj}(s)$.

Step (1) is rather simple. Indeed, the matrix of the operator $\widehat{H}^{yy}(s) : Y^c_d \longrightarrow Y^c_{d-d_H}$ with respect to the basises $\{\lambda_k^{(-d)} w_k \,|\, k \in \mathbb{Z}_0\} \subset Y^c_d$ and $\{\lambda_k^{(-d+d_H)} w_k \,|\, k \in \mathbb{Z}_0\} \subset Y^c_{d-d_H}$ is equal to $\{\lambda_k^{(-d+d_H)} \widehat{H}_{kj}(s)\, \lambda_j^{(-d)}\}$. So by Lemma B1 $\forall j \in \mathbb{Z}_0$

$$\Big|\, |k|^{d-d_H} \widehat{H}_{kj}(s)\, |j|^{-d} \,\Big|^{\theta_m, \mathrm{Lip}}_{\ell^2(k)} \le C \,\|\widehat{H}^{yy}(s)\|^{\theta_m, \mathrm{Lip}}_{d,\, d-d_H} \le$$

$$\le C(m)\, \varepsilon^{-2/3}\, e^{-5/6\, \delta |s|} \, . \tag{3.20}$$

Step (2) results from (3.18) and Lemma 3.1:

$$|\widehat{F}_{kj}(s)|^{\theta_m \setminus (\theta^1 \cup \theta^2), \mathrm{Lip}} \le \frac{|\widehat{H}_{kj}(s)|^{\theta_m, \mathrm{Lip}}\; T_1}{1 + |\lambda_{k0} - \lambda_{j0}|} ,$$

$$T_1 = C_1(m)(1+|s|)^{2c+1}. \tag{3.21}$$

For Step (3) we have to glue the estimates (3.20), (3.21) in order to obtain estimates on $\widehat{F}^{yy}(s)$ and $F^{yy}(q)$. The operator $\widehat{F}^{yy}(s)$ from the space Y_d^c with the basis $\{\lambda_j^{(-d)} w_j\}$ into the space $Y_{d_c}^c$, $d_c = d - d_H + d_1 - 1$, with the basis $\{\lambda_j^{(-d_c)} w_j\}$ has the matrix

$$\{\lambda_k^{(d_c)} \widehat{F}_{kj} \lambda_j^{(-d)}\}. \tag{3.22}$$

Let us denote by $\pi_{k,j}$ $(j,k \in \mathbb{Z}_0)$ the function $\pi_{k,j} = 1 - \delta_{k,j}$. Then by (3.21) we have a trivial estimate for ℓ^1–norm of the column number j and its Lipschitz coefficient:

$$\begin{aligned} &|\lambda_k^{(d)} \widehat{F}_{kj}(s)\, \lambda_j^{(-d)}|_{\ell^1(k)}^{\Theta_{m+1},\mathrm{Lip}} \le \\ &\le T_1 |\ |k|^{d-d_H} \widehat{H}_{kj}(s)\, |j|^{-d}\, |_{\ell^2(k)}^{\Theta_m,\mathrm{Lip}} \ \left| \frac{\pi_{k,j}\ |k|^{d_1-1}}{1+|\lambda_{k0}-\lambda_{j0}|} \right|_{\ell^2(k)} \end{aligned} \tag{3.23}$$

To estimate the r.h.s. we need the following statement:

<u>Lemma 3.2</u>. If j_1 in (1.28) is large enough then $\forall j, k \in \mathbb{N}$

$$C_1\, |j^{d_1} - k^{d_1}| \ge |\lambda_{j0} - \lambda_{k0}| \ge C_1^{-1}\, |j^{d_1} - k^{d_1}|. \tag{3.24}$$

If $j > k > 0$ then

$$|\lambda_{j0} - \lambda_{k0}| \ge C_2 j^{d_1-1}. \tag{3.25}$$

<u>Proof</u>. For $j = k$ the inequalities (3.24) are evident. So we may suppose that $j > k$. Then by the assumption (1.26)

$$\lambda_{j0} - \lambda_{k0} = K_2(j^{d_1} - k^{d_1}) + \Delta(j, k),$$

$$|\Delta(j,k)| \le C \sum_{l=1}^{r-1} (j^{d_{1,l}} - k^{d_{1,l}}) + K_1 j^{d_{1,r}} + K_1 k^{d_{1,r}} \le$$

$$\le C_1(j)(j^{d_1} - k^{d_1}) + K_1 j^{d_{1,r}} + K_1 k^{d_{1,r}} \le C_2(j)(j^{d_1} - k^{d_1})$$

and $C_1(j)$, $C_2(j) \longrightarrow 0$ as $j \longrightarrow \infty$ (one has to mention that $j^{d_1} - k^{d_1} \ge j^{d_1} - (j-1)^{d_1} \ge C j^{d_1 - 1}$ and so $(j^{d_1} - k^{d_1}) j^{-d_{1,r}} \longrightarrow \infty$ $(j \longrightarrow \infty)$ because $d_{1,r} < d_1 - 1$). Now the estimate (3.24) is proved for j greater than some C_*. For $j, k \le C_*$ it is true with some $C_1 >> 1$ because $\inf\{|\lambda_{j0} - \lambda_{k0}| \;|1 \le k < j \le C_*\} > 0$ the assumption (1.28) with $s = 0$ and $-\ell_k = \ell_j = 1$ (one has to take $j_1 \ge C_*$).

Inequality (3.25) results from (3.24). ∎

By this lemma

$$\left| \frac{\pi_{k,j} \, |k|^{d_1 - 1}}{1 + |\lambda_{k0} - \lambda_{j0}|} \right|^2_{\ell^2(k)} \le C \left[\sum_{k=-\infty}^{|j|-1} + \sum_{k=|j|+1}^{\infty} \right] \frac{|k|^{2(d_1-1)}}{\left[|k|^{d_1} - |j|^{d_1} \right]^2}.$$

After a substitution $k = |j| y$ one can estimate the sums in the r.h.s. via integrals. So

$$\left| \frac{\pi_{k,j} \, |k|^{d_1 - 1}}{1 + |\lambda_{k0} - \lambda_{j0}|} \right|^2_{\ell^2(k)} \le \frac{C_1}{|j|} \left[\int_{-\infty}^{1 - |j|^{-1}} + \int_{1 + |j|^{-1}}^{\infty} \right] \frac{|y|^{2(d_1 - 1)} \, dy}{\left[|y|^{d_1} - 1 \right]^2} \le C_2 .$$

By (3.20), (3.23) and by the last estimate, ℓ^1–norm of the column number j of the matrix (3.22) and its Lipschitz constant are no greater than

$$L_1 = C_1(m) \, T_1 \varepsilon^{-2/3} e^{-5/6 \, \delta |s|} .$$

For ℓ^1–norm of the row number k of the matrix (3.22) and its Lipschitz constant we have the estimate:

$$| \lambda_k^{(d_c)} \widehat{F}_{kj}(s)\, \lambda_j^{(-d)} |_{\ell^1(j)}^{\Theta_{m+1},\mathrm{Lip}} \le$$

$$\le C\, T_1 \,|\, |k|^{d-d_H-1} \widehat{H}_{kj}(s) |j|^{1-d} |_{\ell^2(j)}^{\Theta_m,\mathrm{Lip}} \left| \frac{\pi_{k,j}}{|j|(1+|\lambda_{k0}-\lambda_{j0}|)} |k|^{d_1} \right|_{\ell^2(j)} \tag{3.26}$$

As $H^{yy}(q) \in \mathscr{L}^s(Y_d^c;\, Y^c_{d-d_H})$ then by the interpolation theorem (Corollary A2)

$$\|H^{yy}\|_{1-d+d_H,1-d}^{U_m^1,\Theta_m} \le 2\, \|H^{yy}\|_{d\,,\,d-d_H}^{U_m^1,\Theta_m} \le C\, C_*(m)\, \varepsilon^{-2/3}$$

and for the conjugate operator $(H^{yy})^*$ one has the estimate

$$\|(H^{yy})^*\|_{d-1,\,d-d_H-1}^{U_m^1,\Theta_m} \le C\, C_*(m)\, \varepsilon^{-2/3}\,.$$

Thus $\forall s,k$

$$|\, |k|^{d-d_H-1} \widehat{H}_{kj}(s) |j|^{1-d} |_{\ell^2(j)}^{\Theta_m,\mathrm{Lip}} \le C(m)\, \varepsilon^{-2/3} e^{-5/6\, \delta |s|} \tag{3.27}$$

and the first factor in the r.h.s. in (3.26) is estimated. For the second one the following estimate is true:

$$\left| \frac{\pi_{k,j} \, |k|^{d_1}}{|j| \, (1+|\lambda_{k0}-\lambda_{j0}|)} \right|^2_{\ell^2(j)} \le$$

$$\le \frac{C}{|k|} \left[\int_{-\infty}^{-|k|^{-1}} + \int_{|k|^{-1}}^{1-|k|^{-1}} + \int_{1+|k|^{-1}}^{\infty} \right] \frac{dy}{y^2(1-\mathrm{sgn}\ y \ |y|^{d_1})^2} \le C_1 \, .$$

Thus by (3.26), (3.27) ℓ^1–norm of the row number k is bounded above by the constant $L_2 = C_2(m)\, T_1\, \varepsilon^{-2/3} \exp -\frac{5}{6}\, \delta |s|$.

So the matrix (3.22) of the operator $\widehat{F^{yy}}(s) : Y^c_d \longrightarrow Y^c_{d_c}$ has columns and rows bounded in ℓ^1–norm together with their Lipschitz constants by $\max(L_1,L_2)$. Hence the norm of the operator is bounded by the same constant; for this classical result see [HLP, Chap. 8] or [HS]. We have got the estimate

$$\| \widehat{F^{yy}}(s) \|^{\theta_{m+1},\mathrm{Lip}}_{d\,,\,d_c} \le C(m)\, T_1\, \varepsilon^{-2/3} e^{-5/6\ \delta |s|} \, .$$

By it and Lemma B2

$$\| f^{yy} \|^{U^2_m, \theta_{m+1}}_{d\,,\,d_c} + \sum_{j=1}^{n} \| \frac{\partial}{\partial q_j} f^{yy} \|^{U^2_m, \theta_{m+1}}_{d\,,\,d_c} \le C^e_1(m)\, \varepsilon^{-2/3} \tag{3.28}$$

because the norm of f^{yy} is equivalent to the norm of F^{yy} up to a factor $C^e(m)$. So (2.39) is proved for $a = d$. The estimate (2.40) results from the equality (2.33) and from estimate (3.28).

The symmetry of the operators F^{yy} and f^{yy} results from the one of the Fourier coefficients $\widehat{F^{yy}}(s)$ (formula (3.18)). For $q \in \mathbb{T}^n$ the operator $f^{yy}(q)$ is real, i.e. it maps Y_d into Y_{d-d_H} because the operators $h^{yy}(q)$, $q \in \mathbf{T}^n$, are real. So

$f^{yy}(q) \in \mathscr{L}^{s}(Y^c_d\,;Y^c_{d_c})$. Now the validity of the estimate (2.39) $\forall a \in [-d_c,d]$ results from the estimate for $a = d$, from the symmetry of operator f^{yy} and interpolation theorem (Corollary A2) . The assertion c) is proved.

We give now a sketch of a proof of the assertion b). Let us substitute into (2.32) $f^y = W\,F^y$, $h^y = WH^y$. Then

$$\frac{\partial}{\partial \omega'} F^y - \left[J\,A_m - \operatorname{diag}\left[i\,\frac{\partial}{\partial \omega'} W_j\right]\right] F^y = H^y$$

or

$$(i\omega' \cdot s)\,\widehat{F}^y - \operatorname{diag}(i\,\lambda'_j(\theta))\,\widehat{F}^y = \widehat{H}^y . \tag{3.29}$$

Let

$$\widehat{F}^y(s) = \sum_{j\in\mathbb{Z}_0} \widehat{F}_j(s)w_j\,,\ \widehat{H}^y(s) = \sum_{j\in\mathbb{Z}_0} \widehat{H}_j(s)w_j$$

Then by (3.29)

$$\widehat{F}_j(s) = D_1^{-1}(j,s;\theta)\,\widehat{H}_j(s),\ D_1(j,s;\theta) = i(s\cdot\omega' - \lambda'_j) \tag{3.30}$$

By (2.30), (3.10) and Lemma B1

$$\|\widehat{H}^y(s)\|^{\Theta_{m+1},\mathrm{Lip}}_{d-d_H} \le C^e(m)\,\varepsilon^{-1/3}e^{-5/6\,\delta|s|} . \tag{3.31}$$

To estimate D_1^{-1} we use an analog of Lemma 3.1 (it will be proved in § 4):

<u>Lemma 3.3</u>. There exists a Borel subset $\Theta^3 \subset \Theta_m$ with the property (3.1) and such that

$$|D_1^{-1}|^{\Theta_m \setminus \Theta^3, \mathrm{Lip}} \leq C\, C_{**}(m)\, |s|^{2n+3} . \tag{3.32}$$

By equality (3.30) and estimates (3.31), (3.32)

$$\|\widehat{F}^y(s)\|_{d-d_H}^{\Theta_{m+1}, \mathrm{Lip}} \leq C_1^e(m)\, |s|^{2n+5} e^{-5/6\, \delta |s| \varepsilon^{-1/3}} .$$

So by Lemma B2

$$\|f^y\|_{d-d_H}^{U_m^2, \Theta_{m+1}} + \|\nabla_q f^y\|_{d-d_H}^{U_m^2, \Theta_{m+1}} \leq C^e(m)\, \varepsilon^{-1/3}$$

and the estimate (2.37) results from the equality (2.32). ∎

4. Proof of Lemmas 3.1, 3.3 (estimation of small divisors)

The estimate (3.19) results easily from the following one:

$$|D(k,j,s;\theta)| \geq \frac{|\lambda_{k0} - \lambda_{j0}|}{C_1(m)(1+|s|)^c}$$
$$\forall k \neq j \in \mathbb{Z}_0 , \forall s \in \mathbb{Z}^n , \forall \theta \in \Theta_m \setminus (\Theta^1 \cup \Theta^2) . \tag{4.1}$$

Indeed, $\mathrm{Lip}\, D^{-1} \leq (\mathrm{Lip}\, D)(\inf|D|)^{-2}$ and by the estimates (3.6), (1.27) $\mathrm{Lip}\, D(k,j,s;\cdot) \leq C(|s| + 1 + \max\{|j| , |k|\}^{d_1 - 1})$. So (4.1) and (3.25) imply (3.19) .

We may suppose that $|j| \geq |k|$ and $j > 0$ because $|D(k,j,s)| = |D(j,k,s)| = |D(-k,-j,-s)|$. So in what follows

$$j > 0 , |k| \leq j , k \neq j . \tag{4.2}$$

By estimates (3.6)

$$|\lambda_j' - \lambda_{j0}| \le C(\delta_a\,|j|^{d_{1,r}} + \varepsilon_0^\rho\,|j|^{d_H}) \qquad \forall j,\ \forall\theta \tag{4.3}$$

By this estimate and (4.2), (3.25) we have for $\delta_a, \varepsilon_0 << 1$ inequalities

$$|\lambda_k' - \lambda_j'| \ge |\lambda_{k0} - \lambda_{j0}| - |\lambda_k' - \lambda_{k0}| - |\lambda_j' - \lambda_{j0}| \ge \tfrac{1}{2}|\lambda_{k0} - \lambda_{j0}| +$$

$$+ \tfrac{1}{2} C_2 |j|^{d_1 - 1} - C_1(\delta_a + \varepsilon_0^\delta)|j|^{d_1-1} \ge \tfrac{1}{2}|\lambda_{k0} - \lambda_{j0}|\,. \tag{4.4}$$

If $2|\omega' \cdot s| \le |\lambda_k' - \lambda_j'|$ then by (4.4) $|D| \ge \frac{1}{2}|\lambda_k' - \lambda_j'| \ge \frac{1}{4}|\lambda_{k0} - \lambda_{j0}|$ and the estimate (4.1) is obtained. So we may suppose below that

$$2|\omega' \cdot s| \ge |\lambda_k' - \lambda_j'| \tag{4.5}$$

In particular, $s \ne 0$. By (4.4), (4.5) and (3.24), (3.25)

$$j^{d_1-1} \le C|\omega' \cdot s|\,, \tag{4.6}$$

$$|\lambda_{k0} - \lambda_{j0}| \le C|s|\,, \tag{4.7}$$

$$|j^{d_1} - |k|^{d_1}| \le C_1|s|\,. \tag{4.8}$$

Situations $d_1 = 1$ and $d_1 > 1$ have to be considered separately. We start with the more difficult one.

A) $d_1 = 1$. Then in (1.25) $d_H < -\chi$ and in (1.26) $d_{1,r} \le -\chi$ and $d_{1,j} \le 1 - \chi$ $\forall j$ for some $0 < \chi < 1$. Depending on the relation between k and s, we consider three cases.

A1) $|s| \le 9\,K_1|k|^{-\chi} + \frac{1}{2}$. Then

$$|k| \leq (18\,K_1)^{1/\chi}\,,\ |s| \leq 9\,K_1 + \tfrac{1}{2} \tag{4.9}$$

because $|s| \geq 1$, $|k| \geq 1$. By (4.8)

$$j \leq C_1|s| + |k| \leq C_1(9\,K_1 + \tfrac{1}{2}) + (18\,K_1)^{1/\chi} = C_{1*}$$

Let us take in the assumption 3) of the theorem $j_1 \geq C_{1*}$ and $M_1 \geq 9\,K_1 + \frac{1}{2}$. Then by (3.3), (4.3) and (1.28) with $\ell_j = 1$, $\ell_{|k|} = -\operatorname{sgn} k$ (or $\ell_j = 2$ if $k = -j$) we have

$$|D| \geq |\omega_0 \cdot s + \lambda_{k0} - \lambda_{j0}| - (9\,K_1 + \tfrac{1}{2})\,|\omega' - \omega_0| -$$

$$- |\lambda'_k - \lambda_{k0}| - |\lambda'_j - \lambda_{j0}| \geq K_5 - C|\varepsilon_0^{\rho} + \delta_a|\,.$$

Now the estimate (4.1) results from the last one because $0 \leq j \leq C_{1*}$, $|k| \leq (18\,K_1)^{1/\chi}$.

A2) $|s| > 9\,K_1 k^{-\chi} + \frac{1}{2}$, $|k| \leq C_{*3}(m)\ |s|^{m_0}$. Here $m_0 \geq \chi^{-1}(n+3)$ and a function $C_{*3}(m)$ will be chosen later. By (3.6) and (1.27) $\operatorname{Lip}(\lambda'_k - \lambda'_j) \leq 3\,K_1|k|^{-\chi}$ if $\varepsilon_0 << 1$. So

$$|s| \geq 3\,\operatorname{Lip}(\lambda'_k - \lambda'_j) + \tfrac{1}{2}\,. \tag{4.10}$$

Let

$$T = T(k,j,s) = C_{*4}^{-1}(m)|s|^{-m_1}|\lambda_{j0} - \lambda_{k0}|, \qquad m_1 = m_0 + n + 2\,,$$

and

$$\Theta'(k,j,s) = \{\theta \in \Theta_m \mid\ |D(k,j,s;\theta)| \leq T\}\,,$$

$$\Theta^{2,1} = \cup \{\Theta'(k,j,s) \mid (9\, K_1 |k|^{-\chi} + \tfrac{1}{2}) < |s| \;,\; |k| \le j \;,$$

$$|k| \le C_{*3}(m)\, |s|^{m_0} \;,\; |\lambda_{k0} - \lambda_{j0}| \le C|s| \} \,.$$

We shall construct a set Θ^2 as $\Theta^2 = \Theta^{2,1} \cup \Theta^{2,2}$ (the set $\Theta^{2,2}$ will be defined later). Therefore, if $\theta \notin \Theta^2$ then $\theta \notin \Theta^{2,1}$ and $|D| \ge T$. So (4.1) is true.

We have to estimate mes $\Theta^{2,1}$ [I]. For this end we estimate mes Θ' (k,j,s) [I]. By the estimate (3.3$'$) mes Θ' (k,j,s) [I] ≤ 2 mes Ω' (k,j,s) [I]. Here Ω' (k,j,s) [I] is the image of the set Θ' (k,j,s) [I] under the map (3.2). To estimate mes Ω' [I] it is enough to estimate one–dimensional Lebesque measure of the intersection of Ω' [I] with an arbitrary line of a form $\{\omega' = \omega'(t) = \eta + t\, s|s|^{-1} \mid t \in \mathbb{R}\}$, $\eta \in \mathbb{R}^n$. The set of "t" corresponding to this intersection is contained in the set

$$\{t \mid -T \le \Gamma(t) \le T\} \;,\; \Gamma(t) = \eta \cdot s + t|s| + (\lambda_k' - \lambda_j')(\omega'(t)) \qquad (4.11)$$

By (3.3) Lip $(\omega : \omega' \mapsto \omega) \le \frac{3}{2}$ if $\varepsilon_0 << 1$ $\forall I \in \mathcal{J}$. So by (4.10) Lip $(t \mapsto (\lambda_k' - \lambda_j')(\omega'(t)) \le \frac{1}{2}|s| - \frac{1}{4}$. Hence for $t_1 > t_2$

$$\Gamma(t_1) - \Gamma(t_2) \ge |s|(t_1 - t_2) - |(\lambda_k' - \lambda_j')(\omega'(t_1)) -$$

$$- (\lambda_k' - \lambda_j')(\omega'(t_2))| \ge |t_1 - t_2|\, (\tfrac{1}{2}|s| + \tfrac{1}{4})$$

and the measure of the set (4.11) is no greater than $2T(\frac{1}{2}|s| + \frac{1}{4})^{-1}$. Since the set Ω' [I] is bounded and the vector η may be chosen orbitrarily, we have by Fatou lemma: mes Ω' (k,j,s) [I] $\le CT(|s| + 1)^{-1}$. So

$$\text{mes}\, \Theta^{2,1}[I] \le \sum_{k,j,s} \text{mes}\, \Theta'(k,j,s)\ [I] \le 2 \sum_{s \ne 0} \sum_{j,k} \text{mes}\, \Omega'(k,j,s)\ [I] \,.$$

As $|k| \le C_{*3}(m)\,|s|^{m_0}$ and $|j-k| \le C_1|s|$, then we have no more than $C\,C_{*3}(m)\,|s|^{m_0+1}$ admissible pairs (j,k) . As $|\lambda_{k0}-\lambda_{j0}| \le C|s|$ then $T \le C|s|^{1-m_1}\,C_{*4}^{-1}(m)$ and

$$\text{mes}\,\Theta^{2,1}[I] \le C \sum_{s\neq 0} \frac{C_{*3}(m)\;|s|^{m_0+1}}{C_{*4}(m)\;|s|^{m_1}} \le \frac{C_1\;C_{*3}(m)}{C_{*4}(m)} .$$

Therefore, under a suitable choice of the function $C_{*4}(m)$, depending on the choice of $C_{*3}(m)$, mes $\Theta^{2,1}[I]$ is no greater than one–half of the r.h.s. of (3.1) .

A_3) $|k| \ge C_{*3}(m)\,|s|^{m_0}$, $s \neq 0$. Then by (4.2) and (4.7) $j > k > 0$. By (1.26) (with $d_1 = 1$, $d_{1,r} \le -\chi$, $d_{1,j} \le 1-\chi$ $\forall j = 1, \ldots, r-1$) , (4.3) (with $d_{1,r} < -\chi$, $d_H < -\chi$) and (4.7) we have

$$\begin{aligned} |\lambda_j' - \lambda_k' - K_2(j-k)| &\le C\,k^{-\chi}(|j-k|+1) \le \\ &\le C\,C_{*3}{}^{-\chi}(m)\;|s|^{1-\chi m_0}. \end{aligned} \tag{4.12}$$

Let us set

$$\Omega''(s,N) = \left\{ \omega' \;\middle|\; |\omega' - \omega_0| \le 1 \,,\ |\omega' \cdot s - NK_2| \le \frac{|\lambda_{j0} - \lambda_{k0}|}{C_{*2}(m)\;|s|^{n+2}} \right\}$$

(a function $C_{*2}(m)$ will be chosen later) and

$$\Theta^{2,2} = \cup\ \{\theta \in \Theta_m \,|\, \omega'(\theta) \in \Omega''(s,N)\} .$$

Here we take the union over all $s \in \mathbb{Z}_0^n$ and $N \in \mathbb{Z}$. The set $\Omega''(s,N)$ is empty if

$|N| \geq C|s|\ K_2^{-1}$; by (4.7) mes $\Omega''(s,N) \leq C(C_{*2}(m)\ |s|^{n+2})^{-1}$. So by (3.3$'$)

$$\text{mes}\ \Theta^{2,2}[I] \leq \sum_{s\neq 0}\ \sum_{|N|\leq C|s|/K_2} \frac{C_1}{C_{*2}(m)\ \ |s|^{n+2}} \leq \frac{C_2}{C_{*2}(m)}$$

and mes $\Theta^{2,1}[I]$ is no greater than one–half of r.h.s. of (3.1) if the function $C_{*2}(m)$ is large enough.

If $\theta \notin \Theta^{2,2}$ then by (3.25) and (4.12), by the definition of $\Omega''(s,N)$ and by the inequality $m_0 \geq (n+3)/\chi$

$$|D| = |(\lambda'_j - \lambda'_k - K_2(j-k)) + (K_2(j-k) - \omega' \cdot s)| \geq$$

$$\geq |\lambda_{j0} - \lambda_{k0}|\ C_{*2}^{-1}(m)\ |s|^{-n-2} - C\ C_{*3}^{-\chi}(m)\ |s|^{1-\chi\, m_0} \geq$$

$$\geq \tfrac{1}{2} |\lambda_{j0} - \lambda_{k0}|\ C_{*2}^{-1}(m)\ |s|^{-n-2}\ ,$$

if $C_{*3}(m)$ is large enough. The inequality (4.1) for $\theta \in \Theta_m \setminus \Theta^{2,2}$ results from the last one.

Now the lemma is proved for $d = 1$ with $\Theta^2 = \Theta^{2,1} \cup \Theta^{2,2}$.

B) $d_1 > 1$. Let us find $\chi \in (0,1)$ such that $d_1 - 1 > \chi$ and $d_{1,r} \leq d_1 - 1 - \chi$.

By the inequality (4.6)

$$|s| \geq C_*\, j^{d_1 - 1}\ . \tag{4.13}$$

Let us denote $j_* = (12\ K_1\ C_*^{-1})^{1/\chi}$, $j_{**} = 3\, j_*^{d_1-1} (K_1\, j_*^{-\chi} + 1) + 1$ and consider two cases.

B1) $j \le j_*$, $|s| \le j_{**}$. In this case the estimate (4.1) results from (4.3) and assumption 3) of the theorem if $j_1 \ge j_*$, $M_1 \ge j_{**}$ and $\varepsilon_0 << 1$, $\delta_a << 1$.

B2) $j > j_*$ or $|s| > j_{**}$. Let the sets $\Theta'(k,j,s)$ and $\Omega'(k,j,s)[I]$ be the same as in the item A2) and

$$\Theta^2 = \cup \{\Theta'(k,j,s) \mid j > |k| \, , \, |\lambda_{k0} - \lambda_{j0}| \le C\,|s| \, , \, |s| \ge C_* j^{\chi}\} .$$

Then for $\theta \in \Theta_m \backslash \Theta^2$ the estimate (4.1) is true. So we have to estimate mes $\Theta^2[I]$.

By (1.27) and (3.6)

$$\mathrm{Lip}\,(\omega' \mapsto \lambda_r'(\omega')) \le \tfrac{3}{2} j^{d_1-1}(K_1 j^{-\chi} + \varepsilon_0^{\rho}) \, , \, r = k,j .$$

By this estimate and (3.3) we have for the function $\Gamma(t)$ (see (4.11)):

$$\Gamma(t_1) - \Gamma(t_2) \ge |s|\,(t_1 - t_2) - 3\,j^{d_1-1}(K_1 j^{-\chi} + \varepsilon_0^{\rho})\,(t_1 - t_2) .$$

If $j > j_*$ then by (4.13) for $t_1 > t_2$

$$\Gamma(t_1) - \Gamma(t_2) \ge (t_1 - t_2) j^{d_1-1}\,(C_* - 3\,(K_1\,j^{-\chi} + \varepsilon_0^{\rho})) \ge \tfrac{1}{2}\,C_*(t_1 - t_2) ,$$

if $j \le j_*$ then $|s| > j_{**}$ and

$$\Gamma(t_1) - \Gamma(t_2) \ge (t_1 - t_2)(|s| - 3\,j_*^{d_1-1}\,(K_1\,j_*^{-\chi} + 1) \ge t_1 - t_2 .$$

So mes $\Omega'(k,j,s)[I] \le C_1 T$ and mes $\Theta^2[I] \le C \sum\limits_{s \ne 0} \sum\limits_{k,j} T(k,j,s)$.

By (4.13) there are no more than $C|s|^{2\chi}$ admissible pairs (j,k) ; by (4.7) $|\lambda_{k0} - \lambda_{j0}| \le C|s|$. So

$$\text{mes}\,\Theta^2[I] \le C \sum_{s\neq 0} C_{*4}^{-1}(m)|s|^{1+2\chi-m_1} \le \frac{C_1}{C_{*4}(m)},$$

if $m_1 \ge n+2+2\chi$, and the estimate (3.1) is fulfilled if $C_{*4}(m)$ is large enough.

The lemma is proved. ∎

Proof of the Lemma 3.3. Let us define the set Θ^3 as follows:

$$\Theta^3 = \cup\,\{\Theta'(j,s)\,|\,s \in \mathbb{Z}^n,\ j \in \mathbb{Z}_0\},$$

$$\Theta'(j,s) = \{\theta \in \Theta_m\,|\ |s\cdot\omega'(\theta) - \lambda_j'(\theta)| \le C_{**}^{-1}(m)(1+|s|)^{-n-1}\}.$$

By the assumption (1.26) the set Θ' is empty if $|j| \ge C|s|^{1/d_1}$. By (1.26) and (1.28) with $s=0$, $|\ell_1|+\ldots+|\ell_{j_1}|=1$ and M_1 large enough, $|\lambda_j(\theta)| \ge C^{-1}$ $\forall j,\theta$. So by (4.3) this set is empty if $s=0$ provided that $\varepsilon_0 \ll 1$, $\delta_a \ll 1$ and $C_*(m) \gg 1$. Thus we may suppose that

$$|j| \le C|s|^{1/d_1},\ s\neq 0. \tag{4.14}$$

As in the proof of Lemma 3.1 we get that $\text{mes}\,\Theta'(j,s)[I] \le C\,C_{**}^{-1}(m)\,|s|^{-n-2}$. So by (4.14)

$$\text{mes}\,\Theta^3[I] \le \frac{C}{C_{**}(m)} \sum_{s\neq 0}\ \sum_{j\le C|s|} (1+|s|)^{-n-2} \le \frac{C_1}{C_{**}(m)}$$

and (3.1) is true if $C_{**}(m)$ is large enough. If $\theta \notin \Theta^3$ then $|D| \ge C_{**}(m)^{-1}(1+|s|)^{-n-1}$ and (3.32) is proved.

5. Proof of Lemma 2.3 (estimation of the change of variables)

Let us denote by $E^{c\,\sigma}_{s,\varepsilon}$, $s \in \mathbb{R}$, $\sigma = \pm$, the space $E^c_s = \mathbb{C}^{2n} \times Y^c_s$ endowed with the norm $\|\cdot\|_{(\sigma,s,\varepsilon)}$,

$$\|(p,\xi,y)\|^2_{(\pm,s,\varepsilon)} = |p|^2 + \varepsilon^{\pm\frac{4}{3}} |\xi|^2 + \varepsilon^{\pm\frac{2}{3}} \|y\|^2_{\mp s} .$$

The following assertion results from the definition.

Lemma 5.1. For all $s \in \mathbb{R}$ the spaces $E^{c\,\pm}_{s,\varepsilon}$ are dual with respect to the bilinear pairing $<\cdot,\cdot>_E : E^c \times E^c \longrightarrow \mathbb{C}$,

$$\|\mathfrak{h}\|_{(\pm,s,\varepsilon)} = \sup_{\|\mathfrak{h}^*\|_{(\mp,s,\varepsilon)} \leq 1} |< \mathfrak{h},\mathfrak{h}^* >_E| .$$

We denote by $\mathrm{dist}_{(s,\varepsilon)}$ the metric in $\mathscr{Y}^c_s$ induced by $\|\cdot\|_{(-,s,\varepsilon)}$.

Let us write down the system (2.28) in the form:

$$\dot{\mathfrak{h}} = \varepsilon\, \mathscr{F}(\mathfrak{h}) ,\ \mathfrak{h} = \mathfrak{h}(t) = (q(t), \xi(t), y(t)) ,$$
$$\mathscr{F} = (\mathscr{F}^q, \mathscr{F}^\xi, \mathscr{F}^y),\ \mathscr{F}^q = \nabla_\xi F,\ \mathscr{F}^\xi = -\nabla_q F,\ \mathscr{F}^y = J\nabla_y F . \tag{5.1}$$

If $\varepsilon_0 << 1$, then for $j = 1 , \dots , 5$

$$\mathrm{dist}_{(s,\varepsilon)} (O_m^{\ j+1,c} , O_m^{\ c} \setminus O_m^{\ jc}) \geq C^{-1}(m) . \tag{5.2}$$

By Lemma 2.2 and Cauchy estimate

$$\varepsilon\|\mathcal{F}\|^{O_m^{3c}\times\Theta_{m+1},\mathrm{Lip}}_{(-,d\ +\ \Delta d,\varepsilon)} \leq C^e(m)\varepsilon^{1/3} \quad ,\Delta d = d_1 - d_H - 1 . \tag{5.3}$$

By (5.2) and (5.3) for $0 \leq t \leq 1$ and $\varepsilon_0 << 1$ the solution of (5.1) depends analytically on $\mathfrak{h}(0) \in O_m^{4c}$ and stays inside O_m^{3c}. So (2.43) is proved.

For every $\mathfrak{h} \in O_m^{3c}$ the following estimate on the tangent map $\mathcal{F}_*$ results by Lemma 2.2:

$$\|\varepsilon\ \mathcal{F}_*(\mathfrak{h};\cdot)\|^{\Theta_{m+1},\mathrm{Lip}}_{(-,\,a,\varepsilon)\ ,\ (-,a\,+\,\Delta d,\varepsilon)} \leq C^e(m)\ \varepsilon^{1/3}$$
$$\forall a \in D = [-d - \Delta d, d]\ . \tag{5.4}$$

For $t \in [0,1]$ let us set $\eta(t) = S^t{}_*(\mathfrak{h})\eta$. Then $\eta(t)$ is a solution of the Cauchy problem

$$\dot\eta(t) = \varepsilon\ \mathcal{F}_*(\mathfrak{h}(t))\ \eta(t)\ ,\ \eta(0) = \eta\ ,\ \mathfrak{h}(t) = S^t(\mathfrak{h})\ .$$

By (5.4) for $\mathfrak{h} \in O_m^{4c}$ and $a \in D$ we get the estimates:

$$\|S^t_*(\mathfrak{h}) - \mathrm{Id}\ \|^{\Theta_{m+1},\mathrm{Lip}}_{(-,\,a,\varepsilon)\ ,\ (-,a\,+\,\Delta d,\varepsilon)} \leq tC^e(m)\ \varepsilon^{1/3} \tag{5.5}$$

and

$$\|S^t_*(\mathfrak{h}) - \mathrm{Id} - t\varepsilon\ \mathcal{F}_*(\mathfrak{h})\|^{\Theta_{m+1},\mathrm{Lip}}_{(-,\,a,\varepsilon)\ ,\ (-,a\,+\,\Delta d,\varepsilon)} \leq t^2C^e_1(m)\ \varepsilon^{2/3} \tag{5.6}$$

The first of them results from the identity

$$\eta(t) - \eta = \varepsilon \int_0^t \mathcal{F}_*(\mathfrak{h}(\tau))\ \eta(\tau)\ d\tau$$

and the second one results from the identity

$$\eta(t) - \eta - \varepsilon t\ \mathscr{F}_*(\mathfrak{h})\eta = \varepsilon \int_0^t (\mathscr{F}_*(\mathfrak{h}(\tau))\ \eta(t) - \mathscr{F}_*(\mathfrak{h})\ \eta)\ d\tau\ .$$

Let $\mathfrak{h}(t) = (q(t)\ ,\ \xi(t)\ ,\ y(t))$ be a solution of (5.1) with $\mathfrak{h}(0) = \mathfrak{h} = (q,\xi,y)$. Then

$$\dot{q}(\tau) = \varepsilon\ f^{\xi}(q(\tau))\ .$$

So $\Pi_q \circ S^\tau(\mathfrak{h}) = S_q^\tau(q;\theta)$ (i.e. does not depend on ξ and y) and by (2.35)

$$\begin{aligned} |\,S_q^\tau(q) - q\,|^{U_m^3, \Theta_{m+1}} &\le \tau\, C(m)\, \varepsilon^{1/3}\ , \\ |\,S_q^\tau(q) - q - \tau\,\varepsilon\, f^{\xi}(q)\,|^{U_m^3, \Theta_{m+1}} &\le \tau^2\ C_1(m)\ \varepsilon^{2/3} \end{aligned} \tag{5.7}$$

By the first estimate with $\tau = 1$ we get the assertion (2.45).

For $y(t)$ we have the equation

$$\dot{y}(t) = 2\varepsilon\ J\ f^{yy}(q(\tau))y + \varepsilon\ J\ f^{y}(q(\tau))\ . \tag{5.8}$$

Let $z(t) = z(t)\ (q;\theta)$ be a solution of (5.8) with zero Cauchy data. Then by (2.36), (2.39), (2.40) and (5.7)

$$\|z(t)\|^{U_m^4, \Theta_{m+1}}_{d-d_H\ +\ d_1} \le t\ C^e(m)\ \varepsilon^{2/3} \qquad \forall t \in [0,1]\ . \tag{5.9}$$

Let us substitute into (5.8) $y(t) = z(t) + u(t)$. Then

$$\dot{u} = 2\varepsilon\ J\ f^{yy}(q(\tau))u\ ,\ u(0) = y\ . \tag{5.10}$$

So $u(t) = y + U(t)y$, here $U(t)$ is a linear operator and by (2.39), (5.7)

$$\|U(t)\|_{d\,,\,d\,+\,\Delta d}^{U_m^4,\,\theta_{m+1}} \le t\, C^e(m)\, \varepsilon^{1/3} \qquad \forall t \in [0,1] \ . \tag{5.11}$$

So $\prod_y^0 S_m(q,\xi,y) - y = z(1)\,(q;\theta) + U(1)\,(q;\theta)y$ and the estimate (2.47) results from (5.9), (5.11).

The estimate (2.46) results from the equation on $\xi(t)$ and the estimates on $q(t)$, $y(t)$. Now (2.44) results from (2.45) – (2.47) .

The transformation $S_m = S^t|_{t=1}$ is canonical as a shift along the trajectories of a Hamiltonian flow (see Part 1, Theorem 2.4). To investigate the transformed hamiltonian $\mathscr{H}_m \circ S_m$ we start with an analysis of the quadratic term $\mathfrak{A}(t) = \frac{1}{2} < A_m(q(t))\, y(t)\, ,\, y(t) >$ with $y(t) = z(t) + u(t) = z(t)\,(q;\theta) + y + U(t)\,(q,\theta)y$. It is equal to the sum of terms of zero order, first order and second order on y :

$$\mathfrak{A}(t) = \mathfrak{A}_0(t) + < \mathfrak{A}^y(t),y > + < \mathfrak{A}^{yy}(t)y,y > \, , \tag{5.12}$$

$$\mathfrak{A}_0(t)\,(q;\theta) = \tfrac{1}{2} < A_m(q(t);\theta)\, z(t),\, z(t) > \, ,$$

$$\mathfrak{A}^y(t)\,(q;\theta) = (I + U(t))^* A_m(q(t))\, z(t) \, ,$$

$$\mathfrak{A}^{yy}(t)\,(q;\theta) = \tfrac{1}{2}\,(I + U(t))^* A_m(q(t))\,(I + U(t)) \, .$$

Lemma 5.2. The following estimates are valid:

$$\begin{aligned} &\|\mathfrak{A}^{yy}(1) - \mathfrak{A}^{yy}(0) - \tfrac{\varepsilon}{2} f^{\xi}(q) \cdot \nabla_q A_m(q) - \\ &- [J\, A_m(q)\, ,\, \varepsilon f^{yy}(q)]\, \|_{d\,,\,d-d_H}^{U_m^4,\,\theta_{m+1}} \le C^e(m)\, \varepsilon^{2/3} \, , \end{aligned} \tag{5.13}$$

$$\|\mathfrak{A}^{y}(1)-\mathfrak{A}^{y}(0)-J\,A_m(q)f^{y}(q)\|_{d-d_H}^{U_m^4,\Theta_{m+1}} \le C^e(m)\varepsilon\,, \tag{5.14}$$

$$|\mathfrak{A}_0(1)|^{U_m^4,\Theta_{m+1}} \le C^e(m)\,\varepsilon^{4/3}\,. \tag{5.15}$$

Proof. By the definition of $\mathfrak{A}^{yy}(t)$ we get the equality:

$$< (\mathfrak{A}^{yy}(1)-\mathfrak{A}^{yy}(0))\,y,y > = \frac{1}{2}\int_0^1 \frac{d}{dt} < A_m(q(t))\,u(t)\,,\,u(t) > dt =$$

$$= \int_0^t < B(t)\,u(t)\,,\,u(t) > + \frac{1}{2} < (\varepsilon\, f^{\xi}(q(t))\cdot \nabla A_m(q(t)))\,u(t)\,,\,u(t) > dt \tag{5.16}$$

with $B(t) = [J\,A_m(q(t))\,,\,\varepsilon\, f^{yy}(q(t))]$. By (5.7) and (2.40)

$$\|B(t)\|_{d\,,\,d-d_H}^{U_m^4,\Theta_{m+1}} \le C^e(m)\,\varepsilon^{1/3}\,, \tag{5.17}$$

$$\|B(t)-B(0)\|_{d\,,\,d-d_H}^{U_m^4,\Theta_{m+1}} \le t\,C^e(m)\,\varepsilon^{2/3}\,. \tag{5.18}$$

By (5.7) and (2.9), (2.10)

$$\|\,\Big|_{\tau=0}^{t}(\varepsilon\, f^{\xi}(q(\tau))\cdot \nabla A_m(q(\tau)))\|_{d\,,\,d-d_H}^{U_m^4,\Theta_{m+1}} \le t\,C(m)\,\varepsilon^{2/3}\,. \tag{5.19}$$

Now we may replace the integrand in (5.16) by its value at $t = 0$ and get the estimate (5.13) from (5.18), (5.19) and (5.11).

To prove (5.14) we rewrite $<(\mathfrak{A}^y(1)-\mathfrak{A}^y(0))\,,\,y>$ as follows:

$$<(\mathfrak{A}^y(1)-\mathfrak{A}^y(0))\,,\,y> = \int_0^1 \frac{d}{dt} < A_m(q(t))\,z(t)\,,\,u(t) > dt =$$

$$= \int_0^1 \Big(< (\tfrac{d}{dt} A_m(q(t))) + A_m\,(q(t))\,(2\,\varepsilon\,J\,f^{yy}(q(t))z + \varepsilon\,J\,f^y(q(t))\,,\,u(t) > +$$

$$+ < A_m\,(q(t))\,z(t)\,,\,2\,\varepsilon\,J\,f^{yy}(q(t))u(t) > \Big)\,dt\,.$$

If $\|y\|_{-d+d_H} \leq 1$ then by (5.12), (5.8) and estimates on f^{yy}, f^y this integral differs from $< J\,A_m(q)\,\varepsilon\,f^y(q)\,,\,y>$ by $C^e(m)\,\varepsilon$, as stated in (5.14).

The last estimate of the lemma results from (5.10). ∎

By (5.12), $\nabla(\mathfrak{A}(1)-\mathfrak{A}(0)) = (\mathfrak{A}^y(1)-\mathfrak{A}^y(0)) + 2(\mathfrak{A}^{yy}(1)-\mathfrak{A}^{yy}(0))\,y$. So we have the following consequence from this lemma:

<u>Corollary 5.3</u>. For $\mathfrak{h} \in O^c_{m+1}$

$$\|\nabla_y(\mathfrak{A}(1)-\mathfrak{A}(0) - \tfrac{1}{2} < \varepsilon\,f^{\xi}(q)\cdot\nabla_q\,A_m(q)\,y\,,\,y> -$$

$$- < [J\,A_m(q)\,,\,\varepsilon\,f^{yy}(q)]\;y\,,\,y> - < J\,A_m(q)\,f^y(q)\,,\,y>)\|^{U^c_{m+1},\Theta_{m+1}}_{d-d_H} \leq \varepsilon\,.$$

Let $S_m(\mathfrak{h}) = \mathfrak{h} + \varepsilon\mathfrak{h}^1 = (\tilde{q},\tilde{\xi},\tilde{y})$. We write the transformed hamiltonian as follows:

$$\mathscr{H}_m(S_m(\mathfrak{h};\theta);\theta) = (H'_m(\mathfrak{h};\theta) + \varepsilon < \Delta h^{yy}(q;\theta)\,y,y>)$$

$$+ [\tfrac{1}{2} < A_m(\tilde{q})\,\tilde{y},\tilde{y}> - \tfrac{1}{2} < A_m(q)\,y,y> -$$

$$-< [J\, A_m(q)\, ,\varepsilon\, f^{yy}(q)]\; y,y > -\tfrac{1}{2} < \varepsilon\, f^{\xi}(q)\cdot \nabla_q A_m\; y,y > -$$

$$-< J\, A_m(q)\, f^{y}(q)\, ,\, y >]_1 + \varepsilon[(\xi^1 - \mathcal{F}^{\xi})\cdot A_{m+1}]_2 +$$

$$+\,\varepsilon\,[h^q - \frac{\partial}{\partial\omega} f^q]_3 + \varepsilon\,[(h^{\xi} - \frac{\partial}{\partial\omega} f^{\xi})\cdot \xi]_4 -$$

$$-\,\varepsilon[< \frac{\partial}{\partial\omega} f^y - A_m\, J\, f^y - h^y\, ,\, y >]_5 -$$

$$-\,\varepsilon[< (\frac{\partial}{\partial\omega} f^{yy} - [J\, A_m\, ,\, f^{yy}] - h^{yy} - \tfrac{1}{2} f^{\xi}\cdot \nabla_q A_m + \Delta h^{yy})\; y,y >]_6 +$$

$$+\ [(\varepsilon H_{2m} + \varepsilon H_{3m} + H^3)(\mathfrak{h} + \varepsilon\mathfrak{h}^1) - (\varepsilon H_{2m} + \varepsilon H_{3m} + H^3)(\mathfrak{h})]_7 +$$

$$+\,\varepsilon\,[H_{3m}]_8 + H^3 \tag{5.20}$$

We denote by $\Delta_j H$ the functional in the brackets $[\cdot]_j$ (together with the preceding factor).

Lemma 5.4. For $j = 1\, ,\, \dots\, ,\, 8$ the following estimates hold:

$$|\Delta_j H|^{O^c_{m+1},\Theta_{m+1}} \le \tfrac{1}{8}\, C_*(m+1)\, \varepsilon^{\rho+1} \tag{5.21}$$

$$\|\nabla_y\, \Delta_j H\|^{O^c_{m+1},\Theta_{m+1}}_{d\,-\,d_H} \le \tfrac{1}{8}\, C_*(m+1)\, \varepsilon^{\frac{2}{3}(\rho+1)} \tag{5.22}$$

Proof. We prove the more complicated estimates (5.22) only.

j = 1. The estimate is contained in Corollary 5.3.

j = 2. For the natural projection $\Pi_y : E^c_{(+,-d+d_H,\varepsilon)} \longrightarrow Y_{d-d_H}$ we have:

$$\|\Pi_y\|_{(+,-d+d_H,\varepsilon)\,,\,d-d_H} \le \varepsilon^{-1/3}\,. \tag{5.23}$$

By (5.6) with $a = -d + d_H \in D$, $t = 1$ and by Lemma 5.1

$$\|(S_m - \mathrm{Id} - \varepsilon\,\mathcal{F})^*(\mathfrak{h})\|^{\theta_{m+1},\mathrm{Lip}}_{(+\,,\,d_1-d-1,\varepsilon)\,,\,(+,-d+d_H,\varepsilon)} \le$$

$$\le C_1^e(m)\,\varepsilon^{2/3}\;. \tag{5.24}$$

Since

$$\nabla_y(\varepsilon(\xi^1 - \mathcal{F}^\xi)\cdot\Lambda_{m+1}) = \Pi_y \circ (S_m - \mathrm{Id} - \varepsilon\,\mathcal{F})^*(\mathfrak{h})(0,\Lambda_{m+1},0)$$

and $\|(0,\Lambda_{m+1},0)\|_{(+,d_1-d-1,\varepsilon)} \le C\,\varepsilon^{2/3}$, then the estimate (5.22) results from (5.23), (5.24).

j = 3 – 6 . $\Delta_3 H = \dots = \Delta_6 H = 0$.

j = 7 . For arbitrary function H we have the identity:

$$\nabla_y(H(\mathfrak{h} + \varepsilon\mathfrak{h}^1) - H(\mathfrak{h})) = (\nabla_y H(\mathfrak{h})\,|_{\mathfrak{h}=\mathfrak{h}+\varepsilon\mathfrak{h}^1} - \nabla_y H(\mathfrak{h})) +$$

$$+\,\Pi_y(\varepsilon\mathfrak{h}^1)^*(\mathfrak{h})\;\nabla_{\mathfrak{h}} H(\mathfrak{h} + \varepsilon\mathfrak{h}^1)\,.$$

So we have to estimate two terms,

$$\nabla_y H(\mathfrak{h})\,|_{\mathfrak{h}=\mathfrak{h}+\varepsilon\mathfrak{h}^1} - \nabla_y H(\mathfrak{h}) \tag{5.25}$$

and

$$\Pi_y(\varepsilon\mathfrak{h}^1)^*(\mathfrak{h})\;\nabla_{\mathfrak{h}} H(\mathfrak{h} + \varepsilon\mathfrak{h}^1)\,, \tag{5.26}$$

for $H = \varepsilon(H_{2m} + \varepsilon H_{3m})$ and for $H = H^3$. Let us denote $\tilde{\mathfrak{h}} = S_m(\mathfrak{h})$ and mention that $\Pi_q \circ (\mathfrak{h}^1{}_*(\mathfrak{h}))\,(0,0,y) \equiv 0$. So

$$\Pi_y \circ (\varepsilon\mathfrak{h}^1)^*(\mathfrak{h})\, \nabla_{\mathfrak{h}} H(\tilde{\mathfrak{h}}) = \Pi_y \circ (\varepsilon\mathfrak{h}^1)^*(\mathfrak{h})\, (0, \nabla_\xi H(\tilde{\mathfrak{h}}), \nabla_y H(\tilde{\mathfrak{h}}))$$

and by (5.23) and (5.5) with $a = -d + d_H$

$$\|\Pi_y \circ (\varepsilon\mathfrak{h}^1)^*(\mathfrak{h})\, \nabla_{\mathfrak{h}} H(\tilde{\mathfrak{h}})\|^{\Theta_{m+1},\mathrm{Lip}}_{d-d_H} \le$$

$$\le \varepsilon^{-1/3} \|(\varepsilon\mathfrak{h}^1)^*\|^{\Theta_{m+1},\mathrm{Lip}}_{(+,-d+d_1-1,\varepsilon),\,(+,-d+d_H,\varepsilon)} \times$$

$$\times \|(0, \nabla_\xi H(\tilde{\mathfrak{h}}))\,,\, \nabla_y H(\tilde{\mathfrak{h}}))\|^{\Theta_{m+1},\mathrm{Lip}}_{(+,-d+d_1-1,\varepsilon)} \le$$

$$\le C^e(m)\, (\varepsilon^{2/3} |\nabla_\xi H(\tilde{\mathfrak{h}})|^{\Theta_{m+1},\mathrm{Lip}} + \varepsilon^{1/3} \|\nabla_y H(\tilde{\mathfrak{h}})\|^{\Theta_{m+1},\mathrm{Lip}}_{d-d_1+1}) \tag{5.27}$$

Let $H = \varepsilon(H_{2m} + H_{3m})$. Then the estimate (5.22) for the term (5.25) results from (2.12), (2.44) and Cauchy estimate. The estimate for the term (5.26) results from (5.27), (2.11), (2.12).

Let $H = H^3$. The term (5.25) is equal to

$$\int_0^\varepsilon \frac{d}{d\tau} \nabla_y H^3(\mathfrak{h} + \tau\mathfrak{h}^1)\, d\tau = \int_0^\varepsilon (\nabla_y H_3)_*(\mathfrak{h} + \tau\mathfrak{h}^1)\mathfrak{h}^1\, d\tau$$

and its $\|\cdot\|^{\Theta_{m+1},\mathrm{Lip}}_{d-d_H}$–norm is estimated above by

$$|(\nabla_y H_3)_*(\tilde{\mathfrak{h}})|^{\Theta_{m+1},\mathrm{Lip}}_{E^c_{d+d_H-d_H},\, Y^c_{d-d_H}} \; \|\varepsilon\mathfrak{h}^1\|^{\Theta_{m+1},\mathrm{Lip}}_{d+d_H-d_H} .$$

The first factor is no greater than $C\,\varepsilon^{2/3}$ by (1.27) and Cauchy estimate. The second one is no greater than ε^{ρ} by (2.44) as $d_H \le d_1 - 1$. So the term (5.25) is estimated.

The estimate for the term (5.26) results from (5.27), (1.27) and Cauchy estimate because $d - d_1 + 1 \le d - d_H$.

j = 8. The estimate contains in Lemma 2.1, item c). ■

By the equation (5.20) and Lemma 5.4 hamiltonian $\mathscr{H}_m(S_m(\mathfrak{h};\theta);\theta)$ has a form (2.6) with

$$A_{m+1}(q;\theta) = A_m(q;\theta) + 2\varepsilon\, \Delta h^{yy}(q;\theta) . \tag{5.28}$$

Lemma 2.4 is proved.

6. Proof of Statement 1.2.

By the definitions of the maps $\sum^0$ and $\sum^{\varepsilon_0}$, for $\mathfrak{h} = (q,0,0) \in T^n_0$ $\sum^0(\mathfrak{h};\theta) = \Pi_{\mathscr{Y}}(q,0,0;\theta) = (q,0,0) \in \mathscr{Y}$ and $\sum^{\varepsilon_0}(q;\theta) = \sum^0_{\infty}(q,0,0;\theta)$. So we have to prove that

$$\left|\sum^0_{\infty} - \Pi_{\mathscr{Y}}\right|^{T^n_0 \times \Theta_{\varepsilon_0},\mathrm{Lip}}_{E_{d_c}} \le C\,\varepsilon_0 \tag{6.1}$$

By the proof of Theorem 1.1 the map $\sum^0_{\infty}$ is equal to

$$\sum_{\infty}^{0}(\mathfrak{h}\,;\theta) = S_0(\cdot\,;\theta)\circ S_1(\cdot\,;\theta)\circ\ldots\circ S_{m-1}(\cdot\,;\theta)\circ\sum_{\infty}^{m}(\mathfrak{h}\,;\theta) \tag{6.2}$$

and

$$|\sum_{\infty}^{m} - \Pi_{\mathscr{Y}}|_{E_{d_c}}^{O_\infty^c\times\Theta_{\varepsilon_0},\mathrm{Lip}} \le 3\,\varepsilon_m{}^{\rho} \tag{6.3}$$

(Lemma 2.5). The r.h.s. in (6.3) is smaller than ε_0 if $m \ge m(\rho)$. So to prove (6.1) it is enough to check that

$$|S_j - \Pi_{\mathscr{Y}}|_{E_{d_c}}^{O_{j+1}^c\times\Theta_{j+1},\mathrm{Lip}} \le C\,\varepsilon_0 \qquad \forall j \le m(\rho) \tag{6.4}$$

In a similar way,

$$\omega'(\omega,I) = \omega + \varepsilon_0 h_0^{0\xi} + \varepsilon_1 h_1^{0\xi} + \ldots \tag{6.5}$$

(see (2.56)) and $|\varepsilon_j h_j^{0\xi} + \varepsilon_{j+1}\,h_{j+1}^{0\xi} + \ldots| \le C(j)\,\varepsilon_j{}^{1/3}$ (see (2.60) with $m = j$, $p = \infty$). So to get (1.34) we have to prove that

$$|\varepsilon_j h_j^{0\xi}| \le C\,\varepsilon_0 \qquad \forall j \le m(\rho) \tag{6.6}$$

(we increase $m(\rho)$ if there is a need in it).

To prove (6.4), (6.6) we should improve the constants in the r.h.s. of the estimates of Lemmas 2.1, 2.2. For this end we define independent on ε_0 domains $Q_m{}^c$, $Q_m{}^{jc}$ instead of $O_m{}^c$, $O_m{}^{jc}$: $Q_m{}^{jc} = O(\mathbf{T}^n\times\{0\}\times\{0\}\,,\,\delta_m{}^j\,,\,\mathscr{Y}_d{}^c)$,
$Q_m{}^c = Q_m{}^{0c}$ (see (2.4)).

We prove by induction the following statement. Hamiltonian $\mathscr{H}_m$ (see (2.6)) may be written down in the domain $Q_m{}^c$ in the following way:

$$\mathscr{H}_m = H_{0m}(\mathfrak{h};\theta) + \varepsilon_0\, H_{(m)}(\mathfrak{h};\theta) + H^3(\mathfrak{h};\theta)\,. \tag{6.7}$$

Here the function H^3 is the same as in (1.24), $H_{(m)} \in \mathscr{A}^R_{\theta_m}(Q_m{}^c;\mathbb{C})$ and

$$|H_{(m)}|^{Q_m{}^c,\theta_m} \le C_m\,,\ \|\nabla_y H_{(m)}\|^{Q_m{}^c,\,\theta_m}_{d-d_H} \le C_m \tag{6.8}$$

By (2.6) and (2.7) we see that $\varepsilon_0 H_{(m)} = \varepsilon_m H_m$ on $O_m{}^c$. So $\varepsilon_0 H_{(m)}$ is an analytical extension of $\varepsilon_m H_m$ on the domain $Q_m{}^c$.

For $m = 0$ the representation (6.7) coincides with the initial one (see (1.23), (1.24)). Let us suppose that the statement is true for some $0 \le m \le m(\rho) - 1$. We denote the terms $\varepsilon_m H_m$, $\varepsilon_m h^q$, $\varepsilon_m h^{1\xi}$ etc. in the decomposition (2.16) by $\varepsilon_0 H_{(m)}$, $\varepsilon_0 h^q{}_{(m)}$, $\varepsilon_0 h^{1\xi}_{(m)}$ etc. and denote the coefficients $\varepsilon_m f^q$, $\varepsilon_m f^\xi$ etc. of the hamiltonian $\varepsilon_m F$ by $\varepsilon_0 f^q_{(m)}$, $\varepsilon_0 f^\xi_{(m)}$ etc. By repeating the proof of Lemma 2.1 we have for $h^q_{(m)}$, $h^{1\xi}_{(m)}$ etc. the estimates of the items a), b) of Lemma 2.1 with r.h.s. replaced by C_m (we don't controll the rate of increase on m).

In particular,

$$\varepsilon_m |h_m^{0\xi}| = \varepsilon_0 |h^{0\xi}_{(m)}| \le \varepsilon_0\, C_m\,. \tag{6.9}$$

For $H^3_{(m)}$ we have an estimate of the form (6.8).

By repeating the proof of Lemma 2.2 we get for $f^q_{(m)}$, $f^\xi_{(m)}$ etc. the estimates of form (2.35) – (2.40) with the r.h.s. replaced by C^1_m. So after an analytical extension into domain $Q_m{}^{3c}$ the vector–field of equation (2.28) is no larger than $C^2_m\, \varepsilon_0$. So S_m may be (analytically) extended to a map from $Q_m{}^{4c}$ into $Q_m{}^{3c}$ and for this extension the estimates of the item a), Lemma 2.4, hold with r.h.s. replaced by $C_m^3 \varepsilon_0$ (and with Q^c_{m+1,d_c} in the notations of the norms). In particular

$$|S_m - \Pi_{\mathcal{Y}}|_{E_{d_c}}^{Q^c_{m+1,d_c} \times \Theta_{m+1},\mathrm{Lip}} \le C_m^3 \varepsilon_0 . \qquad (6.10)$$

Hence the transformed hamiltonian $\mathcal{H}_m \circ S_m$ may be extended to the domain Q^c_{m+1,d_c} and has there the form (6.7) with $m := m + 1$.

Now the estimates (6.4) and (6.6) result from (6.9), (6.10) with $m = 0 , 1 , \dots , m(\rho)$.

7. On the reducibility of variational equations.

In the statement of Theorem 1.1 we made no use of the estimates (2.9). (2.10), (2.24) on the quadratic on y part of the hamiltonian $\mathcal{H}_m$. These estimates allow us to prove that the variational equations for (1.19) along the solutions $z^{\varepsilon_0}(t)$ are reducible to the constant coefficient ones (this reducibility is a typical by–product of KAM–procedure; see [A1], § 5.5.10).

The variational equations for $\delta z_0 = (\delta q_0 , \delta \xi_0 , \delta y_0) \in E_d$ along the solution $z = z^{\varepsilon_0}(t)$ have the form:

$$\begin{gathered} \delta \dot{q}_0 = \varepsilon_0 (\nabla_\xi H_0(z))_* \delta z_0 , \ \delta \dot{\xi}_0 = - \varepsilon_0 (\nabla_q H_0(z))_* \delta z_0 , \\ \delta \dot{y}_0 = J(A(\omega) \ \delta y_0 + \varepsilon_0 (\nabla_y H_0(z))_* \delta z_0) . \end{gathered} \qquad (7.1)$$

Let us denote by $T^n_{\varepsilon_0} = T^n_{\varepsilon_0}(\omega, I) \equiv \Sigma^{\varepsilon_0}_{(\omega,I)}(\mathbb{T}^n)$ the invariant tori constructed in Theorem

1.1.

Theorem 7.1. Under the assumptions of Theorem 1.1 there exists an analytical mapping $\Phi_1 : T^n_{\varepsilon_0} \longrightarrow \mathscr{L}(E_d, E_d)$ such that the substitution $\delta z_0 = \Phi_1(z(t))\, \delta\mathfrak{h}$, $\delta\mathfrak{h} = (\delta q, \delta\xi, \delta y) \in E_d$, transforms solutions of (7.1) into the solutions of the equations

$$\delta\dot{q} = 0\,,\ \delta\dot{\xi} = 0\,,\ \delta\dot{y} = J\, \overline{A}_\infty(\theta)\ \delta y\,.$$

Here $\overline{A}_\infty(\theta)\, \varphi_j^{\pm} = \overline{\lambda}_j(\theta)\, \varphi_j^{\pm}$ $\forall j$ and $|\overline{\lambda}_j(\theta) - \lambda_j(\omega)| \le \varepsilon_0^\rho\, j^{d_H^0}$ $\forall \rho < \frac{1}{3}$.

The change of variables Φ_1 is constructed in two steps:

1. The substitution

$$z^{\varepsilon_0}(t) = \Sigma^0_\infty(\mathfrak{h}_\infty(t))\,,\ \mathfrak{h}_\infty(t) = (q + \omega' t, 0, 0)$$

$$\delta z_0 = \Sigma^0_\infty(\mathfrak{h}_\infty(t))_*\, \delta\mathfrak{h}_\infty$$

transforms solutions of (7.1) into solutions of the equations

$$\delta\dot{q}_\infty = 0\,,\ \delta\dot{\xi}_\infty = 0\,,\ \delta\dot{y}_\infty = J\, A_\infty(q_\infty(t))\ \delta y_\infty \tag{7.2}$$

2. The equation for δy_∞ in (7.2) may be reduced to the constant–coefficient one via the substitution $\delta y_\infty = W(q_\infty)\ \delta y$, $W = \mathrm{diag}\,(\exp\,(iW_j(q_\infty)))$; see § 3.

We omit the details.

Appendix A. Interpolation theorem.

Let X_1 be a real Hilbert space with a Hilbert basis $\{\eta_j \mid j \in \mathbb{Z}_0\}$ (i.e. $<\eta_j, \eta_k>_{X_1} = \delta_{j,k}$) . Let X_2 be a dense subspace of X_1 with the Hilbert basis $\{\chi_j^{-1}\eta_j\}$, $\chi_j \geq C \quad \forall j$. Then for $0 \leq \tau \leq 1$ the interpolation space $[X_2, X_1]_\tau$ is a Hilbert space with the Hilbert basis $\{\chi_j^{-1+\tau}\eta_j \mid j \in \mathbb{Z}_0\}$. In particular if $X_1 = Y_a$, $X_2 = Y_b$, $b > a$, and Y_a , Y_b are the spaces from the scale $\{Y_s\}$ as in § 1, then by the conditions (1.21)

$$[X_2, X_1]_\tau = [Y_b, Y_a]_\tau = Y_{\tau a+(1-\tau)b}$$

(one has to take $\eta_j = \varphi_j^+$ for $j > 0$ and $\eta_j = \varphi_{-j}^-$ for $j < 0$). The norms in the spaces are equivalent:

$$K^{-1}\|y\|_{\tau a+(1-\tau)b} \leq |y|_{[Y_b, Y_a]_\tau} \leq K\,\|y\|_{\tau a+(1-\tau)b}$$

For complexifications X_1^c and X_2^c of the spaces X_1 , X_2 we set by definition

$$[X_2^c, X_1^c]_\tau = [X_2, X_1]_\tau \underset{\mathbb{R}}{\otimes} \mathbb{C}$$

(i.e. an interpolation of complexifications is equal to the complexification of interpolation). So $[Y_b^c, Y_a^c]_\tau = Y^c_{\tau a+(1-\tau)b}$.

<u>Theorem A1</u> (interpolation theorem). Let a linear operator $L : Y^c_\infty \longrightarrow Y^c_{-\infty}$ may be continued to the continuous maps $Y^c_{s_0} \longrightarrow Y^c_{l_0}$ and $Y^c_{s_1} \longrightarrow Y^c_{l_1}$. Then $\forall \tau \in [0,1]$ it may be continued to the continuous map $Y^c_{s_\tau} \longrightarrow Y^c_{l_\tau}$, $s_\tau = \tau s_0 + (1-\tau) s_1$, $l_\tau = \tau l_0 + (1-\tau) l_1$, and

$$\|L\|_{s_\tau, l_\tau} \leq C \max\{\|L\|_{s_0, l_0}, \|L\|_{s_1, l_1}\} .$$

For the general formulation of the theorem and for a proof see [LM, RS2].

Corollary A2. Let a linear continuous operator $Y_s^c \longrightarrow Y_l^c$ be symmetric with respect to the pairing $< \cdot , \cdot >$ (i.e. $L \in \mathscr{L}^s(Y_s^c, Y_l^c)$). Then $\forall \tau \in [0,1]$ $L \in \mathscr{L}^s(Y_{s_\tau}^c, Y_{l_\tau}^c)$, $s_\tau = \tau(s+1)-1$, $l_\tau = \tau(s+1)-1$, and $\|L\|_{s_\tau, l_\tau} \leq C \, \|L\|_{s,l}$.

Proof. We have equalities: $\|L\|_{-l,-s} = \|L^*\|_{-l,-s} = \|L\|_{s,l}$. Here L^* is the operator, conjugate to L with respect to the pairing $< \cdot , \cdot >$. Now the assertion results from Theorem A1 with $s_0 = s$, $s_1 = -l$, $l_0 = l$, $l_1 = -s$. ∎

Appendix B. Some estimates for Fourier series.

Let B be a Banach space with a norm $\|\cdot\|$, B^c be the complexification of B , $M = \{\mu\}$ be a metric space, $\xi > 0$ and

$$G \in \mathscr{A}_M^R(U(\xi) \, ; B^c) \, , \, \|G\|^{U(\xi),M} \leq 1 \, . \tag{B1}$$

Let us write a Fourier series for G :

$$G(q;\mu) = \sum_{s \in \mathbb{Z}^n} \widehat{G}(s;\mu) \, e^{is \cdot q} \, . \tag{B2}$$

Lemma B1. For every $s \in \mathbb{Z}^n$

$$\|\widehat{G}(s;\cdot)\|^{M,Lip} \leq e^{-\xi |s|} \, . \tag{B3}$$

and

$$\widehat{G}(s,\mu) = \overline{\widehat{G}(-s,\mu)} \qquad \forall s \, , \, \forall \mu \tag{B4}$$

An "almost inverse" statement is true:

Lemma B2. If (B3), (B4) are true $\forall s \in \mathbb{Z}^n$ and $0 < \Delta < \xi$ then the series (B2) converges $\forall q \in U(\xi - \Delta)$, the map G is analytic and

$$G \in \mathcal{A}^R_M(U(\xi - \Delta) \, ; B^c) \, , \|G\|^{U(\xi-\Delta),M} \leq 4^n \Delta^{-n}$$

Lemma B3. If (B1) takes place, $0 < 2\Delta < \xi < 1$ and

$$R_{M_*} G(q) = \sum_{|s| \geq M_*} \widehat{G}(s;\mu) \, e^{is \cdot q} \, ,$$

then

$$\|R_{M_*} G\|^{U(\xi-2\Delta),M} \leq C(n) \, \Delta^{-n-1} \, e^{-\frac{3}{4} M_* \Delta} \, .$$

The proves of the lemmas given in [A2, § 4.2] for $B = \mathbb{R}^n$, are valid for arbitrary Banach space B.

Appendix C. Lipschitz homeomorphisms of Borel sets.

Let $\Omega \subset \mathbb{R}^n$ be a bounded Borel subset and $\Lambda : \Omega \longrightarrow \mathbb{R}^n$ be a Lipschitz map of the form $\Lambda(a) = a + \Lambda_1(a)$,

$$\text{Lip}\, \Lambda_1 \leq \mu < 1 \, . \tag{C1}$$

So

$$\text{Lip}\, \Lambda \leq 1 + \mu \, . \tag{C2}$$

Theorem C1. If (C1) takes place than the inverse map Λ^{-1} is well–defined and

$$\text{Lip}\, \Lambda^{-1} \le (1-\mu)^{-1} . \tag{C3}$$

For arbitrary Borel set $\Omega' \subset \Omega$

$$(1-\mu)^n \,\text{mes}\, \Omega' \le \text{mes}\, \Lambda(\Omega') \le (1+\mu)^n \,\text{mes}\, \Omega \tag{C4}$$

Proof. The first statement is evident. Indeed, if $\Lambda(x_j) = y_j$, $j = 1, 2$, then $(x_1 - x_2) + (\Lambda_1 x_1 - \Lambda_1 x_2) = y_1 - y_2$ and by (C1) $|x_1 - x_2|^2 \le \mu |x_1 - x_2|^2 + |x_1 - x_2|\ |y_1 - y_2|$. So $|x_1 - x_2| \le (1-\mu)^{-1} |y_1 - y_2|$ and (C3) is proved.

To prove (C4) let us continue Λ to a Lipschitz map $\Lambda^c : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with the same Lipschitz constant (Kirszbraun's theorem, see [Fe]). Let $\text{mes}\, \Omega' = a$. Then the upper measure of Ω' is equal to a, too. So $\forall \varepsilon > 0$ the set Ω' may be covered by a countable set of balls $B_j \subset \mathbb{R}^n$, radius of B_j is equal to r_j, and

$$V_1 \sum_{j=1}^{\infty} r_j^n \le (1+\varepsilon)\, a$$

(V_1 is the measure of 1–ball in $\mathbb{R}^n$). As $\text{Lip}\, \Lambda^c = \text{Lip}\, \Lambda \le (1+\mu)$, then $\Lambda(B_j)$ is contained in a ball of the radius $(1+\mu)\, r_j$. As $\Lambda(\Omega') \subset \cup \Lambda(B_j)$, then

$$\text{mes}\, \Lambda(\Omega') \le V_1 \sum (1+\mu)^n r_j^n \le (1+\mu)^n (1+\varepsilon) \,\text{mes}\, \Omega' .$$

The second inequality in (C4) is proved because $\varepsilon > 0$ may be chosen arbitrarily small.

To prove the first inequality we have to consider the map Λ^{-1} and to use (C3).

List of notations

1. Constants.

$C, C_1, C_2, \dots$ – positive constants which arrive in estimates. They are independent on ε and m and are different in different parts of the text.

$K, K_1, \dots$ – constants which characterize initial data in theorems;

m – the number of the iteration;

$C(m), C_1(m), \dots$ – functions of m of the form $C_1 m^{C_2}$;

$C_{*j}, C_{*j}(m)$ – fixed constants and fixed functions of the form $C(m)$;

$C_1^e(m), C_2^e(m), \dots$ – functions of m of the form $\exp C(m)$;

$$e(m) = \frac{1^{-2} + 2^{-2} + \dots + m^{-2}}{2(1^{-2} + 2^{-2} + \dots\ )}, \qquad e(m) < \tfrac{1}{2} \quad \forall m;$$

$\varepsilon_m = \varepsilon_0^{(1+\rho)^m}, \ 0 < \rho < 1/3$;

$\delta_m = \delta_0(1 - e_m), \ \delta_m > \frac{1}{2}\delta_0 \ \forall m$;

$\delta_m^j = (1 - \frac{j}{6})\,\delta_m + \frac{j}{6}\,\delta_{m+1}, \ 0 \le j \le 5$

2. <u>Linear spaces and maps.</u>

Y, Z – Hilbert spaces with norms $\|\cdot\|_Y, \|\cdot\|_Z$ and inner products $<\cdot,\cdot>_Y$, $<\cdot,\cdot>_Z$;

$\{Y_s \mid s \in \mathbb{R}\}$ – a scale of Hilbert spaces $Y_s, |\cdot|_{Y_s} = \|\cdot\|_s, Y_0 = Y, Y_{s_1} \subset Y_{s_2}$ for $s_1 \ge s_2$, Y_s and Y_{-s} are conjugate with respect to the pairing $<\cdot,\cdot> = <\cdot,\cdot>_Y$;

$\{\lambda_j^{(-s)} \varphi_j^{\pm} \mid j \in \mathbb{N}\}$ – a Hilbert basis of Y_s, $\lambda_j^{(-s)} = (\lambda_j^{(s)})^{-1} > \ \forall j, \forall s$;

Y^c, $Y_s^{\,c}$ – complexifications of Y, Y_s, the scalar product $<\cdot,\cdot>$ in Y is continued to a complex–bilinear pairing $Y_s^{\,c} \times Y_{-s}^{\,c} \longrightarrow \mathbb{C}$, $s \in \mathbb{R}$;

$\mathscr{L}(Y_s^{\,c}; Y_l^{\,c})$ a space of linear continuous operators from $Y_s^{\,c}$ to $Y_l^{\,c}$ provided with the operator norm $\|\cdot\|_{s,l}$;

$\mathscr{L}^s(Y_s^{\,c}; Y_l^{\,c})$ – operators from $\mathscr{L}(Y_s^{\,c}; Y_l^{\,c})$ symmetric with respect to $<\cdot,\cdot>$

3. Sets and domains

$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}_0^{\,s} = \mathbb{Z}^s \setminus \{0\}$, $\mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}$, $\mathbb{R}_+ = \{x \in \mathbb{R} \,|\, x \geq 0\}$;

$O(Q,\delta,M)$ – δ–neighborhood of a subset Q of a metric space M;

$O(\delta,Z) = O(0,\delta,Z)$ for a Banach space Z;

$\mathfrak{A} \subset \mathbb{R}^n$ – a set of parameters a;

$\mathfrak{A}(a_0,\delta) = \{a \in \mathfrak{A} \subset \mathbb{R}^n \,|\, |a - a_0| \leq \delta\}$;

Ω_0 – a set of frequencies vectors $(\omega_1, \dots, \omega_n)$;

$\mathscr{I}$ – a set of actions $(I_1, \dots, I_n)$;

$\Theta_j = \{\theta = (\omega,I)\}$, $j = 0, 1, \dots$ – subsets of $\Omega_0 \times \mathscr{I}$;

$\Theta[I] = \{\omega \in \Omega \,|\, (\omega,I) \in \Theta\}$ for a $\Theta \subset \Omega \times \mathscr{I}$ and arbitrary $I \in \mathscr{I}$;

$\mathscr{Y}_s = \mathbb{T}^n \times \mathbb{R}^n \times Y_s$, $\mathscr{Y} = \mathscr{Y}_0$, tangent space to $\mathfrak{h} \in \mathscr{Y}_s$ is identified with $E_s = \mathbb{R}^n \times \mathbb{R}^n \times Y_s$;

$\mathscr{Y}_s{}^c = (\mathbb{C}^n / 2\pi \mathbb{Z}^n) \times \mathbb{C}^n \times Y_s{}^c$;

$U(\delta) = \{\xi \in \mathbb{C}^n / 2\pi \mathbb{Z}^n \mid |\mathrm{Im}\, \xi| < \delta\}$;

$O^c(\xi_0, \xi_1, \xi_2; \mathscr{Y}_s{}^c) = U(\xi_0) \times O(\xi_1, \mathbb{C}^n) \times O(\xi_2, Y_s{}^c)$;

$U_m = U(\delta_m)$, $O_m{}^c = O^c(\delta_m, \varepsilon_m{}^{2/3}, \varepsilon_m{}^{1/3}; \mathscr{Y}_d{}^c)$;

$O_m{}^{jc} = O^c(\delta_m{}^j, (2^{-j}\varepsilon_m)^{2/3}, (2^{-j}\varepsilon_m)^{1/3}; \mathscr{Y}_d{}^c)$, $0 \leq j \leq 5$;

$O_m = O_m{}^c \cap \mathscr{Y}_d$;

4. Maps and functions

For a map $G : Q_1 \longrightarrow Q_2$ (Q_j is a metric space with a distance dist_j , $j = 1,2$)

$$\mathrm{Lip}\, G = \sup_{x_1 \neq x_2} \frac{\mathrm{dist}_2(G(x_1), G(x_2))}{\mathrm{dist}_1(x_1, x_2)} ;$$

$|G|_{Q_2}^{Q_1,\mathrm{Lip}} = \max\{\sup_{q \in Q_1} |G(q)|_{Q_2}, \mathrm{Lip}\, G\}$ if $G : Q_1 \longrightarrow Q_2$ and Q_2 is a Banach space;

$\mathscr{A}^R(O_1{}^c; O_2{}^c)$ is the set of Frechet complex–analytical mappings from $O_1{}^c \subset B_1{}^c$ to $O_2{}^c \subset B_2{}^c$ which map $O_1{}^c \cap B_1$ into B_2 ;

$\mathscr{A}^R{}_M(O_1{}^c; O_2{}^c)$ is the set of mappings $G : O_1{}^c \times M \longrightarrow O_2{}^c$ such that $G(\cdot\, ; m) \in \mathscr{A}^R(O_1{}^c; O_2{}^c)$ $\forall m \in M$ and

$$|G|_{B_2}^{O_1^c;M} = \sup_{b \in O_1^c} |G(b;\cdot)|_{B_2}^{M,\mathrm{Lip}} < \infty \,;$$

$<\bar{J}\,dz,dz>_Z$ is the 2–form in a Hilbert space Z , $<\bar{J}\,dz,dz>_Z[z_1,z_2] = = <\bar{J}\,dz,dz>_Z$.

Bibliography

[A] Arnold, V.I.: Mathematical methods in classical mechanics. Moscow: Nauka, 1974; English transl. Springer, 1978.

[AA] Arnold, V.I., Avez, A.: Ergodic problems of classical mechanics. Addison–Wesley Publishing Co., 1989.

[A1] Arnold, V.I.: Geometric methods in the theory of ordinary differential equation. Moscow: Nauka, 1978; English transl. Springer, 1983.

[A2] Arnold, V.I.: Proof of a theorem of A.N.Kolmogorov on the conservation of quasiperiodic motions under a small change of the Hamiltonian function. Uspekhi Mat. Nauk 18:5, 13-40 (1963); Russ. Math. Surv. 18:5, 9-36 (1963).

[A3] Arnold, V.I.: Small denominators and problems of stability of motions in classical and celestial mechanics. Uspekhi Mat. Nauk 18:6, 91-192 (1963); Russ. Math. Surv. 18:6, 85-191 (1963).

[AbM] Abraham, R., Marsden, J.: Foundations of mechanics. Benjamin, 1967.

[AKN] Arnold, V.I., Kozlov V.V., Neistadt A.I.: Mathematical aspects of classical and celestial mechanics. Encycl. of mathem. scien., vol. 3. Springer, 1989.

[AlFS] Albanese, C., Fröhlich, J., Spencer, T.: Periodic solutions of some infinite-dimensional hamiltonians associated with non-linear partial differential equations, I and II. Comm. Math. Phys. 116, 475-502 (1988) and 119, 677-699 (1988).

[BFG] Benettin, G., Fröhlich, J., Giorgilli, A.: A Nekhoroshev-type theorems for Hamiltonian systems with infinitely many degrees of freedom. Commun. Math Phys. 119, 95-108 (1989).

[BGG] Benettin, G., Galgani, L., Giorgilli, A.: A proof of Nekhoroshev's theorem for the stability times in nearly integrable Hamiltonian systems. Celestial Mechanics 37, 1-25 (1985).

[Bi] Bibikov, Yu. N.: Multifrequencies nonlinear oscillations and their bifurcations (in Russ.). Leningrad: Leningrad Univ. Press, 1991.

[Bre] Brezis, H.: Operateurs maximaux monotones et semigroupes de contractions dans les espaces de Hilbert. North–Nolland, 1973.

[Bru1] Bruno, A.D.: The normal form of a Hamiltonian system. Uspekhi Mat. Nauk 43:1, 23-56 (1988); Russ. Math. Surv. 43:1, 25-66 (1988).

[Bru2] Bruno, A.D.: Normalization of a Hamiltonian system near a cycle or torus. Uspekhi Mat. Nauk 44:2, 49-77 (1989); Russ. Math. Surv. 44:2, 53-89 (1989).

[CFKS] Cycon, H.L, Froese, R.G., Kirsch, W., Simon, B.: Schrödinger operators with applications to quantum mechanics and global geometry. Springer, 1987.

[Ch-B] Choquet-Bruhat, Y., De Witt-Morette, C.: Analysis, manifolds and physics. North Holland, 1982.

[ChM] Chernoff P.R., Marsden J.E.: Properties of infinite-dimensional Hamiltonian systems. Lecture Notes in Math 425 (1974).

[DEGM] Dodd, R.K., Eilbeck, J.C., Gibbon, J.D., Morris, H.C.: Solitons and nonlinear wave equations. Academic Press, 1984.

[DPRV] Dodson, M.M., Pöschel, J., Rynne, B.P., Vickers, J.A.G.: The Hausdorf dimension of small divisors for lower dimensional KAM-tori. To appear.

[El] Eliasson, L.H.: Perturbations of stable invariant tori. Ann. Sc. Super. Pisa, Cl. Sci., IV Ser. 15, 115-147 (1988).

[Fe] Federer, H.: Geometric measure theory. Springer, 1969.

[FPU] Fermi, E., Pasta, J.R., Ulam S.M.: Studies of nonlinear problems. In 'Collected works of E. Fermi, vol.2' Chicago: Univ. Chicago Press, 1965.

[FSW] Fröhlich, J., Spencer, T., Wayne, C.E.: Localization in disordered nonlinear dynamical systems. J. Stat.Phys. 42, 247-274 (1986).

[Gr] Graff, S.M.: On the continuation of hyperbolic invariant tori for Hamiltonian systems. J.Differ. Equations 15, 1-69 (1974).

[HLP] Hardy, G.H., Littlewood, J.E., Polya, G.: Inequalities. Cambridge: Cambridge Univ. Press, 1967.

[HS] Halmos, P., Sunder, V.: Bounded integral operators on L^2-spaces. Springer, 1978.

[K1] Kuksin, S.B.: Perturbation of quasiperiodic solutions of infinite-dimensional Hamiltonian systems. Izv. Akad. Nauk SSSR Ser. Mat. 52, 41-63 (1988); Math. USSR Izvestiya 32:1, 39-62 (1989).

[K2] Kuksin, S.B.: Hamiltonian perturbations of infinite-dimensional linear sustems with an imaginary spectrum. Funts. Anal. Prilozh. 21:3, 22-37 (1987); Funct. Anal. Appl. 21, 192-205 (1987).

[K3] Kuksin, S.B.: Conservative perturbations of infinite-dimensional linear systems depending on a vector parameter. Funkts. Anal. Prilozh. 23:1, 72-73 (1989); Funct. Anal. Appl. 23, 62-63 (1989).

[K4] Kuksin, S.B.: An averaging theorem for distributed conservative systems and its applications to the von Karman equations. Prikl. Matem. Mekhan. 53:2, 196-205 (1988); P.M.M. USSR 53:2, 150-157 (1989).

[K5] Kuksin, S.B.: Perturbation theory for quasiperiodic solutions of infinite-dimensional Hamiltonian systems, and its applications to the Korteweg–de Vries equation. Matem. Sbornik 136(178):3 (1988); Math. USSR Sbornik 64, 397-413 (1989).

[K6] Kuksin, S.B.: Perturbation theory for quasiperiodic solutions of infinite-dimensional Hamiltonian systems, parts 1-3. Preprints MPI/90-99, 90-100, 90-107 of Max-Planck-Institut für Mathematik, Bonn (1990).

[K7] Kuksin, S.B.: An infinitesimal Liouville–Arnold theorem as criterion of reducibility for variational Hamiltonian equations. Preprint MPI/91-3 of Max-Planck-Institut für Mathematik, Bonn (1991).

[K8] Kuksin, S.B.: Reducible variational equations and the perturbation of invariant tori of Hamiltonian systems. Matem. Zametki 45:5, 38-49 (1989); Mathematical Notes 45:5, 373-381 (1989).

[Ka] Kato, T.: Quasi-linear equations of evolutions, with applications to partial differential equations. Lecture Notes Math. 448, 25-70 (1975).

[Kol] Kolmogorov, A.N.: On the conservation of conditionally periodic motions for a small change in Hamilton's function (in Russ.). Dokl. Acad. Nauk SSSR 98, 525-530 (1954); English transl. in Lecture Notes in Physics 93, 51-56 (1979).

[Lax] Lax, P.D.: Development of singularities of solutions of nonlinear hyperbolic partial differential equations. J. Math. Phys. 5, 611-613 (1964).

[Laz] Lazutkin, V.F.: KAM-theory and semiclassical approximation to eigenfunctions. To appear in "Springer".

[Lio] Lions, J.-L.: Quelques methodes de resolution des problemes aux limites nonlineaires. Paris: Dunod, 1969.

[LL] Lichtenberg, A.J., Lieberman, M.A.: Regular and stochastic motions. Springer, 1983.

[LM] Lions, J.-L., Magenes, E.: Non-homogeneous boundary value problems and applications. Springer, 1972.

[Lo] Lochak, P.: Canonical perturbation theory via simultaneous approximation. Preprint of CMA ENS, Paris 1991; submitted to Russ. Math. Surveys.

[Ma] Marchenko, V.A.: Sturm–Liouville operators and applications. Kiev: Naukova Dimka, 1977; English transl. Basel: Birkhäuser, 1986.

[Mel] Melnikov, V.K.: On some cases of conservation of conditionally periodic motions under a small change of the Hamiltonian function. Dokl. Akad Nauk SSSR 165:6,1245-1248 (1965); Sov. Math. Dokl. 6, 1592-1596 (1965).

[Me2] Melnikov, V.K.: A family of conditionally periodic solutions of a Hamiltonian system. Dokl. Akad. Nauk SSSR 181:3, 546-549 (1968); Sov. Math. Dokl. 9, 882-886 (1968).

[Mo] Moser, J.: Stable and random motions in dynamical systems. Princeton: Princeton Univ. Press, 1973.

[Mo1] Moser, J.: Convergent series expansions for quasi-periodic motions. Math. Ann. 169, 136-176 (1967).

[N] Nekhoroshev, N.N.: An exponential estimate of the time of stability of nearly-integrable Hamiltonian systems. Uspekhi Mat. Nauk 32:1, 5-66 (1977); Russ. Math. Surveys 32, 1-65 (1977).

[Nik] Nikolenko, N.N.: The method of Poincare normal forms in problems of integrability of equations of evolution type. Uspekhi Mat. Nauk 41:5, 109-152 (1986); Russ. Math. Surveys 41:5, 63-114 (1986).

[Nov] Novikov, S.P.(ed): Theory of solitons. Moscow: Nauka, 1980; English transl. New York: Plenum Press, 1984.

[P1] Pöschel, J.: On elliptic lower dimensional tori in Hamiltonian systems. Math. Z. 202, 559-608 (1989).

[P2] Pöschel, J.: Small divisors with spatial structurte in infinite dimensional Hamiltonian systems. Comm. Math. Phys. 127, 351-393 (1990).

[P3] Pöschel, J.: Integrability of Hamiltonian systems on Cantor sets. Comm. Pure. Appl. Math. 35, 653-695 (1982).

[PasF] Pastur, L.A., Figotin, A.L.: Random and quasiperiodic selfadjoint operators (in Russ.). Moscow: Nauka, 1991.

[PT] Pöschel, J., Trubowitz, E.: Inverse spectral theory. Boston: Academic Press, 1987.

[RS1] Reed, M., Simon, B.: Methods of modern mathematical physics, Vol.1. Academic

Press, 1980.

[RS2] Reed, M., Simon, B.: Methods of modern mathematical physics, Vol.2. Academic Press, 1980.

[Ru] Rüssmann, H.: Über invariante Kurven differenzierbarer Abbildungen eines Kreisringes. Nachr. Akad. Wiss., Gottingen II, Math. Phys. Kl., 67-105 (1970).

[SM] Siegel, C.L., Moser, J.: Lectures on celestial mechanics. Springer, 1971.

[U] Ulam, S.: Problems in modern mathematics. John Wiley and Sons, Inc., 1964.

[VB] Vittot, M., Bellissard, J.: Invariant tori for an infinite lattice of coupled classical rotators. Preprint CPT-Marseille, 1985.

[W1] Wayne, C.E.: Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory. Commun. Math. Phys. 127, 479-528 (1990).

[W2] Wayne, C.E.: The KAM theory of systems with short range interactions, 1 and 2. Comm. Math. Phys. 96, 311-329 (1984); 331-344 (1984).

[W3] Wayne, C.E.: Bounds on the trajectories of a system of weakly coupled rotators. Comm. Math. Phys. 104, 21-36 (1986).

[War] Ware, B.: Infinite-dimensional versions of two theorems of Carl Ziegel. Bull. Amer. Math. Soc. 82, 613-615 (1976).

[Z1] Zehnder, E.: Generalized implicit function theorem with applications to some small divisor problems, I and II. Commun. Pure Appl. Math. 28, 91-140 (1975); 29, 49-111 (1976).

[Z2] Zehnder, E.: Siegel's linearization theorem in infinite dimensions. Manus. Math. 23, 363-371 (1978).

[ZIS] Zakharov, V.E., Ivanov, M.F., Shur. L.N.: On the abnormally slow stochastisation in some two-dimensional field theory models . Pis'ma. Zh. Eksp. Teor. Fiz. 30:1, 39-44 (1979); JETP Letters 30:1 (1979).