SOME EXAMPLES OF THREEFOLDS WITH TRIVIAL CANONICAL BUNDLE

Lectures: F. Hirzebruch

Notes: J. Werner

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
D - 5300 Bonn 3

Sonderforschungsbereich 40
Theoretische Mathematik
Beringstraße 4
D - 5300 Bonn 1

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Some examples of threefolds with trivial canonical bundle

Ausarbeitung und Ergänzung von Vorträgen von
F. Hirzebruch durch J. Werner.

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I.

Let X be a compact Kähler manifold of complex dimension n with $c_1^{\mathbf{R}} = 0$. Then we can change the Kähler metric in the same cohomology class to get a Ricci flat metric: the Ricci tensor represents the first Chern class, and because the first Chern class of X is cohomologically zero, Yau's solution of the Calabi conjecture gives us the possibility of replacing the given metric by an Einstein-Kähler metric which is Ricci flat.

Which manifolds of this type exist?

The structure theorem, as given in Beauville's paper [1](see also [11]) which is based on results of Calabi [5] and Bogomolov [2], tells us that the universal covering of such an X is always isomorphic to a product of the form

$$C^k \times \Pi V_i \times \Pi X_j \quad .$$

The V_i's are simply connected Kähler manifolds of dimension greater than or equal to three with trivial canonical class. The V_i's have the following Hodge numbers:

$$h^{0,0} = 1, h^{0,1} = 0, h^{0,2} = 0, \ldots, h^{0,n-1} = 0, h^{0,n} = 1 \quad .$$

The X_j's are simply connected holomorphic symplectic even dimensional Kähler manifolds with trivial canonical bundle (also called hyperkählerian), this implies that there exists a holomorphic two-form φ , such that $\varphi, \varphi^2, \ldots, \varphi^{k/2}$ give all holomorphic forms up to factors, where k is the dimension of X_j and $\varphi^{k/2}$ is the non-vanishing section of the canonical bundle. The described decomposition of the universal covering of X is unique. The proof of this structure theorem uses the Einstein-Kähler metric [17]. Actually, the theorem says a little more:

There exists some <u>finite</u> unramified cover X^f of X ,

$$X^f = T^k \times \Pi V_i \times \Pi X_j \quad , \text{ with a Torus } T^k \quad .$$

What are the consequences of this fact in dimension three?

Suppose that we have a Kähler manifold X of dimension three with $c_1^{\mathbf{R}} = 0$ and $e(X) \neq 0$. Then $e(X^f) \neq 0$, so no torus can occur in the decomposition of X^f as above. Because there are no factors X_j either (they are evendimensional), there exists some finite cover X^f of X of the form $X^f = V$, where V is of the type described above.
Therefore X^f and also X have the following invariants:

$$h^{0,0} = 1 = h^{0,3} \; , \; h^{0,1} = 0 = h^{0,2} \quad .$$

By duality, the only variable Hodge numbers are $h^{1,1}$ and $h^{2,1}$. From general theory on Hodge numbers the Euler number $e(X)$ of X is $2h^{1,1} - 2h^{2,1}$.

The most obvious examples are complete intersections of k smooth hypersurfaces in $\mathbf{P}^{3+k}(\mathbf{C})$ in general position:

a) A Quintic in $\mathbf{P}^4$ with Euler number -200 .

b) The intersection of a Quartic and a Quadric resp. two Cubics in $\mathbf{P}^5$ with Euler number -176 resp. -144 .

c) The intersection of a Cubic and two Quadrics in $\mathbf{P}^6$ with Euler number -144.

d) The intersection of four Quadrics in $\mathbf{P}^7$; its Euler number is -128 .

These are the only examples we can get by taking smooth complete intersections, because the first Chern class of a smooth intersection of k hypersurfaces of degree $d_1,\ldots,d_k$

in $\mathbf{P}^{3+k}(\mathbf{C})$ vanishes if and only if

$$\sum_{i=1}^{k} d_i = k + 4 \quad .$$

For these examples see [6], p.11.

Other important examples are double (resp. triple) coverings of $\mathbf{P}^3(\mathbf{C})$, branched along smooth octic (resp. sextic) surfaces; the Euler numbers of these threefolds are equal to -296 (resp. -204).

Physicists studying superstring theory are interested in 3 dimensional Kähler manifolds which have $c_1^{\mathbf{R}} = 0$ and absolute value of the Euler number as small as possible, but not equal to zero. In order to get such manifolds, they take the examples given above, look for some groups acting freely on the manifolds, and divide by these group-actions.

The Euler number of the quotient is then the Euler number of the given manifold divided by the order of the group.

For example, the group $\mathbf{Z}_5 \times \mathbf{Z}_5$ acts freely on the Quintic in $\mathbf{P}^4(\mathbf{C})$ given by

$$\sum_{i=0}^{4} z_i^5 = 0 \quad ;$$

the quotient is a manifold with Euler number -8 ; see [6], p.16.

Another method for getting new examples with different Euler numbers will be described in this paper: we introduce singularities and then resolve them in different ways.

II.

Let us consider the double covering of $\mathbf{P}^3(\mathbf{C})$, branched along an octic surface which is allowed to have singularities. Assume first, that these singularities are locally of type $g(u,v,z) = 0$, where g is a homogenous polynomial defining a smooth curve of degree 4 in the plane at infinity with u,v,z as homogenous coordinates. Therefore the surface singularity is the vertex of the cone over the curve g . The threefold singularity is given in local affine coordinates by

$$w^2 + g(u,v,z) = 0 \quad .$$

Blowing up the singular point of the branch divisor in $\mathbf{P}^3$, the exceptional divisor D is isomorphic to $\mathbf{P}^2$. The proper transform $\tilde{B}$ of the branching surface B cuts out a curve of degree 4 of this exceptional divisor. A singular point p of the threefold is then resolved into a double cover of $\mathbf{P}^2$, branched along the curve of degree 4. This is a del-Pezzo-surface, which is isomorphic to $\mathbf{P}^2$ blown up in seven points. So the Euler number of the surface, which replaces the singular point of the threefold, is 10.

The Milnor number of a hypersurface singularity of type

$$X_1^{n_1} + X_2^{n_2} + X_3^{n_3} + X_4^{n_4} = 0$$

is $\prod_{i=1}^{4}(n_i-1)$. Thus the third Betti number of the Milnor fiber of the given singularity $g(u,v,z)=0$ is 27, at least in the case $g = u^4 + v^4 + z^4$, but this holds also for any g . When the singular point is taken out of the threefold, the Euler number changes in the same manner as if the Milnor fiber is taken out of a smooth model. So in our example the Euler number descreases by 1-27, that means: it increases by 26. Gluing in the del Pezzo-surface enlarges the Euler number again by 10. So with every singularity of the described type the Euler number of $\tilde{X}$, which is the double covering of the blown up $\tilde{\mathbf{P}}^3$, branched along $\tilde{B}$, increases by 36.

What is the canonical class of $\tilde{X}$?

Let ρ be the blowing up and π the covering map from $\tilde{X}$ to $\tilde{\mathbf{P}}^3$. Then we have the following formulas:

$$K_{\tilde{\mathbf{P}}^3} = -4\rho^* H + 2D$$

$$K_{\tilde{X}} = \pi^*\left(K_{\tilde{\mathbf{P}}^3} + \frac{1}{2}\tilde{B}\right)$$

$$\tilde{B} = \rho^* B - \operatorname{ord}_p B \cdot D \quad ;$$

H is a generic hyperplane in $\mathbf{P}^3$. Because $B \sim 8H$ and $\operatorname{ord}_p B = 4$, $K_{\tilde{X}}$ is trivial.

Now let us construct an octic surface with 8 singularities of type (4,4,4). The Euler number of $\tilde{X}$ is then equal to $-296 + 8 \cdot 36 = -8$. In homogenous coordinates $X_0, \ldots, X_3$ consider the quartic

$$X_1^4 + X_2^4 + X_3^4 = 0$$

with one singularity. Now change the coordinates to $T_0, \ldots, T_3$ in the following way:

$$X_1 - X_0 = T_1^2$$
$$X_2 - X_0 = T_2^2$$
$$X_3 - X_0 = T_3^2$$
$$X_0 = T_0^2 \quad .$$

The transformation map is of degree 8, the inverse image of the quartic $\sum_{i=1}^{3} X_i^4 = 0$ is an octic with 8 singularities of the type we require.

A similar example is given by the triple covering of $\mathbf{P}^3(\mathbf{C})$, branched along a sextic surface with singularities locally of type $g(u,v,z) = 0$, where g is a homogeneous polynomial defining a smooth curve of degree 3. For example, the threefold singularities are locally given by

$$w^3 + u^3 + v^3 + z^3 = 0 \quad .$$

Blowing up, these points are replaced by cubic del Pezzo-surfaces: they are triple coverings of $\mathbf{P}^2$, branched along a cubic curve, and have Euler number 9. Because the Milnor number of such singularities is 16, every resolution of a singularity of this type increases the Euler number, compared with the smooth case, by 24. Analogous to the octic case, the canonical bundle again remains trivial, because $\mathrm{ord}_p B = 3$ for all singular points p and

$$K_{\tilde{X}} = \pi^*\left(K_{\tilde{P}^3} + \frac{2}{3}\tilde{B}\right) \quad .$$

Constructing a sextic surface with 8 such singularities, we reach the Euler number $-204 + 8 \cdot 24 = -12$. Here we proceed as before: the sextic (in affine coordinates)

$$\sum_{i=1}^{3} (x_i^2 - 1)^3 = 0$$

has 8 singularities of type (3,3,3) .
The group, generated by

$$(x_1, x_2, x_3, w) \longmapsto (x_2, x_3, x_1, w\xi_3)$$

with ξ_3 a primitive third root of unity, operates on the singular threefold

$$-\sum_{i=1}^{3} (x_i^2 - 1)^3 + w^3 = 0 \quad .$$

The only fixpoints are the singular points $P_1 = (1,1,1,0)$ and $P_2 = (-1,-1,-1,0)$. The group operation induces an operation on the resolution, which is a fixpoint-free automorphism of the cubic surfaces resolving P_1 and P_2 . The quotient is a smooth Kähler manifold with trivial canonical bundle and Euler number $-12/3 = -4$.

III.

Let us return to the double covering of $\mathbf{P}^3(\mathbf{C})$, branched along an octic surface and assume that the singularities of the branching surface now are ordinary nodes, described locally by the equation

$$\sum_{i=1}^{3} z_i^2 = 0 \quad .$$

Blowing up a singular point of the octic in P^3, the proper transform of this surface cuts out an irreducible conic curve of the exceptional $\mathbf{P}^2$. So the singularity of the threefold is resolved into a double covering of $\mathbf{P}^2$, branched along an irreducible conic curve. This surface is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. But now the canonical bundle of the new smooth threefold is no longer trivial. So we must look for another way of resolving the singularities.

We proceed by taking the so called "small resolution", which can be described as follows:

In suitable local analytic coordinates, a threefold singularity of the type we want to have is given by the equation

$$\sum_{i=1}^{4} u_i^2 = 0 \quad .$$

In other coordinates this can be written as

$$\phi_1 \phi_2 = \phi_3 \phi_4 \quad .$$

The local meromorphic function

$$\frac{\phi_1}{\phi_3} = \frac{\phi_4}{\phi_2}$$

has a point of indeterminancy at the critical point $\phi_1 = \phi_2 = \phi_3 = \phi_4 = 0$. The graph of this meromorphic function is smooth and contains a $\mathbf{P}^1$ at that critical point; the singularity is replaced by a set of codimension two. Therefore this kind of resolution does not influence the canonical class. For details see [10]. The meromorphic function

$$\frac{\phi_1}{\phi_4} = \frac{\phi_3}{\phi_2} .$$

gives us a different small resolution. So globally there exist 2^s different small resolutions (s=# of double points). Another way to get these resolutions is to blow down each $\mathbf{P}^1$ of one of the rulings of the exceptional $\mathbf{P}^1 \times \mathbf{P}^1$ we constructed before. This can be done, since both fibre and base of the $\mathbf{P}^1 \times \mathbf{P}^1$ have intersection (-1) with the $\mathbf{P}^1 \times \mathbf{P}^1$.

Computing the Milnor number we see that the Euler number increases by 2 with every small resolution of a singularity. Now a different problem comes into the game:

It is uncertain whether or not at least some of the small resolutions $\hat{X}$ are still Kähler. This depends on the number of double points and their special position. In general the manifolds $\hat{X}$ are only Moišezon : the transcendence degree of the function field is 3.

The results of Moišezon [12] tell us that a manifold is projective algebraic if and only if it is Moišezon and Kähler. So in our examples the properties "Kähler" and "projective algebraic" are equivalent.

If we take, for example, the Čmutov octic with 108 nodes as branching surface - this octic is given in affine coordinates by the equation

$$\sum_{i=1}^{3} T_8(x_i) - 1 = 0 \quad ,$$

T_8 the Čebyšev polynomial of degree 8 - there is no small resolution which is Kähler; all exceptional curves are homologous to zero. But if we take the Čmutov octic with 144 nodes, given by

$$\sum_{i=1}^{3} T_8(x_i) + 1 = 0 \quad ,$$

there exist some $\hat{X}$ which are projective algebraic; see [16]. These are Kähler manifolds with trivial canonical bundle and Euler number $-296 + 2 \cdot 144 = -8$.
For the general construction of Čmutov hypersurfaces see [15] or the appendix of this paper.

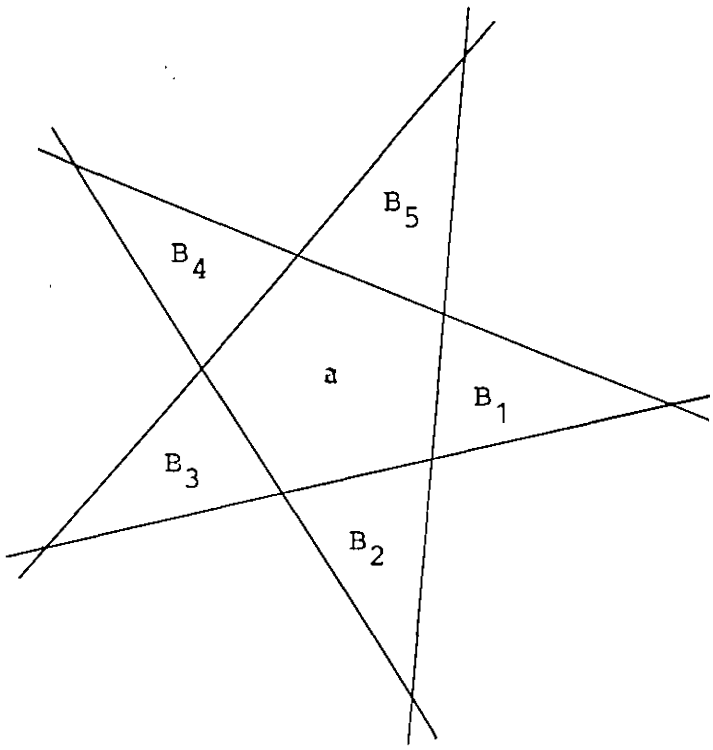
IV.

A similar example is given by the quintic Čmutov hypersurface in $\mathbf{P}^4(\mathbf{C})$:

$$\sum_{i=1}^{4} T_5(x_i) = 0 \quad .$$

This threefold has 96 nodes, and because the Euler number of a smooth quintic in P^4 is -200, the small resolutions of that singular variety have Euler number -8. Again there exist some projective algebraic small resolutions.

Now let $f(x,y) = 0$ be the quintic curve in the affine plane, which is given by the product of the five lines of a regular pentagon.



As a function of two real variables x and y, f has relative extrema in the center a of the pentagon and in one point b_i of each triangle B_i. So both partial derivatives of f vanish at these six points and at the ten intersection points of the five lines. The function f can be normalized, such that $f(b_i) = -1$ for all $i = 1,\dots,5$; by symmetry $f(b_i) = f(b_j)$ for all i and j. Consider the threefold given in four affine coordinates by the equation

$$f(u,v) - f(z,w) = 0 \quad .$$

At the singular points all partial derivatives vanish, so if (u,v,z,w) is a singular point of the threefold, (u,v) and (z,w) are critical points of f. There are three possibilities:

$$\begin{array}{lll} f(u,v) = 0 = f(z,w) & (100 \text{ points}) & \\ f(u,v) = -1 = f(z,w) & (\ 25 \text{ points}) & \\ f(u,v) = a = f(z,w) & (\ \ 1 \text{ point}\ ), & a > 0 \quad . \end{array}$$

So our threefold has 126 nodes, the Euler number of a small resolution is $-200 + 2 \cdot 126 = +52$. In this case there exist projective algebraic small resolutions, as Chad Schoen pointed out. The hypersurface in $\mathbb{P}^4(\mathbb{C})$

$$f(u,v) - 2f(z,w) = 0$$

has only 100 singularities coming from the intersection points

of the lines. Here the small resolutions have Euler number 0, so this example is not interesting for the physicists.
We can get other examples by

$$f(u,v) - \tilde{f}(z,w) = 0 \quad ,$$

where $\tilde{f}$ arises from a little perturbation of the line configuration of f. A suitable choice of $\tilde{f}$ gives us a threefold with 101 nodes: $\tilde{f}$ must have the value $+a$ at the critical point inside the pentagon and values different from -1 at all critical points inside the triangles. The Euler number of a small resolution is $+2$. It might be, that in this special case there is no projective algebraic small resolution.

𝔳.

In our next example a family of threefolds is described by the affine equation

$$f(x,y,z) + t^{m\cdot h} = 0 \quad (m \in \mathbb{N}) \quad ,$$

where f has some surface singularities of type
$A_r (r \geq 1)$, $D_r (r \geq 4)$, E_6 , E_7 and E_8 .
These singularities are given by the local equation

$$z^2 + g(x,y) = 0 \quad ,$$

where g has a curve singularity of type a_r, d_r, e_6, e_7, or e_8. These are

$$a_r : x^2 + y^{r+1} = 0$$
$$d_r : x(y^2 + x^{r-2}) = 0$$
$$e_6 : x^3 + y^4 = 0$$
$$e_7 : x(x^2 + y^3) = 0$$
$$e_8 : x^3 + y^5 = 0 .$$

In [3] and [4] Brieskorn investigates threefolds which fiber into surfaces and looks for resolutions of the surface singularities of singular fibers. Because the parameter t gives us a fibration of our threefold into a family of surfaces, the results of Brieskorn give us small resolutions of our threefold singularities in those cases, where h is the so called Coxeter number of the singularity. That means:
If the exponent of t in the equation of the threefold is a multiple of the Coxeter number of a singularity occuring in f, then there exist small resolutions of this singularity.
The Coxeter numbers are

$$h(A_r) = r + 1$$
$$h(D_r) = 2r - 2$$
$$h(E_6) = 12$$
$$h(E_7) = 18$$
$$h(E_8) = 30 .$$

The Milnor number of every singularity is a product of the Milnor number of the surface singularity of f - which is the index r - and $(m \cdot h - 1)$, because the variable t does not appear in the polynomial f. So every singularity enlarges the Euler number by

$$r \cdot (m \cdot h - 1) - 1 + (r + 1) = r \cdot m \cdot h \quad .$$

The Dynkin diagram of the singularity shows, that the resolving curve is a line configuration consisting of r projective lines with $(r-1)$ intersections points; its Euler number is equal to

$$r \cdot 2 - (r - 1) = r + 1 \quad .$$

Now let us have a look at a more special case of this general example:

$$f(x,y) + z^2 + t^8 = 0$$

with $f = 0$ a curve of degree 8.
If f is smooth, the threefold is smooth, too, and has trivial canonical bundle and Euler number -296.
Now f is allowed to have singularities of type a_1, a_3, a_7, and d_5. In these cases we get small resolutions of the singularities of the threefold, because the Coxeter numbers of

A_1, A_3, A_7 , and D_5 are divisors of 8 . The Euler number of a small resolution is

$$e = -296 + 8 \cdot \Sigma r \quad ;$$

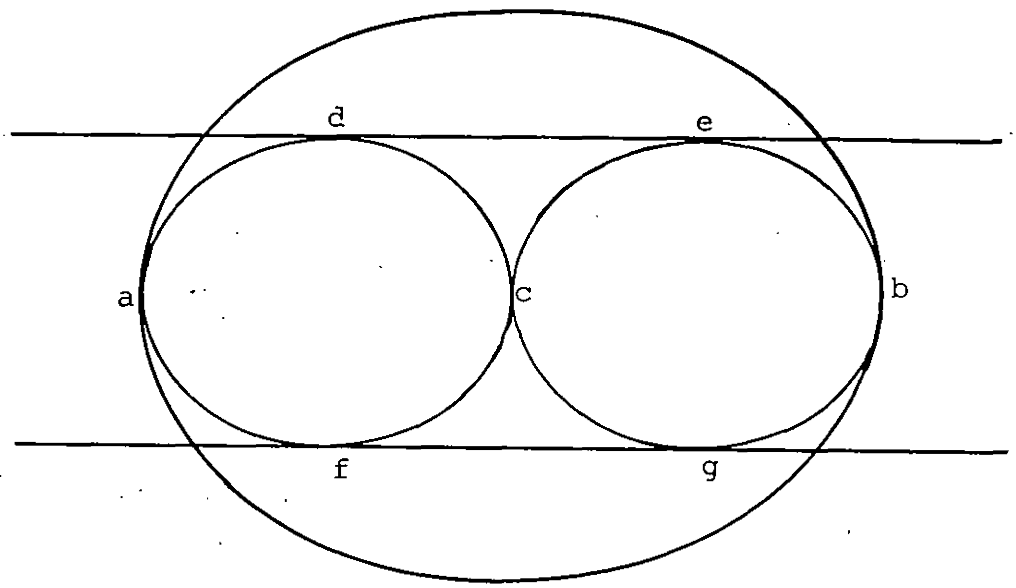
Σr is the sum of the indices of all singularities of f . This sum is smaller than or equal to 37 ; see [13], p.291. We give two examples with

$$\Sigma r = 36 \quad .$$

In this case we reach

$$e = -8 \quad .$$

First f consists of four conics which touch each other such that f has 12 singularities of type a_3 . In the other example f is a product of Persson's tri-conical configuration ([13] p.292) and two lines as in the following picture:



At the points a and b we have a_7-singularities; those at the points c to g are of type a_3 . Finally we get 7 a_1-singularities: the four intersection points of the two lines and the big conic, then two imaginary points, where the small conics intersect, and the intersection point of the two lines. Again the sum of the indices is 36 .

VI.

In this chapter we want to construct some threefolds with large positive Euler number. First let us have a look at the intersection of four quadrics in $\mathbf{P}^7$ with Euler number -128 . To construct an example of such a threefold, we take 8 planes

$$\ell_1 = 0, \ldots, \ell_8 = 0$$

in $\mathbf{P}^3(\mathbf{C})$ in general position. The functions

$$\sqrt[2]{\ell_i/\ell_1} \qquad (i = 2,\dots,8)$$

give us a smooth threefold in $\mathbf{P}^7$, a branched covering of $\mathbf{P}^3$ of degree 2^7. The quadrics in $\mathbf{P}^7$ are given by the equations

$$\sum_{i=1}^{8} \alpha_{ij} t_i^2 = 0 \quad (j = 1,\dots,4) \quad ,$$

the four 8-tuples $(\alpha_{1j},\dots,\alpha_{8j})$ are linear independent solutions of

$$\sum_{i=1}^{8} \alpha_{ij} \ell_i = 0 \quad .$$

Now we want to introduce singularities; the eight planes are chosen in special position. We allow, that through any point may pass up to four of the planes; however assume that no three of the planes have a line in common. If we have a plane configuration with t points belonging to four planes, the threefold has $8 \cdot t$ double points.

Take, for example, the plane configuration of a regular octahedron. Each of the six points of the octahedron belongs to four planes. The eight planes divide into four pairs which are parallel. To each pair belongs an intersection line in the plane of infinity. These four lines intersect in six points. So t is equal to 12, the threefold has 96 singularities. A small resolution gives us a manifold with trivial canonical bundle and Euler number

$$-128 + 2 \cdot 96 = +64 \quad .$$

Again it is an open problem whether the small resolutions are projective algebraic.

Next we consider the intersection of two cubics in $\mathbf{P}^5(\mathbf{C})$ with Euler number -144 . We construct such a smooth threefold by functions

$$\sqrt[3]{\ell_i/\ell_1} \quad (i = 2,\ldots,6) \quad ,$$

where

$$\ell_1 = 0,\ldots,\ell_6 = 0$$

are six planes in $\mathbf{P}^3(\mathbf{C})$ in general position. We get a covering of $\mathbf{P}^3(\mathbf{C})$ of degree 3^5 . To introduce singularities, again we choose the plane configuration so, that up to four planes pass through an arbitrary point. Again we assume, that no three of the planes have a line in common. A configuration with t of such points gives $3 \cdot t$ threefold singularities of type $(3,3,3,3)$; the resolution of every singularity is a cubic del Pezzo-surface (see Chapter II).

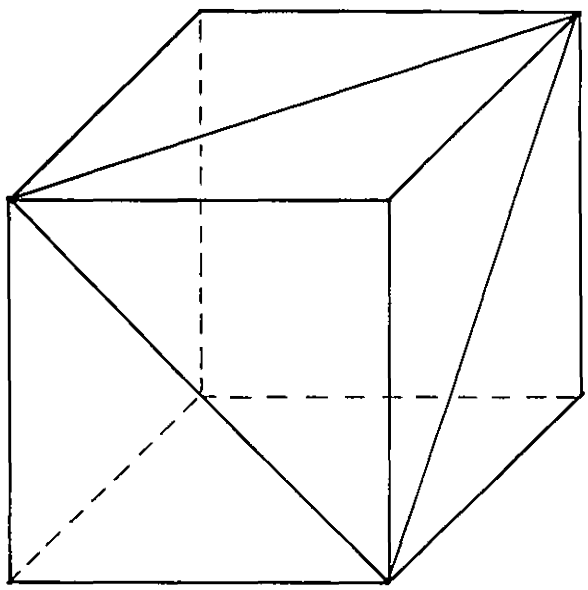
The canonical bundle remains trivial, the Euler number increases by $24 \cdot 3 \cdot t$.

Let the plane configuration be that one of a regular cube: in this situation we have three pairs of parallel planes and therefore three intersection lines in the plane of infinity.

These three lines intersect in three points, so t is equal to three and the threefold has 9 singularities. The Euler number of the resolution is equal to

$$-144 + 9 \cdot 24 = +72 \quad .$$

Another example with Euler number $+72$ is described in [6], p.12. I know it from E. Calabi.
In this context a very interesting example is constructed by B. Hunt. He considers a covering of degree 2^7 of $\mathbf{P}^3(\mathbf{C})$ branched along the plane-configuration of the following special type: six planes are the planes of a regular cube, the seventh is a plane passing through three edges of the cube as in the picture, and the eighth is the plane at infinity.



This construction leads to a plane configuration with 3 fivetuple and 3 quadruple points. The threefold has 24 ordinary double points and 12 singularities, given in $\mathbf{C}^5$ by the two equations

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 = 0$$

$$c_1 z_1^2 + c_2 z_2^2 + c_3 z_3^2 + c_4 z_4^2 + c_5 z_5^2 = 0 \quad ,$$

$c_i \neq c_j$ if $i \neq j$. To compute the Milnor number of these singularities, we consider a small deformation

$$\sum_{i=1}^{5} z_i^2 = \alpha$$

$$\sum_{i=1}^{5} c_i z_i^2 = \beta \qquad ;$$

its Euler number is equal to $e(X) - e(Y)$, where X is the complete intersection of two quadrics in $\mathbf{P}^5$ and Y is the complete intersection of two quadrics in $\mathbf{P}^4$; X is described by

$$\sum_{i=1}^{5} z_i^2 = \alpha z_0^2$$

$$\sum_{i=1}^{5} c_i z_i^2 = \beta z_0^2 \qquad ,$$

Y is a $\mathbf{P}^2$ blown up in 5 points (a del Pezzo-surface).

$e(Y) = 8$, and because $\beta_3(X) = 4$ (see [8] p.465) we get $e(X) = 0$. So the Euler number increases by 8, when the singularity is taken out. The singularity can be resolved by a (2,2) complete intersection in $\mathbf{P}^4$; the resolving surface is again a $\mathbf{P}^2$ blown up in 5 points with Euler number 8. This resolution does not influence the canonical class, so a resolution of all singularities of this type and a simultaneous small resolution of the double points gives a smooth threefold with trivial canonical bundle and Euler number

$$-128 + 12 \cdot 16 + 24 \cdot 2 = +112 \ .$$

It is not known whether there exists such a "mixed" resolution which is Kähler. For details see [9].

VII.

To construct our last example, let

$$g(z_3, z_4, z_5) = 0$$

be a smooth curve of degree 10 in $\mathbf{P}^2$. Consider the threefold

$$z_1^2 + z_2^5 + g(z_3, z_4, z_5) = 0$$

in a weighted projective space.
Choose new variables $u_1^5 = z_1$ and $u_2^2 = z_2$ and look at Y, given in $\mathbf{P}^4$ by the equation

$$u_1^{10} + u_2^{10} + g = 0 \quad .$$

The transformations

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \longmapsto \begin{pmatrix} \xi \cdot u_1 \\ \alpha \cdot u_2 \end{pmatrix}$$

with $\xi^5 = 1$, $\alpha^2 = 1$ form a group $\mathfrak{G}$ of order 10,

$$\left\{ z_1^2 + z_2^5 + g = 0 \right\} = Y/\mathfrak{G}$$

is our given threefold in the weighted projective space.
This group operation is not free: the subgroups, consisting of those transformations with $\alpha = 1$ and $\xi = 1$ respectively leave the surfaces $\{u_1 = 0\}$ and $\{u_2 = 0\}$ respectively fixed; the curve $\{u_1 = 0, u_2 = 0\}$ is the fixpoint set of the full group $\mathfrak{G}$. So

$$e(Y/\mathfrak{G}) = \frac{e(Y) + [(5-1) + (2-1)]\,e(F) + (5-1)\cdot(2-1)\,e(C)}{10} \quad ,$$

where F is a smooth surface of degree 10 in $\mathbf{P}^3$ and C is a smooth curve of degree 10 in $\mathbf{P}^2$.

$$e(Y) = -5900, \; e(F) = 660 \;, \; e(C) = -70 \quad ,$$

so

$$e(Y/G) = -288 \quad .$$

The canonical divisor on Y/G is computed as follows:
Let f be the quotient map $Y \longrightarrow Y/G$ and H a generic hyperplane in $\mathbf{P}^4$. Then

$$5H|_Y = K_Y = f^*\left(K_{Y/G} + \frac{1}{2}D + \frac{4}{5}B\right) \quad ,$$

where $D = \{z_2 = 0\}|_{Y/G}$ and $B = \{z_1 = 0\}|_{Y/G}$. Then

$$f^*D = \left\{u_2^2 = 0\right\}|_Y \sim 2H|_Y \quad ,$$

$$f^*B = \left\{u_1^5 = 0\right\}|_Y \sim 5H|_Y \quad .$$

So

$$f^*(K_{Y/G}) \sim 0 \quad ,$$

therefore the canononical bundle on Y/G must be trivial; see also [14], p.21.
If $g = 0$ has t singularities of type a_4, the threefold has t singularities of the form

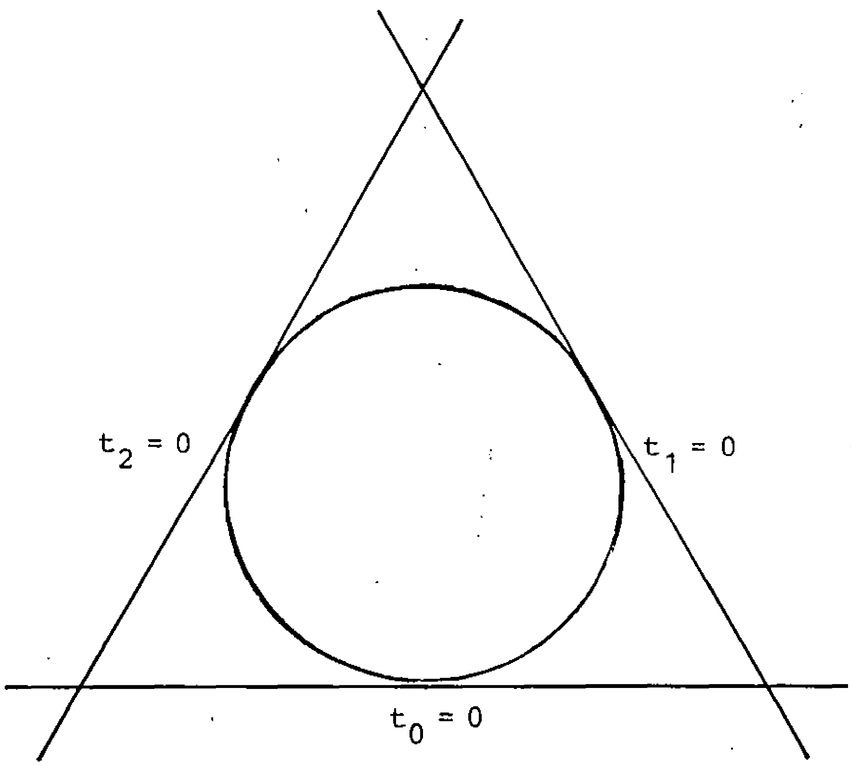
$$u^2 + v^2 + w^5 + t^5 = 0 \quad .$$

As described in chapter D, the Euler number increases by 20,

if we resolve such a singularity by a small resolution.
The equation

$$\left(z_3^5 + z_4^5 + z_5^5\right)^2 - 4\left(z_3^5 z_4^5 + z_3^5 z_5^5 + z_4^5 z_5^5\right) = 0$$

gives us a curve of degree 10 with 15 a_4 singularities; see [13] , p.311. This equation is of the form $f(z_3^5, z_4^5, z_5^5) = 0$, where $f(t_0, t_1, t_2) = 0$ is the conic in the following picture:



The small resolutions of the threefold have Euler number $-288 + 15 \cdot 20 = +12$. Are they projective algebraic?

APPENDIX

Let $T_d(x)$ be the Čebyšev polynomial in one variable of degree d , defined by

$$T_d(\cos\alpha) = \cos(d\alpha) \quad .$$

We have

$$T_d(x) = \sum_{j=0}^{[d/2]} \left(-1\right)^j \binom{d}{2j} x^{d-2j}\left(1-x^2\right)^j \quad .$$

At the points

$$\alpha_k := \cos\frac{k\pi}{d} , \quad 1 \leq k \leq d-1 \quad ,$$

$T'_d(x)$ has simple zeros; they are maxima (if k is even) and minima (if k is odd) of T_d as a real function. The values at these points are

$$T_d(\alpha_k) = \begin{cases} +1 & \text{if } k \text{ is even} \\ -1 & \text{if } k \text{ is odd} \end{cases} \quad .$$

The Čmutov hypersurface of degree d in $\mathbf{P}^n(\mathbf{C})$ is given in n affine coordinates by the equation

$$\sum_{j=1}^{n} T_d(x_j) = \begin{cases} 0 & \text{if } n \text{ is even} \\ +1 & \text{if } n \text{ is odd} \end{cases} \quad .$$

What are the singularities of this hypersurface?
All partial derivatives vanish if and only if

$$\frac{dT_d}{dx}(x_j) = 0 \quad \text{for all } j \;;$$

this is the case iff for all $j = 1,\ldots,n$ there exists a $k \in \{1,\ldots,d-1\}$, such that

$$x_j = \cos\left(\frac{k\pi}{d}\right) = \alpha_k \quad .$$

So in affine coordinates the singularities are the points

$$\left(\alpha_{i_1},\ldots,\alpha_{i_n}\right) \quad ,$$

where $[n/2]$ of the indices i_j are odd and the other are even. The singularities are nodes, because

$$\frac{d^2T_d}{dx^2}\left(\alpha_k\right) \neq 0 \quad \text{for all } k \quad .$$

Homogenizing does not influence the singularities, all of them are contained in the affine piece described above.
The Čmutov hypersurfaces have many nodes, for fixed n asymptotically

$$\binom{n}{[n/2]}\left(\frac{d}{2}\right)^n \quad \text{nodes for} \quad d \longrightarrow \infty \quad .$$

If $n=3$, the number of nodes is $\frac{3}{8}d\,(d-2)^2$ if d is even and $\frac{3}{8}(d-1)^3$ if d is odd; asymptotically we get $\frac{3}{8}d^3$ ordinary double points, the best example known.

We can generalize our construction and consider all cases

$$\sum_{j=1}^{n} (-1)^{\beta_j} T_d(x_j) = \begin{cases} 0 & \text{if } n \text{ is even} \\ (-1)^{\beta_0} & \text{if } n \text{ is odd} \end{cases}$$

with $\beta_0, \beta_1, \ldots, \beta_n \in \{-1,+1\}$. If $n \geq 3$, all these hypersurfaces are irreducible. We get the largest number of double points for the hypersurfaces

$$\sum_{j=1}^{n} T_d(x_j) = -1 \qquad \text{if } n \text{ is odd}$$

and

$$\sum_{j=1}^{n} (-1)^j T_d(x_j) = 0 \qquad \text{if } n \text{ is even.}$$

If $n = 3$ and d is even, these hypersurfaces have $\frac{3}{8}d^2(d-2)$ nodes: the octic has 144 double points.

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