Relative Elliptic Theory and Sobolev Problems

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Abstract

An algebra of operators associated with a smooth embedding $i : X \to M$ is constructed. For elliptic elements of this algebra the finiteness theorem (Fredholm property) is proved. The connection with elliptic Sobolev problems is indicated.

Introduction

1.By the relative elliptic theory we mean the elliptic theory associated with the pair (M, X) where M is a smooth closed manifold and X is its submanifold. Thus, the relative elliptic theory is an elliptic theory in the category of smooth embeddings.

The trivial (and non interesting) example of a relative elliptic operator is given by a pair of elliptic operators given on manifolds M and X, correspondingly. Such an 'operator' can be written down in the form of a diagonal matrix operator

$$\begin{pmatrix} D_M & 0 \\ 0 & D_X \end{pmatrix} : \begin{pmatrix} H^{s_1}(M) \\ H^{s_2}(X) \end{pmatrix} \to \begin{pmatrix} H^{\sigma_1}(M) \\ H^{\sigma_2}(X) \end{pmatrix}$$

with elliptic (pseudodifferential) elements D_M and D_X acting, for example, in Sobolev spaces on the manifolds M and X.

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This trivial example, however, shows that in the general situation a relative elliptic operator must be, probably, represented by some (quadratic) matrix which, certainly, must not to be a diagonal one. We first note that a posteriory this suggestion is quite correct. However, it needs a serious explanation. The matter is that, by their sense, the elements standing in the lower left and upper right corners of the supposed matrix must be an operators acting from the space of functions given on one of the manifolds into that on the another one. Hence, this matrix we can write down in the form

$$\begin{pmatrix} D_{MM} & D_{MX} \\ D_{XM} & D_{XX} \end{pmatrix} \tag{1}$$

where indices show the direction of action of the corresponding operators (from the right to the left). Apparently, even a non experienced reader will wonder what the operators D_{MX} and D_{XM} are. It is more or less clear that these operators are not in any case pseudodifferential ones, at least if this notion is meant in the classical sense. Actually, classical pseudodifferential operators act from the space of functions on some smooth manifold into that given on one and the same manifold. The mentioned operators do not possess this property just by their definition. What are these operators? What is their nature?

Let us try to propose some candidates to the role of these operators. To begin with, let us consider the operator D_{XM}. This operator acts from the space of functions on the whole manifold into that on the submanifold. The most natural operator of the kind is the restriction operator (or boundary operator) induced by the embedding $i : X \hookrightarrow M$. If one takes into account that operators of the form (1) should form an algebra with involution, it becomes clear that the operator D_{MX} must be dual to the previous one that is, to be an operator of corestriction or, as we shall call it, a coboundary operator. Certainly, in the general case one should use as the operators D_{MX} and D_{XM} a composition of the two above operators with some pseudodifferential ones. Thus, at the first glance, our future theory is a theory of the operators of the form

$$\begin{pmatrix} D_{MM} & C_{MM}i_* \\ i^*B_{MM} & D_{XX} \end{pmatrix} \tag{2}$$

where the operators i^* and i_* are elementary boundary and coboundary operators, and the operators D_{MM}, B_{MM}, C_{MM}, and D_{XX} are pseudodifferential operators on the corresponding manifolds. Unfortunately, however, operators of the form (2) do not form an algebra, what is certainly ineligible fact from the viewpoint of the elliptic theory. Actually, the proof of the finiteness theorem (Fredholm property) is carried out in the convenient and natural form with the help of constructing of the so-called regulizers, that is, the inverse (up to the compact ones) operators. Of course, this method requires the algebra structure in the set of the operators considered and, certainly, effective enough calculus.

As we have already mentioned above, unfortunately the set of operators of the type (2) do not form an algebra. The reason is (and this is shown in the paper) that the calculation of the product of the two operators of the kind gives in the upper left corner of the resulting matrix the operator of more general structure than a pseudodifferential one. This fact is not surprising since the factors included in this operator are a fortiory non pseudodifferential operators and one cannot in general expect the result to be a pseudodifferential one[1]. Thus, it seems to be natural, in order to introduce the algebra structure in the set of the considered operators, to extend the class of pseudodifferential operators by adding to the latter ones the operators of the type

$$C_{MM} i_* i^* B_{MM}$$

arising in the composition. The remarkable fact is that after such an extension the set of operators is closed with respect to the composition and, hence, the question of constructing of the needed algebraic structure is in principle solved.

Certainly, this is not the end. It would be desirable to give the description of the recently introduced operators (which, as we have already mentioned, are not pseudodifferential ones) in the known terms. Of course, the positive solution of the stated problem is just a good luck. In principle, the introduced operators can have absolutely new nature and be not known in the literature. Fortunately, in this case it is not so. All the operators of the above type admit a representation in the form of the operators which are rather well-known. For example, these operators can be interpreted as certain *Fourier integral operators* on special Lagrangian manifolds. In this interpretation each type of operators (boundary, coboundary, etc) corresponds to its own Lagrangian manifold. These operators admit also an adequate interpretation in quite different terms. Namely, they can be represented as a special class of pseudodifferential operators acting in sections of infinitely-dimensional bundles (ψDO's with operator-valued symbols). In different situations different treatments can be useful. For example, in constructing a calculus the convenient interpretation is Fourier integral operators, in formulation of the ellipticity conditions and in computation of the index these operators can be interpreted as *pseudodifferential operators in sections of infinite-dimensional bundles* and so on.

Thus, in the present paper the relative theory of pseudodifferential operators is constructed as the theory associated with the smooth embedding $i : X \hookrightarrow M$. In the framework of this theory the notion of ellipticity of the corresponding operator is introduced and the corresponding finiteness theorem is proved. Finally, the index formula for the relative elliptic operators is written down.

2. The relative elliptic theory allows one to solve one of the important and interesting problems in the theory of differential equations – the so-called *Sobolev problem.* This problem is that in situation of a pair $i : X \hookrightarrow M$ one searches a solution to the equation

$$Du \equiv f \pmod X,$$

[1]In this connection we note that, nevertheless, the operator arising in composition in the lower right corner of the result in fact is a pseudodifferential operator.

where the comparison means that the equation is fulfilled everywhere on M except for points of the submanifold X. Besides, some 'boundary conditions' are given on X and the problem is, first, to present a correct statement of such a problem and, second, to investigate its solvability (Fredholm property). The problem of such kind was formerly considered by S. L. Sobolev [1] for the polyharmonic equation. The general statement and the investigation of these problems was carried out by B. Sternin [2] who give them the name Sobolev problems.

We remark that these problems possess a series of interesting features. For example, the number of 'boundary conditions' of the problem essentially depends on the index of the Sobolev space in which the solution is searched for. In particular, in the space of sufficiently smooth functions the Sobolev problem becomes, in essence, trivial. Later on, a solution to an elliptic Sobolev problem is not in general an infinitely smooth function even if the right-hand part of the equation is infinitely smooth. The solution shall have singularities on the submanifold X.

3. The present paper develops ideas, methods and results contained in [2], [3], [4]. More exactly, this paper gives a new glance on the theory of Sobolev's elliptic problems and on the relative elliptic theory from the viewpoint of the modern theory of differential equations. In particular, the interpretation of boundary and coboundary operators in terms of Fourier integral operators presented in this paper is new, and we used the results of papers [5], [6], [7] containing an extension of the algebra of pseudodifferential operators.

In the period of almost thirty years which passed from the time when the first paper [2] on the Sobolev problem had appeared, a lot of remarkable papers on the elliptic theory were published. The notion of the coboundary operator introduced in [3] is at present in general usage in the general theory of differential equations. In these papers, independently and in different situations a lot of constructions close to those of the relative elliptic theory had appeared. In particular, the well-known at present Boutet de Monvel's algebra [8] was created, which plays the most important role in the theory of solvability of elliptic pseudodifferential operators on a compact manifold with boundary. The like structures appear also in construction of algebra of Mellin ψDO's created by B.-W. Schulze [9], [10].

1 From Sobolev problem to operator morphisms

1.1 Two physical examples

To illustrate the appearance of the notion of Sobolev problem, we begin with rather simple physical examples. If we consider a thin film stretched on a one-dimensional contour and try to sustain this film by a thin needle, then *the film will on notice the needle and the needle will come through the film without changing its form* (or, maybe,

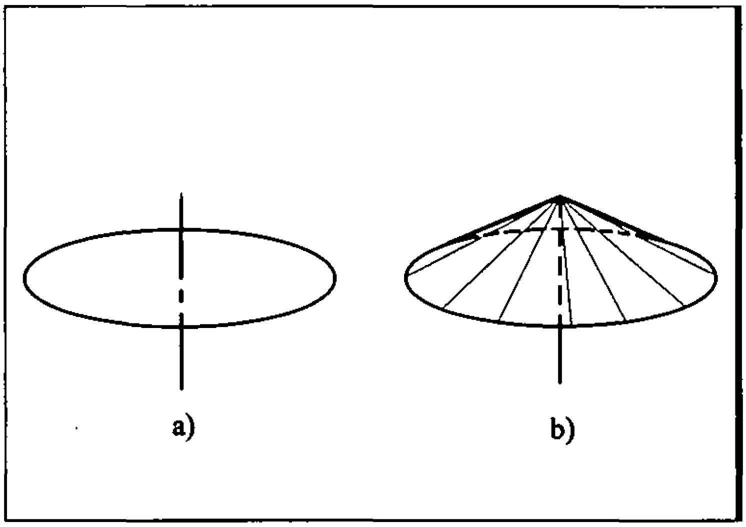

Figure 1: a) Needle goes through thin film. b) Needle supports thick membrane.

the film will be broken down). However, if we try to sustain a thick membrane by the same needle then we shall see that the form of this membrane shall be changed. In other words, *one cannot pose 'boundary conditions' for the equation for thin film in a single point, but can pose such conditions for the equation of the thick membrane* (see Figure 1).

Mathematically, the nature of this phenomenon is as follows.

The problem in the space $\mathbf{R}^2$ describing a form of a two-dimensional thin film stretched on a one-dimensional contour reads

$$\begin{cases} \Delta u = 0, \\ u|_\Gamma = \varphi, \end{cases} \tag{3}$$

where the function φ defined on a plane closed curve Γ describes the form of the contour on which the considered film is stretched. If we try to sustain this film by a needle in some point $(x_0, y_0) \in \mathbf{R}^2$, this means that:

1) We suppose that the equation involved in the problem (3) is valid not in all points of the domain bounded by Γ but in all its points *except for the point* (x_0, y_0). This will be in the sequel written down in the form

$$\Delta u \equiv 0 \mod (x_0, y_0) \tag{4}$$

and means exactly that the distribution Δu is supported in the single point (x_0, y_0).

2) We supply problem (3) by the additional 'boundary condition' of the form

$$u|_{(x_0,y_0)} = u_0 \tag{5}$$

with some fixed value of u_0.

However, as it follows from the well-known theorem on removable singularity for harmonic functions, any solution of the comparison (4) which is bounded at the point (x_0, y_0) (the latter follows from the condition (5) will be a harmonic function in the whole domain (including the point (x_0, y_0)) and, hence, the value of this function at this point is uniquely determined by the data of problem (3). Hence, the condition (5) is quite superfluous and must not be posed.

Quite another situation takes place for the problem describing (say, rigidly fixed) thick membrane

$$\begin{cases} \Delta^2 u = 0, \\ u|_\Gamma = \varphi, \quad \left.\dfrac{\partial u}{\partial n}\right|_\Gamma = \varphi, \end{cases}$$

where $\partial/\partial n$ is a derivative along the outer normal direction to the contour Γ. If we replace the equation involved into the latter problem by the comparison

$$\Delta^2 u \equiv 0 \mod (x_0, y_0), \tag{6}$$

we can pose the *additional condition* of the type (5) since this comparison possesses a nontrivial continuous solution (which behaves as $r^2 \ln r$ near the point (x_0, y_0), r being a distance from this point; the function $r^2 \ln r$ is an exact solution to comparison (6) in the whole space $\mathbf{R}^2$).

Thus, we see that for equations of enough high order one can pose problems involving 'boundary conditions' on manifolds of enough high codimension. The problems with such conditions are called *Sobolev problems.*

1.2 Boundary and coboundary operators

Let us describe now the *general statement* of the Sobolev problem. To avoid the unessential difficulties, we shall consider such a problem on a smooth compact manifold M without boundary (in this case one does not need any boundary conditions).

Let us introduce important notions in the theory of pseudodifferential operators which will be of use in what follows. Let M be a smooth compact manifold without boundary and let

$$i : X \hookrightarrow M \tag{7}$$

be a smooth embedding of codimension ν. Later on, let $E \to M$ be a complex vector bundle over M. By $C^\infty(M, E)$ we denote the space of smooth sections of this bundle over the manifold M.

The imbedding (7) induces a mapping

$$i^* : C^\infty(M, E) \to C^\infty(X, E|_X)$$

which will be called *an elementary boundary operator* induced by the embedding i. From the Sobolev embedding theorem follows that this mapping extends up to the continuous mapping

$$i^* \,:\, H^s(M,E) \,\to\, H^{s-\nu/2}(X,\, E|_X)$$

for $s > \nu/2$ where by $H^s(M,E)$ we denoted the Sobolev space of sections of the bundle E, corresponding to a real index s.

The elementary coboundary operator is defined by the duality:

$$i_* \,:\, H^{-s+\nu/2}(X,\, E|_X) \,\to\, H^{-s}(M,E)\,,$$

which in certainly continuous for $s > \nu/2$.

The above introduced elementary boundary and coboundary operators certainly are not (classical) pseudodifferential operators. This follows at least from the fact that these operators do not act in the *whole* Sobolev scale of spaces. The nature of these operators will be discussed in detail a few lines below.

Let now E and F be two bundles over M and

$$B \,:\, C^\infty(M,E) \,\to\, C^\infty(M,F)$$

be a pseudodifferential operator of order b. Then the composition $i^* \circ B$ is called a *boundary operator* associated with the operator B. This operator extends up to the continuous operator

$$i^* \circ B \,:\, H^s(M,E) \,\to\, H^{s-b-\nu/2}(X,\, F|_X) \tag{8}$$

for $s > b + \nu/2$.

The *coboundary operator* associated with some pseudodifferential operator C of order c is defined by duality:

$$C \circ i_* \,:\, H^{-s+c+\nu/2}(X,\, F|_X) \,\to\, H^{-s}(M,E)\,. \tag{9}$$

This mapping is evidently continuous for $s > c + \nu/2$.

Let us try to find out what boundary and coboundary operators are like. We have already remarked that these operators are not pseudodifferential ones.

1. *Coboundary operator.* For investigating the nature of these operators let us try to make the things clear for, say, coboundary operator. First of all we notice that outside some (arbitrary) neighbourhood of the submanifold X this operator is an infinitely-smoothing one, so that in essential part this operator is just a germ on the manifold X. Thus, we consider a neighbourhood of an arbitrary point on the submanifold X and introduce in this neighbourhood the coordinates (x,t) such that the equation of X is $t = 0$. We have

$$Ci_*f = \left(\frac{1}{2\pi}\right)^{n+\nu} \int\!\!\int e^{i[\tau t + p(x-x')]} C(x,t,p,\tau) f(x')\, d\tau dp dx'$$

where $C(x,t,p,\tau)$ is the symbol of the operator C (p,τ are dual variables to x,t, correspondingly). The integral over τ in the latter expression can be computed and we come to the following expression for the operator Ci^*:

$$Ci_*f = \left(\frac{1}{2\pi}\right)^n \int e^{ip(x-x')}\widetilde{C}(x,t,p)f(x')\,dpdx' \tag{10}$$

where

$$\widetilde{C}(x,t,p) = \left(\frac{1}{2\pi}\right)^\nu \int e^{i\tau t}C(x,t,p,\tau)\,d\tau. \tag{11}$$

Operator (10) has the form of a pseudodifferential operator or, more strictly, a family of pseudodifferential operators parameterized by t. However, its symbol (11) *is not a smooth function of the parameter*: it has singularities at $t = 0$. The nature of this phenomenon is quite clear. Operator (10) is, for example, a Green operator of a boundary value problem (a potential), that is, an integral operator whose kernel is 'a Green function'. Such an operator always has singularities as $t \to 0$. Here is the concrete example which confirms this fact.

Let the symbol of the operator C is equal to

$$C(x,t,p,\tau) = \frac{1}{1+p^2+\tau^2}$$

(we suppose, for simplicity, that t and τ are one-dimensional variables). Then, using the residue theorem, one obtains

$$\widetilde{C}(x,t,p) = \frac{1}{2\pi}\int \frac{e^{i\tau t}\,d\tau}{1+p^2+\tau^2} = \frac{1}{2\sqrt{1+p^2}}e^{-|t|\sqrt{1+p^2}}.$$

Evidently, the latter function has the singularity in t at $t = 0$, as required. Besides, from the latter relation one can see that the operator with the symbol $\widetilde{C}(x,t,p)$ is concentrated on the manifold X since outside this manifold (that is, for $t \neq 0$) the symbol $\widetilde{C}(x,t,p)$ decreases more rapidly than any power of $|p|$.

2. *Boundary operator*. Similar to the case of coboundary operator, the boundary operator is also concentrated on the manifold X. Thus, we consider also a neighbourhood of X and, using the above notation, write down the boundary operator in the form

$$i^*Bf = \left(\frac{1}{2\pi}\right)^{n+\nu} \int\int e^{i[p(x-x')-\tau' t']}B(x,0,p,\tau')f(x',t')\,dpd\tau dx'dt' \tag{12}$$

where $B(x,t,p,\tau')$ is the symbol of the operator B. Let us rewrite this operator in the form of a pseudodifferential operator. We have

$$i^*Bf = \left(\frac{1}{2\pi}\right)^{n+\nu} \int\int e^{i[p(x-x')+\tau'(t-t')]}\widetilde{B}(x,p,\tau')f(x',t')\,dpd\tau dx'dt' \tag{13}$$

where

$$\widetilde{B}(x,p,\tau') = e^{-i\tau' t} B(x,0,p,\tau'). \tag{14}$$

Thus, the boundary operator (12) is a pseudodifferential operator of the form (13) whose symbol (14) is a *rapidly oscillating* one. It is easy to see that the non-smoothness of a coboundary operator and the rapid oscillations of a boundary one are dual notions.

3. *Lagrangian uniformization.* The above considerations, of course, exactly express the situation for the considered questions. In principle, we can try to introduce the calculus with the above representations of the considered operators. Unfortunately, this is not quite convenient in practice. The manipulation with non smooth or (and) rapidly oscillating symbols supplies us with a lot of purely technical difficulties. For example, it is necessary to describe the admissible character of non-smoothness of symbols, the character of oscillations and so on.

In other words, while determining a class of coboundary operators one needs

– first, to fix the type of singularities,

- and, second, to derive the rules of maintaining with such operators (calculus), that is, to introduce the structure of the module over the ring of pseudodifferential operators.

This program can be fulfilled with the help of the idea of the so-called Lagrangian uniformization which, for example, can be realized by Maslov's canonical operator method or with the help of Fourier integral operators theory. Namely, the above operators can be represented as Fourier integral operators (see [11], [12], [13], and others) on special Lagrangian manifolds which encounter the character of singularities (oscillations) with *smooth* symbols.

Let us consider first the boundary operator. The corresponding Lagrangian manifold is

$$N^*(\Delta(X)) \subset T^*(X \times M) \tag{15}$$

where

$$\Delta : X \to X \times M$$

is a 'diagonal' embedding given by $\Delta(x) = (x,x)$ and N^* is a conormal bundle of a submanifold.

Let us write down the local expression of this operator. It acts from the space of functions given on one smooth manifold (M in our case) to that given on another smooth manifold (X in our case). We shall mark the coordinates on the first manifold by primes. Thus, (x',t') are coordinates on M (such that $t'=0$ is an equation of the submanifold X) and x are coordinates on X. By (p',τ',p) we denote dual coordinates for (x',t',x) and call them *impulse coordinates* dual to *physical coordinates* (x',t',x).

Now, the operator i^* can be written down in the form

$$(i^*Bu)(x) = \left(\frac{1}{2\pi}\right)^{n+\nu} \int e^{i(p'(x-x')-\tau' t')} B(x,0,p',\tau) u(x',t')\, dp'd\tau'dx'dt'. \tag{16}$$

The equations of Lagrangian manifold (15) in these coordinates are

$$x = x', p = p', t' = 0$$

Let us describe the procedure of computing the phase function

$$\varphi = p'(x - x') - \tau' t'$$

of integrals of the type (16) in terms of the corresponding Lagrangian manifold.

We remark that on each Lagrangian manifold there exist an atlas consisting of *canonical charts*. The coordinates in such charts are partly physical coordinates of the phase space, partly its impulse coordinates, but the set of these coordinates must not contain any pair of coordinates dual to each other. For example, canonical coordinates on $\Delta(X)$ are (x, p', τ') (the choice of canonical coordinates is not unique).

Now we can describe the phase function. It consists of the two terms

$$\varphi = (-p'x' - t'\tau') + S(x, p', \tau')$$

where the first term is the sum of products of pairs of dual coordinates over all pairs for which the impulse term of the pair is involved into the set of canonical coordinates (the products corresponding to the primed variables are taken with the sign $-$) and the function S is a solution of the equation

$$dS = pdx + x'dp' + t'd\tau'|_{\Delta(X)}.$$

The structure of the right-hand part of the latter relation is clear from the considered example; we note only that we change the sign of the term when this term corresponds to an impulse coordinate or in case when it corresponds to the primed one (so that the sign of the term $x'p'$ is changed twice).

Evidently, the solution of the latter equation is $S = xp'$ and we come to the phase function

$$p'(x - x') - \tau' t',$$

as required.

Similar, the elementary coboundary operator i_* can be written down as a Fourier integral operator associated with the Lagrangian manifold

$$N^*(\Delta^*(X)) \subset T^*(M \times X) \tag{17}$$

where

$$\Delta^* : X \to M \times X$$

is a corresponding 'diagonal' embedding. The local expression for the corresponding Fourier integral operator is

$$(Ci_*u)(x) = \left(\frac{1}{2\pi}\right)^{n+\nu} \int e^{i[p'(x-x')+\tau t]} C(x, p', \tau)\, u(x')\, dp' d\tau dx'.$$

The equations of the corresponding Lagrangian manifold are

$$t = 0, x = x', p = p',$$

and the corresponding phase function is equal to

$$\varphi = -p'x' + \tau t + S(x, p', \tau)$$

where

$$dS(x, p', \tau) = pdx + x'dp' - td\tau|_{\Delta^*(X)} = p'dx + xdp' = d(xp'),$$

as required.

Thus, we have shown that boundary and coboundary operators, (8) and (9) can be written down in the form of Fourier integral operators associated with Lagrangian manifolds (15) and (17), respectively. The symbols of these Fourier integral operators coincide with the symbols of the pseudodifferential operators B and C. We have come to the following definition.

Definition 1.1 A *general boundary operator* is a Fourier integral operator associated with the Lagrangian manifold (15). A *general coboundary operator* is a Fourier integral operator associated with the Lagrangian manifold (17).

The classes of symbols involved in general boundary and coboundary operators will be discussed below (see Section 2).

1.3 Sobolev problems

Let us consider now the following problem for the pair $X \overset{i}{\hookrightarrow} M$.

$$\begin{cases} Du \equiv f \mod H^{s-m}(M, X; E_2), \\ i^* Bu = g, \end{cases} \tag{18}$$

Here D is a pseudodifferential operator of order m in sections of bundles E_1 and E_2 on the manifold M, B is a pseudodifferential operator of order b acting in sections of bundles E_1 and F on the manifold M, and $H^{s-m}(M, X; E_2)$ is a subspace of the Sobolev space $H^{s-m}(M, E_2)$ which consists of functions (distributions) supported in X. It follows from (8) that for the trace i^*Bu to be correctly defined, we must suppose that $s - b > \nu/2$.

Let us now investigate the solvability of problem (18).

Evidently, the comparison

$$Du \equiv f \mod H^{s-m}(M, X; E_2)$$

which is an equation on the manifold $M \backslash X$ with respect to the unknown u is equivalent to the following equation

$$Du + i_*^L v = f$$

on the *whole* manifold M with respect to *the two* unknowns u and v. Here i_*^L is a coboundary operator corresponding to the jet-operator in transversal variables (in local coordinates this operator is a *string* containing all derivatives in transversal direction up to the order L). Therefore, the following proposition is valid.

Proposition 1.1 *Problem* (18) *with respect to u is equivalent to a system of equations*

$$\begin{cases} Du + i_*^L v = f, \\ i^* B u = g, \end{cases} \tag{19}$$

with respect to the functions u and v defined on M and X correspondingly.

Here

$$L = \begin{cases} \left[m - s - \frac{\nu}{2}\right] & \text{if } m - s - \frac{\nu}{2} \text{ is noninteger,} \\ m - s - \frac{\nu}{2} - 1 & \text{if } m - s - \frac{\nu}{2} \text{ is integer.} \end{cases} \tag{20}$$

The proof of the stated assertion is quite simple. Actually, suppose that u is a solution to the problem (18). Then the function $Du - f$ is a distribution supported in X. By the Schwartz theorem [14], such a distribution can be represented as a sum of the delta function on X and its derivatives up to the order L whose coefficients are functions on X belonging to the corresponding Sobolev spaces. Denoting the set of these coefficients by v we obtain that system of equations (19) is valid for such choice of v. We remark that the vector v is chosen in the unique way. The inverse assertion is quite evident.

System (19) can be rewritten in the matrix form

$$\begin{pmatrix} D & i_*^L \\ i^* B & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

Thus, we had come to the necessity to consider the operators of the type

$$\begin{pmatrix} D & i_*^L \\ i^* B & 0 \end{pmatrix} \tag{21}$$

which we call *the operator morphisms.* These morphisms (as well as their generalizations) shall be investigated in the next section. Let us discuss now the number of 'boundary conditions' which must be involved to the stated problem. It is almost evident that for the problem (18) to be well-posed one must, at least, require that the number of 'boundary conditions' in (18) is equal to the degree of freedom allowed by the comparison included into this problem. As we have seen, the difference $Du - f$, being a distribution from the space $H^{s-m}(M, E_2)$ supported in X, can be represented in the form

$$Du - f = \sum_{|\alpha| \le L} c_\alpha \delta_X^{(\alpha)}$$

where $\delta_X^{(\alpha)}$ are transversal derivatives of the delta function concentrated on X of order α and c_α are some functions on X. The number of these functions is exactly the above mentioned degree of freedom.

Therefore, the number of unknown functions allowed by the considered comparison is equal to the number of multiindices α with $|\alpha| \leq L$. Thus, the dependence of the number of 'boundary conditions' necessary for problem (18) to be well-posed on the index s of the Sobolev space is given by relations (20).

Certainly, the fact that problem (18) involves the correct number of 'boundary conditions' does not guarantee that the problem is well-posed. One should require also that this problem must be an elliptic one. The exact formulation of this notion for Sobolev problems will be given below (see Section 3). There is one more question which is worth investigating in the above context. This question is what is the *maximal* number of 'boundary conditions' if we require that *all these conditions are given by differential operators.* The matter is that, as we have seen above, the number of conditions required for the well-posed Sobolev problem in the space $H^s(M)$ is given by formula (20). From the other hand, the maximal order L_1 of derivative admitting the restriction to the manifold X is equal to

$$L_1 = \begin{cases} \left[s - \frac{\nu}{2}\right], & \text{if } s - \frac{\nu}{2} \text{ is noninteger,} \\ s - \frac{\nu}{2} - 1, & \text{if } s - \frac{\nu}{2} \text{ is integer.} \end{cases} \tag{22}$$

Evidently, for existence of the Sobolev problem with purely differential 'boundary conditions one must require that $L_1 \leq L$. However, the function (20) is a decreasing one, whence the function (22) is an increasing one. Hence, the maximal number of boundary conditions in a purely differential Sobolev problem is reached for such values of s that $L = L_1$. One can easily verify that this equality takes place for $s = m/2$. Now we see that any Sobolev problem with differential conditions in the half-interval containing $m/2$ is equivalent to the *Sobolev-Dirichlet problem*, in which one prescribes values of all derivatives of u in transversal direction up to the order L_D where

$$L_D = \frac{m}{2} - \left[\frac{\nu}{2}\right] - 1$$

Certainly, there exists Sobolev problem for values of s to the left from the interval containing $m/2$. However, for such values of s one must use *pseudodifferential* 'boundary conditions'. On the opposite, to the right of the number $m - \nu/2$ *there exists no Sobolev problem.* In other words, the following theorem on removable singularities is valid.

Proposition 1.2 *If*

$$Du \equiv f \mod H^{s-m}(M, X; E_2)$$

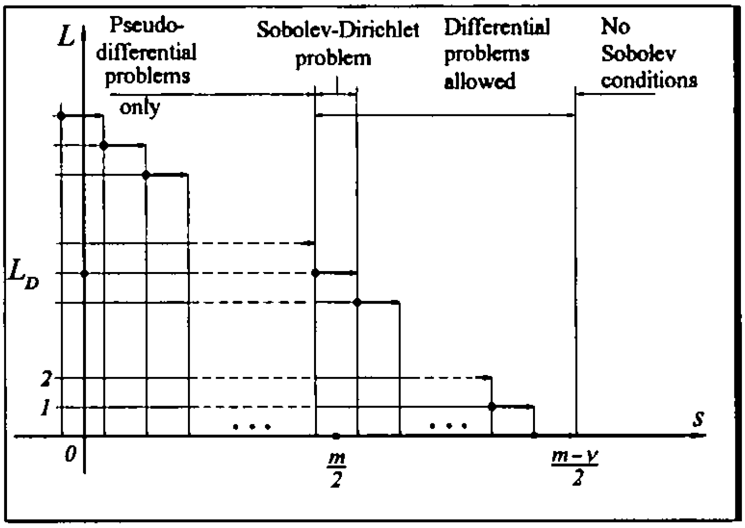

Figure 2: The number of 'boundary conditions' depend on s.

for $s \geq m - \nu/2$, *then the equation*

$$Du = f$$

is valid on the whole manifold M.

The results of our considerations are shown on Figure 2. On this Figure L is the 'number' of 'boundary conditions' (more exactly, the maximal order $L = |\alpha|$ of derivatives of delta function included into the left-hand part of equation (19)) and s is the index of the considered Sobolev space. The graph on this Figure illustrates the dependence $L = L(s)$. On this Figure we have also marked the region in which purely differential Sobolev problems are allowed, the interval where the Sobolev-Dirichlet problem is well-posed and the region where in order to construct a well-posed Sobolev problem one must use pseudodifferential 'boundary problems'.

2 Algebra of operator morphisms

2.1 Calculus

To investigate the Fredholm properties of a Sobolev problem, or, what is the same, of an operator morphism, it is useful to introduce the structure of an algebra on the set of morphisms of the type (21). The main reason for the necessity of introducing of such a structure is that, as it is well-known, the Fredholm property of an operator follows from the existence of the so-called regulizer to this operator, that is, of the almost inverse[2] operator. Let us try to find out the general form of operators which form a minimal algebra containing operators of the form (21).

First of all, we note that the set of matrices of this form *do not form an algebra.* Actually, the product of the two operators of this form is

$$\begin{pmatrix} D_1 & i_* \\ i^*B_1 & 0 \end{pmatrix} \begin{pmatrix} D_2 & i_* \\ i^*B_2 & 0 \end{pmatrix} = \begin{pmatrix} D_1D_2 + i_*i^*B_2 & D_1 i_* \\ i^*B_1D_2 & i^*B_1 i_* \end{pmatrix}.$$

The result differs from the initial form (21) by the following.

1) There exist an operator of the form $i_*i^*B_2$ as a term of the upper left element of the obtained matrix.

2) In the upper right corner of the result stands the operator $D_1 i_*$ with some pseudodifferential operator D_1 instead of i_*.

3) The right lower element $i^*B_1i_*$ of the obtained matrix is not equal to zero.

Let us investigate the situation in more detail.

First of all, we note that the operator $i^*B_1i_*$ is a *pseudodifferential* operator on the manifold X (see [15], [16]).This assertion can be verified in the following way. It is quite evident that the operator $i^*B_1i_*$ does not depend on values of the symbol of the operator B_1 outside any neighbourhood of the manifold X. Therefore, we can consider this operator as pseudodifferential operator on X whose symbol is a pseudodifferential operator acting in transversal directions. Now we have

$$i^*B_1i_*e^{i\lambda S}\varphi = i^*B_1e^{i\lambda S}\varphi \otimes \delta_X$$

where δ_X is the delta function concentrated on the manifold X. Using L. Hörmander's definition of a pseudodifferential operator [17], we obtain

$$i^*B_1i_*e^{i\lambda S}\varphi = e^{i\lambda S}\left\{i^*\left[e^{-i\lambda S}B_1e^{i\lambda S}\delta_X\right]\right\}\varphi$$

and, hence, the operator $i^*B_1i_*$ is a pseudodifferential operator with the principal symbol

$$\lim_{\lambda\to\infty} \lambda^{-b}i^*e^{-i\lambda S}B_1e^{i\lambda S}\delta_X = \lim_{\lambda\to\infty}\lambda^{-b}i^*\,\mathrm{smbl}_X\ B_1\delta_X,$$

where b is the order of the operator B_1 and $\mathrm{smbl}_X\ B_1$ is a pseudodifferential operator in transversal directions mentioned above. It is now easy to compute that

$$\mathrm{smbl}\,(i^*B_1i_*) = \int \mathrm{smbl}\,B_1\,d\tau$$

where the integration is taken over fibers of the conormal bundle to X. Thus, in the lower right corner of the general morphism some pseudodifferential operator on the manifold X should stand.

[2]That is, the inverse operator in the quotient algebra with respect to the (two-sided) ideal of compact operators.

Later on, if we compose the operators of the obtained form, the number of terms containing operators i^* and i_* in the upper left corner will increase. Thus, we must have some representations of such sums in order to write them down in some general form. This form can be obtained as some *Fourier integral operator* (we recall that, due to the results of Section 1 above, the operators i^* and i_* are Fourier integral operators).

After these preliminary considerations we are able to write down the general form of an operator morphism:

$$\begin{pmatrix} D_{11}+\widehat{\Phi}_{11} & \widehat{\Phi}_{21} \\ \widehat{\Phi}_{12} & D_{11} \end{pmatrix}$$

where D is a pseudodifferential operator on M, E is a pseudodifferential operator on X, and the operators $\widehat{\Phi}_{11}$, $\widehat{\Phi}_{21}$, $\widehat{\Phi}_{12}$ are Fourier integral operators associated with the following Lagrangian manifolds

$$\begin{aligned} L_{11} &= N^*(Y_{11}), \\ L_{21} &= N^*(Y_{21}), \\ L_{12} &= N^*(Y_{12}), \end{aligned} \tag{23}$$

correspondingly. Here the manifolds Y_{11}, Y_{21}, and Y_{12} are given by

$$\begin{aligned} Y_{11} &= \{(\alpha,\beta)\in M\times M \,:\, \alpha=\beta\in X\}, \\ Y_{21} &= \{(\alpha,\beta)\in M\times X \,:\, \alpha=\beta\in X\}, \\ Y_{12} &= \{(\alpha,\beta)\in X\times M \,:\, \alpha=\beta\in X\}. \end{aligned}$$

The origin of the second and the third of Lagrangian manifolds (23) is clear. They are Lagrangian manifolds associated with (co)boundary operators. What concerns the first of manifolds (23), its form can be obtained from the following considerations.

Similar to the above considered (co)boundary operators, the operator of the form $\widehat{\Phi}_{11}$ is not a classical pseudodifferential operator just from the reason that this operator is also concentrated on the manifold X. Now, following the uniformization scheme carried out for (co)boundary operators, we shall investigate an operator of the type $\widehat{\Phi}_{11}$ and shall try to represent it in the form of a Fourier integral operator on the corresponding Lagrangian manifold.

For simplicity, we consider the typical example of the operator of the form $\widehat{\Phi}_{11}$ which is a composition of coboundary and boundary operators. It has the form

$$\begin{aligned} Ci_*Ai^*Bf &= \left(\frac{1}{2\pi}\right)^{n+\nu}\int e^{i[\tau t+p(x-x')-\tau' t']}C(x,t,p,\tau)A(x,p) \\ &\times\ B(x,t,p,\tau')f(x',t')\,dpd\tau d\tau' dx' dt', \end{aligned} \tag{24}$$

we use the coordinate notation of Section 1. This operator can be written down in the form of a pseudodifferential operator on the manifold M:

$$Ci_*Ai^*Bf = \left(\frac{1}{2\pi}\right)^{n+\nu}\int e^{i[\tau'(t-t')+p(x-x')]}F(x,t,p,\tau')f(x',t')\,dpd\tau' dx' dt'$$

with the symbol

$$F(x,t,p,\tau') = e^{-it\tau'} \int e^{it\tau} C(x,t,p,\tau) A(x,p) B(x,t,p,\tau')\, d\tau.$$

As one should expect, the symbol of this operator which incorporates all 'defects' both boundary and coboundary operators is a non-smooth rapidly oscillating symbol. The operator $\widehat{\Phi}_{11}$ can be uniformized, that is, represented in the form of Fourier integral operator on a Lagrangian manifold L_{11} which was introduced above. This is a Lagrangian manifold such that its restriction (in the natural sense) leads to the manifolds which correspond to boundary and coboundary operators.

Let us consider now the question of computation of the phase function of integral (24). Using the coordinates $(x,t,p,\tau; x',t',p',\tau')$ we can write down the equations of the corresponding Lagrangian manifold in the form

$$x = x', p = p', t = 0, t' = 0.$$

Let us choose (x, p', τ, τ') as canonical coordinates. Then the phase function must be given by the equality

$$\varphi = -x'p' + t\tau - t'\tau' + S(x,p',\tau,\tau'),$$

where

$$dS = pdx + x'dp' - td\tau + t'd\tau'|_{L_{11}} = d(xp').$$

Therefore, we obtain $S = xp'$, and the phase function

$$\varphi = p'(x - x') + t\tau - t'\tau',$$

as required.

Now let us describe the classes of symbols used in the corresponding Fourier integral operators which are simply the functions on the above Lagrangian manifolds.

The symbols Φ_{11}, Φ_{21}, and Φ_{12} of the operators $\widehat{\Phi}_{11}$, $\widehat{\Phi}_{21}$, and $\widehat{\Phi}_{12}$ must satisfy the following inequalities (usual in the theory of pseudodifferential operators):

$$\begin{aligned} &\left|D_x^\alpha D_p^\beta D_\tau^\gamma D_{\tau'}^\delta \Phi_{11}(x,p,\tau,\tau')\right| \leq \\ &\leq C_{\alpha\beta\gamma\delta} \left(1 + |(p,\tau)|^2\right)^{\frac{m-|\gamma|}{2}} \left(1 + |p|^2\right)^{\frac{k-|\beta|}{2}} \left(1 + |(p,\tau')|^2\right)^{\frac{l-|\delta|}{2}}, \end{aligned}$$

for the operator $\widehat{\Phi}_{11}$. The corresponding symbol class is denoted by $\mathrm{Smbl}_{11}(m,k,l)$. For operators of the type $\widehat{\Phi}_{21}$ symbols must satisfy the estimates

$$\left|D_x^\alpha D_p^\beta D_\tau^\gamma \Phi_{21}(x,p,\tau)\right| \leq C_{\alpha\beta\gamma} \left(1 + |(p,\tau)|^2\right)^{\frac{m-|\gamma|}{2}} \left(1 + |p|^2\right)^{\frac{k-|\beta|}{2}};$$

The corresponding symbol class will be denoted by $\mathrm{Smbl}_{21}(m,k)$. Finally, the estimates for symbols of operators of the type $\widehat{\Phi}_{12}$ are

$$\left|D_x^\alpha D_p^\beta D_{\tau'}^\delta \Phi_{12}(x,p,\tau')\right| \leq C_{\alpha\beta\delta}\left(1+|p|^2\right)^{\frac{k-|\beta|}{2}}\left(1+|(p,\tau')|^2\right)^{\frac{l-|\delta|}{2}},$$

the corresponding symbol classes being denoted by $\mathrm{Smbl}_{12}(k,l)$.

Thus, in accordance to Definition 1.1 above, the operators of the type $\widehat{\Phi}_{12}$ and $\widehat{\Phi}_{21}$ are general boundary and coboundary operators, respectively. The operators of the type $\widehat{\Phi}_{11}$ has quite a different nature. Its appearance is caused by an algebraic structure. As we shall see below, operators of this kind are also involved into regulizers for corresponding morphisms (see Section 3 above).

We present here the local expressions of all three types of mentioned above Fourier integral operators:

$$\widehat{\Phi}_{11}f = \left(\frac{1}{2\pi}\right)^{2n+\nu}\int e^{i\{\tau t+p(x-x')-\tau' t'\}}\Phi_{11}(x,p,\tau,\tau')f(x',t')\,dpd\tau d\tau' dx' dt', \quad (25)$$

$$\widehat{\Phi}_{21}f = \left(\frac{1}{2\pi}\right)^{n+\nu}\int e^{i\{\tau t+p(x-x')\}}\Phi_{21}(x,p,\tau)f(x')\,dpd\tau dx', \quad (26)$$

$$\widehat{\Phi}_{12}f = \left(\frac{1}{2\pi}\right)^{n+\nu}\int e^{i\{p(x-x')-\tau' t'\}}\Phi_{12}(x,p,\tau')f(x',t')\,dpd\tau' dx' dt', \quad (27)$$

where we have used the local coordinates introduced above, n is a dimension of the manifold X and ν is a codimension of X in M.

Below we formulate theorems describing properties of the introduced operators [3]. First, we present three theorems describing the action of the constructed operators in the Sobolev spaces.

Theorem 2.1 *Operator* (25) *is a continuous operator in spaces*

$$\widehat{\Phi}_{11} : H^s(M) \to H^{s-r}(M)$$

for the values of s satisfying the inequalities

$$l+\frac{\nu}{2} < s < k+l+\frac{\nu}{2}.$$

Here the order[4] r of this operator is equal to

$$r = m+k+l+\nu.$$

[3]The introduced operators as well as the above symbol classes are similar to those introduced in the papers [5], [6], [7]. Therefore, we shall not present below the proofs of the theorems concerning the properties of these operators and restrict ourselves only by the corresponding formulations.

[4]The (Sobolev) order of an operator $\widehat{A}$ in the Sobolev scale H^s is the exact lower bound of the set of numbers a for which the mapping

$$\widehat{A} : H^s \to H^{s-a}$$

is bounded. Here and below the order of operator is its Sobolev order.

Theorem 2.2 *Operator* (26) *is a continuous operator in spaces*

$$\widehat{\Phi}_{21} \;:\; H^s(X) \;\rightarrow\; H^{s-r}(M)$$

for the values of s satisfying the inequality

$$s < k.$$

Here the order r of this operator is equal to

$$r = k + m + \nu/2.$$

Theorem 2.3 *Operator* (27) *is a continuous operator in spaces*

$$\widehat{\Phi}_{12} \;:\; H^s(M) \;\rightarrow\; H^{s-r}(X)$$

for the values of s satisfying the inequality

$$s > l + \frac{\nu}{2}.$$

here the order r of this operator is equal to

$$r = l + k + \nu/2.$$

All inequalities involved in the formulations of the above theorems can be easily understood if one takes into account the particular form Ci_*Ai^*B, Ci_*, and i^*B of the operators $\widehat{\Phi}_{11}$, $\widehat{\Phi}_{21}$,and $\widehat{\Phi}_{12}$, respectively. For these particular forms all mentioned inequalities are immediate subsequences of the Sobolev embedding theorem.

The following affirmation describes the compositions of operators of the introduced type.

Theorem 2.4 *Compositions of operators of the type* (25) - (27) *are as follows:*

1) *The composition of the two operators $\widehat{\Phi}'_{11}$ and $\widehat{\Phi}''_{11}$ is an operator of the type* (25). *The symbol $\Phi(x,p,\tau,\tau')$ of this operator is given by*

$$\Phi(x,p,\tau,\tau') = \left(\frac{1}{2\pi}\right)^{\nu} \int \Phi'(x,p,\tau,\tau'')\Phi''(x,p,\tau'',\tau')\,d\tau''.$$

This symbol belongs to the class $\mathrm{Smbl}_{11}(m', k'+k''+l'+m''+\nu, l'')$ *if the factors have symbols from* $\mathrm{Smbl}_{11}(m',k',l')$ *and* $\mathrm{Smbl}_{11}(m'',k'',l'')$ *respectively.*

2) *The composition of the two operators $\widehat{\Phi}'_{11}$ and $\widehat{\Phi}''_{21}$ is an operator of the type* (26). *The symbol $\Phi(x,p,\tau)$ of this operator is given by*

$$\Phi(x,p,\tau) = \left(\frac{1}{2\pi}\right)^{\nu} \int \Phi'(x,p,\tau,\tau'')\Phi''(x,p,\tau'')\,d\tau''.$$

This symbol belongs to the class $\mathrm{Smbl}_{21}(m', k' + k'' + l' + m'' + \nu)$ *if the factors have symbols from* $\mathrm{Smbl}_{11}(m', k', l')$ *and* $\mathrm{Smbl}_{21}(m'', k'')$ *respectively.*

3) *The composition of the two operators* $\widehat{\Phi}'_{12}$ *and* $\widehat{\Phi}''_{11}$ *is an operator of the type* (27). *The symbol* $\Phi(x, p, \tau')$ *of this operator is given by*

$$\Phi(x, p, \tau') = \left(\frac{1}{2\pi}\right)^{\nu} \int \Phi'(x, p, \tau'')\Phi''(x, p, \tau'', \tau')\, d\tau''.$$

This symbol belongs to the class $\mathrm{Smbl}_{12}(k' + k'' + l' + m'' + \nu.l'')$ *if the factors have symbols from* $\mathrm{Smbl}_{12}(k', l')$ *and* $\mathrm{Smbl}_{11}(m'', k'', l'')$ *respectively.*

4) *The composition of the two operators* $\widehat{\Phi}'_{21}$ *and* $\widehat{\Phi}''_{12}$ *is an operator of the type* (25). *The symbol* $\Phi(x, p, \tau, \tau')$ *of this operator is given by*

$$\Phi(x, p, \tau') = \Phi'(x, p, \tau)\Phi''(x, p, \tau').$$

This symbol belongs to the class $\mathrm{Smbl}_{11}(m', k' + k'', l'')$ *if the factors have symbols from* $\mathrm{Smbl}_{21}(m', k')$ *and* $\mathrm{Smbl}_{12}(k'', l'')$ *respectively.*

5) *The composition of the two operators* $\widehat{\Phi}'_{12}$ *and* $\widehat{\Phi}''_{21}$ *is a pseudodifferential operator on the manifold* X. *The symbol* $P(x, p)$ *of this operator is given by*

$$P(x, p) = \left(\frac{1}{2\pi}\right)^{\nu} \int \Phi'(x, p, \tau'')\Phi''(x, p, \tau'')\, d\tau''.$$

The order of this operator is equal to $k' + l' + m'' + k''$ *if the factors have symbols from* $\mathrm{Smbl}_{12}(k', l')$ *and* $\mathrm{Smbl}_{21}(m'', k'')$ *respectively.*

Now we can write down the general form of an operator morphism. Namely, we shall consider morphisms of the form

$$\begin{pmatrix} D_{11} + \widehat{\Phi}_{11} & \widehat{\Phi}_{21} \\ \widehat{\Phi}_{12} & D_{22} \end{pmatrix} \tag{28}$$

where D_{11} is a pseudodifferential operator on the manifold M, D_{22} is a pseudodifferential operator on the manifold X, the operator $\widehat{\Phi}_{11}$ has the type (25) and $\widehat{\Phi}_{21}$, $\widehat{\Phi}_{12}$ are general coboundary and boundary operators (26) and (27), respectively.

Theorems 2.1 - 2.3 allows one to write down the relations which are necessary for operator (28) to be a continuous operator in the following Sobolev spaces:

$$\begin{pmatrix} D_{11} + \widehat{\Phi}_{11} & \widehat{\Phi}_{21} \\ \widehat{\Phi}_{12} & D_{22} \end{pmatrix} : \begin{pmatrix} H^{s_1}(M) \\ H^{s_2}(X) \end{pmatrix} \to \begin{pmatrix} H^{\sigma_1}(M) \\ H^{\sigma_2}(X) \end{pmatrix}. \tag{29}$$

Theorem 2.5 *Let the types* (m_{11}, k_{11}, l_{11}), (m_{21}, k_{21}), (k_{12}, l_{12}) *of operators* $\widehat{\Phi}_{11}$, $\widehat{\Phi}_{21}$, $\widehat{\Phi}_{12}$, *and orders* m *and* l *of the operators* D_{11} *and* D_{22} *satisfy the following relations:*

$$\begin{cases} \sigma_1 = s_1 - m, \\ \sigma_2 = s_2 - l, \\ \sigma_1 = s_1 - m_{11} - k_{11} - l_{11} - \nu, \\ \sigma_1 = s_2 - m_{21} - k_{21} - \nu/2, \\ \sigma_2 = s_1 - k_{12} - l_{12} - \nu/2, \end{cases} \tag{30}$$

and the following inequalities:

$$\begin{cases} \max\left\{\frac{\nu}{2} - l_{11}, \frac{\nu}{2} + l_{12}\right\} < s_1 < \frac{\nu}{2} + k_{11} + l_{11}, \\ s_2 < k_{21}. \end{cases} \tag{31}$$

Then operator (28) *is a continuous operator in spaces* (29).

We remark that the orders of all the operators in (28) are chosen in such a way (with the help of relations (30)), that if the orders of the operators D_{11} and D_{22} are equal to zero, then the orders of all the rest operators also equals zero. With this observation, we see that the following affirmation is a direct corollary of the preceding Theorem.

Corollary 2.1 *The set of operator morphisms* (28) *with* $m = l = 0$, *and fixed values of* m_{ij}, k_{ij}, *and* l_{ij} *such that* $m_{11} + k_{11} + l_{11} + \nu = 0$, $m_{21} + k_{21} + \nu/2 = 0$, *and* $k_{12} + l_{12} + \nu/2 = 0$ *is an algebra of operator morphisms acting continuously in spaces* (29) *with* $s_1 = \sigma_1$ *and* $s_2 = \sigma_2$ *provided that the numbers* s_1 *and* s_2 *satisfy inequalities* (31).

Corollary 2.2 *The set of operators of the form* $1 + \widehat{\Phi}_{11}$ *where* $\widehat{\Phi}_{11}$ *is an operator of zeroth order form an algebra.*

To conclude this section we remark that all the notions introduced here are invariant with respect to the conjugation. In particular, *the conjugate to the operator morphism is, in turn, an operator morphism* and, hence, the constructed algebra is a *-algebra (an algebra with involution) where the involution is given by the operation of conjugation of operators acting in Banach spaces.

2.2 Ellipticity and finiteness theorems for morphisms

In this section we shall derive the conditions under which operator (29) determined by operator morphism (28) possesses the Fredholm property. As it is well known, to prove that it is sufficient to construct a regulizer for this operator, that is, the inverse operator modulo compact ones. Thus, our first aim is to construct a regulizer for operator (29). We remark that all our considerations will be carried out on the level of principal terms of the considered operators in Sobolev spaces.

To construct the almost inverse for operator (29) one must construct the resolving operator for the system of equations

$$\begin{pmatrix} D_{11}+\widehat{\Phi}_{11} & \widehat{\Phi}_{21} \\ \widehat{\Phi}_{12} & D_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix},$$

or, in the another form

$$\begin{cases} \left(D_{11}+\widehat{\Phi}_{11}\right) u + \widehat{\Phi}_{21} v = f, \\ \widehat{\Phi}_{12} u + D_{22} v = g, \end{cases} \tag{32}$$

up to the principal term. This system will be solved by excluding of unknowns. Namely, (it will be our first condition) we require first that the operator standing in the upper left corner of matrix (28) is elliptic in the sense of the paper [5], [6], [7].

We remark that operators of the type $\widehat{\Phi}_{11}$, localized (up to the infinitely smoothing operators in a neighbourhood of the manifold X) can be treated as pseudodifferential operators on the manifold X with operator-valued coefficients in the space of operators of special type in fibers of a tubular neighbourhood of X in M (see also the last Section). These coefficients are simply integral operators in the variables τ which are dual to the variables transversal to X with respect to the Fourier transform. Later on, Fourier integral operators form a module over the ring of pseudodifferential operators, and the operator

$$D_{11}+\widehat{\Phi}_{11}$$

can be rewritten in the form

$$D_{11}+\widehat{\Phi}_{11} = D_{11}\left(1+D_{11}^{-1}\widehat{\Phi}_{11}\right)$$

Hence, it is clear that the invertibility[5] of this operator is reduced to the two requirements:

1) the operator D_{11} is invertible on the manifold M;

2) the operator $\left(1+D_{11}^{-1}\widehat{\Phi}_{11}\right)$ is invertible on the manifold M.

We remark that the first condition means that the operator D_{11} is an elliptic one. Let us consider the second condition in more detail. Since, as we had mentioned above, the operators of the type $\widehat{\Phi}_{11}$ form a module over the ring of pseudodifferential operators, the operator $D_{11}^{-1}\widehat{\Phi}_{11}$ is also the operator of the type $\widehat{\Phi}_{11}$. Later on, due to Corollary 2.2 the inverse to $\left(1+D_{11}^{-1}\widehat{\Phi}_{11}\right)$ can be found in the form $\left(1+\widehat{\Phi}_1\right)$. Due to Theorem 2.4 we obtain the following equation

$$\Phi_1(\tau,\tau') + \left(\frac{1}{2\pi}\right)^{\nu} \int \Phi(\tau,\tau'')\Phi_1(\tau'',\tau')\,d\tau' = -\Phi(\tau,\tau') \tag{33}$$

[5]Here and below by invertibility we mean invertibility up to compact operators.

for the symbol $\Phi_1(\tau,\tau')$ of the operator $\widehat{\Phi}_1$. This equation must be fulfilled outside some compact neighbourhood of the zero section in T^*X. Actually, due to equation (33) the composition

$$\left(1+D_{11}^{-1}\widehat{\Phi}_{11}\right)\left(1+\widehat{\Phi}_1\right)$$

have the form $\left(1+\widehat{\Phi}_2\right)$ where $\widehat{\Phi}_2$ is an operator of the type $\widehat{\Phi}_{11}$ having zero order with principal symbol vanishing outside a compact neighbourhood of the zero section in T^*X. The symbol of this operator a priory belongs to the space $\mathrm{smbl}_{11}(m,k,l)$ with $m+k+l+\nu=0$. However, since this symbol has a compact support in fibers of T^*X, it belongs also to the space $\mathrm{smbl}_{11}(m,k-\epsilon,l)$ for some $\epsilon>0$. Hence, the corresponding operator $\widehat{\Phi}_2$ has negative Sobolev order.

To understand what the condition of solvability of equation (33) means, we shall show that this equation is equivalent to the classical Fredholm integral equation of the second kind with an integrable kernel. Actually, integral equation (33) must be solvable in the class of symbols $\Phi_1(\tau,\tau')$ which satisfy the estimates

$$|\Phi_1(\tau,\tau')|\le C\left(1+|\tau|^2\right)^{\frac{m}{2}}\left(1+|\tau'|^2\right)^{\frac{l}{2}}.$$

Hence, if we denote

$$\widetilde{\Phi}(\tau,\tau')=\frac{\Phi(\tau,\tau')}{\left(1+|\tau|^2\right)^{\frac{m}{2}}\left(1+|\tau'|^2\right)^{\frac{l}{2}}},$$

$$\widetilde{\Phi}_1(\tau,\tau')=\frac{\Phi_1(\tau,\tau')}{\left(1+|\tau|^2\right)^{\frac{m}{2}}\left(1+|\tau'|^2\right)^{\frac{l}{2}}},$$

the *bounded* functions corresponding to the functions $\Phi(\tau,\tau')$ and $\Phi_1(\tau,\tau')$, then we come to the classical integral equation

$$\widetilde{\Phi}_1(\tau,\tau')+\left(\frac{1}{2\pi}\right)^{\nu}\int\left(1+|\tau''|^2\right)^{\frac{m+l}{2}}\widetilde{\Phi}(\tau,\tau'')\widetilde{\Phi}_1(\tau'',\tau')\,d\tau'=-\widetilde{\Phi}(\tau,\tau'),$$

which has an *integrable kernel* due to the inequality $l+m<-\nu$ obtained from estimates of Theorem 2.5 taken for $s_i=\sigma_i$, $i=1,2$ (see also Corollary 2.1). Thus, this equation, which must to be solved in the space of bounded functions, is a usual Fredholm integral equation of the second kind.

Thus, there exist a regulizer $\left(D_{11}+\widehat{\Phi}_{11}\right)^{-1}$ for this operator which has the same form:

$$\left(D_{11}+\widehat{\Phi}_{11}\right)^{-1}=D_{11}^{-1}+\widehat{\Phi}'_{11}$$

where the operator $\widehat{\Phi}'_{11}$ is of the type (25) with the symbol from the symbol space $\mathrm{Smbl}_{11}(-l_{11}, -k_{11}, -m_{11})$ if the operator $\widehat{\Phi}_{11}$ has the symbol from $\mathrm{Smbl}_{11}(m_{11}, k_{11}, l_{11})$. Therefore, from the first equation (32) we can derive u via v:

$$u = \left(D_{11}^{-1} + \widehat{\Phi}'_{11}\right)\left(f - \widehat{\Phi}_{21} v\right). \tag{34}$$

Substituting the latter relation in the second equation (32) we obtain the equation for the unknown function v:

$$\widehat{\Phi}_{12}\left(D_{11}^{-1} + \widehat{\Phi}'_{11}\right)\left(f - \widehat{\Phi}_{21} v\right) + D_{22} v = g,$$

or, in another form

$$\left(D_{22} - \widehat{\Phi}_{12}\left(D_{11}^{-1} + \widehat{\Phi}'_{11}\right)\widehat{\Phi}_{21}\right) v = g - \widehat{\Phi}_{12}\left(D_{11}^{-1} + \widehat{\Phi}'_{11}\right) f. \tag{35}$$

Due to Theorem 2.4 the operator acting on the unknown v in the left-hand side of the latter relation is a pseudodifferential operator whose symbol can be easily computed ba means of the mentioned theorem. We denote this operator by

$$\Delta \overset{\mathrm{def}}{=} D_{22} - \widehat{\Phi}_{12}\left(D_{11}^{-1} + \widehat{\Phi}'_{11}\right)\widehat{\Phi}_{21}. \tag{36}$$

The second condition is the invertibility of operator (29) (which is equivalent to its ellipticity as a pseudodifferential operator on X). Under such a condition we can find the function v from (35)

$$v = -\Delta^{-1}\widehat{\Phi}_{12}\left(D_{11}^{-1} + \widehat{\Phi}'_{11}\right) f + \Delta^{-1} g, \tag{37}$$

and substitute this expression in (34)

$$\begin{aligned} u = &\left(\left(D_{11}^{-1} + \widehat{\Phi}'_{11}\right) + \left(D_{11}^{-1} + \widehat{\Phi}'_{11}\right)\widehat{\Phi}_{21}\Delta^{-1}\widehat{\Phi}_{12}\left(D_{11}^{-1} + \widehat{\Phi}'_{11}\right)\right) f \\ &- \left(D_{11}^{-1} + \widehat{\Phi}'_{11}\right)\widehat{\Phi}_{21}\Delta^{-1} g. \end{aligned} \tag{38}$$

Equations (37) and (38) show that the regulizer for operator (29) has the form

$$\begin{pmatrix} \widehat{R}_{11}\left(1 + \widehat{\Phi}_{21}\Delta^{-1}\widehat{\Phi}_{12}\widehat{R}_{11}\right) & -\widehat{R}_{11}\widehat{\Phi}_{21}\Delta^{-1} \\ -\Delta^{-1}\widehat{\Phi}_{12}\widehat{R}_{11} & \Delta^{-1} \end{pmatrix}, \tag{39}$$

where $\widehat{R}_{11} = \left(D_{11}^{-1} + \widehat{\Phi}'_{11}\right)$ is an inverse to $D_{11} + \widehat{\Phi}_{11}$.

Now we can formulate the main statement of this Section. First of all, we introduce the following definition.

Definition 2.1 The operator morphism (29) is called to be *elliptic morphism* if:

1) The operator $D_{11} + \widehat{\Phi}_{11}$ is an invertible one on the manifold M;

2) The (pseudodifferential) operator (36) is an invertible operator on the manifold X.

We remark that, in the case when the operator $\widehat{\Phi}_{11}$ vanishes (as it takes place, for example, for morphisms associated with Sobolev problems) part 1) of the above Definition is reduced simply to the ellipticity of the pseudodifferential operator D_{11} as well as operator (36).

The finiteness theorem is stated now as follows.

Theorem 2.6 *If operator morphism* (29) *is elliptic, then it possesses the Fredholm property, that is, the kernel and the cokernel of this morphism are finite-dimensional spaces.*

Proof. One can easily verify by the straightforward computations that operator morphism (39) is a two-sided regulizer for operator morphism (28) (these computations are quite simple but a little bit long; we leave them to the reader). This completes the proof.

In essence, we had shown that elliptic morphisms are exactly those which are invertible in the above constructed algebra up to operators of lower order.

2.3 Index of elliptic morphisms

The last aim of this section is to construct the index formula for elliptic morphism. Let

$$\mathcal{A} = \begin{pmatrix} D_{11} + \widehat{\Phi}_{11} & \widehat{\Phi}_{21} \\ \widehat{\Phi}_{12} & D_{22} \end{pmatrix} \tag{40}$$

be an elliptic morphism. To compute its index we construct a homotopy which will connect the general elliptic morphism with the diagonal one. The mentioned homotopy is as follows:

$$\begin{pmatrix} D_{11} + \widehat{\Phi}_{11} & \sqrt{1-t}\widehat{\Phi}_{21} \\ \sqrt{1-t}\widehat{\Phi}_{12} & D_{22} - t\widehat{\Phi}_{12}\left(D_{11}^{-1} + \widehat{\Phi}'_{11}\right)\widehat{\Phi}_{21} \end{pmatrix}, \quad t \in [0,1]. \tag{41}$$

It is easy to verify that:

1) morphism (41) coincides with the initial morphism (40) for $t = 0$ and is a diagonal one for $t = 1$;

2) the operator of the type (36) involved in the definition of ellipticity of morphism (40) does not depend on t and coincide with that for the initial morphism.

Thus, we see that the homotopy (41) is a homotopy in the class of elliptic morphisms connecting (40) with the following diagonal morphism

$$\begin{pmatrix} D_{11}+\widehat{\Phi}_{11} & 0 \\ 0 & D_{22}-\widehat{\Phi}_{12}\left(D_{11}^{-1}+\widehat{\Phi}'_{11}\right)\widehat{\Phi}_{21} \end{pmatrix}.$$

Thus, we come to the following statement.

Theorem 2.7 *The index of any elliptic morphism* (40) *is equal to*

$$\begin{aligned} \operatorname{index}\mathcal{A} &= \operatorname{index} D_{11} + \operatorname{index}\left(1+D_{11}^{-1}\widehat{\Phi}_{11}\right) \\ &+ \operatorname{index}\left(D_{22}-\widehat{\Phi}_{12}\left(D_{11}^{-1}+\widehat{\Phi}'_{11}\right)\widehat{\Phi}_{21}\right). \end{aligned}$$

We remark that the first and the last summands in the latter expression are indices of pseudodifferential operators on the smooth compact manifolds without boundary M and X correspondingly. The second summand is the index of a pseudodifferential operator on the manifold X with coefficients in the algebra of operators of special form (see discussion preceding formula (33) above).

3 The Sobolev problem

In this section we apply the above developed theory to the investigation of the Sobolev problem (18). As in Section 1 it was shown that this problem is equivalent to the operator morphism of the form

$$\begin{pmatrix} D & i_*^L \\ i^*B & 0 \end{pmatrix} \tag{42}$$

(see (19), (21); here D is a pseudodifferential operator of order m and B is a pseudodifferential operator of order b), it suffices to investigate the finiteness properties of morphisms of this particular form. However, to do this we need to make some (though quite simple and natural) modification of the above theory. The matter is that operators $\widehat{\Phi}_{21}$ and $\widehat{\Phi}_{12}$ corresponding to morphisms of the form (42) are, in general, *matrix* operators. Actually, the operator $\widehat{\Phi}_{21}$ corresponding to (42) for $L \geq 1$ is a string of Fourier integral operators of the type (26) with the corresponding symbol string (τ^α) with α running over the set of all natural multiindices with $|\alpha| \leq L$.

Let us write down the conditions under which morphism (42) determines a continuous mapping

$$\begin{pmatrix} D & i_*^L \\ i^*B & 0 \end{pmatrix} : \begin{pmatrix} H^{s_1}(M) \\ H^{s_2}(X) \end{pmatrix} \to \begin{pmatrix} H^{\sigma_1}(M) \\ H^{\sigma_2}(X) \end{pmatrix}. \tag{43}$$

Since we have for the considered problem

$$\begin{aligned}
&\Phi_{11}(x,p,\tau,\tau')=0,\\
&\Phi_{21}(x,p,\tau)=(\tau^{\alpha}),\\
&\Phi_{12}(x,p,\tau')=B(x,p,\tau')
\end{aligned} \tag{44}$$

(where $B(x,p,\tau')$, as well as the corresponding operator $\widehat{\Phi}_{12}$ evidently must be treated as a column consisting of pseudodifferential operators B_j of orders b_j, $j=1,\dots,J$), we see that:

1) In (43) the numbers s_2 and σ_2 are not numbers[6] but columns:

$$s_2=(s_2^{\alpha},\ |\alpha|\le L),\quad \sigma_2=\left(\sigma_2^j,\ j=1,\dots,J\right)$$

where J is a number of 'boundary ' conditions in problem (18), that is, the height of the column $\widehat{\Phi}_{12}$.

2) All the equalities and inequalities in the statement of Theorem 2.5 including numbers m_{11}, k_{11}, and l_{11} must be cancelled out, as well as those including l.

3) All the equalities and inequalities in the statement of Theorem 2.5 including numbers m_{21}^{α} and k_{21}^{α} must be repeated for all values of α with $|\alpha|\le L$.

4) All the equalities and inequalities in the statement of Theorem 2.5 including numbers k_{12}^j and l_{12}^j must be repeated for all values of j with $1\le j\le J$.

We remark that due to the particular form of morphism (42) one has

$$m_{21}^{\alpha}=|\alpha|,\ k_{21}^{\alpha}=0,\ k_{12}^j=0,\ l_{12}^j=b_j$$

where b_j are orders of the operators B_j, $j=1,\dots,J$. Therefore, the system of equalities (30) and inequalities (31) becomes in this concrete case

$$\left\{\begin{aligned}
&\sigma_1=s_1-m,\\
&\sigma_1=s_2^{\alpha}-|\alpha|-\tfrac{\nu}{2},\ |\alpha|\le L,\\
&\sigma_2^j=s_1-b_j-\tfrac{\nu}{2},\ 1\le j\le J,
\end{aligned}\right. \tag{45}$$

and

$$\left\{\begin{aligned}
&\tfrac{\nu}{2}+b_j<s_1,\ 1\le j\le J,\\
&s_2^{\alpha}<0,\ |\alpha|\le L.
\end{aligned}\right. \tag{46}$$

Using equalities (45) one can exclude all indices of the Sobolev spaces involved to problem (18) expressing them via $s=s_1$:

$$\left\{\begin{aligned}
&\sigma_1=s-m,\\
&s_2^{\alpha}=s-m+|\alpha|+\tfrac{\nu}{2},\ |\alpha|\le L,\\
&\sigma_2^j=s-b_j-\tfrac{\nu}{2},\ j\le J.
\end{aligned}\right.$$

[6]In fact, we must nesessarily work in the situation of Douglis – Nirenberg systems [18].

Substituting these expressions to the system of inequalities (46), we obtain the final bounds for s such that the mapping (43) is a bounded operator:

$$\frac{\nu}{2} + \max_{1\le j\le J} b_j < s < m - L - \frac{\nu}{2}.$$

Now we can write down the ellipticity conditions for Sobolev problem (18). Since, as we have already mentioned above, the operator $\widehat{\Phi}_{11}$ is identically zero, the first ellipticity condition in Definition 2.1 is reduced to the condition of ellipticity of the pseudodifferential operator D. Let us write down the second condition. It claims, that the operator

$$\Delta \stackrel{def}{=} -\widehat{\Phi}_{12} D^{-1} \widehat{\Phi}_{21} \tag{47}$$

must be an elliptic pseudodifferential operator on the manifold X. Here $\widehat{\Phi}_{12}$ and $\widehat{\Phi}_{21}$ are operators of the type (26), (27) with symbols given by (44). In particular, it means that operator (47) (which is a *matrix* operator) must be *quadratic*. This means, that the number J of the 'boundary conditions' must be equal to the number of 'coboundary conditions', that is, to the number of the multiindices α such that $|\alpha| \le L$.

Now Definition 2.1 of ellipticity of an operator morphism leads us to the following refined formulation of the notion of ellipticity for a Sobolev problem (see [4]).

Definition 3.1 Sobolev problem (18) is called to be *elliptic* if

1) The operator D, involved in this problem is an elliptic pseudodifferential operator on the manifold M.

2) The operator given by (47) is an elliptic operator on the manifold X.

The following statement is a direct consequence of Theorem 2.6.

Theorem 3.1 *If Sobolev problem* (18) *is elliptic, then this problem is a Fredholm one, that is, it has finite-dimensional kernel and cokernel.*

Let us discuss now the operator statement of a Sobolev problem. If

$$\begin{cases} Du \equiv f \mod H^{s-m}(M, X; E_2), \\ i^* Bu = g, \end{cases}$$

is a Sobolev problem, then there exists an evident operator

$$\mathcal{S} \,:\, H^s(M, E_1) \to \left(H^{s-m}(M, E_2) / H^{s-m}(M, X; E_2)\right) \oplus H^{s-b-\nu/2}(M, F), \tag{48}$$

corresponding to this problem, where b is the order of the operator B. From the latter formula one can construct the *conjugated* operator and, hence, to write down the problem which is conjugated to the considered Sobolev problem.

Namely, if one takes into account that the space $H^{s-m}(M,X;E_2)$ is simply an image of the coboundary operator i_*^L, then it becomes clear that the conjugated operator is

$$\mathcal{S}^* \ : \ \mathrm{Ker}\,(i_L^*) \oplus H^{-s+b+\nu/2}(M,F) \ \to \ H^{-s}(M,E_1)\,,$$

and, hence, the conjugated problem is

$$\begin{cases} Du + Bi_*v = f, \\ i_L^* u = 0, \end{cases}$$

where i_L^* is a composition of taking a jet of order L in the transversal direction and elementary boundary operator. It is easy to see also that the (almost) inverse to operator (48) has the form

$$S^{-1} = \left(D^{-1}\left(1 + i_*^L\left(-i^*BD^{-1}i_*^L\right)^{-1} i^*BD^{-1}\right), D^{-1}i_*^L\left(-i^*BD^{-1}i_*^L\right)^{-1}\right). \qquad (49)$$

The fact that any Sobolev problem can be reduced to the operator morphism together with the result of Theorem 2.7 leads us to the following statement.

Theorem 3.2 *Let*

$$\begin{cases} Du \equiv f \mod H^{s-m}(M,X;E_2)\,, \\ i^*Bu = g, \end{cases}$$

be an elliptic Sobolev problem. Then the following formula takes place

$$\mathrm{index}\,\mathcal{S} = \mathrm{index}\,D + \mathrm{index}\,\Delta$$

where the operator Δ *given, in general by the formula* (47) *can be here written down in the form*

$$\Delta = -i^*BD^{-1}i_*^L.$$

Thus, we have reduced the computation of the index for a Sobolev problem to the computation of two indices of pseudodifferential operators on smooth compact manifold without boundary.

4 Concluding remarks

1. The general form of operator morphism

$$\begin{pmatrix} D_{11} + \widehat{\Phi}_{11} & \widehat{\Phi}_{21} \\ \widehat{\Phi}_{12} & D_{22} \end{pmatrix}$$

shows that there exists a generalization of the notion of Sobolev problem for the operators of the form $D_{11} + \widehat{\Phi}_{11}$. We shall not consider here this new class of Sobolev problems in detail, but shall only remark that the obtained class is a class of non-local problems. This fact follows from a *non-local* nature of the operator $\widehat{\Phi}_{11}$. It is important to note that such class of Sobolev problems does not lead to the further extension of the class of operator morphisms. Thus, the class of Sobolev problems of the mentioned general kind is closed not only with respect to the conjugation, but also with respect to taking an almost inverse operator (see formula (49) above).

2. From the algebraic point of view the class of operators of the form

$$D_{11} + \widehat{\Phi}_{11}$$

is an extension of the class of pseudodifferential operators with the help of (non pseudodifferential) operators of special form. Such kind of an extension (associated with the embedding $i : X \hookrightarrow M$ is a particular case of more general construction of extension of algebra of pseudodifferential operators (associated not only with the embedding but also with the pointed bundle $\pi : M \to X$) was introduced by authors in their recent papers [5], [6], [7]. The algebra of operator morphisms constructed on the base of this extension as well as the theory of the corresponding 'Sobolev problems' gives a new interesting class of non-local problems (cf. with the problems in the above cited papers).

3.Taking into account the importance of the above introduced operators $\widehat{\Phi}_{ij}$, we present here one more treatment of these operators which is not connected with the Lagrangian uniformization. This interpretation is based on the fact that, as it was already mentioned above, all these operators are concentrated on the submanifold X. The natural question arises: can one interpret these operators (which are pseudodifferential operators on M with non regular symbols) as pseudodifferential operators of more general nature on the manifold X. The answer to this question is 'yes' if one considers the category of pseudodifferrential operators on X acting in sections of infinite-dimensional bundles. This interpretation occurs to be useful for clarifying the ellipticity conditions of Subsection 2.2 where, in essence, the ellipticity of operators acting in sections of infinite-dimensional bundles was considered. Let us illustrate this on the example of the operator $\widehat{\Phi}_{11}$. As we have mentioned above, this operator has the form

$$\widehat{\Phi}_{11} f = \left(\frac{1}{2\pi}\right)^{n+\nu} \int e^{i[\tau t + px]} \Phi(x, p, \tau, \tau') \widetilde{f}(p, \tau')\, dp d\tau d\tau'$$

(see formula (25) above). This operator can be represented in the form of pseudodifferential operator on the manifold X:

$$\widehat{\Phi}_{11} f = \left(\frac{1}{2\pi}\right)^{n} \int e^{ixp} \widehat{K}(x, p) \widetilde{f}(p)\, dp$$

with the symbol $\widehat{K}(x,p)$ at each point (x,p) is an operator of the form

$$\widehat{K}(x,p) = F_{\tau\to t} K_{(x,p)} F_{t\to\tau}$$

where

$$\left(K_{(x,p)}f\right)(\tau) = \left(\frac{1}{2\pi}\right)^{\nu} \int \Phi_{(x,p)}(\tau,\tau') f(\tau')\, d\tau'$$

and $\Phi_{(x,p)}(\tau,\tau') = \Phi(x,p,\tau,\tau')$.

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