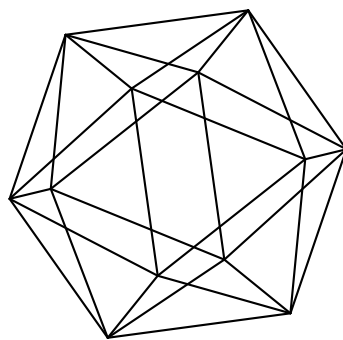


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THE TOPOLOGY OF MEROMORPHIC LOOP SPACES OF SMOOTH CURVES

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ABSTRACT. We study the topology of the \mathbb{C} -points of the algebraic loop space of a smooth affine curve. In particular we will compute their sets of connected components. Our results can be stated somewhat suggestively in terms of a certain *non-abelian weight filtration* on the fundamental group of the curve.

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2. INTRODUCTION

If M is, say, a manifold, then we can consider the free loop space, $LM = \text{Map}(S^1, M)$, of unbased maps from a circle into M . If M is connected then the set of connected components of LM is identified with the fundamental group $\pi_1(M)$. There is an algebro-geometric version of LM , gotten heuristically by replacing the circle S^1 with a *formal punctured disc*, $\text{spec}(\mathbb{C}((z)))$. The resulting object is an ind-scheme if we take our target to be affine.

The question of the topology of the \mathbb{C} -points of this ind-scheme now arises fairly naturally. We address this in the case of a smooth affine curve. To the best of our knowledge, the first non-trivial such computation is due to Contou-Carrère, who studied the algebraic loop space of \mathbb{G}_m in his work on what is now called the Contou-Carrère symbol, (cf. [1]). In the case of \mathbb{G}_m the answer turns out to *agree* with the topological analogue. We will see that this is very much not the case in general.

3. BASICS

For the reader unfamiliar with algebraic loop spaces we include some background here. For more a more detailed introduction we refer the reader to the work of Kapranov and Vasserot in [1].

Definition 3.1. • If $S = \text{spec}(A)$ is an affine scheme then we write \mathcal{D}_S for the affine scheme $\text{spec}(A[[z]])$ and \mathcal{D}_S^* for the affine scheme $\text{spec}(A((z)))$. We write $\mathcal{D}_{\mathbb{C}}^*$ for $\mathcal{D}_{\text{spec}(\mathbb{C})}^*$, and we sometimes shorten this further to just \mathcal{D}^* .

- If X is a scheme then we define the pre-sheaves, $X(\mathcal{D})$ and $X(\mathcal{D}^*)$, respectively by $X(\mathcal{D})(S) = X(\mathcal{D}_S)$ and $X(\mathcal{D}^*) = X(\mathcal{D}_S^*)$. We refer to them as the arc space of X and the loop space of X respectively. We will sometimes refer to them as holomorphic (resp. meromorphic) loops into X . The space $X(\mathcal{D})$ is a closed subspace of $X(\mathcal{D}^*)$.
- There is a covering action of the monoid \mathbb{N}^\times on the space $X(\mathcal{D}^*)$ gotten by pre-composition with the etale covers, $z \mapsto z^n$.

We will briefly summarise the representability properties of these pre-sheaves:

Lemma 3.1. • If X is a scheme then the pre-sheaf $X(\mathcal{D})$ is representable by a scheme, affine if X is.

- If X is an affine scheme then $X(\mathcal{D}^*)$ is an ind-affine scheme, i.e. a filtered colimit (taken inside the pre-sheaf category) of affine schemes with transition maps closed inclusions.

Proof. This is well known, we refer the reader to [3]. One reduces to the case of \mathbb{A}^1 using compatibility with limits and then deals with the \mathbb{A}^1 case explicitly. \square

We give a couple of examples of loop spaces of affine curves. The reader is cautioned that we will sometimes write \mathcal{D}^* for what should more properly be denoted $\mathcal{D}_{\mathbb{C}}$.

Example. • We will briefly describe the loop space of \mathbb{A}^1 . It can easily be seen to be the ind-scheme $\varinjlim_n \mathbb{A}^{[-n, \infty)}$. A Laurent power series $\gamma(z) = \sum_{i \geq -n} \gamma_i z^i$ corresponds to the element $(\gamma_i)_i \in \mathbb{A}^{[-n, \infty)}$ in the obvious way. The sub-space of arcs is $\mathbb{A}^{[0, \infty)}$. Endowing the \mathbb{C} -points of this space with the ind-topology we note that it is contractible. We shall see below that \mathbb{A}^1 is the *only* smooth curve for which this is the case. It is instructive to consider the following \mathbb{A}^1 -family of loops

into \mathbb{A}^1 . Define $\gamma : \mathbb{A}^1 \rightarrow \mathbb{A}^1(\mathcal{D}^*)$ as follows: by the functor of points definition it corresponds to a map, $\mathcal{D}_{\mathbb{A}^1}^* \rightarrow \mathbb{A}^1$, i.e. to a function on the affine scheme $\mathcal{D}_{\mathbb{A}^1}^*$, and so to an element of the algebra $\mathbb{C}[t]((z))$. We take this element to be $z + tz^{-1}$. The fibre at $t = 0$ is holomorphic but the other fibres have poles. We will see below that \mathbb{A}^1 is the only smooth curve for which this is possible. Indeed for every other smooth curve, X , the complex points of the arc space are *open* in the complex points of the loop space.

- We consider now the case of \mathbb{G}_m . According to work of Contou-Carrère, [2], if A is a \mathbb{C} -algebra, then an invertible element of $A((z))$ admits a unique expression of the form,

$$\alpha(z) = \alpha_0 z^{\nu(\alpha)} \prod_{0 < i < \infty} (1 - \alpha_{-i} z^{-i}) \prod_{0 < j} (1 - \alpha_j z^j),$$

where $\nu(\alpha)$ is the order, α_0 is a unit of A and all the α_{-i} are nilpotents. This implies an isomorphism,

$$\mathbb{G}_m(\mathcal{D}^*) \cong \mathbb{Z} \times \mathcal{D}^\infty \times \mathbb{G}_m \times \mathbb{A}^\infty.$$

In particular we see that $\pi_0(\mathbb{G}_m(\mathcal{D}_{\mathbb{C}}^*)) = \mathbb{Z}$, corresponding to pole/ zero order. We will find it useful, with a view to generalising this result, to decompose \mathbb{Z} as $(\infty \times \mathbb{Z}_{<0}) \sqcup \{0\} \sqcup (0 \times \mathbb{Z}_{>0})$. Recall that we have a covering action of the monoid \mathbb{N}^\times on $\mathbb{G}_m(\mathcal{D}_{\mathbb{C}}^*)$. This descends to an action on π_0 . It is easily seen that it acts trivially on the summand, $\{0\}$, of π_0 and via multiplication on the summands $(\infty \times \mathbb{Z}_{<0})$ and $(0 \times \mathbb{Z}_{<0})$. Taking the quotient by this action, and denoting it $\pi_0^{\mathbb{N}^\times}$, we see that this has size 3; one component for each puncture and one for the holomorphic loops. It is in this form that the computation will generalise.

Before we state our main theorem we will fix some notation:

Definition 3.2. *If X is a smooth curve with smooth proper model \overline{X} , we write*

$$\mathcal{W}_0^{na} \pi_1(X) := \ker(\pi_1(X(\mathbb{C})) \rightarrow \pi_1(\overline{X}(\mathbb{C}))),$$

and we call it the non-abelian weight 0 piece of π_1 .

Remark. This is a free group, being a subgroup of a free group. The definition is merely an ad-hoc adaptation of the usual mixed Hodge theory for open curves, recalling that the weight 0 subspace is spanned by co-cycles pulled back from the smooth proper model, cf ([2]). It should not be confused with more sophisticated notions of mixed Hodge structures

on fundamental groups. We choose to formulate the result in this manner partly to make contact with a very interesting remark of Bhatt, cf. <https://mathoverflow.net/a/16650/10899>.

We are now in a position to state our main result:

Theorem 3.2. *Let X be a smooth complex affine curve, then we compute the number of connected components, up to coverings, of its loop space as*

$$\#\pi_0^{\mathbb{N}^\times} X(\mathcal{D}_{\mathbb{C}}^*) = 1 + rk(\mathcal{W}_0^{na} \pi_1(X)).$$

Remark. We can re-state our main result more explicitly as follows: let X be a smooth affine curve. If $X = \mathbb{A}^1$ then its loop space is connected. Otherwise the set of connected components of its loop space is naturally identified with $\mathbb{N}^\times \times (\overline{X} - X) \sqcup *$.

4. THE PROOF

We will break the proof into a couple of cases, namely those curves X with $\#(\overline{X} - X) > 1$, and those with $\#(\overline{X} - X) = 1$ - the second case proving to be the trickier one. Our strategy will be to find invariants of loops which do not change as we continuously deform the loop, and then to use these to differentiate between components.

Definition 4.1. *If $\gamma : \mathcal{D}_{\mathbb{C}}^* \rightarrow X$ is a loop, we denote by $\overline{\gamma}$ the extension to a holomorphic loop, $\overline{\gamma} : \mathcal{D}_{\mathbb{C}} \rightarrow \overline{X}$.*

Remark. The existence of such an extension is guaranteed by the valuative criterion. We caution the reader that this *does not* correspond to a morphism of spaces, $X(\mathcal{D}^*) \rightarrow \overline{X}(\mathcal{D})$. This can be seen for example by considering the \mathbb{A}^1 -family of loops, $z + tz^{-1}$, defined above, indeed the extensions to the central element $0 \in \mathcal{D}$ vary discontinuously with t .

The most natural invariant associated to a loop is the pull-back on degree 1 de Rham cohomology. In fact it makes more sense to consider *continuous de Rham cohomology* of topological \mathbb{C} -algebras. We record a simple lemma:

Lemma 4.1. *Let S be a connected affine scheme with \mathbb{C} -points s_0 and s_1 . Let*

$$\gamma : \mathcal{D}_S^* \rightarrow X,$$

be an S -family of loops. Write γ_0 and γ_1 for the fibres at s_0 and s_1 . Then the pull-backs,

$$s_i^* : H_{dR}^*(X) \rightarrow H_{dR}^*(\mathcal{D}^*),$$

agree for $i = 0, 1$, where H_{dR}^ denotes continuous de Rham cohomology.*

Proof. This is a simple generalisation of the corresponding statement for discrete de Rham cohomology. \square

Remark. We remind the reader that the first continuous de Rham cohomology of $\mathcal{D}_{\mathbb{C}}^*$ is one dimensional, generated by $d \log(z)$. Suppose we have a loop, γ , into X , with $\bar{\gamma}(0) = p \in \bar{X}$. Suppose further that we have a meromorphic 1-form ω on X , then the pull-back in first continuous de Rham cohomology corresponds to a multiple (given by the order at 0) of the residue.

We show now that if two non-holomorphic loops admit extensions with different central value then they must lie in different components. In order to do this we first record a simple lemma:

Lemma 4.2. *Let $p \neq q$ be two points of a smooth proper curve X . Then there exists a meromorphic 1-form, ω , on X , which is holomorphic away from $\{p, q\}$ and which satisfies $res_p(\omega) = 1$, $res_q(\omega) = -1$.*

Proof. This is a simple computation with the Riemann-Roch Theorem. Let Ω_X denote the canonical bundle. We apply Riemann-Roch to the bundle $\Omega_X(p+q)$. We note

$$deg(\Omega_X(p+q)) = 2g(X),$$

and thus we compute $h^0(\Omega_X(p+q)) = 1 + g > h^0(\Omega_X)$. Taking a non-zero element,

$$\omega \in H^0(X, \Omega_X(p+q)) / H^0(X, \Omega_X),$$

we may assume without loss of generality that it is not holomorphic at p . It has a pole of order at most one at p and thus a pole of order exactly one. Rescaling we may assume $res_p(\omega) = 1$, the only other point with potentially non-zero residue is q whence we deduce $res_q(\omega) = -1$ by the Residue Theorem, completing the proof. \square

We deduce the following:

Corollary 4.3. *Let X be an affine smooth curve whose smooth proper model \bar{X} satisfies $\#(\bar{X} - X) > 1$. Let γ_0 and γ_1 be non-holomorphic loops into X and let $\bar{\gamma}_0$ and $\bar{\gamma}_1$ be their extensions to holomorphic loops into \bar{X} . Assume that the values at $0 \in \mathcal{D}$ of these extensions differ. Then $[\gamma_0] \neq [\gamma_1]$ in $\pi_0 X(\mathcal{D}_{\mathbb{C}}^*)$. Further, if γ_2 is a holomorphic loop into X then its class in π_0 differs from either of the classes of the $\gamma_{0/1}$.*

Proof. . The lemma above furnishes a meromorphic 1-form ω , so that the three residues $\text{res}_{\overline{\gamma}_i(0)}(\omega)$ all differ. Recalling that these residues correspond to de Rham pull-backs and then combining the lemmas above it suffices to show that if two loops, γ_0 and γ_1 , are in the same connected component of $X(\mathcal{D}_{\mathbb{C}}^*)$ then there is a connected affine S and an S -family of loops interpolating them.

Let the connected component containing them be called S . It is ind-affine and thus we find an affine closed sub-scheme containing the loops γ_0 and γ_1 . They must lie in the same component of a suitably large such sub-scheme, which we call S . The inclusion $S \hookrightarrow X(\mathcal{D}^*)$ provides the tautological S -family, $\gamma : \mathcal{D}_S^* \rightarrow X$ and we conclude. \square

Remark. The reader will note that this argument breaks down without the assumption that $\#(\overline{X} - X) > 1$. Indeed if $\#(\overline{X} - X) = 1$ then any holomorphic form, ω , on X will have trivial residue at the puncture of \overline{X} .

The next step is to identify when two loops, γ_0 and γ_1 , both extending to arcs with central value $p \in \overline{X}$, lie in the same connected component. There is an obvious guess, and it is the correct one. We first note that in a purely local situation, the computation of the connected components of a loop space is easy:

Lemma 4.4. *Let $\mathfrak{X} = \mathcal{D}^*(\mathcal{D}^*)$. Then connected components of $\mathfrak{X}(\mathbb{C})$ are labelled by $\mathbb{Z}_{>0}$, corresponding to the degree of the morphism $\mathcal{D}^* \rightarrow \mathcal{D}^*$.*

Proof. First note that if $\gamma : \mathcal{D}^* \rightarrow \mathcal{D}^*$ is a loop of order n then $\gamma^*(d \log(z)) = nd \log(z)$ so that loops of different degree must lie in different connected components according to the lemma above. Writing a loop of order n in the form $\gamma(z) = z^n \gamma_+(z)$ gives a homeomorphism with $\mathbb{C}^\times \times \mathbb{C}^\infty$ and thus we conclude the lemma. With more effort we could describe the space \mathfrak{X} in the manner of Contou-Carrère (for \mathbb{G}_m) but this is not needed for this note. \square

Corollary 4.5. *It follows that after we quotient by the covering action we have just one component, i.e. we have $\pi_0^{\mathbb{N}^\times} \mathfrak{X} = *$.*

Corollary 4.6. *If two loops, γ_0 and γ_1 , both extend to arcs having the same central value, $p \in \overline{X}$, then they lie in the same component of $X(\mathcal{D}_{\mathbb{C}})$ iff they have the same order at p .*

Proof. Write $\mathcal{D}_p^* \hookrightarrow X$ for the inclusion of the formal punctured neighbourhood of p . It suffices to note that our loops γ_i admit unique factorisations through \mathcal{D}_p^* and then to use the lemma above. \square

Remark. This suffices to prove the main theorem in the case of curves with at least two punctures once we note that the arc space is also connected. In summary we see that in this case we have one component corresponding to the arcs, and one for each pair consisting of a positive integer and a puncture of the curve X . After quotienting by the \mathbb{N}^\times -action, it is not hard to see that this gives $1 + rk(\mathcal{W}_0^{na}\pi_1(X))$ components.

Remark. Our methods above, dealing with the case of curves X with $\#(\overline{X} - X) > 1$, were all essentially abelian. Abelian invariants, say $H_{et}^*(-, \mathbb{Z}/l)$, will not distinguish X from \overline{X} in the case of $\#(\overline{X} - X) = 1$. Indeed the set of etale \mathbb{Z}/l -torsors for X and \overline{X} agree as they have the same *abelianised* etale fundamental groups. This suggests that we attempt to distinguish non-holomorphic loops from holomorphic ones via the induced map on π_1^{et} . Note that we know from above that there are *at most* two components (up to covers), our goal below is to show that there are exactly two as soon as $g(\overline{X}) > 0$. We remind the reader that the Newton-Puiseux theorem implies that we have $\pi_1^{et}(\mathcal{D}^*) = \widehat{\mathbb{Z}}$. Further the Riemann Existence theorem implies that we have $\pi_1^{et}(X) = \widehat{\pi}_1(X(\mathbb{C}))$, the profinite completion of the topological fundamental group.

We begin with the following simple lemma:

Lemma 4.7. *Let X be a smooth curve with $\#(\overline{X} - X) = 1$, and $g(\overline{X}) > 0$, then there exists an etale cover, $Y \rightarrow X$, which is not pulled back from an etale cover of \overline{X} , i.e. does not extend to an etale cover of \overline{X} .*

Proof. Let us take standard generators for $\pi_1(\overline{X}(\mathbb{C}))$ coming from viewing a genus g real surface as a $4g$ -gon with appropriate edge identifications. We will call the generators $\{\alpha_1, \beta_1, \dots, \alpha_g, \beta_g\}$. $\pi_1(\overline{X}(\mathbb{C}))$ is generated by these elements subject to the one relation $\prod_i [\alpha_i, \beta_i] = 1$. $\pi_1(X(\mathbb{C}))$ is freely generated by them.

It suffices, by Riemann Existence, to find a finite group G such that the natural map,

$$Hom(\pi_1(\overline{X}(\mathbb{C})), G) \rightarrow Hom(\pi_1(X(\mathbb{C})), G),$$

is not surjective. We take $G = S_3$. We define an element of the right hand side by sending $\alpha_1 \mapsto (12), \beta_1 \mapsto (123)$ and the rest of the generators to the unit. This maps the element $\prod_i [\alpha_i, \beta_i] = 1$ to a non-trivial permutation and thus we are done. \square

Corollary 4.8. *Let X be as above, and fix any non-holomorphic loop, $\gamma \in X(\mathcal{D}_{\mathbb{C}}^*)$. Then the induced map of etale fundamental groups, $\widehat{\mathbb{Z}} = \pi_1^{et}(\mathcal{D}^*) \rightarrow \pi_1^{et}(X)$, is non-zero.*

Proof. Let $\infty \in \overline{X}$ denote the puncture point. We may assume the extension, $\overline{\gamma} : \mathcal{D} \rightarrow \overline{X}$, maps 0 to ∞ to order 1, since otherwise the result is simply to pre-compose with the map induced on $\pi_1^{et}(\mathcal{D}^*)$ by a degree n cover, which is of course not zero.

To show that the induced map, $\widehat{\mathbb{Z}} = \pi_1^{et}(\mathcal{D}^*) \rightarrow \pi_1^{et}(X)$, is non-zero it suffices to show that there is a non-trivial etale cover of X which pulls back to a non-trivial etale cover of \mathcal{D}^* under γ .

However, let us note that if γ pulls back a cover, $Y \rightarrow X$, to a trivial cover of \mathcal{D}^* , then the pulled-back cover extends over the puncture point $0 \in \mathcal{D}$, whence the original cover extends over the puncture $\infty \in \overline{X}$. We have seen above that there exist covers not extendable in this manner, whence the proof is complete. \square

Corollary 4.9. *Let γ_0 be a holomorphic and γ_1 a non-holomorphic element of $X(\mathcal{D}_{\mathbb{C}}^*)$, for X as above, then the classes of γ_0 and γ_1 differ in $\pi_0(X(\mathcal{D}_{\mathbb{C}}^*))$. Further, two non-holomorphic loops extending to arcs with the same central value lie in the same component iff they have the same degree.*

Proof. Note that a holomorphic loop induces zero on π_1^{et} , further we can detect the order from the induced map on π_1^{et} . As such, using arguments similar to Lemma 4.1 above, it suffices to show the following: if

$$\gamma : \mathcal{D}_S^* \rightarrow X,$$

is an S -family of loops with γ_0 and γ_1 the fibres at s_0 and s_1 , then maps induced on π_1^{et} by the γ_i agree. This follows from the results of SGA, specifically it can be deduced from the results of [SGA 1, Exp. X, Cor 2.4, Thm 3.8, Cor. 3.9 & Section 2 of Exp. XIII], many thanks to Jason Starr for the detailed reference. \square

Putting this all together we deduce the main theorem, which we re-state below:

Theorem 4.10. *Let X be a smooth complex affine curve, then we compute the number of connected components, up to coverings, of its loop space as*

$$\#\pi_0^{\mathbb{N}^\times} X(\mathcal{D}_{\mathbb{C}}^*) = 1 + rk(\mathcal{W}_0^{na} \pi_1(X)).$$

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