Polar Actions on Hilbert Space

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§0. Introduction.

In finite dimensions, the theory of symmetric spaces, polar representations, and isoparametric submanifolds of Euclidean spaces are all closely related. For example, the following are known:

(1) The isotropy representation of symmetric space G/K at eK (which is called an *s-representation*) is polar. Moreover, Dadok proved in [D] that up to orbital equivalence these are the only polar actions on Euclidean spaces.
(2) The orbits of s-representations are generalized full or partial flag manifolds.
(3) The principal orbits of a polar representation are isoparametric ([Te1]).
(4) Thorbergsson proved in [Th] that an irreducible, full, compact, isoparametric submanifold of codimension $r \geq 3$ in Euclidean space must be a principal orbit of some s-representation.
(5) There are many codimension two isoparametric submanifolds ([OT1,2], [FKM]) that are not orbits of any orthogonal representations. Nevertheless these submanifolds share many of the geometric and topological properties of flag manifolds.

So it is natural to ask whether there are analoguous results in infinite dimensions. A start in this direction is made in [Te2], where we proved an infinite dimensional analogue of (3). In this paper, we give a general construction for polar actions on Hilbert space, study the submanifold geometry of the infinite dimensional orbits of these actions, and discuss their relations to "infinite dimenisonal symmetric spaces".

Let us first recall some definitions from transformation group theory ([PT2]). A smooth isometric action of a Hilbert Lie group $\hat{G}$ on a Hilbert manifold M is called *proper* if $g_n \cdot x_n \to y_0$ and $x_n \to x_0$ in M implies that $\{g_n\}$ has a convergent subsequence in $\hat{G}$; and it is called *Fredholm* if given any $x \in M$ the orbit map $\hat{G} \to M$ defined by $g \mapsto g \cdot x$ is a Fredholm map. Note that isotropy subgroups of proper actions are compact, and the orbits of Fredholm actions are of finite codimension. An isometric, proper Fredholm action is called *polar* if there exists a closed submanifold S of M, which meets all orbits and meets them orthogonally. Such an S is called a *section* for the action. We call

$$N(S, \hat{G}) = \{g \in \hat{G} \mid g \cdot S \subset S\}, \quad Z(S, G) = \{g \in \hat{G} \mid g \cdot s = s \;\; \forall s \in S\}$$

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the normalizer and centralizer of S in G respectively. The quotient group

$$W(S,\hat{G}) = N(S,\hat{G})/Z(S,\hat{G}),$$

is a discrete group, called the *generalized Weyl group* associated to the polar action.

In [Te2], we obtained an infinite dimensional analogue of (3), and also showed that the orbits of polar actions on Hilbert spaces have many of the special properties of the finite dimensional complex or real flag manifolds. In particular, for a polar action of a Hilbert Lie group $\hat{G}$ on a Hilbert space V, we proved that the generalized Weyl group W of the action is an affine Weyl group, the focal points of a principal orbit are just the points on singular orbits, and the focal set meets the section in the union of the reflection hyperplanes of W. The focal multiplicity of points on a reflection hyperplane is invariant under W, so we can use them as markings for the Dynkin diagram of W, thereby associating to each polar action on a Hilbert space a marked affine Dynkin diagram. Moreover, we also proved that:

(i) If M is a principal orbit in V then M is an isoparametric submanifold of V, i.e., the normal bundle $\nu(M)$ is globally flat, and the shape operators $A_{v(x)}$ and $A_{v(y)}$ are conjugate along any parallel normal field v.

(ii) the restriction of the distance function $f_a(x) = \| x - a \|^2$ to any G-orbit is a perfect Morse function, and the homology of an orbit can be computed directly from W and its multiplicities in the same way as in [BS] for the finite dimensional case.

There are two known families of polar actions on Hilbert space. The first family is given in [Te2]: the action of the Hilbert Lie group $H^1(S^1, G)$ of H^1-loops of a compact connected, semi-simple Lie group G on the Hilbert space $H^0(S^1, \mathfrak{g})$ given by the gauge transformation

$$g \cdot u = gug^{-1} - g'g^{-1}$$

is polar. These orbits can be viewed as infinite dimenisonal analogue of complex full or partial flag manifolds ([PS]). A second family was constructed by Pinkall and Thorbergsson ([PiT]): Let G/K be a symmetric space of compact type, and

$$\hat{K} = \{g \in H^1([0,\pi], G) \mid g(0), g(\pi) \in K\}.$$

Then the action of the Hilbert Lie group $\hat{K}$ on $H^0([0,\pi], \mathfrak{g})$ by gauge transformations is again polar. The orbits can be viewed as infinite dimensional analogues of real flag manifolds (see [Te3]).

Note that the first example is related to the Adjoint action of G on G, and the second example is related to the $K \times K$-action on G. Since both the Adjoint action and the $K \times K$-action on G are polar with flat sections, this leads naturally to a general construction of polar actions on Hilbert spaces. Namely, if H is a closed subgroup of $G \times G$, and the action of H on G defined by $(h_1, h_2) \cdot x = h_1 x h_2^{-1}$ is polar with flat sections, then the action of the group $P(G,H)$ of H^1-paths g in G with $(g(0),g(1)) \in H$ on $V = H^0([0,1],\mathfrak{g})$ by gauge transformations

$$g \cdot u = gug^{-1} - g'g^{-1}$$

is polar.

Next we dicuss the relation between polar actions on Hilbert space and involutions of affine Kac-Moody groups. Let $L(G) = H^1(S^1,G)$ denote the Hilbert Lie group of Sobelov H^1-loops in a compact, connected, simple Lie group G. Then the affine Kac-Moody group $\hat{L}(G)$ of type 1 is a 2-torus bundle $\pi : \hat{L}(G) \to L(G)$ (see [PS]). There is a non-degenerate, Ad-invariant bilinear form on the affine algebra $\hat{L}(\mathfrak{g})$ ([K],[PS]), and the Adjoint representation of $\hat{L}(G)$ on its Lie algebra gives rise to the first example of polar action on Hilbert space above. We show that every involution ρ on G induces an involution ρ^* on $\hat{L}(G)$ such that $\hat{K}$ is the fixed point set of ρ^* and the Adjoint representation of $\hat{K}$ on $\hat{\mathfrak{p}}$ gives rise to the second example of polar action on Hilbert space above, where K is the fixed point set of ρ and $\hat{\mathfrak{p}}$ is the -1 eigenspace of $d\rho^*$ on $\hat{L}(\mathfrak{g})$. Let σ be an outer automorphism of G of order k, and let $L(G,\sigma)$ denote the subgroup of $L(G)$ consisting of loops g that are Z_k-equivariant, i.e., satisfy $g(t+2\pi/k) = \sigma(g(t))$. Then the subgroup $\hat{L}(G,\sigma) = \pi^{-1}(L(G,\sigma))$ is the affine Kac-Moody group of type k. Moreover, we show that the Adjoint actions of affine groups of type k on their Lie algebras give rise to the polar action of $P(G,H)$ on $H^0([0,1],\mathfrak{g})$, where $H = \{(x,\sigma(x)) \mid x \in G\}$. Following the terminology from the theory of finite dimenisonal symmetric spaces, we call $\hat{L}(G)$ and $\hat{L}(G,\sigma)$ symmetric spaces of type II, and $\hat{L}(G)/\hat{K}$ a symmetric space of type I.

The above discussion suggests that, as in finite dimensions, a close relationship does exist between polar actions on Hilbert spaces and isotropy representations of symmetric spaces. However, the precise nature of this relation is still not well-understood.

The paper is organized as follows: In section 1, we associate to each polar H-action with flat sections on a compact Lie group G, a polar action on the Hilbert space $H^0([0,1],\mathfrak{g})$, and we study the relation between the H-action and the associated infinite dimensional action. In section 2, we give description of the principal curvatures of the principal orbits of the associated infinite dimenisonal action in terms of the H-action. Finally, in section 3, we discuss the relation between the isotropy representations of infinite dimensional symmetric spaces and polar actions on Hilbert spaces.

§1. $P(G,H)$-actions.

In this section, we give a general construction of polar actions on Hilbert spaces from polar actions on compact Lie groups with flat sections.

We first set up notations. Let G be a connected, compact, semi-simple Lie group, equipped with a bi-invariant metric. Let

$$\hat{G} = H^1([0,1],G), \qquad V = H^0([0,1],\mathfrak{g})$$

denote the Hilbert Lie group of all Sobolev H^1-paths from $[0,1]$ to G and the Hilbert space of H^0-maps from $[0,1]$ to $\mathfrak{g}$ respectively. (One should think of elements of V as representing connections (automatically flat) for the product principal G-bundle over $[0,1]$). Let $\hat{G}$ act on V isometrically via gauge transformations:

$$g \cdot u = gug^{-1} - g'g^{-1}.$$

It is obvious that given any $u \in V$ there exists $g \in \hat{G}$ such that $u = -g'g^{-1}$. This proves that $\hat{G}$ acts transitively on V. Using the same proof as in [Te2], we see that this action is proper and Fredholm. So one way to produce interesting proper Fredholm, isometric actions on Hilbert space is to find closed, finite codimension subgroups of $\hat{G}$, whose orbits in V give interesting geometric and topological manifolds. In the following, we will give a method of finding subgroups of $\hat{G}$, whose action on V is polar.

Let H be a closed, connected subgroup of $G \times G$, acting on G by

$$(h_1, h_2) \cdot g = h_1 g h_2^{-1}.$$

Let

$$P(G,H) = \{g \in \hat{G} = H^1([0,1],G) \mid \ (g(0), g(1)) \in H\}.$$

Note that the homomorphism

$$\Psi : \hat{G} \to G \times G, \qquad g \mapsto (g(0), g(1))$$

is a submersion and $P(G,H) = \Psi^{-1}(H)$. It follows that $P(G,H)$ is closed and has finite codimension in $\hat{G}$, hence the action of $P(G,H)$ is also proper and Fredholm.

Next we define two operators

$$E : H^0([0,1],\mathfrak{g}) \to H^1([0,1],G), \qquad \Phi : H^0([0,1],\mathfrak{g}) \to G$$

as follows: let $E(u)(t)$ be the parallel translation in the trivial principal bundle $I \times G$ over $I = [0,1]$ defined by the connection $u(t)dt$, and $\Phi(u)$ the holonomy; i.e., $E(u) : [0,1] \to G$ is the unique solution of the initial value problem:

$$\begin{cases} E^{-1}E' = u \\ E(0) = e, \end{cases}$$

and

$$\Phi(u) = E(u)(1).$$

1.1 Proposition. *With notations as above,*

(i) $E(g \cdot u) = g(0)E(u)g^{-1}, \quad \Phi(g \cdot u) = (g(0), g(1)) \cdot \Phi(u),$

(ii) $v \in P(G,H)\cdot u$ *if and only if* $\Phi(v) \in H\cdot\Phi(u)$.

PROOF. It is easy to prove (i) by a direct computation. To see (ii), let $v \in P(G,H)\cdot u$. Then it follows from (i) that $\Phi(v) \in H\cdot\Phi(u)$. Conversely, suppose $\Phi(v) = h_1\Phi(u)h_2^{-1}$ for some $(h_1,h_2) \in H$. Let $g(t) = E(v)^{-1}(t)h_1E(u)(t)$. Then $(g(0),g(1)) = (h_1,h_2)$, and

$$E(g\cdot u) = g(0)E(u)g^{-1} = h_1E(u)E(u)^{-1}h_1^{-1}E(v) = E(v),$$

which implies that $g\cdot u = v$. ∎

We recall ([PT1]) that a section for a polar action is automatically totally geodesic. Moreover it is well-known that a compact, flat, totally geodesic submanifold of G containing e is a torus subgroup.

1.2 Theorem. *Let the notation be as above, and suppose the H-action on G is polar with flat sections. Let A be a torus section through e and let $\mathfrak{a}$ denote its Lie algebra. Then:*

(1) the $P(G,H)$-action on V is polar, and the space $\hat{\mathfrak{a}} = \{\hat{a} \mid a \in \mathfrak{a}\}$ is a section, where $\hat{a} : [0,1] \to \mathfrak{g}$ denotes the constant map with value a,

(2) the generalized Weyl group $\hat{W} = W(\hat{\mathfrak{a}}, P(G,H))$ is an affine Weyl group,

(3) if $\Psi : P(G,H) \to H$ is the homomorphism defined by $\Psi(g) = (g(0),g(1))$, then $\Psi(N(\hat{\mathfrak{a}},P(G,H)) = N(A,H)$, Ψ maps $Z(\hat{\mathfrak{a}},P(G,H))$ isomorphically onto $Z(A,H)$, and

$$\Lambda = \mathrm{Ker}(\Psi)\cap N(\hat{\mathfrak{a}},P(G,H)) = \{g(t) = \exp(\lambda t) \mid \lambda \in \Lambda(A)\},$$

where $\Lambda(A)$ is the unit lattice of A:

$$\Lambda(A) = \{\lambda \in \mathfrak{a} \mid \exp(\lambda) = e\},$$

(4) Ψ induces a surjective homomorphism from $\hat{W}$ to $W = W(A,H)$, W is isomorphic to $\hat{W}/\Lambda$, and $\hat{W}$ is the semi-direct product of W with Λ,

(5) Ψ maps the isotropy subgroup $P(G,H)_{\hat{a}}$ isomorphically to $H_{\exp(a)}$; in fact its inverse is the map $\varphi : H_{\exp(a)} \to P(G,H)_{\hat{a}}$ defined by $(h_1,h_2) \mapsto g(t) = \exp(-at)h_1\exp(at)$,

(6) $u \in V$ is a singular point of the $P(G,H)$-action if and only if $\Phi(u) \in G$ is a singular point of the H-action.

PROOF. We prove each statement of the theorem seperately below.

(1) To see that $\hat{\mathfrak{a}}$ meets every $P(G,H)$-orbit, we let $u \in V$. Since A meets every H-orbit, there is $a \in \mathfrak{a}$ such that $\Phi(u) \in H\cdot\exp(a)$. But $E(\hat{a}) = \exp(ta)$, so $\Phi(\hat{a}) = \exp(a)$. By 1.1 (ii), we have $\hat{a} \in P(G,H)\cdot u$. To prove $\hat{\mathfrak{a}}$ is orthogonal to $P(G,H)\cdot\hat{a}$ for all $a \in \mathfrak{a}$, we note that

$$T(P(G,H)\cdot\hat{a})_{\hat{a}} = \{[u,\hat{a}] - u' \mid u \in H^1([0,1],\mathfrak{g}),\quad (u(0),u(1)) \in \mathfrak{h}\}.$$

For $b \in \mathfrak{a}$, we have $[a, b] = 0$ and

$$\langle [u, \hat{a}] - u', \hat{b} \rangle = \int_0^1 ([u, a] - u', b)\, dt$$

$$= \int_0^1 (u, [a, b]) - (u, b)' dt = 0 - (u(1) - u(0), b) = 0,$$

where the last equality follows from the fact that

$$T(H \cdot e)_e = \{x - y \,|\, (x, y) \in \mathfrak{h}\} \ \perp \ \mathfrak{a}.$$

(2) It was proved in [Te2] that the generalized Weyl group of a polar action on Hilbert space is an affine Weyl group.

(3) First we prove $\Psi(N(\ \hat{\mathfrak{a}}, P(G, H)) \subset N(A, H)$. Let $g \in N(\ \hat{\mathfrak{a}}, P(G, H))$. Then for $a \in \mathfrak{a}$ there exists $b \in \mathfrak{a}$ such that $g \cdot \hat{a} = \hat{b}$. Using 1.1 (i) and the fact that $E(\hat{a}) = \exp(ta)$, $E(\hat{b}) = \exp(bt)$, we have $\exp(b) = g(0)\exp(a)g(1)^{-1}$. This proves that $(g(0), g(1)) \in N(A, H)$.

Next we prove that $N(A, H) \subset \Psi(N(\ \hat{\mathfrak{a}}, P(G, H))$. Let $(h_1, h_2) \in N(A, H)$. Then $h_1 A h_2^{-1} = A$. In particular, this implies that $h_1 h_2^{-1} \in A$, $h_1 A = A h_2$, $h_1 A h_1^{-1} = A h_2 h_1^{-1} = A$, and $h_1 \mathfrak{a} h_2^{-1} = \mathfrak{a}$. Now given $a_1 \in \mathfrak{a}$, there exists $b_1 \in \mathfrak{a}$ such that $h_1 \exp(a_1) h_2^{-1} = \exp(b_1)$. Set

$$g(t) = \exp(-b_1 t) h_1 \exp(a_1 t).$$

Then it is easily seen that $(g(0), g(1)) = (h_1, h_2)$, and a direct computation shows that

$$g \cdot \hat{a} = b_1 + \exp(-b_1 t) h_1 (a - a_1) h_1^{-1} \exp(b_1 t).$$

Since $h_1 \mathfrak{a} h_1^{-1} = \mathfrak{a}$ and $\mathfrak{a}$ is abelian, $g \cdot \hat{a} = b_1 + h_1(a - a_1)h_1^{-1} \in \ \hat{\mathfrak{a}}$.

To compute the intersection of $\mathrm{Ker}(\Psi)$ and $N(\ \hat{\mathfrak{a}}, P(G, H))$, we assume that $g \in N(\ \hat{\mathfrak{a}}, P(G, H))$ and $g(0) = g(1) = e$. Then for $a \in \mathfrak{a}$, there exists $b \in \mathfrak{a}$ such that $g \cdot \hat{a} = \hat{b}$, so 1.1 (i) implies that

$$g(t) = E(\hat{b})^{-1} g(0) E(\hat{a}) = \exp((a - b)t).$$

But $g(1) = \exp(a - b) = e$, i.e., $(a - b) \in \Lambda$. Conversely, let $\lambda \in \Lambda(A)$, and $g(t) = \exp(\lambda t)$. Then $g \in P(G, H)$, $g \in \mathrm{Ker}(\Psi)$, and $g \cdot \hat{a}$ is the constant map with constant equal to $a - \lambda$. This proves that $g \in \mathrm{Ker}(\Psi) \cap N(\hat{\mathfrak{a}}, P(G, H))$.

Now let $g \in Z(\ \hat{\mathfrak{a}}, P(G, H))$. Since $g \cdot \hat{0} = \hat{0}$, $g' g^{-1} = 0$. So $g(t) = h_1$ is a constant map. But $g \cdot \hat{a} = h_1 a h_1^{-1} = a$. This implies that $\Psi(g) = (h_1, h_1) \in Z(A, H)$, and Ψ maps $Z(\ \hat{\mathfrak{a}}, P(G, H))$ injectively into $Z(A, H)$. To prove surjectivity we let $(h_1, h_2) \in Z(A, H)$. Then $h_1 e h_2^{-1} = e$ implies that $h_1 = h_2$. Set $g(t) = h_1$. Since $h_1 a h_1^{-1} = a$, $g \cdot \hat{a} = \hat{a}$ for all $a \in \mathfrak{a}$.

(4) is a consequence of (3).

(5) If $g \in P(G,H)_{\hat{a}}$, then from $E(\hat{a})(t) = \exp(ta)$ and 1.1 we have $\exp(at) = g(0)\exp(at)g^{-1}$. So $\exp(a) = g(0)\exp(a)g(1)^{-1}$, i.e., $\Psi(g) \in H_{\exp(a)}$. Conversely, if $h_1 \exp(a)h_2^{-1} = \exp(a)$, then set $g(t) = \exp(-at)h_1\exp(at)$, a direct computation as above implies that $g \cdot \hat{a} = \hat{a}$.

(6) follows from (5). ∎

Let N be a submanifold of G, and $\Omega(G,e,N)$ the set of all H^1-paths in G such that $\gamma(0) = e$ and $\gamma(1) \in N$. It is known that (cf. [P1]) $\Omega(G,e,N)$ is a Riemannian Hilbert manifold,

$$T(\Omega(G,e,N))_\gamma = \{u\gamma \mid u \in H^1([0,1],\mathfrak{g})\},$$

and the Riemannian inner product on $T(\Omega(G,e,N))_\gamma$ is

$$(u\gamma, v\gamma) = \int_0^1 (u'(t), v'(t))\, dt.$$

If N is an H-orbit in G, then $P(G,H)$ acts on $\Omega(G,e,N)$ by $g * \gamma = g(0)\gamma g^{-1}$, and this action is isometric and transitive.

1.3 Corollary. *With the same notation and assumption as in 1.2, let N be the H-orbit through $\exp(a)$ in G, and*

$$F : \Omega(G,e,N) \to H^0([0,1],\mathfrak{g}), \qquad F(\gamma) = \gamma^{-1}\gamma'.$$

Then F is an isometric, equivariant embedding, and the image of F is equal to $P(G,H) \cdot \hat{a}$.

1.4 Remark. Let M be a Riemannian manifold, G a compact Lie group acting on M isometrically, $p \in M$ a regular point, and N a G-orbit in M. Let

$$\mathcal{E} : \Omega(M,p,N) \to R, \quad \mathcal{E}(\gamma) = \int_0^1 \| \gamma'(t) \|^2\, dt$$

denote the energy functional. Bott and Samelson proved that (cf. [BS]) if the G-action on M is variationally complete, then $\mathcal{E}$ is a perfect Morse function, and the homology of $\Omega(M,p,N)$ can be computed explicitly in terms of the singular data of the G-action. Moreover, they showed that the following H-actions on G are variationally complete:

(i) the action of the diagonal group $H = \Delta(G) = \{(g,g) \mid g \in G\}$ on G, i.e., the Adjoint action of G on G,

(ii) the action of $H = K \times K$ on G, where G/K is a symmetric space.

By results of Hermann ([He]) and Conlon ([C1]) there are two more families of variational complete actions on G:

(iii) the action of $H = K_1 \times K_2$ on G, where both G/K_1 and G/K_2 are symmetric spaces,

(iv) the action of $H = G(\sigma) = \{(g, \sigma(g)) \mid g \in G\}$ on G, where σ is an automorphism of G; the action of $G(\sigma)$ on G will be called the *σ-action.*

Conlon noted that the above four families of actions are polar with flat sections, and he proved that in general polar actions with flat sections are variationally complete (Conlon called these sections K-transversal domains [C2]).

1.5 Examples. Applying 1.2 to the examples (i)–(iv) in 1.4 gives many polar actions on Hilbert spaces. In fact, the first and second families of examples described in the introduction are the $P(G,H)$-action on V corresponding to examples (i) and (ii) in 1.4 respectively. Note also that under the isometric embedding F in 1.3, the path space $\Omega(G, e, H \cdot \exp(a))$ is embedded in the Hilbert space $V = H^0([0,1], \mathfrak{g})$ as a taut submanifold with constant principal curvatures, and the energy functional $\mathcal{E}$ corresponds to the square of the norm in the Hilbert space V, i.e., $\mathcal{E}(g) = \| F(g) \|^2$.

1.6 Theorem. *(Conlon [C2]) Let M be a simply connected, complete Riemannian manifold, K a compact Lie group acting on M isometrically, and M_s the set of singular points. Suppose the K-action on M is polar with a flat section Σ, and the K-action on M is not transitive. Then*

(i) $M_s \neq \emptyset$,

(ii) $M_s \cap \Sigma$ is the union of finitely many totally geodesic hypersurfaces $\{P_1, \ldots, P_r\}$,

(iii) the generalized Weyl group $W(\Sigma, K)$ is a Coxeter group generated by the reflections of Σ in the P_i.

In the following, we will discuss further structures of the generalized Weyl group for polar H-action on G with a flat section A (here G need not be simply connected). First we consider the rank of the affine Weyl group $\hat{W}$ of the $P(G,H)$-action on V. Let $W \subset \mathrm{Iso}(R^k)$ be a Coxeter group, $\{\ell_i \mid i \in I\}$ the set of reflection hyperplanes of W, and $u_i \in R^k$ the unit normal to ℓ_i. Then the *rank* of W is the dimension of the linear span of $\{u_i \mid i \in I\}$. In particular, the Dynkin diagram of a rank k, finite Coxeter group has k vertices, and that of a rank k infinite Coxeter group has $k+1$ vertices. Recall also that the codimension of the principal orbits of an action is called the *cohomogeneity*. For a polar action the cohomogeneity is the dimension of a section [PT1].

1.7 Theorem. *Let the assumptions and notations be as in Theorem 1.2.*

(i) the rank of the affine Weyl group $\hat{W} = W(\hat{\mathfrak{a}}, P(G,H))$ is equal to the cohomogeneity of the H-action on G, which is equal to the dimension of A,

(ii) if the H-action is not transitive, then the H-action on G always has singular points, or equivalently the $P(G,H)$-action has singular points.

Proof. We may assume that $M = P(G,H) \cdot \hat{o}$ is a principal orbit. Suppose that the rank of $\hat{W}$ is less than the dimension of $\mathfrak{a}$. Then there exists $0 \neq b \in \mathfrak{a}$ such that the line $R\hat{b}$ does not meet any reflection hyperplane of $\hat{W}$. So there is no focal point for M on the line $R\hat{b}$. This implies that the shape operator $A_{\hat{b}} = 0$. But

$$TM_{\hat{0}} = \{u' \mid u \in H^1([0,1], \mathfrak{g}),\ (u(0), u(1)) \in \mathfrak{h}\},$$

$$A_{\hat{b}}(u') = [u, \hat{b}] = 0.$$

So $[u, b] = 0$ for all $u \in TM_{\hat{0}}$, which implies that $[x, b] = 0$ for all $x \in \mathfrak{g}$. Since $\mathfrak{g}$ is semi-simple, $b = 0$, which is a contradiction. This proves (i).

(ii) follows directly from the proof of (i). Since the set of focal points of M is the set of singular points for the $P(G, H)$-action ([Te2]), if the action has no singular points then M has no focal points. This implies that all shape operators $A_{\hat{b}} = 0$ for $\hat{b} \in \nu(M)_{\hat{0}}$. Then the above computation implies that $b = 0$. Hence $\dim(\ \hat{\mathfrak{a}}) = 0$, i.e., the H-action is transitive. This proves (ii). ∎

1.8 Theorem. *Let the assumptions and notations be as in Theorem 1.2. The slice representation of the H-action at $\exp(a)$ and the slice representation of the $P(G, H)$-action at $\hat{a}$ are equivalent polar representations. In fact,*

(1) the slice representation of the H-action at $\exp(a)$ is given by

$$(h_1, h_2) \cdot (b \exp(a)) = (h_1 b h_1^{-1}) \exp(a),$$

(2) the normal plane $\nu(P(G, H) \cdot \hat{a})_{\hat{a}}$ is

$$\{v(t) = \exp(-at) b \exp(at) \mid b \exp(a) \in \nu(H \cdot \exp(a))_{\exp(a)}\},$$

and the slice representation at $\hat{a}$ is given by $g \cdot v = g v g^{-1}$,

(3) $f : \nu(H \cdot \exp(a))_{\exp(a)} \to \nu(P(G, H) \cdot \hat{a})_{\hat{a}}$ defined by

$$b \exp(a) \mapsto v(t) = \exp(-at) b \exp(at)$$

is an equivariant isomorphism, i.e.,

$$\varphi((h_1, h_2)) \cdot f(b \exp(a)) = f((h_1, h_2) \cdot b \exp(a)),$$

where φ is as in 1.2 (5).

PROOF. It is known ([PT1]) that the slice representations of a polar action are also polar. The proof of (1)–(3) will be given seperately below.

(1) Given $(h_1, h_2) \in H_{\exp(a)}$, then we have $h_1 \exp(a) h_2^{-1} = \exp(a)$. So if $b \exp(a) \in \nu(H \cdot \exp(a))_{\exp(a)}$, then we have

$$(h_1, h_2) \cdot b \exp(a) = h_1 b \exp(a) h_2^{-1} = h_1 b h_1^{-1} \exp(a).$$

(2) Let $M = P(G, H) \cdot \hat{a} \subset V = H^0([0,1], \mathfrak{g})$. By a direct computation, we obtain that

$$TM_{\hat{a}} = \{[u, \hat{a}] - u' \mid u \in H^1([0,1], \mathfrak{g}),\ (u(0), u(1)) \in \mathfrak{h}\},$$

$$\nu(M)_{\hat{a}} = \{v \in V \mid v' = [v, \hat{a}],\ (v(1), y) = (v(0), x)\ \forall (x, y) \in \mathfrak{h}\}.$$

It is easy to see that if $v' = [v, \hat{a}]$ then there exists $b \in \mathfrak{g}$ such that $v(t) = \exp(-at) b \exp(at)$. In order for $v \in \nu(M)_{\hat{a}}$, v must also satisfy the condition $(v(0), x) = (v(1), y)$ for all $(x, y) \in \mathfrak{h}$, so

$$\begin{aligned}(v(1), y) &= (\exp(-a) b \exp(a), y) = (b \exp(a), \exp(a) y) \\ &= (v(0), x) = (b, x) = (b \exp(a), x \exp(a)).\end{aligned}$$

It follows that

$$(b \exp(a),\ x \exp(a) - \exp(a) y) = 0,\ \ \forall (x, y) \in \mathfrak{h}.$$

But

$$T(H \cdot \exp(a))_{\exp(a)} = \{x \exp(a) - \exp(a) y \mid (x, y) \in \mathfrak{h}\},$$

so (2) follows.

(3) is a consequence of (1) and (2). ∎

1.9 Theorem. *Let the assumptions and notations be as in Theorem 1.2. Then,*

(1) there exists $a \in \mathfrak{a}$ such that the affine Weyl group $\hat{W} = W(\hat{\mathfrak{a}}, P(G, H))$ is the semi-direct product of the finite Weyl group $\hat{W}_{\hat{a}}$ and a lattice group Λ_1, where $\hat{W}_{\hat{a}}$ is isotropy subgroup of $\hat{W}$ at $\hat{a}$, and Λ_1 is a lattice group containing the unit lattice $\Lambda(A)$ and is invariant under $\hat{W}_{\hat{a}}$,

(2) the Weyl group of the slice representation of the H-action at $\alpha = \exp(a)$ is the isotropy subgroup W_α of $W = W(A, H)$, and the Weyl group of the slice representation of the $P(G, H)$-action at $\hat{a}$ is $\hat{W}_{\hat{a}}$,

(3) $\hat{W}_{\hat{a}}$ is isomorphic to W_α,

(4) the generalized Weyl group $W(A, H)$ of the H-action on G is the semi-direct product of W_α and the abelian group $\Lambda_1/\Lambda(A)$.

PROOF. It follows from the standard theory of affine Weyl groups ([Bo]) that there exists $\hat{a} \in \hat{\mathfrak{a}}$ such that

(i) $\hat{W}$ is the semi-direct product of the isotropy subgroup $\hat{W}_{\hat{a}}$ and a lattice group Λ_1,

(ii) if $\hat{W}$ is of rank k, then $\hat{W}_{\hat{a}}$ is a finite Weyl group of rank k,

(iii) $\hat{a}$ is a vertex of a Weyl chamber of $\hat{W}$, and $\hat{W}_{\hat{a}}$ is maximal among all isotropy subgroups of $\hat{W}$

It is known that ([PT1]) the slice representation of a polar action is polar, and the Weyl group of the slice representation at x is equal to the isotropy subgroup at x of the generalized Weyl group of the polar action. So W_α and $\hat{W}_{\hat{a}}$ are the Weyl groups of the slice representations of the H-action and $P(G, H)$-action at α and $\hat{a}$ respectively. It follows from 1.2 (3)–(5) that $\Psi(g) = (g(0), g(1))$ induces an isomorphism from $\hat{W}_{\hat{a}}$ to W_α. This proves (1)–(3), and statement (4) is then a consequence of (1) and 1.2 (4). ∎

Let X be a Riemannian K-manifold. Then a submanifold Δ (possibly with boundary) of X is called a *fundamental domain* for the K-action if it meets each K-orbit in exactly one point. For example, a Weyl chamber in $\hat{\mathfrak{a}}$ is a fundamental domain of the action of the affine Weyl group $\hat{W}$ on $\hat{\mathfrak{a}}$.

1.10 Proposition. *With the same notation and assumption as in 1.2. Let τ be a Weyl chamber in $\hat{\mathfrak{a}}$. Then*

(1) τ is a fundamental domain for the $P(G,H)$-action on V,
(2) τ and $\exp(\tau)$ are isometric,
(3) $\exp(\tau)$ is a fundamental domain for the H-action on G.

PROOF. The orbit space of a polar action is isometric to the orbit space of its section under the action of the generalized Weyl group ([Te2]). Since τ is a fundamental domain for the $\hat{W}$-action on $\hat{\mathfrak{a}}$, (1) follows.

Let Φ be as in 1.1 and Ψ be as in 1.2. Then $\Phi(\hat{\mathfrak{a}}) = \exp(a)$. It follows form 1.1 that $\Phi \mid \tau$ is injective, and $\Phi(\tau) = \exp(\tau)$ meets every H-orbit. This proves (3). Since A is flat, $\exp : \mathfrak{a} \to A$ is a local isometry. So τ is isometric to $\exp(\tau)$. ∎

1.11 Proposition. *Suppose the H-action on G is polar with a torus A as a section, and the action is not transitive. Let k denote the cohomogeneity of the H-action, i.e., $k = \dim(A)$.*

(1) If $k = 1$, then $W(A,H)$ is Z_2 or a dihedral group.
(2) If $k > 1$, then $W(A,H)$ is a rank k or $k+1$ crystallographic group (i.e., a Weyl group).

PROOF. By 1.7 (i), the affine Weyl group $\hat{W} = W(\hat{\mathfrak{a}}, P(G,H))$ has rank k, so $\hat{W}$ is generated by $(k+1)$ reflections. By 1.2 (4), Ψ induces a homomorphism $\hat{\Psi}$ from $\hat{W}$ onto $W(A,H)$, so $W(A,H)$ is a finite group generated by $(k+1)$ order two elements, which implies that it is a Coxeter group of rank at most $k+1$. But 1.9 (1) and (2) imply that the rank of $W(A,H)$ is at least k. So the rank of $W(A,H)$ is either k or $k+1$. But $\hat{W}$ is an affine Weyl group, so $\hat{W}$ satisfies the crystallographic condition if $k > 1$. Hence the image $\hat{\Psi}(\hat{W}) = W(A,H)$ also satisfies the crystallographic condition if $k > 1$. ∎

We end of this section, with two conjectures:

1.12 Conjecture. If the H-action on G is of cohomogeneity 1, then the generalized Weyl group is crystallographic. More generally if M is an isoparametric hypersurface of G, then the number g of distinct principal curvatures of M must be $1, 2, 3, 4$ or 6.

1.13 Conjecture. Given any polar action of a Hilbert Lie group L on a Hilbert space V, one can find some compact Lie group G and a closed subgroup H of $G \times G$ such that:

(1) the action of H on G is polar with flat sections,
(2) the L-action is orbital equivalent to the $P(G,H)$-action, i.e., there exists an isometry from V to $H^0([0,1], \mathfrak{g})$ that maps L-orbits onto the $P(G,H)$-orbits.

§2. The submanifold geometry of $P(G,H)$-orbits.

It is a consequence of section 1 that the submanifold geometry of the orbits of a polar H-action on G with flat sections determines the submanifold geometry of orbits of the associated $P(G,H)$-action in the Hilbert space $V = H^o([0,1],\mathfrak{g})$. For example, the curvature distributions, the curvature normals, and the marked Dynkin diagram of M can be explicitly computed from the data of the H-action on G. These geometric data of M have been given in [Te2] when the H-action is the Adjoint action on G, and in [PiT] when the H-action is the $K \times K$-action on G (here K is the fixed point set of an involution on G). They are all obtained from well-known root space decompositons with respect to the flat section. In the following, we will give the relation between the geometry of M and the H-action.

To simplify the notation, we may assume that $e \in G$ is a regular point for the H-action. So $M = P(G,H)\cdot\hat{0}$ is a principal orbit, and is isoparametric in V ([Te2]). Note that $\nu(M)_{\hat{o}} = \hat{\mathfrak{a}}$, and given $b \in \mathfrak{a}$ the vector field $\tilde{b}(g\cdot\hat{o}) = g\hat{b}g^{-1}$ is a parallel normal field on M. Moreover, there exist a smooth subbundle E_0 of TM, finite rank smooth vector subbundles $\{E_i \mid i \in I\}$ (the $E_i^{\prime}s$ are called the curvature distributions), $\{v_i \mid i \in I\} \subset \mathfrak{a}$, and parallel normal fields $\{\tilde{v}_i \mid i \in I\}$ ($\tilde{v}_i$'s are called curvature normals) such that

(1) $TM = E_0 \oplus \Sigma\{E_i \mid i \in I\}$,
(2) the shape operator $A_{\tilde{b}} \mid E_i = (b, v_i)\ \mathrm{id}_{E_i}$ for $i \in I$, and $A_b \mid E_0 = 0$,
(3) E_0 is integrable, and leaves of E_0 are affine subspaces of V,
(4) each E_i, $i \in I$, is integrable, and the leaf $S_i(x)$ of E_i through $x \in M$ is a standard sphere, called the curvature sphere for E_i,
(5) if $\ell_i = \{\hat{b} \in \hat{\mathfrak{a}} \mid (b, v_i) = 1\}$, then $\{\ell_i \mid i \in I\}$ is the set of reflection hyperplanes of the affine Weyl group $\hat{W}$ for the $P(G,H)$-action, (this determines v_i in terms of $\hat{W}$),
(6) if m_i denotes the rank of E_i, then the m_i's are invariant under $\hat{W}$,
(7) the marked affine Dynkin diagram associated to M is the diagram for $\hat{W}$ with the vertex corresponding to a simple root ℓ_i marked by $m_i = \text{rank } E_i$.

Let τ be a Weyl chamber in $\hat{\mathfrak{a}}$ for the $\hat{W}$-action on $\hat{\mathfrak{a}}$. Then τ determines $\{\ell_i \mid i \in I\}$. By 1.10, τ is isometric to a fundamental domain of the H-action on G. So $\{\ell_i \mid i \in I\}$ and $\{v_i \mid i \in I\}$ can be computed explicitly from the Euclidean geometry of the fundamental domain of H-action on G, which is a k-simplex ($k = \dim(\mathfrak{a})$). Since $E_i(g\cdot\hat{0}) = gE_i(\hat{0})g^{-1}$, to find the relation between E_i and the H-action it suffices to find the relation between $E_i(\hat{0})$ and the H-action.

2.1 Proposition. *With the same notation as above, choose $a_i \in \mathfrak{a}$ such that $\hat{a}_i \in \ell_i$ and $\hat{a}_i$ does not lie in ℓ_j for any $j \in I$, $j \neq i$. Let $\alpha_i = \exp(a_i)$. Then*

(1) $\mathfrak{h}_e \subset \mathfrak{h}_{\alpha_i} = \{(x, \alpha_i^{-1}x\alpha_i) \mid x \in \mathfrak{g}\} \cap \mathfrak{h}$,
(2) for $i \in I$, the map $(x, \alpha_i^{-1}x\alpha_i) \mapsto v(t) = \exp(-a_it)[x, a_i]\exp(a_it)$ gives a well-defined isomorphism from $\mathfrak{h}_{\alpha_i}/\mathfrak{h}_e$ to $E_i(\hat{0})$,
(3) $[\mathfrak{h}_e, \mathfrak{a}] = 0$,

(4) $E_0(\hat{0}) = \{u \in H^0([0,1], \mathfrak{z}_0) \mid \int_0^1 u(t)dt \ \in T(He)_e\}$, *where $\mathfrak{z}_o$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$, i.e.,* $\mathfrak{z}_0 = \{x \in \mathfrak{g} \mid [x, \mathfrak{a}] = 0\}$.

PROOF. Since e is assumed to be a regular point for the H-action, H_e fixes A pointwise. So we obtain (3), and $\mathfrak{h}_e \subset \mathfrak{h}_\alpha$ for any $\alpha \in A$. Note that $(h_1, h_2) \in H_\alpha$ if and only if $h_2 = \alpha^{-1} h_1 \alpha$. This proves (1).

It is known that (cf. [Te2]) $P(G,H)_{\hat{a}_i} \cdot \hat{0}$ is the curvature sphere $S_i(\hat{o})$ of E_i through $\hat{0}$, so $E_i(\hat{0}) = T(P(G,H)_{\hat{a}_i} \cdot \hat{0})_{\hat{o}}$. By (1) and 1.2 (5), we have

$$P(G,H)_{\hat{a}} = \{g(t) = \exp(-at) h_1 \exp(at) \mid (h_1, \exp(-a) h_1 \exp(a)) \in H\},$$

an (2) follows by a straightforward computation.

To prove (4), we recall that

$$\begin{aligned} TM_{\hat{o}} &= \{u' \mid u \in H^1([0,1], \mathfrak{g}),\ (u(0), u(1)) \in \mathfrak{h}\}, \\ &= \{v \in H^o([0,1], \mathfrak{g}) \mid \int_0^1 v(t)dt \ \in T(He)_e\}, \\ A_{\hat{b}}(u') &= [u, b]. \end{aligned}$$

So for $v = u' \in TM_{\hat{o}}$, we have $v \in E_0$ if and only if $[u, b] = 0$ for all $b \in \mathfrak{a}$. This implies that $u(t) \in \mathfrak{z}_0$, which proves (4). ∎

In the following we give more detailed geometric description for the cohomogeneity one actions, and for the σ-actions.

2.2 Cohomogeneity one actions.

Suppose the H-action on G is of cohomogeneity 1, $W(A, H)$ is the dihedral group of order $2n$, and A is a circle of length ℓ. Let $b \in \mathfrak{a}$ be a unit vector such that $\exp(\ell b) = e$. We may assume that $\ell = 1$ and e is a singular point for the H-action. Then

$$\{\beta_j = \exp(jb/2n) \mid 0 \le j < 2n\}$$

is the set of singular points on A, $\{\exp(tb) \mid 0 \le t \le 1/2n\}$ is a fundamental domain for the H-action, and the affine Weyl group $W = W(\hat{\mathfrak{a}}, P(G,H))$ is the semi-direct product of Z_2 with the lattice group $\Lambda_1 = \{j\hat{b}/n \mid j \in Z\}$. For $0 \le t \le 1/2n$, let M_t denote the $P(G,H)$-orbit through the constant path $t\hat{b}$. Let m_1 and m_2 denote the dimension of H_e/H_α and H_{β_1}/H_α respectively, where $\alpha = \exp(tb)$ for some $0 < t < 1/2n$. Then $\{M_t \mid 0 < t < 1/2n\}$ is a family of isoparametric hypersurfaces of V, and the marked Dynkin diagram associated to M_t is $\hat{A}_1$ marked with multiplicity (m_1, m_2), the non-zero principal curvatures of M_t are $\{\lambda_j = ((j/2n) - t)^{-1} \mid j \in Z\}$, and the multiplicity of λ_j is m_1 if j is even, and is m_2 if j is odd. Moreover,

(1) M_0 and $M_{1/2n}$ are smooth submanifolds of V with codimension $1 + m_1$ and $1 + m_2$ respectively, and $M_0 \cup M_{1/2n}$ is the set of focal points of M_t in V for $0 < t < 1/2n$,

(2) given $0 < t < 1/2n$, V can be written as the union of $B_0 \cup B_1$ such that
   (i) B_0 is the normal disk bundle of radius t of M_0 in V,
   (ii) B_1 is the normal disk bundle of radius $1/2n - t$ of $M_{1/2n}$ in V,
   (iii) $B_0 \cap B_1 = \partial B_0 = \partial B_1 = M_t$,
   (iv) let $b_i = \dim(H_i(M_t, Z_2))$, and $p(x) = \sum b_i x^i$ the Poincaré series of M, then

$$\begin{aligned} p(x) &= \frac{(1+x^{m_1})(1+x^{m_2})}{1-x^{m_1+m_2}} \\ &= \sum_{k=0}^{\infty} x^{k(m_1+m_2)}(1+x^{m_1}+x^{m_2}+x^{m_1+m_2}). \end{aligned}$$

2.3 The σ-actions.

Let $H = G(\sigma) = \{(g, \sigma(g)) \mid g \in G\}$ be as in 1.4 (iv), where σ is an outer automorphism of order r on G ($r = 2$ or 3). Let $\mathfrak{K}$ be the fixed point set of σ on $\mathfrak{g}$, and $\mathfrak{p} = \mathfrak{K}^{\perp}$. Note that H_e is the diagonal group of K, and the slice representation at e is the Adjoint representation of K on $\mathfrak{K}$. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{K}$. Then $\hat{\mathfrak{a}}$ is a section of the $P(G,H)$-action.

Since $\{\mathrm{ad}(a) : \mathfrak{g} \to \mathfrak{g} \mid a \in \mathfrak{a}\}$ is a family of commuting skew-adjoint operators, there exist ([H]) positive root systems Δ_0 and Δ_1 (subset of $\mathfrak{a}^*$), $x_\alpha, y_\alpha \in \mathfrak{K}$ for $\alpha \in \Delta_0$, $r_\beta, s_\beta \in \mathfrak{p}$ for $\beta \in \Delta_1$, and a linear subspace $\mathfrak{p}_0$ of $\mathfrak{p}$, which give the following decompositions of $\mathfrak{K}$ and $\mathfrak{p}$:

$$\mathfrak{K} = \mathfrak{a} + \sum_{\alpha\in\Delta_0} Rx_\alpha + Ry_\alpha, \qquad \mathfrak{p} = \mathfrak{p}_0 + \sum_{\beta\in\Delta_1} Rr_\beta + Rs_\beta,$$

such that

$$\begin{aligned} [a, x_\alpha] &= -\alpha(a) y_\alpha, \quad [a, y_\alpha] = \alpha(a) x_\alpha, \quad \forall \alpha \in \Delta_0, \\ [a, r_\beta] &= -\beta(a) s_\beta, \quad [a, s_\alpha] = \beta(a) r_\beta, \quad \forall \beta \in \Delta_1, \\ \sigma(u_\alpha) &= u_\alpha, \quad \sigma(p_\beta) = e^{2\pi i/r} p_\beta, \end{aligned} \tag{$*$}$$

where $u_\alpha = x_\alpha + iy_\alpha$, $p_\beta = r_\beta + is_\beta$, and $r =$ order (σ). Let $M = P(G,H)\cdot \hat{a}$ be a principal orbit for some $a \in \mathfrak{a}$. Then the normal plane of M at $\hat{a}$ is $\hat{\mathfrak{a}}$, and for $\hat{b} \in \mathfrak{a}$, the shape operator of M in the direction of $\hat{b}$ is

$$A_{\hat{b}}([u, \hat{a}] - u') = -[u, \hat{b}].$$

Let $a_1, \ldots, a_m$ be a basis for $\mathfrak{a}$. Then we have:

Case (1) order $(\sigma) = 2$. Let $\{q_j\}$ be a basis for $\mathfrak{p}_0$. Then by (*) and a direct computation, we see that the real and imaginary parts of $u_\alpha e^{2nit}$, $p_\beta e^{(2n+1)it}$, $a_j e^{2nit}$, and $q_j e^{(2n+1)t}$ form an eigenbasis for $A_{\hat{b}}$ with $\alpha(b)/(\alpha(a)+2n)$, $\beta(b)/(\beta(a)+2n+1)$, 0 and 0 as eigenvalues respectively. So the reflection hyperplanes of $W(\hat{\mathfrak{a}}, P(G,H))$ in $\mathfrak{a}$ are the affine hyperplanes defined by $\alpha(t) = 2n$, $\beta(t) = 2n+1$, for $\alpha \in \Delta_0$, $\beta \in \Delta_1$ and $n \in Z$.

Case (2) order(σ) = 3. By a direct computation we obtain that the real and imaginary parts of $u_\alpha e^{3nit}$, $\bar{p}_\beta e^{(3n+1)it}$, $p_\beta e^{(3n+2)it}$, $a_j e^{3nit}$, $\bar{q}_j e^{(3n+1)it}$, and $q_j e^{(3n+2)it}$ form an eigenbasis for $A_{\hat{b}}$ with $\alpha(b)/(\alpha(a)+3n)$, $\beta(b)/(\beta(a)+3n+1)$, $\beta(b)/(\beta(a)+3n+2)$, $0, 0$ and 0 as eigenvalues respectively, where $\{q_j, \bar{q}_j\}$ is a basis of the complexify of $(\mathfrak{p}_0)_C$ such that $\sigma(q_j) = e^{2\pi i/3} q_j$ (hence $\sigma(\bar{q}_j) = e^{-2\pi/3}\bar{q}_j$). So the reflection hyperplanes of $W(\hat{\mathfrak{a}}, P(G,H))$ in $\mathfrak{a}$ are the affine hyperplanes defined by $\alpha(t) = 3n$, $\beta(t) = 3n+1$, $\beta(t) = 3n+2$ for $\alpha \in \Delta_0$, $\beta \in \Delta_1$ and $n \in Z$.

§3. Involutions of Kac-Moody algebras and the $P(G,H)$-actions.

In this section, we review the construction of affine Kac-Moody algebras ([K]) and affine groups ([PS]). We also construct automorphisms on affine groups from given automorphisms on compact Lie groups and discuss their relations with the $P(G,H)$-actions of section 2.

Let G be a compact, simply connected, simple Lie group, and $(\,,\,)$ the normalized bi-invariant inner product on $\mathfrak{g}$, i.e., $(h_\alpha, h_\alpha) = 2$ for the shortest simple coroot of $\mathfrak{g}$. Let $L(\mathfrak{g}) = H^0(S^1, \mathfrak{g})$, t the angle variable in S^1, and let u' denote du/dt. Let $[u,v]_o(t) = [u(t), v(t)]$. Then $L(\mathfrak{g})$ is a Lie algebra. The affine Kac-Moody algebra of type 1 is a two dimensional extension over $L(\mathfrak{g})$:

$$\hat{L}(\mathfrak{g}) = L(\mathfrak{g}) + Rc + Rd$$

with the bracket operation defined by

$$\begin{aligned}
[u,v] &= [u,v]_o + \omega(u,v)c,\\
[d,u] &= u'\\
[c,u] &= [c,d] = 0,
\end{aligned}$$

where ω is the 2-cocyle defined on $L(\mathfrak{g})$ by

$$\omega(u,v) = \frac{1}{2\pi}\int_0^{2\pi} (u(t), v'(t))\, dt.$$

(Note that $[\,,\,]$ is only defined on a dense subspace $H^1(S^1, \mathfrak{g})$ of $L(\mathfrak{g})$; see 3.4 for a discussion of this point.) Let $\langle\,,\,\rangle$ denote the bi-linear form on $\hat{L}(\mathfrak{g})$ defined as follows:

$$\begin{aligned}
\langle u, v\rangle &= \int_0^{2\pi} (u(t), v(t))\, dt,\\
\langle c, d\rangle &= 1, \quad \langle u, c\rangle = \langle c, c\rangle = \langle d, d\rangle = 0.
\end{aligned}$$

Then $\langle,\rangle$ is Ad-invariant, i.e.,

$$\langle [\xi, \eta], \zeta\rangle = \langle \xi, [\eta, \zeta]\rangle, \quad \forall \xi, \eta, \zeta \in \hat{L}(\mathfrak{g}).$$

Next we construct two types of automorphisms on $\hat{L}(\mathfrak{g})$ from automorphisms on G: Given $\sigma : G \to G$, an automorphism of order k, we will use the symbol σ also to denote the induced automorphism on $\mathfrak{g}$. Let

$$\hat{\sigma} : \hat{L}(\mathfrak{g}) \to \hat{L}(\mathfrak{g})$$

be the linear map defined by

$$\hat{\sigma}(u)(t) = \sigma(u(-2\pi/k + t)), \quad \hat{\sigma}(c) = c, \quad \hat{\sigma}(d) = d.$$

Let ρ be an involution on G, and

$$\rho^* : \hat{L}(\mathfrak{g}) \to \hat{L}(\mathfrak{g})$$

the linear map defined by

$$\rho^*(u)(t) = \rho(u(-t)), \quad \rho^*(c) = -c, \quad \rho^*(d) = -d.$$

3.1 Proposition. *Let $\sigma : \mathfrak{g} \to \mathfrak{g}$ be an automorphism of order k on the simple Lie algebra $\mathfrak{g}$, and $(\ ,\ )$ the normalized Ad-invariant form on $\mathfrak{g}$. Then*
(i) $(\sigma(x), \sigma(y)) = (x, y)$,
(ii) $\omega(\hat{\sigma}(u), \hat{\sigma}(v)) = \omega(u, v)$,
(iii) $\omega(\sigma^*(u), \sigma^*(v)) = -\omega(u, v)$, *if σ is an involultion.*

PROOF. Since σ is an automorphism of $\mathfrak{g}$, $\langle x, y \rangle = (\sigma(x), \sigma(y))$ is an Ad-invariant inner product on $\mathfrak{g}$. But $\mathfrak{g}$ is simple implies that $(\sigma(x), \sigma(x)) = c(x, x)$ for some $c > 0$. Since $\sigma^k = \mathrm{id}$, $c^k = 1$. This implies $c = 1$, which proves (i). Using (i), the definition of the cocycle ω, and a direct computation, (ii) and (iii) follows. ∎

As a consequence, we have

3.2 Corollary. *If σ and ρ are automorphisms of $\mathfrak{g}$ of order k and 2 respectively, then $\hat{\sigma}$ and ρ^* are automorphisms of $\hat{L}(\mathfrak{g})$ of order k and 2 respectively.*

The fixed point set $\hat{L}(\mathfrak{g}, \sigma)$ of $\hat{\sigma}$ is called the *affine Kac-Moody algebra of type k* if σ is an order k outer automorphism of $\mathfrak{g}$. It is obvious that

$$\hat{L}(\mathfrak{g}, \sigma) = L(\mathfrak{g}, \sigma) + Rc + Rd,$$

where

$$L(\mathfrak{g}, \sigma) = \{u \in L(\mathfrak{g}) \mid u(t) = \sigma(u(-2\pi/k + t))\}.$$

It can also be easily seen that

$$\hat{L}(\mathfrak{g}) = \hat{\mathfrak{k}} + \hat{\mathfrak{p}},$$

is the decomposition into the 1 and -1 eigenspaces of ρ^*, where

$$\hat{\mathfrak{k}} = \{u \in L(\mathfrak{g}) \mid u(-t) = \rho(u(t))\},$$
$$\hat{\mathfrak{p}} = \{u \in L(\mathfrak{g}) \mid u(-t) = -\rho(u(t))\} + Rc + Rd.$$

3.3 Construction of $\hat{L}(G)$.

We review briefly the construction of $\hat{L}(G)$ as given in Chapter 4 of [PS]. The first step is to construct the central extension of the loop group $L(G) = H^1(S^1, G)$ given by the 2-cocycle ω. The left invariant 2-form on $L(G)$ determined by the 2-cocycle ω will also be denoted by ω. Since the inner product on $\mathfrak{g}$ is normalized, the 2-form $\frac{1}{2\pi i}\omega$ is an integral cohomology class of $L(G)$, so there exists a principal S^1-bundle, $\varphi : P \to L(G)$ with a connection 1-form β such that the curvature of β is $\varphi^*\omega$, (i.e., the Chern class of this principal bundle is $[\omega/2\pi i]$). Then the group $\tilde{L}(G)$ of bundle isomorphisms of P that preserve the connection β and cover a left translation ℓ_g for some $g \in L(G)$ is the central extension by S^1 given by the cocyle ω.

We fix a base point $y_0 \in \varphi^{-1}(e)$. For a curve $\gamma : [0,1] \to L(G)$, let

$$\Pi_\gamma : \varphi^{-1}(\gamma(0)) \to \varphi^{-1}(\gamma(1))$$

denote the parallel translation along γ given by the connection β. Suppose $F \in \tilde{L}(G)$ and F covers the left translation ℓ_g on $L(G)$. Since $F^*(\beta) = \beta$,

$$F \circ \Pi_\gamma = \Pi_{g\gamma} \circ F.$$

So F is uniquely determined by $g \in L(G)$ and $F(y_0)$, i.e., $\tilde{L}(G)$ is the set of pairs (g, y), where $g \in L(G)$ and $y \in \varphi^{-1}(g)$.

There is also an explicit construction of $\tilde{L}(G)$ given in [PS] (p. 47), which we will use to lift automorphisms $\hat{\sigma}$ and ρ^* to $\hat{L}(G)$: Let (g, p, z) be a triple with $g \in L(G)$, p a path in $L(G)$ joining e to g and $z \in S^1$. Then F can be constructed by using parallel translation. For $y \in \varphi^{-1}(h)$, we choose some curve γ in $L(G)$ joining h to e, let

$$F(y) = \Pi_{g\gamma}^{-1} z(\Pi_p(\Pi_\gamma(y))).$$

This is well-defined and in $\tilde{L}(G)$. For if γ_1 and γ_2 are curves joining x to e, then since $L(G)$ is simply connected, $\gamma_1^{-1} * \gamma_2$ (the loop obtained by the inverse of γ_1 followed by γ_2) bounds a surface S in $L(G)$. But S^1 is abelian, so "curvature is infinitesimal holonomy", i.e., we have $C(\gamma_1^{-1} * \gamma_2)\Pi_{\gamma_1}(y) = \Pi_{\gamma_2}(y)$ for all $y \in \varphi^{-1}(x)$, where

$$C(\gamma_1^{-1} * \gamma_2) = \exp\,(i \int_S \omega).$$

This also proves that (g_1, p_1, z_1) and (g_2, p_2, z_2) gives the same F if and only if

$$g_1 = g_2, \quad z_1 = C(p_2 * p_1^{-1})z_2;$$

and we call two such triples equivalent. Define multiplication of triples by

$$(g_1, p_1, z_1) \cdot (g_2, p_2, z_2) = (g_1 g_2,\ p_1 * (g_1 \cdot p_2),\ z_1 z_2).$$

Then the group of equivalence classes of these triples is the Hilbert Lie group $\tilde{L}(G)$, whose Lie algebra is the central extension $\tilde{L}(\mathfrak{g}) = L(\mathfrak{g}) + Rc$ of $L(\mathfrak{g})$ by ω.

It is easily seen that $e^{is} \cdot (g, p, z) = (e^{is} \cdot g, e^{is} \cdot p, z)$ defines a continuous action of S^1 on $\tilde{L}(G)$, where $(e^{is} \cdot g)(t) = g(s+t)$ and $e^{is} \cdot p$ denotes the path $(e^{is} \cdot p)(r) = e^{is} \cdot p(r)$. Then the semi-direct product $\hat{L}(G) = S^1 \ltimes \tilde{L}(G)$ given by this S^1-action is a topological group, which is the group model for the affine algebra $\hat{L}(\mathfrak{g})$.

3.4 Remark. Although $\hat{L}(G)$ is a Hilbert manifold, and a topological group, (i.e., the group multiplication is continuous), it is *not* a Hilbert Lie group, because the map

$$H^1(S^1, G) \times S^1 \to H^1(S^1, G), \quad (f, g) \mapsto f \circ g$$

while smooth in the second variable is only continuous in the first variable ([P2], [PS]). As a result, the Lie bracket of the corresponding "Lie algebra" $\hat{L}(\mathfrak{g})$ is only defined on a dense subspace. It should also be noted that $\hat{L}(G)$ is a Hilbert manifold locally modeled on the product of the H^1-space and R^2, but the Ad-invariant metric is the product of the H^0 metric on the H^1-space and the Lorentz metric on R^2, so $\hat{L}(G)$ is only a Lorentz manifold in the weak sense. Our goal in reviewing this construction of affine groups is to suggest a good infinite dimensional analogue of s-representations and their relation to polar actions on Hilbert spaces. As we will now see, while the above problems make it difficult to give a rigorous definition of infinite dimensional symmetric spaces, they do not interfere with the construction of the desired polar actions.

3.5 The Adjoint action of $\hat{L}(G)$.

Let

$$\pi : \hat{L}(G) \to L(G), \quad (e^{is}, (g, p, z)) \mapsto g$$

denote the natural projection. Then the Adjoint action of $\hat{L}(G)$ on its Lie algebra $\hat{L}(\mathfrak{g})$ ([K], [PS]) is given by

$$\begin{aligned} \mathrm{Ad}(\hat{g})(u) &= gug^{-1} + \langle gug^{-1}, g'g^{-1}\rangle\, c, \\ \mathrm{Ad}(\hat{g})(d) &= -g'g^{-1} + d - \frac{1}{2}\langle g'g^{-1}, g'g^{-1}\rangle\, c, \end{aligned}$$

if $\hat{g} = (1, (g, p, z))$. Note that the intersection R^∞ of the sphere of radius -1 with the hyperplane $\{u + rc + sd \in \hat{L}(\mathfrak{g}) \mid s = 1\}$, i.e.,

$$R^\infty = \{u + rc + d \mid u \in L(\mathfrak{g}),\ r = -(1 + \tfrac{1}{2}\langle u, u\rangle)\},$$

is a horosphere of the infinite dimensional hyperbolic space. Hence R^∞ is invariant under the Adjoint action of $\hat{L}(G)$ on $\hat{L}(\mathfrak{g})$, and R^∞ is isometric to the Hilbert space $L(\mathfrak{g}) = H^0(S^1, \mathfrak{g})$ via $u + rc + d \mapsto u$. Moreover, the action on R^∞ factors through $L(G)$, and the corresponding action on $L(\mathfrak{g})$ is equivalent to the action of $L(G) \simeq P(G, \Delta(G))$ on $L(\mathfrak{g}) \simeq H^0([0,1], \mathfrak{g})$ by gauge group transformations.

3.6 The automorphism $\hat{\sigma}$ on $\hat{L}(G)$.

Let σ be an automorphism of order k on G. Then

$$\hat{\sigma} : L(G) \to L(G), \quad \hat{\sigma}(g)(t) = \sigma(g(-2\pi/k + t))$$

is an automorphism on $L(G)$. It follows from the construction of $\hat{L}(G)$ in 3.3 that

$$(e^{is}, (g, p, z)) \mapsto (e^{is}, (\hat{\sigma}(g), \hat{\sigma}(p), z))$$

is a well-defined automorphism on $\hat{L}(G)$, which will still be denoted by $\hat{\sigma}$. Then the fixed point set $\hat{L}(G, \sigma)$ of $\hat{\sigma}$ is the affine group of type k, and $\hat{L}(\mathfrak{g}, \sigma)$ is its Lie algebra. Again

$$R_\sigma^\infty = \{u + rc + d \mid u \in L(\mathfrak{g}, \sigma),\, r = -(1 + \tfrac{1}{2}(u, u))\}$$

is invariant under the Adjoint action of $\hat{L}(G, \sigma)$, and R_σ^∞ is isometric to $L(\mathfrak{g}, \sigma)$ via the map $u + rc + d \mapsto u$. Moreover, the action on R_σ^∞ factors through $L(G, \sigma)$, and the corresponding action on $L(\mathfrak{g}, \sigma)$ is equivalent to the action of $L(G, \sigma) \simeq P(G, G(\sigma))$ on $L(\mathfrak{g}, \sigma) \simeq H^0([0, 1], \mathfrak{g})$ by gauge group transformations, where $G(\sigma) = \{(x, \sigma(x)) \mid x \in G\}$.

3.7 Involution ρ^* on $\hat{L}(G)$.

Let ρ be an involution on G, and K the fixed point set of ρ. Then

$$\rho^* : L(G) \to L(G), \quad \rho^*(g)(t) = \rho(g(-t))$$

is an involution on $L(G)$. Note that $\rho^*(e^{is} \cdot g) = e^{-is} \cdot \rho^*(g)$. Using this equality and the construction of $\hat{L}(G)$ in 3.3, it follows that

$$(e^{is}, (g, p, z)) \mapsto (e^{-is}, (\rho^*(g), \rho^*(p), z^{-1}))$$

is a well-defined involution on $\hat{L}(G)$, which will still be denoted by ρ^*. Then the fixed point set of ρ^* on $\hat{L}(G)$ is

$$\hat{L}(G)_\rho = \{(1, (g, p, 1)) \mid \rho^*(g) = g,\ \rho^*(p(r)) = p(r)\},$$

which is isomorphic to

$$\{g \in L(G) \mid \rho(g(-t)) = g(t)\} \simeq P(G, K \times K).$$

Let $\hat{L}(\mathfrak{g}) = \hat{\mathfrak{k}} + \hat{\mathfrak{p}}$ denote the $1, -1$ eigendecomposition of ρ^* on $\hat{L}(\mathfrak{g})$ as before. Then $\hat{\mathfrak{k}}$ is the Lie algebra of $\hat{L}(G)_\rho$, and the Adjoint action of $\hat{L}(G)_\rho$ leaves $\hat{\mathfrak{p}}$ invariant, and induces on $\hat{\mathfrak{p}}$ the isotropy representation of the "symmetric space" $\hat{L}(G)/\hat{L}(G)_\rho$ at the identity coset. This isotropy action also leaves $R_\rho^\infty = R^\infty \cap \hat{\mathfrak{p}}$ invariant. Moreover, R_ρ^∞ is isometric to $\{u \in L(\mathfrak{g}) \mid \rho(u(-t)) = -u(t)\}$ via the map $u + rc + d \mapsto u$, and the the action of $\hat{L}(G)_\rho$ on R_ρ^∞ is equivalent to the $P(G, K \times K)$-action on $H^0([0, 1], \mathfrak{g})$.

3.8 Natural embedding of $\hat{L}(G)/\hat{L}(G)_\rho$ into $\hat{L}(G)$.

Let ρ be an involution on G, and K the fixed point set of ρ. Then the finite dimensional symmetric space G/K can be naturally embedded in G as the orbit N through e of the action of $G(\rho) = \{(x, \rho(x)) \mid x \in G\}$ on G, i.e., $N = \{x\rho(x)^{-1} \mid x \in G\}$. Similarly in infinite dimension, the involution ρ^* induces an action of $\hat{L}(G)$ on $\hat{L}(G)$ via $h * y = hy\rho^*(h)^{-1}$, and the orbit $\tilde{M}$ of this action through the identity gives a natural embedding of $\hat{L}(G)/\hat{L}(G)_\rho$ into $\hat{L}(G)$. We also note that $e \in \tilde{M}$, the restriction τ of the isometry $x \mapsto x^{-1}$ on $\hat{L}(G)$ to $\tilde{M}$ induces an isometry of $\tilde{M}$, $\tau(e) = e$, and $d\tau_e$ is $-$id on $T\tilde{M}_e$. So if $p = g * e \in \tilde{M}$, then $\tau_p = g\tau g^{-1}$ has the property that $\tau_p(p) = p$ and $d(\tau_p)_p$ is $-$id on $T\tilde{M}_p$. This is the characteristic property of a globally symmetric space. To describe $\tilde{M}$ explicitly, we compute directly and see that

$$\tilde{M} = \{(e^{is}, (g, p, z)) \mid g(t) = h(t)\rho(h(-s-t))^{-1} \text{ for some } h \in L(G)\}.$$

Note that for $g(t) = h(t)\rho(h(-s-t))^{-1}$ we have $g = e^{-is/2}\cdot((e^{is/2}\cdot h)\rho^*(e^{is/2}\cdot h)^{-1})$. So the loop part $\pi(\tilde{M}) = S^1 \cdot M$, where

$$M = \left\{h\rho^*(h)^{-1} \mid h \in L(G)\right\} \subset L(G).$$

Next we claim that

$$M = \{g \in L(G) \mid g(t)^{-1} = \rho(g(-t)),\ g(0), g(\pi) \in N\} \simeq \Omega(G, N, N),$$

where $\Omega(G, N, N)$ denote the Hilbert manifold of H^1-paths in G with end points in N. It is easy to see that M is contained in the right hand side. Now suppose given $g \in L(G)$ such that $g(t)^{-1} = \rho(g(-t))$ and $g(0), g(\pi) \in N$. Then there exist $x, y \in G$ such that $x\rho(x)^{-1} = g(0)$ and $y\rho(y)^{-1} = g(\pi)$. Let $r : [0, \pi] \to G$ be any H^1-map such that $r(0) = x, r(\pi) = y$, and define

$$h(t) = \begin{cases} r(t), & \text{if } t \in [0, \pi]; \\ g(t)\rho(r(2\pi - t)), & \text{if } t \in [\pi, 2\pi]. \end{cases}$$

Then $h \in L(G)$ and $h\rho^*(h)^{-1} = g$, i.e., $g \in M$.

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