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Abstract. We study $\mathscr{D}$-elliptic sheaves in terms of their associated modules, which we call Drinfeld-Stuhler modules. We prove some basic results about Drinfeld-Stuhler modules and their endomorphism rings, and then examine the existence and properties of Drinfeld-Stuhler modules with large endomorphism algebras, which are analogous to CM and supersingular Drinfeld modules. Finally, we examine the fields of moduli of Drinfeld-Stuhler modules.

1. Introduction

The idea of $\mathscr{D}$-elliptic sheaves was proposed by Ulrich Stuhler, as a natural generalization of Drinfeld's elliptic sheaves [9], [4]. The moduli varieties of $\mathscr{D}$-elliptic sheaves were studied by Laumon, Rapoport and Stuhler in [17], with the aim of proving the local Langlands correspondence for GL_d in positive characteristic. In this paper, we study some of the basic arithmetic properties of $\mathscr{D}$-elliptic sheaves, and in particular their endomorphism rings.

Let C be a smooth, projective, geometrically connected curve over the finite field $\mathbb{F}_q$. Let F be the function field of C. Let $\infty \in C$ be a fixed closed point, and $A \subset F$ be the ring of functions regular outside ∞. Denote by F_∞ the completion of F at ∞. Let D be a central division algebra over F of dimension d^2, which is split at ∞, i.e., $D \otimes_F F_\infty$ is isomorphic to the matrix algebra $M_d(F_\infty)$. Fix a maximal A-order O_D in D. An *A-field* is a field L equipped with an A-algebra structure, i.e., with a homomorphism $\gamma : A \to L$. A $\mathscr{D}$-elliptic sheaf over an A-field L is essentially a vector bundle of rank d^2 on $C \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(L)$ equipped with an action of O_D and with a meromorphic O_D-linear Frobenius satisfying certain conditions (see Section 3). One can think of these objects as being analogous to abelian varieties equipped with an action of an order in a central division algebra over $\mathbb{Q}$.

In this paper, we study $\mathscr{D}$-elliptic sheaves in terms of their associated modules, which we call *Drinfeld-Stuhler modules*. The relationship between $\mathscr{D}$-elliptic sheaves and Drinfeld-Stuhler modules is very similar to the relationship between elliptic sheaves and Drinfeld modules; cf. [9], [4]. Let L be an A-field. Let τ be the Frobenius endomorphism relative to $\mathbb{F}_q$, i.e., the map $x \mapsto x^q$. The ring of $\mathbb{F}_q$-linear endomorphisms $\mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a,L})$ of the additive algebraic group scheme $\mathbb{G}_{a,L}$ over L is canonically isomorphic to the skew polynomial ring $L[\tau]$ with the commutation relation $\tau b = b^q\tau$, $b \in L$. A Drinfeld-Stuhler O_D-module over L is an embedding

$$\phi : O_D \to M_d(L[\tau])$$

satisfying certain conditions (see Definition 2.2). This concept implicitly appears in [17, §3], although it does not play an important role in that paper since its "shtuka" incarnation

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(the $\mathscr{D}$-elliptic sheaf) seems better suited for the study of moduli spaces. The advantage of the concept of Drinfeld-Stuhler module is that it is relatively elementary, and one can easily write down explicit examples of these objects (see Section 2). We expect that the reader familiar with the theory of Drinfeld modules, but not necessarily with [17], will find it easier to understand the results of this paper in terms of Drinfeld-Stuhler modules, rather than $\mathscr{D}$-elliptic sheaves.

Some of the properties of Drinfeld-Stuhler modules are very similar to, and in fact can be deduced from, the properties of Drinfeld modules, e.g., uniformizability and CM theory. There are also some notable differences. The most significant is probably the fact that the modular varieties of Drinfeld-Stuhler modules are projective [17], unlike the Drinfeld modular varieties, which are affine [8]. Another difference is that Drinfeld-Stuhler modules can be defined only over fields which split D (see Lemma 2.3), so there are no Drinfeld-Stuhler modules over F itself, even in the simplest case when $A = \mathbb{F}_q[T]$.

The main results of this paper concern the endomorphism ring $\mathrm{End}_L(\phi)$ of a Drinfeld-Stuhler O_D-module ϕ over L, and its field of moduli. By definition, $\mathrm{End}_L(\phi)$ is the centralizer of $\phi(O_D)$ in $M_d(L[\tau])$. In Section 4, we prove that $\mathrm{End}_L(\phi)$ is a projective A-module of rank $\leq d^2$ such that $\mathrm{End}_L(\phi) \otimes_A F_\infty$ is isomorphic to a subalgebra of the central division algebra over F_∞ with invariant $-1/d$. Moreover, if $\gamma : A \to L$ is injective, then $\mathrm{End}_L(\phi)$ is an A-order in an *imaginary* field extension K of F which embeds into D, so, in particular, $\mathrm{End}_L(\phi)$ is commutative and its rank over A divides d. ("Imaginary" in this context means that there is a unique place ∞' of K over ∞.) Next, we study the Drinfeld-Stuhler modules with large endomorphism rings, namely the appropriate analogues of complex multiplication and supersingularity. The results here are similar to those for Drinfeld modules; cf. [10], [11], [13]. We prove that if K is an imaginary field extension of F of degree d which embeds into D, then, up to isomorphism, the number of Drinfeld-Stuhler O_D-modules over $\overline{F}$ with $\mathrm{End}_{\overline{F}}(\phi) = O_K$ is finite and non-zero, and any such module can be defined over the Hilbert class field of K(=the maximal unramified abelian extension of K in which ∞' totally splits); see Theorem 4.9. In Section 5, we give several equivalent conditions for a Drinfeld-Stuhler O_D-module to be "supersingular". For a non-zero prime ideal $\mathfrak{p} \lhd A$, the endomorphism ring of a supersingular Drinfeld-Stuhler module over the algebraic closure of $A/\mathfrak{p}$ is a maximal A-order in the central division algebra over F with invariants equal to the negatives of invariants of D, except at $\mathfrak{p}$ and ∞, where the invariants are $1/d$ and $-1/d$, respectively. In Section 6, we prove a Hilbert's 90-th type theorem for $M_d(L^{\mathrm{sep}}[\tau])$, and use this theorem and our classification of automorphism groups of Drinfeld-Stuhler modules to show that if d and $q^d - 1$ are coprime, then a field of moduli for a Drinfeld-Stuhler module is a field of definition. This implies that the coarse moduli scheme of Drinfeld-Stuhler O_D-modules has no L-rational points for field extensions L/F which do not split D, assuming d and $q^d - 1$ are coprime.

2. Basic properties and examples

Notation and Terminology 2.1. Let F be the field of rational functions on a smooth and geometrically irreducible projective curve C defined over the finite field $\mathbb{F}_q$ of q elements, where q is a power of a prime number. Fix a place ∞ of F (equiv. a closed point of C), and let A be the subring of F consisting of functions which are regular away from ∞. A is a

Dedekind domain. An *imaginary* field extension of F is an extension K/F in which ∞ does not split. For a field L we denote by L^{alg} (resp. L^{sep}) its algebraic (resp. separable) closure.

For a place v of F, we denote by F_v, O_v, $\mathbb{F}_v$ the completion of F at v, the ring of integers in F_v, and the residue field at v, respectively. If $v \neq \infty$, so corresponds to a non-zero prime ideal $\mathfrak{p}$ of A, we sometimes write $A_{\mathfrak{p}}$ or A_v instead of O_v, and $\mathbb{F}_{\mathfrak{p}}$ instead of $\mathbb{F}_v$.

Given a unitary ring R, we denote by $R^\times$ the group of multiplicative units in R. Let $M_d(R)$ be the ring of $d \times d$ matrices with entries in R; the group of units in $M_d(R)$ is denoted by $\mathrm{GL}_d(R)$. Given $r_1, \dots, r_d \in R$, we denote by $\mathrm{diag}(r_1, \cdots, r_d) \in M_d(R)$ the matrix which has r_i as the (i,i)-th entry, $1 \leq i \leq d$, and zeros everywhere else.

Let D be a central division algebra over F. Let $\mathrm{Ram}(D)$ be the set of places of F which ramify in D, i.e., $v \in \mathrm{Ram}(D)$ if and only if $D_v := D \otimes_F F_v$ is not isomorphic to $M_d(F_v)$. From now on we assume that $\infty \notin \mathrm{Ram}(D)$, so that the places in $\mathrm{Ram}(D)$ correspond to prime ideals of A. We denote

$$\mathfrak{r}(D) = \prod_{\mathfrak{p} \in \mathrm{Ram}(D)} \mathfrak{p}.$$

Fix a maximal A-order O_D in D. Note that A is the center of O_D.

Let L be an A-field, i.e., a field equipped with an A-algebra structure $\gamma : A \to L$. The *A-characteristic of* L is the prime ideal $\mathrm{char}_A(L) := \ker(\gamma) \lhd A$; we say that L has *generic A-characteristic* if $\ker(\gamma) = 0$. We assume throughout that $\mathrm{char}_A(L)$ does not divide $\mathfrak{r}(D)$.

We have a canonical isomorphism (cf. [26, Prop. 1.1])

$$\mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a,L}^d) \cong M_d(L[\tau]).$$

We can write the elements of $M_d(L[\tau])$ as finite sums $\sum_{i\geq 0} B_i \tau^i$, where $B_i \in M_d(L)$ and $\tau : L^d \to L^d$ is the map given by $\tau : (x_1, \dots, x_d)^t \mapsto (x_1^q, \dots, x_d^q)^t$. An element $S = \sum_{i \geq 0} B_i \tau^i \in M_d(L[\tau])$ acts on the tangent space $\mathrm{Lie}(\mathbb{G}_{a,L}^d) \cong L^d$ via $\partial(S) := B_0$. It is clear that

$$\partial : M_d(L[\tau]) \to M_d(L), \quad S \mapsto \partial(S)$$

is a surjective homomorphism.

Definition 2.2. A *Drinfeld-Stuhler O_D-module* defined over L is an embedding

$$\begin{aligned} \phi : O_D &\to M_d(L[\tau]) \\ b &\mapsto \phi_b \end{aligned}$$

satisfying the following conditions:

(i) For any non-zero $b \in O_D$ the kernel $\phi[b] := \ker \phi_b$ of the endomorphism ϕ_b of $\mathbb{G}_{a,L}^d$ is a finite group scheme over L of order $\#(O_D/O_D \cdot b)$.

(ii) The composition
$$A \to O_D \xrightarrow{\phi} M_d(L[\tau]) \xrightarrow{\partial} M_d(L)$$
maps $a \in A$ to $\mathrm{diag}(\gamma(a), \dots, \gamma(a))$.

The action of $\phi(O_D)$ on the tangent space $\mathrm{Lie}(\mathbb{G}_{a,L}^d)$ gives a homomorphism

$$\partial_\phi : O_D \to M_d(L),$$

which extends linearly to a homomorphism

$$\partial_{\phi,L} : O_D \otimes_A L \to M_d(L).$$

Lemma 2.3. *$\partial_{\phi,L}$ is an isomorphism.*

Proof. Both sides are rings with 1, so $\partial_{\phi,L}$ is non-zero, as it maps 1 to 1. If L has generic A-characteristic, then L is an extension of F, hence $O_D \otimes_A L$ is a central simple algebra over L. Therefore, $\partial_{\phi,L}$ is injective, and comparing the dimensions we see that it is in fact an isomorphism. Now assume that $\mathrm{char}_A(L) = \mathfrak{p} \neq 0$. Then $O_D \otimes_A L$ is obtained by extension of scalars from $O_D \otimes_A A_\mathfrak{p} \to O_D \otimes_A \mathbb{F}_\mathfrak{p}$. On the other hand, $O_D \otimes_A A_\mathfrak{p} \cong M_d(A_\mathfrak{p})$ since $\mathfrak{p} \nmid \mathfrak{r}(D)$. Now it is clear that $O_D \otimes_A \mathbb{F}_\mathfrak{p} \cong M_d(\mathbb{F}_\mathfrak{p})$, hence $O_D \otimes_A L \cong M_d(\mathbb{F}_\mathfrak{p}) \otimes_{\mathbb{F}_\mathfrak{p}} L \cong M_d(L)$. Since $M_d(L)$ is a central simple algebra over L, the previous argument again implies that $\partial_{\phi,L}$ is an isomorphism. $\square$

Remarks 2.4. (1) If $d = 1$, so that $D = F$, then Definition 2.2 becomes the definition of a Drinfeld A-module of rank 1. We will implicitly assume from now on that $d \geq 2$.

(2) Let $b \in O_D$. If we consider D as a vector space over F, then the left multiplication by b induces a linear transformation. Let $\det(b)$ denote the determinant of this linear transformation. Note that O_D is an A-lattice in D in the sense of [22, Ch. III, §1]. By Proposition 3 in [22, Ch. III, §1], we have $\#(O_D/O_Db) = \#(A/\det(b)A)$. Finally, recall that $(-1)^d \det(b) =: \mathrm{Nr}(b)$ is the *non-reduced norm* of b; cf. [21, §9a]. Hence condition (i) is equivalent to saying that $\phi[b]$ is a finite group scheme of order $\#(A/\mathrm{Nr}(b)A)$.

(3) The characteristic polynomial of $\partial_{\phi,L}(b)$is the polynomial that one obtains by applying γ to the coefficients of the *reduced characteristic polynomial* of b; cf. [21, p. 113].

(4) We recall some necessary and sufficient conditions for a finite field extension L of F to *split* D, i.e., $D \otimes_F L \cong M_d(L)$, since, by Lemma 2.3, if there is a Drinfeld-Stuhler O_D-module defined over L, then L necessarily splits D. (In particular, a Drinfeld-Stuhler module cannot be defined over F itself.) Let $\mathfrak{p} \lhd A$. The Wedderburn structure theorem says that $D \otimes_F F_\mathfrak{p} \cong M_{\kappa_\mathfrak{p}}(D'_\mathfrak{p})$, where $D'_\mathfrak{p}$ is a central division algebra of dimension $d_\mathfrak{p}^2 = (d/\kappa_\mathfrak{p})^2$. By [21, (32.15)], L splits D if and only if for each prime $\mathfrak{p} \lhd A$ and for all primes $\mathfrak{P}$ of L lying above $\mathfrak{p}$, $d_\mathfrak{p}$ divides $[L_\mathfrak{P} : K_\mathfrak{p}]$. Moreover, if $[L : F] = d$, then L splits D if and only if L embeds into D. Finally, by [21, (7.15)], every maximal subfield L of D contains F and $[L : F] = d$.

Definition 2.5. Let ϕ, ψ be Drinfeld-Stuhler O_D-modules over L. A *morphism* $u : \phi \to \psi$ over L is $u \in M_d(L[\tau])$ is such that $u\phi_b = \psi_b u$ for all $b \in O_D$. We say that u is an *isomorphism* if u is invertible in the ring $M_d(L[\tau])$. We say that u is an *isogeny* if $\ker(u)$ is a finite group scheme over L. We say that a Drinfeld-Stuhler O_D-module ϕ over L *can be defined* over a subfield K of L (equiv. K is a *field of definition for* ϕ) if there is a Drinfeld-Stuhler O_D-module ψ over K which is isomorphic to ϕ over L. The set of morphisms $\phi \to \psi$ over L is an A-module $\mathrm{Hom}_L(\phi, \psi)$, where A acts by $a \circ u := u\phi_a$. (Using the fact that $a \in A$ is in the center of O_D, it is easy to check that $u\phi_a \in \mathrm{Hom}_L(\phi, \psi)$.) We denote $\mathrm{End}_L(\phi) = \mathrm{Hom}_L(\phi, \phi)$; this is a subring of $M_d(L[\tau])$. For an arbitrary field extension $\mathcal{L}$ of L we can consider ϕ, ψ as Drinfeld-Stuhler O_D-modules over $\mathcal{L}$, so we have the corresponding module $\mathrm{Hom}_\mathcal{L}(\phi, \psi)$ of morphisms over $\mathcal{L}$. We will denote $\mathrm{Hom}(\phi, \psi) = \mathrm{Hom}_{L^{\mathrm{alg}}}(\phi, \psi)$ and $\mathrm{End}(\phi) = \mathrm{End}_{L^{\mathrm{alg}}}(\phi)$.

Lemma 2.6. *If $u \in \mathrm{Hom}_L(\phi, \psi)$ is non-zero, then u is an isogeny.*

Proof. The ring $O_D \otimes_A L \cong M_d(L)$ acts on the tangent space $\mathrm{Lie}(\mathbb{G}_{a,L}^d)$ via $\partial_{\phi,L}$ and $\partial_{\psi,L}$. Suppose $u \in \mathrm{Hom}_L(\phi, \psi)$ is non-zero and has infinite kernel. Since $\ker(u) \subset \mathbb{G}_{a,L}^d$ is an algebraic subgroup with infinitely many geometric points, the connected component $\ker(u)^0$ of the identity has positive dimension. Then u acts on the tangent space by a linear transformation ∂_u which has non-trivial kernel $0 \subsetneq \ker(\partial_u) \subsetneq L^d$. Since $\partial_{\phi,L}\partial_u = \partial_u\partial_{\psi,L}$, the space $\ker(\partial_u)$ is invariant under $M_d(L)$ acting via $\partial_{\psi,L}$, which leads to a contradiction. $\square$

Lemma 2.7. *Let ϕ and ψ be Drinfeld-Stuhler O_D-modules over L. Assume L has generic A-characteristic. Then:*

(1) *The map $\partial : \mathrm{Hom}_L(\phi, \psi) \to M_d(L)$ is injective.*
(2) *$\mathrm{End}_L(\phi)$ is a commutative ring.*

Proof. Suppose $u \in \mathrm{Hom}_L(\phi, \psi)$ is non-zero but $\partial(u) = 0$. Then $u = B_m\tau^m + B_{m+1}\tau^{m+1} + \cdots$, where $m \geq 1$ is the smallest index such that $B_m \neq 0$. For $a \in A$, the equality $u\phi_a = \psi_a u$ leads to $B_m\gamma(a)^{q^m} = \gamma(a)B_m$. Since $B_m \in M_d(L)$ has at least one non-zero entry, we must have $\gamma(a)^{q^m} = \gamma(a)$. Since a was arbitrary, this implies $\gamma(A) \subseteq \mathbb{F}_{q^m}$. On the other hand, since L has generic A-characteristic, $\gamma(A)$ is infinite, which leads to a contradiction.

By the first claim, ∂ maps $\mathrm{End}_L(\phi)$ isomorphically to its image in $M_d(L)$. On the other hand, $\partial(\mathrm{End}_L(\phi))$ is in the centralizer of $\partial_\phi(O_D)$. By Lemma 2.3, $\partial_\phi(O_D)$ contains a basis of $M_d(L)$, so $\partial(\mathrm{End}_L(\phi))$ is in the centrer of $M_d(L)$, which consists of diagonal matrices. Hence ∂ identifies $\mathrm{End}_L(\phi)$ with an A-subalgebra of L. $\square$

Lemma 2.8. *Let ϕ and ψ be Drinfeld-Stuhler O_D-modules over L. Let $\mathcal{L}$ be a field extension of L in which L is separably closed. Then any morphism $u : \phi \to \psi$ over $\mathcal{L}$ is already defined over L. In particular, $\mathrm{Hom}_{L^{\mathrm{sep}}}(\phi, \psi) = \mathrm{Hom}(\phi, \psi)$.*

Proof. This statement is the analogue of a well-known theorem of Chow for abelian varieties. Let $R = \mathcal{L} \otimes_L \mathcal{L}$. Since L is separably closed in $\mathcal{L}$, $\mathrm{Spec}(R)$ is irreducible. We can consider ϕ and ψ as Drinfeld-Stuhler O_D-modules over R, i.e., as embeddings $O_D \to \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a,R}^d)$, which we will denote as $\phi_{/R}$, $\psi_{/R}$. By a theorem of Grothendieck on descent of morphisms [7, Thm. 3.1], it is enough to show that the two pullbacks $p_j^*(u) : \mathbb{G}_{a,R}^d \to \mathbb{G}_{a,R}^d$ of u along the projections $p_1, p_2 : \mathrm{Spec}(R) \rightrightarrows \mathrm{Spec}(\mathcal{L})$ are equal. Let $0 \neq a \in A$ be an element coprime to $\mathrm{char}_A(L)$. Following the scheme-theoretic proof of Chow's theorem [7, Thm. 3.19], we conclude that $p_1^*(u)$ and $p_2^*(u)$ coincide on each $\phi_{/R}[a^n]$ for all $n \geq 1$. Hence the kernel of the morphism $p_1^*(u) - p_2^*(u) : \phi_{/R} \to \psi_{/R}$ is not a finite group scheme over R. On the other hand, the proof of Lemma 2.6 can be extended to $\mathrm{Hom}_R(\phi, \psi)$, which leads to a contradiction. $\square$

Lemma 2.9. *Let ϕ be a Drinfeld-Stuhler O_D-module over L and $b \in O_D$. The kernel of ϕ_b is étale over L if and only if $\mathrm{Nr}(b)$ is coprime to $\mathrm{char}_A(L)$.*

Proof. This follows from [17, Prop. 3.10]. $\square$

If $u \in \mathrm{Hom}_L(\phi, \psi)$, then it is clear that the group scheme $\ker(u) \subset \mathbb{G}_{a,L}^d$ is invariant under $\phi(O_D)$. Conversely, we have the following:

Lemma 2.10. *Assume L is algebraically closed. Let $H \subset \mathbb{G}_{a,L}^d$ be a finite étale subgroup scheme which is invariant under $\phi(O_D)$. There is a Drinfeld-Stuhler O_D-module ψ and an isogeny $u : \phi \to \psi$ whose kernel is H.*

Proof. From the discussion on page 155 in [12] it follows that there is $u \in \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a,L}^d)$ with $\ker(u) = H$. Let $b \in O_D$. Consider the endomorphism $u\phi_b$ of $\mathbb{G}_{a,L}^d$. Since H is invariant under $\phi(O_D)$, we have $H \subseteq \ker(u\phi_b)$. Then we can factor $u\phi_b$ as $\psi_b u$ for some $\psi_b \in \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a,L}^d)$. It is easy to see that $b \mapsto \psi_b$ gives an embedding $O_D \to M_d(L[\tau])$ and $\#\psi[b] = \#\phi[b]$. Since $\partial(u) \in M_d(L)$ is an invertible matrix, and $\partial_\phi(a)$ $(a \in A)$ is the scalar matrix $\mathrm{diag}(\gamma(a), \dots, \gamma(a))$, we also get that $\partial_\psi(a) = \mathrm{diag}(\gamma(a), \dots, \gamma(a))$. □

Lemma 2.11. *Assume L is algebraically closed and has generic A-characteristic. Let $u : \phi \to \psi$ be an isogeny. There is an isogeny $w : \psi \to \phi$ such that $wu = \phi_a$ and $uw = \psi_a$ for some $a \in A$. This implies that $\mathrm{End}_L(\phi) \otimes_A F$ is a field extension of F (see also Theorem 4.1).*

Proof. Let $H = \ker(u)$. Then H is invariant under $\phi(O_D)$, and in particular, under $\phi(A)$. We can consider H as a finite A-module, and as such, there is $a \in A$ which annihilates H. Therefore, $H \subseteq \phi[a]$. Let H' be the image of $\phi[a]$ under u. We claim that H' is invariant under $\psi(O_D)$. To see this let $b \in O_D$ and $h \in H'$. We need to show that $\psi_b(h) \in H'$. Now $h = u(x)$ for some $x \in \phi[a]$, so $\psi_b(h) = \psi_b u(x) = u\phi_b(x)$. On the other hand, using the fact that a is in the center of O_D, we have $\phi_a\phi_b(x) = \phi_{ab}(x) = \phi_{ba}(x) = \phi_b\phi_a(x) = 0$. Thus, $\phi_b(x) \in \phi[a]$ and $\psi_b(h) \in H'$. Using Lemma 2.10, we get an isogeny $w : \psi \to \varphi$ of Drinfeld-Stuhler O_D-modules with kernel H'. The composition $wu : \phi \to \varphi$ has kernel $\phi[a]$, hence $\phi \cong \varphi$. Finally, note that $uwu = u\phi_a = \psi_a u$, which implies $uw = \psi_a$, as u is an isogeny. The existence of w implies that $\mathrm{End}_L(\phi) \otimes_A F$ is a division algebra over F. Since $\mathrm{End}_L(\phi) \otimes_A F$ is also commutative by Lemma 2.7, it is a field. □

As a consequence of the Grunwald-Wang theorem, every central simple F-algebra is cyclic; see [21, (32.10)]. This means that there is a Galois extension K/F with $\mathrm{Gal}(K/F) \cong \mathbb{Z}/d\mathbb{Z}$, a generator σ of $\mathrm{Gal}(K/F)$, and $f \in F^\times$ such that

$$D \cong (K/F, \sigma, f) = \bigoplus_{i=0}^{d-1} Kz^i, \qquad z \cdot y = \sigma(y)z, \quad z^d = f, \quad y \in K, \tag{2.1}$$

where we identify z^0 with the identity element of D. Moreover, one can choose f to be in A; cf. [21, (30.4)].

Assume K/F is imaginary and let O_K be the integral closure of A in K. Consider the A-order

$$O_D = \bigoplus_{i=0}^{d-1} O_K z^i \tag{2.2}$$

in D. This order is not necessarily maximal. It is not hard to compute that its discriminant is equal to $f^{d(d-1)}\mathrm{disc}(K/F)^d$; see [5, Cor. 7]. For an A-order in D to be maximal, it is necessary and sufficient for its discriminant to be equal to the discriminant of a maximal order. The discriminant of a maximal order in D can be computed from the invariants of D; see [21, Thm. 32.1] and [5, Prop. 25]. For $\mathfrak{p} \in \mathrm{Ram}(D)$, let the reduced fraction $s_\mathfrak{p}/r_\mathfrak{p} \in \mathbb{Q}/\mathbb{Z}$ be the invariant of D at $\mathfrak{p}$. Set $r = \mathrm{lcm}(r_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Ram}(D))$. Then a maximal order in D has discriminant $\left(\prod_{\mathfrak{p}\in\mathrm{Ram}(D)} \mathfrak{p}^{r-\frac{r}{r_\mathfrak{p}}}\right)^r$. For example, if d is prime, then the discriminant of a maximal order is equal to $\mathfrak{r}(D)^{d(d-1)}$. Comparing the discriminant of O_D with the discriminant of a maximal order gives an explicit criterion for the order O_D to be maximal; see [5, Cor. 26].

Example 2.12. Assume the order O_D in (2.2) is maximal. Let $\varphi : O_K \to L[\tau]$ be a Drinfeld O_K-module of rank 1 defined over some field L. Observe that the restriction of φ to A defines a Drinfeld A-module of rank d over L. Let

$$\phi : O_D \to M_d(L[\tau])$$

be defined as follows:

$$\phi_\alpha = \mathrm{diag}(\varphi_\alpha, \varphi_{\sigma\alpha}, \dots, \varphi_{\sigma^{d-1}\alpha}), \quad \alpha \in O_K,$$
$$\phi_z = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ \varphi_f & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Using the fact that $\varphi_\alpha\varphi_f = \varphi_f\varphi_\alpha$, it is easy to check that $\phi_z\phi_\alpha = \phi_{\sigma\alpha}\phi_z$ and $\phi_z^d = \phi_f$. Thus, ϕ is an embedding. Moreover, for $a \in A$, we have $\phi_a = \mathrm{diag}(\varphi_a, \dots, \varphi_a)$, which maps under ∂ to $\mathrm{diag}(\gamma(a), \dots, \gamma(a))$ by the definition of Drinfeld modules. Finally,

$$\#\phi[z] = \#\ker\varphi_f = \#(A/fA)^d = \#(A/f^dA) = \#(A/\mathrm{Nr}(z)A),$$

and

$$\#\phi[\alpha] = \#(O_K/O_K\alpha)^d = \#(A/\mathrm{Nr}(\alpha)A).$$

Thus, ϕ is a Drinfeld-Stuhler O_D-module.

Example 2.13. As a more explicit version of Example 2.12, let $A = \mathbb{F}_q[T]$ and $F = \mathbb{F}_q(T)$. Let $\mathbb{F}_{q^d}$ denote the degree d extension of $\mathbb{F}_q$. Let $K = \mathbb{F}_{q^d}(T)$, which is a cyclic imaginary extension as ∞ is inert in K. In this case, $O_K = \mathbb{F}_{q^d}[T]$ and the Galois group $\mathrm{Gal}(K/F) \cong \mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)$ has a canonical generator σ given by the Frobenius automorphism (i.e., σ induces the qth power morphism on $\mathbb{F}_{q^d}$). Let $\mathfrak{r} \in A$ be a monic square-free polynomial with prime decomposition $\mathfrak{r} = \mathfrak{p}_1 \cdots \mathfrak{p}_m$. Assume the degree of each prime $\mathfrak{p}_i$ is coprime to d. Let D be the cyclic algebra $D = (K/F, \sigma, \mathfrak{r})$. Then, by [12, Thm. 4.12.4], for any prime $\mathfrak{p} \lhd A$ one has

$$\mathrm{inv}_\mathfrak{p}(D) = \frac{\mathrm{ord}_\mathfrak{p}(\mathfrak{r})\deg(\mathfrak{p})}{d} \in \mathbb{Q}/\mathbb{Z}. \tag{2.3}$$

Since the sum of the invariants of D over all places of F is 0, if we assume that $\sum_{i=1}^m \deg(\mathfrak{p}_i)$ is divisible by d, then D will be split at ∞ and will ramify only at the primes of A dividing $\mathfrak{r}$.

The order $O_D = \bigoplus_{i=0}^{d-1} O_K z^i$ is maximal in D, since its discriminant is equal to $\mathfrak{r}^{d(d-1)}$. Let L be an O_K-field and $\gamma : A \to O_K \to L$ be the composition homomorphism. Let $\varphi : O_K \to L[\tau]$ be defined by $\varphi_T = \gamma(T) + \tau^d$; this is a rank-1 Drinfeld O_K-module and a rank-d Drinfeld A-module. Then

$$\phi : O_D \to M_d(L[\tau])$$

given by

$$\phi_T = \mathrm{diag}(\varphi_T, \dots, \varphi_T),$$
$$\phi_h = \mathrm{diag}(h, h^q, \dots, h^{q^{d-1}}), \quad h \in \mathbb{F}_{q^d},$$
$$\phi_z = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ \varphi_{\mathfrak{r}} & 0 & 0 & \cdots & 0 \end{pmatrix},$$

is a Drinfeld-Stuhler module.

Remark 2.14. It is easy to see from the previous example that for general $b \in O_D$ the kernel $\phi[b]$ is not necessarily O_D-invariant, hence condition (i) in the definition of Drinfeld-Stuhler modules cannot be stated in the stronger form of isomorphism of left O_D-modules: $\phi[b] \cong O_D/O_D \cdot b$. Indeed, take $d = 2$ and $b = h + z$ with $h \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. A non-zero element $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{G}^2_{a,K}(\overline{K})$ is in $\phi[b]$ only if $h\alpha + \beta = 0$. On the other hand, $\phi_h \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} h\alpha \\ h^q\beta \end{pmatrix}$, so $\phi_h \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \phi[b]$ only if $h^2\alpha + h^q\beta = 0$. This implies $h^2\alpha = h^{q+1}\alpha$. Since $h^{q-1} \neq 1$, we must have $\alpha = 0$, but then $\beta = 0$.

3. O_D-motives, $\mathscr{D}$-elliptic sheaves and O_D-lattices

We keep the notation and assumptions of Section 2. In particular, L is an A-field such that $\mathrm{char}_A(L) \nmid \mathfrak{r}(D)$. Let O_D^{opp} denote the opposite ring of O_D (see [21, p. 91]). There are three categories closely related with the category of Drinfeld-Stuhler modules. These alternative points of view on Drinfeld-Stuhler modules will be important for the proofs of the main results of this paper. The first category is a variant of Anderson's motives.

Definition 3.1. An *O_D-motive* is a left $O_D^{\mathrm{opp}} \otimes_{\mathbb{F}_q} L[\tau]$-module M with the following properties (cf. [25, p. 68], [17, p. 228]):

(i) M is a locally free $O_D^{\mathrm{opp}} \otimes_{\mathbb{F}_q} L$-module of rank 1.
(ii) M is a free $L[\tau]$-module of rank d.
(iii) For all $a \in A$,
$$(a \otimes 1 - 1 \otimes \gamma(a))\overline{M} \subset \tau\overline{M},$$
where $\overline{M} := M \otimes_L L^{\mathrm{alg}}$ is considered as a left $A \otimes_{\mathbb{F}_q} L^{\mathrm{alg}}[\tau]$-module.

The morphisms between O_D-motives are the homomorphisms of $O_D^{\mathrm{opp}} \otimes_{\mathbb{F}_q} L[\tau]$-modules. We denote the corresponding category by **DMot**. (An O_D-motive is a pure Anderson A-motive, in the sense of [26] or [6], of rank d^2, dimension d, and weight $1/d$; see [25, §9.2].)

Given a Drinfeld-Stuhler O_D-module ϕ over L, let $M(\phi)$ be the group

$$\mathrm{Hom}_{\mathbb{F}_q}(\mathbb{G}^d_{a,L}, \mathbb{G}_{a,L}) \cong L[\tau]^d$$

equipped with the unique $O_D^{\mathrm{opp}} \otimes_{\mathbb{F}_q} L[\tau]$-module structure such that

$$(\ell m)(e) = \ell(m(e)), \quad (\tau m)(e) = m(e)^q, \quad (bm)(e) = m(\phi(b)e),$$

for all $e \in \mathbb{G}_{a,L}^d$, $\ell \in L$, $b \in O_D$, and morphisms $m : \mathbb{G}_{a,L}^d \to \mathbb{G}_{a,L}$. It is easy to see that $M(\phi)$ is an O_D-motive.

Theorem 3.2. *The functor $\phi \mapsto M(\phi)$ gives an anti-equivalence of categories between the category of Drinfeld-Stuhler O_D-modules and* **DMot**.

Proof. This can be proven by a slight modification of Anderson's method; see [26, Thm. 2.3]. □

The second category arises from $\mathscr{D}$-elliptic sheaves mentioned in the introduction.

Definition 3.3. Fix a maximal $\mathcal{O}_C$-order $\mathscr{D}$ in D such that $H^0(C-\infty, \mathscr{D}) = O_D$. A *$\mathscr{D}$-elliptic sheaf over L* is a sequence $\mathbb{E} = (\mathcal{E}_i, j_i, t_i)_{i\in\mathbb{Z}}$, where $\mathcal{E}_i$ is a locally-free $\mathcal{O}_{C\otimes_{\mathbb{F}_q}L}$-module of rank d^2 equipped with a right action of $\mathscr{D}$ which extends the $\mathcal{O}_C$-action, and

$$j_i : \mathcal{E}_i \hookrightarrow \mathcal{E}_{i+1}$$
$$t_i : {}^\tau\mathcal{E}_i := (\mathrm{Id}_C \otimes \mathrm{Frob}_q)^*\mathcal{E}_i \hookrightarrow \mathcal{E}_{i+1}$$

are injective $\mathscr{D}$-linear homomorphisms. Moreover, for each $i \in \mathbb{Z}$ the following conditions hold:

(i) The diagram

$$\begin{tikzcd} \mathcal{E}_i \arrow[r, "j_i"] & \mathcal{E}_{i+1} \\ {}^\tau\mathcal{E}_{i-1} \arrow[u, "t_{i-1}"] \arrow[r, "{}^\tau j_{i-1}"] & {}^\tau\mathcal{E}_i \arrow[u, "t_i"'] \end{tikzcd}$$

commutes;

(ii) $\mathcal{E}_{i+d\cdot\deg(\infty)} = \mathcal{E}_i \otimes_{\mathcal{O}_C} \mathcal{O}_C(\infty)$, and the inclusion

$$\mathcal{E}_i \xrightarrow{j_i} \mathcal{E}_{i+1} \xrightarrow{j_{i+1}} \cdots \to \mathcal{E}_{i+d\cdot\deg(\infty)} = \mathcal{E}_i \otimes_{\mathcal{O}_C} \mathcal{O}_C(\infty)$$

is induced by $\mathcal{O}_C \hookrightarrow \mathcal{O}_C(\infty)$;

(iii) $\dim_L H^0(C\otimes L, \mathrm{coker} j_i) = d$;

(iv) $\mathcal{E}_i/t_{i-1}({}^\tau\mathcal{E}_{i-1}) = z_*\mathcal{V}_i$, where $\mathcal{V}_i$ is a d-dimensional L-vector space, and z is the morphism induced by γ:

$$z : \mathrm{Spec}(L) \to \mathrm{Spec}(A) \to C.$$

A *morphism* between two $\mathscr{D}$-elliptic sheaves over L

$$\psi = (\psi_i)_{i\in\mathbb{Z}} : \mathbb{E} = (\mathcal{E}_i, j_i, t_i)_{i\in\mathbb{Z}} \to \mathbb{E}' = (\mathcal{E}_i', j_i', t_i')_{i\in\mathbb{Z}}$$

is a sequence of sheaf morphisms $\psi_i : \mathcal{E}_i \to \mathcal{E}_{i+n}'$ for some fixed $n \in \mathbb{Z}$ which are compatible with the action of $\mathscr{D}$ and commute with the morphisms j_i and t_i:

$$\psi_{i+1} \circ j_i = j_{i+n}' \circ \psi_i \quad \text{and} \quad \psi_i \circ t_{i-1} = t_{i+n-1}' \circ {}^\tau\psi_{i-1}.$$

Note that the group $\mathbb{Z}$ acts freely on the objects of the category of $\mathscr{D}$-elliptic sheaves by "shifting the indices":

$$n \cdot (\mathcal{E}_i, j_i, t_i)_{i\in\mathbb{Z}} = (\mathcal{E}_i', j_i', t_i')_{i\in\mathbb{Z}}$$

with $\mathcal{E}_i' = \mathcal{E}_{i+n}$, $j_i' = j_{i+n}$, $t_i' = t_{i+n}$. Let $\mathbf{DES}/\mathbb{Z}$ be the quotient of the category of $\mathscr{D}$-elliptic sheaves by this action of $\mathbb{Z}$.

Let $\mathbb{E} = (\mathcal{E}_i, j_i, t_i)_{i\in\mathbb{Z}}$ be a $\mathscr{D}$-elliptic sheaf over L. Consider

$$M(\mathbb{E}) := H^0((C - \{\infty\}) \otimes L, \mathcal{E}_i).$$

This is independent of i since $\mathrm{supp}(\mathcal{E}_i/\mathcal{E}_{i-1}) \subset \{\infty\} \times \mathrm{Spec}(L)$. It is an $L[\tau]$-module, where the operation of τ is induced from $t_i : {}^{\tau}\mathcal{E}_i \to \mathcal{E}_{i+1}$. In fact, $M(\mathbb{E})$ is an O_D-motive; see [17, (3.17)].

Theorem 3.4. *The functor $\mathbb{E} \mapsto M(\mathbb{E})$ gives an equivalence of* $\mathbf{DES}/\mathbb{Z}$ *with* $\mathbf{DMot}$.

Proof. This is implicitly proven in [17, (3.17)] and explicitly in [25, 10.3.5]. We outline the main steps of the proof since part of this argument will be used later in the paper.

First note that since $M(\mathbb{E})$ does not depend on the choice of $\mathcal{E}_i$, the map is indeed a functor from $\mathbf{DES}/\mathbb{Z}$ to $\mathbf{DMot}$. Next, let $W_\infty := H^0(\mathrm{Spec}(O_\infty \hat{\otimes} L), \mathcal{E}_0)$. From the definition of $\mathscr{D}$-elliptic sheaf one deduces that W_∞ has a natural structure of a free $L[\![\tau^{-1}]\!]$-module of rank d; see [17, p. 231]. In addition, W_∞ is a right $\mathscr{D}_\infty$-module so that we get an injective $\mathbb{F}_q$-algebra homomorphism

$$\varphi_\infty : \mathscr{D}_\infty^{\mathrm{opp}} \to \mathrm{End}_{L[\![\tau^{-1}]\!]}(W_\infty),$$

and if we denote by π_∞ a uniformizer of O_∞ and $\tau_\infty = \tau^{\deg(\infty)}$, then W_∞ has the property that $\tau_\infty^{-d} W_\infty = \pi_\infty W_\infty$.

The pair $(M(\mathbb{E}), W_\infty)$ is a vector bundle of rank d over the non-commutative projective line over L in the sense of [17, (3.13)]. Hence, by [17, (3.16)],

$$(M(\mathbb{E}), W_\infty) \cong \mathcal{O}(n_1) \oplus \cdots \oplus \mathcal{O}(n_d),$$

where $\mathcal{O}(n) = (L[\tau], \tau^n L[\![\tau^{-1}]\!])$. Since $(M(\mathbb{E}), W_\infty)$ is equipped with a coherent right $\mathscr{D}$-action (cf. [17, (3.14)]), we have $n_1 = \cdots = n_d$. Hence $(M(\mathbb{E}), W_\infty) \cong \mathcal{O}(n)^{\oplus d}$ for some $n \in \mathbb{Z}$. If we define $W'_\infty = H^0(\mathrm{Spec}(O_\infty \hat{\otimes} L), \mathcal{E}_i)$, then $(M(\mathbb{E}), W'_\infty)$ is again a vector bundle of rank d over the non-commutative projective line. Moreover $(M(\mathbb{E}), W'_\infty) = (M(\mathbb{E}), \tau^i W_\infty)$; see [17, p. 235]. Hence, up to the action of $\mathbb{Z}$, $M(\mathbb{E})$ uniquely determines the vector bundle $(M(\mathbb{E}), W_\infty)$. On the other hand, by [17, (3.17)], the vector bundle $(M(\mathbb{E}), W_\infty)$ with its coherent $\mathscr{D}$-action uniquely determines $\mathbb{E}$ and any O_D-motive is isomorphic to $M(\mathbb{E})$ for some $\mathbb{E}$. This proves that the functor in question is fully faithful and essentially surjective. $\square$

The third category arises in the theory of analytic uniformization of Drinfeld-Stuhler modules. Let $\mathbb{C}_\infty$ be the completion of an algebraic closure of F_∞. Let ϕ be a Drinfeld-Stuhler O_D-module over $\mathbb{C}_\infty$. By fixing an isomorphism $\mathrm{Lie}(\mathbb{G}^d_{a,\mathbb{C}_\infty}) \cong \mathbb{C}^d_\infty$, we get an action of O_D on $\mathbb{C}^d_\infty$ via ∂_ϕ.

Theorem 3.5. *There is a discrete O_D-submodule Λ_ϕ of $\mathbb{C}^d_\infty$, which is locally free of rank 1, and an entire $\mathbb{F}_q$-linear function $\exp_\phi : \mathbb{C}^d_\infty \to \mathbb{C}^d_\infty$, which is surjective with kernel Λ_ϕ, such that for any $b \in O_D$ the following diagram is commutative:*

$$\begin{tikzcd}
0 \arrow[r] & \Lambda_\phi \arrow[r] \arrow[d, "\partial_\phi(b)"'] & \mathbb{C}^d_\infty \arrow[r, "\exp_\phi"] \arrow[d, "\partial_\phi(b)"'] & \mathbb{C}^d_\infty \arrow[r] \arrow[d, "\phi_b"'] & 0 \\
0 \arrow[r] & \Lambda_\phi \arrow[r] & \mathbb{C}^d_\infty \arrow[r, "\exp_\phi"] & \mathbb{C}^d_\infty \arrow[r] & 0.
\end{tikzcd}$$

Proof. The exponential function $\exp_\phi$ is the function constructed by Anderson in [1, §2]. The existence of Λ_ϕ (which is equivalent to the surjectivity of $\exp_\phi$ by [1, Thm. 4]) was proved by Taelman [25, §§9-10] in the terminology of O_D-motives. A starting point in Taelman's proof is a clever use of Tsen's theorem, which via the Morita equivalence reduces the proof to the analytic uniformization of Drinfeld modules (already known by the work of Drinfeld [8]). □

Corollary 3.6. *The ring* $\mathrm{End}(\phi)$ *is canonically isomorphic to the ring*

$$\mathrm{End}(\Lambda_\phi) := \{c \in \mathbb{C}_\infty \mid c\Lambda_\phi \subseteq \Lambda_\phi\}.$$

Proof. The functorial properties of $\exp_\phi$ (cf. [1, p. 473]) imply that ∂ maps $\mathrm{End}(\phi)$ isomorphically to the ring

$$\{P \in M_d(\mathbb{C}_\infty) \mid P\Lambda \subseteq \Lambda, P\partial_\phi(b) = \partial_\phi(b)P \text{ for all } b \in O_D\}.$$

Since any matrix which commutes with $\partial_\phi(O_D)$ must be a scalar, we get the desired isomorphism. □

Now suppose $\mathbb{C}_\infty^d$ is equipped with an action of O_D via some embedding $\iota : O_D \to M_d(\mathbb{C}_\infty)$. Suppose there is a discrete $\iota(O_D)$-submodule $\Lambda \subset \mathbb{C}_\infty^d$ which is locally free of rank one. Then there is a unique Drinfeld-Stuhler O_D-module such that $\iota = \partial_\phi$ and $\Lambda = \Lambda_\phi$; this follows from [25, §10.1.3], which itself crucially relies on [1, Thm. 6]. Hence the category of Drinfeld-Shuhler modules over $\mathbb{C}_\infty$ is equivalent to the category of O_D-lattices as above. One can use this equivalence to give an analytic description of the set of isomorphism classes of Drinfeld-Shuhler modules over $\mathbb{C}_\infty$ as follows: Let

$$\Omega^d = \mathbb{P}^{d-1}(\mathbb{C}_\infty) - \bigcup_H H(\mathbb{C}_\infty)$$

be the Drinfeld symmetric space, where H runs through the set of F_∞-rational hyperplanes in $\mathbb{P}^{d-1}(\mathbb{C}_\infty)$. Similar to the ring of finite adèles

$$\mathbb{A}_f = \{(a_v) \in \prod_{v\neq\infty} F_v \mid a_v \in A_v \text{ for almost all } v\},$$

define

$$D(\mathbb{A}_f) = \{(a_v) \in \prod_{v\neq\infty} D_v \mid a_v \in O_D \otimes_A A_v \text{ for almost all } v\}.$$

Let $\hat{A} := \prod_{v\neq\infty} A_v$ and $\widehat{O}_D := \prod_{v\neq\infty} O_D \otimes_A A_v$. We embed D in $D(\mathbb{A}_f)$ diagonally. Fixing an isomorphism $D_\infty \cong M_d(F_\infty)$, identifies $D^\times$ with a subgroup of $\mathrm{GL}_d(F_\infty)$ and therefore induces an action of $D^\times$ on Ω.

Proposition 3.7. *There is a one-to-one correspondence between the set of isomorphism classes of Drinfeld-Shuhler* O_D*-modules over* $\mathbb{C}_\infty$ *and the double coset space*

$$D^\times \setminus \Omega^d \times D(\mathbb{A}_f)^\times / \widehat{O}_D^\times,$$

where $D^\times$ *acts on both* Ω^d *and* $D(\mathbb{A}_f)^\times$ *on the left, and* $\widehat{O}_D^\times$ *acts on* $D(\mathbb{A}_f)^\times$ *on the right:*

$$\gamma \cdot (z, \alpha) \cdot k = (\gamma z, \gamma\alpha k), \quad \gamma \in D^\times, \quad z \in \Omega^d, \quad \alpha \in D(\mathbb{A}_f)^\times, \quad k \in \widehat{O}_D^\times.$$

Proof. This can be proved by a standard argument [25, p. 74] (see also [4, Thm. 4.4.11]). We recall this argument, since we will use it later on. Let $\Lambda \subset \mathbb{C}_\infty^d$ be an O_D-lattice, where D acts on $\mathbb{C}_\infty^d$ via the fixed isomorphism $D_\infty \cong M_d(F_\infty)$. The F-span $F\Lambda$ is a free module over D of rank 1. A choice of generator of this module defines a point in $\mathbb{P}^{d-1}(\mathbb{C}_\infty)$. One checks that this point lies in Ω^d if and only if Λ is discrete. The embedding $\Lambda \subset F\Lambda = D$ can be tensored to an embedding $\hat{A}\Lambda \subset D(\mathbb{A}_f)$ and the former can be recovered from the latter as $\Lambda = \hat{A}\Lambda \cap D$. Now $\hat{A}\Lambda$ is a locally free module over $\widehat{O}_D$. Since all such modules are free, we conclude that the locally free O_D-submodules $\Lambda \subset D$ of rank one are in bijection with the free rank one $\widehat{O}_D$-submodules of $D(\mathbb{A}_f)$ and the latter are in bijection with $D(\mathbb{A}_f)^\times/\widehat{O}_D^\times$. Finally, moding out by the choice of the generator of $F\Lambda$, that is, by $D^\times$, we get the desired one-to-one correspondence. □

4. Complex multiplication

Theorem 4.1. *Let ϕ is a Drinfeld-Stuhler O_D-module over an A-field L. Then:*

(1) $\mathrm{End}_L(\phi)$ *is a projective A-module of rank $\leq d^2$.*
(2) $\mathrm{End}_L(\phi) \otimes_A F_\infty$ *is isomorphic to a subalgebra of the central division algebra over F_∞ with invariant $-1/d$.*
(3) *If L has generic A-characteristic, then* $\mathrm{End}_L(\phi)$ *is an A-order in an imaginary field extension of F which embeds into D. In particular,* $\mathrm{End}_L(\phi)$ *is commutative and its rank over A divides d.*
(4) *The automorphism group* $\mathrm{Aut}_L(\phi) := \mathrm{End}_L(\phi)^\times$ *is isomorphic to $\mathbb{F}_{q^r}^\times$ for some r dividing d.*

Proof. It is enough to prove (1), (2) and (3) after extending L to its algebraic closure, so we will assume that L is algebraically closed.

Since the O_D-motive $M(\phi)$ associated to ϕ is an Anderson A-motive of dimension d and rank d^2, the argument in [1, §1.7] implies that $\mathrm{End}_{A\otimes L[\tau]}(M(\phi))$ is a projective A-module of rank $\leq d^4$ (see also [6, Thm. 9.5]). Hence, thanks to Theorem 3.2, $\mathrm{End}(\phi)$ is a projective A-module of rank $\leq d^4$.

Let W_∞ be the $\mathscr{D}_\infty \otimes L[\![\tau^{-1}]\!]$-module attached to ϕ in the proof of Theorem 3.4. As we discussed, W_∞ is well-defined up to the shifts $W_\infty \mapsto \tau W_\infty$. Since $\mathscr{D}_\infty \cong \mathbb{M}_d(O_\infty)$, using the Morita equivalence [17, p. 262], one concludes that W_∞ is equivalent to an $O_\infty \otimes L[\![\tau^{-1}]\!]$-module W'_∞ which is free of rank 1 over O_∞, free of rank 1 over $L[\![\tau^{-1}]\!]$, and $\tau_\infty^{-d}W'_\infty = \pi_\infty W'_\infty$. From W'_∞ we get an $\mathbb{F}_q$-algebra homomorphism

$$\phi_\infty : O_\infty \to \mathrm{End}_{L[\![\tau^{-1}]\!]}\left(L[\![\tau^{-1}]\!]\right) = L[\![\tau^{-1}]\!], \quad \phi_\infty(\pi_\infty) = \tau_\infty^{-d}.$$

Thus,

$$\mathrm{End}_{O_\infty\otimes L[\![\tau^{-1}]\!]}(W'_\infty)^{\mathrm{opp}} = \mathrm{End}(\phi_\infty) = \{f \in L[\![\tau^{-1}]\!] \mid f\phi_\infty(b) = \phi_\infty(b)f \text{ for all } b \in O_\infty\}.$$

Since $O_\infty = \mathbb{F}_{q^{\deg(\infty)}}[\![\pi_\infty]\!]$, the image of O_∞ under ϕ_∞ is the subring $\mathbb{F}_{q^{\deg(\infty)}}[\![\tau_\infty^{-d}]\!]$ of $L[\![\tau^{-1}]\!]$. Now it is easy to see that

$$\mathrm{End}(\phi_\infty) \cong \mathbb{F}_{q^{d\deg(\infty)}}[\![\tau_\infty^{-1}]\!],$$

which is the maximal order in the central division algebra over F_∞ with invariant $-1/d$; cf. [17, Appendix B]. Definition 3.14 and Theorem 3.17 in [17] imply that $\mathrm{End}(\phi)$ acts faithfully on W'_∞, and this action gives an embedding $\mathrm{End}(\phi)\otimes_A F_\infty \hookrightarrow \mathrm{End}(\phi_\infty)\otimes_{O_\infty} F_\infty$ (see also [6, Thm. 8.6]). Since $\mathrm{rank}_{O_\infty}\mathrm{End}(\phi_\infty)=d^2$, we get $\mathrm{rank}_A\mathrm{End}(\phi)\le d^2$. This proves (1) and (2).

To prove (3), note that ϕ is defined over some finitely generated subfield of L which can be embedded into $\mathbb{C}_\infty$. So, without loss of generality, we assume $L=\mathbb{C}_\infty$. Combining (1) and (2) with Lemma 2.11 already implies that $\mathrm{End}(\phi)$ is an A-order in an imaginary field extension of F. We need to show that $\mathrm{End}(\phi)$ embeds into D. Let Λ_ϕ be the O_D-lattice associated to ϕ by Theorem 3.5. By Corollary 3.6, $\alpha\in\mathrm{End}(\phi)$ corresponds to $c\in\mathbb{C}_\infty$ such that $c\Lambda_\phi\subseteq\Lambda_\phi$. On the other hand, the F-span $F\Lambda_\phi$ is a free module over D of rank 1, so c corresponds to a unique element of D. Mapping α to that element, gives an embedding $\mathrm{End}(\phi)\hookrightarrow D$. Finally, the rank of $\mathrm{End}(\phi)$ over A is equal to the degree of $\mathrm{End}(\phi)\otimes_F F$ over F, and it is well-known that a subfield of D has degree over F dividing d; cf. [21, (7.15)].

To prove (4), suppose we have proved this claim over L^{alg}. Then $\mathrm{Aut}_L(\phi)$ is a subgroup of the finite cyclic group $\mathrm{Aut}(\phi)$, so has a generator α of finite order. This element α is algebraic over $\mathbb{F}_q$. Since $\mathbb{F}_q\subset\mathrm{End}_L(\phi)$, we get $\mathbb{F}_q(\alpha)\subset\mathrm{End}_L(\phi)$. On the other hand, $\mathbb{F}_q(\alpha)$ is a finite field extension of $\mathbb{F}_q$. Hence $\mathrm{Aut}_L(\phi)=\mathbb{F}_q(\alpha)^\times\cong\mathbb{F}_{q^r}^\times$ for some r dividing d. It remains to prove the claim assuming L is algebraically closed. Let H be the central division algebra over F_∞ with invariant $-1/d$. It is known that the valuation ord_∞ on F_∞ extends to a discrete valuation w on H; see [21, (12.6)]. Moreover,

$$\mathcal{H}=\{\alpha\in H\mid w(\alpha)\ge 0\}=\{\alpha\in H\mid \mathrm{Nr}_{H/F_\infty}(\alpha)\in O_\infty\}$$

is the unique maximal order of H, $\mathcal{M}=\{\alpha\in H\mid w(\alpha)>0\}$ is the unique maximal two-sided ideal of $\mathcal{H}$, and $\mathcal{H}/\mathcal{M}\cong\mathbb{F}_{q^d}$; see (12.8), (13.2), (14.3) in [21]. We know that $\mathrm{End}(\phi)\subset H(F_\infty)$ is discrete. Since any element of $\mathrm{Aut}(\phi)$ has norm 1, we get that $\mathrm{Aut}(\phi)=(\mathcal{H}\cap\mathrm{End}(\phi))^\times$. But $\mathcal{H}$ is compact in ∞-adic topology, so $\mathcal{H}\cap\mathrm{End}(\phi)$ is a finite subfield of $\mathcal{H}$, which under the reduction map $\mathcal{H}\to\mathcal{H}/\mathcal{M}$ embeds into $\mathbb{F}_{q^d}$. □

Example 4.2. Let ϕ be the Drinfeld-Stuhler O_D-module over an algebraically closed field L of generic A-characteristic. From Theorem 4.1 we know that $\mathrm{End}(\phi)$ is an A-order in an imaginary field extension of F of degree dividing d. We show that this bound is the best possible. Consider ϕ from Example 2.12. Let φ be the rank-1 Drinfeld O_K-module over L from the same example. Let

$$E:=\{\mathrm{diag}(\varphi_\alpha,\dots,\varphi_\alpha)\mid \alpha\in O_K\}\subset M_d(L[\tau]).$$

It is clear that $E\cong O_K$. One easily checks that the elements of E commute with ϕ_α, $\alpha\in O_K$, and ϕ_z. Therefore $E\subseteq\mathrm{End}(\phi)$. Since O_K is a maximal A-order in K, Theorem 4.1 implies that $\mathrm{End}(\phi)\cong O_K$.

Definition 4.3. Let K be an imaginary field extension of F of degree d, and E an A-order in K. We say that a Drinfeld-Stuhler O_D-module ϕ over a field of generic A-characteristic has *complex multiplication* by E (or has *CM*, for short) if $E=\mathrm{End}(\phi)$. Note that in that case K necessarily embeds into D. A *CM subfield* of D is a commutative subfield of D which is an imaginary extension of F of degree d.

Lemma 4.4. *Let ϕ be a Drinfeld-Stuhler O_D-module over an algebraically closed field L having CM by E. Let O_K be the maximal A-order in the fraction field K of E. Then there is a Drinfeld-Stuhler O_D-module ψ which is isogenous to ϕ and has CM by O_K.*

Proof. Let $\mathfrak{c}$ be the conductor of E, i.e., the largest ideal of O_K which is also an ideal of E. Let $H := \cap_{c\in\mathfrak{c}} \ker(c)$, where the intersection is taken in $\mathbb{G}_{a,L}^d$. Since the action of O_D on $\mathbb{G}_{a,L}^d$ commutes with E, the finite étale subgroup scheme H of $\mathbb{G}_{a,L}^d$ is invariant under $\phi(O_D)$. Thus, by Lemma 2.10, there is a Drinfeld-Stuhler O_D-module ψ over L and an isogeny $u : \phi \to \psi$ whose kernel is H. Now one can apply the argument in the proof of [12, Prop. 4.7.19] to deduce that $\mathrm{End}(\psi) \cong O_K$. □

We further investigate the properties of Drinfeld-Stuhler modules with CM using analytic uniformization. We fix an embedding $D^\times \to \mathrm{GL}_d(F_\infty)$ through which $D^\times$ acts on Ω. For $(z, \alpha) \in \Omega^d \times D(\mathbb{A}_f)^\times$, let $\Lambda_{(z,\alpha)}$ be the O_D-lattice corresponding to (z, α); see Proposition 3.7. Let $K_z^\times := \{\gamma \in D^\times \mid \gamma z = z\}$.

Lemma 4.5. *$K_z := K_z^\times \cup \{0\}$ is a subfield of D and* $\mathrm{End}(\Lambda_{(z,\alpha)}) = K_z \cap \alpha\widehat{O}_D\alpha^{-1}$.

Proof. Let $\tilde{z} \in \mathbb{C}_\infty^d$ be an element mapping to z; such $\tilde{z}$ is well-defined up to a scalar multiple. Denote $O = D \cap \alpha\widehat{O}_D\alpha^{-1}$. The lattice $\Lambda = O\tilde{z}$ is in the isomorphism class of $\Lambda_{(z,\alpha)}$. We have

$$c \in \mathrm{End}(\Lambda) \Leftrightarrow c\Lambda \subset \Lambda \Leftrightarrow cO\tilde{z} \subset O\tilde{z} \Leftrightarrow Oc\tilde{z} \subset O\tilde{z},$$

where c acts on $\tilde{z}$ as a scalar matrix. The inclusion $Oc\tilde{z} \subset O\tilde{z}$ is equivalent to the existence of $\gamma \in O$ such that $\gamma\tilde{z} = c\tilde{z}$. This γ obviously fixes z, and since $\gamma \in \alpha\widehat{O}_D\alpha^{-1}$, we get $\gamma \in K_z \cap \alpha\widehat{O}_D\alpha^{-1} =: E_{(z,\alpha)}$. Conversely, suppose $\gamma \in E_{(z,\alpha)}$, so $\gamma \in O$ and $\gamma\tilde{z} = c\tilde{z}$ for some non-zero $c \in \mathbb{C}_\infty$. Reversing the previous argument we see that $c \in \mathrm{End}(\Lambda)$.

Observe that $E_{(z,\alpha)}$ is a subring of D since for $\gamma, \gamma' \in E_{(z,\alpha)}$ with $\gamma\tilde{z} = c\tilde{z}$, $\gamma'\tilde{z} = c'\tilde{z}$, we have $(\gamma + \gamma')\tilde{z} = (c + c')\tilde{z}$. Hence $K := E_{(z,\alpha)} \otimes_A F$ is a commutative subalgebra of D, i.e., K is a subfield of D. Since the map $E_{(z,\alpha)} \to \mathrm{End}(\Lambda)$, $\gamma \mapsto c$, is a homomorphism which extends to $K \to \mathbb{C}_\infty$, it must be injective. But we have seen that $E_{(z,\alpha)} \to \mathrm{End}(\Lambda)$ is also surjective, thus it is an isomorphism. □

Remark 4.6. For any $\alpha \in D(\mathbb{A}_f)^\times$ and a CM field $K \subset D$, the intersection $K \cap \alpha\widehat{O}_D\alpha^{-1}$ is an A-order in K. To prove this, first observe that $D \cap \alpha\widehat{O}_D\alpha^{-1}$ is a maximal order in D. Hence it is enough to prove that for any maximal order $\mathcal{M}$ in D the intersection $E := K \cap \mathcal{M}$ is an A-order. It is clear that $A \subset E$. By Exercise 4, p. 131, [21], there is a maximal order $\mathcal{M}'$ in D such that $K \cap \mathcal{M}' = O_K$. It is easy to see that $\mathcal{M}'' := \mathcal{M} \cap \mathcal{M}'$ is an A-order in D (in fact, it is a hereditary order; cf. [21, §40]). Hence $\mathcal{M}''$ has finite index in $\mathcal{M}'$. On the other hand, since $E = O_K \cap \mathcal{M}''$, under the natural homomorphism $\mathcal{M}' \to \mathcal{M}'/\mathcal{M}''$ the module O_K maps onto O_K/E. Thus, E has finite index in O_K, i.e., is an order.

Lemma 4.7. *Let K be a CM subfield of D. The number of fixed points of $K^\times$ in Ω^d is non-zero and is at most d.*

Proof. Since F has transcendence degree 1 over $\mathbb{F}_q$, we can find a primitive element $\gamma \in K$ such that $K = F(\gamma)$; cf. [2]. It is enough to prove that γ has at least one and at most d fixed points in Ω^d. The characteristic polynomial of γ, as an element of $\mathrm{GL}_d(F_\infty)$, is irreducible. Since the minimal polynomial of γ divides its characteristic polynomial, it must be equal to the characteristic polynomial. The claim then follows from the fact that a matrix in $\mathrm{GL}_d(\mathbb{C}_\infty)$, whose characteristic and minimal polynomials are equal, has at least one and at most d eigenvectors, up to scaling. □

Let K be a CM subfield of D, and E be an A-order in K. Let

$$\mathbb{T}_E := \{\alpha \in D(\mathbb{A}_f)^\times \mid K \cap \alpha \widehat{O}_D \alpha^{-1} = E\}.$$

It is easy to check that $K^\times$ acts on $\mathbb{T}_E$ from the left by multiplication and $\widehat{O}_D^\times$ acts from the right. It is known that $\mathbb{T}_E$ is non-empty and the double coset space $K^\times \setminus \mathbb{T}_E/\widehat{O}_D^\times$ has finite cardinality divisible by the class number of E; cf. [27, pp. 92-93].

Remark 4.8. The elements of $\mathbb{T}_E$ correspond to optimal embeddings of K into the maximal orders of D with respect to E. For example, if $d = 2$, $E = O_K$, and K is a separable quadratic extension of F if the characteristic of F is 2, then

$$\#(K^\times \setminus \mathbb{T}_{O_K}/\widehat{O}_D^\times) = \#\mathrm{Pic}(O_K) \prod_{\mathfrak{p}|\mathfrak{r}} \left(1 - \left(\frac{K}{\mathfrak{p}}\right)\right),$$

where $\left(\frac{K}{\mathfrak{p}}\right) = -1$ (resp. $= 0$) if $\mathfrak{p}$ remains inert (resp. ramifies) in K; see [27, p. 94].

Theorem 4.9. *Let S_K be the set of fixed points of $K^\times$ in Ω^d. We have:*

1. *Up to isomorphism, the number of Drinfeld-Stuhler O_D-modules over $\mathbb{C}_\infty$ having CM by E is equal to $\#(K^\times \setminus S_K \times \mathbb{T}_E/\widehat{O}_D^\times)$. In particular, that number is finite and non-zero.*
2. *A Drinfeld-Stuhler O_D-module having CM by O_K can be defined over the Hilbert class field of K.*

Proof. (1) In our set-up, we have fixed an embedding of K into D. For each $(z, \alpha) \in S_K \times \mathbb{T}_E$ we have $\mathrm{End}(\Lambda_{(z,\alpha)}) = K_z \cap \alpha\widehat{O}_D\alpha^{-1} = K \cap \alpha\widehat{O}_D\alpha^{-1} = E$. Note that for any $\gamma \in D^\times$, we have $K_{\gamma z} = \gamma K_z \gamma^{-1}$, and so

$$K_{\gamma z} \cap \gamma\alpha\widehat{O}_D\alpha^{-1}\gamma^{-1} = \gamma(K \cap \alpha\widehat{O}_D\alpha^{-1})\gamma^{-1} = \gamma E \gamma^{-1} \cong E,$$

which implies $\mathrm{End}(\Lambda_{\gamma(z,\alpha)}) \cong \mathrm{End}(\Lambda_{(z,\alpha)})$. Now suppose $(z, \alpha) \in \Omega^d \times D(\mathbb{A}_f)^\times$ is such that $\mathrm{End}(\Lambda_{(z,\alpha)}) \cong E$. Then K_z must be isomorphic to K, so K_z is another embedding of K into D. By the Skolem-Noether theorem [21, (7.21)], two embeddings $K \rightrightarrows D$ differ by an inner automorphism of D. Thus, there is $\gamma \in D^\times$ such that $K_z = \gamma K \gamma^{-1}$. This implies that we can find $z' \in S_K$ such that $\gamma z' = z$. We also have $\gamma K \gamma^{-1} \cap \alpha\widehat{O}_D\alpha^{-1} = \gamma E \gamma^{-1}$, which implies $\gamma^{-1}\alpha \in \mathbb{T}_E$. Hence we can find $\alpha' \in \mathbb{T}_E$ such that $\alpha = \gamma\alpha'$. Overall, we conclude that $(z, \alpha) = \gamma(z', \alpha')$ for some $(z', \alpha') \in S_K \times \mathbb{T}_E$. The stabilizer in $D^\times$ of any $z \in S_K$ is $K^\times$. Hence the set of images in $D^\times \setminus \Omega^d \times D(\mathbb{A}_f)^\times/\widehat{O}_D^\times$ of $(z, \alpha) \in \Omega^d \times D(\mathbb{A}_f)^\times$ with CM by E is the double coset space $K^\times \setminus S_K \times \mathbb{T}_E/\widehat{O}_D^\times$.

(2) Let ϕ be a Drinfeld-Stuhler O_D-module with $\mathrm{End}(\phi) \cong O_K$. Let $M(\phi)$ be the O_D-motive associated to ϕ. By definition, the action of O_K on $\mathbb{G}^d_{a,\mathbb{C}_\infty}$ commutes with $\phi(O_D)$, hence $M(\phi)$ is an $O_D^{\mathrm{opp}} \otimes_A O_K$-module. On the other hand, $O_D^{\mathrm{opp}} \otimes_A O_K$ is an A-order in $D^{\mathrm{opp}} \otimes_F K \cong M_d(K)$; cf. Exercise 6 on page 131 of [21]. Computing the discriminants, one checks that $O_D^{\mathrm{opp}} \otimes_A O_K$ is a maximal order in $M_d(K)$. By the Morita equivalence (cf. [17, p. 262] and [25, p. 68]), $M(\phi)$ is equivalent to an O_K-motive M' of rank 1 and dimension 1 (as defined in [26]). Through a generalization of Anderson's result (cf. [26, Thm. 2.9]), M' corresponds to a Drinfeld O_K-module φ of rank 1. Since φ can be defined over H (see [13, §8]), ϕ also can be defined over H. $\square$

5. Supersingularity

In this section we fix a prime ideal $\mathfrak{p} \lhd A$ such that $\mathfrak{p} \notin \mathrm{Ram}(D)$. Let L be a field extension of $\mathbb{F}_\mathfrak{p} := A/\mathfrak{p}$ of degree m, so L is a finite field of order q^n, where $n = m \cdot \deg(\mathfrak{p})$. Let $\pi = \tau^n$ be the associated Frobenius morphism. With abuse of notation, denote by π also the diagonal matrix $\mathrm{diag}(\pi, \dots, \pi) \in M_d(L[\tau])$. Note that π is in the center of $M_d(L[\tau])$ since $\tau^n \ell = \ell \tau^n$ for all $\ell \in L$.

Theorem 5.1. *Let ϕ be a Drinfeld-Stuhler O_D-module defined over L. Since π commutes with $\phi(O_D)$, we have $\pi \in \mathrm{End}_L(\phi)$. Let $\widetilde{F} := F(\pi)$ be the subfield of $D' := \mathrm{End}_L(\phi) \otimes_A F$ generated over F by π. Then:*

1. *$[\widetilde{F} : F]$ divides d, and ∞ does not split in $\widetilde{F}/F$.*
2. *Let $\widetilde{\infty}$ be the unique place of $\widetilde{F}$ over ∞. There is a unique prime $\tilde{\mathfrak{p}} \neq \widetilde{\infty}$ of $\widetilde{F}$ that divides π. Moreover, $\tilde{\mathfrak{p}}$ lies above $\mathfrak{p}$.*
3. *D' is a central division algebra over $\widetilde{F}$ of dimension $(d/[\widetilde{F} : F])^2$ and with invariants*
$$\mathrm{inv}_{\tilde{v}}(D') = \begin{cases} -[\widetilde{F} : F]/d & \text{if } \tilde{v} = \widetilde{\infty}, \\ [\widetilde{F} : F]/d & \text{if } \tilde{v} = \tilde{\mathfrak{p}}, \\ -[\widetilde{F}_{\tilde{v}} : F_v] \cdot \mathrm{inv}_v(D) & \text{otherwise,} \end{cases}$$
for each place v of F and each place $\tilde{v}$ of $\widetilde{F}$ dividing v.

Proof. Observe that $D' \cong \mathrm{End}(M(\phi) \otimes_A F)^{\mathrm{opp}}$. The theorem then follows from [17, (9.10)] and the equivalences of Section 3. (We should mention that in Section 9 of [17] the $\mathscr{D}$-elliptic sheaves are considered over the algebraic closure of $\mathbb{F}_\mathfrak{p}$. On the other hand, the arguments in that section apply also over L with our choice of $(\widetilde{F}, \pi)$ in place of a "φ-pair" in [17], since Theorem A.6 in [17] can be proved for $(\widetilde{F}, \pi)$ as in [16, §2.2].) $\square$

Example 5.2. Let L be the degree d extension of $A/TA \cong \mathbb{F}_q$. Let ϕ be the Drinfeld-Stuhler O_D-module from Example 2.13. Assume (T) does not divide $\mathfrak{r}$. Fix a generator h of $\mathbb{F}_{q^d}$ over $\mathbb{F}_q$. Our Drinfeld-Stuhler module ϕ is generated over $\mathbb{F}_q$ by $\phi_T = \mathrm{diag}(\tau^d, \dots, \tau^d) = \pi$, ϕ_h, and ϕ_z, which satisfy the relations
$$\phi_T\phi_h = \phi_h\phi_T, \quad \phi_T\phi_z = \phi_z\phi_T, \quad \phi_z\phi_h = \phi_{h^q}\phi_z, \quad \phi_z^d = \phi_\mathfrak{r} = \mathrm{diag}(\varphi_\mathfrak{r}, \dots, \varphi_\mathfrak{r}).$$
With abuse of notation, for $i \geq 1$ let
$$\tau^i := \mathrm{diag}(\tau^i, \dots, \tau^i) \qquad \text{and} \qquad h := \mathrm{diag}(h, \dots, h).$$
Define
$$\kappa_i = \phi_z^i \tau^{d-i}, \quad 1 \leq i \leq d-1.$$
Note that, since the image of φ is in $\mathbb{F}_q[\tau]$, we have $\phi_z^i\tau^{d-i} = \tau^{d-i}\phi_z^i$. In particular, h and κ_i commute with ϕ_z. It is clear that these elements also commute with $\phi_T = \tau^d$. Finally, h obviously commutes with ϕ_h, and so does κ_i:
$$\kappa_i\phi_h = \phi_z^i\tau^{d-i}\phi_h = \phi_z^i\phi_{h^{q^{d-i}}}\tau^{d-i} = \phi_{h^{q^d}}\phi_z^i\tau^{d-i} = \phi_h\kappa_i.$$
We conclude that $E := \mathbb{F}_q[\phi_T, h, \kappa_1, \dots, \kappa_{d-1}] \subseteq \mathrm{End}_L(\phi)$.

Note that h and κ_i do not commute,

$$\kappa_i h = h^{q^{d-i}}\kappa_i = \sigma^{-i}(h)\kappa_i, \tag{5.1}$$

where σ is the Frobenius automorphism in $\mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)$. Let $E_i := \mathbb{F}_q[\phi_T, h, \kappa_i] \subset E$. Since $\mathbb{F}_q[\phi_T, h] \cong \mathbb{F}_{q^d}[T]$, we have $E_i = O_K[\kappa_i]$, where $K = \mathbb{F}_{q^d}(T)$. Denote $D_i = E_i \otimes_A F$. Combining the relation (5.1) with $\kappa_i^d = \phi_{\mathfrak{r}^i T^{d-i}}$, we see that for i coprime to d we have

$$D_i \cong (K/F, \sigma^{-i}, \mathfrak{r}^i T^{d-i})$$

(see (2.1) for the notation). By [21, (30.4)], for i coprime to d, we have

$$(K/F, \sigma^{-i}, \mathfrak{r}^i T^{d-i}) \cong (K/F, \sigma, \mathfrak{r}^{-1}T).$$

Hence for $1 \leq i, i' \leq d-1$ coprime to d we have $D_i \cong D_{i'}$, and we denote this cyclic algebra by $\overline{D}$. The invariants of $\overline{D}$ are easy to compute using (2.3) or [21, (31.7)]:

$$\mathrm{inv}_v(\overline{D}) = \begin{cases} 1/d & \text{if } v = (T), \\ -1/d & \text{if } v = \infty, \\ -\mathrm{inv}_v(D) & \text{otherwise.} \end{cases}$$

Let $D' := \mathrm{End}_L(\phi) \otimes_A F$. By Theorem 4.1, we have $\dim_F(D') \leq d^2$. Since $\dim_F \overline{D} = d^2$, we conclude that $D' \cong \overline{D}$. Note that the invariants of $\overline{D}$ agree with the invariants of D' given by Theorem 5.1, since in this case $\pi \in F$.

Next, we claim that $\mathrm{End}_L(\phi)$ is a maximal A-order in D'. One can argue as follows: The discriminant of $E_1 \subset \mathrm{End}_L(\phi)$ is $(\mathfrak{r}T^{d-1})^{d(d-1)}$ (cf. Example 2.12), so $\mathrm{End}_L(\phi) \otimes_A A_\mathfrak{p}$ is a maximal $A_\mathfrak{p}$-order in $\overline{D}_\mathfrak{p}$ for all $\mathfrak{p} \neq (T)$. On the other hand, the discriminant of E_{d-1} is $(\mathfrak{r}^{d-1}T)^{d(d-1)}$, so $\mathrm{End}_L(\phi) \otimes_A A_T$ is a maximal A_T-order in $\overline{D}_T$. Since an A-order in $\overline{D}$ is maximal if and only if it is locally maximal at all primes $\mathfrak{p} \lhd A$ (see [21, (11.6)]), we conclude that $\mathrm{End}_L(\phi)$ is a maximal order.

Finally, note that $\mathbb{F}_{q^d}^\times \cong \mathrm{Aut}_L(\phi)$. Indeed, $\mathbb{F}_{q^d}^\times \cong \mathbb{F}_q(h)^\times \subseteq \mathrm{Aut}_L(\phi)$, so the equality holds by part (4) of Theorem 4.1.

This example shows that the bounds on the rank of $\mathrm{End}_L(\phi)$ and the order of $\mathrm{Aut}_L(\phi)$ given by Theorem 4.1 cannot be improved.

Proposition 5.3. *Let ϕ be a Drinfeld-Stuhler O_D-module over L. The following are equivalent:*

(1) $\dim_F(\mathrm{End}(\phi) \otimes_A F) = d^2$;
(2) *some power of* π *lies in* A;
(3) *there is a unique prime* $\tilde{\mathfrak{p}}$ *in* $\widetilde{F}$ *lying over* $\mathfrak{p}$;
(4) $\phi[\mathfrak{p}]$ *is connected.*

Proof. Let L' be a finite extension of L of degree c. The Frobenius of L' is π^c. Applying Theorem 5.1, we see that $\dim_F(\mathrm{End}_{L'}(\phi) \otimes_A F) = d^2$ is equivalent to $F(\pi^c) = F$, and since π is integral over A, this last condition is equivalent to $\pi^c \in A$. This shows that (1) and (2) are equivalent.

Assume (2), i.e., $\pi^c \in A$ for some $c \geq 1$. By Theorem 5.1, $\mathrm{ord}_\mathfrak{p}(\pi^c) \neq 0$. This implies $\mathrm{ord}_\mathfrak{P}(\pi^c) \neq 0$ for any prime $\mathfrak{P}$ in $\widetilde{F}$ lying over $\mathfrak{p}$, and hence also $\mathrm{ord}_\mathfrak{P}(\pi) \neq 0$. Applying Theorem 5.1 again, we conclude that $\mathfrak{P} = \tilde{\mathfrak{p}}$ is unique, which is (3). To prove (3)⇒(2), let

$f = \mathrm{Nr}_{\widetilde{F}/F}(\pi)$. We have $\mathrm{ord}_{\mathfrak{p}}(f) > 0$ and $\mathrm{ord}_{\mathfrak{p}'}(f) = 0$ for any prime $\mathfrak{p}' \lhd A$ not equal to $\mathfrak{p}$. Let $\mathrm{ord}_{\tilde{\mathfrak{p}}}(\pi) = u$ and $\mathrm{ord}_{\tilde{\mathfrak{p}}}(f) = w$. The element $\pi^w/f^u \in \widetilde{F}$ has no zeros or poles away from $\widetilde{\infty}$, since $\tilde{\mathfrak{p}}$ is the unique prime over $\mathfrak{p}$ by assumption. This implies that π^w/f^u lies in the algebraic closure $\mathbb{F}$ of $\mathbb{F}_q$ in $\widetilde{F}$. Therefore, $\pi^{w\kappa} = f^{u\kappa} \in A$, where $\kappa = \#\mathbb{F} - 1$.

Assume (2). Then π^c generates $\mathfrak{p}^h$ for some $c, h \geq 1$. This implies that $\phi[\mathfrak{p}]$ is connected, since $\phi[\mathfrak{p}] \subseteq \phi[\mathfrak{p}^h] = \ker(\pi^c)$, and $\ker(\pi^c)$ is obviously connected. Thus, (2)⇒(4). Conversely, assume $\phi[\mathfrak{p}]$ is connected. Then $\phi[\mathfrak{p}^h]$ is connected for all $h \geq 1$. Choose h such that $\mathfrak{p}^h = (a)$ is principal. The assumption that $\phi[a]$ is connected is equivalent to the action of τ on $M(\phi)/aM(\phi)$ being nilpotent, i.e., $\tau^r M(\phi) \subset aM(\phi) \subset \mathfrak{p}M(\phi)$ for all large enough integers r. This last condition implies that $\dim_F(\mathrm{End}(\phi) \otimes_A F) = d^2$; see [19, §6]. Hence (4)⇒(1). □

Definition 5.4. A Drinfeld-Stuhler O_D-module ϕ over $\overline{\mathbb{F}}_{\mathfrak{p}}$ satisfying the equivalent conditions of Proposition 5.3 is called *supersingular*. (In particular, the Drinfeld-Stuhler module ϕ in Example 5.2 is supersingular.)

Theorem 5.5. *Let ϕ be a supersingular Drinfeld-Stuhler O_D-module over $\overline{\mathbb{F}}_{\mathfrak{p}}$. We have:*

1. $\mathrm{End}(\phi)$ *is a maximal A-order in* $\mathrm{End}(\phi) \otimes F$*;*
2. *ϕ can be defined over the extension of $\mathbb{F}_{\mathfrak{p}}$ of degree $d \cdot \#\mathrm{Pic}(A)$;*
3. *the number of isomorphism classes of supersingular Drinfeld-Stuhler O_D-modules over $\overline{\mathbb{F}}_{\mathfrak{p}}$ is equal to the class number of* $\mathrm{End}(\phi)$*;*
4. *all supersingular Drinfeld-Stuhler O_D-modules are isogenous over $\overline{\mathbb{F}}_{\mathfrak{p}}$.*

Proof. (1) and (3) are proved in [19, Thm. 6.2], (2) follows from [18, §5], (4) follows from [17, (9.13)]. □

6. Fields of moduli

Let L be an arbitrary A-field. We will need a Hilbert's 90-th type theorem for $\mathrm{GL}_d(L^{\mathrm{sep}}[\tau])$. This is probably known to specialists, but in absence of a convenient reference we will prove this fact following the argument of the corresponding statement for simple algebras; cf. [3, §III.8.7].

Lemma 6.1. *We have:*

1. *Every left ideal of $L[\tau]$ is principal.*
2. *Every finitely generated torsion-free left $L[\tau]$-module is free.*

Proof. (1) follows from the existence of the right division algorithm for $L[\tau]$ (see [12, Cor. 1.6.3]), and (2) essentially follows from the same fact (see [12, Cor. 5.4.9]). □

Let K be a finite Galois extension of L of degree n. Let $\sigma_1, \sigma_2, \dots, \sigma_n$ be the elements of $G := \mathrm{Gal}(K/L)$. The Galois group G acts on $K[\tau]$ via the obvious action on the coefficients of polynomials, and it acts on the ring $M_d(K[\tau])$ by acting on the entries of matrices. Let $M := K[\tau]^d$ be the free left $K[\tau]$-module of rank d. Then $\mathrm{GL}_d(K[\tau])$ can be identified with the group $\mathrm{Aut}_{K[\tau]}(M)$ of automorphism of M, where $g \in \mathrm{GL}_d(K[\tau])$ acts on M from the right as on row vectors. (Of course, $\mathrm{GL}_d(K[\tau])$ also acts on M from the left as on column vectors, but that action is not $K[\tau]$-linear.) From this identification it is easy to see the validity of the following:

Lemma 6.2. *If $v_1, \dots, v_d \in M$ form a left $K[\tau]$-basis of M, then the matrix S whose rows are $v_1, \dots, v_d$ is in $\mathrm{GL}_d(K[\tau])$. Conversely, the rows of $S \in \mathrm{GL}_d(K[\tau])$ form a left $K[\tau]$-basis of M.*

Remark 6.3. There are matrices in $M_d(K[\tau])$ whose rows are left linearly independent but whose columns are left linearly dependent over $K[\tau]$, e.g., $\begin{pmatrix} 1 & \tau \\ \alpha+\tau & \tau(\alpha+\tau) \end{pmatrix}$ where $\alpha \in K$ is such that $\alpha^q \neq \alpha$.

Lemma 6.4. *The inclusion $L[\tau] \subset K[\tau]$ makes $K[\tau]$ into a left $L[\tau]$-module. As such, $K[\tau]$ is a free left $L[\tau]$-module of rank n.*

Proof. It is obvious that $K[\tau]$ has no torsion elements for the action of $L[\tau]$. Let $\alpha_1, \dots, \alpha_n \in K$ be an L-basis of K. It is enough to show that $K[\tau] = \sum_{i=1}^n L[\tau]\alpha_i$. By the Dedekind's theorem on the independence of characters, $\{\alpha_1, \dots, \alpha_n\}$ form an L-basis of K if and only if $\det(\sigma_i \alpha_j) \neq 0$. On the other hand, $\det(\sigma_i\alpha_j) \neq 0$ if and only if $\det(\sigma_i\alpha_j)^{q^r} = \det(\sigma_i\alpha_j^{q^r}) \neq 0$ for any $r \geq 0$. Hence $\{\alpha_1^{q^r}, \dots, \alpha_n^{q^r}\}$ is also an L-basis of K. Let $f = a_0 + a_1\tau + \dots + a_k\tau^k \in K[\tau]$. For each a_i we can find $b_{i,1}, \dots, b_{i,n} \in L$ such that $\sum_{j=1}^n b_{i,j}\alpha_j^{q^i} = a_i$. Thus, $f = \sum_{j=1}^n g_j\alpha_j$, where $g_j := \sum_{i=1}^k b_{i,j}\tau^i \in L[\tau]$. □

We say that G acts on M by *semi-linear automorphisms* (cf. [3, p. 110]), $G \times M \to M$, $(\sigma, m) \mapsto \sigma * m$, if for all $m, m' \in M$, $\sigma \in G$, and $\lambda \in K[\tau]$ we have

(i) $\sigma * (m + m') = \sigma * m + \sigma * m'$,
(ii) $\sigma * (\lambda m) = \sigma\lambda \cdot \sigma * m$,

where $\sigma\lambda$ denotes the usual action of G on $K[\tau]$, and the dot denotes the action of $K[\tau]$ on M. Let

$$M^G := \{m \in M \mid \sigma * m = m \text{ for all } \sigma \in G\}.$$

It is easy to see that M^G is a left $L[\tau]$-module.

Lemma 6.5. *The left $L[\tau]$-module M^G is free of rank d, i.e., $M^G \cong L[\tau]^d$. Moreover, the map $K \otimes_L M^G \to M$, $\alpha \otimes m \mapsto \alpha m$, is an isomorphism.*

Proof. Since every left ideal of $L[\tau]$ is principal (Lemma 6.1), every submodule of a free left $L[\tau]$-module of finite rank is also free of finite rank (cf. [21, Thm. 2.44]). On the other hand, by Lemma 6.4, the left $L[\tau]$-module M is free of finite rank. Hence the $L[\tau]$-submodule M^G of M is also free of finite rank. To show that the rank of M^G over $L[\tau]$ is d, it is enough to show that the map $K \otimes_L M^G \to M$, $\alpha \otimes m \mapsto \alpha m$, is an isomorphism. This last isomorphism follows from the Galois descent for vector spaces; see [3, Lem. III.8.21]. □

Proposition 6.6. *Let $c : G \to \mathrm{GL}_d(K[\tau])$, $\sigma \mapsto c_\sigma$, be a map which satisfies $c_{\sigma\delta} = \sigma(c_\delta)c_\sigma$ for all $\sigma, \delta \in G$. Then there is a matrix $S \in \mathrm{GL}_d(K[\tau])$ such that $c_\sigma = (\sigma S)^{-1}S$ for all $\sigma \in G$.*

Proof. Define a (twisted) action of G on M:

$$\sigma * m = (\sigma m)c_\sigma \quad \text{for all } m \in M, \sigma \in G.$$

One easily checks that $(\sigma\delta) * m = \sigma * (\delta * m)$ for all $\sigma, \delta \in G$ and $m \in M$, so this is indeed an action. Moreover, this action is semi-linear. Using Lemma 6.5, we can choose a basis $v_1, \dots, v_d$ of the left $L[\tau]$-module $M^G \cong L[\tau]^d$ such that $\sum_{i=1}^d K[\tau]v_i = M$. Since M is a free left $K[\tau]$-module of rank d, the elements $v_1, \dots, v_d$ form a left $K[\tau]$-basis of M. Let S be the matrix whose rows are $v_1, \dots, v_d$. By Lemma 6.2, $S \in \mathrm{GL}_d(K[\tau])$. The relations

$$v_i = \sigma * v_i = (\sigma v_i)c_\sigma \text{ for all } i = 1, \dots, d,$$

are equivalent to the matrix equality $S = (\sigma S)c_\sigma$ for all $\sigma \in G$, and this implies the claim of the lemma. □

Lemma 6.7. *Let ϕ be a Drinfeld-Stuhler O_D-module over K with $\mathrm{Aut}_K(\phi) \cong \mathbb{F}_{q^r}^\times$ (cf. Theorem 4.1). Then*

$$\partial : \mathrm{Aut}_K(\phi) \to \mathrm{GL}_d(K)$$

gives an isomorphism from the group $\mathrm{Aut}_K(\phi)$ to the group $\mathcal{A} := \{\mathrm{diag}(\alpha, \dots, \alpha) \mid \alpha \in \mathbb{F}_{q^r}^\times\}$.

Proof. By Lemma 2.3, $\partial(\mathrm{Aut}_K(\phi))$ lies in the center of $\mathrm{GL}_d(K)$. Since the center of $\mathrm{GL}_d(K)$ consists of diagonal matrices, and the $(q^r - 1)$-th roots of 1 in K are the elements of $\mathbb{F}_{q^r}^\times$, the restriction of ∂ to $\mathrm{Aut}_K(\phi)$ is indeed a homomorphism into $\mathcal{A}$. Since $\mathrm{Aut}_K(\phi) \cong \mathcal{A}$, to prove that ∂ is an isomorphism it is enough to prove that it is injective. Let $h := q^r - 1$. Assume $\alpha \in \mathrm{Aut}_K(\phi)$ is such that $\partial(\alpha) = 1$. Then we can write $\alpha = 1 + \sum_{i=1}^n B_i \tau^n$ for some $n \geq 1$. Suppose not all B_i are zero, and let m be the smallest index such that $B_m \neq 0$. Then

$$1 = \alpha^h = 1 + hB_m\tau^m + \cdots ,$$

which implies $hB_m = 0$. Since h is coprime to the characteristic of K, we must have $B_m = 0$, which is a contradiction. □

Remark 6.8. It is not generally true that the elements of $\mathrm{Aut}_K(\phi)$ are diagonal matrices in $\mathrm{GL}_d(K[\tau])$. For example, suppose $d = 2$, $\mathrm{diag}(\alpha, \alpha) \in \mathrm{Aut}_K(\phi)$, and $\alpha \notin \mathbb{F}_q$. Let $S = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(K[\tau])$. Then $\begin{pmatrix} \alpha & (\alpha^q - \alpha)\tau \\ 0 & \alpha \end{pmatrix} \in \mathrm{Aut}_K(\psi)$, where ψ is the Drinfeld-Stuhler module $S\phi S^{-1}$.

Definition 6.9. Let ϕ be a Drinfeld-Stuhler O_D-module over L^{sep}. For $\sigma \in \mathrm{Gal}(L^{\mathrm{sep}}/L)$, let ϕ^σ be the composition

$$\phi^\sigma : O_D \xrightarrow{\phi} M_d(L^{\mathrm{sep}}[\tau]) \xrightarrow{\sigma} M_d(L^{\mathrm{sep}}[\tau]).$$

It is easy to check that ϕ^σ is again a Drinfeld-Stuhler O_D-module. We say that L is a *field of moduli for* ϕ if for all $\sigma \in \mathrm{Gal}(L^{\mathrm{sep}}/L)$ the Drinfeld-Stuhler module ϕ^σ is isomorphic to ϕ.

If L is a field of definition for ϕ, then L is obviously a field of moduli.

Theorem 6.10. *Let ϕ be a Drinfeld-Stuhler O_D-module over L^{sep} with $\mathrm{Aut}(\phi) \cong \mathbb{F}_{q^r}^\times$. Assume L is a field of moduli for ϕ. If d and $q^r - 1$ are coprime, then L is a field of definition for ϕ.*

Proof. We can find a finite Galois extension K of L such that ϕ is defined over K and all isomorphisms of ϕ to ϕ^σ for every $\sigma \in \mathrm{Gal}(K/L)$ are defined over K. (Take, for example, K such that ϕ and $\phi[a]$ are defined over K, where $a \in A$ is coprime with $\mathrm{char}_A(L)$ and $\mathfrak{r}(D)$.) In particular, $\mathrm{Aut}_K(\phi) = \mathrm{Aut}(\phi)$. Denote $G = \mathrm{Gal}(K/L)$. For each $\sigma \in G$, choose an isomorphism $\lambda_\sigma : \phi \to \phi^\sigma$. Then

$$\lambda_{\sigma\delta}\phi\lambda_{\sigma\delta}^{-1} = \phi^{\sigma\delta} = (\phi^\delta)^\sigma = (\lambda_\delta\phi\lambda_\delta^{-1})^\sigma = \sigma(\lambda_\delta)\phi^\sigma\sigma(\lambda_\delta)^{-1} = \sigma(\lambda_\delta)\lambda_\sigma\phi\lambda_\sigma^{-1}(\sigma\lambda_\delta)^{-1}. \tag{6.1}$$

Hence
$$\lambda_{\sigma\delta} = \sigma(\lambda_\delta)\lambda_\sigma\alpha_{\sigma,\delta}$$
with $\alpha_{\sigma,\delta} \in \mathrm{Aut}(\phi)$.

Let $\underline{\det} : \mathrm{GL}_d(K[\tau]) \to K^\times$ be the composition
$$\underline{\det} : \mathrm{GL}_d(K[\tau]) \xrightarrow{\partial} \mathrm{GL}_d(K) \xrightarrow{\det} K^\times.$$
The assumption that d and $q^r - 1$ are coprime, combined with Lemma 6.7, implies that $\underline{\det} : \mathrm{Aut}(\phi) \xrightarrow{\sim} \mathbb{F}_{q^r}^\times$ is an isomorphism. Denote $\mu_\sigma = \underline{\det}(\lambda_\sigma)$ and $h = q^r - 1$. Then $\mu_{\sigma\delta} = \sigma(\mu_\delta)\mu_\sigma\underline{\det}(\alpha_{\sigma,\delta})$, and $\mu_{\sigma\delta}^h = \sigma(\mu_\delta^h)\mu_\sigma^h$. Hence $G \to K^\times$, $\sigma \mapsto \mu_\sigma^h$, is a 1-cocycle. By Hilbert's Theorem 90 for $K^\times$, there is $b \in K^\times$ such that $\mu_\sigma^h = b/\sigma(b)$ for all $\sigma \in G$. Let a be an element of L^{sep} such that $a^h = b$. The extension $K' := K(a)$ is Galois over L. Put $G^* = \mathrm{Gal}(K'/L)$, and let $\pi : G^* \to G$ be the natural homomorphism. For every $\sigma \in G^*$, we see that $\mu_{\pi(\sigma)}\sigma(a)/a$ is an h-th root of unity, hence there is a unique $\alpha_\sigma \in \mathrm{Aut}(\phi)$ such that $\mu_{\pi(\sigma)}\underline{\det}(\alpha_\sigma) = a/\sigma(a)$.

Put $c_\sigma = \lambda_{\pi(\sigma)}\alpha_\sigma$. Then $c_\sigma : \phi \to \phi^\sigma$ is an isomorphism and $\underline{\det}(c_\sigma) = a/\sigma(a)$. Repeating the calculation (6.1) for c_σ, we arrive at the relations $c_{\sigma\delta} = \sigma(c_\delta)c_\sigma\beta_{\sigma,\delta}$ for some $\beta_{\sigma,\delta} \in \mathrm{Aut}(\phi)$. But now, taking $\underline{\det}$ of both sides, we have
$$\frac{a}{\sigma\delta(a)} = \frac{\sigma(a)}{\sigma(\delta(a))}\frac{a}{\sigma(a)}\underline{\det}(\beta_{\sigma,\delta}).$$
Thus $\underline{\det}(\beta_{\sigma,\delta}) = 1$. Since $\underline{\det} : \mathrm{Aut}(\phi) \to K^\times$ is injective, we must have $\beta_{\sigma,\delta} = 1$. Therefore, $c_{\sigma\delta} = \sigma(c_\delta)c_\sigma$ for all $\sigma, \delta \in G^*$. By Proposition 6.6, there is $S \in \mathrm{GL}_d(K'[\tau])$ such that $c_\sigma = (\sigma S)^{-1}S$ for all $\sigma \in G^*$. Put $\psi = S\phi S^{-1}$; this is a Drinfeld-Stuhler module isomorphic to ϕ over K'. For any $\sigma \in G^*$ we have
$$\psi^\sigma = (S\phi S^{-1})^\sigma = (\sigma S)(\phi^\sigma)(\sigma S^{-1}) = (\sigma S)(c_\sigma\phi c_\sigma^{-1})(\sigma S)^{-1} = S\phi S^{-1} = \psi,$$
so ψ is defined over L. □

Remarks 6.11. (1) Recall from Theorem 4.1 that $\mathrm{Aut}(\phi) \cong \mathbb{F}_{q^r}^\times$ for some r dividing d. Therefore, the assumption in Theorem 6.10 can be replaced by a universal but stronger assumption that d and $q^d - 1$ are coprime. Note that if $d = p^e$ is a power of the characteristic of F, then the assumption of Theorem 6.10 is always satisfied. On the other hand, if $d = \ell$ is a prime different from p, then the assumption is satisfied if and only if ℓ does not divide $q - 1$.

(2) The proof of Theorem 6.10 specialized to $d = 1$ implies that the fields of moduli for Drinfeld A-modules of arbitrary rank are fields of definition. This can be considered as the analogue of the well-known fact that the fields of moduli for elliptic curves are fields of definition; cf. [24, Prop. I.4.5]. The proof for elliptic curves uses the j-invariant, an invariant which is not available for Drinfeld modules if A is not the polynomial ring or the rank is greater than 2.

In [17], $\mathscr{D}$-elliptic sheaves are defined over any $\mathbb{F}_q$-scheme S. The functor which associates to S the set of isomorphism classes of $\mathscr{D}$-elliptic sheaves over S modulo the action of $\mathbb{Z}$ possesses a coarse moduli scheme $X^{\mathscr{D}}$ which is a smooth proper scheme over $C' := C - \mathrm{Ram}(D) - \{\infty\}$ of relative dimension $(d-1)$; this follows from Theorems 4.1 and 6.1 in [17], combined with the Keel-Mori theorem. Thanks to Theorems 3.2 and 3.4, the fibre of this moduli scheme over a point x of C' is the coarse moduli space of isomorphism classes of Drinfeld-Stuhler O_D-modules over fields L such that $z(\mathrm{Spec}(L)) = x$.

Corollary 6.12. *Let $X^{\mathscr{D}}_F := X^{\mathscr{D}} \otimes_{C'} \mathrm{Spec}(F)$. Assume d and $q^d - 1$ are coprime. If L is a field extension of F which does not split D, then $X^{\mathscr{D}}_F(L) = \emptyset$.*

Proof. Given a non-zero $a \in A$ coprime with $\mathfrak{r}(D)$, one can consider the problem of classifying Drinfeld-Stuhler modules with level-a structures, i.e., classifying pairs (ϕ, ι), where ϕ is a Drinfeld-Stuhler O_D-module and ι is an isomorphism $\iota : \phi[a] \cong O_D/aO_D$.This moduli problem is representable; see [17, (5.1)]. Denote the corresponding moduli scheme by $X^{\mathscr{D}}_a$. The forgetful map $(\phi, \iota) \mapsto \phi$ gives a Galois covering $X^{\mathscr{D}}_{a,F} \to X^{\mathscr{D}}_F$. Suppose there is an L-rational point P on $X^{\mathscr{D}}_F$. Then a preimage P' of P in $X^{\mathscr{D}}_{a,F}$ is defined over a Galois extension L' of L. Since $X^{\mathscr{D}}_{a,F}$ is a fine moduli scheme, there is a Drinfeld-Stuhler O_D-module ϕ defined over L' which corresponds to P'. For any $\sigma \in \mathrm{Gal}(L'/L)$, the Drinfeld-Stuhler modules ϕ and ϕ^σ are isomorphic over L', since ϕ arises from an L-rational point on $X^{\mathscr{D}}_F$. Hence L is a field of moduli of ϕ. By Theorem 6.10 and Remark 6.11 (1), ϕ can be defined over L. Now Lemma 2.3 implies that L splits D. □

Remark 6.13. Theorem 6.10 is the analogue for Drinfeld-Stuhler modules of a theorem of Shimura for abelian varieties [23, Thm. 9.5]. It is known that in general the fields of moduli for abelian varieties are not necessarily fields of definition. For example, let B be an indefinite quaternion division algebra over $\mathbb{Q}$, and let X^B be the associated Shimura curve over $\mathbb{Q}$, which is the coarse moduli scheme of abelian surfaces equipped with an action of B. The main result in [15] provides examples of nonarchimedean local fields L failing to split B with $X^B(L) \neq \emptyset$ (see also [14, §1]); a necessary condition for this phenomenon is that 2 ramifies in B. If we let $A = \mathbb{F}_q[T]$, $F = \mathbb{F}_q(T)$, and $d = 2$, then $X^{\mathscr{D}}_F$ is the function field analogue of X^B; cf. [17], [20]. However, examples similar to those constructed by Jordan and Livné do not exist in this setting since for any finite extension L of F_v, $v \in \mathrm{Ram}(D)$, which does not split D we have $X^D(L) = \emptyset$ by Theorem 4.1 in [20]. This leaves open the interesting question whether in general the fields of moduli of Drinfeld-Stuhler modules are fields of definition.

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