Neron-Severi group for torus quasi bundles over curves

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0. Introduction

By the Neron-Severi group of a compact complex manifold X we mean the kernel of the natural homomorphism $H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X)$. It is a subgroup of $H^2(X,\mathbb{Z})$ generated by the first Chern classes of line bundles on X. In this paper we shall study the Neron-Severi group for torus quasi bundles over curves. Firstly, we study the case of torus principal bundles $X \xrightarrow{\pi} B$ over a (complex, compact, connected, smooth) curve B, whose structure group is a compact complex torus $T = V/\Lambda$. A T-principal bundle $X \xrightarrow{\pi} B$ is defined by a cohomology class $\xi \in H^1(\mathcal{O}_B(T))$, where $\mathcal{O}_B(T)$ is the sheaf of germs of locally holomorphic maps from B to T. The cohomology class ξ determines a characteristic class $c(\xi) \in H^2(B,\Lambda)$. By a Theorem of Blanchard ([1]), the total space X of such a T-principal bundle is a non-Kähler manifold if and only if $c(\xi) \neq 0$. In the first two parts of the paper we present some basic facts on torus principal bundles (see [7]) and we compute Leray spectral sequences for the sheaves $\mathbb{Z}_X$ and $\mathcal{O}_X$. In the third part we define for any line bundle $L \in Pic(T)$ an *associated* $T^\vee$*-principal bundle*, described by an element $\tilde{\varphi}_L(\xi) \in H^1(\mathcal{O}_B(T^\vee))$, where $T^\vee$ is the dual torus, and we compute the Neron-Severi group for torus principal bundles. We state the main result (Theorem 5):

"For a T-principal bundle $X \xrightarrow{\pi} B$, defined by a cohomology class

$$\xi \in H^1(\mathcal{O}_B(T)),$$

we have an exact sequence of free groups

$$0 \to Hom(J_B, T^\vee) \to NS(X)/F_2 \to \tilde{N}(X) \to 0 \ ,$$

where $F_2 = \pi^* NS(B)$ and $\tilde{N}(X)$ is the subgroup of the Neron-Severi group of the torus T defined by

$$\tilde{N}(X) = \{c_1(L) \in NS(T) \mid \tilde{\varphi}_L(\xi) \text{ is the trivial torus bundle } \} \ ,$$

J_B is the Jacobian variety of the curve B and $T^\vee$ is the dual torus. If X is Kähler F_2 is isomorphic to $NS(B) \simeq \mathbb{Z}$ and if X is non-Kähler, F_2 is the torsion subgroup of $NS(X)$ "

In the fourth part we reinterpret the obtained results geometrically (see Theorem 6).
Then, in the fifth part, we study the case of torus quasi bundles. By a quasi T-bundle $\pi : X \to B$ over a curve B we mean that π is a T-principal bundle over $B \setminus \{b_1, b_2, ..., b_\ell\}$ and that the fibre $\pi^{-1}(b_i)$ over the point b_i is of the form $m_i T_i$ where $m_i \geq 2$ and T_i is a torus (the fibre $m_i T_i$ is called a multiple fibre of the multiplicity m_i). In the Appendix we show that all torus quasi bundles are obtained from $B \times T$ by means of generalized logarithmic transformations. We associate, canonically, a T_0- principal bundle $\pi_0 : Y \to B$ to a quasi T-bundle $\pi : X \to B$ and a holomorphic mapping $f : X \to Y$, with $T_0 = T/H$, where H is a finite subgroup of the torus T. Then we extend the computation of the Neron-Severi group for torus quasi bundles (see Theorem 17).
For the case of elliptic surfaces see [3], [4].

Acknowledgements. The first named author would like to thank the Alexander von Humboldt-Stiftung for support and both authors would like to express their thanks to Professor F. Hirzebruch and Max-Planck-Institut für Mathematik in Bonn for hospitality. This paper was prepared at the time we visited this institution.

1. Basic facts on torus principal bundles

Let $T = V/\Lambda$ be an n-dimensional compact complex torus, defined by a lattice $\Lambda \subset V$ in the n-dimensional complex vector space V. Canonical notation concerning the torus T will be used:

$$T_0(T) = H^0(T, \Theta_T) = V \ , \ H^i(T, \Theta_T) = H^i(T, \mathcal{O}_T) \otimes V \ ,$$

$$H^0(T, \Omega^1_T) = H^0(T, \Theta_T)^\vee = V^\vee \ , \ \Lambda = H_1(T, \mathbb{Z}) \ , \ H^1(T, \mathbb{Z}) = \Lambda^\vee \ .$$

If B is a compact complex manifold of dimension m, then $X \xrightarrow{\pi} B$ denotes a T-principal bundle over B. Let $\mathcal{O}_B(T)$ denote the sheaf of germs of locally holomorphic maps from B to T. The T-principal bundles are described by cohomology classes ξ of $H^1(B, \mathcal{O}_B(T))$ (see [6]). For a Čech 1-cocycle (ξ_{ij}) the function

$$\xi_{ij} : U_i \cap U_j \to T$$

identifies $(z, t) \in U_i \times T$ with $(z, t') = (z, \xi_{ij}(z) + t) \in U_j \times T$ for all $z \in U_i \cap U_j$.

Taking local sections of the constant sheaves

$$0 \to \Lambda \to V \to T \to 0$$

one gets an exact sequence of sheaves on the manifold B

(1)
$$0 \to \Lambda \to \mathcal{O}_B \otimes V \to \mathcal{O}_B(T) \to 0 \ ,$$

with the induced exact cohomology sequence

(2)
$$\ldots \to H^0(\mathcal{O}_B(T)) \to H^1(B,\Lambda) \to H^1(B,\mathcal{O}_B) \otimes V \to$$
$$\to H^1(\mathcal{O}_B(T)) \xrightarrow{c} H^2(B,\Lambda) \to H^2(B,\mathcal{O}_B) \otimes V \to \ldots \ .$$

The cohomology class ξ of the bundle in $H^1(\mathcal{O}_B(T))$ determines a characteristic class $c(\xi) \in H^2(B,\Lambda) = H^2(B,\mathbb{Z}) \otimes \Lambda$.

Because transition functions of the T-principal bundle $X \xrightarrow{\pi} B$ act trivially on the cohomology of fibre, we get natural identifications:

(3)
$$R^q\pi_*\mathbb{Z}_X = \mathbb{Z}_B \otimes_{\mathbf{Z}} H^q(T,\mathbb{Z}) \ ; \ R^q\pi_*\mathcal{O}_X = \mathcal{O}_B \otimes_{\mathbf{C}} H^q(T,\mathcal{O}_T) \ .$$

The transgression of the fibre bundle in integral cohomology is a map

$$\delta : H^1(T,\mathbb{Z}) \to H^2(B,\mathbb{Z}) \ .$$

Under the identification

$$H^1(T,\mathbb{Z}) = Hom(\Lambda,\mathbb{Z}) = \Lambda^\vee \ ,$$

the characteristic class $c(\xi) \in H^2(B,\mathbb{Z}) \otimes \Lambda$ and the mapping $\delta : H^1(T,\mathbb{Z}) \to H^2(B,\mathbb{Z})$ coincide (see [7], 6.1). The first possibly nontrivial d_2-homomorphism

$$H^0(B, R^1\pi_*\mathcal{O}_X) \to H^2(B,\pi_*\mathcal{O}_X)$$

in the Leray spectral sequence of $\mathcal{O}_X$ is denoted by

$$\varepsilon : H^1(T,\mathcal{O}_T) \to H^2(B,\mathcal{O}_B) \ .$$

Recall for convenience the following result of Höfer (see [7], 7.1 and 7.2):

Proposition *There is an injective map*

$$\Phi : Pic(B) \otimes_{\mathbf{Z}} \Lambda = H^1(\mathcal{O}_B^*) \otimes_{\mathbf{Z}} \Lambda \to H^1(\mathcal{O}_B(T))$$

compatible with taking characteristic classes, i.e. if $\Sigma\ \mathcal{L}_k \otimes \lambda_k$ *is a combination of line bundles in* $Pic(B) \otimes_{\mathbf{Z}} \Lambda$, *then the characteristic class* $c(\xi)$ *of* $\Phi(\Sigma\ \mathcal{L}_k \otimes \lambda_k)$ *equals* $\Sigma\ c_1(\mathcal{L}_k) \otimes \lambda_k \in H^2(B,\Lambda)$.

$$\begin{array}{ccc} Pic(B)\otimes_{\mathbb{Z}}\Lambda & \xrightarrow{\Phi} & H^1(\mathcal{O}_B(T)) \\ {\scriptstyle c_1\otimes id}\downarrow & & \downarrow{\scriptstyle c} \\ H^2(B,\mathbb{Z})\otimes\Lambda & \xrightarrow{=} & H^2(B,\Lambda) \end{array}$$

Moreover, if $H^2(B,\mathbb{C})$ has a Hodge decomposition, then the image of Φ, i.e. the set of isomorphism classes of principal bundles constructed above, equals

$$im\Phi = \{Isom.\ classes\ of\ T-principal\ bundles\ with\ \varepsilon\ = 0\}\ .$$

Remark. If B is a curve, then ε vanishes for dimension reasons.Thus, every T-principal bundle over B comes (in an unique way) from the above construction. The construction itself is a *generalized logarithmic transformation* applied to the trivial T-principal bundle $B\times T$ (see [9]). Indeed, we can write $\mathcal{L}_k = \mathcal{O}_B(D_k)$, with D_k a divisor on B; by choosing a sufficiently fine open covering (U_i) of B the transition functions of each $\mathcal{L}_k$ are expressed by a cocycle $(f_{ij}^{(k)})$. Now, identify $(z,t_i)\in U_i\times T$ with $(z,t_j)\in U_j\times T$ if and only if

$$t_i = t_j + \left[\Sigma\ \frac{\lambda_k}{2\pi\sqrt{-1}}\ \log\left(f_{ij}^{(k)}\right)\right]\ ,$$

for all $z\in U_i\cap U_j$ (this is exactly Höfer's morphism Φ).

Also we can construct a T-principal bundle over B by using logarithmic transformations similar to the case of elliptic surfaces. Express the divisor D_k as

$$D_k = \sum_{j=1}^{n_k} m_j^{(k)} b_j^{(k)}\ .$$

Let $U_j^{(k)}$ be a coordinate neighbourhood of $b_j^{(k)}$ with local coordinate $t_j^{(k)}$. We may assume

$$U_j^{(k)} = \{t_j^{(k)}\in\mathbb{C}\ |\ |t_j^{(k)}|<\varepsilon\}\ ,$$

for a sufficiently small positive number ε. Let us consider a holomorphic mapping

$$l_j^{(k)} : U_j^{(k)*}\times T \longrightarrow U_j^{(k)*}\times T$$

$$(t_j^{(k)},[\zeta]) \to (t_j^{(k)},\left[\zeta - \frac{m_j^{(k)}\lambda_k}{2\pi\sqrt{-1}}\log t_j^{(k)}\right]).$$

Note that the mapping is an isomorphism. Hence, we can patch $U_j^{(k)} \times T$'s and $(B \setminus \{b_1^{(1)}, ..., b_j^{(k)}, ...\}) \times T$ by the isomorphisms $l_j^{(k)}$ and obtain a T- principal bundle over B. We denote the T-principal bundle obtained in this way by

$$L_{b_1^{(1)}}(m_1^{(1)}\lambda_1, 1)...L_{b_{n_l}^{(l)}}(m_{n_l}^{(l)}\lambda_l, 1)(B \times T)$$

or by

$$L_{D_1}(\lambda_1, 1)...L_{D_l}(\lambda_l, 1)(B \times T)$$

Remark. By the above proposition and Blanchard's theorem ([1]) we can easily show that a T-principal bundle

$$L_{b_1}(a_1, 1)...L_{b_l}(a_l, 1)(B \times T)$$

is Kähler if and only if $\sum_{i=1}^{l} a_i = 0$.

2. Leray spectral sequences

Let $X \xrightarrow{\pi} B$ be a T-principal bundle over the manifold B. We consider the Leray spectral sequences:

$$E_2^{pq} = H^p(B, R^q\pi_*\mathbb{Z}_X) \Longrightarrow H^{p+q}(X, \mathbb{Z}) \tag{4}$$

$$\tilde{E}_2^{pq} = H^p(B, R^q\pi_*\mathcal{O}_X) \Longrightarrow H^{p+q}(X, \mathcal{O}_X) . \tag{5}$$

By the results of Höfer (see [7]) the first spectral sequence (4) degenerates at E_3-level (i.e. $d_r = 0$ for $r > 2$) and the d_2-differential is determined by the map $\delta : H^1(T, \mathbb{Z}) \to H^2(B, \mathbb{Z})$ (i.e. by $c(\xi)$).

Now, we suppose that B is a curve. By (3) we have:

$$E_\infty^{02} = E_3^{02} = ker(E_2^{02} \xrightarrow{d_2} E_2^{21}) =$$

$$= ker(H^0(B, \mathbb{Z}) \otimes H^2(T, \mathbb{Z}) \xrightarrow{d_2} H^2(B, \mathbb{Z}) \otimes H^1(T, \mathbb{Z})) .$$

With the natural identifications

$$H^0(B, \mathbb{Z}) = \mathbb{Z} \ , \ H^2(B, \mathbb{Z}) = \mathbb{Z} \ , \ H^2(T, \mathbb{Z}) = \bigwedge^2 H^1(T, \mathbb{Z}) ,$$

we obtain

$$E_\infty^{02} = ker(H^2(T, \mathbb{Z}) \xrightarrow{d_2} H^1(T, \mathbb{Z})) ,$$

where

$$d_2(\varphi_1 \wedge \varphi_2) = \delta(\varphi_1)\varphi_2 - \delta(\varphi_2)\varphi_1 \ , \forall \varphi_1, \varphi_2 \in H^1(T, \mathbb{Z}) \ .$$

Obviously, we have

$$E_\infty^{11} = E_2^{11} = H^1(B, \mathbb{Z}) \otimes H^1(T, \mathbb{Z}) = H^1(B, \mathbb{Z}) \otimes \Lambda^\vee \ .$$

Finally, we get

$$E_\infty^{20} = E_3^{20} = coker(H^0(B, \mathbb{Z}) \otimes H^1(T, \mathbb{Z}) \xrightarrow{d_2} H^2(B, \mathbb{Z})) =$$

$$= coker(H^1(T, \mathbb{Z}) \xrightarrow{\delta} H^2(B, \mathbb{Z})) \ .$$

The cohomology class $\xi \in H^1(\mathcal{O}_B(T))$ of the T- principal bundle $X \xrightarrow{\pi} B$ has the form $\Phi(\Sigma\, \mathcal{L}_k^0 \otimes \lambda_k^0)$ and its characteristic class has the form

$$c(\xi) = \Sigma\, c_1(\mathcal{L}_k^0) \otimes \lambda_k^0 = m\lambda^0 \in \Lambda = H^2(B, \Lambda) \ , \tag{6}$$

where $\mathcal{L}_k^0 \in Pic(B)$, $\lambda_k^0 \in \Lambda$ is a primitive element (i.e. there exists no positive integer $l \geq 2$ with $\lambda_k^0 = l\tilde{\lambda}_k^0$, $\tilde{\lambda}_k^0 \in \Lambda), m \in \mathbb{N}, m = g.c.d.(c_1(\mathcal{L}_k^0))$ and $\lambda^0 \in \Lambda$. It follows that for any $\varphi \in H^1(T, \mathbb{Z})$ we have the equality $\delta(\varphi) = m\varphi(\lambda^0)$, under the identification $H^1(T, \mathbb{Z}) = \Lambda^\vee = Hom(\Lambda, \mathbb{Z})$.We get

$$E_\infty^{20} = \begin{cases} \mathbb{Z}_m & \text{for } c(\xi) \neq 0 \\ \mathbb{Z} & \text{for } c(\xi) = 0 \end{cases} \ .$$

The second spectral sequence (5) degenerates at E_2-level for torus principal bundles with $\varepsilon = 0$, since the d_2- differential is determined by ε (see [7], 4. and [2]). With natural identifications, by (3) we get:

$$\tilde{E}_\infty^{20} = \tilde{E}_2^{20} = H^0(B, \mathcal{O}_B) \otimes H^2(T, \mathcal{O}_T) = H^2(T, \mathcal{O}_T) \ .$$

$$\tilde{E}_\infty^{11} = \tilde{E}_2^{11} = H^1(B, \mathcal{O}_B) \otimes H^1(T, \mathcal{O}_T) \ .$$

$$\tilde{E}_\infty^{20} = \tilde{E}_2^{20} = 0 \ .$$

3. Neron-Severi group for torus principal bundles

Let $X \xrightarrow{\pi} B$ be a T-principal bundle over the curve B, defined by $\xi \in H^1(\mathcal{O}_B(T))$ with $c(\xi) \neq 0$ (i.e. X is non-Kähler). Let

$$0 \subset F_2 \subset F_1 \subset F_0 = H^2(X, \mathbb{Z})$$

be the filtration induced by the first spectral sequence (4). Then $F_2 = E_\infty^{20} \cong \mathbb{Z}_m$ is a torsion subgroup of $H^2(X, \mathbb{Z})$. Since both $F_1/F_2 = E_\infty^{11}$ and $F_0/F_1 = E_\infty^{02}$ are free, it follows $Tors\ H^2(X, \mathbb{Z}) = F_2 \cong \mathbb{Z}_m$. We get the exact sequence:

$$0 \to H^1(B, \mathbb{Z}) \otimes H^1(T, \mathbb{Z}) \to H^2(X, \mathbb{Z})/Tors\ H^2(X, \mathbb{Z}) \to \tag{7}$$

$$\to ker(H^2(T,\mathbb{Z}) \xrightarrow{d_3} H^1(T,\mathbb{Z})) \to 0.$$

Let

$$0 \subset \tilde{F}_2 \subset \tilde{F}_1 \subset \tilde{F}_0 = H^2(X,\mathcal{O}_X)$$

be the filtration induced by the second spectral sequence (5). Then, we get the exact sequence:

$$(8) \qquad 0 \to H^1(B,\mathcal{O}_B) \otimes H^1(T,\mathcal{O}_T) \to H^2(X,\mathcal{O}_X) \to H^2(T,\mathcal{O}_T) \to 0.$$

The Neron-Severi group, denoted by $NS(X)$, is the kernel of the map in cohomology $H^2(X,\mathbb{Z}) \xrightarrow{i} H^2(X,\mathcal{O}_X)$, induced by the natural map $\mathbb{Z}_X \xrightarrow{i} \mathcal{O}_X$. Since $F_2 \xrightarrow{i} \tilde{F}_2 = 0$, we have $F_2 \subset NS(X)$ and

$$(9) \qquad TorsNS(X) = F_2 = TorsH^2(X,\mathbb{Z}) \cong \mathbb{Z}_m \, .$$

Using the exact sequence of small terms of the first spectral sequence (4) we get

$$TorsNS(X) = im(H^2(B,\mathbb{Z}) \xrightarrow{\pi^*} H^2(X,\mathbb{Z}))\, .$$

By functoriality of the spectral sequences we get the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_1/F_2 & \longrightarrow & F_0/F_2 & \longrightarrow & F_0/F_1 & \longrightarrow & 0 \\ & & \downarrow i' & & \downarrow i & & \downarrow i'' & & \\ 0 & \longrightarrow & \tilde{F}_1 & \longrightarrow & \tilde{F}_0 & \longrightarrow & \tilde{F}_0/\tilde{F}_1 & \longrightarrow & 0 \end{array}$$

where the first line is the exact sequence (7) and the second line is the exact sequence (8). Since $NS(X)/TorsNS(X) \cong ker(i)$, we obtain the exact sequence:

$$(10) \qquad 0 \to ker(i') \to NS(X)/TorsNS(X) \to ker(i'') \xrightarrow{\beta} coker(i').$$

Lemma 1 *We have $ker(i') \cong Hom(J_B, T^\vee)$, where J_B is the Jacobian variety of the curve B, $T^\vee$ is the dual torus of the torus T and $Hom(J_B, T^\vee)$ is the group of homomorphisms of group varieties.*

Proof: By [8], Chap.I, 2, we have the exact sequence

$$0 \to \Lambda^\vee \to \overline{V}^\vee \to T^\vee \to 0\, ,$$

where

$$\Lambda^\vee = H^1(T,\mathbb{Z})\ ,\ \overline{V}^\vee = H^1(T,\mathcal{O}_T)\ ,\ T^\vee = Pic^0(T)\ .$$

Taking local sections of these constant sheaves one gets an exact sequence of sheaves on B

$$0 \to \Lambda^\vee \to \mathcal{O}_B \otimes \overline{V}^\vee \to \mathcal{O}_B(T^\vee) \to 0\ , \tag{11}$$

with the induced exact cohomology sequence:

(12) $$0 \to H^0(B,\Lambda^\vee) \to H^0(B,\mathcal{O}_B)\otimes \overline{V}^\vee \to H^0(\mathcal{O}_B(T^\vee)) \to H^1(B,\Lambda^\vee) \xrightarrow{j}$$

$$\xrightarrow{j} H^1(B,\mathcal{O}_B)\otimes \overline{V}^\vee \to H^1(\mathcal{O}_B(T^\vee)) \xrightarrow{c^\vee} H^2(B,\Lambda^\vee) \to 0.$$

But

$$H^1(B,\Lambda^\vee) = H^1(B,\mathbb{Z})\otimes H^1(T,\mathbb{Z}),$$
$$H^1(B,\mathcal{O}_B)\otimes \overline{V}^\vee = H^1(B,\mathcal{O}_B)\otimes H^1(T,\mathcal{O}_T)$$

and $j = i'$ by naturality. It follows

$$ker(i') = ker(H^1(B,\Lambda^\vee) \xrightarrow{j} H^1(B,\mathcal{O}_B)\otimes \overline{V}^\vee) \cong$$

$$\cong im(H^0(\mathcal{O}_B(T^\vee)) \to H^1(B,\Lambda^\vee)) \cong$$
$$\cong coker(H^0(B,\mathcal{O}_B)\otimes \overline{V}^\vee \to H^0(\mathcal{O}_B(T^\vee))).$$

But $H^0(\mathcal{O}_B(T^\vee))$ is the group of global holomorphic maps $B \to T^\vee$ and

$$im(H^0(B,\mathcal{O}_B)\otimes \overline{V}^\vee \to H^0(\mathcal{O}_B(T^\vee))) \cong \overline{V}^\vee/\Lambda^\vee = T^\vee$$

is the subgroup of constant maps $B \to T^\vee$, which can be identified with the points of $T^\vee$ (or, with the translations of $T^\vee$). Let $B \to J_B$ be the canonical holomorphic map (determined up to a translation of J_B). Given any holomorphic map $B \to T^\vee$ then, if we choose the proper origin on $T^\vee$, the holomorphic map $B \to T^\vee$ is the composition of the canonical map $B \to J_B$ and an homomorphism from J_B to $T^\vee$ (the universal property of the Jacobian). It follows the isomorphism

$$ker(i') \cong Hom(J_B,T^\vee).\ \diamond$$

Lemma 2 *We have*

$$ker(i'') = \{c_1(L) \in NS(T) \mid c_1(L)(\lambda^0) = 0\},$$

where $c(\xi) = m\lambda^0 \in \Lambda$.

Proof: From the previous diagram we get

$$ker(i'') = \{c_1(L) \in NS(T) \mid d_2(c_1(L)) = 0\}.$$

Let $\{e_1, ..., e_{2n}\}$ be a basis of the lattice Λ and let $\{e^1, ..., e^{2n}\}$ be the dual basis in the lattice $\Lambda^\vee$. Any element $E = c_1(L) \in NS(T)$ can be written in the form

$$E = \Sigma_{1\leq i<j\leq 2n}\, a_{ij} e^i \wedge e^j \ , \ a_{ij} \in \mathbb{Z}$$

(see [8], Chap. I, 2). By direct computation we obtain

$$d_2(c_1(L)) = \Sigma_{i<j}\, a_{ij} d_2(e^i \wedge e^j) = \Sigma_{i<j}\, a_{ij}(\delta(e^i)e^j - \delta(e^j)e^i) =$$

$$= m\Sigma_{i<j}\, a_{ij}(e^i(\lambda^0)e^j - e^j(\lambda^0)e^i) = mc_1(L)(\lambda^0),$$

where we made the natural identifications

$$Bil(\Lambda \times \Lambda, \mathbb{Z}) = Hom_{\mathbb{Z}}(\Lambda \otimes \Lambda, \mathbb{Z}) = Hom_{\mathbb{Z}}(\Lambda, \Lambda^\vee).$$

The assertion follows. $\diamond$

For any line bundle $L \in Pic(T)$ we have the homomorphism

$$(13) \qquad \varphi_L : T \to Pic^0(T) = T^\vee,\ \varphi_L(x) = isom.class\ of\ T_x^* L \otimes L^{-1},$$

where $T_x : T \to T$ is the translation with $x \in T$ (see [8]). The T-principal bundle $X \xrightarrow{\pi} B$ being fixed, we can associate to any line bundle $L \in Pic(T)$ an element in $H^1(\mathcal{O}_B(T^\vee))$ in the following way: For the Čech 1-cocycle (ξ_{ij}) defining our T-principal bundle, $\xi_{ij} : U_i \cap U_j \to T$, we put

$$\eta_{ij}^L := \varphi_L \circ \xi_{ij} : U_i \cap U_j \to T^\vee.$$

Then (η_{ij}^L) is a Čech 1-cocycle (φ_L is a homomorphism) and defines a cohomology class in $H^1(\mathcal{O}_B(T^\vee))$, denoted by $\tilde{\varphi}_L(\xi)$.

Definition Let $\xi \in H^1(\mathcal{O}_B(T))$ be fixed. For any $L \in Pic(T)$ the $T^\vee$-principal bundle described by $\tilde{\varphi}_L(\xi)$ will be called the *associated $T^\vee$-bundle to L.*

Lemma 3 *Let $L \in Pic(T)$ be a line bundle. Then, the obstruction to extend L to a line bundle on the total space of the fixed T-principal bundle $X \xrightarrow{\pi} B$ is the associated $T^\vee$ -bundle to L, $\tilde{\varphi}_L(\xi)$.*

Proof: Let $\mathcal{L}_i$ be a line bundle on $U_i \times T$ such that for each point $x \in U_i$, we have

$$(14) \qquad c_1(\mathcal{L}_i|_{x\times T}) = c_1(L).$$

Then, for each point $x \in U_i$,

$$\mathcal{M}_x = (\mathcal{L}_i|_{x \times T}) \otimes L^{-1}$$

is a line bundle of degree zero on T, hence determines a point of $Pic^0(T) = T^\vee$. In this way, the line bundle $\mathcal{L}_i$ defines a holomorphic mapping

$$\varphi_i : U_i \to T^\vee,$$

such that the line bundle

$$p_i^*(L) \otimes (\varphi_i \times id_T)^*(\mathcal{P}) \tag{15}$$

is isomorphic to $\mathcal{L}_i$, where $p_i : U_i \times T \to T$ is the natural projection to the second factor and $\mathcal{P}$ is the Poincaré bundle of $T^\vee$ (which is a line bundle on $T^\vee \times T$). Conversely, if a holomorphic mapping $\varphi_i : U_i \to T^\vee$ is given, then (15) defines a line bundle $\mathcal{L}_i$ on $U_i \times T$ with the property (14). Patching together $\mathcal{L}_i$'s to obtain a line bundle on X, we need to have isomorphisms

$$T_{\xi_{ij}}^* \mathcal{L}_j|_{U_{ij} \times T} \cong \mathcal{L}_i|_{U_{ij} \times T} \tag{16}$$

for all $U_{ij} = U_i \cap U_j \neq \emptyset$, where $T_{\xi_{ij}}$ is an automorphism of $U_{ij} \times T$ induced by the translation of T by $\xi_{ij}(x)$ for each $x \in U_{ij}$.
Since we may assume that $\mathcal{L}_i$ has the form (15), the isomorphism (16) can be rewritten as

$$T_{\xi_{ij}}^*(p_j^* L) \otimes (\varphi_j \times id_T)^*(\mathcal{P})|_{U_{ij} \times T} \cong (p_i^* L) \otimes (\varphi_i \times id_T)^*(\mathcal{P})|_{U_{ij} \times T} \ . \tag{17}$$

Note that for any line bundle M of degree zero on T, we have an isomorphism $T_a^* M \cong M$ for any translation T_a of the torus T.
On the other hand, for each $x \in U_{ij}$, the line bundle

$$T_{\xi_{ij}(x)}^*(L) \otimes L^{-1}$$

defines an element of $T^\vee$ and we have a holomorphic mapping of U_{ij} to $T^\vee$. This holomorphic mapping is nothing but

$$\eta_{ij}^L = \varphi_L \circ \xi_{ij} : U_{ij} \to T^\vee \ .$$

Then, the existence of an isomorphism (17) is equivalent to the equality

$$\eta_{ij}^L + \varphi_j = \varphi_i \ , \tag{18}$$

as the equality in $H^0(U_{ij}, \mathcal{O}_{U_{ij}}(T^\vee))$.
If there exists a line bundle $\mathcal{L}$ on X such that for a point $y \in B$, $\mathcal{L}|_{\pi^{-1}(y)}$ is isomorphic to L, then

$$\mathcal{L}_i := \mathcal{L}|_{U_i \times T} \quad i \in I \ ,$$

satisfy (14) and (16). Therefore, the equality holds for (i,j) with $U_{ij} \neq \emptyset$. Hence, the cocycle $\tilde{\varphi}_L(\xi)$ is zero in $H^1(B, \mathcal{O}_B(T^\vee))$. Conversely, if $\tilde{\varphi}_L(\xi)$ is zero in $H^1(B, \mathcal{O}_B(T^\vee))$, by chossing a suitable open covering $\{U_i\}$ of B, we may assume that the equality (18) holds. Define a line bundle $\mathcal{L}_i$ on $U_i \times T$ by

$$\mathcal{L}_i = p_i^* L \otimes (\varphi_i \times id_T)^*(\mathcal{P}).$$

By (18) we have an isomorphism

$$g_{ij} : \mathcal{L}_j|_{U_{ij}\times T} \to \mathcal{L}_i|_{U_{ij}\times T}.$$

Note that g_{ij} is uniquely determined up to the multiplication of an element of $H^0(U_{ij}, \mathcal{O}^*_{U_{ij}})$. For $i < j$ choose an isomorphism g_{ij} and fix it. Put

$$\begin{aligned} g_{ji} &= g_{ij}^{-1}, \quad i<j \\ g_{ii} &= id. \end{aligned}$$

For $U_{ijk} = U_i \cap U_j \cap U_k \neq \emptyset$, put

$$g_{ijk} = g_{ki} \circ g_{ij} \circ g_{jk}.$$

Since there is a canonical isomorphism of $\mathrm{Aut}(\mathcal{L}|_{\pi^{-1}(U)})$ to $H^0(\pi^{-1}(U), \mathcal{O}^*_{\pi^{-1}(U)}) = H^0(U, \mathcal{O}^*_U)$, the automorphism g_{ijk} of $\mathcal{L}_k|_{U_{ijk}\times T}$ determines and element $\sigma(g_{ijk}) \in H^0(U_{ijk}, \mathcal{O}^*_{U_{ijk}})$. Note that we have equalities:

$$\begin{aligned} \sigma(g_{\ell k} \circ g_{ijk} \circ g_{k\ell}) &= \sigma(g_{ijk}) \quad \text{on } U_{ijk\ell} \\ \sigma(g_{ijk} \circ g_{\ell mk}) &= \sigma(g_{ijk})\sigma(g_{\ell mk}) \quad \text{on } U_{ijk\ell m}. \end{aligned}$$

By using these equalities, it is easy to show that $\{\sigma(g_{ijk})\}$ is a two-cocycle with values in $\mathcal{O}^*_B$. Since we have $H^2(B, \mathcal{O}^*_B) = 0$, if necessarily, by choosing a finer open covering of B and changing the isomorphism g_{ij} by the multiplication of a nowhere vanishing function, we may assume that

$$\sigma(g_{ijk}) = 1.$$

This means that $g_{ijk} = id$ and we can patch together the line bundles $\mathcal{L}_i$ by the isomorphism g_{ij} to obtain a line bundle $\mathcal{L}$ on X. We may also assume that for a point $x \in U_i$ we have $\varphi_i(x) = 0$. Then, we have an isomorphism $\mathcal{L}|_{\pi^{-1}(x)} \cong L$. This proves the lemma. $\diamond$

Lemma 4 *The homomorphism $\beta : ker(i'') \to coker(i')$ is given by the correspondence $c_1(L) \longmapsto \tilde{\varphi}_L(\xi)$.*

Proof: Let $L \in Pic(T)$ be a line bundle. By Appel-Humbert Theorem (see [8]. Chap.I, 2) one has $L = L(H, \alpha)$, where H is a hermitian form on V with $E(\Lambda \times \Lambda) \subset \mathbb{Z}$ ($E = ImH$) and $\alpha : \Lambda \to U(1)$ is a map with

$$\alpha(\lambda_1 + \lambda_2) = e^{i\pi E(\lambda_1, \lambda_2)}\alpha(\lambda_1)\alpha(\lambda_2),\ \lambda_i \in \Lambda.$$

Let us denote by p the canonical projection $V \to T$. By [8], Chap.II, 9, if $a \in V$ with $p(a) = x \in T$, we have

$$\varphi_{L(H,\alpha)}(x) = isom.class\ of\ L(0, \gamma_a),$$

where $\gamma_a : \Lambda \to U(1)$ is the map

$$\gamma_a(\lambda) = e^{2\pi i E(a,\lambda)}\ , \lambda \in \Lambda. \tag{19}$$

From the exact sequence (12) we get

$$coker(i') \cong ker(H^1(\mathcal{O}_B(T^\vee)) \xrightarrow{c^\vee} H^2(B, \Lambda^\vee)).$$

By the previous lemmas it remains to show that the condition $c_1(L)(\lambda^0) = 0$ implies the condition $c^\vee(\eta) = 0$, where $\eta = \tilde{\varphi}_L(\xi)$. For any $z \in U_i \cap U_j$ we choose $a_{ij}(z) \in V$ such that $p(a_{ij}(z)) = \xi_{ij}(z) \in T$.Then

$$\eta_{ij}^L(z) = \varphi(\xi_{ij}(z)) = L(0, \gamma_{a_{ij}(z)}),$$

where $\gamma_{a_{ij}(z)}$ is given by the formula (19) for $c_1(L) = E$.
Since (ξ_{ij}) is a cocycle we have $a_{jk}(z) - a_{ik}(z) + a_{ij}(z) \in \Lambda$. More precisely, we have

$$cls(a_{jk}(z) - a_{ik}(z) + a_{ij}(z)) = m\lambda^0 = c(\xi) \in \Lambda = H^2(B, \Lambda).$$

Let us denote by $p^\vee$ the canonical projection $\overline{V}^\vee \to T^\vee$ and recall that

$$\overline{V}^\vee = Hom_{\mathbb{C}-antilin.}(V, \mathbb{C}).$$

If $l \in \overline{V}^\vee$ then $p^\vee(l) = L(0, \alpha_l)$, where $\alpha_l : \Lambda \to U(1)$ is the map

$$\alpha_l(\lambda) = e^{2\pi i Iml(\lambda)}\ , \lambda \in \Lambda,$$

(see [8], Chap.II, 9). In order to define $c^\vee(\eta)$ in Čech cohomology we can choose $l_{ij;z} \in \overline{V}^\vee$ such that

$$Iml_{ij;z} = E(a_{ij}(z), \cdot).$$

Then, the characteristic class $c^\vee(\eta)$ is given by the 2-cocycle $(\rho_{ijk;z})$,where

$$\rho_{ijk;z} = l_{jk;z} - l_{ik;z} + l_{ij;z} \in \Lambda^\vee = H^2(B, \Lambda^\vee).$$

But, for all $\lambda \in \Lambda$, we have

$$Im\rho_{ijk;z}(\lambda) = E(a_{jk}(z) - a_{ik}(z) + a_{ij}(z), \lambda) = E(m\lambda^0, \lambda) = 0.$$

Since a linear form $l \in \overline{V}^\vee$ is uniquely determined by its imaginary part, we get $c^\vee(\eta) = 0$ in $H^2(B, \Lambda^\vee)$.◇

We have proved the following result:

Theorem 5 *Let $X \xrightarrow{\pi} B$ be a T-principal bundle over the curve B, defined by a cohomology class $\xi \in H^1(\mathcal{O}_B(T))$ with $c(\xi) \neq 0$ (i.e. X is non-Kähler). Then we have an exact sequence of free abelian groups*

$$0 \to Hom(J_B, T^\vee) \to NS(X)/TorsNS(X) \to \tilde{N}(X) \to 0 \ ,$$

where $\tilde{N}(X)$ is the subgroup of the Neron-Severi group of the torus T defined by

$$\tilde{N}(X) = \{c_1(L) \in NS(T) \mid \tilde{\varphi}_L(\xi) \text{ is the trivial torus bundle } \}. \ \diamond$$

Remark. In the case T is an elliptic curve we have $\tilde{N}(X) = 0$ (see [3]).

Remark. Clearly, a similar result holds in the case of a Kähler torus principal bundle for the group $NS(X)/\pi^* NS(B)$ (see also the last section).

Example. Let T be a two-dimensional complex torus with period matrix Ω, where

$$\Omega^t = \begin{pmatrix} 1 & 0 & \tau_1 & \alpha \\ 0 & 1 & 0 & \tau_2 \end{pmatrix}$$

with $Im\tau_j > 0$, $j = 1, 2$. If the complex numbers τ_1, τ_2, α are algebraically independent over the rational numbers $\mathbb{Q}$ then, it is well-known that T is not algebraic, that is, T is not an abelian variety. Let E_j be an elliptic curve with period matrix $(1, \tau_j)$, $j = 1, 2$. Then, there exists a holomorphic mapping

$$\pi : T \to E_2$$

such that π is an E_1-principal bundle over E_2.
The lattice Λ of T is generated by vectors $(1,0), (0,1), (\tau_1, 0), (\alpha, \tau_2)$. Put $\lambda^0 = (\tau_1, 0)$. Choose a point b of a curve B and make a logarithmic transformation to obtain a T-principal bundle

$$X = L_b(m\lambda^0, 1)(B \times T),$$

where m is an arbitrary positive integer. Then, we have $c(X) = m\lambda^0$ and X is non-Kähler.
Since the second coordinate of λ^0 is zero, there exists a holomorphic mapping

$$\mu : X \to B \times E_2 \ .$$

Then, any line bundle L on T, which is the pull-back of a line bundle L_2 on E_2 by π, can be extended holomorphically to the one on X, since L_2 can be extended to a line bundle on $B \times E_2$. Hence, for our T-principal bundle X, we have $\tilde{N}(X) \neq 0$.
Similarly, we can also construct a T-principal bundle over B with $\tilde{N}(X) \neq 0$ from a period matrix Ω

$$\Omega^t = \begin{pmatrix} I_m & 0 & \tau_m & \alpha \\ 0 & I_n & 0 & \tau_n \end{pmatrix},$$

where $(I_m, \tau_m)^t$ and $(I_n, \tau_n)^t$ are period matrix of tori and α is an $m \times n$ matrix.

4. A filtration on $Pic(X)$

In this section we reinterpret the results in the previous section geometrically. We use freely the notation in the previous section. Let $\pi : X \to B$ be a T-principal bundle as in the previous section. Choose a general point $b \in B$ and fix it. In the following we identify the torus T with the fiber $\pi^{-1}(b)$. Restricting a line bundle $\mathcal{L}$ on X to the fiber $\pi^{-1}(b)$, we have a natural group homomorphism

$$(20) \qquad Pic(X) \xrightarrow{r} Pic(\pi^{-1}(b)) = Pic(T)$$

Then $ker\ r$ consists of isomorphism classes of line bundles whose restriction to the fibre $\pi^{-1}(b)$ is trivial, hence the restriction to each fiber of π is a line bundle of degree 0 on the torus under identification of the torus with each fiber.

Let $\{U_j\}$ be an open covering of B with trivialization

$$(21) \qquad \pi^{-1}(U_j) \simeq U_j \times T$$

For each line bundle $\mathcal{L}$ belonging to $ker\ r$ there exists a holomorphic mapping

$$\varphi_j : U_j \to Pic^0(T) = T^\vee$$

with

$$\mathcal{L}|_{\pi^{-1}(U_j)} \simeq (\varphi_j \times id_T)^*(\mathcal{P}),$$

where $\mathcal{P}$ is the Poincaré bundle on $Pic^0(T) \times T$. Since any line bundle of degree 0 on the torus is invariant by the translations, on $U_j \cap U_k \neq \emptyset$ we have

$$\varphi_j = \varphi_k.$$

Hence, the line bundle $\mathcal{L}$ defines a holomorphic mapping

$$(22) \qquad \varphi : B \to T^\vee.$$

Since the restriction $\mathcal{L}|_{\pi^{-1}(b)}$ is trivial, the holomorphic mapping (22) satisfies

$$\varphi(b) = [0]. \tag{23}$$

The line bundle $\mathcal{L}$ and the holomorphic mapping φ are related by

$$\mathcal{L} \simeq \pi^*(M) \otimes \varphi^*(\mathcal{P}),$$

where M is a line bundle on the curve B and $\varphi^*(\mathcal{P})$ is the line bundle on X whose restriction to $\pi^{-1}(U_j)$ is $(\varphi_j \times id_T)^*(\mathcal{P})$. Note that by the argument of the proof of Lemma 3 we can patch together $(\varphi_j \times id_T)^*(\mathcal{P})$'s to get $\varphi^*(\mathcal{P})$, since the line bundle of degree 0 on a torus is invariant under the translations. Also note that there is a one to one correspondence between the set of holomorphic mappings (22) with property (23) and $Hom(J_B, T^\vee)$.

Let us consider a group homomorphism

$$R : Pic(X) \xrightarrow{r} Pic(T) \xrightarrow{c} H^2(T, \mathbb{Z}). \tag{24}$$

The homomorphism R is essentially equivalent to a natural homomorphism

$$Pic(X) \to Pic(T)/Pic^0(T)$$

induced by the homomorphism r. A line bundle $\mathcal{L}$ belonging to $ker\ R$ is the one whose restriction to each fiber of π is of degree 0. Note that by the proof of Lemma 3 each line bundle $L \in Pic^0(T)$ can be extended to a line bundle $\mathcal{L}$ on X in such a way that its restriction to each fiber is isomorphic to L. Hence, there is an isomorphism

$$ker\ R/ker\ r \simeq Pic^0(T). \tag{25}$$

Define subgroups P_j of $Pic(X)$ by

$$P_2 = \pi^* Pic(B), \quad P_1 = ker\ r, \quad P_0 = Pic(X). \tag{26}$$

Then, $\{P_\bullet\}$ defines an decreasing filtration of $Pic(X)$. By the above consideration and the arguments of the previous section we have the following theorem.

Theorem 6 *We have the following isomorphisms.*

$$P_1/P_2 \simeq Hom(J_B, Pic^0(T)) \tag{27}$$

$$P_0/P_1 \simeq \{\ L \in Pic(T) \mid \tilde{\varphi}_L(\xi) = 0\ \} \tag{28}$$

where $\xi \in H^1(B, \mathcal{O}_B(T))$ is the cohomology class corresponding to the T-principal bundle $\pi : X \to B$ and $\tilde{\varphi}_L(\xi) = 0$ is defined in §3.$\diamond$

Remark. Taking the Chern classes, we have

$$c_1(P_2) = F_2, \quad c_1(P_1) = F_1. \tag{29}$$

5. Neron-Severi group for torus quasi bundles

Let $T = V/\Lambda$ be an n-dimensional torus. By a quasi T-bundle $\pi : X \to B$ over a curve B we mean that π is a T-principal bundle over $B \setminus \{b_1, b_2, \dots, b_\ell\}$ and that the fiber $\pi^{-1}(b_j)$ over the point b_j is of the form $m_j T_j$ where $m_j \geq 2$ and T_j is a torus. The fiber $m_j T_j$ is called a multiple fiber of the multiplicity m_j. To construct such a quasi T-bundle we first generalize the notion of logarithmic transformation.

Choose points $b_1, b_2 \dots, b_k$ on B and put $B' = B - \{b_1, b_2, \dots, b_k\}$. For each point b_i fix a positive integer m_i. We let a_i be an element of $\frac{1}{m_i}\Lambda$ such that the order of the point $[a_i]$ of the torus T corresponding to a_i is precisely m_i. Let

$$D_i = \{ \, t_i \in \mathbb{C} \mid |t_i| < \epsilon \, \}$$

be a coordinate neighbourhood of the point b_i and put

$$\widehat{D}_i = \{ \, s_i \in \mathbb{C} \mid |s_i| < \epsilon^{1/m_i} \, \}.$$

By the mapping

$$\begin{aligned} \lambda_i \ : \ \widehat{D}_i &\to D_i \\ s_i &\mapsto s_i^{m_i}, \end{aligned} \tag{30}$$

$\widehat{D}_i$ is an m_i-sheeted ramified covering of D_i. A holomorphic mapping g_i : $\widehat{D}_i \times T \to \widehat{D}_i \times T$ defined by

$$g_i : (s_i, [\zeta]) \mapsto (e_{m_i} s_i, \ [\zeta + a_i]) \tag{31}$$

is an analytic automorphism of order m_i and generates the cyclic group $G_i = \langle g_i \rangle$ of order m_i where

$$e_{m_i} = \exp(2\pi\sqrt{-1}/m_i).$$

Since the automorphism g_i has no fixed points, the quotient $\widehat{D}_i \times T/G_i$ is a complex manifold. Let

$$\mu_i : \widehat{D}_i \times T \to \widehat{D}_i \times T/G_i \tag{32}$$

be the canonical quotient mapping. By $[s_i, [\zeta]]$ we denote the point of the quotient manifold $\widehat{D}_i \times T/G_i$ corresponding to a point $(s_i, [\zeta])$ of $\widehat{D}_i \times T$. We have a holomorphic mapping

$$\begin{aligned} \pi_i \ : \ \widehat{D}_i \times T/G_i &\to D_i \\ [s_i, [\zeta]] &\mapsto \ s_i^{m_i}. \end{aligned}$$

Over the punctured disk D_i^* the holomorphic mapping π_i gives a T-principal bundle, and over the origin 0 the equation

$$\pi_i = 0$$

defines a divisor of a form m_iT_i where $T_i = T/\langle[a_i]\rangle$ is a torus obtained as the quotient by a finite subgroup generated by the point $[a_i]$.

The mapping

$$\begin{aligned} \ell_{a_i} \quad &: \quad \widehat{D}_i^* \times T/G \to D_i^* \times T \\ (33) \qquad & \qquad [s_i, [\zeta]] \mapsto (s_i^m, [\zeta - \frac{m_i a_i}{2\pi\sqrt{-1}} \log s_i]) \end{aligned}$$

is a well-defined holomorphic mapping and isomorphic. Therefore, we can patch together $\widehat{D}_i \times T/G_i$, $i = 1, 2, \ldots, k$ and $B' \times T$ by the isomorphisms ℓ_{a_i} to obtain a compact complex manifold X which is denoted by

$$(34) \qquad L_{b_1}(a_1, m_1)L_{b_2}(a_2, m_2) \cdots L_{b_k}(a_k, m_k)(B \times T)$$

and is called the manifold obtained from $B \times T$ by means of logarithmic transformations. There is a natural holomorphic mapping $\pi : X \to B$ given by π_i on $\widehat{D}_i \times T/G_i$ and the projection to the first factor on $B' \times T$. The fiber space $\pi : X \to B$ is a T-principal bundle over B' and has multiple fibres with multiplicity m_i, if $m_i \geq 2$. In the Appendix we shall show that all quasi T-bundle are obtained in this manner.

In the following let us consider a quasi T-bundle $\pi : X \to B$ of the form (34) and we assume that

$$m_i \geq 2, \quad i = 1, 2, \ldots, \ell, \quad m_{\ell+1} = \cdots = m_k = 1.$$

Let us consider geometrically line bundles on X. Choose a general point b and consider a natural restriction homomorphism

$$(35) \qquad r : Pic(X) \to Pic(\pi^{-1}(b)) = Pic(T)$$

Let us first consider the structure of $ker\ r$. Note that for the multiple fiber m_iT_i the line bundle $[T_i]$ associated with the divisor T_i of X is an element of $ker\ r$ and $[T_i]^{\otimes m_i} = [m_iT_i]$ is the pull-back of the line bundle $[b_i]$ on the curve B.

Let P_2 be a subgroup of $Pic(X)$ generated by $\pi^*Pic(B)$ and $[T_i]$, $i = 1, 2, \ldots, \ell$. A line bundle $\mathcal{L}$ belonging to P_2 is characterized by the fact that the restriction of $\mathcal{L}$ to each fiber $\pi^{-1}(c)$, $c \in B'$ is the trivial line bundle.

To a line bundle $\mathcal{L} \in ker\ r$, by the same argument as in §4, we can associate a holomorphic mapping

$$\varphi' : B' \to Pic^0(T) = T^\vee.$$

The pull-back $\mu_i^*(\mathcal{L}|_{\pi^{-1}(D_i)})$ defines also a holomorphic mapping

$$\widehat{\varphi}_i : \widehat{D}_i \to Pic^0(T),$$

where $\mu_i : \widehat{D} \times T \to \pi^{-1}(D_i) = \widehat{D}_i \times T/G_i$ is a natural quotient mapping(32). Then, on $\widehat{D}_i^*$ we have

$$\widehat{\varphi}_i = \varphi' \circ \lambda_i,$$

where $\lambda_i : \widehat{D}_i \to D_i$ is defined in (30). This implies that the holomorphic mapping φ' can be extended to a holomorphic mapping

$$\varphi : B \to Pic^0(T) = T^{\vee}. \tag{36}$$

As $\mathcal{L}|_{\pi^{-1}(b)}$ is a trivial bundle, we have

$$\varphi(b) = [0]. \tag{37}$$

Note that the set of holomorphic mappings (36) with property (37) are canonically isomorphic to $Hom(J_B, Pic^0(X))$. If $\mathcal{L}$ and $\mathcal{M}$ in $ker\ r$ give the same holomorphic mapping (36), then the restriction of the line bundle $\mathcal{L} \otimes \mathcal{M}^{-1}$ to each fiber $\pi^{-1}(c)$, $c \in B'$ is the trivial bundle, hence is an element of P_2.

Lemma 7 *There exists a natural group isomorphism*

$$j : ker\ r/P_2 \simeq Hom(J_B, Pic^0(T)). \tag{38}$$

Proof: To each line bundle $\mathcal{L} \in ker\ r$ we can associate a holomorphic mapping (36) with property (37). This defines an element of $Hom(J_B, Pic^0(T))$. If the mapping φ gives the zero element of $Hom(J_B, Pic^0(T))$, φ is the zero map. Hence, the restriction of $\mathcal{L}$ to each fiber $\pi^{-1}(c)$, $c \in B'$ is the trivial bundle. Hence, $\mathcal{L}$ belongs to P_2. This shows the injectivity.

Conversely, let $\varphi : B \to T^{\vee}$ be a non-constant holomorphic mapping with $\varphi(b) = [0]$. Then, on $X' = \pi^{-1}(B')$ we can construct a line bundle $\mathcal{L}'$ such that $\mathcal{L}'|_{\pi^{-1}(c)}$ is a line bundle of degree zero corresponding to the point $\varphi(c)$ for each $c \in B'$. For $\widehat{D}_i$, $i = 1, 2 \ldots, k$, put

$$\widehat{\varphi}_i = \varphi \circ \lambda_i.$$

Then, $\widehat{\varphi}_i$ defines a line bundle $\widehat{\mathcal{L}}_i$ such that $\widehat{\mathcal{L}}_i|_{s_i \times T}$ corresponds to $\widehat{\varphi}_i(s_i)$, As the line bundle $\widehat{\mathcal{L}}_i$ is invariant under the group G_i, it defines a line bundle $\mathcal{L}_i$ on $\widehat{D}_i \times T/G_i$. By our construction, $\mathcal{L}_i|_{\pi^{-1}(D_i^*)}$ and $\mathcal{L}'|_{\pi^{-1}(D_i^*)}$ are isomorphic. Hence, $\mathcal{L}_i$'s and $\mathcal{L}'$ define a line bundle $\mathcal{L}$ on X which corresponds to the mapping φ. This shows the surjectivity of the mapping j. $\diamond$

Next let us consider the image of the homomorphism r.

Lemma 8 *If a line bundle L of T can be extended to a line bundle $\mathcal{L}$ on X, then L is invariant by the translations $T_{[a_i]}$, $i = 1, 2, \ldots, \ell$.*

Proof. The pull-back $\tilde{\mathcal{L}}_i := \mu_i^*(\mathcal{L}|_{\pi_i^{-1}(D_i)})$ is invariant by the action of the group G_i, where $\mu_i : \widehat{D}_i \times T \to \widehat{D}_i \times T/G_i = \pi^{-1}(D_i)$ is the natural quotient mapping(32). In particular, the restriction $\tilde{\mathcal{L}}_i|_{0\times T}$ is invariant by the group generated by the translation $T_{[a_i]}$. Since $\tilde{\mathcal{L}}_i|_{0\times T}$ has a form $L \otimes M$ with degree zero line bundle M on T and M is invariant by all the translations, the line bundle L is invariant by the translation $T_{[a_i]}$.$\diamond$

Let H be a subgroup of the torus T generated by $[a_1]$, $[a_2], \ldots, [a_\ell]$. The group H is isomorphic to Λ_0/Λ where Λ_0 is the lattice generated by Λ and a_i's. To any H-invariant line bundle L on the torus T, we associate a cohomology class $\{\eta_{ij}^L\}$ in $H^1(B, \mathcal{O}_B(T^\vee))$ as follows.

Let $\{U_j\}$ be an open covering of the curve B such that $U_i = D_i$ for $i = 1, 2, \ldots, \ell$ and that $b_i \notin U_i \cap U_j$ for $i \neq j$. Since the line bundle L is invariant by the translation $T_{[a_i]}$, though $[\frac{a_i}{2\pi\sqrt{-1}} \log t_i]$ is multivalued

$$T^*_{[\frac{a_i}{2\pi\sqrt{-1}} \log t_i]} L \otimes L^{-1} \tag{39}$$

is a well-defined line bundle on $\pi^{-1}(U_i \cap U_j)$ for $i = 1, 2, \ldots, \ell$ and $j \neq i$. Then there exists a holomorphic mapping φ_{ij} from $U_{ij} = U_i \cap U_j$ to $T^\vee$ such that the line bundle (39) is the pull-back $(\varphi_{ij} \times id_T)^*(\mathcal{P})$ of the Poincaré bundle. Put

$$\eta_{ij}^L := \begin{cases} \varphi_{ij} & \text{if } 1 \leq i \leq \ell,\ \ell < j \\ 0 & \text{if } \ell < i, j \end{cases} . \tag{40}$$

Then, it is easy to show that $\{\eta_{ij}^L\}$ is a one cocycle and defines a cohomology class $[\{\eta_{ij}^L\}] \in H^1(B, \mathcal{O}_B(T^\vee))$.

Lemma 9 *An H-invariant line bundle L on the torus $T = \pi^{-1}(b)$ can be extended to the one on X if and only if the cohomology class $[\{\eta_{ij}^L\}]$ is zero.*

Proof: Assume that there exists a line bundle $\mathcal{L}$ on X which is an extension of L. Then, the pull-back $\mu_i^*(\mathcal{L}|_{\pi^{-1}(U_i)})$ of the restriction of $\mathcal{L}$ on $\pi^{-1}(U_i)$, $i = 1, 2, \ldots, \ell$, to $\widehat{D}_i \times T$ can be expressed as

$$L \otimes (\hat{\varphi}_i \times id_T)^*(\mathcal{P}), \tag{41}$$

where $\hat{\varphi}_i : \widehat{D}_i \to T^\vee$ is a holomorphic mapping. Since the line bundle $\mu_i^*(\mathcal{L}|_{\pi^{-1}(U_i)})$ is invariant under the group G_i, we have

$$\hat{\varphi}_i(s_i) = \hat{\varphi}_i(e_{m_i} s_i).$$

Hence, there exists a holomorphic mapping $\varphi_i : U_i \to T^\vee$ with

$$\widehat{\varphi}_i(s_i) = \varphi_i(s_i^{m_i}). \tag{42}$$

Since $\mathcal{L}$ is a global line bundle, on $U_{ij} \neq \emptyset$ we have

$$T^*_{[\frac{a_i}{2\pi\sqrt{-1}}\log t_i]} L \otimes (\varphi_i \times id_T)^*(\mathcal{P}) = L \otimes (\varphi_j \times id_T)^*(\mathcal{P}). \tag{43}$$

This implies that we have

$$\eta^L_{ij} = \varphi_j - \varphi_i. \tag{44}$$

Hence, the cohomology class is zero.

Conversely assume that the cohomology class is zero, hence we have holomorphic mappings $\varphi_j : U_j \to T^\vee$ which satisfy (44). For $i = 1, 2, \dots, \ell$ define $\widehat{\varphi}_i$ by (42). Then the line bundle $\widehat{\mathcal{L}}_i = L \otimes (\widehat{\varphi}_i \times id_T)^*(\mathcal{P})$ is invariant by the action of the group G_i, hence defines a line bundle $\mathcal{L}_i$ on $\pi^{-1}(U_i)$. For $j > \ell$ put $\mathcal{L}_j = L \otimes (\varphi_j \times id_T)^*(\mathcal{P})$. Since we have the equality (43), we can patch together these line bundles and obtain a line bundle $\mathcal{L}$ which is an extension of L. $\diamond$

Now as in §4 we introduce a decreasing filtration $\{P_\bullet\}$ of $Pic(X)$ by

$$P_2 = \text{the subgroup generated by } \pi^* Pic(B) \text{ and } [T_i]\text{'s}, \tag{45}$$

$$P_1 = ker\ r, \quad P_0 = Pic(X), \tag{46}$$

where $m_i T_i$, $i = 1, 2, \dots, \ell$ are all the multiple fibers of the quasi T-bundle $\pi : X \to B$. By the above arguments we have the following theorem.

Theorem 10 *We have the following isomorphisms.*

$$P_1/P_2 \simeq Hom(J_B, Pic^0(T)) \tag{47}$$

$$P_0/P_1 \simeq \{ L \in Pic(T)^H \mid [\{\eta^L_{ij}\}] = 0 \}.\diamond \tag{48}$$

Let us reinterpret the group $\{ L \in Pic(T)^H \mid [\{\eta^L_{ij}\}] = 0 \}$ by means of a torus principal bundle associated with the quasi T-bundle $\pi : X \to B$.

Let Λ_0 be a lattice in the vector space V generated by Λ and a_i, $i = 1, 2, \dots, \ell$ and put

$$T_0 = V/\Lambda_0. \tag{49}$$

Then, we have

$$T_0 = T/H,$$

where H is a subgroup of the torus T generated by $[a_1]$, $[a_2]$, $\dots [a_\ell]$. There is a canonical surjective homomorphism

$$h : T \to T_0 \tag{50}$$

of complex tori. The following lemma is well-known and easy to prove.

Lemma 11 *A line bundle L on the torus T is invariant by the translations $T_{[a_i]}$, $i = 1, 2, \ldots, \ell$, if and only if there exists a line bundle L_0 on T_0 with*

$$L = h^* L_0. \diamond$$

Put

(51) $$Y = L_{b_1}(a_1, 1) L_{b_2}(a_2, 1) \cdots L_{b_\ell}(a_\ell, 1)(B \times T_0)$$

with structure morphism $\pi_0 : Y \to B$, which is a T_0-principal bundle.

Lemma 12 *There exists a holomorphic mapping*

$$f : X \to Y$$

such that the following diagram is commutative.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \pi \downarrow & & \downarrow \pi_0 \\ B & = & B \end{array}$$

Moreover, f is unramified outside the multiple fibers.

Proof: There is a natural unramified holomorphic mapping

$$f' : B' \times T \to B' \times T_0.$$

We need to show that f' can be extended to a holomorphic mapping f of X to Y. On $\widehat{D}_i \times T/G_i$ let us define a holomorphic mapping f_i by

$$\begin{array}{rcll} f_i & : & \widehat{D}_i \times T/G_i \to D_i \times T_0 & \\ & & [s_i, [\zeta]] & \mapsto (s_i^{m_i}, h([\zeta])). \end{array}$$

We need to show that these holomorphic mappings are compatible to f'. By our definition of the logarithmic transformation we have the following commutative diagram.

$$\begin{array}{ccccc} \ell_i & : & \widehat{D}_i^* \times T/G_i & \to & D_i^* \times T \\ & & [s_i,\ [\zeta]] & \mapsto & (s_i^{m_i}, [\zeta - \frac{m_i a_i}{2\pi\sqrt{-1}} \log s_i]) \\ & f' \downarrow & \downarrow & \downarrow & \downarrow f_i \\ & & (s_i^{m_i},\ [\zeta]_0) & \mapsto & (s_i^{m_i}, [\zeta - \frac{a_i}{2\pi\sqrt{-1}} \log(s_i^{m_i})]_0) \\ \ell_i^{(0)} & : & D_i^* \times T_0 & \to & D_i^* \times T_0 \end{array}$$

Here, $[\zeta]_0$ means the point of the torus T_0 corresponding to ζ. The commutativity of the above diagram shows that the mappings f' and f_i's are compatible and define a holomorphic mapping $f : X \to Y$ over B. $\diamond$

Lemma 13 *The quasi T-bundle X is Kähler if and only if Y is Kähler. The condition is equivalent to the equality*

$$\sum_{i=1}^{k} a_i = 0 \tag{52}$$

Proof: Assume that the equality (52) holds, hence, Y is Kähler. Let ω be a Kähler form of Y. Note that $f : X \to Y$ is an abelian covering ramified along the support of T_i of the multiple fibers. Hence, the pull-back $f^*\omega$ is positive definite on $X \setminus \cup_{i=1}^{\ell} T_i$ and at each point of T_i it is positive semi-definite. Near the multiple fiber $m_i T_i$, X is isomorphic to $\widehat{D}_i \times T/G_i$. As a (1,1)-form

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}(\sum_{\nu=1}^{n} |\zeta_\nu|^2 + |s_i|^2)$$

is G_i-invariant, it defines a Kähler form on $\widehat{D}_i \times T/G_i$. Let ρ_i be a non-negative C^∞-function in $|s_i|^2$ satisfying

$$\rho_i(t) = \begin{cases} 1 & |t| < \epsilon^{2/m_i}/3 \\ 0 & |t| \geq 2\epsilon^{2/m_i}/3. \end{cases}$$

Then, a form

$$\omega_i = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\{\rho_i(|s_i|^2)(\sum_{\nu=1}^{n} |\zeta_\nu|^2 + |s_i|^2)\}$$

is positive definite on $\pi^{-1}(D_i(\epsilon^{2/m_i}/3)$ and $\omega_i \equiv 0$ on $\pi^{-1}(D_i(2\epsilon^{2/m_i}/3)$, where we put $D_i(r) = \{\ s_i \mid |s_i| \ < r\ \}$. Hence, we may regard ω_i as a global (1,1)-form on X. Since, $f^*\omega$ is positive definite on $X \setminus \cup_{i=1}^{\ell} T_i$, and ω_i is positive definite in a neighbourhood of T_i and zero outside a certain neighbourhood of T_i, the form

$$\alpha f^*\omega + \sum_{i=1}^{\ell} \omega_i$$

is positive definite on X, if we choose α sufficiently large. Hence, X is Kähler.

Conversely, assume that X is Kähler. Put

$$d = m_1 \cdot m_2 \cdots m_\ell, \quad m_0 = LCM\{m_1, m_2, \ldots, m_\ell\}.$$

We can always find a d-fold abelian covering $\sigma : \tilde{B} \to B$ of the curve B branched at $b_1, b_2, \ldots, b_\ell$ and a point $b_0 \in B \setminus \{b_1, b_2, \ldots, b_\ell\}$ such that σ has

d/m_i ramification points $\{b_i^{(m)}\}$, $m = 1, 2, \dots, d/m_i$, $i = 0, 1, 2, \dots, \ell$. Over the points b_j, $\ell < j \le k$, σ is unramified. Put $\sigma^{-1}(b_j) = \{b_j^{(1)}, b_j^{(2)}, \dots, b_j^{(d)}\}$. Then, the normalization $\widetilde{X}$ of $X \times_B \tilde{B}$ has a natural structure of a principal T-bundle over $\tilde{B}$ and it is isomorphic to

$$\prod_{i=1}^{k} \prod_{m=1}^{d/m_i} L_{b_i^{(m)}}(m_i a_i, 1)(\tilde{B} \times T) \tag{53}$$

The natural holomorphic mapping $\tilde{\sigma} : \widetilde{X} \to X$ is only branched over $\pi^{-1}(b_0)$. By the similar argument as above we can show that $\widetilde{X}$ is Kähler if X is Kähler. Then, by (52), $\widetilde{X}$ is Kähler if and only if

$$\sum_{i=1}^{k} \sum_{m=1}^{d/m_i} m_i a_i = 0.$$

The equality can be rewritten as

$$\sum_{i=1}^{k} \frac{d}{m_i} m_i a_i = d \sum_{i=1}^{k} a_i = 0.$$

Hence, the equality (52) holds and Y is also Kähler. This proves the lemma. $\diamond$

Lemma 14 *The subgroup $\pi^* H^2(B, \mathbb{Z})$ of $H^2(X, \mathbb{Z})$ is a finite group if and only if*

$$\sum_{i=1}^{k} a_i \neq 0.$$

Proof: Since the holomorphic mapping $f : X \to Y$ is finite, $\pi^* H^2(B, \mathbb{Z})$ is finite if and only if the subgroup $\pi_0^* H^2(B, \mathbb{Z})$ in $H^2(Y, \mathbb{Z})$ is finite. The latter group is finite if and only if Y is non-Kähler. On the other hand, Y is non-Kähler if and only if

$$\sum_{i=1}^{k} a_i \neq 0.$$

This proves the lemma. $\diamond$

Put

$$N(X) = \{ L \in Pic(T)^H \mid [\{\eta_{ij}^L\}] = 0 \} \tag{54}$$

$$N(Y) = \{ L_0 \in Pic(T_0) \mid \tilde{\varphi}_{L_0}(\xi_0) = 0 \} \tag{55}$$

where $\xi_0 \in H^1(B, \mathcal{O}_B(T_0))$ is the cohomology class corresponding to the T_0-principal bundle $\pi_0 : Y \to B$. Taking the dual of the homomorphism $h : T \to T_0$ (50) we have an exact sequence

$$0 \to T_0^\vee \stackrel{h^\vee}{\to} T^\vee \to H^\vee \to 0, \tag{56}$$

where $H^\vee$ is a finite abelian group. Sheafifying the exact sequence (56) and taking the cohomology, we obtain the following exact sequence.

$$(57) \qquad 0 \to H^1(B, \mathcal{O}_B(T_0^\vee)) \xrightarrow{h^\vee} H^1(B, \mathcal{O}_B(T^\vee)) \to H^1(B, H^\vee) \to .$$

Lemma 15 *For a line bundle L_0 on the torus T_0 put $L = h^* L_0$. Then we have*

$$h^\vee(\tilde{\varphi}_{L_0}(\xi_0)) = [\{\eta_{ij}^L\}].$$

Proof: We use the same open covering $\{U_j\}$ of the curve B defined above. Then, the cohomology class ξ_0 is given by a cocycle

$$(58) \qquad \zeta_{ij} := \begin{cases} \frac{a_i}{2\pi\sqrt{-1}} \log t_i & \text{if } 1 \le i \le \ell,\ \ell < j \\ 0 & \text{if } \ell < i, j. \end{cases}$$

Hence $\tilde{\varphi}_{L_0}(\xi_0))$ is given by a cocycle

$$\zeta_{ij}^L := \begin{cases} \phi_{ij} & \text{if } 1 \le i \le \ell,\ \ell < j \\ 0 & \text{if } \ell < i, j \end{cases}$$

where ϕ_{ij} is given by

$$T^*_{[\frac{a_i}{2\pi\sqrt{-1}} \log t_i]} L_0 \otimes L_0^{-1} = (\phi_{ij} \times id_T)^*(\mathcal{P}_0).$$

Here $\mathcal{P}_0$ is the Poincaré bundle on $Pic^0(T_0) \times T_0$. Then it is easy to show that we have

$$h^\vee(\phi_{ij}) = \varphi_{ij}.$$

This is the desired result. ◇

Lemma 16

$$h^*(N(Y)) = N(X).$$

Proof: For a line bundle $L_0 \in N(Y)$ we let $\mathcal{L}_0$ be a line bundle on Y which is an extension of L_0. Then, $f^*\mathcal{L}_0$ is a line bundle on X which is an extension of the line bundle $h^* L_0$, where $f : X \to Y$ is the holomorphic mapping in Lemma 12. Hence, we have $h^*(N(Y)) \subset N(X)$.

Conversely, take a line bundle $L \in N(X)$ and choose a line bundle L_0 on T_0 with $h^* L_0 = L$. By the above Lemma 15 and the exact sequence (57), $\tilde{\varphi}_{L_0}(\xi_0) = 0$. Hence, $L_0 \in N(Y)$. This shows $N(X) \subset h^*(N(Y))$. ◇

By the above argument and the arguments in the previous sections we have the following exact sequences.

$$(59) \qquad 0 \to Hom(J_B, T^\vee) \to Pic(X)/P_2 \to N(X) \to 0$$

$$(60) \qquad 0 \to Hom(J_B, T_0^\vee) \to Pic(Y)/\pi_0^* Pic(B) \to N(Y) \to 0.$$

Taking the Chern classes of the line bundles, finally we obtain the following theorem.

Theorem 17 *There exists an exact sequence*

$$(61)\qquad 0 \to Hom(J_B, T^\vee) \to NS(X)/\tilde{F}_2 \to \widetilde{N}(X) \to 0,$$

where $\tilde{F}_2$ *is a subgroup of* $H^2(X,\mathbb{Z})$ *generated by* $c_1([T_i])$, $i = 1, 2, \ldots, \ell$, *and*

$$(62)\qquad \widetilde{N}(X) = \{\ c_1(L) \mid L \in Pic(X)^H,\quad [\{\eta^L_{ij}\}] = 0\ \}.$$

The subgroup $\tilde{F}_2$ *is finite if and only if* X *is non-Kähler. Moreover, we have*

$$\widetilde{N}(X) = h^* \widetilde{N}(Y)$$

where

$$\widetilde{N}(Y) = \{\ c_1(L_0) \mid L_0 \in Pic(Y),\quad \tilde{\varphi}_{L_0}(\xi_0) = 0\ \}.$$

Proof: To each homomorphism

$$\varphi \in Hom(J_B, T^\vee)$$

we can associate a line bundle $\mathcal{L}$ on X such that for each point $c \in B'$ the restriction $\mathcal{L}|_{\pi_{-1}}(c)$ corresponds to $\varphi(c)$. Let us consider the first Chern class $c_1(\mathcal{L})$ of $\mathcal{L}$. Note that we have an exact sequence

$$0 \to R^1\pi_*\mathbb{Z} \to R^1\pi_*\mathcal{O}_X \to \mathcal{O}_B(T^\vee) \to 0$$

and from this exact sequence we have the exact sequence

$$(63)\qquad \to H^0(B, R^1\pi_*\mathcal{O}_X) \to H^0(B, \mathcal{O}_B(T^\vee)) \xrightarrow{c} H^1(B, R^1\pi_*\mathbb{Z}) \to .$$

The element $\varphi \in Hom(J_B, \mathcal{O}_B(T^\vee))$ gives an element $\tilde{\varphi} \in H^0(B, \mathcal{O}_B(T^\vee))$ with $\tilde{\varphi}(b) = [0]$. Then the image of $c(\tilde{\varphi}) \in H^1(B, R^1\pi_*\mathbb{Z})$ to $H^2(X,\mathbb{Z})/\pi^*H^2(B,\mathbb{Z})$ is $c_1(\mathcal{L}) \mod \pi^*H^2(B,\mathbb{Z})$. Since we have an isomorphism

$$H^0(B, \mathcal{O}_B(T^\vee))/\mathrm{Im}H^0(B, R^1\pi_*\mathcal{O}_X) \simeq Hom(J_B, T^\vee),$$

by the exact sequence (63) we have an inclusion

$$Hom(J_B, \mathcal{O}_B(T^\vee)) \hookrightarrow H^1(B, R^1\pi_*\mathbb{Z}).$$

To show that the natural mapping

$$H^1(B, R^1\pi_*\mathbb{Z}) \to H^2(X,\mathbb{Z})/\pi^*H^2(B,\mathbb{Z})$$

is injective, we need to consider the spectral sequence

$$E_2^{p,q} = H^p(B, R^q\pi_*\mathbb{Z}) \Longrightarrow H^{p+q}(X,\mathbb{Z}).$$

By the dimension reason, we have

$$\begin{aligned} E_\infty^{0,2} &= E_3^{0,2} = ker\ \{H^0(B, R^2\pi_*\mathbb{Z}) \to H^2(B, R^1\pi_*\mathbb{Z})\} \\ E_\infty^{1,1} &= E_2^{1,1} = H^1(B, R^1\pi_*\mathbb{Z}) \\ E_\infty^{2,0} &= E_3^{2,0} = coker\ \{H^0(B, R^1\pi_*\mathbb{Z}) \to H^2(B,\mathbb{Z})\}. \end{aligned}$$

The spectral sequence defines the filtration $\{F_\bullet\}$ on the cohomology group $H^2(X, \mathbb{Z})$ such that there are canonical isomorphisms

$$E_\infty^{2,0} \simeq F_2, \tag{64}$$

$$E_\infty^{1,1} \simeq F_1/F_2, \tag{65}$$

$$E_\infty^{0,2} \simeq F_0/F_1. \tag{66}$$

It is easy to see that $F_2 = \pi^* H^2(B, \mathbb{Z})$, hence by the above isomorphism (65) the natural mapping

$$H^1(B, R^1\pi_*\mathbb{Z}) \to H^2(X, \mathbb{Z})/\pi^* H^2(B, \mathbb{Z})$$

is injective. Therefore, the natural mapping

$$Hom(J_B, \mathcal{O}_B(T^\vee)) \to H^1(B, R^1\pi_*\mathbb{Z}) \to H^2(X, \mathbb{Z})/\pi^* H^2(B, \mathbb{Z})$$

is also injective. The rest of the statements follow from the above arguments. This proves the theorem. ◇

Remark. By the similar arguments as in [5, Chap. II, Lemma 1.6 and Lemma 7.3], the structure of the first homology group $H_1(X, \mathbb{Z})$ is given by

$$H_1(X, \mathbb{Z}) \simeq \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_g \oplus \mathbb{Z}\beta_1 \oplus \cdots \oplus \mathbb{Z}\beta_g \oplus (\Lambda_0/(\sum_{i=1}^k a_i)),$$

where Λ_0 is the lattice in the vector space V generated by Λ and a_i's and

$$H_1(B, \mathbb{Z}) \simeq \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_g \oplus \mathbb{Z}\beta_1 \oplus \cdots \oplus \mathbb{Z}\beta_g.$$

By virtue of Lemma 13 , $H_1(X, \mathbb{Z})$ has torsion if and only if X is non-Kähler. Moreover, if X is non-Kähler, there is a non-canonical isomorphism

$$Tor\ H^2(X, \mathbb{Z}) \simeq Tor\ \Lambda_0/(\sum_{i=1}^k a_i).$$

Thus, in this case, since $R^1\pi_*\mathbb{Z}$ and $R^2\pi_*\mathbb{Z}$ are constant sheaves of finite free $\mathbb{Z}$-modules, by the isomorphisms (64), (65) and (66), we conclude that

$$Tor\ H^2(X, \mathbb{Z}) = \pi^* H^2(B, \mathbb{Z}).$$

Appendix

In this appendix we shall show that all quasi T-bundle over a curve B are obtained from the product $B \times T$ by means of logarithmic transformations. Let $\pi : X \to B$ be a quasi T-bundle over the curve B. We let $m_1T_1, m_2T_2, \dots, m_\ell T_\ell$ be all the multiple fibers of π. Put

$$b_i = \pi(T_i), \quad i = 1, 2, \dots, \ell.$$

Choose a coordinate neighbourhood D_i of b_i and a local coordinate t_i with center b_i. We may assume

$$D_i = \{ t_i \in \mathbb{C} \mid |t_i| < \epsilon \}.$$

Put

$$\widehat{D}_i = \{ s_i \in \mathbb{C} \mid |s_i| < \epsilon^{1/m_i} \}.$$

Then a homomorphism

$$\begin{array}{ccc} \widehat{D}_i & \to & D_i \\ s_i & \mapsto & s_i^{m_i} \end{array}$$

is an m_i-sheeted cyclic covering. We let $\widehat{X}_i$ be the normalization of the fiber product $X|_{D_i} \times_{D_i} \widehat{D}_i$ with a natural holomorphic mapping

$$\mu_i : \widehat{X}_i \to X_i = \pi^{-1}(D_i).$$

At a point $p \in \pi^{-1}(b_i)$ we can choose local coordinates $(x, y_1, \dots, y_n)$ where the holomorphic mapping π is expressed as

$$t_i = \pi((x, y_1, \dots, y_n)) = x^{m_i}.$$

Then, $\widehat{X}_i$ is locally given by the normalization of

$$s_i^{m_i} - x^{m_i} = 0.$$

Hence, μ_i is a unramified covering. Also the complex manifold $\widehat{X}_i$ has a structure of a fiber space

$$\widehat{\pi}_i : \widehat{X}_i \to \widehat{D}_i$$

over $\widehat{D}_i$ which is smooth over $\widehat{D}_i$. Since $X_i \to D_i$ is a T-principal bundle over the punctured disk D_i^*, it is easy to show that $\widehat{\pi}_i$ is a T-principal bundle, hence $\widehat{\pi}_i$ is isomorphic to the product $D_i \times T$ with the projection to the first factor.

By our construction $\mu_i : \widehat{X}_i \to X_i$ is an m_i-sheeted cyclic unramified covering and the cyclic G_i of order m_i operates on $\widehat{X}_i$. A generator g_i of the group G_i has a form

$$\begin{array}{rrl} g_i : & \widehat{D}_i \times T & \to \widehat{D}_i \times T \\ & (s_i, [\zeta]) & \mapsto (e_{m_i} s_i, [\zeta + a_i]) \end{array} \tag{67}$$

where $[a_i]$ is a point of the torus T of order m_i. Then, the quotient manifold $\widehat{D}_i \times T/G_i$ is isomorphic to $X_i = \pi^{-1}(D_i)$. There is an analytic isomorphism

$$\begin{aligned} \ell_{a_i} : \widehat{D}_i^* \times T/G_i &\to D_i^* \times T \\ [s_i, [\zeta]] &\to (s_i^{m_i}, [\zeta - \frac{m_i a_i}{2\pi\sqrt{-1}} \log s_i]) \end{aligned} \tag{68}$$

We let $\widetilde{X}$ be a complex manifold obtained by patching together $X - \cup_{i=1}^{\ell} \pi^{-1}(b_i)$ and $D_i \times T$'s by the isomorphisms $\ell_{a_i}^{-1}$:

$$\widetilde{X} = (X \setminus \cup_{i=1}^{\ell} \pi^{-1}(b_i)) \bigcup_{i=1}^{\ell} D_i \times T. \tag{69}$$

Then, the complex manifold $\widetilde{X}$ has a natural structure $\tilde{\pi} : \widetilde{X} \to B$ of a T-principal bundle over the curve B.

Conversely, the quasi T-bundle $\pi : X \to B$ is obtained from the T-principal bundle $\tilde{\pi} : \widetilde{X} \to B$ by means of the logarithmic transformations:

$$X = L_{b_1}(a_1, m_1) L_{b_2}(a_2, m_2) \cdots L_{b_\ell}(a_\ell, m_\ell)(\widetilde{X}), \tag{70}$$

by patching together $(\widetilde{X} \setminus \cup_{i=1}^{\ell} \tilde{\pi}^{-1}(b_i))$ and $\widehat{D}_i^* \times T/G_i$'s by the isomorphisms ℓ_{a_i}.

By the remark in §1, the T-principal bundle $\tilde{\pi} : \widetilde{X} \to B$ is obtained from $B \times T$ by means of logarithmic transformations

$$\widetilde{X} = L_{b_{\ell+1}}(a_{\ell+1}, 1) L_{b_{\ell+2}}(a_{\ell+2}, 1) \cdots L_{b_k}(a_k, 1)(B \times T). \tag{71}$$

Hence, by (70) and (71) the quasi T-bundle $\pi : X \to B$ is obtained from $B \times T$ by means of logarithmic transformations

$$X = L_{b_1}(a_1, m_1) \cdots L_{b_\ell}(a_\ell, m_\ell) L_{b_{\ell+1}}(a_{\ell+1}, 1) \cdots L_{b_k}(a_k, 1)(B \times T).$$

Thus, any quasi T-bundle over the curve B is obtained from $B \times T$ by means of logarithmic transformations.

References

[1] Blanchard, A. : Espaces fibrés kählériens compacts, C.R. Acad. Sci., **238**, 2281–2283 (1954)

[2] Borel, A. : A spectral sequence for complex analytic bundles, Appendix II to [6].

[3] Brînzănescu, V. : Neron-Severi group for nonalgebraic elliptic surfaces I: elliptic bundle case, Manuscripta Math. **79**, 187–195 (1993)

[4] Brînzănescu, V. : Neron-Severi group for nonalgebraic elliptic surfaces II: non-kählerian case, to appear in Manuscripta Math. (1994)

[5] Friedman, F. & Morgan, J.W. : Smooth Four- Manifolds and Complex Surfaces, Springer-Verlag, 1994

[6] Hirzebruch, F. : Topological Methods in Algebraic Geometry. 3 rd Edition, Springer-Verlag: Berlin-Heidelberg-New York, 1966

[7] Höfer, T. : Remarks on torus principal bundles, J. Math. Kyoto Univ. (JMKYAZ), **33**-1, 227–259 (1993)

[8] Mumford, D. : Abelian Varieties, Oxford University Press, 1974

[9] Ueno, K. : Degeneration of elliptic surfaces and certain non-Kähler manifolds, Progr. Math. Vol. **39**, 545–566, Birkhäuser: Boston-Basel-Stuttgart, 1983

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