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MULTI-SENSITIVITY, LYAPUNOV NUMBERS AND ALMOST AUTOMORPHIC MAPS

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Abstract. In this paper we study several stronger forms of sensitivity for continuous surjective selfmaps on compact metric spaces and relations between them. The main result of the paper is an analog of the Auslander-Yorke dichotomy theorem, which states that a minimal system is either multi-sensitive or an almost one-to-one extension of its maximal equicontinuous factor. For minimal dynamical systems, we also show that all notions of thick sensitivity, multi-sensitivity and thickly syndetical sensitivity are equivalent, and all of them are much stronger than sensitivity.

1. Introduction

Throughout this paper (X, T) denotes a *topological dynamical system*, where X is a compact metric space with metric ϱ and $T : X \to X$ is a continuous surjective map. If X is a singleton then we call (X, T) *trivial*.

The notion of sensitivity (sensitive dependence on initial conditions) was first used by Ruelle [31]. According to the works by Guckenheimer [20], Auslander and Yorke [7] a dynamical system (X, T) is called *sensitive* if there exists a positive δ such that for every $x \in X$ and every neighborhood U_x of x, there exist $y \in U_x$ and a nonnegative integer n with $\varrho(T^n(x), T^n(y)) > \delta$.

Recently several authors studied different properties related to sensitivity (cf. [1], [5], [30], [22]). The following proposition holds according to [5].

Proposition 1.1. *The following conditions are equivalent*:

1. (X, T) *is sensitive.*
2. *There exists a positive δ such that for every $x \in X$ and every neighborhood U_x of x, there exists $y \in U_x$ with* $\limsup_{n\to\infty} \varrho(T^n(x), T^n(y)) > \delta$.
3. *There exists a positive δ such that in any opene*[1] *U in X there are $x, y \in U$ and a nonnegative integer n with $\varrho(T^n(x), T^n(y)) > \delta$.*
4. *There exists a positive δ such that in any opene $U \subset X$ there are $x, y \in U$ with* $\limsup_{n\to\infty} \varrho(T^n(x), T^n(y)) > \delta$.

According to these properties were defined the following Lyapunov numbers [25] (here we set $\sup \varnothing = 0$ by convention):

$$\mathbb{L}_r = \sup\{\delta : \text{ for every } x \in X \text{ and every open neighborhood } U_x \text{ of } x \text{ there exist } y \in U_x \text{ and a nonnegative integer } n \text{ with } \varrho(T^n(x), T^n(y)) > \delta\};$$

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[1] Because we so often have to refer to open, nonempty subsets, we will call such subsets *opene*.

$$
\begin{aligned}
\overline{\mathbb{L}}_r &= \sup\{\delta : \text{ for every } x \in X \text{ and every open neighborhood } U_x \text{ of } x \text{ there} \\
&\qquad \text{exists } y \in U_x \text{ with } \limsup_{n\to\infty} \varrho(T^n(x), T^n(y)) > \delta\}; \\
\mathbb{L}_d &= \sup\{\delta : \text{ in any opene } U \subset X \text{ there exist } x, y \in U \text{ and there is} \\
&\qquad \text{a nonnegative integer } n \text{ with } \varrho(T^n(x), T^n(y)) > \delta\}; \\
\overline{\mathbb{L}}_d &= \sup\{\delta : \text{ in any opene } U \subset X \text{ there exist } x, y \in U \text{ with} \\
&\qquad \limsup_{n\to\infty} \varrho(T^n(x), T^n(y)) > \delta\}.
\end{aligned}
$$

So, various definitions of sensitivity, formally give us different Lyapunov numbers – quantitative measures of these sensitivities. Nevertheless, as was shown in [25], for topologically weakly mixing minimal systems all these Lyapunov numbers are the same.

Some another way to measure the sensitivity of a system, by checking how large is the set of nonnegative integers for which the sensitivity still happens, was initiated by Moothathu in [30]. This is the main subject of this paper.

Let $\mathcal{S}$ be a subset of the set of all natural numbers (positive integers) $\mathbb{N}$. $\mathcal{S}$ is *thick* if for each $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that $\{n_k, n_k+1, \dots, n_k+k\} \subset \mathcal{S}$. $\mathcal{S}$ is *syndetic* if there exists $m \in \mathbb{N}$ such that $\mathcal{S} \cap \{n, n+1, \dots, n+m\} \neq \varnothing$ for each $n \in \mathbb{N}$. $\mathcal{S}$ is *thickly syndetic* if $\{n \in \mathbb{N} : \{n, n+1, \dots, n+k\} \subset \mathcal{S}\}$ is syndetic for each $k \in \mathbb{N}$. $\mathcal{S}$ is *cofinite* if $\mathcal{S} \supset \{m, m+1, m+2, \dots\}$ for some $m \in \mathbb{N}$. Observe that each syndetic set and any thick set has a nonempty intersection, which is called a *piecewise syndetic* set.

Let $\delta > 0$. For an opene $U \subset X$ define

$$N_T(U, \delta) = \{n \in \mathbb{N} : \text{ there are } x_1, x_2 \in U \text{ such that } \varrho(T^n x_1, T^n x_2) > \delta\}.$$

It is easy to see from Proposition 1.1 that (X, T) is sensitive iff $N_T(U, \delta)$ is infinite for some δ and every opene set $U \subset X$.

Following [30] and [29], recall the following definitions of some stronger versions of sensitivity. A topological dynamical systems (X, T) is called

(1) *thickly sensitive* if there exists $\delta > 0$ such that for any opene $U \subset X$, $N_T(U, \delta)$ is thick;
(2) *thickly syndetically sensitive* if there exists $\delta > 0$ such that for any opene $U \subset X$, $N_T(U, \delta)$ is thickly syndetic;
(3) *cofinitely sensitive* if there exists $\delta > 0$ such that for any opene $U \subset X$, $N_T(U, \delta)$ is cofinite;
(4) *multi-sensitive* if there exists $\delta > 0$ such that for any positive integer k and any opene $U_1, \dots, U_k \subset X$, $\bigcap_{i=1}^{k} N_T(U_i, \delta) \neq \varnothing$.

Inspired by [25], we may introduce the following *Lyapunov numbers*

$$
\begin{aligned}
\mathbb{L}_{m,r} &= \sup\{\delta : \text{ for any positive integer } k, \text{ for every } x_i \in X \text{ and any open} \\
&\qquad \text{neighborhood } U_i \text{ of } x_i,\ i = 1, \dots, k, \text{ there exist } y_i \in U_i \text{ and} \\
&\qquad \text{a nonnegative integer } n \text{ with } \min_{1 \le i \le k} \varrho(T^n x_i, T^n y_i) > \delta\}; \\
\overline{\mathbb{L}}_{m,r} &= \sup\{\delta : \text{ for any positive integer } k, \text{ for every } x_i \in X \text{ and any open} \\
&\qquad \text{neighborhood } U_i \text{ of } x_i,\ i = 1, \dots, k, \text{ there exist } y_i \in U_i \\
&\qquad \text{with } \limsup_{n\to\infty} \min_{1 \le i \le k} \varrho(T^n x_i, T^n y_i) > \delta\};
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{L}_{m,d} &= \sup\{\delta: \text{ for any positive integer } k \text{ and any opene } U_i\subset X, \text{ there} \\
&\qquad \text{exist } x_i,y_i\in U_i \text{ and a nonnegative integer } n \text{ with} \\
&\qquad \min_{1\le i\le k}\varrho(T^n x_i,T^n y_i)>\delta\}; \\
\overline{\mathbb{L}}_{m,d} &= \sup\{\delta: \text{ for any positive integer } k \text{ and any opene } U_i\subset X, \text{ there} \\
&\qquad \text{exist } x_i,y_i\in U_i \text{ with } \limsup_{n\to\infty}\min_{1\le i\le k}\varrho(T^n x_i,T^n y_i)>\delta\}.
\end{aligned}
$$

As we show in Section 2, these new Lyapunov numbers $\mathbb{L}_{m,r},\overline{\mathbb{L}}_{m,r},\mathbb{L}_{m,d}$ and $\overline{\mathbb{L}}_{m,d}$ are all related to each other. In particular, for a nontrivial weakly mixing system $\mathbb{L}_{m,d}=\overline{\mathbb{L}}_{m,d}=\mathrm{diam}(X)$ and $\mathbb{L}_{m,r}=\overline{\mathbb{L}}_{m,r}>0$ (Proposition 2.4); for a system with a dense set of distal points $\mathbb{L}_{m,r}=\overline{\mathbb{L}}_{m,r}$ (Proposition 2.6). An analogue of Proposition 1.1 also holds for multi-sensitive systems (see Proposition 2.2).

The Lyapunov stability or, in another word, equicontinuity is the opposite side of sensitivity. Recall that a point $x\in X$ is called *Lyapunov stable* if for every $\varepsilon>0$ there exists a $\delta>0$ such that $\varrho(x,x')<\delta$ implies $\varrho(T^n x,T^n x')<\varepsilon$ for any $n\in\mathbb{N}$. This condition says exactly that the sequence of iterates $\{T^n: n\ge 0\}$ is equicontinuous at x. The system (X,T) is called *equicontinuous* if $\{T^n: n\ge 0\}$ is equicontinuous at any point of X. The well-known Auslander-Yorke dichotomy theorem states that a minimal dynamical system is either sensitive or equicontinuous [7] (see also [2]). Equicontinuity can be localized easily by introducing equicontinuity points. Later the Auslander-Yorke dichotomy theorem was refined in [3],[17]: a transitive system is either sensitive or almost equicontinuous (in the sense of containing some equicontinuity points).

Such a dichotomy can also be found in the study of stronger versions of sensitivity. By using Veech's characterization of equicontinuous structure relation of a system [33, Theorem 1.1], we show that an invertible minimal system is either multi-sensitive or almost automorphic (Corollary 3.2). Recall that the concept of almost automorphy, as a generalization of almost periodicity, was first introduced by Bochner in 1955 (in the context of differential geometry [10]) and studied by many authors starting from [11], [32], [34].

We may also measure the equicontinuity of (a point in) a system by checking how large is the set of nonnegative integers where equicontinuity happens. More precisely, we introduce the concept of syndetically equicontinuous points. It turns out that this new notion of local equicontinuity is very useful. In fact, the refined Auslander-Yorke dichotomy theorem [3], [17] also holds in our setting (Theorem 5.4): a transitive system is either thickly sensitive or containing syndetically equicontinuous points. Observe that for transitive systems thick sensitivity is equivalent to multi-sensitivity (Proposition 4.1). Moreover, any nonminimal M-system is thickly syndetically sensitive and for minimal dynamical systems all notions of thick sensitivity, multi-sensitivity and thickly syndetical sensitivity are equivalent and much stronger than sensitivity (Theorem 4.6). We also present three diagrams, which illustrate a comparison between stronger forms of sensitivity for dynamical systems.

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2. Lyapunov numbers

In this section we are interested in relationships between those introduced Lyapunov numbers $\mathbb{L}_{m,r}, \overline{\mathbb{L}}_{m,r}, \mathbb{L}_{m,d}$ and $\overline{\mathbb{L}}_{m,d}$.

It is easy to see that (X, T) is multi-sensitive iff $\mathbb{L}_{m,d} > 0$, and

$$\mathbb{L}_{m,d} \geq \mathbb{L}_{m,r} \geq \overline{\mathbb{L}}_{m,r} \text{ and } \mathbb{L}_{m,d} \geq \overline{\mathbb{L}}_{m,d} \geq \overline{\mathbb{L}}_{m,r}. \tag{2.1}$$

Moreover, we have the following

Lemma 2.1. $\mathbb{L}_{m,d} \leq 2\overline{\mathbb{L}}_{m,r}$.

Proof. We only need consider the case $\mathbb{L}_{m,d} > 0$. The proof follows [25, Proposition 2.1] and we provide it for completeness.

Let $\varepsilon > 0$ be small enough with $\mathbb{L}_{m,d} > 2\varepsilon$. Now, let $k \in \mathbb{N}$ and $x_i \in X$ with a neighborhood U_i for each $i = 1, \ldots, k$. Set $V_{0,1} = U_1, \ldots, V_{0,k} = U_k$. Take

$$n_0 \in \bigcap_{i=1}^{k} N_T(V_{0,i}, \mathbb{L}_{m,d} - \frac{\varepsilon}{2})$$

and then there exist $y_{0,1} \in V_{0,1}, \ldots, y_{0,k} \in V_{0,k}$ such that

$$\min_{1\leq i\leq k} \varrho(T^{n_0}x_i, T^{n_0}y_{0,i}) > \frac{\mathbb{L}_{m,d} - \varepsilon}{2}. \tag{2.2}$$

Moreover, we can choose open neighborhoods $V_{1,1}$ of $y_{0,1}$ (with $\overline{V_{1,1}} \subset V_{0,1}$), $\ldots$, $V_{1,k}$ of $y_{0,k}$ (with $\overline{V_{1,k}} \subset V_{0,k}$) such that

$$\max_{0\leq n\leq n_0} \max_{1\leq i\leq k} \operatorname{diam}(T^n V_{1,i}) \leq \frac{\varepsilon}{2}. \tag{2.3}$$

Again take

$$n_1 \in \bigcap_{i=1}^{k} N_T(V_{1,i}, \mathbb{L}_{m,d} - \frac{\varepsilon}{2})$$

and hence $n_1 > n_0$ by (2.3). We continue the process and define recursively (for each $m \geq 2$) open neighborhoods $V_{m,1}$ of some $y_{m-1,1}$ (with $\overline{V_{m,1}} \subset V_{m-1,1}$), $\ldots$, $V_{m,k}$ of some $y_{m-1,k}$ (with $\overline{V_{m,k}} \subset V_{m-1,k}$) and $n_m > n_{m-1}$ such that

$$\min_{1\leq i\leq k} \varrho(T^{n_{m-1}}x_i, T^{n_{m-1}}y_{m-1,i}) > \frac{\mathbb{L}_{m,d} - \varepsilon}{2} \tag{2.4}$$

and

$$\max_{0\leq n\leq n_{m-1}} \max_{1\leq i\leq k} \operatorname{diam}(T^n V_{m,i}) \leq \frac{\varepsilon}{2}, n_m \in \bigcap_{i=1}^{k} N_T(V_{m,i}, \mathbb{L}_{m,d} - \frac{\varepsilon}{2}). \tag{2.5}$$

Since by the construction, for each $i = 1, \ldots, k$, $\bigcap\limits_{m\geq 1} V_{m,i} \neq \varnothing$, we can take a point y_i from the intersection (and so $y_i \in U_i$). Directly from (2.4) and (2.5) we have

$$\limsup_{n\to\infty} \min_{1\leq i\leq k} \varrho(T^n x_i, T^n y_i) \geq \frac{\mathbb{L}_{m,d}}{2} - \varepsilon.$$

Thus the conclusion follows from the arbitrariness of $\varepsilon > 0$. $\square$

As a consequence, we have the following

Proposition 2.2. *The following conditions are equivalent:*

1. (X,T) *is multi-sensitive.*
2. *There exists a positive* δ *such that for any positive integer* k*, for every* $x_i \in X$ *and any open neighborhood* U_i *of* x_i*,* $i = 1,\dots,k$*, there exist* $y_i \in U_i$ *and a nonnegative integer* n *with* $\min_{1\le i\le k} \varrho(T^n x_i, T^n y_i) > \delta$.
3. *There exists a positive* δ *such that for any positive integer* k*, for every* $x_i \in X$ *and any open neighborhood* U_i *of* x_i*,* $i = 1,\dots,k$*, there exist* $y_i \in U_i$ *with* $\limsup_{n\to\infty} \min_{1\le i\le k} \varrho(T^n x_i, T^n y_i) > \delta$.
4. *There exists a positive* δ *such that for any positive integer* k *and any opene* $U_1,\dots,U_k \subset X$*, there exist* $x_i, y_i \in U_i$ *with* $\limsup_{n\to\infty} \min_{1\le i\le k} \varrho(T^n x_i, T^n y_i) > \delta$.

Recall that (X,T) is called (*topologically*) *transitive* if $N_T(U_1,U_2) = \{n \in \mathbb{N} : U_1 \cap T^{-n}U_2 \neq \varnothing\}$ for any opene subsets $U_1, U_2 \subset X$, and *weakly mixing* if $(X \times X, T \times T)$ is transitive. A point $x \in X$ is called *transitive* if its *orbit* $\mathrm{orb}_T(x) = \{T^n x : n = 0,1,2,...\}$ is dense in X. Denote by $\mathrm{Tran}(X,T)$ the set of all transitive points of (X,T). Since T is surjective, (X,T) is transitive iff $\mathrm{Tran}(X,T) \neq \varnothing$.

The system is called *minimal* if every point has a dense orbit or, equivalently, if $\mathrm{Tran}(X,T) = X$. In general, a subset A of X is called *invariant* if $TA = A$. If A is a closed, nonempty, invariant subset then $(A, T|_A)$ is called the associated *subsystem.* A *minimal subset* of X is a nonempty, closed, invariant subset such that the associated subsystem is minimal. Clearly, (X,T) is minimal iff it admits no proper, nonempty, closed, invariant subset. A point $x \in X$ is called a *minimal point* if it lies in some minimal subset. Since Zorn's Lemma implies that every closed, nonempty invariant set contains a minimal set. If (X,T) is a transitive system with a dense set of minimal points, then we call it an *M-system* [17].

Lemma 2.3. *Let* (X,T) *be a transitive system. Then* $\mathbb{L}_{m,d} = \overline{\mathbb{L}}_{m,d}$.

Proof. It suffices to show that $\mathbb{L}_{m,d} \le \overline{\mathbb{L}}_{m,d}$ in the case of $\mathbb{L}_{m,d} > 0$. Let $k \in \mathbb{N}$ and take opene $U_1,\dots,U_k \subset X$. Let $\delta > 0$ be small enough with $\mathbb{L}_{m,d} > \delta$. By the definition there exist $n \in \mathbb{N}$ and $x_i', y_i' \in U_i$ for each $i = 1,\dots,k$ with $\min\limits_{1\le i\le k} \varrho(T^n x_i', T^n y_i') > \delta$. Then, for each $i = 1,\dots,k$ we could find opene $x_i' \in V_i \subset U_i$ and $y_i' \in W_i \subset U_i$ such that both $\mathrm{diam}(T^n V_i)$ and $\mathrm{diam}(T^n W_i)$ are small enough, thus $\min\limits_{1\le i\le k} \mathrm{dist}(T^n V_i, T^n W_i) > \delta$.

Since (X,T) is transitive, take $z \in \mathrm{Tran}(X,T)$ and then choose $s_i, t_i \in \mathbb{N}$ with $T^{s_i} z \in V_i$ and $T^{t_i} z \in W_i$ for each $i = 1,\dots,k$. Observe that once $m \in \mathbb{N}$ such that $T^m z$ is sufficiently close to z then $T^{s_i+m} z \in V_i$ and $T^{t_i+m} z \in W_i$ for each $i = 1,\dots,k$, and hence $\min\limits_{1\le i\le k} \varrho(T^{n+s_i+m} z, T^{n+t_i+m} z) > \delta$. Since $z \in \mathrm{Tran}(X,T)$, clearly there are infinitely many $m_1 < m_2 < \dots$ in $\mathbb{N}$ such that each $T^{m_j} z$ is close enough to z, and hence we obtain $\overline{\mathbb{L}}_{m,d} \ge \delta$ by taking $x_i = T^{s_i} z \in U_i$ and $y_i = T^{t_i} z \in U_i$ for each $i = 1,\dots,k$, finishing the proof. $\square$

Clearly that any nontrivial weakly mixing system is multi-sensitive [30], and a classic result of Gottschalk states that $x \in X$ is minimal iff $N_T(x,U) = \{n \in \mathbb{N} : T^n x \in U\}$ is a syndetic set for any neighborhood U of x.

By the same proof of [25, Theorem 4.1] one has:

Proposition 2.4. *Let (X,T) be a nontrivial weakly mixing system. Then $\mathbb{L}_{m,d} = \overline{\mathbb{L}}_{m,d} = \operatorname{diam}(X)$ and $\mathbb{L}_{m,r} = \overline{\mathbb{L}}_{m,r} > 0$.*

Recall that $\mathcal{S} \subset \mathbb{N}$ is an *IP set* (the family of all IP sets we denote by $\mathcal{F}_{ip}$) if there exists $\{p_k : k \in \mathbb{N}\} \subset \mathbb{N}$ with $\{p_{i_1} + \cdots + p_{i_k} : k \in \mathbb{N} \text{ and } i_1 < \cdots < i_k\} \subset \mathcal{S}$, and is an *IP* set* if $\mathcal{S} \cap \mathcal{T} \neq \varnothing$ for each IP set $\mathcal{T} \subset \mathbb{N}$. It is easy to see that the intersection of an IP set and an IP* set is an infinite set. Remark that for an IP set $\mathcal{S} \subset \mathbb{N}$, $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ implies that either $\mathcal{S}_1$ or $\mathcal{S}_2$ is an IP set by Hindman's theorem (see for example [15, Theorem 8.12]), and from this it is not hard to see that the intersection of any finitely many IP* sets is an IP* set (see for example [2, Corollary 7.5]).

Lemma 2.5. *Let $\delta > 0$, k be a positive integer, and $x_i \in X$ with a neighborhood U_i for each $i = 1, \dots, k$. If $\mathbb{L}_{m,r} > \delta$, then*

$$\mathcal{N} = \left\{ n \in \mathbb{N} : \min_{1 \le i \le k} \varrho(T^n x_i, T^n y_i) > \delta \text{ for some } y_1 \in U_1, \dots, y_k \in U_k \right\} \in \mathcal{F}_{ip}.$$

Proof. By the assumption, $\mathcal{N} \neq \varnothing$ as $\delta < \mathbb{L}_{m,r}$. Now assume that

$$\mathcal{A} = \{p_{i_1} + \cdots + p_{i_j} : 1 \le i_1 < \cdots < i_j \le l\} \subset \mathcal{N}$$

for some $\{p_1, \dots, p_l\} \subset \mathbb{N}$ with $l \in \mathbb{N}$. We shall find $p_{l+1} \in \mathbb{N}$ such that $p_{l+1} + \mathcal{A}_0 \subset \mathcal{N}$ with $\mathcal{A}_0 = \{0\} \cup \mathcal{A}$, and then obtain the conclusion by induction.

Take $x_{s,i} \in X$ with $T^{p_1 + \cdots + p_l - s} x_{s,i} = x_i$ for each $s \in \mathcal{A}_0$ and any $i = 1, \dots, k$. Since $\delta < \mathbb{L}_{m,r}$, obviously we may choose $q_l > p_1 + \cdots + p_l$ and $y_{s,i} \in T^{-(p_1 + \cdots + p_l - s)} U_i$ for each $s \in \mathcal{A}_0$ and any $i = 1, \dots, k$ such that

$$\min_{s \in \mathcal{A}_0} \min_{1 \le i \le k} \varrho(T^{q_l} x_{s,i}, T^{q_l} y_{s,i}) > \delta. \tag{2.6}$$

Set $p_{l+1} = q_l - (p_1 + \cdots + p_l) \in \mathbb{N}$ and $x'_{s,i} = T^{p_1 + \cdots + p_l - s} y_{s,i} \in U_i$ for each $s \in \mathcal{A}_0$ and any $i = 1, \dots, k$. Then (2.6) means equivalently

$$\min_{s \in \mathcal{A}_0} \min_{1 \le i \le k} \varrho(T^{p_{l+1} + s} x_i, T^{p_{l+1} + s} x'_{s,i}) > \delta,$$

that is, $p_{l+1} + \mathcal{A}_0 \subset \mathcal{N}$, which finishes the proof. □

A pair of points $x \in X$ and $y \in X$ is called *proximal* if $\liminf_{n \to \infty} \varrho(T^n x, T^n y) = 0$. In this case each of points from the pair is said to be also *proximal* to another. We will say that a point $x \in X$ is *distal* if it is not proximal to any another point from the orbit closure $\overline{\operatorname{orb}_T(x)}$. Note that by [15, Theorem 9.11]: $x \in X$ is distal iff $N_T(x, U)$ is an IP* set for any neighborhood U of x (and hence any distal point is minimal); and for distal points $x_i \in X_i$ of the system $(X_i, T_i), i = 1, \dots, k$, point $(x_1, \dots, x_k) \in X_1 \times \cdots \times X_k$ is also distal of the system $(X_1 \times \cdots \times X_k, T_1 \times \cdots \times T_k)$.

Proposition 2.6. *If the system (X,T) contains a dense set of distal points. Then $\mathbb{L}_{m,r} = \overline{\mathbb{L}}_{m,r}$.*

Proof. By (2.1) and Lemma 2.1 we have $2\overline{\mathbb{L}}_{m,r} \ge \mathbb{L}_{m,r} \ge \overline{\mathbb{L}}_{m,r}$. Thus we only need prove $\mathbb{L}_{m,r} \le \overline{\mathbb{L}}_{m,r}$ in the case of $\mathbb{L}_{m,r} > 0$.

Let $\delta > 0$ be small enough with $\mathbb{L}_{m,r} > \delta$, and we take an open cover $\{V_1, \dots, V_p\}$ of X with $\max_{1 \le i \le p} \operatorname{diam}(V_i) < \delta$. Now let $k \in \mathbb{N}$ and $x_i \in X$ with a neighborhood U_i for each $i = 1, \dots, k$, and for each $s = (s_1, \dots, s_k) \in \{1, \dots, p\}^k$ we set

$$\mathcal{N}_s = \{n \in \mathbb{N} : T^n x_1 \in V_{s_1}, \dots, T^n x_k \in V_{s_k}\}.$$

Observe that

$$\mathcal{N}=\left\{n\in\mathbb{N}:\min_{1\le i\le k}\varrho(T^n x_i,T^n y_i)>\mathbb{L}_{m,r}-\delta \text{ for some } y_1\in U_1,\dots,y_k\in U_k\right\}$$

is an IP set by Lemma 2.5, and then $\mathcal{N}\cap\mathcal{N}_t$ is also an IP set for some $t\in\{1,\dots,p\}^k$, because $\mathcal{N}=\bigcup_{s\in\{1,\dots,p\}^k}(\mathcal{N}\cap\mathcal{N}_s)$. Choose $\{q_0,q_1,q_2,\dots\}\subset\mathbb{N}$ with $\{q_{i_1}+\cdots+q_{i_j}: j\in\mathbb{N}\text{ and }0\le i_1<\cdots<i_j\}\subset\mathcal{N}\cap\mathcal{N}_t$, and hence $q_0+\mathcal{T}\subset\mathcal{N}\cap\mathcal{N}_t$ for some $\mathcal{T}\in\mathcal{F}_{ip}$.

Since $q_0\in\mathcal{N}$, there exist $y_i\in U_i$ for each $i=1,\dots,k$ such that

$$\min_{1\le i\le k}\varrho(T^{q_0}x_i,T^{q_0}y_i)>\mathbb{L}_{m,r}-\delta. \tag{2.7}$$

Note that the set of distal points is dense in X, we may assume that all points $y_1,\dots,y_k$ are distal. Then all of $T^{q_0}y_1,\dots,T^{q_0}y_k$ (and hence $(T^{q_0}y_1,\dots,T^{q_0}y_k)$) are also distal points. In particular,

$$\mathcal{M}=\left\{n\in\mathbb{N}:\max_{1\le i\le k}\varrho(T^{q_0}y_i,T^{q_0+n}y_i)<\delta\right\}$$

is an IP* set. Thus $\mathcal{M}\cap\mathcal{T}\neq\varnothing$, which is in fact an infinite set. Observing (2.7), it is easy to check from the construction that $\varrho(T^{q_0+r}x_i,T^{q_0+r}y_i)>\mathbb{L}_{m,r}-3\delta$ for each $r\in\mathcal{M}\cap\mathcal{T}$ and any $i=1,\dots,k$ (as $r\in\mathcal{M}$ and $q_0,q_0+r\in\mathcal{N}_t$). Then the conclusion follows from the arbitrariness of $\delta>0$. □

As will be shown in Example 4.2, we can not require $\mathbb{L}_{m,r}>0$ under the assumption of Proposition 2.6 even for a minimal system with positive topological entropy.

3. Dichotomy of multi-sensitivity for minimal systems

The Auslander-Yorke dichotomy theorem states that a minimal dynamical system is either sensitive or equicontinuous (see [2], [3], [7], [17]). The main goal of this section is to prove an analog of the Auslander-Yorke theorem for multi-sensitivity (see Theorem 3.1), which is the main result of this paper.

A continuous map $\phi:X\to Y$ is called *almost open* if $\phi(U)$ has a nonempty interior in Y for any opene $U\subset X$. Recall that if a dynamical system (X,T) is minimal then the map $T:X\to X$ is almost open ([27]).

Let (X,T) and (Y,S) be topological dynamical systems. By a *factor map* $\pi:(X,T)\to(Y,S)$ we mean that $\pi:X\to Y$ is a continuous surjection with $\pi\circ T=S\circ\pi$. In this case, we also call (X,T) to be an *extension* of (Y,S) and (Y,S) to be a *factor* of (X,T), sometimes we also call $\pi:(X,T)\to(Y,S)$ to be an *extension*. It is easy to see that all of sensitivity, thick sensitivity, thickly syndetical sensitivity, cofinite sensitivity and multi-sensitivity can be lifted from a factor to an extension by an almost open factor map by the method used in the proof of [17, Lemma 1.6]. Note that any factor map from a system containing a dense set of minimal points to a minimal system is almost open, as each factor map between minimal systems is also almost open [6, Theorem 1.15].

Each dynamical system admits a maximal equicontinuous factor. In fact, this factor is related to the regionally proximal relation of the system. The *regionally proximal relation* $Q_+(X,T)$ of (X,T) is defined as: $(x_1,x_2)\in Q_+(X,T)$ iff for any $\varepsilon>0$ there exist $x_1',x_2'\in X$ and $m\in\mathbb{N}$ with $\max\{\varrho(x_1,x_1'),\varrho(x_2,x_2'),\varrho(T^m x_1',T^m x_2')\}<\varepsilon$. Observe that $Q_+(X,T)\subset X\times X$ is closed and positively invariant (in the sense that if $(x_1, x_2) \in Q_+(X,T)$ then $(Tx_1, Tx_2) \in Q_+(X,T)$), which induces the maximal equicontinuous factor $(X_{\mathrm{eq}}, S_{\mathrm{eq}})$ of (X,T). And if (X,T) is minimal, then $Q_+(X,T)$ is in fact an equivalence relation by [6, 9, 13, 33] and [23, Proposition A.4]. Denote by $\pi_{\mathrm{eq}} : (X,T) \to (X_{\mathrm{eq}}, S_{\mathrm{eq}})$ the corresponding factor map. Remark that $(X_{\mathrm{eq}}, S_{\mathrm{eq}})$ is invertible, when (X,T) is transitive, because each transitive equicontinuous system is uniformly rigid [17, Lemma 1.2] and hence invertible.

Let X be a compact metric space. Recall that the function $f : X \to \mathbb{R}_+$ is *upper semi-continuous* if $\limsup\limits_{x \to x_0} f(x) \leq f(x_0)$ for each $x_0 \in X$. Let $\phi : X \to Y$ be a continuous surjective map. If there exists a dense subset $Y_0 \subset Y$ such that $\pi^{-1}(y)$ is a singleton for each $y \in Y_0$, then we call ϕ *almost one-to-one*. Note that such a set Y_0 can be always presented as a G_δ subset of Y, because

$$Y_0 = \{y \in Y : \pi^{-1}(y) \text{ is a singleton}\} = \bigcap_{n \in \mathbb{N}} \left\{ y \in Y : \mathrm{diam}(\pi^{-1}(y)) < \frac{1}{n} \right\}.$$

If $\pi : (X,T) \to (Y,S)$ is an almost one-to-one factor map between topological dynamical systems, then we also call (X,T) *almost one-to-one extension of* (Y,S). Recall also that if a dynamical system (X,T) is minimal, where X is a compact metric space, then the map $T : X \to X$ is almost one-to-one [27, Theorem 2.7].

The main result of this paper is the following dichotomy for multi-sensitive minimal systems, whose proof will be presented at the end of this section.

Theorem 3.1. *Let (X,T) be a minimal system. Then (X,T) is either multi-sensitive or an almost one-to-one extension of (X_{eq}, S_{eq}). Furthermore, (X,T) is not multi-sensitive iff (X,T) is an almost one-to-one extension of (X_{eq}, S_{eq}).*

Let (X,T) be an invertible system. Recall that $x \in X$ is an *almost automorphic point* of (X,T) if $T^{n_k}x \to x'$ implies $T^{-n_k}x' \to x$ for any $\{n_k : k \in \mathbb{N}\} \subset \mathbb{Z}$. (X,T) is said to be *almost automorphic* if $X = \overline{\mathrm{orb}_T(x)}$ for an almost automorphic point $x \in X$ ([32]). The structure of almost automorphic systems was characterized in [32]: a minimal invertible system is almost automorphic iff it is an almost one-to-one extension of its maximal equicontinuous factor $(X_{\mathrm{eq}}, S_{\mathrm{eq}})$.

Thus, directly from Theorem 3.1, we have the following

Corollary 3.2. *Let (X,T) be an invertible minimal system. Then (X,T) is not multi-sensitive iff it is almost automorphic.*

Let us also remark that by Theorem 4.6 all notions of thick sensitivity, multi-sensitivity and thickly syndetical sensitivity are equivalent for minimal dynamical systems, therefore one can apply Theorem 3.1 and Corollary 3.2 to any of them.

Now we will use the following concepts of Furstenberg [15]. Let $\mathcal{S} \subset \mathbb{N}$. $\mathcal{S}$ is a *central set* if there exists a topological dynamical system (X,T) with $x \in X$ and opene $U \subset X$ containing a minimal point y of (X,T) such that the pair (x,y) is proximal and $N_T(x,U) \subset \mathcal{S}$. $\mathcal{S}$ is a *difference set* if there exists $\{s_1 < s_2 < \dots\} \subset \mathbb{N}$ with $\mathcal{S} = \{s_i - s_j : i > j\}$. $\mathcal{S}$ is a Δ^**-set* if $\mathcal{S}$ has a nonempty intersection with any difference set. We also call a difference set as a Δ*-set*. Note that each central set is an IP set [15, Proposition 8.10], and hence contains a Δ-set [15, Lemma 9.1]; and if (X,T) is a minimal system, then $N_T(U,U)$ is a Δ^*-set for any opene $U \subset X$ by [15, Page 177].

Let $\pi : (X,T) \to (Y,S)$ be a factor map between dynamical systems. We call π *proximal* if any pair of points $x_1, x_2 \in X$ is proximal whenever $\pi(x_1) = \pi(x_2)$.

Proposition 3.3. *Let $\pi : (X,T) \to (Y,S)$ be a factor, not almost one-to-one map between minimal systems, where (Y,S) is invertible. Then $\inf\limits_{y\in Y} \operatorname{diam}(\pi^{-1}y) > 0$. Moreover, if π is also proximal, then (X,T) is thickly sensitive.*

Proof. Since (Y,S) is an invertible minimal system, it is not hard to show that $\pi^{-1}(y)$ is not a singleton for any $y \in Y$. So, let us first prove that $d := \inf\limits_{y\in Y} \operatorname{diam}(\pi^{-1}y) > 0$.

Let $\psi : Y \to [0, \operatorname{diam}(X)]$ be given by $y \mapsto \operatorname{diam}(\pi^{-1}y)$, and hence for each $y \in Y$ one has $\psi(y) > 0$ as $\pi^{-1}(y)$ is not a singleton. Since the function ψ is upper semi-continuous, $E_c(\psi)$ - the set of all points of continuity of ψ, is a residual subset of Y (see for example [15, Lemma 1.28]). Suppose that $d = 0$. So, there exists a sequence of points $y_i \in Y$ such that $\lim\limits_{i\to\infty} \psi(y_i) = 0$.

Let $y_c \in E_c(\psi)$ and $\varepsilon > 0$. There exists opene $V \subset Y$ containing y_c such that $|\psi(y_c) - \psi(y)| \leq \varepsilon$ whenever $y \in V$. Since (Y,S) is minimal, there exists $m \in \mathbb{N}$ with $\bigcup\limits_{j=0}^{m} S^{-j}V = Y$. By taking a subsequence (if necessary) we may assume that $\{y_i : i \in \mathbb{N}\} \subset S^{-k}V$ for some $k \in \{0, 1, \ldots, m\}$. Since $\lim\limits_{i\to\infty} \psi(y_i) = 0$, in other words, the diameter of $\pi^{-1}(y_i)$ tends to zero, the diameter of $\pi^{-1}(S^k y_i) = T^k \pi^{-1}(y_i)$ also tends to zero. Therefore $\lim\limits_{i\to\infty} \psi(S^k y_i) = 0$, which implies that $\psi(y_c) \leq \varepsilon$ by the construction of V and m. Finally $\psi(y_c) = 0$, a contradiction.

Take $0 < \delta < \dfrac{d}{6}$. Now assume that π is proximal. We shall prove that (X,T) is thickly sensitive with a sensitive constant δ. Let $x_* \in X$ and $m \in \mathbb{N}$ and take opene $U \subset X$ containing x_*. Let $V \subset U$ be an opene set containing x_* with $\max\limits_{0\leq i\leq m} \operatorname{diam}(T^i V) < \delta$. Since a factor map between minimal systems is almost open [6, Theorem 1.15], therefore for each $i = 0, 1, \ldots, m$: we can choose $y_i \in \operatorname{int}(\pi(T^i V))$ (the interior of $\pi(T^i V)$), $u_i \in \pi^{-1}(y_i)$ with $\operatorname{dist}(u_i, T^i V) > \dfrac{d}{2} - \delta$ because $\operatorname{diam}(T^i V) < \delta$, and set

$$W_i = \{x \in X : \varrho(x, u_i) < \delta\} \cap \pi^{-1}(\operatorname{int}(\pi(T^i V))) \ni u_i.$$

Obviously $\operatorname{dist}(W_i, T^i V) > \dfrac{d}{2} - 2\delta > \delta$ for each $i = 0, 1, \ldots, m$.

Note that since (X,T) is minimal, the set of all minimal points of the system $(X^{m+1}, T^{(m+1)})$, the product system of $m+1$ copies of (X,T), is dense in X^{m+1}. Hence we can take a minimal point $(v_0, v_1, \ldots, v_m) \in W_0 \times W_1 \times \cdots \times W_m$ of the system $(X^{m+1}, T^{(m+1)})$, and let $x_i \in T^i V$ with $\pi(x_i) = \pi(v_i)$ (because $\pi(v_i) \in \pi(T^i V)$) for each $i = 0, 1, \ldots, m$. Since the factor map $\pi : (X,T) \to (Y,S)$ is proximal, it is not hard to show that

$$\pi' : (X^{m+1}, T^{(m+1)}) \to (Y^{m+1}, S^{(m+1)}), (x_i' : 0 \leq i \leq m) \mapsto (\pi(x_i') : 0 \leq i \leq m)$$

is also a proximal factor map. In particular, $((x_0, x_1, \ldots, x_m), (v_0, v_1, \ldots, v_m))$ is proximal (under the action $T^{(m+1)}$), and thus

$$\mathcal{S} = N_{T^{(m+1)}}((x_0, x_1, \ldots, x_m), W_0 \times W_1 \times \cdots \times W_m)$$

is a central set and contains a Δ-set [15]. Finally $\mathcal{S} \cap \mathcal{N} \neq \varnothing$ where $\mathcal{N} = N_T(V,V) \subset N_T(TV, TV) \subset \cdots \subset N_T(T^m V, T^m V)$ is a Δ^*-set [15]. Now for any $n \in \mathcal{S} \cap \mathcal{N}$ and each $i = 0, 1, \ldots, m$: on one hand $T^n x_i \in W_i$ as $n \in \mathcal{S}$, and hence $T^{n+i} V \cap W_i \ni$

$T^n x_i$ as $x_i \in T^i V$; on the other hand $T^{n+i}V \cap T^i V \neq \varnothing$ as $n \in \mathcal{N}$, therefore $\operatorname{diam}(T^{n+i}V) \geq \operatorname{dist}(W_i, T^i V) > \delta$. Thus

$$N_T(U,\delta) \supset N_T(V,\delta) \supset \{n+i : n \in \mathcal{S} \cap \mathcal{N}, i = 0,1,\dots,m\},$$

which implies that (X,T) is thickly sensitive by the arbitrariness of U and m. $\square$

The following lemma is just a reformulation of Theorem 1.1 in [33].

Lemma 3.4. *Let (X,T) be an invertible minimal system. Let $x, x' \in X$ and $\pi_{eq} : (X,T) \to (X_{eq}, S_{eq})$ as introduced at the beginning of this section. Then $\pi_{eq}(x) = \pi_{eq}(x')$ iff for every opene $U, V \subset X$ containing x and x', respectively, there exist $n_1, m_1 \in \mathbb{Z}$ such that $T^{n_1}x, T^{n_1+m_1}x \in U$ and $T^{m_1}x \in V$.*

Now let us show that it is also true for any (not only invertible) continuous minimal map. Recall that the *natural extension* $(\widehat{X}, \widehat{T})$ of (X,T) is defined as

$$\widehat{X} = \{(x_1, x_2, \dots) : T(x_{i+1}) = x_i \text{ and } x_i \in X \text{ for each } i \in \mathbb{N}\},$$

$$\widehat{T} : (x_1, x_2, \dots) \mapsto (Tx_1, x_1, \dots).$$

Then $(\widehat{X}, \widehat{T})$ is an invertible extension of (X,T) with a factor map $\widehat{\pi} : (\widehat{X}, \widehat{T}) \mapsto (X,T), (x_1, x_2, \dots) \mapsto x_1$. It is not hard to check from the definitions that (X,T) is minimal (sensitive, thickly sensitive, thickly syndetically sensitive, cofinitely sensitive, multi-sensitive) iff $(\widehat{X}, \widehat{T})$ is minimal (sensitive, thickly sensitive, thickly syndetically sensitive, cofinitely sensitive, multi-sensitive, respectively).

Lemma 3.5. *Let (X,T) be a minimal system and $x, y \in X$. Then $(x,y) \in Q_+(X,T)$ iff for every opene $U, V \subset X$ containing x and y, respectively, there exist $n, m \in \mathbb{N}$ such that $T^n x, T^{n+m}x \in U$ and $T^m x \in V$.*

Proof. From the definition $(x,y) \in Q_+(X,T)$ once for every opene $U, V \subset X$ containing x and y, respectively, there exist $n, m \in \mathbb{N}$ such that $T^n x, T^{n+m}x \in U$ and $T^m x \in V$.

Now assume $(x,y) \in Q_+(X,T)$ and take opene $U, V \subset X$ containing x and y, respectively. Let $(\widehat{X}, \widehat{T})$ be the natural extension of (X,T) with the factor map $\widehat{\pi} : (\widehat{X}, \widehat{T}) \to (X,T)$. Since (X,T) is minimal, $(\widehat{X}, \widehat{T})$ is an invertible minimal system. Hence $Q_+(\widehat{X}, \widehat{T})$ is a closed invariant equivalence relation which induced the maximal equicontinuous factor $(\widehat{X}_{\mathrm{eq}}, \widehat{T}_{\mathrm{eq}})$ of $(\widehat{X}, \widehat{T})$ and $Q_+(X,T) = (\widehat{\pi} \times \widehat{\pi})Q_+(\widehat{X}, \widehat{T})$ by [23, Lemma A.3 and Proposition A.4]. In particular, there exist $(x_*, y_*) \in Q_+(\widehat{X}, \widehat{T})$ and opene $U_*, V_* \subset \widehat{X}$ containing x_* and y_*, respectively, such that $\widehat{\pi}(x_*) = x, \widehat{\pi}(y_*) = y$ and $\widehat{\pi}(U_*) \subset U, \widehat{\pi}(V_*) \subset V$. Let $\pi'_{\mathrm{eq}} : (\widehat{X}, \widehat{T}) \to (\widehat{X}_{\mathrm{eq}}, \widehat{T}_{\mathrm{eq}})$ be the corresponding factor map. Then $\pi'_{\mathrm{eq}}(x_*) = \pi'_{\mathrm{eq}}(y_*)$, and by applying Lemma 3.4 there exist $n_1, m_1 \in \mathbb{Z}$ such that $\widehat{T}^{n_1}x_*, \widehat{T}^{n_1+m_1}x_* \in U_*$ and $\widehat{T}^{m_1}x_* \in V_*$. Moreover, we choose opene $W \subset \widehat{X}$ containing x_* such that $\widehat{T}^{n_1}W \subset U_*, \widehat{T}^{n_1+m_1}W \subset U_*$ and $\widehat{T}^{m_1}W \subset V_*$. Since $(\widehat{X}, \widehat{T})$ is minimal, x_* is recurrent in the sense that $\widehat{T}^{l_k}x_*$ tends to x_* for a sequence of positive integers $l_1 < l_2 < \dots$, and so $N_{\widehat{T}}(x_*, W)$ is an IP set by [15, Theorem 2.17]. Hence there exists $p_1, q_1 \in \mathbb{N}$ such that

$$n = n_1 + p_1 > 0, m = m_1 + q_1 > 0 \text{ and } \{p_1, q_1, p_1 + q_1\} \subset N_{\widehat{T}}(x_*, W).$$

Thus $\widehat{T}^n x_*, \widehat{T}^{n+m}x_* \in U_*$ and $\widehat{T}^m x_* \in V_*$. Therefore $T^n x, T^{n+m}x \in U$ and $T^m x \in V$ by the above construction. This finishes the proof. $\square$

Using an idea of the proof of [32, Lemma 2.1.2] we obtain the following result, which is of independent interest.

Proposition 3.6. *Let (X,T) be a minimal system and $x,y\in X$. Then $(x,y)\in Q_+(X,T)$ iff $N_T(x,U)$ contains a Δ-set for any opene $U\subset X$ containing y.*

Proof. Sufficiency. Since $N_T(x,U)$ contains a Δ-set, there exist $\{s_1<s_2<s_3\}\subset\mathbb{N}$ with $T^{s_3-s_2}x, T^{s_2-s_1}x, T^{s_3-s_1}x\in U$. Let $x'=x, y'=T^{s_2-s_1}x$ and $m=s_3-s_2\in\mathbb{N}$. Then $T^m x', T^m y'\in U$ and $(x,y)\in Q_+(X,T)$ by the arbitrariness of opene $U\subset X$ containing y.

Necessity. Assume $(x,y)\in Q_+(X,T)$ and take opene $U\subset X$ containing y. Choose positive real numbers η and $\eta_k, k\in\mathbb{N}$ such that $\eta=\sum\limits_{k\in\mathbb{N}}\eta_k$ and $B_\eta(y)\subset U$, where $B_\eta(y)$ denotes the open ball of radius η centered at y. By applying Lemma 3.5 to $B_{\eta_1}(x)$ and $B_{\eta_1}(y)$, there exist $n_1,m_1\in\mathbb{N}$ such that

$$T^{n_1}x, T^{n_1+m_1}x\in B_{\eta_1}(x) \text{ and } T^{m_1}x\in B_{\eta_1}(y).$$

Let $\delta>0$ be small enough and applying Lemma 3.5 to $B_\delta(x)$ and $B_{\eta_1}(y)$, we have $n_2,m_2\in\mathbb{N}$ such that $T^{n_2}x, T^{n_2+m_2}x\in B_\delta(x)$ and $T^{m_2}x\in B_{\eta_2}(y)$. Since δ is small enough, we can require additionally

$$\max_{0\le r\le n_1+m_1}\varrho(T^{r+n_2}x,T^r x)<\eta_2 \text{ and } \max_{0\le r\le n_1+m_1}\varrho(T^{r+n_2+m_2}x,T^r x)<\eta_2.$$

We continue the process by induction. Put $l_k=\sum\limits_{i=1}^{k}(n_i+m_i)$ for each $k\in\mathbb{N}$. Then there exist $n_{k+1},m_{k+1}\in\mathbb{N}$ such that $T^{m_{k+1}}x\in B_{\eta_{k+1}}(y)$,

$$\max_{0\le r\le l_k}\varrho(T^{r+n_{k+1}}x,T^r x)<\eta_{k+1} \text{ and } \max_{0\le r\le l_k}\varrho(T^{r+n_{k+1}+m_{k+1}}x,T^r x)<\eta_{k+1}. \tag{3.1}$$

Set $p_k=m_k+n_{k+1}$ and $s_k=p_1+\cdots+p_k$ for every $k\in\mathbb{N}$. Then

$$\begin{aligned}
\varrho(T^{p_i+\cdots+p_j}x,y) &= \varrho(T^{m_i+\sum_{k=i+1}^{j}(n_k+m_k)+n_{j+1}}x,y)\\
&\le \varrho(T^{m_i+\sum_{k=i+1}^{j}(n_k+m_k)+n_{j+1}}x,T^{m_i+\sum_{k=i+1}^{j}(n_k+m_k)}x)+\\
&\quad\cdots+\varrho(T^{m_i+(n_{i+1}+m_{i+1})}x,T^{m_i}x)+\varrho(T^{m_i}x,y)\\
&< \eta_{j+1}+\cdots+\eta_i \text{ (using (3.1))} <\eta,
\end{aligned}$$

for all $i\le j$. So, $N_T(x,U)\supset\{s_j-s_i: i<j\}$ from the construction. $\square$

Proposition 3.7. *Let (X,T) be a minimal system and let $\pi_{eq}:(X,T)\to(X_{eq},S_{eq})$ be not proximal. Then (X,T) is thickly sensitive.*

Proof. Since $\pi_{\rm eq}:(X,T)\to(X_{\rm eq},S_{\rm eq})$ is not proximal, there exist a not proximal pair of points $x_1,x_2\in X$ such that $\pi_{\rm eq}(x_1)=\pi_{\rm eq}(x_2)$ (and hence $(x_1,x_2)\in Q_+(X,T)$, as (X,T) is minimal). Then $d:=\inf\limits_{n\in\mathbb{N}}\varrho(T^n x_1,T^n x_2)>0$, and take $0<\delta<\dfrac{d}{3}$.

We are going to prove that (X,T) is thickly sensitive with a sensitive constant $\delta>0$. Since (X,T) is minimal, it suffices to show that $N_T(U,\delta)$ is thick for any opene $U\subset X$ containing x_1.

For any $m\in\mathbb{N}$ take opene $V,W\subset U$ containing x_1 and x_2, respectively, such that $\max\limits_{0\le i\le m}\max\{\mathrm{diam}(T^iV),\mathrm{diam}(T^iW)\}<\delta$. By the above construction

$\min_{0\le i\le m} \operatorname{dist}(T^iV, T^iW) > \delta$. Since $(x_1, x_2) \in Q_+(X,T)$, $N_T(x_1, W)$ contains a Δ-set by Proposition 3.6, and hence has a nonempty intersection with $\mathcal{N}$, where $\mathcal{N} = N_T(V,V) \subset N_T(TV,TV) \subset \cdots \subset N_T(T^mV, T^mV)$ is a Δ^*-set [15]. Therefore for every $n \in N_T(x_1, W) \cap \mathcal{N}$ and $i = 0, 1, \dots, m$ we have: $T^{n+i}V \cap T^iW \ni T^{n+i}x_1$, because $T^nx_1 \in W$, and $T^{n+i}V \cap T^iV \neq \varnothing$, because $n \in \mathcal{N}$. That gets $\operatorname{diam}(T^{n+i}V) \geq \operatorname{dist}(T^iW, T^iV) > \delta$. Thus

$$N_T(U,\delta) \supset N_T(V,\delta) \supset \{n + i : n \in N_T(x_1, W) \cap \mathcal{N}, i = 0, 1, \dots, m\},$$

which implies that (X,T) is thickly sensitive. $\square$

Recall once more that a point $x \in X$ is called Lyapunov stable if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\varrho(x, x') < \delta$ implies $\varrho(T^nx, T^nx') < \varepsilon$ for any $n \in \mathbb{N}$. This condition says exactly that the sequence of iterates $\{T^n : n \geq 0\}$ is equicontinuous at x. Hence, such a point is also called an *equicontinuity point*. Denote by $\operatorname{Eq}(X,T)$ the set of all equicontinuity points of (X,T).

Clearly if (X,T) is sensitive then $\operatorname{Eq}(X,T) = \varnothing$, and (X,T) is equicontinuous iff $\operatorname{Eq}(X,T) = X$, and a thickly sensitive system is sensitive. Recall that a transitive non sensitive system (X,T) has zero topological entropy, $\operatorname{Tran}(X,T) = \operatorname{Eq}(X,T)$, and moreover a minimal system (X,T) is equicontinuous iff it is not sensitive ([4, Theorem 4.1] and [3, 17, 19]).

Lemma 3.8. *Let $\pi : (X,T) \to (Y,S)$ be a factor map and let y_0 be a minimal, equicontinuity point of (Y,S) such that $\pi^{-1}(y_0)$ is a singleton. Then (X,T) is not thickly sensitive.*

Proof. Assume, to the contrary, that (X,T) is thickly sensitive with a sensitive constant $\delta > 0$. Let ϱ be a compatible metric over Y. Since $\pi^{-1}(y_0)$ is a singleton, we can take opene $W \subset X$ containing $\pi^{-1}(y_0)$ such that $\operatorname{diam}(W) < \delta$. There exists opene $V \subset Y$ containing y_0 such that $\pi^{-1}(V) \subset W$. Let $\varepsilon > 0$ be small enough such that $\{y \in Y : \varrho(y, y_0) < 2\varepsilon\} \subset V$. Since $y_0 \in \operatorname{Eq}(Y,S)$, there exists $\varepsilon \geq \kappa > 0$ such that $\varrho(S^ny, S^ny_0) < \varepsilon$ whenever $\varrho(y, y_0) < \kappa$ and $n \in \mathbb{N}$.

Take $V' = \{y \in Y : \varrho(y, y_0) < \kappa\}, U = \pi^{-1}(V')$ and set $\mathcal{S} = N_S(y_0, V')$. If $n \in \mathcal{S}$ and $y \in V'$, then $S^ny_0 \in V'$ and $\varrho(S^ny_0, S^ny) < \varepsilon$, and so $\varrho(y_0, S^ny) < 2\varepsilon$, that gives $S^nV' \subset V$. Note that $\mathcal{S}$ is syndetic, because y_0 is a minimal point, and

$$T^nU = T^n\pi^{-1}(V') \subset \pi^{-1}(S^nV') \subset \pi^{-1}(V) \subset W$$

for each $n \in \mathcal{S}$, which implies $N_T(U,\delta) \cap \mathcal{S} = \varnothing$, a contradiction to the assumption. Thus (X,T) is not thickly sensitive. $\square$

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let $\pi_{\text{eq}} : (X,T) \to (X_{\text{eq}}, S_{\text{eq}})$ be the factor map as introduced at the beginning of this section. If π is almost one-to-one, then (X,T) is not thickly sensitive by Lemma 3.8. Now assume that (X,T) is not thickly sensitive, then π is proximal by Proposition 3.7, and then π is almost one-to-one by Proposition 3.3 (observing that $(X_{\text{eq}}, S_{\text{eq}})$ is an invertible minimal system). This finishes the proof, because a minimal system (X,T) is thickly sensitive iff it is multi-sensitive by Theorem 4.6. $\square$

4. Multi-sensitivity, thick sensitivity and thickly syndetical sensitivity

In this section we prove that for minimal systems all of the following notions: thickly syndetical sensitivity, multi-sensitivity and thick sensitivity are equivalent, and show that all of them are much stronger than sensitivity. We begin with the following

Proposition 4.1. *If (X,T) is multi-sensitive, then (X,T) is thickly sensitive. Moreover, if (X,T) is transitive, then the converse also holds.*

Proof. First assume that (X,T) is multi-sensitive with a sensitive constant $\delta > 0$. Then $\overline{\mathbb{L}}_{m,r} \geq \dfrac{\delta}{2}$ by Lemma 2.1. Now take any opene $U \subset X$ and $k \in \mathbb{N}$. By the definition of $\overline{\mathbb{L}}_{m,r}$ one has that $\bigcap_{i=0}^{k} N_T(T^{-i}U, \dfrac{\delta}{3})$ is an infinite set and we may choose n_k from it with $n_k \geq k$. Obviously $\{n_k - k, \ldots, n_k - 1, n_k\} \subset N_T(U, \dfrac{\delta}{3})$, which implies that (X,T) is thickly sensitive with a sensitive constant $\dfrac{\delta}{3}$.

Now we assume that a transitive system (X,T) is thickly sensitive with a sensitive constant $\delta > 0$. Let $k \in \mathbb{N}$ and $U_1, \ldots, U_k$ be opene sets in X. Take a transitive point $x \in \mathrm{Tran}(X,T)$. Then there exists $n_i \in \mathbb{N}$ such that $T^{n_i}x \in U_i$, where $i = 1, \ldots, k$. So, we may get an opene $U \subset X$ such that $T^{n_i}U \subset U_i$ for every $i = 1, \ldots, k$. By assumption there exists $s \in \mathbb{N}$ with $\{s, s+1, \ldots, s+n_1+\cdots+n_k\} \subset N_T(U, \delta)$, and then one has $s \in \bigcap\limits_{i=1}^{k} N_T(U_i, \delta)$. That shows that (X,T) is multi-sensitive with a sensitive constant δ. $\square$

Observe that Moothathu pointed out firstly in [30] that multi-sensitivity implies thick sensitivity. As a direct corollary of Lemma 3.8, one has:

Example 4.2. *There exists a minimal sensitive invertible system containing dense distal points, which is in fact an almost one-to-one extension of an equicontinuous system (hence not thickly sensitive by Lemma 3.8, and then not multi-sensitive). Moreover, we can require the constructed system to have either zero topological sequence entropy or positive topological entropy.*

Construction. It is easy to check that the Denjoy minimal system $(\mathcal{D}, R)$ is a minimal sensitive invertible system containing dense distal points, which is an almost one-to-one extension of an irrational rotation over the circle and has zero topological sequence entropy. Where $(\mathcal{D}, R)$ is constructed as follows: Let $(\mathbb{S}, S)$ be an irrational rotation over the circle and $x_0 \in \mathbb{S}$. We identify $(x, 0)$ and $(x, 1)$ for all $x \notin \{S^k x_0 : k \in \mathbb{Z}\}$. Then set $\mathcal{D}$ to be the quotient space $\mathbb{S} \times \{0, 1\}$ equipped with this identification, where R acts naturally on $\mathcal{D}$ induced from the action S.

Now we consider a Toeplitz flow which is a minimal invertible system and has positive topological entropy (thus it is sensitive), which in fact is an almost one-to-one extension of an odometer (and hence contains dense distal points). See [12] for the definition of a Toeplitz flow and a detailed construction of such a system. $\square$

We say that (X,T) is *topologically ergodic* (*thickly syndetically transitive*, respectively) if the set $N_T(U,V)$ is syndetic (thickly syndetic, respectively) for any opene $U, V \subset X$. Recall that (X,T) is weakly mixing iff $N_T(U,V)$ is a thick set for any opene $U, V \subset X$ [14], and (X,T) is thickly syndetically transitive iff (X,T) is not only weakly mixing, but also topologically ergodic [24, Theorem 4.7].

A nonminimal M-system is thickly syndetically sensitive [29, Theorem 8]. Observe also that the intersection of finitely many thickly syndetic sets is also thickly syndetic (and hence nonempty).

Proposition 4.3. *If (X,T) is thickly syndetically transitive, then (X,T) is thickly syndetically sensitive.*

Proof. Assume (X,T) is a thickly syndetically transitive system. We take opene $V_1, V_2 \subset X$ with $\delta = \mathrm{dist}(V_1, V_2) > 0$. Now let $U \subset X$ be an opene subset. By the assumption, both $N_T(U, V_1)$ and $N_T(U, V_2)$ are thickly syndetic, and hence $N_T(U, \delta) \supset N_T(U, V_1) \cap N_T(U, V_2)$ is also thickly syndetic. □

In fact, we have the following property of topologically ergodic systems and provide a proof of it for completeness (see also [29, Theorem 8]).

Lemma 4.4. *Let (X,T) be a topologically ergodic system with two different minimal subsets M_1 and M_2 of X. Then (X,T) is thickly syndetically sensitive.*

Proof. Let $\mathrm{dist}(M_1, M_2) > \delta > 0$. Take opene subsets $U \subset X$ and $V_i \subset X$ containing M_i, $i = 1, 2$ with $\mathrm{dist}(V_1, V_2) > \delta$. We shall prove that every $N_T(U, V_i), i = 1, 2$ are thickly syndetic, and hence $N_T(U, V_1) \cap N_T(U, V_2)$ is also thickly syndetic. In fact, for any $m \in \mathbb{N}$, we may choose opene $W_i \subset V_i$ with $T^j W_i \subset V_i$ for all $j = 0, 1, \ldots, m$. Then

$$N_T(U, V_i) \supset \bigcup_{j=0}^{m} N_T(U, T^j W_i) \supset \{n + j : n \in N_T(U, W_i), j = 0, 1, \ldots, m\}$$

is thickly syndetic, because $N_T(U, W_i)$ is syndetic. □

By the same reason a thickly syndetically sensitive system is multi-sensitive.

The following Figure 1 presents a comparison between stronger forms of sensitivity for general topological dynamical systems.

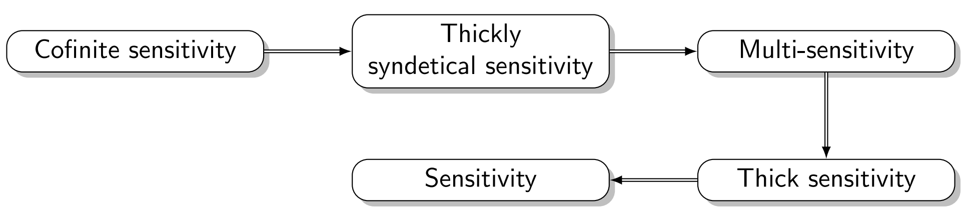

Figure 1. General case.

Proposition 4.5. *If (X,T) is a thickly sensitive M-system, then (X,T) is thickly syndetically sensitive.*

Proof. Recall again that if (X,T) is an M-system, then $(X^k, T^{(k)})$, the product system of k copies of (X,T), contains a dense set of minimal points for any $k \in \mathbb{N}$. In fact we can say more. Let $x \in \mathrm{Tran}(X,T)$ and $U_1, \ldots, U_k$ be opene subsets in X. There are $n_1, \ldots, n_k \in \mathbb{N}$ such that $T^{n_1} x \in U_1, \ldots, T^{n_k} x \in U_k$. Since (X,T) is an M-system, there is a minimal point $x_0 \in X$ sufficiently close to x such that

$T^{n_1}x_0 \in U_1, \ldots, T^{n_k}x_0 \in U_k$. Thus $U_1 \times \cdots \times U_k$ contains the minimal point $(T^{n_1}x_0, \ldots, T^{n_k}x_0)$ of $(X^k, T^{(k)})$.

Let (X,T), which is thickly sensitive, has a sensitive constant $\delta > 0$. Let U be an opene subset in X. Since $N_T(U,\delta)$ is a thick set,

$$\bigcap_{i=0}^{k} N_T(T^{-i}U,\delta) \supset \{n \geq k : \{n-k,\ldots,n-1,n\} \subset N_T(U,\delta)\}.$$

Therefore there are a positive integer $n_0 \in \bigcap\limits_{i=0}^{k} N_T(T^{-i}U,\delta)$, $n_0 \geq k$, and $x_i, y_i \in T^{-i}U$ with $\varrho(T^{n_0}x_i, T^{n_0}y_i) > \delta$, $i = 0,1,\ldots,k$. Moreover, we can choose opene subsets $U_i, V_i \subset T^{-i}U$ such that $\varrho(T^{n_0}x_i', T^{n_0}y_i') > \delta$ for all $x_i' \in U_i$ and $y_i' \in V_i$, $i = 0,1,\ldots,k$. Again, since (X,T) is an M-system, the system $(X^{2k+2}, T^{(2k+2)})$ contains a dense set of minimal points, and there is a minimal point $(z_0, z_0', z_1, z_1', \ldots, z_k, z_k') \in U_0 \times V_0 \times U_1 \times V_1 \times \cdots \times U_k \times V_k$. Obviously

$$\mathcal{S} = \bigcap_{i=0}^{k} (N_T(z_i, U_i) \cap N_T(z_i', V_i))$$

is a syndetic set. From the construction we get that $N_T(U,\delta) \supset \{m + n_0 - i : m \in \mathcal{S}, i = 0,1,\ldots,k\}$, which is a thickly syndetic set. $\square$

Combining Proposition 4.1, Example 4.2 and Proposition 4.5 we have the following

Theorem 4.6. *Let (X,T) be a topological dynamical system. Then*

(1) *Thickly syndetical sensitivity $\Longrightarrow$ multi-sensitivity $\Longrightarrow$ thick sensitivity.*
(2) *Multi-sensitivity $\Longleftrightarrow$ thick sensitivity, when (X,T) is transitive.*
(3) *If (X,T) is an M-system, then these three sensitivities are equivalent, and all of them are much stronger than sensitivity, even when (X,T) is minimal.*
(4) *If (X,T) is a nonminimal M-system, then it is thickly syndetically sensitive.*

The following Figure 2 presents a comparison between stronger forms of sensitivity for M-systems.

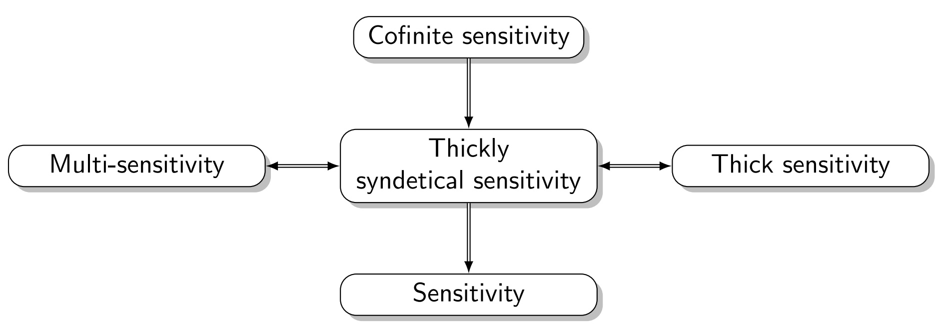

Figure 2. M-systems.

Let us mention about some another stronger form of sensitivity. Recall that a pair of points $x, y \in X$ is called a *Li-Yorke pair* if $\liminf\limits_{n\to\infty} \varrho(T^n x, T^n y) = 0$ while $\limsup\limits_{n\to\infty} \varrho(T^n x, T^n x) > 0$. A dynamical system (X,T) is called *spatio-temporally* *chaotic* if for any point $x \in X$ and its neighborhood U_x there is a point $y \in U_x$ such that the pair x, y is Li-Yorke [8], (X,T) is called *Li-Yorke sensitive* if there exists $\delta > 0$ such that for any point $x \in X$ and its neighborhood U_x there is a point $y \in U_x$ with $\liminf_{n\to\infty} \varrho(T^n x, T^n y) = 0$ while $\limsup_{n\to\infty} \varrho(T^n x, T^n x) > \delta$ [5], and (X,T) is called *distal* if any point of X is distal. Clearly that Li-Yorke sensitivity is much stronger than sensitivity and a distal system contains no Li-Yorke pairs.

The following Figure 3 presents a comparison between stronger forms of sensitivity for topological transitive systems.

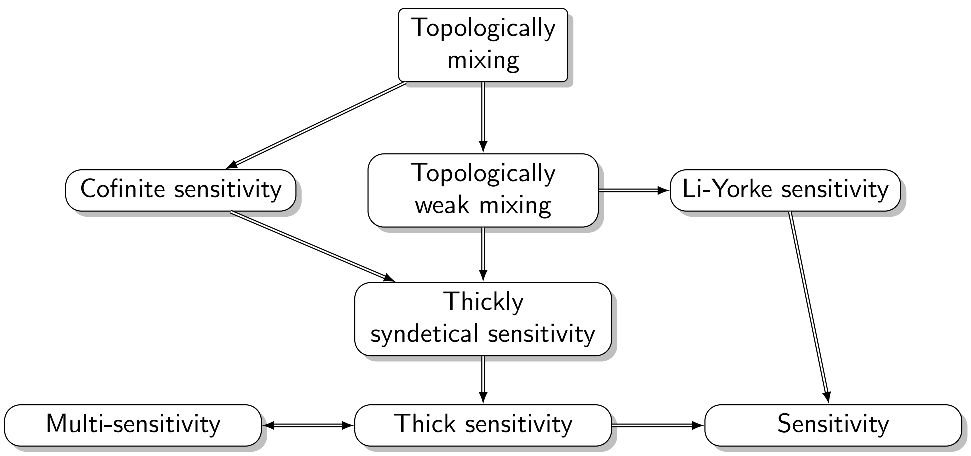

Figure 3. Topologically transitive systems.

Remarks: 1. Even for minimal systems cofinite sensitivity does not imply spatio-temporal chaos, and hence Li-Yorke sensitivity. In fact, Example 4.7 provides a cofinitely sensitive invertible minimal system containing no Li-Yorke pairs.

2. When a system (X,T) is minimal, spatio-temporal chaos (and hence Li-Yorke sensitivity) does imply multi-sensitivity. Assume that (X,T) is a minimal system which is not multi-sensitive. Then by the dichotomy theorem (Theorem 3.1) (X,T) is an almost one-to-one extension of its maximal equicontinuous factor. Take $x \in X$ such that $\pi^{-1}(x)$ is a singleton, where π is the factor map from (X,T) to its maximal equicontinuous factor. It is easy to see that (x,z) can not be proximal for any $z \in X$ $(z \neq x)$. Therefore (X,T) is not spatio-temporally chaotic.

3. In general, even for transitive systems, Li-Yorke sensitivity does not imply thick sensitivity. In fact, there is a nonminimal E-system (and hence sensitive system), such that 1) it contains a fixed point as its unique minimal set, and hence the system is Li-Yorke sensitive by [5, Corollary 3.7]; 2)it is not thickly sensitive. We will discuss more about such systems in the next section

For example, let (X,T) be the system as in Example 5.2. Then by collapsing the unique minimal set in (X,T) into a fixed point we obtain a system (Y,S), which is the system with the required properties. By Example 5.2, (X,T) is not thickly sensitive, and by the construction $\pi : (X,T) \mapsto (Y,S)$ is almost open. Then (Y,S) should be not thickly sensitive.

Question. Are all nonminimal M-systems Li-Yorke sensitive?

Example 4.7. *There exists an invertible minimal distal system (and hence containing no Li-Yorke pairs) which is cofinitely sensitive.*

Construction. Let $\alpha \notin \mathbb{Q}$ and (X,T) be given by $X = \mathbb{R}^2/\mathbb{Z}^2$ and $T : (x,y) \mapsto (x+\alpha, x+y)$. It is well known that (X,T) is an invertible minimal distal system (see [15, Chapter 1]). Now for any opene $U \subset X$ take $x_0, y_0 \in \mathbb{R}/\mathbb{Z}$ and $\delta > 0$ with $(x_0 - \delta, x_0 + \delta) \times \{y_0\} \subset U$. Since

$$T^n(x,y) = \left(x + n\alpha, nx + \frac{n(n-1)}{2}\alpha + y\right)$$

for any point $(x,y) \in X$ and any positive integer n, the diameter of $T^n U$ is at least the length of the circle $\mathbb{R}/\mathbb{Z}$ when n is large enough. From which one has directly that (X,T) is cofinitely sensitive. □

5. More about thick sensitivity

This section is mostly devoted to the transitive, not thickly sensitive systems. Recall that non sensitivity of a system is related to equicontinuity of points in the system. More precisely, $\mathrm{Tran}(X,T) = \mathrm{Eq}(X,T)$ for a transitive non sensitive system (X,T) and a minimal system (X,T) is not sensitive iff $\mathrm{Eq}(X,T) = X$. In this section, we link thick sensitivity of a system with another kind of equicontinuity (i.e. syndetical equicontinuity) of points in the system.

We begin this section with recalling a definition of the topological sequence entropy for the system (X,T) by using one of the classical Bowen-Dinaburg definition of topological entropy $h(T)$. For an increasing sequence $\mathcal{N} = n_1 < n_2 < \dots$ of $\mathbb{N}$ and $n_0 = 0$. For any integer $k \geq 1$ the function $\varrho_k(x,y) = \max_{0 \leq j \leq k-1} \varrho(T^{n_j}x, T^{n_j}y)$ defines a metric on X equivalent with ϱ. Now fix an integer $k \geq 1$ and $\varepsilon > 0$. A subset $E \subset X$ is called (k,T,ε)*-separated* (with respect to $\mathcal{N}$), if for any two distinct points $x, y \in E$, $\varrho_k(x,y) > \varepsilon$. Denote by $\mathrm{sep}(k,T,\varepsilon)$ the maximal cardinality of a (k,T,ε)-separated set in X and $h_{\mathcal{N}}(T,\varepsilon) = \limsup_{k\to\infty} \frac{1}{k} \log \mathrm{sep}(k,T,\varepsilon)$. Obviously that $h_{\mathcal{N}}(T,\varepsilon_1) \geq h_{\mathcal{N}}(T,\varepsilon_2)$, when $\varepsilon_1 < \varepsilon_2$. *The topological sequence entropy* of (X,T) along the sequence $\mathcal{N}$ is defined by

$$h_{\mathcal{N}}(T) = \lim_{\varepsilon\to 0} \limsup_{k\to\infty} \frac{1}{k} \log \mathrm{sep}(k,T,\varepsilon).$$

As another corollary of Theorem 3.1 and [21, Theorem 4.3], one has that any minimal thickly sensitive system (and hence multi-sensitive by Proposition 4.1) has positive topological sequence entropy. In fact, we can obtain the following

Proposition 5.1. *Let (X,T) be a multi-sensitive system. Then (X,T) has positive topological sequence entropy.*

Proof. We are going to define an increasing sequence of positive integers $\mathcal{N} = n_1 < n_2 < \cdots < n_i < \dots$ and a sequence of $(k+1, f, \varepsilon)$-separated subsets of X (with respect to $\mathcal{N}$) with cardinality $2^k, k = 1, 2, \ldots$. Then obviously we will have that $h_{\mathcal{N}}(T) \geq \log 2$.

Let (X,T) be a multi-sensitive system with a sensitive constant $2\delta > 0$. Take opene $U_{(1)}, U_{(2)} \subset X$ with $\mathrm{dist}(U_{(1)}, U_{(2)}) > \delta$, and define $V_{(1)} = T^{-n_1}U_{(1)}, V_{(2)} = T^{-n_1}U_{(2)}$ for a positive integer n_1. Obviously that any two points $x_1 \in V_{(1)}, x_2 \in V_{(2)}$ are $(2,T,\delta)$-separated. Since (X,T) is multi-sensitive and has the sensitive constant 2δ, the Lyapunov number $\overline{\mathbb{L}}_{m,r} \geq \delta$ by Lemma 2.1 and hence there exist a positive integer $n_2 > n_1$ and 4 points $x_{(1,1)}, x_{(1,2)} \in V_{(1)}$ and $x_{(2,1)}, x_{(2,2)} \in V_{(2)}$ with $\min_{i\in\{1,2\}} \varrho(T^{n_2}x_{(i,1)}, T^{n_2}x_{(i,2)}) > \delta$. Therefore they are $(3, T, \delta)$-separated. More precisely $\min_{i,j\in\{1,2\}} \varrho(T^{n_1}x_{(1,i)}, T^{n_1}x_{(2,j)}) > \delta$ and $\min_{i\in\{1,2\}} \varrho(T^{n_2}x_{(i,1)}, T^{n_2}x_{(i,2)}) > \delta$. Take a small enough neighborhood $U_{(i,j)}$ of point $T^{n_2}x_{(i,j)}$ such that $\min_{i\in\{1,2\}} \mathrm{dist}(U_{(i,1)}, U_{(i,2)}) > \delta$ and set $V_{(i,j)} := T^{-n_2}U_{(i,j)} \cap V_{(i)}$ for $i, j \in \{1, 2\}$.

Now assume that by induction we have defined the sequence of positive integers $n_1 < \cdots < n_k$ and 2^k points x_s in opene subsets V_s, $s \in \{1, 2\}^k$ such that

$$\min_{s,s'\in\{1,2\}^k, s\neq s'} \max_{1\leq q\leq k} \mathrm{dist}(T^{n_q}V_s, T^{n_q}V_{s'}) > \delta,$$

and therefore the set of points $\{x_s \in V_s, s \in \{1, 2\}^k\}$ is $(k + 1, T, \delta)$-separated.

Since (X, T) is multi-sensitive, there exit $n_{k+1} > n_k$ and two points $x_{(s_1,\dots,s_k,1)} \in V_{(s_1,\dots,s_k)}, x_{(s_1,\dots,s_k,2)} \in V_{(s_1,\dots,s_k)}$ with $\varrho(T^{n_{k+1}}x_{(s_1,\dots,s_k,1)}, T^{n_{k+1}}x_{(s_1,\dots,s_k,2)}) > \delta$ for any $s_1, \dots, s_k \in \{1, 2\}^k$. So, the set of all these 2^{k+1} points is $(k + 2, T, \delta)$-separated (with respect to $\mathcal{N}$), because by the induction hypothesis any two different points $x_{(s_1,\dots,s_i,\dots,s_k,l)} \neq x_{(s_1,\dots,s'_i,\dots,s_k,l)}$ are also $(k + 1, T, \delta)$-separated (with respect to $\mathcal{N}$) for any $l \in \{1, 2\}$. This finishes the proof. □

Recall that (X, T) is a *E-system* [17] if (X, T) is a transitive system admitting an invariant probability Borel measure μ with full support, that is $T\mu = \mu$ and $\mu(U) > 0$ for all opene $U \subset X$. Note that any E-system is topologically ergodic as shown in [18, Theorem 4.4] (and hence any E-system containing two different minimal subsystems is thickly syndetically sensitive by Lemma 4.4), and a non sensitive E-system is minimal equicontinous [17, Theorem 1.3]. It is easy to see that any M-system is a E-system, and then conclude again that any nonminimal M-system is thickly syndetically sensitive.

In fact, we have the following

Example 5.2. *There exists a nonminimal E-system (X, T) (and hence sensitive) with positive topological entropy which is not thickly sensitive. Moreover (X, T) admits an ergodic measure with full support and contains a unique minimal subsystem.*

Construction. We take a minimal invertible system (Y, S) with positive topological entropy from Example 4.2, which is not thickly sensitive. By the classical variational principle (see for example [35, Theorem 8.6]) we choose an ergodic measure ν of (Y, S) with positive measure-theoretic ν-entropy $h_\nu(Y, S)$. Let $\nu = \int_Z \nu_z d\eta(z)$ be the disintegration of ν over the Pinsker factor $(Z, \mathcal{D}, \eta, R)$ of $(Y, \mathcal{B}_\nu, \nu, S)$, where $(Y, \mathcal{B}_\nu, \nu, S)$ is the completion of $(Y, \mathcal{B}_Y, \nu, S)$ and $\mathcal{B}_Y$ denotes the Borel σ-algebra of Y (for the construction of such a disintegration see for example [15, Chapter 5, §4]). Set $\lambda = \int_Z \nu_z \times \nu_z d\eta(z)$, which is in fact an ergodic measure of $(Y \times Y, S \times S)$ with positive measure-theoretic λ-entropy and $\lambda(X \setminus \Delta_Y) > 0$ by [16], where $\Delta_Y = \{(y, y) : y \in Y\}$ and $X \subset Y \times Y$ is the support of λ, that is the smallest closed subset of $Y \times Y$ with $\lambda(X) = 1$. It is easy to see that $(X, S \times S)$ forms a transitive system having a nonempty intersection with Δ_Y, denoted by (X, T), and then $X \supsetneq \Delta_Y$ by the minimality of (Y, S).

Now we shall finish the construction by proving that (X, T) is not thickly sensitive (and hence not thickly syndetically sensitive, which implies from Lemma 4.4

that (X,T) contains a unique minimal subsystem Δ_Y). The proof of it is direct. Let ϱ be a compatible metric over Y, and then over $Y\times Y$ we take the compatible metric $\varrho_1((y_1,y_2),(y_1',y_2'))=\max\{\varrho(y_1,y_1'),\varrho(y_2,y_2')\}$ for $y_1,y_1',y_2,y_2'\in Y$. Thus $\mathrm{diam}(U_1\times U_2)=\max\{\mathrm{diam}(U_1),\mathrm{diam}(U_2)\}$, from which it is easy to show that, if (X,T) is thickly sensitive, then (Y,S) is also thickly sensitive. Assume (X,T) is thickly sensitive with a sensitive constant $\delta>0$. Then $N_S(U,\delta)=N_{S\times S}((U\times U)\cap X,\delta)$ is a thick set for any opene $U\subset Y$ (reminder $\Delta_Y\subset X$). We have a contradiction with the selection of (Y,S). $\square$

In the following we introduce the concept of syndetically equicontinuous points of a system and investigate it for transitive, not thickly sensitive systems. We say that $x\in X$ is a *syndetically equicontinuous point* of (X,T) if for any $\varepsilon>0$ there exist opene $U\subset X$ containing x and a syndetic set $\mathcal{N}\subset\mathbb{N}$ such that $\varrho(T^nx,T^nx')\le\varepsilon$ whenever $x'\in U$ and $n\in\mathcal{N}$. Denote by $\mathrm{Eq}_{\mathrm{syn}}(X,T)$ the set of all syndetically equicontinuous points of (X,T). Then $\mathrm{Eq}_{\mathrm{syn}}(X,T)\subset\mathrm{Eq}(X,T)$.

Since a thick set has a nonempty intersection with a syndetic set, one has readily that, if (X,T) is thickly sensitive, then $\mathrm{Eq}_{\mathrm{syn}}(X,T)=\varnothing$. Equivalently, if $\mathrm{Eq}_{\mathrm{syn}}(X,T)\neq\varnothing$, then (X,T) is not thickly sensitive. Note that $\mathrm{Tran}(X,T)=\mathrm{Eq}(X,T)$ for a transitive non sensitive system (X,T). Similarly, we have the following

Proposition 5.3. *Let (X,T) be a transitive, not thickly sensitive system. Then* $\mathrm{Tran}(X,T)\subset Eq_{syn}(X,T)$.

Proof. Let $\delta>0$. Since the system (X,T) is not thickly sensitive, there exists opene $U'\subset X$ such that $N_T(U',\delta)$ is not thick. Or equivalently, there exists a syndetic set $\mathcal{N}\subset\mathbb{N}$ such that $\varrho(T^nx_1,T^nx_2)\le\delta$ whenever $x_1,x_2\in U'$ and $n\in\mathcal{N}$. Now for any $x\in\mathrm{Tran}(X,T)$ there exists $m\in\mathbb{N}$ with $T^mx\in U'$ and hence there exists opene $U\subset X$ containing x with $T^mU\subset U'$. In particular, $\varrho(T^{m+n}x,T^{m+n}x')\le\delta$ whenever $x'\in U$ and $n\in\mathcal{N}$. That implies $x\in\mathrm{Eq}_{\mathrm{syn}}(X,T)$, because the set $m+\mathcal{N}$ is syndetic. $\square$

A direct corollary of Proposition 5.3 is the following

Theorem 5.4. *Let (X,T) be a topological dynamical system.*

(1) *Assume the system (X,T) is transitive. Then (X,T) is not thickly sensitive iff $Eq_{syn}(X,T)\neq\varnothing$.*
(2) *Assume the system (X,T) is minimal. Then (X,T) is not thickly sensitive iff $Eq_{syn}(X,T)=X$.*

Let us show it may happen that $\mathrm{Tran}(X,T)\subsetneq\mathrm{Eq}_{\mathrm{syn}}(X,T)$ for a transitive not thickly sensitive system.

Example 5.5. *There exists a nonminimal E-system (X',T') (and hence sensitive) with positive topological entropy which is not thickly sensitive and contains a unique minimal subsystem such that:*

$$\textit{either } \mathrm{Tran}(X',T')\subsetneq Eq_{syn}(X',T')=X' \textit{ or } \mathrm{Tran}(X',T')\subsetneq Eq_{syn}(X',T')\subsetneq X'.$$

Construction. Let (Y,S), (X,T) and ergodic measure λ (over (X,T)) be as in Example 5.2. Both (Y,S) and (X,T) are not thickly sensitive, Δ_Y is the unique minimal subsystem of (X,T), X is the support of λ with $X\supsetneq\Delta_Y$ (and hence $\lambda(\Delta_Y)<1$) and (X,T) has positive measure-theoretic λ-entropy.

1. *Case of* $\mathrm{Tran}(X',T') \subsetneq Eq_{syn}(X',T') = X'$. Set $(X',T') = (X,T)$. We need show that $\mathrm{Eq_{syn}}(X,T) = X$.

Since (Y,S) is not thickly sensitive, $\mathrm{Eq_{syn}}(Y,S) \neq \varnothing$ by Proposition 5.3. Let $y_0 \in \mathrm{Eq_{syn}}(Y,S)$. Thus, for each $\delta > 0$ there exists opene $U_Y \subset Y$ containing y_0 and syndetic $\mathcal{N} \subset \mathbb{N}$ such that $\varrho(S^n y, S^n y_0) \leq \delta$ whenever $y \in U_Y$ and $n \in \mathcal{N}$. Hence $\varrho(S^n y_1, S^n y_2) \leq 2\delta$ whenever $y_1, y_2 \in U_Y$ and $n \in \mathcal{N}$, and finally $\varrho_1(T^n x_1, T^n x_2) \leq 2\delta$ by the construction of ϱ_1 whenever $x_1, x_2 \in (U_Y \times U_Y) \cap X$ and $n \in \mathcal{N}$. Now take any $x \in X$ and a positive integer m with $T^m x \in (U_Y \times U_Y) \cap X$ as Δ_Y is the unique minimal subsystem of (X,T). Then, in fact, there exists opene $U \subset X$ containing x with $T^m U \subset (U_Y \times U_Y) \cap X$, thus $\varrho_1(T^{m+n}x, T^{m+n}x') \leq 2\delta$ by the construction of ϱ_1 whenever $x' \in U$ and $n \in \mathcal{N}$. That implies $x \in \mathrm{Eq_{syn}}(X,T)$, because $m + \mathcal{N}$ is also a syndetic set.

2. *Case of* $\mathrm{Tran}(X',T') \subsetneq Eq_{syn}(X',T') \subsetneq X'$. Now we take (X',T') to be the system constructed by collapsing Δ_Y into a fixed point p_0 of (X,T). Let $\pi : (X,T) \to (X',T')$ be the corresponding factor map. It is easy to see that (X',T') is an invertible nonminimal E-system and π is an almost open factor map. It implies that (X',T') is not thickly sensitive, because (X,T) is not thickly sensitive.

Observe that λ is an ergodic measure with full support $X \supsetneq \Delta_Y$, and $\lambda(\Delta_Y) = 0$ by the ergodicity of λ (note $T\Delta_Y = \Delta_Y$). Now take a measure λ' as the projection of measure λ over (X',T') (with respect to π). It is not hard to show that (X,T,λ) and (X',T',λ') are measure-theoretic isomorphic. Therefore the measure-theoretic λ'-entropy of (X',T') is equal to the measure-theoretic λ-entropy of (X,T). In particular, (X',T') has positive topological entropy.

Moveover, from the above construction it follows that λ' is an ergodic measure of (X',T') such that X' is the support of λ' (and hence the system (X',T') is topologically ergodic). (X',T') has positive measure-theoretic λ'-entropy and X' contains no isolated points. Thus $X' \setminus \mathrm{Tran}(X',T')$ is a dense subset of X', in particular, $X' \setminus \mathrm{Tran}(X',T') \supsetneq \{p_0\}$ (see [26]). In fact, $\pi : X \setminus \Delta_Y \to X' \setminus \{p_0\}$ is a homeomorphism. Therefore we obtain that $X' \setminus \{p_0\} \subset \mathrm{Eq_{syn}}(X',T')$, as $\mathrm{Eq_{syn}}(X,T) = X$.

Finally we are going to show that $p_0 \notin \mathrm{Eq_{syn}}(X',T')$ and hence $\mathrm{Tran}(X',T') \subsetneq X' \setminus \{p_0\} = \mathrm{Eq_{syn}}(X',T') \subsetneq X'$. Recall that (X',T') is an invertible topologically ergodic system and $(X',(T')^{-1})$ is also a topologically ergodic system from the definition. Let $x_* \in \mathrm{Tran}(X',(T')^{-1})$ and take $0 < \delta < \mathrm{dist}(\{x_*\},\{p_0\})$. Choose opene U_* containing x_* and opene U_0 containing p_0. By the proof of Lemma 4.4 one has that $N_{(T')^{-1}}(U_*, U_0)$ $(= N_{T'}(U_0, U_*))$ is thickly syndetic, which implies directly from the definition of δ that $N_{T'}(U_0, \delta)$ is thickly syndetic, as p_0 is a fixed point of the system (X',T'). Hence $p_0 \notin \mathrm{Eq_{syn}}(X',T')$. The construction is done. $\square$

Moreover, as a conclusion we have the following

Proposition 5.6. *Let (X,T) be a topological dynamical system. Assume that (X,T) satisfies one of the following conditions.*

1. *(X,T) is equicontinous.*
2. *For every $\varepsilon > 0$ there exist a $\delta > 0$ and a syndetic subset $\mathcal{A} \subset \mathbb{N}$ such that $\varrho(x,y) < \delta$ implies $\varrho(T^n x, T^n y) < \varepsilon$ for any $x, y \in X$ and $n \in \mathcal{A}$.*
3. *$Eq_{syn}(X,T) = X$.*
4. *For every $\varepsilon > 0$ there exist a $\delta > 0$ and $m \in \mathbb{N}$ such that $\varrho(x,y) < \delta$ implies $\min_{0 \leq i \leq m} \varrho(T^{n+i}x, T^{n+i}y) < \varepsilon$ for any $x, y \in X$ and $n \in \mathbb{N}$.*

Then (1) $\Longleftrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4).

Proof. In fact, it suffices to prove (2) $\Longrightarrow$ (1) and (3) $\Longrightarrow$ (4).
(2) $\Longrightarrow$ (1): Let $\varepsilon > 0$. By the condition (2) there exist $\varepsilon > \delta > 0$ and a syndetic set $\mathcal{A} \subset \mathbb{N}$ such that $\varrho(x, y) < \delta$ implies $\varrho(T^n x, T^n y) < \varepsilon$ for any $x, y \in X$ and $n \in \mathcal{A}$. Since syndetic sets have "bounded gaps" in $\mathbb{N}$, there exists $m \in \mathbb{N}$ such that $\{n + i : n \in A, i = 0, 1, \dots, m\} \supset \{m + 1, m + 2, \dots\}$. Therefore there exists $\delta' > 0$ such that $\varrho(x, y) < \delta'$ implies $\varrho(T^i x, T^i y) < \delta < \varepsilon$ (hence $\varrho(T^{n+i} x, T^{n+i} y) < \varepsilon$, $n \in \mathcal{A}$) for any $x, y \in X$ and $i = 0, 1, \dots, m$. So, $\varrho(x, y) < \delta'$ implies $\varrho(T^j x, T^j y) < \varepsilon$ for any $x, y \in X$ and $j \in \mathbb{N}$, i.e. (X, T) is equicontinuous.
(3) $\Longrightarrow$ (4): Let $\varepsilon > 0$. Since $\mathrm{Eq}_{\mathrm{syn}}(X, T) = X$, for any $x \in X$ there exist opene $U_x \subset X$ containing x and a syndetic set $\mathcal{A}_x \subset \mathbb{N}$ such that $\varrho(T^n x, T^n x') < \varepsilon$ for all $x' \in U_x$ whenever $n \in \mathcal{A}_x$ (and hence $\varrho(T^n x', T^n x'') < 2\varepsilon$ whenever $x', x'' \in U_x$). We take $m_x \in \mathbb{N}$ such that $\{n, n + 1, \dots, n + m_x\} \cap \mathcal{A}_x \neq \varnothing$ for each $n \in \mathbb{N}$, and therefore $\min\limits_{0 \le i \le m_x} \varrho(T^{n+i} x', T^{n+i} x'') < 2\varepsilon$ for any $x', x'' \in U_x$ and $n \in \mathbb{N}$. Observe that X is a compact metric space. So, we can take a set of points $\{x_1, \dots, x_s\} \subset X$ such that $\{U_{x_j} : j = 1, \dots, s\}$ forms an open cover of X. Then there exists $\delta > 0$ such that any points $x, y \in X$ with $\varrho(x, y) < \delta$ are contained in some U_{x_j}. Set $m = \max\{m_{x_j} : j = 1, \dots, s\}$. So, $\min_{0 \le i \le m} \varrho(T^{n+i} x, T^{n+i} y) \le \min_{0 \le i \le m_{x_j}} \varrho(T^{n+i} x, T^{n+i} y) < 2\varepsilon$ for each $n \in \mathbb{N}$. This finishes the proof. $\square$

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