Functional difference analogues of Darboux-Pöshl-Teller potentials

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Abstract

We consider the functional-difference deformation of the Schrödinger equation. The main goal of this article is to construct some integrable potentials representing a natural difference deformation of the so called two parametric Darboux-Pöshl-Teller model and to describe explicitly the solutions of the related difference Schrödinger equation. In the limit when the difference step tends to zero the related formulas reproduce well known results concerning the Schrödinger operator with DPT potential. We also describe the solutions of the difference KdV (DKdV) equation with the nonsingular "difference DPT" initial data.

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1 Introduction

In 1882 Darboux [1] proved the integrability of the following Stourm-Liouville equation [1]

$$-y'' + \left(\frac{n(n+1)q^2}{\cos^2 qx} + \frac{m(m+1)q^2}{\sin^2 qx} \right) y = \lambda y \tag{1}$$

Later in 1933 in a frame of study of the quantum theory of the two atomic molecules Pöshl and Teller, (the creator of the US hydrogenous bombe), rediscovered [9] the hyperbolic version of (1) and proved independently from Darboux the integrabilty of the equation obtained from (1) by transformation $q \to iq$. Darboux proposed two independent methods for solving (1). First, taking $\sin^2 qx$ as a new independent variable he reduced (1) to Gauss hypergeometric equation and explicitely expressed the solutions in terms of the hypergeometric functions. Next, exploring the fact that the functions $y(m,n) := \cos^{n+1} qx \sin^{m+1} qx$ represent the particular solutions of (1) with $\lambda = q^2(m+n+2)^2$, namely taking them as generating functions of some sequence of Darboux transformations he obtained the global solution of the same equation in terms of the elementary functions. Comparison of these two solutions provides the notrivial case of the reduction of the Gauss hypergeometrical function to the elementary functions. The third representation for the solutions of the same equation in terms of Wronskian determinants was recently obtained in [6].

In this article we consider the following functional-difference deformation of the Schrödinger equation

$$f(x+h)\,v(x,h) + f(x-h) = 2\cosh ikh f(x), \tag{2}$$

where h is a non negative parameter. We can obviously rewrite (10) in a following form.

$$h^{-2}[v(x,h)f(x+h) + f(x-h) - 2f(x)] = h^{-2}2(\cosh ikh - 1)f(x). \tag{3}$$

Assume now that, when $h \to 0$ the related potential v has the following asymptotics:

$$v(x,h) = 1 + h^2 g(x) + O(h^3). \tag{4}$$

[1] Darboux also considered and solved the equation which is an elliptic function generalization of (1):see [2] and the concluding remarks.

Under this assumption, when $h \to 0$, the limit of (3) is exactly the Schrödinger equation with h-independent potential $g(x)$:

$$f'' + g(x)f(x) = -k^2 f(x) \tag{5}$$

In a sequel we use the following abreviations

$$c(x) := \cosh qx \qquad\qquad s(x) := \sinh qx$$

Below we consider special class $v_{nm}(x,h,q)$ of the potentials satisfying (4):

$$v_{nm}(x) = \frac{c(x+(n+1)h)c(x-nh)s(x+(m+1)h)s(x-mh)}{c(x)c(x+h)s(x)s(x+h)} \tag{6}$$

$$= v_{n0}(x)v_{0m}(x) \tag{7}$$

We see that in (6) the additive structure of the original DPT potentials is replaced by the multiplicative structure. For any C^3 function l(x) the short calculation proves the following asymptotic estimate :

$$\frac{l(x+(n+1)h)l(x-nh)}{l(x)l(x+h)} = 1 + h^2 n(n+1)\frac{d^2}{dx^2}\log l(x) + 0(h^3).$$

This makes obvious that the potential $v_{nm}(x)$, defined above, when $h \to 0$, has the following asymtotics:

$$v_{nm}(x) = 1 + h^2\left(\frac{n(n+1)q^2}{\cosh^2 qx} - \frac{m(m+1)q^2}{\sinh^2 qx}\right) + 0(h^3) \tag{8}$$

Therefore the limit of (3) with $v = v_{nm}(x)$ is just a Schrödinger equation with a hyperbolic Darboux-Pöshl-Teller potential:

$$-y'' + \left(-\frac{n(n+1)q^2}{\cosh^2 qx} + \frac{m(m+1)q^2}{\sinh^2 qx}\right) y = k^2 y. \tag{9}$$

It is enough to replace $c(x)$ and $s(x)$ by $\cos qx$ and $\sin qx$ in order to recover from (6) the original trigonometric Darboux potentials (1) in the limit $h \to 0$. Below we show that

$$f(x+h)\, v_{nm}(x,h) + f(x-h) = 2\cosh ikhf(x), \tag{10}$$

is an integrable equation and its global solution can be expressed by means of the elementary functions. In the limit $h \to 0$ all known results, concerning the Schrödinger equation, are easily recovered. The cases $n = 0$, or $m = 0$ were studied in the recent papers [10, 3, 11, 12] using the different tools and leading to the formulas different from ours having more complicated combinatorial structure. The Lattice specialisation of the case m=0, corresponding to the choice $h = 1, x = j \in Z$ was studied in [11, 12]. In these articles various interesting connections with q- ultraspherical polynomials , Askey-Vilson polynomials and q-spherical functions of the $m = 0$ case were discovered. Similar connections exist also for general $v_{nm}(x)$ potentials. We expect to discuss them elsewhere.

The article is organized as follows. In the second section we collected a few results concerning the functional-difference version of the Darboux dressing applicable to any equation of the form (10) as well as for the closely related functional difference KdV equation and the related hierarchie. The related results represent a very special case of much more general statements proved in [8]. They can also be considered as a natural functional difference extrapolation of the results of [7], p.84 concerning the lattice KdV equation. Therefore, in this part we omited all the proofs. The third section contains the construction of the solutions of (10) with potential $v_{nm}(x)$ defined in (6), based on a proper generalization of the Darboux approach for the Schrödinger equation, combined with the results of the second section. The special solutions corresponding to the products $c^{m+1}(x)s^{n+1}(x)$, used in original Darboux construction [1], are replaced by the appropriate products of the shifts of the same functions c and s, playing the same role in our case. In particular, this leads to especially simple formulas for the discrete eigenvalues and bound states eigenfunctions in the case of v_{n0} potentials. In the last section we explain how to get the same potentials and their eigenfunctions from Casorati determinants by appropriate reduction of the general determinant Darboux dressing formulae. This leads immediatly to the solution of the difference KdV equation corresponding to $v_{n0}(x)$ taken as initial data.

In the concluding remarks we mention some possible developments and extensions of the results of this article.

2 Functional difference Schrödinger and KdV equations and Darboux dressing

The functional-difference KdV equation reads

$$\dot{v}(x,t) = v(x)[v(x-h) - v(x+h)]. \tag{11}$$

Its Lax representation can be written as a compatibility condition of the difference Schrödinger equation:

$$v(x)f(x+h) + f(x-h) = \lambda f(x), \tag{12}$$

and of the following evolution equation:

$$\dot{f} = -v(x,t)v(x+h,t)f(x+2h) \tag{13}$$

This Lax pair is a natural functional interpolation of the lattice Lax pair used in [7], p. 84. For the trivial solution (potential) $v = 1$, two important real solutions of (12) with $\lambda = 2\cosh\beta h$ are

$$f = \cosh(\beta x + \eta), \qquad\qquad f = \sinh(\beta x + \eta),$$

where η and β are two real parameters. They can be easily extended to the solutions of the system (12)-(13):

$$f = e^{t\cosh(\beta h)}\cosh(\beta x + t\sinh(\beta h) + \eta),$$
$$f = e^{t\cosh(\beta h)}\sinh(\beta x + t\sinh(\beta h)\eta)$$

The space of the solutions of (12) is infinite dimensional since for any h-periodic function g (i.e $g(x+h) = g(h)$), $g(x)f(x)$ is again the solution of (12). The same is true for (13) under the assumption that $g(x)$ is a t-independent h periodic function of x.

Multiplying $f(x)$ by any h-antiperiodic function $p(x)$, (i.e $p(x \pm h) = -p(x)$), for instance taking $p(x) = e^{\pm\frac{i\pi}{h}}$, we see that the product function $\hat{f}(x) := f(x)p(x)$ is the solution of the equation which differs from (12) only by the different sign of λ:

$$v(x)\hat{f}(x+h) + \hat{f}(x-h) = -\lambda\hat{f}(x), \hat{f}(x) = p(x)f(x). \tag{14}$$

Therefore, the spectrum of the difference Schrödinger equation is symmetric for any given $v(x)$.

The difference KdV equation (11) admits an infinit dimensional space of the stationary solutions which are t-independent $2h$-periodic functions of x, which makes again certain difference with its lattice reduction, $x = j \in Z, h = 1$, considered for instance in [7], p. 84.

2.1 Darboux dressing

Darboux transform of an arbitrary solution $f(x)$ of (12) generated by the fixed solution $f_1(x)$ of the same system with $\lambda = \lambda_1$, is defined by the formula

$$\psi_1(x) = f(x-2h) - \sigma(x) f(x) \tag{15}$$

$$= \frac{\begin{vmatrix} f(x-2h) & f_1(x-2h) \\ f(x) & f_1(x) \end{vmatrix}}{f_1(x)}, \tag{16}$$

$$\sigma_1(x) = \frac{f_1(x-2h)}{f_1(x)}, \tag{17}$$

where $f_1(x)$ is a fixed solution of (12) with $\lambda = \lambda_1$.
Proposition 1(Darboux covariance property).
Darboux transform of $f(x)$ represents a general solution of the following equation

$$v_1(x)\psi_1(x+h) + \psi_1(x-h) = \lambda\psi_1(x), \tag{18}$$

$$v_1(x) = v(x)\frac{\sigma_1(x)}{\sigma_1(x+h)} = \frac{f_1(x-2h)f_1(x+h)}{f_1(x)f_1(x-h)}. \tag{19}$$

Providing that $f_1(x)$ is also a solution of (13) and $v(x)$ is a solution of (11), (hence both depend also on t), $v_1(x,t)$ described by the formula (19) is a new solution of DKdV equation

In other words, Darboux transform maps (12) into equation of the same form, with the same value of spectral parameter λ but with a new potential constructed in terms of the initial potential $v(x)$ and a fixed solution $f_1(x)$ of (12). The same transformation maps the given solution of the DKdV equation into a new solution which differs from the old one by the factor explicitely constructed in terms of the fixed solution of the Lax system (**??**). One step Darboux dressing of the trivial solution $v = 1$ with the generating function $f = e^{t\cosh(\beta h)}\cosh(\beta x + t\sinh(\beta h) + \eta)$ obviously produces the smooth 1-soliton of the DKdV equation:

$$v = \frac{\cosh(\beta(x-2h) + t\sinh(\beta h) + \eta) \cdot \cosh(\beta(x+h) + t\sinh(\beta h) + \eta)}{\cosh(\beta x + t\sinh(\beta h) + \eta) \cdot \cosh(\beta(x-h) + t\sinh(\beta h) + \eta)},$$

representing a wave propagating with constant negative velocity $-\frac{\sinh \beta h}{\beta}$.

2.2 Iterated Darboux transform

In a sequel the notation $\Delta_n(x) = \Delta_n[f_1(x),)f_2(x), \ldots, f_n(x)]$ will be used for the following Casorati determinant:

$$\Delta_n(x) = \begin{vmatrix} f_1(x-2(n-1)h) & f_2(x-2(n-1)h) & \ldots & f_n(x-2(n-1)h) \\ f_1(x-2(n-2)h) & f_2(x-2(n-2)h) & \ldots & f_n(x-2(n-2)h) \\ \vdots & \vdots & \vdots & \vdots \\ f_1(x) & f_2(x) & \ldots & f_n(x) \end{vmatrix}$$

We also use below the following notations

$$\psi_n(x) = \frac{\Delta_{n+1}[f(x), f_1(x),)f_2(x), \ldots, f_n(x)]}{\Delta_n(x)}, \tag{20}$$

$$F_j(x) = \psi_{j-1}(x)|_{f(x)=f_j(x)},\ j = 1, 2, \ldots, n. \tag{21}$$

$$\sigma_j(x) = \frac{F_j(x-2h)}{F_j(x)}, \tag{22}$$

$$v(x)f_j(x+h) + f_j(x-h) = \lambda_j f_j(x), \tag{23}$$

$$v(x)f(x+h) + f(x-h) = \lambda f(x). \tag{24}$$

Prposition 2.
The function ψ_n represents the general solution of the functional difference equation

$$v_n(x)\psi_n(x+h) + \psi_n(x-h) = \psi_n(x), \tag{25}$$

$$v_n(x) = v(x)\,\frac{\Delta_n(x-2h)\Delta_n(x+h)}{\Delta_n(x)\Delta_n(x-h)} \tag{26}$$

Providing that $v(x,t)$ is a solution of (11), and $f_j(x)$ are also the solutions of (13) the RHS of (39) is again the solution of DKdV equation. In particular, setting $v(x,t) = 1$ and

$$f_{2j+1}(x,t) = e^{t\cosh(\beta_{2j+1}h)}\cosh(\beta_{2j+1}x + t\sinh(\beta_{2j+1}h) + \eta_{2j+1}),$$
$$f_{2j} = e^{t\cosh(\beta_{2j}h)}\sinh(\beta_{2j}x + t\sinh(\beta_{2j}h) + \eta_{2j}), \eta_j \in R, 0 < \beta_1 < \ldots < \beta_n, \tag{27}$$

we obtain from (39) *the real nonsingular multi solitons solutions of the DKdV equation. While substitunig (27) into (26) the exponential factors in the RHS of (27) can be omitted .*

The function $\psi_n(x)$ can be also represented in a following factorized form

$$\psi_n(x) = (T^{-2} + \sigma_n(x)) \dots (T^{-2} + \sigma_1(x)) f(x, \lambda) \tag{28}$$

, where T is the shift operator : $T^{\pm k} f(x) = f(x \pm kh)$.

3 Difference DPT potentials and the related solutions of the difference Schrödinger equation

3.1 Shifted Darboux dressing

The above version of the Darboux dressing exposed in section 2 is convinient in a sense that it shows how to construct the new integrable potential (or the new solution of DKdV equation) from the given one, keeping the later as a stable factor entering into the new potential, (or a new solution of DKdV equation). Somehow in a sequel it will be more convinient to use the same formulas corresponding to the result of the n-fold Darboux transform considered at the reference point $x + nh$ rather then at the point x. It is clear from above that the function $\phi_1(x)$ defined by the formula

$$\phi_1(x) = f(x-h) - \kappa_1(x) f(x+h) = \frac{\begin{vmatrix} f(x-h) & f_1(x-h) \\ f(x+h) & f_1(x+h) \end{vmatrix}}{f_1(x+h)} \tag{29}$$

$$\kappa_1(x) = \frac{f_1(x-h)}{f_1(x+h)}, \tag{30}$$

satisfies the equation of the DS type with potential

$$u_1(x) = v_1(x+h) = v(x+h)\frac{\kappa_1(x)}{\kappa_1(x+h)} = v(x+h)\frac{f_1(x-h) f_1(x+2h)}{f_1(x+h) f_1(x)}. \tag{31}$$

As before we will call the mapping

$$D_{f_1} : f(x) \to \phi_1(x) \tag{32}$$

Darboux transform of the solution $f(x,\lambda)$ generated by $f_1(x,\lambda_1)$. It transforms the solution of the DS equation with potential $v(x)$ to the solutions of the DS equation with potential $u_1(x)$ with the same value of the spectral papameter λ.

In a sequel the notation $\delta_n(x) = \delta_n[f_1,)f_2,\ldots,f_n](x)$ will be used for the following Casorati determinant:

$$\delta_n(x) = \Delta_n(x+(n-1)h) = \begin{vmatrix} f_1(x-(n-1)h) & f_2(x-(n-1)h) & \ldots & f_n(x-(n-1)h) \\ f_1(x-(n-3)h) & f_2(x-(n-3)h) & \ldots & f_n(x-(n-3)h) \\ \vdots & \vdots & \vdots & \vdots \\ f_1(x+(n-1)h) & f_2(x+(n-1)h) & \ldots & f_n(x+(n-1)h) \end{vmatrix}.$$

We also use below the following notations

$$\phi_n(x) = \frac{\delta_{n+1}[f,f_1,f_2,\ldots,f_n](x)}{\delta_n(x+h)}, \tag{33}$$

$$\Phi_j(x) = \phi_{j-1}(x)|_{f(x)=f_j(x)},\ j = 1,2,\ldots,n. \tag{34}$$

$$\kappa_j(x) = \frac{\Phi_j(x-h)}{\Phi_j(x+h)}, \tag{35}$$

$$v(x)f_j(x+h) + f_j(x-h) = \lambda_j f_j(x), \tag{36}$$

$$v(x)f(x+h) + f(x-h) = \lambda f(x). \tag{37}$$

Proposition 2a *The function ϕ_n represents the general solution of the functional difference equation*

$$v_n(x)\phi_n(x+h) + \phi_n(x-h) = \phi_n(x), \tag{38}$$

$$v_n(x) = v(x+nh)\,\frac{\delta_n(x-h)\delta_n(x+2h)}{\delta_n(x)\delta_n(x+h)} \tag{39}$$

Providing that $v(x,t)$ is a solution of (11), that $f_j(x)$ are also the solutions of (13) the RHS of (39) is again the solution of DKdV equation.

The function $\phi_n(x)$ can be also represented in a following factorized form

$$\phi_n(x) = (T^{-1} + \kappa_n(x))\ldots(T^{-1} + \kappa_1(x))f(x,\lambda) \tag{40}$$

, where T is the shift operator : $T^{\pm k}f(x) = f(x \pm kh)$.

3.2 Potentials $v_{nm}(x)$ as the result of the multiple Darboux transform

Here we obtain the main result of this article - global solution of the difference Schrödinger equation

$$v_{mn}(x)\psi(x+h)+\psi(x-h)=\lambda(x). \tag{41}$$

, where $v_{mn}(x)$, is defined by (6) and n,m are two nonegative integers. We suppose also that $n \geq m$. The case $n < m$ can be treated in a same way without any difficulties and we less this case as an easy exercise for the reader.

First we prove the following statement.

Proposition 3. *The following 2 functions*

$$F_1(x,n) := \prod_{k=0}^{n} c(x-kh), \qquad n \geq 0 \tag{42}$$

$$F_2(x,n) := \left(\prod_{k=1}^{n} c(x+kh)\right)^{-1}, \qquad n \geq 1 \tag{43}$$

are the solutions of (41) with $m=0$ $\lambda=\lambda_j, j=1,2$

$$\lambda_1 := 2\cosh qh(n+1),\ \lambda_2 := 2\cosh qhn,$$

respectively.

Proof.

Substituting $\psi(x)=F_j(x,n)$ in (41) and removing the commun factors in the LHS and HS of the obtained relation, we reduce the proof to checking the following 2 identities:

$$c(x+(n+1)h)+c(x-(n+1)h))=2\cosh qh(n+1)\,c(x),$$

$$c(x-nh)+c(x+nh)=2\cosh qhn\,c(x)$$

Replacing n by $n-1$ we transform the first identity to the second one, which can be trivialy checked. This completes the proof.□

Remark. Replacing $c(x)$ by $s(x)$ in the proposition proved above we obtaine the similar statement concerning the potential v_{0n}.

Proposition 4. *The following 4 functions*

$$F_1(x,n,m) = \prod_{k=0}^{n} c(x-kh) \prod_{j=0}^{m} s(x-jh),$$

$$F_2(x,n,m) = \Big(\prod_{k=1}^{n} c(x+kh) \prod_{j=1}^{m} s(x+jh)\Big)^{-1}, \qquad m,n \geq 1,$$

$$F_3(x,n,m) = \prod_{k=0}^{m} s(x-kh) \Big(\prod_{j=1}^{n} c(x+jh)\Big)^{-1}, \qquad n \geq 1,$$

$$F_4(x,n,m) = \prod_{k=0}^{n} c(x-kh) \Big(\prod_{j=1}^{m} s(x+jh)\Big)^{-1}, \qquad m \geq 1,$$

are the solutions of (41) with $\lambda = \lambda_j, j = 1,2,3,4$

$$\lambda_1 := 2\cosh qh(n+m+2),\ \lambda_2 := 2\cosh qh(n+m),$$
$$\lambda_3 := 2\cosh qh(n-m-1),\ \lambda_4 := 2\cosh qh(m-n-1),$$

respectively.

Proof.

Substituting $\psi(x) = F_j(x,n,m)$ in (41) and removing the commun factors in the LHS and HS of the obtained relation, we reduce the proof to checking the following 4 identities:

$$c(x+(n+1)h)s(x+(m+1)h) + c(x-(n+1)h))s(x-(m+1)h) = \lambda_1 c(x)s(x),$$

$$c(x-nh)s(x-mh) + c(x+nh)s(x+mh) = \lambda_2 c(x)s(x)$$

$$c(x-nh)s(x+(m+1)h) + c(x+nh)(s(x-(m+1)h) = \lambda_3 c(x)s(x),$$

$$s(x-mh)c(x+(n+1)h) + s(x+mh)(c(x-(n+1)h) = \lambda_4 c(x)s(x),$$

Replacing n by $n-1$ and m by $m-1$ we transform the first identity to the second one, which can be checked by the direct calculation. The proof of the third and of the fourth identities is also straightforward and complets the proof of Proposition 3.□

The special solutions listed above become particularly simple in the cases $m = 0$ or $n = 0$, corresponding to the potentials $v_{n0}(x)$ or v_{0m}: for these cases they are different from the solutions given by Proposition 2 and they correspond to the different values of spectral parameter λ.

Remark. Formulas similar to those of Propositions 3,4 can be obtained for the trigonometric version of the potentials v_{nm} replacing cosh by cos in the expressions for λ_j and replacing $c(x), s(x)$ by $\cos qx$ and $\sin qx$ respectively, which corresponds to replace q by iq in the formulas listed above.

Now we are in a position to show that the potentials $v_{nm}(x)$ can be obtained from the trivial starting potential $v(x) = 1$ by the action of the n-fold Darboux dressing assuming that $m \leq n$, (otherwise we should use an appropriately choosen m-fold Darboux dressing). This can be done in two steps. First we perform the (n-m)-fold consequtive Darboux dressing using the functins $F_1(x, k, 0), k = 0, \ldots, n-m-1$, as the generating functions of the consequtive single Darboux transforms , thus generating the potential $v_{(n-m)0}(x)$ together with the general solution of the related DS equation. Next, using $F_1(x, n-m+j), j), j = 1, \ldots, m$, as a generating functions of the consequtive Darboux transforms we obtaine the potential $v_{nm}(x)$ together with the general solution of the related Schrödinger equation. Due to the Darboux-covariance statement formulated above we only have to check that the following proposition holds.

Proposition 5.
The potential $v_{j0}(x)$ is mapped to $v_{j+1,0}(x)$ by the Darboux transform (32) with the generating function $F_1(x, j)$ and $v_{jk}(x)$ is mapped to $v_{j+1,k+1}(x)$ by the DT with generating function $F_1(x, j, k)$.

Proof.
The proof obviously reduces to check two identities:

$$v_{j+1,0}(x) = v_{j,0}(x+h)\frac{F_1(x-h, j)F_1(x+2h, j)}{F_1(x+h, j)F_1(x, j)},$$

$$v_{j+1,k+1}(x) = v_{j,k}(x+h)\frac{F_1(x-h, j, k)F_1(x+2h, j, k)}{F_1(x+h, j, k)F_1(x, j, k)}, \tag{44}$$

following immediatly from the definition of the potentials $v_{jk}(x)$ and the functions $F_1(x, j), F_1(x, j, k)$ after removing the commun factors in nominators and denominators of the RHS of (44).□

Now the general solution of (41)in the case $n \geq m$ is given by the formula

$$\phi_n(x) = (T^{-1} + \kappa_n(x)) \ldots (T^{-1} + \kappa_1(x))f(x, \lambda) \tag{45}$$

, where $\kappa_j(x)$ is defined by the formula

$$\kappa_j(x) = \frac{F_1(x-h,k,0)}{F_1(x+h,k,0)}, k = 1, \dots, n-m, \tag{46}$$

$$\kappa_j(x) = \frac{F_1(x-h,n-m+j,j)}{F_1(x+h,n-m+j,j)}, j = 1, \dots, m. \tag{47}$$

The special solutions of the form $F_2(x,j), F_3(x,j,k)$ represent smooth bound state eigenfunctions exponentially decreasing at infinity. They can be used to construct the new bound states for the same potentials.

3.3 Bound state eigenfunctions for difference Schrödinger equation with potential $v_{n0}(x)$.

From the results derived above we can immediatly deduce the following statement:

Proposition 6.
Operator $v_{n0}(x)T + T^{-1}$ has $2n$ discrete eigenvalues:

$$\pm\lambda_j, \lambda_j = 2\cosh qhj,\ j = 1, \dots n.$$

The eigenfunctions $\psi_{n,j}$ corresponding to the eigenvalues λ_{n-j} are given by the formulas

$$\psi_{n,j} = D_{F_1(x,n-1)} \cdot D_{F_1(x,n-2)} \cdots D_{F_1(x,n-j+1)} F_3(x, n-j, 0),$$
$$j = 2, \dots, n-1,$$

where the operations $D_{F_1(x,j)}$ are defined as in subsection 3.1.

$$\psi_{n,0} = F_2(x,n), \qquad \psi_{n,1} = F_3(x,n,0).$$

The infinitedimensional subspace of eigenfunctions corresponding to the same eigenvalue is obtained by multiplying $\psi_{n,j}$ by any h periodic function of x. The eigenfunctions corresponding to the eigenvalues $-\lambda_j$ are obviously obtained from the eigenfunctions, corresponding to the eigenvalues λ_j, multiplying $\psi_{n,j}$ by $\exp(i\pi x/h)$.

Different formulas for the bound states eigenfunctions follows also from the determinant representation of the general solution of (12) with potential $v_{n0}(x)$ given in the last section.

One more form for the bound states eigenfunctions quite different from given here can be found in [3].

4 Getting v_{n0} potentials from Casorati determinants

In this section we use the following solutions of (12) with

$$v = 1, \lambda = 2, 2\cosh qh, 2\cosh 2qh, \ldots, 2\cosh qnh :$$

$$f_1(x) = 1, f_2(x) = \sinh q(x+h), f_3 = \cosh 2q(x+h), \ldots, f_{n+1} = \cosh nq(x+h). \tag{48}$$

$f(x, \lambda)$ as before denotes any solution of (12) with $v = 1$.

The statement formulated below provides the Casorati determinant representation for the potentials $v_{n0}(x)$ and the general solution of the related DSE equation :

Proposition 7. *Potential $v_{m0}(x)$ can be represented in a following determinant form*

$$v_{n0}(x) = \frac{\Delta_{n+1}(x-2h)\Delta_{n+1}(x+h)}{\Delta_{n+1}(x)\Delta_{n+1}(x-h)}, \ \Delta_{n+1}(x) = \Delta_{n+1}[f_1, \ldots, f_{n+1}](x) \tag{49}$$

The general solution of the DSE equation with the potential $v_{n0}(x)$ is given by the formula

$$\frac{\Delta_{n+2}[f(x, \lambda, f_1, \ldots, f_{n+1}](x)}{\Delta_{n+2}[f_1, \ldots, f_{n+1}](x)},$$

The solution of DKdV equation with the initial condition $v(x, 0) = v_{n0}(x)$ is given by the formula (49) *where the functions $f_j(x, t)$ are defined as follows*

$$f_1(x) = 1, f_2(x) = \sinh[q(x+h) + t\sinh qh],$$

$$f_3 = \cosh[2q(x+h) + t\sinh 2qh], \ldots, f_{n+1} = \cosh[nq(x+h) + t\sinh nqh],$$

if n is even number and ending by $f_{n+1} = \sinh[nq(x+h) + t\sinh nqh]$ if n is an odd number.

Proof.

It is clear that the formulas (6) represent the special reductions of (26) with $v(x, t) = 1$ corresponding to the special choice of λ_j described above and to the particular selection of the solutions $f_j(x)$ of (12), (with $v(x, t) = 1$) for which F_k defined in (42) coincide with $F_1(x, k)$ up to the x independent factor. Assume for instance that

$$f_1(x) = 1, f_2(x) = \cosh qx, f_3 = \cosh 2qx, \ldots, f_{n+1} = \cosh nqx.$$

In this case the determinant $\Delta_{n+1}(x)$ can be easily computed:

$$\Delta_{n+1}(x) = 2^{\frac{n(n-1)}{2}} \prod_{n+1\geq j>k\geq 1} (\cosh q(x-2(n-j)) - \cosh q(x-2(n-k))$$

Therefore for the function F_n we get the formula

$$F_n(x) = 2^{n-1} \prod_{k=1}^{n} (\cosh qx - \cosh q(x-2kh)) \tag{50}$$

$$= 2^{2n-1} \prod_{k=1}^{n} \sinh q(x-kh) \sinh qkh. \tag{51}$$

Taking into account, that $\sinh q(x-i\pi/2q-kh) = i\cosh q(x-kh)$, we realize, that the x-dependent factor in $F_n(x+i\pi/2q)$ is given by the formula:

$$\prod_{k=1}^{n} \cosh q(x-kh) = \prod_{k=1}^{n} s(x-kh) = F_1(x-h, n-1).$$

In other words taking the functions $f_j(x)$ in the form

$$f_1(x) = 1, f_2(x) = \sinh q(x+h), f_3 = \cosh 2q(x+h), \ldots, f_{n+1} = \cosh nq(x+h), \tag{52}$$

if n is even number, and ending by $f_{n+1} = \sinh nq(x+h)$, if n is an odd number, we obtain the relation

$$F_n(x) = cF_1(x, n-1), \tag{53}$$

where the LHS of (53) is defined by (42) and the RHS of the same formula was defined in the Proposition 2. Since the LHS and RHS of (53) are the solutions of (12) with the same value of $\lambda = 2\cosh qnh$ the related potentials are the same. Therefore, it is clear that the related global solution of the discrete Shrödinger equation with potential $v_{n0}(x)$ can be written as in the Prposition 7. above. The structure of the global solution and the formula for the solution of the difference KdV equation with the initial condition $v(x,0) = v_{n0}(x)$, given above, now follows immediately from the Proposition 2.□

Of course it is also possible to get by the reduction of Casorati determinants the whole family of the potentials $v_{nm}(x)$ as it was done for the DPT potentials in [6]. We postpone the corresponding details to a more detailed publication.

5 Conclusion

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The difference Schrödinger equation with potential $v_{nm}(x)$, considered above, merits some further investigations along the lines developped for its one parametric reductions [3, 10] and further lattice reductions [11, 12], considered before and corresponding to the potentials $v_{n0}(x)$ and their trigonometric versions or to their lattice restrictions.

Let us emphazise that even in this reduced case our formulas for the solutions of the related DSE equation are different from given in the aforementioned works and have more simple combinatorial structure.

Recently an elliptic analog of the potentials $v_{n0}(x)$, representing also the functional difference generalization of the Lamé equation was considered by Ruijenaars,Krichever,Zabrodin, Felder and Varchenko (see for instance [4, 15]).

The related potential is obtained from $v_{n0}(x)$ simply by setting c(x) = $\theta_1(qx)$, where $\theta_1(x)$ is one of Jacobi elliptic theta functions:

$$\theta_1(x|\tau) := -\sum_{k\in Z} \exp\Big(i\pi\tau(k+1/2)^2 + 2i\pi(x+1/2)(k+1/2)\Big).$$

Their exists the beautyfull 4 parametric generalization of the Lamé finite gap elliptic potential, discovered first by Darboux, [2]. [2] It was rediscovered later by Trebich and Verdier [13, 14] in a context of study of elliptic reductions of the Its-Matveev formula [5] for the general finite gap potentials. Its difference analogue until now was not discussed in the literature. The degenerate case, considered above, suggests the conjecture that the difference integrable elliptic analog of the DVT potential has the form

$$v(x, n_1, n_2, n_3, n_4) = \prod_{j=1}^{4} \frac{c_j(x - n_j h) +) \, c_j(x + (n_j+1)h) j}{c_j(x)\, c_j(x+h)},$$

where $c_j(x)$ are the 4 Jacobi theta functions

$$c_j(x) = \theta_j(qx).$$

When $n_2 = n_3 = n_4 = 0$, $v(x, n_1, n_2, n_3, n_4$ reduces to the Krichever-Zabrodin-Ruijenaars-Felder-Varchenko difference Lamé potential. In the limit

[2] Darboux proved its integrability via explicit construction of the related solutions of the Sturm-Liouville equation.

$h \to 0$, $v(x, n_1, n_2, n_3, n_4)$ obviously has the following asymptotics

$$v = 1 + h^2 \sum_{j=1}^{4} n_j(n_j + 1) \frac{d^2}{dx^2} \log \theta_j(qx) + 0(h^3),$$

where the coefficient of h^2 is exactly the famous Darboux-Verdier-Trebich potential. It is clear, from the remarks made in the introduction, that the related difference Schrödinger equation in a same limit transforms to the Schrödinger equation with the DVT potential. It is obvious that the potential $v(x, n_1, n_2, n_3, n_4)$ represent a double periodic elliptic function with the periods q^{-1} and $q^{-1}\tau$. When $Im\, \tau \to +\infty$ and q is a real parameter $v(x, n_1, n_2, n_3, n_4)$ tends to the trigonometric version of the potential $v_{nm}(x)$ with $c(x) = \cos qx, s(x) = \sin qx$. Replacing q by iq, in a same limit, we obtain the hyperbolic potentials $v_{nm}(x)$ defined by (6). The investigation of the solutions corresponding to the DSE with potential $v(x, n_1, n_2, n_3, n_4)$ is actually in progress.

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