Stratified fibre bundles

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Abstract

A stratified bundle is a fibered space in which strata are classical bundles and in which attachment of strata is controlled by a structure category $\mathfrak{F}$ of fibers. Well known results on fibre bundles are shown to be true for stratified bundles; namely the pull back theorem, the bundle theorem and the principal bundle theorem.

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1 Introduction

A stratified bundle is a filtered fibered space $X = \{X_i, i \geq 0\}$ for which the complements $X_i \smallsetminus X_{i-1}$, termed strata, are fibre bundles. Moreover the attachment of strata is controlled by a structure category $\mathfrak{F}$ of fibers. For example the tangent bundle of a stratified manifold is a stratified vector bundle, see [4]. Moreover for a compact smooth G-manifold M the projection $M \to M/G$ to the orbit space is a stratified bundle by results of Davis [7]. In this paper we prove three basic properties: The *pull back theorem* shows that certain pull backs of stratified bundles are again stratified bundles, see (4.7). The *bundle theorem* (5.1) states that stratified bundles are the same as classical fibre bundles in case the structure category is a groupoid. Moreover the *principal bundle theorem* (6.12) shows that the function space of admissible maps, X^V, is a stratified bundle and the $\mathfrak{F}^{\mathrm{op}}$-diagram $V \mapsto X^V$ plays the role of the associated principal bundle, see (6.17). In the proofs of the results we have to consider the intricate compatibility of quotient topology, product topology, and compact-open topology in function spaces. As application we study in [4] algebraic constructions on stratified vector bundles (like the direct sum, tensor product, exterior product) and we show that stratified vector bundles lead to a stratified K-theory generalizing the Atiyah-Hirzebruch K-theory. This was the main motivation for the proof of the basic properties in this paper.

2 Fibre families

Let $\mathfrak{F}$ be a small category together with a functor $F\colon \mathfrak{F} \to \mathbf{Top}$ to the category **Top** of topological spaces. We assume that the functor F satisfies the assumption:

> $(*)$ For every object V in $\mathfrak{F}$ the space $F(V)$ is locally compact, second-countable and Hausdorff.

Then $\mathfrak{F}$ is termed *structure category* and F is a *fibre functor* on $\mathfrak{F}$. If F is a faithful functor then $\mathfrak{F}$ is a topological enriched category in which morphism sets have the compact open topology. In many examples the functor F is actually the inclusion of a subcategory $\mathfrak{F}$ of **Top** so that in this case we need not to mention the fibre functor F.

A *fibre family* with fibres in $\mathfrak{F}$ (or a $(\mathfrak{F}, F)$-family) is a topological space X, termed *total space*, together with a map $p_X\colon X \to \bar{X}$, termed *projection* to the *base space* $\bar{X}$, and for every $b \in \bar{X}$ a selected homeomorphism $\Phi_b\colon p_X^{-1}b \approx FX_b$ where X_b is an object in $\mathfrak{F}$, called *fibre*, depending on $b \in \bar{X}$. The homeomorphism Φ_b is termed *chart* at b. The family $(p_X\colon X \to \bar{X}, X_b, \Phi_b, b \in \bar{X})$ is denoted simply by X. Each object V in $\mathfrak{F}$ yields the *point family* also denoted by V given by the map $p_V\colon FV \to *$ where $*$ is the singleton space.

Given two $\mathfrak{F}$-families X and Y a $\mathfrak{F}$-*map* from X to Y is a pair of maps $(f, \bar{f})$ such that the following diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle p_Y} \\ \bar{X} & \xrightarrow{\bar{f}} & \bar{Y}, \end{array}$$

commutes, and such that for every $b \in \bar{X}$ the composition given by the dotted arrow of the diagram

$$\begin{array}{ccc} p_X^{-1}(b) & \xrightarrow{f|p^{-1}(b)} & p_Y^{-1}(\bar{f}b) \\ {\scriptstyle \Phi_b^{-1}}\uparrow & & \downarrow{\scriptstyle \Phi_{\bar{f}b}} \\ FX_b & \dashrightarrow & FY_{\bar{f}b} \end{array}$$

is a morphism in the image of the functor F. That is, there exists a morphism $\phi\colon X_b \to Y_{\bar{f}b}$ in $\mathfrak{F}$ such that the dotted arrow is equal to $F(\phi)$. We will often denote FX_b by X_b and it will be clear from the context whether X_b denotes an object in $\mathfrak{F}$ or a space in **Top** given by the functor F. If a $\mathfrak{F}$-map $f = (f, \bar{f})$ is a $\mathfrak{F}$-isomorphism then f and $\bar{f}$ are homeomorphisms but the converse need not be true.

If X is a $\mathfrak{F}$-family and Z is a topological space, then $X \times Z$ is a $\mathfrak{F}$-family with projection $p_{X\times Z} = p \times 1_Z\colon X \times Z \to \bar{X} \times Z$. The fibre over a point $(b, z) \in \bar{X} \times Z$ is equal to $p_X^{-1}(b) \times \{z\}$; using the chart $\Phi_b : p_X^{-1}(b) \to X_b$ the chart $\Phi_{(b,z)}$ is defined by $(x, z) \mapsto \Phi_b(x) \in X_b$ where of course we set $(X \times Z)_{(b,z)} = X_b$. In particular, by taking $Z = I$ the unit interval we obtain the cylinder object $X \times I$ and therefore the notion of *homotopy*: two $\mathfrak{F}$-maps $f_0, f_1\colon X \to Y$ are $\mathfrak{F}$-homotopic (in symbols $f_0 \sim f_1$) if there is a $\mathfrak{F}$-map $F\colon X \times I \to Y$ such that $f_0 = Fi_0$ and $f_1 = Fi_1$. Here i_0 and i_1 are the inclusions $X \to X \times I$ at the levels 0 and 1 respectively.

Let $\mathfrak{F}$-**Top** be the category consisting of $\mathfrak{F}$-families $p_X : X \to \bar{X}$ in **Top** and $\mathfrak{F}$-maps. Homotopy of $\mathfrak{F}$-maps yields a natural equivalence relation $\sim$ on $\mathfrak{F}$-**Top** so that the homotopy category $(\mathfrak{F}\text{-}\mathbf{Top})/_\sim$ is defined.

(2.1) Remark. Consider a structure category $\mathfrak{F}$ with fibre functor F. In general it is not assumed that F is faithful, but it is easy to see that the category $\mathfrak{F}$-**Top** is equivalent to $\mathfrak{F}'$-**Top**, where $\mathfrak{F}' = \mathfrak{F}/_\equiv$ is the category with the same objects as $\mathfrak{F}$ and morphisms given by equivalence classes of morphisms, with $f \equiv f' \iff F(f) = F(f')$. We call $\mathfrak{F}'$ the *faithful image* category of F.

(2.2) Definition. If V is a fibre in $\mathfrak{F}$ and $\bar{X}$ is a space in **Top**, then the projection onto the first factor $p_1\colon X = \bar{X} \times V \to \bar{X}$ yields the *product family* with fibre V; the charts $\Phi_b\colon \{b\} \times V \to V = X_b$ are given by projection and $X_b = V$ for all $b \in \bar{X}$. If X is a $\mathfrak{F}$-family $\mathfrak{F}$-isomorphic to a product family then X is said to be a *trivial* $\mathfrak{F}$*-bundle*. In general a $\mathfrak{F}$*-bundle* is a locally trivial family of fibres, i.e. a family X over $\bar{X}$ such that every $b \in \bar{X}$ admits a neighborhood U for which $X|U$ is trivial. Here $X|U$ is the *restriction* of the family X defined by $U \subset \bar{X}$.

Given a family Y with projection $p_Y\colon Y \to \bar{Y}$ and a map $\bar{f}\colon \bar{X} \to \bar{Y}$, the pull-back $X = \bar{f}^*Y$ is the total space of a family of fibres given by the vertical dotted arrow of the following pull-back diagram.

(2.3)
$$\begin{tikzcd} X = \bar{f}^*Y \arrow[r, dotted] \arrow[d, dotted] & Y \arrow[d, "p"] \\ \bar{X} \arrow[r, "\bar{f}"] & \bar{Y}. \end{tikzcd}$$

The charts are defined as follows: For every $b \in \bar{X}$ let $X_b = Y_{\bar{f}b}$, and $\Phi_b : p_X^{-1}(b) \to X_b$ the composition $p_X^{-1}(b) \to p_Y^{-1}(\bar{f}b) \approx Y_{\bar{f}b} = X_b$ where the map $p_X^{-1}(b) \to p_Y^{-1}(\bar{f}b)$ is a homeomorphism since X is a pull-back.

A $\mathfrak{F}$-map $i\colon A \to Y$ is termed a *closed inclusion* if $\bar{\imath}\colon \bar{A} \to \bar{Y}$ is an inclusion, $\bar{\imath}\bar{A}$ is closed in $\bar{Y}$ and the following diagram is a pull-back:

(2.4)
$$\begin{tikzcd}
i^*Y = A \arrow[r, "i"] \arrow[d, "p_A"'] & Y \arrow[d, "p_Y"] \\
\bar{A} \arrow[r, "\bar{\imath}"] & \bar{Y}
\end{tikzcd}$$

Hence a closed inclusion $i\colon A \to Y$ induces homeomorphisms on fibres.

The push-out construction can be extended to the category $\mathfrak{F}$-**Top**, provided the push-out is defined via a closed inclusion.

(2.5) Lemma. *Given $\mathfrak{F}$-families A, X, Y and $\mathfrak{F}$-maps $f\colon A \to X$, $i\colon A \to Y$ with i a closed inclusion the push-out diagram in $\mathfrak{F}$-***Top**

(2.6)
$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d] \\
Y \arrow[r] & Z
\end{tikzcd}$$

exists and $X \to Z$ is a closed inclusion.

Proof. The push-out is obtained by the following commutative diagram,

$$\begin{tikzcd}
A \arrow[rrr, "f"] \arrow[dddd, tail, "i"'] \arrow[dr, "p_A"] & & & X \arrow[dl, "p_X"'] \arrow[dddd] \\
& \bar{A} \arrow[r, "\bar{f}"] \arrow[dd, tail, "\bar{\imath}"] & \bar{X} \arrow[dd] & \\
& & & \\
& \bar{Y} \arrow[r] & \bar{Z} & \\
Y \arrow[rrr] \arrow[ur, "p_Y"] & & & Z \arrow[ul, dotted, "p_Z"']
\end{tikzcd}$$

in which the spaces Z and $\bar{Z}$ are push-outs in **Top** of f, i and $\bar{f}$, $\bar{\imath}$ respectively; the map p_Z exists and is unique because of the push-out property. Moreover, the charts of fibres of Z are given as follows: $\bar{Z}$ is the union of the two subsets $\bar{X}$ and $\bar{Y} \smallsetminus \bar{A}$. For $b = \bar{x} \in \bar{X} \subset \bar{Z}$ there is a canonical composition of homeomorphisms $p_Z^{-1}(b) \approx p_X^{-1}(\bar{x}) \approx X_{\bar{x}}$ that yields the chart

$$\Phi_b : p_Z^{-1}(b) \to X_{\bar{x}} = Z_b.$$

In fact, $i : A \to Y$ being a closed inclusion, for every $\bar{a} \in \bar{A}$ the fibre over $\bar{a}$ in A is canonically isomorphic to the fibre over $\bar{a}$ in Y and hence for every $\bar{x} \in X$ the fibre over $\bar{x}$ in X is canonically isomorphic to the fibre over $\bar{x}$ in Z. Next, for $b = \bar{y} \in \bar{Y} \setminus \bar{A} \subset Z$ the homeomorphism is

$$\Phi_b : p_Z^{-1}(b) \approx p_Y^{-1}(\bar{y}) \approx Y_{\bar{y}} = Z_b.$$

The $\mathfrak{F}$-map $j : Y \to Z$ induced by the push-out is a closed inclusion, since $\bar{j} : \bar{Y} \to \bar{Z}$ is the inclusion of a closed subspace of $\bar{Z}$; furthermore, j is an isomorphism on fibres and an inclusion, and therefore $Y = j^*Z$. *q.e.d.*

3 $\mathfrak{F}$-complexes

For an object V in $\mathfrak{F}$ the family $F(V) \to *$ with base space a singleton is termed a *$\mathfrak{F}$-point* and is also denoted by V. A disjoint union of $\mathfrak{F}$-points is called a *$\mathfrak{F}$-set*. This is a $\mathfrak{F}$-family for which the base space has the discrete topology. Let D^n be the unit disc in $\mathbb{R}^n$ and S^{n-1} its boundary with base point $* \in S^{n-1}$. The complement $e^n = D^n \setminus S^{n-1}$ is the *open cell* in D^n. An $\mathfrak{F}$-cell is a product family $V \times e^n \to e^n$ with $V \in \mathfrak{F}$.

We say that a $\mathfrak{F}$-family X is obtained from a $\mathfrak{F}$-family D by *attaching* n-cells if a $\mathfrak{F}$-set Z together with a $\mathfrak{F}$-map f is given, such that the following diagram

$$\begin{tikzcd}
Z \times S^{n-1} \arrow[r, "f"] \arrow[d] & D \arrow[d] \\
Z \times D^n \arrow[r, "\Phi"] & X = D \cup_f (Z \times D^n)
\end{tikzcd} \tag{3.1}$$

is a push-out in $\mathfrak{F}$-**Top**. The inclusion $Z \times S^{n-1} \to Z \times D^n$ is a closed inclusion, therefore the push-out exists and the induced map $D \to X$ is a closed inclusion and $X \setminus D = Z \times e^n$ is a union of open $\mathfrak{F}$-cells. If Z is a $\mathfrak{F}$-point then we say that X is obtained from D by attaching a $\mathfrak{F}$-cell and Φ is the *characteristic map* of the $\mathfrak{F}$-cell.

(3.2) Definition. A *relative $\mathfrak{F}$-complex* (X, D) is a family X and a filtration

$$D = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n \subset X_{n+1} \subset \cdots \subset X$$

of $\mathfrak{F}$-families X_n, $n \geq -1$, such that for every $n \geq 0$ the $\mathfrak{F}$-family X_n is obtained from X_{n-1} by attaching n-cells and

$$X = \lim_{n \geq 0} X_n.$$

The spaces X_n are termed *n-skeleta* of (X, D). If D is empty we call X a $\mathfrak{F}$-*complex*. Then X is a union of $\mathfrak{F}$-cells. It is not difficult to show that a relative $\mathfrak{F}$-complex (X, D) is Hausdorff and normal provided that D is Hausdorff and normal.

(3.3) Example. Let G be a compact Lie group and let $\mathfrak{F}$ be the *category of orbits* of G, that is, $\mathfrak{F}$ is the subcategory of **Top** consisting of spaces G/H, where H is a closed subgroup of G, and G-equivariant maps $G/H \to G/H'$. Then each G-CW-complex (see [16]) is a $\mathfrak{F}$-complex.

4 Stratified bundles

Let A be a closed subset of a space X. We say that (X, A) is a *CW-pair* if there exists a homeomorphism $(X, A) \approx (X', A')$ of pairs where X' is a CW-complex and A' a subcomplex of X'. A *CW-space* is a space homeomorphic to a CW-complex.

(4.1) Definition. We call a space X a *stratified* space if a filtration

$$X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots \subset \lim_{n\to\infty} X_n = X$$

is given and if for every $i \geq 1$ there is given a CW-pair (M_i, A_i) and a map $h_i \colon A_i \to X_{i-1}$ with the following properties: The subspace X_i is obtained by attaching M_i to X_{i-1} via the attaching map h_i, i.e. there is a push-out diagram:

$$\begin{tikzcd} A_i \arrow[r, tail] \arrow[d, "h_i"'] & M_i \arrow[d] \\ X_{i-1} \arrow[r, tail] & X_i. \end{tikzcd}$$

Moreover X_0 is a CW-space. The complements $X_i \setminus X_{i-1}$ are termed *strata*, while the filtration $\emptyset \subset X_0 \subset X_1 \subset \cdots \subset X_n \subset \ldots$ is called *stratification* of X. The strata $X_i \setminus X_{i-1}$ coincide with the complements $M_i \setminus A_i$. We call the pairs (M_i, A_i) and the space X_0 the *attached spaces* of X. If all attached spaces (M_i, A_i) are CW-complexes with subcomplexes A_i and if all attaching maps are cellular then X is a CW-complex and X_i is a subcomplex of X. In this case we say that the stratified space X is a *stratified CW-complex*. In particular CW-complexes with the *skeletal* filtration are stratified CW-complexes. We say that a stratified space is *finite* if the number of non-empty strata is finite. We always assume that a stratified space which is *not finite is a stratified CW-complex*. The cellular approximation theorem shows that a stratified space is homotopy equivalent to a stratified CW-complex. We point out that a stratified space X is a Hausdorff and regular space, see Gray [9].

A map

$$f\colon X \to X' \tag{4.2}$$

between stratified spaces is a filtration preserving map $f = \{f_n\}_{n\geq 0}$ together with commutative diagrams

$$\begin{tikzcd} X_{i-1} \arrow[d, "f_{i-1}"] & A_i \arrow[l] \arrow[r, tail] \arrow[d] & M_i \arrow[d, "g_i"] \\ X'_{i-1} & A'_i \arrow[l] \arrow[r, tail] & M'_i \end{tikzcd}$$

such that $g_i \cup f_{i-1} = f_i$ for $i \geq 1$. A map is termed *stratum-preserving* if for every i

$$f_i(X_i \smallsetminus X_{i-1}) \subset X'_i \smallsetminus X'_{i-1}.$$

Let **Stra** be the category of stratified spaces and stratum-preserving maps.

Consider a manifold with boundary $(M, \partial M)$, a manifold N, and a submersion $h\colon \partial M \to N$. The push-out of h and the inclusion $\partial M \subset M$

$$\begin{tikzcd} \partial M \arrow[r, tail] \arrow[d, "h"'] & M \arrow[d] \\ N \arrow[r] & M \cup_h N. \end{tikzcd} \tag{4.3}$$

yields a stratified space $X = M \cup_h N$ with stratification $X_0 = N \subset X_1 = X$. For example if a diffeomorphism $\partial M \cong Z \times P$ is given where Z and P are manifolds and if $h : \partial M \cong Z \times P \to Z$ is defined by the projection then $X = M \cup_h N$ is a *manifold with singularities*, see Rudyak [13], Baas [2], Botvinnik [5], Sullivan [15], Vershinin [17]. Also the stratified manifolds (stratifolds) of Kreck [11] are stratified spaces. Manifolds with singularities and stratifolds have "tangent bundles" which are stratified vector bundles. These are important examples of stratified fibre bundles introduced in the next definition.

Recall that $\mathfrak{F}$ denotes a structure category together with a fibre functor $F\colon \mathfrak{F} \to \mathbf{Top}$.

(4.4) Definition. A $\mathfrak{F}$-*stratified fibre bundle* is a stratified space $\bar{X}$ together with a $\mathfrak{F}$-family

$$X \to \bar{X}$$

with the following properties. For $i \geq 1$ the restriction $X_i = X|\bar{X}_i = X_{i-1} \cup_{A_i} M_i$ is the push-out of $\mathfrak{F}$-maps

$$\begin{tikzcd} M_i \arrow[d] & A_i \arrow[l, tail] \arrow[r, "h_i"] \arrow[d] & X_{i-1} \arrow[d] \\ \bar{M}_i & \bar{A}_i \arrow[l, tail] \arrow[r, "\bar{h}_i"] & \bar{X}_{i-1} \end{tikzcd}$$

where $M_i \to \bar{M}_i$ is a $\mathfrak{F}$-bundle and $A_i = M_i|\bar{A}_i$. Moreover, $X_0 \to \bar{X}_0$ is a $\mathfrak{F}$-bundle and $X = \lim_{i\to\infty} X_i$. Hence the strata $X_i \smallsetminus X_{i-1} \to \bar{X}_i \smallsetminus \bar{X}_{i-1}$ are $\mathfrak{F}$-bundles.

(4.5) Example. By comparing definition (4.4) and definition (3.2) it is easy to see that a $\mathfrak{F}$-complex with the skeletal filtration is a $\mathfrak{F}$-stratified bundle since the spaces $(V \times D^n, V \times S^{n-1})$ are trivial $\mathfrak{F}$-bundles over the CW-pair (D^n, S^{n-1}). Moreover, if X is a $\mathfrak{F}$-stratified bundle such that all the attaching maps are cellular, then X is a $\mathfrak{F}$-complex. Here we use the bundle theorem in section 5 below. In general, a $\mathfrak{F}$-stratified bundle has the $\mathfrak{F}$-homotopy type of a $\mathfrak{F}$-complex (this is a consequence of a cellular approximation theorem for $\mathfrak{F}$-complexes, see [3]).

(4.6) Example. Let G be a compact Lie group and M a compact smooth G-manifold. Let $\mathfrak{F}$ be the orbit category of G. Then the *augmented $\mathcal{G}$-normal system* associated to M, as defined by Davis in [7], yields naturally a $\mathfrak{F}$-stratified bundle $M \to \bar{M} = M/G$ via the corresponding assembling functor. The strata are defined exactly as in the proof of theorem 4.9 of [7], by $M_n = M(n-1)$ and $A_n = \partial M_n$. Thus each open stratum contains the disjoint union of all the open strata given by the stratification of M (or the corresponding stratification on $\bar{M}$) by normal orbit type of depth n in the poset of the normal orbit types.

A *$\mathfrak{F}$-stratified map* $f\colon X \to X'$ between $\mathfrak{F}$-stratified bundles is is given by sequences $\{f_n\}_{n\geq 0}$ and $\{g_n\}_{n\geq 1}$ of $\mathfrak{F}$-maps such that, given the commutative diagrams

$$\begin{tikzcd} X_{i-1} \arrow[d, "f_{i-1}"] & A_i \arrow[l] \arrow[r, tail] \arrow[d] & M_i \arrow[d, "g_i"] \\ X'_{i-1} & A'_i \arrow[l] \arrow[r, tail] & M'_i \end{tikzcd}$$

for every i, we have $g_i \cup f_{i-1} = f_i$ for $i \geq 1$. If for every i the inclusion $f(X_i \smallsetminus X_{i-1}) \subset X'_i \smallsetminus X'_{i-1}$ holds, f is termed *stratum-preserving.*

(4.7) Pull back theorem. *Let $\bar{X}$ and $\bar{X}'$ be finite stratified spaces with attached spaces which are locally finite and countable CW-complexes. Let $\bar{f}\colon \bar{X} \to \bar{X}'$ be a stratum-preserving map (in* **Stra***) and $X' \to \bar{X}'$ a $\mathfrak{F}$-stratified bundle. Then the pull-back $\bar{f}^*X' \to \bar{X}$ is a $\mathfrak{F}$-stratified bundle.*

Proof. Let n be the number of strata of $\bar{X}$ and $X_i = \bar{f}^*X_i'$ for $i = 1, \dots n$. Clearly $\bar{f}_0^*X_0'$ is a $\mathfrak{F}$-bundle on $\bar{X}_0$. Since X' is a $\mathfrak{F}$-stratified bundle there are $\mathfrak{F}$-bundles M_i' and $\mathfrak{F}$-maps $h_i'\colon A_i' \subset M_i' \to X_{i-1}'$ such that $X_i' = X_{i-1}' \cup_{h_i'} M_i'$. Let the attached spaces of $\bar{X}$ be denoted by $(\bar{M}_i, \bar{A}_i)$ and $\bar{h}_i$ the attaching maps. The stratified map $\bar{f}$ yields maps $\bar{g}_i\colon \bar{M}_i \to \bar{M}_i'$ such that $\bar{g}_i \cup \bar{f}_{i-1} = \bar{f}_i$. For every $i \geq 1$ let M_i be the pull-back $\bar{g}_i^* M_i' \to \bar{M}_i$. It is a $\mathfrak{F}$-bundle. Let A_i be its restriction to $\bar{A}_i$. By assumption $\bar{M}_i$ and $\bar{M}_i'$ are locally finite and countable; furthermore since $\bar{f}$ is stratum-preserving, for every i we have $\bar{f}(\bar{X}_i \smallsetminus \bar{X}_{i-1}) \subset \bar{X}_i' \smallsetminus \bar{X}_{i-1}'$. Thus, by applying proposition (7.4) below, the pull-back family X_i is obtained by attaching the $\mathfrak{F}$-bundle M_i to X_i via a $\mathfrak{F}$-map $h_i\colon A_i \subset M_i \to X_i$, where h_i is a suitable $\mathfrak{F}$-map induced by the construction. This is true for $i = 1, \dots, n$, hence $X_n = X$ is a $\mathfrak{F}$-stratified bundle. *q.e.d.*

A similar theorem was proved by Davis [7] (theorems 1.1 and 1.3), in the case of the pull-back f^*M of a smooth G-manifold M, where f is a (weakly) stratified map and G is a compact Lie group.

Let $(\mathfrak{F}, F)$ and $(\mathfrak{F}', F')$ be structure categories. Then we define the structure category $(\mathfrak{F} \times \mathfrak{F}', F \times F')$ by the functor

$$F \times F' \colon \mathfrak{F} \times \mathfrak{F}' \to \mathbf{Top}.$$

Here $\mathfrak{F} \times \mathfrak{F}'$ is the product category consisting of pairs (V, V') of objects $V \in \mathfrak{F}$ and $V' \in \mathfrak{F}'$ and pairs of morphisms. The functor $F \times F'$ carries (V, V') to the product space $F(V) \times F'(V')$. For a $\mathfrak{F}$-family X and a $\mathfrak{F}'$-family X' with the base space $\bar{X} = B = \bar{X}'$ we define the *fiberwise product* $X \times_B X'$ by the pull-back diagram in **Top**

$$\begin{tikzcd} X \times_B X' \arrow[r] \arrow[d] \arrow[dr, dotted, "p"] & X' \arrow[d] \\ X \arrow[r] & B \end{tikzcd}$$

Then it is clear that $X \times_B X' \to B$ is a $(\mathfrak{F} \times \mathfrak{F}', F \times F')$-family. Moreover we get compatibility with stratifications as follows.

(4.8) Proposition. *Let X be a $\mathfrak{F}$-stratified bundle and let X' be a $\mathfrak{F}'$-stratified bundle such that the stratified spaces $\bar{X} = B = \bar{X}'$ coincide. Moreover assume that X and X' are locally compact. Then $X \times_B X' \to B$ is a $\mathfrak{F} \times \mathfrak{F}'$-stratified bundle.*

Proof. By construction X is the quotient space of a projection

$$q\colon X_0 \cup \coprod_{i\geq 0} M_i \to X,$$

and the same holds for X', with a projection q'. By the Whitehead theorem the map $q \times q'$ is a quotient map onto $X \times X'$, since X and X' are locally compact. Now, $X \times_B X \subset X \times X'$ is a closed subspace, hence the restricted projection

$$q \times q' \colon (q \times q')^{-1} X \times_B X \to X' \times_B X'$$

is a quotient map. We can consider now the union

$$(X_0 \times_{\bar{X}_0} X'_0) \coprod_i (M_i \times_{\bar{M}_i} M'_i) \subset (q \times q')^{-1} X \times_B X \to X'.$$

Then it is possible to see that the projection $q \times q'$ yields a quotient map

$$(X_0 \times_{\bar{X}_0} X'_0) \coprod_i (M_i \times_{\bar{M}_i} M'_i) \to X \times_B X',$$

which gives the $\mathfrak{F} \times \mathfrak{F}'$-stratification of $X \times_B X'$. *q.e.d.*

5 The bundle theorem

A *groupoid* is a category in which all morphisms are isomorphisms. If the structure category $\mathfrak{F}$ is a groupoid then $\mathfrak{F}$-complexes and $\mathfrak{F}$-stratified bundles are actually $\mathfrak{F}$-bundles. More precisely we show the following result which we could not find in the literature though special cases are well known like the clutching construction of Atiyah [1].

(5.1) Bundle theorem. *Let $\mathfrak{F}$ be a structure category which is a groupoid. Then a $\mathfrak{F}$-stratified bundle $X \to \bar{X}$ is a $\mathfrak{F}$-bundle over $\bar{X}$. Conversely, given a stratified space $\bar{X}$ and a $\mathfrak{F}$-bundle X over $\bar{X}$ then X is a $\mathfrak{F}$-stratified bundle. In particular, a $\mathfrak{F}$-complex $X \to \bar{X}$ is a $\mathfrak{F}$-bundle over $\bar{X}$ and, given a CW-complex $\bar{X}$ and a $\mathfrak{F}$-bundle X over $\bar{X}$, then X is $\mathfrak{F}$-isomorphic to a $\mathfrak{F}$-complex.*

Proof. If X is a $\mathfrak{F}$-complex then the theorem is a consequence of corollary (5.4) and lemma (8.5) below. Otherwise, assume that X is a $\mathfrak{F}$-stratified bundle. Since X is obtained by attaching $\mathfrak{F}$-bundles to $\mathfrak{F}$-bundles via $\mathfrak{F}$-maps, we can apply a finite number of times corollary (5.4) below and obtain the result. On the other hand assume that $\bar{X}$ is a stratified space and that $X \to \bar{X}$ is a $\mathfrak{F}$-bundle. By applying a finite number of times lemma (8.3) it is possible to show that X is a $\mathfrak{F}$-stratified bundle. *q.e.d.*

(5.2) Example. Let $\mathfrak{F}$ be the category consisting of finite sets $\{1, 2, \ldots, n\}$ and permutations of these sets. A $\mathfrak{F}$-family X is a finite-to-one map $X \to \bar{X}$. By theorem (5.1), if X is a $\mathfrak{F}$-complex, then p_X is locally trivial and hence a covering map.

(5.3) Lemma. *Let $\mathfrak{F}$ be a groupoid. Let $Y \to \bar{Y}$ be a $\mathfrak{F}$-bundle, $\bar{Z}$ a set and $h\colon V \times \bar{Z} \times S^{n-1} \to Y$ a $\mathfrak{F}$-map which is the attaching map of $X = Y \cup_h (V \times \bar{Z} \times D^n)$. Given a set $\bar{U} \subset \bar{Y}$ open in $\bar{Y}$ such that $Y|\bar{U}$ is trivial, there is an open set $\bar{U}'' \subset \bar{X}$ such that $\bar{U}'' \cap \bar{Y} = \bar{U}$ and such that $X|\bar{U}''$ is trivial. As a consequence, the push-out space $X = Y \cup_h (V \times \bar{Z} \times D^n)$ is a $\mathfrak{F}$-bundle.*

Proof. Let Φ denote the characteristic map $\Phi\colon V \times \bar{Z} \times D^n \to X$ and $\bar{\Phi}$, $\bar{h}$ the maps induced on the base spaces. The restriction of $V \times \bar{Z} \times S^{n-1}$ to $\bar{h}^{-1}\bar{U}$ is trivial, and with a suitable change of coordinates (which exists since $\mathrm{Aut}_{\mathfrak{F}}(V)$ is a topological group) we can assume that the restriction

$$h_0\colon V \times \bar{h}^{-1}\bar{U} \to Y|\bar{U} \approx V \times \bar{U}$$

is of the form $1_V \times \bar{h}$. Now, there is an open set $\bar{U}' \subset \bar{Z} \times D^n$ with the property that $\bar{U}' \cap \bar{Z} \times S^{n-1} = \bar{h}^{-1}\bar{U}$. Since $\bar{X} = \bar{Y} \cup_{\bar{h}} (\bar{Z} \times D^n)$ the image $\bar{U}'' = \bar{\Phi}(\bar{U}') \cup \bar{U}$ is open in $\bar{X}$. Moreover, since V is locally compact Hausdorff the following diagram is a push-out.

$$\begin{tikzcd} V \times \bar{h}^{-1}\bar{U} \arrow[r] \arrow[d] & V \times \bar{U} \arrow[d] \\ V \times \bar{U}' \arrow[r] & V \times \bar{U}'' \end{tikzcd}$$

But by assumption X is the push-out $Y \cup_h (V \times \bar{Z} \times D^n)$, and the restriction of the push-out to the pre-images of $\bar{U}''$ yields a push-out

$$\begin{tikzcd} V \times \bar{h}^{-1}\bar{U} \arrow[r] \arrow[d] & V \times \bar{U} \arrow[d] \\ V \times \bar{U}' \arrow[r] & X|\bar{U}''. \end{tikzcd}$$

Therefore $X|\bar{U}'' \approx V \times \bar{U}''$ by an isomorphism which is an extension of the chosen isomorphism $Y|\bar{U} \approx V \times \bar{U}$. This implies that every point in $\bar{Y}$ has a neighborhood $\bar{U}''$ in $\bar{X}$ such that $X|\bar{U}''$ is trivial. On the other hand, if x is a point in $\bar{X} \setminus \bar{Y}$, then there is a neighborhood $\bar{U}''$ of x contained in $\bar{X} \setminus \bar{Y}$ and in this case $X|\bar{U}''$ is trivial because it is $\mathfrak{F}$-homeomorphic to $V \times \bar{\Phi}^{-1}\bar{U}'' \subset V \times \bar{Z} \times e^n$. *q.e.d.*

(5.4) Corollary. *Let (X,Y) a relative $\mathfrak{F}$-complex. Assume that $\mathfrak{F}$ is a groupoid and that $Y \to \bar{Y}$ is a $\mathfrak{F}$-bundle. Then $X \to \bar{X}$ is a $\mathfrak{F}$-bundle.*

Proof. Consider the cellular filtration

$$Y = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots \subset X = \lim_n X_n.$$

By applying inductively lemma (5.3) we see that X_n is a $\mathfrak{F}$-bundle over $\bar{X}_n$ for every $n \geq 0$. Moreover, there is a sequence of sets $\bar{U}_n \subset \bar{X}_n$ open in $\bar{X}_n$ such that $\bar{U}_n \cap \bar{X}_{n-1} = \bar{U}_{n-1}$ and $\mathfrak{F}$-isomorphisms $\alpha_n \colon X_n|\bar{U}_n \approx V \times \bar{U}_n$. Moreover α_n can be chosen to be extension of α_{n-1} (see the proof of (5.3)). Since V is locally compact Hausdorff we can take limits, and obtain a $\mathfrak{F}$-homeomorphism

$$X|\bar{U} \approx V \times \bar{U},$$

where $\bar{U} = \lim_n \bar{U}_n$. *q.e.d.*

6 The principal bundle theorem

It is well known that each fibre bundle $X \to \bar{X}$ with fibre V and structure group G yields the associated principal bundle $X_G \to \bar{X}$ with fibre G such that X_G is a right G-space for which there is an isomorphism of bundles.

(6.1)
$$\begin{tikzcd} X \arrow[rr, "\cong"] \arrow[dr] & & X_G \times_G V \arrow[dl] \\ & \bar{X} & \end{tikzcd}$$

Moreover Steenrod [14] p. 39 points out that in case G is a subgroup of the group of homeomorphisms of V with the compact open topology then the principal bundle

(6.2) $$X_G = X^V$$

is the function space X^V of "admissible" maps $V \to X$ with the compact open topology. The action of $g \in G$ on $\alpha \in X^V$ is given by composition

$$\alpha \cdot g = V \xrightarrow{g} V \xrightarrow{\alpha} X \in X^V.$$

In this section we generalize these results for $\mathfrak{F}$-complexes and $\mathfrak{F}$-stratified bundles.

(6.3) Definition. Given a faithful fibre functor $F\colon \mathfrak{F} \to \mathbf{Top}$ and an object $V \in \mathfrak{F}$ we obtain a new fibre functor

$$F^V\colon \mathfrak{F} \to \mathbf{Top}$$

which carries an object W in $\mathfrak{F}$ to the space $F^V(W) = \hom_{\mathfrak{F}}(V, W)$ with the compact open topology. The functor F^V need not be faithful, even though F is faithful.

(6.4) Remark. Since for every V the fibre $F(V)$ is Hausdorff, we see that for each $W \in \mathfrak{F}$ the fibre $F^V(W) = W^V$ of F^V is Hausdorff. Moreover, since W and V are 2° countable, also W^V is 2° countable (see e.g. [8]). We will show that if $(\mathfrak{F}, F)$ is a NKC structure category then W^V is locally compact so that F^V fulfills assumption $(*)$.

The definition of a NKC structure category is the following. First, a family $\mathcal{K}$ of compact sets of V is termed *generating* if for every $\mathfrak{F}$-family Y the subsets

$$N_{K,U} = \{f \in Y^V, f(K) \subset U\}$$

with $K \in \mathcal{K}$ and U open in Y yield a sub-basis for the topology of Y^V.

(6.5) Definition. A structure category $\mathfrak{F}$ with faithful fibre functor F has the NKC-*property* if for every $V \in \mathfrak{F}$ there is a generating family of compact sets $\mathcal{K}$ such that for every $K \in \mathcal{K}$, every $W \in \mathfrak{F}$ and every compact subset $C \subset W$ the subspace

$$N_{K,C} \subset \hom_{\mathfrak{F}}(V, W) = W^V$$

is compact. Examples are given in section 11 below.

(6.6) Remark. Since the spaces W in $\mathfrak{F}$ are assumed locally compact this implies that for a NKC category the function spaces $W^V = \hom_{\mathfrak{F}}(V, W)$ are locally compact. Furthermore, it is not difficult to see that a closed subcategory of a NKC category is also a NKC subcategory.

(6.7) Example. Let $\mathfrak{F}$ denote the category consisting of a finite dimensional vector space V and morphisms given by elements in a closed subgroup G of $GL(V)$. The fibre functor is the embedding $F\colon \mathfrak{F} \to \mathbf{Top}$. Then, by (5.1), a $\mathfrak{F}$-complex X is a vector bundle with structure group G and fibre V. We will see in theorem (11.2) that $(\mathfrak{F}, F)$ has the NKC property. Therefore by theorem (6.12) the principal bundle X^V is a $(\mathfrak{F}, F^V)$-complex and hence a $(\mathfrak{F}, F)$-bundle by (5.1). This is the result of Steenrod in (6.2).

(6.8) Example. Let G be a compact Lie group and $\mathfrak{F}$ its orbit category (see (3.3)). The fibre functor $F\colon \mathfrak{F} \to \mathbf{Top}$ is the embedding. Then a smooth G-manifold X has a structure of a $\mathfrak{F}$-complex where $\bar{X} = X/G$ and $p_X\colon X \to \bar{X}$ is the projection onto the orbit space. For every closed subgroup $H \subset G$ the orbit $V = G/H$ is an object of $\mathfrak{F}$ and the function space X^V is homeomorphic to the fixed subspace $X^H \subset X$ via the evaluation map at $1 \in G$. We will see as a consequence of proposition (11.1) that the orbit category $\mathfrak{F}$ with the embedding functor $F\colon \mathfrak{F} \to \mathbf{Top}$ has the NKC property. Hence by theorem (6.12) the space X^H is a $\mathfrak{F}$-complex with fibre functor $F^{G/H}$.

We now define for every $(\mathfrak{F}, F)$-family Y the associated $(\mathfrak{F}, F^V)$-family Y^V as follows. Let Y^V denote the space of all $\mathfrak{F}$-maps $V \to Y$ with the compact-open topology. Then we obtain the projection

$$Y^V \to \bar{Y} \tag{6.9}$$

which sends a $\mathfrak{F}$-map $a\colon V \to Y$ to the point $p_Y a(v_0) \in \bar{Y}$, where $v_0 \in V$ is an arbitrary point of the fibre V. The pre-image of a point $b \in \bar{Y}$ under this projection is equal to $F\hom(V, Y_b) = Y_b^V$. Thus $Y^V \to \bar{Y}$ is a $(\mathfrak{F}, F^V)$-family.

We recall some properties of the function spaces X^V and $W^V = \hom_{\mathfrak{F}}(V, W)$. By assumption $(*)$ the images under the fibre functor F of all the objects in $\mathfrak{F}$ are locally compact second-countable and Hausdorff. This implies that they are metrizable. Since every object V is metrizable, the automorphism group $\mathrm{Aut}(V)$ is a topological group with the compact-open topology induced by the inclusion $\mathrm{Aut}(V) \subset \hom_{\mathfrak{F}}(V, V)$.

Given a $\mathfrak{F}$-map $f\colon X \to Y$, let f^V denote the map $f^V\colon X^V \to Y^V$ defined by composing with f the $\mathfrak{F}$-maps $V \to X$. Moreover for $\varphi\colon V \to W$ in $\mathfrak{F}$ let $X^\varphi\colon X^W \to X^V$ be the map induced by φ. Let us recall that given a $\mathfrak{F}$-map $f\colon Z \times V \to X$ the adjoint $\hat{f}$ of f is the function $\hat{f}\colon Z \to X^V$ defined by $\hat{f}(z)(v) = f(z, v)$ for every $z \in Z$ and $v \in V$.

(6.10) Lemma. *For every space Z a $\mathfrak{F}$-function $f\colon Z \times V \to X$ is continuous if and only if the adjoint $\hat{f}\colon Z \to X^V$ is continuous.*

Proof. Because V is locally compact and Hausdorff, a function $f\colon Z \times V \to X$ is continuous if and only if its adjoint $\hat{f}\colon Z \to \mathrm{Map}(V, X)$ is continuous, where $\mathrm{Map}(V, X)$ denotes the space of all (not necessarily controlled by $\mathfrak{F}$) maps from V to X with the compact-open topology. But a function $f\colon Z \times V \to X$ is a $\mathfrak{F}$-map if and only if its adjoint sends Z into the subspace $X^V \subset \mathrm{Map}(V, X)$. This implies the lemma. *q.e.d.*

(6.11) Lemma. *The evaluation map $X^V \times V \to X$ which sends (g, v) to $g(v)$ is continuous. Moreover, if $f\colon X \to Y$ is continuous then the induced* *function $f^V\colon X^V \to Y^V$ is continuous. Also, given $\varphi\colon V \to W$ the induced map $X^\varphi\colon X^W \to X^V$ is continuous.*

Proof. The evaluation map is the adjoint of the identity of X^V, hence continuous by lemma (6.10). The evaluation at $v \in V$ is the restriction of the evaluation to the subspace $X^V \times \{v\}$ of $X^V \times V$ and hence continuous. The projection $p\colon X^V \to \bar{X}$ is the composition of the evaluation at any $v_0 \in V$ with the projection p_X and hence continuous. The induced function f^V is the adjoint of the composition $X^V \times V \to X \to Y$ where the first arrow is the evaluation and the second is f. To see that X^φ is continuous it is enough to see that it is the adjoint of the composition $X^W \times V \to X^W \times W \to X$, where the first arrow is $1 \times \varphi$ while the second is the evaluation. *q.e.d.*

The next result is a crucial observation which shows that the function space X^V of a stratified bundle X is again a stratified bundle.

(6.12) Theorem. *Let $\mathfrak{F}$ be a NKC category with fibre functor F. If X is a $(\mathfrak{F}, F)$-stratified bundle and V an object in $\mathfrak{F}$ then the function space X^V is also a $(\mathfrak{F}, F^V)$-stratified bundle.*

Proof. If X is a $(\mathfrak{F}, F)$-complex then this follows from corollary (10.16) below. In the general case one can apply a finite number of times corollary (10.18) and lemma (9.4). *q.e.d.*

Let $\mathbf{Top}_{\bar{X}}$ be the category of spaces over $\bar{X}$; objects are maps $X \to \bar{X}$. Given a $\mathfrak{F}$-family X, the function spaces X^V for every object V in $\mathfrak{F}$ yield a functor X° which carries V to $X^\circ(V) = X^V$. Moreover, given a $\mathfrak{F}$-map $f\colon X \to Y$ there is a natural transformation $f^\circ : X^\circ \to Y^\circ$ defined by $f^V\colon X^V \to Y^V$. Let $\mathfrak{F}^{\mathrm{op}}$-$\mathbf{Diag}_{\bar{X}}$ denote the diagram category in which the objects are functors $\mathfrak{F}^{\mathrm{op}} \to \mathbf{Top}_{\bar{X}}$ and the morphisms are the natural transformations between functors. The operator $(-)^\circ$ sends a $\mathfrak{F}$-family X to the $\mathfrak{F}^{\mathrm{op}}$-diagram X° and a $\mathfrak{F}$-map f to the natural transformation f°. Thus $(-)^\circ$ is a functor

$$(-)^\circ\colon \mathfrak{F}\text{-}\mathbf{Top}_{\bar{X}} \to \mathfrak{F}^{\mathrm{op}}\text{-}\mathbf{Diag}_{\bar{X}}.$$

The $\mathfrak{F}$-diagram X° is the generalization of the concept of principal bundle. In fact, if $X \to \bar{X}$ is a fibre bundle with structure group G and fibre V and $\mathfrak{F} = G \to \mathbf{Top}$ is given by the G-space V then the diagram $X^\circ\colon \mathfrak{F}^{\mathrm{op}} \to \mathbf{Top}$ coincides with the G-space X_G given by the principal bundle $X_G \to \bar{X}$ associated to X, see (6.2).

For a topological enriched category $\mathfrak{F}$ there is a naturally associated functor $\widehat{\hom}\colon \mathfrak{F} \to \mathfrak{F}^{\mathrm{op}}$-$\mathbf{Diag}$ which sends an object W of $\mathfrak{F}$ to the $\mathfrak{F}^{\mathrm{op}}$-diagram

$\hom_{\mathfrak{F}}(-,W) = W^\circ$. In case X is a $\mathfrak{F}$-stratified bundle and $(\mathfrak{F},F)$ is a NKC structure category then by theorem (6.12) the function space X^V is not only a space over $\bar{X}$ but a $(\mathfrak{F},F^V)$-stratified bundle. Moreover this holds for every object $V \in \mathfrak{F}$ in a compatible way. Therefore the diagram X° given by stratified bundles X^V leads to the concept of a *stratified bundle diagram.* For this consider the definition of a stratified bundle in (4.4) and replace the category **Top** by the category of diagrams in **Top**. This yields a notion of a stratified bundle diagram. If the stratification is given by the skeletal filtration of the base space $\bar{X}$ then a $(\mathfrak{F},\widehat{\hom})$-stratified bundle diagram is the same as a "free CW-complex" in the sense of Davis–Lück [6].

(6.13) Definition. If $\mathfrak{F}$ is a topologically enriched category, then a $(\mathfrak{F},\widehat{\hom})$-stratified bundle diagram in the category $\mathfrak{F}^{\mathrm{op}}\text{-}\mathbf{Diag}_{\bar{X}}$ is termed *principal stratified bundle diagram.* This is a $\mathfrak{F}^{\mathrm{op}}$-diagram of stratified $\mathfrak{F}$-bundles over a stratified space $\bar{X}$. For example $X^\circ\colon \mathfrak{F}^{\mathrm{op}} \to \mathbf{Top}_{\bar{X}}$ is such a principal stratified bundle diagram.

If $\mathfrak{F}$ has a single object V and $\hom_{\mathfrak{F}}(V,V)$ is a topological group, then a $(\mathfrak{F},\widehat{\hom})$-stratified bundle diagram is nothing but a principal G-bundle with structure group $G = \hom_{\mathfrak{F}}(V,V)$ over $\bar{X}$, see theorem (5.1).

Now consider a principal stratified bundle diagram $P\colon \mathfrak{F}^{\mathrm{op}} \to \mathbf{Top}_{\bar{X}}$ over $\bar{X}$ and a fibre functor $F\colon \mathfrak{F} \to \mathbf{Top}$. We can build a $(\mathfrak{F},F)$-stratified bundle $X \to \bar{X}$ in **Top** as the coend of the two functors P and F:

$$X = P \otimes_{\mathfrak{F}} F = \int^{V\in\mathfrak{F}} P(V) \times F(V).$$

We recall the coend construction: the space $P \otimes_{\mathfrak{F}} F$ is the quotient space of the coproduct

$$\coprod_{V\in\mathfrak{F}} P(V) \times F(V)$$

with the identification $(P(\alpha)(x),y) \sim (x,F(\alpha)(y))$ for every morphism $\alpha \in \hom_{\mathfrak{F}}(V,W)$, $x \in P(W)$ and $y \in F(V)$. There is a well-defined map $X \to \bar{X}$ and it is not difficult to show:

(6.14) Lemma. *$P \otimes_{\mathfrak{F}} F$ is a $(\mathfrak{F},F)$-family.*

Proof. See e.g. [10], page 44, for the case of a groupoid $\mathfrak{F}$. It suffices to show it in the case P is a $(\mathfrak{F},\widehat{\hom})$-point, that is a diagram P over a point. Since P is principal, there is an object W of $\mathfrak{F}$ such that $P(V) = \hom_{\mathfrak{F}}(V,W)$ for every $V \in \mathfrak{F}$, or, equivalently, $P = W^\circ$. If $g\colon V \to V'$ is a morphism in $\mathfrak{F}$, then $P(g)\colon W^{V'} \to W^V$ is defined by $P(g)(\alpha) = \alpha g$. Thus the coend $W^\circ \otimes_{\mathfrak{F}} F$ is defined as

$$(6.15) \qquad \left(\coprod_{V \in \mathfrak{F}} W^V \times F(V) \right) / \sim,$$

where $(\alpha g, x) \sim (g, F(\alpha)(x))$ for every $g \in \hom_{\mathfrak{F}}(V, V')$, $\alpha \in W^V$ and $x \in F(V)$. Now it is easy to see that for every W

$$W^\circ \otimes_{\mathfrak{F}} F = F(W)$$

and that given a morphism $\alpha^\circ\colon W^\circ \otimes_{\mathfrak{F}} F \to W'^\circ \otimes F$ the following diagram commutes:

$$(6.16) \qquad \begin{array}{ccc} W^\circ \otimes_{\mathfrak{F}} F & \xrightarrow{\alpha^\circ} & W'^\circ \otimes_{\mathfrak{F}} F \\ \Big\downarrow{=} & & \Big\downarrow{=} \\ F(W) & \xrightarrow{F(\alpha)} & F(W') \end{array}$$

q.e.d.

(6.17) Theorem. *Let $\mathfrak{F}$ be a structure category with a faithful fibre functor F so that $\mathfrak{F}$ is topological enriched by the compact open topology. Let $(\mathfrak{F}, F)$ be a NKC structure category and X a $(\mathfrak{F}, F)$-stratified bundle. Then the associated diagram X° is a principal stratified bundle diagram with fibre functor $\widehat{\hom}$. Conversely, if P is a principal $(\mathfrak{F}, \widehat{\hom})$-stratified bundle diagram and F a fibre functor, then the coend $P \otimes_{\mathfrak{F}} F$ is a stratified bundle in* **Top** *with structure category $\mathfrak{F}$ and fibre functor F. Moreover*

$$X^\circ \otimes_{\mathfrak{F}} F = X.$$

This result generalizes the classical equation

$$X_G \times_G V = X$$

for the principal bundle X_G associated to X in (6.1). This construction can be generalized. Let $\varphi\colon G \to H$ be a continuous homomorphism between topological groups and let G (H) be a topological transformation group for V (W). Then a (G, V)-*bundle* X (i.e. a bundle with fibre V and structure group G) yields the principal bundle X_G which in turn yields the φ-*associated bundle*

$$\varphi_\#(X) = X_G \times_G \varphi^* W \to \bar{X}.$$

Here φ^*W is the G-space with the G-action induced by φ, that is $g \cdot w = \varphi(g) \cdot w$ for $g \in G$ and $w \in W$. The associated bundle $\varphi_\#(X)$ has fibre W and structure group H. Hence φ induces the functor

$$\varphi_\#\colon (G, V)\text{-}\mathbf{Bundles}_{\bar{X}} \to (H, W)\text{-}\mathbf{Bundles}_{\bar{X}}. \tag{6.18}$$

We now generalize this functor for stratified bundles.

Let $(\mathfrak{F}, F)$ and $(\mathfrak{G}, G)$ be structure categories with faithful fibre functor F, G so that $\mathfrak{F}$ and $\mathfrak{G}$ have the compact open topology. Let

$$\varphi\colon \mathfrak{F} \to \mathfrak{G}$$

be a continuous functor. If $(\mathfrak{F}, F)$ is a NKC structure category and X a $(\mathfrak{F}, F)$-stratified bundle then the principal stratified bundle diagram X° is defined and we obtain the *φ-associated stratified bundle*

$$\varphi_\#(X) = X^\circ \otimes_{\mathfrak{F}} (G\varphi) \to \bar{X}.$$

Here $\varphi_\#(X)$ is a $(\mathfrak{G}, G)$-stratified bundle. Hence φ induces the functor

$$\varphi_\#\colon (\mathfrak{F}, F)\text{-}\mathbf{Stra}_{\bar{X}} \to (\mathfrak{G}, G)\text{-}\mathbf{Stra}_{\bar{X}}.$$

Proof of theorem (6.17). We have seen that, as a consequence of theorem (6.12), for each $V \in \mathfrak{F}$ the function space X^V is obtained by gluing W^V-bundles M_i^V along with maps $h_i^V\colon A_i^V \subset M_i \to X_{i-1}^V$, for suitable objects $W \in \mathfrak{F}$. In other words, the diagram X° is obtained by gluing the $\mathfrak{F}$-diagrams M_i° (which are $(\mathfrak{F}, \widehat{\mathrm{hom}})$-bundles) along with $\mathfrak{F}$-diagram maps $h_i^\circ\colon A_i^V \to X_{i-1}^\circ$. Thus X° is a $(\mathfrak{F}, \widehat{\mathrm{hom}})$-stratified bundle, that is a $\mathfrak{F}$-stratified bundle with diagram-fibre functor $\widehat{\mathrm{hom}}$. By definition this is a principal stratified bundle diagram.

Conversely, let P be a principal stratified bundle diagram and F a fibre functor. It is not assumed that F is faithful, but that the spaces FV are second-countable locally compact Hausdorff spaces, see assumption $(*)$. Hence, if the i-skeleton of P is obtained by attaching a $(\mathfrak{F}, \widehat{\mathrm{hom}})$-bundle $\hat{M}_i$ to P_{i-1} via a $\mathfrak{F}$-diagram map $\hat{h}_i\colon \hat{A}_i \subset \hat{M}_i \to P_{i-1}$, i.e. $P_i = \hat{M}_i \cup_{\hat{h}_i} P_{i-1}$, the coends are corners of a push-out diagram in $\mathfrak{F}$-$\mathbf{Top}$:

$$\begin{tikzcd}
\hat{A}_i \otimes_{\mathfrak{F}} F \arrow[r, "\hat{h}_i \times_{\mathfrak{F}} F"] \arrow[d] & P_{i-1} \otimes_{\mathfrak{F}} F \arrow[d] \\
\hat{M}_i \otimes_{\mathfrak{F}} F \arrow[r] & P_i \otimes_{\mathfrak{F}} F
\end{tikzcd}$$

Thus X has a filtration given by $X_i = P_i \otimes_{\mathfrak{F}} F$, for all $i \geq 0$, and each X_i is obtained by attaching $M_i = \hat{M}_i \otimes_{\mathfrak{F}} F$ to X_{i-1} via the map

$$h_i = \hat{h}_i \otimes_{\mathfrak{F}} F \colon A_i = \hat{A}_i \otimes_{\mathfrak{F}} F \to X_{i-1}.$$

Moreover, since

$$\lim_{i \geq 0} (P_i \otimes_{\mathfrak{F}} F) = \left(\lim_{i \geq 0} P_i \right) \otimes_{\mathfrak{F}} F,$$

X is equal to the colimit $X = \lim_{i \geq 0} X_i$. To show that it is a $(\mathfrak{F}, F)$-stratified bundle we need only to show that the attaching maps are $\mathfrak{F}$-maps. But $\hat{h}_i$ are $(\mathfrak{F}, \widehat{\hom})$-maps, and by diagram (6.16) the coends of such maps are $(\mathfrak{F}, F)$-maps. The proof is hence complete. *q.e.d.*

7 Pull-back of a push-out $\mathfrak{F}$-family

In this section we prove the pull-back theorem (4.7).

Consider a $\mathfrak{F}$-family Y', a $\mathfrak{F}$-CW-pair (M', A') and a $\mathfrak{F}$-map $h' \colon A' \to Y'$. Then we can glue M' to Y' and obtain a $\mathfrak{F}$-family X', as in the following diagram.

(7.1)
$$\begin{tikzcd}
A' \arrow[rrr, "h'"] \arrow[dr, "p_{A'}"] \arrow[ddd, "j_{A'}"'] & & & Y' \arrow[dl, "p_{Y'}"'] \arrow[ddd, "j_{Y'}"] \\
& \bar{A}' \arrow[r, "\bar{h}'"] \arrow[d, "j_{\bar{A}'}"] & \bar{Y}' \arrow[d, "j_{\bar{Y}'}"] & \\
& \bar{M}' \arrow[r, "\bar{\Phi}'"] & \bar{X}' & \\
M' \arrow[ur, "p_{M'}"] \arrow[rrr, "\Phi'"] & & & X' \arrow[ul, "p_{X'}"']
\end{tikzcd}$$

Now consider a space $\bar{Y}$, a CW-pair $(\bar{M}, \bar{A})$ and a map $\bar{h} \colon \bar{A} \to \bar{Y}$. Let $\bar{X}$ be the push-out space $\bar{X} = \bar{Y} \cup_{\bar{h}} \bar{M}$. Assume that maps $\tilde{f}_A$, $\tilde{f}$, $\bar{f}_{\bar{Y}}$ and $\bar{f}$

are given, such that the following diagram commutes.

$$
\begin{tikzcd}
\bar{A}' \arrow[rrr, "\bar{h}'"] \arrow[ddd, "j_{A'}"] & & & \bar{Y}' \arrow[ddd, "j_{\bar{Y}'}"] \\
& \bar{A} \arrow[ul, "\tilde{f}_A"'] \arrow[r, "\bar{h}"] \arrow[d, "j_{\bar{A}}"] & \bar{Y} \arrow[ur, "\bar{f}_{\bar{Y}}"] \arrow[d, "j_{\bar{Y}}"] & \\
& \bar{M} \arrow[dl, "\tilde{f}"'] \arrow[r, "\bar{\Phi}"] & \bar{X} \arrow[dr, "\bar{f}"] & \\
\bar{M}' \arrow[rrr, "\bar{\Phi}'"'] & & & \bar{X}'
\end{tikzcd}
\tag{7.2}
$$

Then the pull-back families $X = \bar{f}^*X'$, $Y = \bar{f}^*_{\bar{Y}}Y'$, $A = \tilde{f}^*_A A'$ and $M = \tilde{f}^*M'$ are defined. Moreover, we can define the maps $h = (\bar{h}, h')\colon A \to Y$, $j_Y = (j_{\bar{Y}}, j_{Y'})\colon Y \to X$, $j_A = (j_{\bar{A}}, j_{A'})\colon A \to M$ and $\Phi = (\bar{\Phi}, \Phi')\colon M \to X$. They fit into the following diagram.

$$
\begin{tikzcd}
\bar{A} \times A' & A \arrow[l, tail] \arrow[r, "h"] \arrow[d, "j_A"] & Y \arrow[r, tail] \arrow[d, "j_Y"] & \bar{Y} \times Y' \\
\bar{M} \times M' & M \arrow[l, tail] \arrow[r, "\Phi"] & X \arrow[r, tail] & \bar{X} \times X'.
\end{tikzcd}
\tag{7.3}
$$

The aim of this section is to prove the following proposition.

(7.4) Proposition. *Assume that $\bar{M}$ and $\bar{M}'$ are locally finite and countable CW-complexes and that $\bar{f}(\bar{X} \smallsetminus \bar{Y}) \subset \bar{X}' \smallsetminus \bar{Y}'$. Then the pull-back family $X = \bar{f}^*X'$ is obtained by attaching the pull-back $\mathfrak{F}$-family $M = \tilde{f}^*M'$ to $Y = \bar{f}^*_{\bar{Y}}Y'$ via the $\mathfrak{F}$-map $h\colon A \subset M \to Y$ induced by $(\bar{h}, h')$, i.e. the middle square in diagram (7.3) is a push-out.*

The proof of proposition (7.4) will be the content of the rest of the section.

(7.5) Lemma. *The maps in (7.3) are well-defined and the diagram commutes.*

Proof. Consider $(\bar{a}, a') \in A$. Then $\tilde{f}_A(\bar{a}) = p_{A'}(a')$, hence

$$\bar{f}_{\bar{Y}}\bar{h}(\bar{a}) = \bar{h}'\tilde{f}_A(\bar{a}) = \bar{h}'p_{A'}(a') = p_{Y'}h'(a').$$

Hence $(\bar{h}(\bar{a}), h'(a')) \in Y$. On the other hand, if $(\bar{y}, y') \in Y$, then $\bar{f}_{\bar{Y}}(\bar{y}) = p_{Y'}(y')$, hence

$$\bar{f}j_{\bar{Y}}(\bar{y}) = j_{\bar{Y}'}\bar{f}_{\bar{Y}}(\bar{y}) = j_{\bar{Y}'}p_{Y'}(y') = p_{X'}j_{Y'}(y').$$

That is, $j_Y(\bar{y}, y') = (j_{\bar{Y}}(\bar{y}), j_{Y'}(y')) \in X$. The same argument can be applied literally to j_A and Φ. The diagram commutes since it is the restriction of a commutative diagram on the Cartesian products. *q.e.d.*

(7.6) Lemma. *The maps $j_A \colon A \to M$ and $j_Y \colon Y \to X$ are a closed inclusions of $\mathfrak{F}$-families.*

Proof. Consider the following diagram. The left square is a pull-back (by definition) and the right square is a pull-back (the pair (M', A') is a $\mathfrak{F}$-CW-pair).

(7.7)
$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & M' \arrow[d] \\
\bar{A} \arrow[r] & \bar{A}' \arrow[r] & \bar{M}'
\end{tikzcd}$$

Now consider this diagram.

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M' \arrow[d] \\
\bar{A} \arrow[r] & \bar{M} \arrow[r] & \bar{M}'
\end{tikzcd}$$

The composition of the two squares is equal to the composition of the two squares in diagram (7.7), hence it is a pull-back. The right square is a pull-back by definition, hence the left square is a pull-back. (see e.g. exercise 8 of [12], page 72). Since $\bar{A} \subset \bar{M}$ is a closed inclusion, j_A is a closed inclusion. The proof is the same for j_Y. *q.e.d.*

Now consider the maps $q = \Phi \sqcup j_Y$, $\bar{q} = \bar{\Phi} \sqcup j_{\bar{Y}}$, $\bar{q}' = \bar{\Phi}' \sqcup j_{\bar{Y}'}$ and $q' = \Phi' \sqcup j_{Y'}$. They can be arranged in the following diagram.

(7.8)
$$\begin{tikzcd}
M \sqcup Y \arrow[rrr] \arrow[dr] \arrow[ddd, "q"] & & & M' \sqcup Y' \arrow[dl] \arrow[ddd, "q'"] \\
& \bar{M} \sqcup \bar{Y} \arrow[r] \arrow[d, "\bar{q}"] & \bar{M}' \sqcup \bar{Y}' \arrow[d, "\bar{q}'"] & \\
& \bar{X} \arrow[r] & \bar{X}' & \\
X \arrow[rrr] \arrow[ur] & & & X' \arrow[ul]
\end{tikzcd}$$

By definition $\bar{q}$, $\bar{q}'$ and $\bar{q}'$ are quotient maps. We want to show that under suitable conditions q is a quotient map. If this is the case, then proposition (7.4) is proved, since q is exactly the projection defining the topology of the push-out.

(7.9) Lemma. *Assume that $\bar{f}(\bar{X} \smallsetminus \bar{Y}) \subset \bar{X}' \smallsetminus \bar{Y}'$. Then q is onto, $q|Y$ and $q|(M \smallsetminus A)$ are mono.*

Proof. Let $x = (\bar{x}, x') \in X$, so that then $\bar{f}(\bar{x}) = p_{X'}(x')$. If $\bar{x} \in \bar{Y}$, then $\bar{f}(\bar{x}) \in \bar{Y}'$, therefore $x' \in p_{X'}^{-1}\bar{Y}' = Y'$. Thus there is $y = (\bar{y}, y') \in Y$ such that $q(y) = x$. On the other hand, if $\bar{x} \notin \bar{Y}$, since assumption $\bar{f}\bar{x} \notin \bar{Y}'$, necessarily $x' \notin Y'$ hence there is a unique $m' \in M' \smallsetminus A'$ such that $q'(m') = x'$. For the same reason there is a unique $\bar{m} \in \bar{M} \smallsetminus \bar{A}$ such that $\bar{q}(\bar{m}) = \bar{x}$. Now consider the since $\bar{q}'$ is mono in $\bar{M}' \smallsetminus A'$, the chain of equalities

$$\bar{q}'\tilde{f}(\bar{m}) = \bar{f}\bar{q}(\bar{m}) = \bar{f}(\bar{x}) = p_{X'}(x') = p_{X'}q'(m') = \bar{q}'p_{M'}(m')$$

implies that $\tilde{f}(\bar{m}) = p_{M'}(m')$, hence that $(\bar{m}, m') \in M$. We have shown that q is onto. The restriction of q to Y is mono since $j_{\bar{Y}} \times j_{Y'}$ is mono. To see that the restriction of q to $M \smallsetminus A$ is mono, consider that if $(\bar{m}, m') \in M$ then $\bar{m} \notin \bar{A}$ implies $m' \notin A'$, hence $M \smallsetminus A \subset (\bar{M} \smallsetminus \bar{A}) \times (M' \smallsetminus A')$. Now, $\bar{q}$ and q' are mono when restricted to $\bar{M} \smallsetminus \bar{A}$ and $M' \smallsetminus A'$ respectively, hence $q|(M \smallsetminus A)$ is mono. *q.e.d.*

From now on, we will assume that the condition of lemma (7.9) is fulfilled.

(7.10) Lemma. *The restriction $q|(M \smallsetminus A)\colon M \smallsetminus A \to X \smallsetminus Y$ is a homeomorphism.*

Proof. The map $q|(M \smallsetminus A)$ is bijective and continuous. We need to show that it is open. Consider an open set U in $M \smallsetminus A$, and a point $x = (\bar{x}, x') \in U$. There are neighborhood $U_{\bar{M}} \subset \bar{M} \smallsetminus \bar{A}$ and $U_{M'} \subset M' \smallsetminus A'$ such that

$$x \in (U_{\bar{M}} \times U_{M'}) \cap M \subset U.$$

Since $\bar{q}$ and q' are homeomorphisms of $\bar{M} \smallsetminus \bar{A}$ and $M' \smallsetminus A'$ onto their images, the sets $U_{\bar{X}} = \bar{q}U_{\bar{M}}$ and $U_{X'} = q'U_{M'}$ are open subsets of $\bar{X} \smallsetminus \bar{Y}$ and $X' \smallsetminus Y'$. Hence $U_{\bar{X}} \times U_{X'} \cap X$ is open in X and contained in $X \smallsetminus Y$. Moreover, since q is mono in $M \smallsetminus A$, we have that $q\left[(U_{\bar{M}} \times U_{M'}) \cap M\right] = U_{\bar{X}} \times U_{X'} \cap X$, hence $U_{\bar{X}} \times U_{X'} \cap X \subset q(U)$. But x is arbitrary, therefore $q(U)$ is open and so the map $q|M \smallsetminus A$ is open. *q.e.d.*

(7.11) Lemma. *A subset $S \subset M \sqcup Y$ is saturated (i.e. $q^{-1}q(S) = S$) if and only if $S \cap A = h^{-1}(S \cap Y)$.*

(7.12) Lemma. *Assume that $\bar{M}$ and M' are metrizable. If $U_{\bar{M}} \subset \bar{M}$ and $U_{A'} \subset A'$ are open sets with compact closure such that*

$$(\overline{U_{\bar{M}}} \times \overline{U_{A'}}) \cap M \subset q^{-1}U,$$

then there is an open subset $U_{M'} \subset M'$ with compact closure such that

$$U_{M'} \cap A' = U_{A'}$$
$$(\overline{U_{\bar{M}}} \times \overline{U_{M'}}) \cap M \subset q^{-1}U.$$

In the same way, if $U_{\bar{A}} \subset \bar{A}$ and $U_{M'} \subset M'$ are open sets with compact closure such that

$$(\overline{U_{\bar{A}}} \times \overline{U_{M'}}) \cap M \subset q^{-1}U,$$

then there is an open subset $U_{\bar{M}} \subset \bar{M}$ with compact closure such that

$$U_{\bar{M}} \cap \bar{A} = U_{\bar{A}}$$
$$(\overline{U_{\bar{M}}} \times \overline{U_{M'}}) \cap M \subset q^{-1}U.$$

Proof. By assumption the set $q^{-1}U \cap M$ is open in $M \subset \bar{M} \times M'$, therefore there exists an open set $\widetilde{q^{-1}U} \subset \bar{M} \times M'$ such that $\widetilde{q^{-1}U} \cap M = q^{-1}U \cap M$. We need to find the open set $U_{M'} \subset M'$ with the desired properties. Consider the following function $\eta \colon \bar{M} \times M' \to \mathbb{R}$ defined by

$$\eta(x) = d(x, \widetilde{q^{-1}U}^c) + d(x, M), \tag{7.13}$$

where $\widetilde{q^{-1}U}^c$ is the complement of $\widetilde{q^{-1}U}$ in $\bar{M} \times M'$ and d is the distance given by a metric on $\bar{M} \times M'$. Consider a point x in the compact closure $\overline{U_{\bar{M}}} \times \overline{U_{A'}}$. If $x \in M$ as well, then by the assumption $x \in \widetilde{q^{-1}U}$, and since $\widetilde{q^{-1}U}$ is open this implies $\eta(x) \geq d(x, \widetilde{q^{-1}U}^c) > 0$. On the other hand, since M is closed in $\bar{M} \times M'$, if $x \notin M$ then $\eta(x) \geq d(x, M) > 0$. Thus $\eta(x) > 0$ for every $x \in (\overline{U_{\bar{M}}} \times \overline{U_{A'}})$, and so there is an open set $U_{M'} \subset M'$ with compact closure such that $U_{M'} \cap A' = U_{A'}$ and

$$x \in \overline{U_{\bar{M}}} \times \overline{U_{M'}} \implies \eta(x) > 0.$$

This means that if $x \in (\overline{U_{\bar{M}}} \times \overline{U_{M'}}) \cap M$ then $d(x, \widetilde{q^{-1}U}^c) > 0$, thus $x \in \widetilde{q^{-1}U}$, and this implies that

$$(\overline{U_{\bar{M}}} \times \overline{U_{M'}}) \cap M \subset q^{-1}U.$$

For the second part of the lemma, the proof is exactly the same, it is only necessary to exchange the roles of $\bar{A}$ and M'. *q.e.d.*

(7.14) Lemma. *Let X be a $\mathfrak{F}$-bundle over a locally finite and countable CW-complex $\bar{X}$. Then X is metrizable. Moreover, every open subset $O \subset X$ is the union of an ascending sequence of open subsets $O_n \subset X$ with compact closure $\overline{O_n} \subset O_{n+1}$.*

Proof. Since $\bar{X}$ is locally finite and countable, it is metrizable and 2° countable. Since the fibres are 2° countable by assumption, X is 2° countable, since it is obtained by attaching a countable number of $\mathfrak{F}$-cells $V \times D^n$, which are 2° countable. Now, $\bar{X}$ and the $\mathfrak{F}$-cells $V \times D^n$ are completely regular, hence X is regular; therefore by Urysohn metrization theorem X is metrizable. Let d denote its metric. Now consider an ascending chain of finite subcomplexes $\bar{X}_n \subset \bar{X}$ such that $\bar{X}_n$ is contained in the interior of $\bar{X}_{n+1}$ and $\bar{X} = \bigcup_n \bar{X}_n$ (such a sequence exists since $\bar{X}$ is locally finite and countable). Using the fact that the fibres are locally compact and second countable, it is possible to show that X can be written as the union of an ascending sequence of subspaces $X_n \subset X_{n+1} \subset \dots$ where each X_n is compact and is contained in the interior $\mathring{X}_{n+1}$. For every n define the following open set

$$O_n = \{x \in X : d(x, O^c) > \frac{1}{n}\} \bigcap \mathring{X}_n,$$

where O^c denotes the complement of O. If x is an element of the closure $\overline{O_n}$ (closure of O_n in X), then $x \in O_{n+1} \bigcap X_n \subset X_n$, hence $\overline{O_n}$ is a closed subset of the compact space X_n, hence $\overline{O_n}$ is compact. Moreover, since $X_n \subset \mathring{X}_{n+1}$, we have that $\overline{O_n} \subset O_{n+1}$ as claimed. We need to show that $O = \bigcup_n O_n$: if $x \in O$, then $d(x, O^c) > 0$, therefore there is $n_1 \geq 1$ such that $d(x, O^c) > \frac{1}{n}$ for every $n \geq n_1$. Since $\bigcup_n X_n = X$, there is $n_2 \geq n_1$ such that $x \in X_{n_2} \subset \mathring{X}_{n_2+1}$. But these conditions imply that $x \in O_n$ for $n = n_2 + 1$. Thus $O = \bigcup O_n$. *q.e.d.*

(7.15) Lemma. *Assume that $\bar{M}$ and M' are metrizable. If $U_{\bar{A}} \subset \bar{A}$ and $U_{A'} \subset A'$ are open sets such that*

$$(U_{\bar{A}} \times U_{A'}) \cap M \subset q^{-1}U,$$

then there are open sets $U_{\bar{M}} \subset \bar{M}$ and $U_{M'} \subset M'$ such that

$$\begin{gathered} U_{\bar{M}} \cap \bar{A} = U_{\bar{A}} \\ U_{M'} \cap A' = U_{A'} \\ (U_{\bar{M}} \times U_{M'}) \cap M \subset q^{-1}U. \end{gathered}$$

Proof. By lemma (7.14), it is possible to find increasing sequences $U^k_{\bar{A}} \subset U_{\bar{A}}$ and $U^k_{A'} \subset U_{A'}$ with compact closures and such that

$$\begin{gathered} \bigcup_{k \geq 1} U^k_{\bar{A}} = U_{\bar{A}} \\ \bigcup_{k \geq 1} U^k_{A'} = U_{A'}. \end{gathered}$$

Now, by applying lemma (7.12) twice it is possible to define compact subsets $U^1_{\bar M}$ and $U^1_{M'}$ such that

$$(\overline{U^1_{\bar M}} \times \overline{U^1_{M'}}) \cap M \subset q^{-1}U.$$

By induction, we will show that it is possible to find two sequences of open sets $U^k_{\bar M}$ and $U^k_{M'}$ with compact closures, with the property that

$$U^k_{\bar M} \cap \bar A = U^k_{\bar A}$$
$$U^k_{M'} \cap A' = U^k_{A'}$$

for every k, $k' \geq 1$ and

$$(\overline{U^k_{\bar M}} \times \overline{U^{k'}_{M'}}) \cap M \subset q^{-1}U \tag{7.16}$$

for every k, $k' \geq 1$. Assume that the sequences are defined for $j \leq k-1$. By applying $k-1$ times lemma (7.12) and taking the intersection of the resulting open sets, we can show that there is $U^k_{\bar M}$ such that $U^k_{\bar M} \cap \bar A = U^k_{\bar A}$ and

$$(\overline{U^k_{\bar M}} \times \overline{U^{k'}_{M'}}) \cap M \subset q^{-1}U$$

for every $k' \leq k-1$. Now we can apply k times lemma (7.12) and take the intersection of the resulting open sets, to finally find and open set with compact closure $U^k_{M'}$ such that $U^k_{M'} \cap A' = U^k_{A'}$ and

$$(\overline{U^j_{\bar M}} \times \overline{U^k_{M'}}) \cap M \subset q^{-1}U$$

for every $j \leq k$. Thus equation (7.16) holds for every k, $k' \geq 1$. But this implies that the open sets

$$U_{\bar M} = \bigcup_{k \geq 1} U^k_{\bar M}$$
$$U_{M'} = \bigcup_{k \geq 1} U^k_{M'}$$

have the desired property. *q.e.d.*

(7.17) Lemma. *If $\bar M$ and $\bar M'$ are locally finite and countable, then the map q is a quotient map.*

Proof. Consider a subset $U \subset X$ such that $q^{-1}U$ is open in $M \sqcup Y$. We want to show that U is open. Let $x \in U \smallsetminus Y$. Since $q^{-1}U \cap (M \smallsetminus A)$ is open, by lemma (7.10) there is an open neighborhood of x in $X \smallsetminus Y$ contained in U.

If $U \cap Y = \emptyset$ then we have proved that U is open. Otherwise, let $x \in Y \cap U$. Since $q^{-1}U$ is open, $q^{-1}U \cap Y$ is open and contains $y \in q^{-1}(x) \cap Y$. Therefore there are two open sets $U_{\bar{Y}}$ and $U_{Y'}$ such that

$$y \in (U_{\bar{Y}} \times U_{Y'}) \cap Y \subset q^{-1}Y \cap Y.$$

Let

$$\begin{aligned} U_{\bar{A}} &= \bar{h}^{-1}U_{\bar{Y}} \\ U_{A'} &= {h'}^{-1}U_{Y'} \end{aligned}$$

Since $\bar{h}$ and h' are continuous, they are open subsets of $\bar{A}$ and A' respectively. By lemma (7.15) there are open sets $U_{\bar{M}} \subset \bar{M}$ and $U_{M'} \subset M'$ such that

$$\begin{aligned} U_{\bar{M}} \cap \bar{A} &= U_{\bar{A}} = \bar{h}^{-1}U_{\bar{Y}} \\ U_{M'} \cap A' &= U_{A'} = {h'}^{-1}U_{Y'}, \end{aligned}$$

and

$$(U_{\bar{M}} \times U_{M'}) \cap M \subset q^{-1}U.$$

Consider the sets

$$\begin{aligned} U_{\bar{X}} &= \bar{q}(U_{\bar{M}}) \cup U_{\bar{Y}} \subset \bar{X} \\ U_{X'} &= q'(U_{M'}) \cup U_{Y'} \subset X'. \end{aligned}$$

Since $\bar{q}$ and q' are quotient maps (by definition), and since $U_{\bar{M}} \cup U_{\bar{Y}}$ and $U_{M'} \cup U_{Y'}$ are saturated with respect to the maps $\bar{q}$ and q' (see lemma (7.11)), $U_{\bar{X}}$ and $U_{X'}$ are open in $\bar{X}$ and X'. Therefore $U_x = U_{\bar{X}} \times U_{X'} \cap X$ is open in X and contains x. It is left to show that $U_x \subset U$. Consider $t \in U_x \cap Y$. Then $q^{-1}t = h^{-1}t \sqcup \{t\} \subset M \sqcup Y$. But since j_Y is mono, $t \in (U_{\bar{Y}} \times U'_Y) \cap Y$ and therefore $t \in U$. Otherwise, if $t \in X \smallsetminus Y$ then $q^{-1}t = m$, unique point belonging to $(U_{\bar{M}} \times U_{M'}) \cap M \subset q^{-1}U$. Thus $q^{-1}t \subset q^{-1}U$ and therefore $t \in U$. We have proved that $U_x \subset U$ and the proof is now complete. *q.e.d.*

(7.18) Remark. Since a pull-back of a $\mathfrak{F}$-bundle is a $\mathfrak{F}$-bundle, if M' is a $\mathfrak{F}$-bundle, then M is a $\mathfrak{F}$-bundle. Furthermore, by (7.6), A is the restriction of M to $\bar{A}$. Thus, in this case, proposition (7.4) implies that if $\bar{M}$ and $\bar{M}'$ are locally finite countable CW-complexes and M' a $\mathfrak{F}$-bundle, if X' is obtained by attaching the $\mathfrak{F}$-bundle M' to a $\mathfrak{F}$-family Y', then the pull-back $\mathfrak{F}$-family $\bar{f}^*X'$ is obtained by attaching the pull-back $\mathfrak{F}$-bundle M to the pull-back $\mathfrak{F}$-family $\tilde{f}^*Y'$.

8 Push-out of pull-back $\mathfrak{F}$-bundles

In this section we prove the bundle theorem (5.1). In case of the pull-back of $\mathfrak{F}$-bundles, the following properties hold.

(8.1) Lemma. *Let $\bar{Y}$ be a space, $\bar{h}\colon S^{n-1} \to \bar{Y}$ a map and $\bar{X} = \bar{Y} \cup_{\bar{h}} D^n$. Let $\bar{\Phi}\colon D^n \to \bar{X}$ denote the characteristic map. If $X \to \bar{X}$ is a $\mathfrak{F}$-bundle and Y is the restriction of X to $\bar{Y}$, then the following diagram is a push-out*

$$\begin{array}{ccc} \bar{h}^*Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \bar{\Phi}^*X & \longrightarrow & X \end{array} \tag{8.2}$$

Proof. Since the diagram commutes, one gets the map $t\colon P \to X$, where P denotes the push-out space $\bar{\Phi}^*X \cup Y$. It is easy to see that t is bijective and covers the identity of $\bar{X}$. Now we show that t is a local homeomorphism when restricted to the space over $\bar{U}$, where $\bar{U} \subset \bar{X}$ is an open set such that $X|\bar{U}$ is trivial. In fact, in this case the diagram (8.2) is reduced as

$$\begin{array}{ccc} \bar{h}^*(W \times (\bar{U} \cap \bar{Y})) & \longrightarrow & W \times (\bar{U} \cap \bar{Y}) \\ \downarrow & & \downarrow \\ W \times \bar{\Phi}^{-1}\bar{U} & \longrightarrow & W \times \bar{U}, \end{array}$$

which is a push-out since W is locally compact Hausdorff and $\bar{X} = \bar{Y} \cup_{\bar{h}} D^n$. Thus $P|\bar{U}$ and $X|\bar{U}$ are homeomorphic. But $P|\bar{U}$ and $X|\bar{U}$, when $\bar{U}$ ranges over all the trivializing neighborhoods of $\bar{X}$, are open covers of P and X respectively, with $t(P|\bar{U}) = X|\bar{U}$ for each $\bar{U}$. Hence t is an open map. *q.e.d.*

Since exactly the same argument can be applied to the attaching of a set $\bar{Z}$ of n-cells, we have the following generalization of lemma (8.1).

(8.3) Lemma. *Let $\bar{Y}$ be a space, $\bar{Z}$ a set and $\bar{h}\colon \bar{Z} \times S^{n-1} \to \bar{Y}$ an attaching map with $\bar{X} = \bar{Y} \cup_{\bar{h}} (\bar{Z} \times D^n)$. Let $\bar{\Phi}\colon \bar{Z} \times D^n \to \bar{X}$ denote the characteristic map. If $X \to \bar{X}$ is a $\mathfrak{F}$-bundle and Y is the restriction of X to $\bar{Y}$ then the following diagram is a push-out*

$$\begin{array}{ccc} \bar{h}^*Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \bar{\Phi}^*X & \longrightarrow & X. \end{array}$$

We recall now the following important property. Let $M_0 \subset M_1 \subset \cdots \subset M_n \subset \lim_{n\geq 0} M_n$ be a sequence of spaces, and $M = \lim\limits_{n\to\infty} M_n$. Then for every locally compact Hausdorff space W

$$\lim_{n\to\infty} (W \times M_n) = W \times \left(\lim_{n\to\infty} M_n\right) = W \times M. \tag{8.4}$$

(8.5) Lemma. *Let $(\bar M, \bar A)$ be a CW-pair; let $M \to \bar M$ be a $\mathfrak{F}$-bundle and A the restriction of M to $\bar A$. Then (M, A) is a $\mathfrak{F}$-CW-pair.*

Proof. Consider the filtration on skeleta

$$\bar A \subset \bar M_1 \subset \bar M_2 \subset \cdots \subset \bar M_n \subset \cdots \subset \bar M.$$

It induces a filtration on restrictions of the bundles $M \to \bar M$

$$A \subset M_1 \subset M_2 \subset \cdots \subset M_n \subset \cdots \subset M.$$

By lemma (8.3) for every $n \geq 0$ the diagram

$$\begin{tikzcd} \bar h_n^* M_{n-1} \arrow[r] \arrow[d] & M_{n-1} \arrow[d] \\ \bar\Phi_n^* M_n \arrow[r] & M_n, \end{tikzcd}$$

is a push-out, where $\bar h_n \colon \bar Z_n \times S^{n-1} \to \bar M_{n-1}$ is the attaching map on $\bar M_{n-1}$ and $\bar\Phi_n$ is the corresponding characteristic map. Since D^n is contractible, there exists a $\mathfrak{F}$-isomorphism of (pair of) bundles

$$(\bar\Phi_n^* M_n, \bar h_n^* M_{n-1}) \cong W \times (\bar Z_n \times D^n, \bar Z_n \times S^{n-1}) \tag{8.6}$$

for a suitable object $W \in \mathfrak{F}$. This means that M_n is obtained by attaching $W \times \bar Z_n \times D^n$ to M_{n-1} with as attaching map the composition of the map induced by pull-back $\bar h^*$ and the isomorphism of (8.6). Thus (M_n, A) there are skeleta $M_n \subset M$ such that

$$A \subset M_0 \subset M_1 \subset \ldots M.$$

To show that (M, A) is a relative $\mathfrak{F}$-complex (and hence a $\mathfrak{F}$-CW-pair) we need to show that $\lim\limits_{n\to\infty} M_n = M$. Since there is a continuous bijection $t \colon \lim\limits_{n\to\infty} M_n \to M$ covering the identity of $\bar M$, we need to show that M has the topology of the limit. It suffices to restrict M to the trivializing neighborhoods $\bar U \subset \bar M$, and this is true as a consequence of (8.4). *q.e.d.*

Lemma (8.5) implies the following corollary, by taking $A = \emptyset$. Let $\bar M$ be a CW-complex; let $M \to \bar M$ be a $\mathfrak{F}$-bundle. Then M is a $\mathfrak{F}$-complex. This is the second part of (5.1).

9 Push-out of function spaces

We now start the proof of the principal bundle theorem which is achieved in the next two sections.

Let V be an object of $\mathfrak{F}$. We recall that a family $\mathcal{K}$ of compact sets of V is termed *generating* if for every $\mathfrak{F}$-family Y the subsets $N_{K,U}$ with $K \in \mathcal{K}$ and U open in Y yield a sub-basis for the topology of Y^V.

(9.1) Definition. Let $Y \subset X$ a pair of $\mathfrak{F}$-families and a closed inclusion. We say that Y has the *N-neighborhood extension property in X* (NNEP) if for every $V \in \mathfrak{F}$ there exists a generating family $\mathcal{K}$ of compact subsets of V such that the following is true: let U be open in X^V and for $i = 1, \dots, l$ let $K_i \subset V$ be compact sets in $\mathcal{K}$ and $U_i \subset Y$ be open such that

$$\bigcap_{i=1}^{l} N_{K_i,U_i} \subset U \cap Y^V;$$

then there are l open subsets $U_i' \subset X$ such that $U_i' \cap Y = U_i$ and

$$\bigcap_{i=1}^{l} N_{K_i,U_i'} \subset U.$$

An important easy property of the sets $N_{K,U}$ is the following: if $U \subset Y$ and $h\colon A \to Y$ is a map, then

$$N_{K,h^{-1}U} = (h^V)^{-1} N_{K,U}.$$

(9.2) Lemma. *Let Z be a set and (M_z, A_z) be an NNEP-pair for every $z \in Z$. Then the coproduct of inclusions*

$$\coprod_{z\in Z} A_z \subset \coprod_{z \in Z} M_z$$

yields a NNEP-pair.

(9.3) Lemma. *Assume that A has the NNEP in M and that $h\colon A \to Y$ is a $\mathfrak{F}$-map to a $\mathfrak{F}$-family Y. Then Y has the NNEP in the push-out space $X = M \cup_h Y$.*

Proof. By lemma (2.5) $Y \to X$ is a closed inclusion. Let $\Phi\colon M \to X$ denote the characteristic map of the push-out. Let $U \subset X^V$ be open, and let

$K_i \subset V$ and $U_i \subset Y$ be compact subsets of a generating family $\mathcal{K}$ of V and open subsets such that

$$\bigcap_{i=1}^{l} N_{K_i,U_i} \subset U \cap Y^V$$

as in definition (9.1). The map Φ^V is continuous, hence $U'' = (\Phi^V)^{-1}U \subset M^V$ is open in M^V. Moreover, the intersection

$$(h^V)^{-1} \bigcap_{i=1}^{l} N_{K_i,U_i} = \bigcap_{i=1}^{l} N_{K_i,h^{-1}U_i}$$

is contained in U''. Since A has the NNEP in X, there are l open sets U_i'' in M such that $U_i'' \cap A = h^{-1}U_i$ and

$$\bigcap_{i=1}^{l} N_{K_i,U_i''} \subset U''.$$

Now consider the subsets $U_i' = U_i \bigcup \Phi(U_i'') \subset X$. Since $U_i' \cap Y = U_i$, $\Phi^{-1}U_i' = U_i''$ and X is the push-out of M and Y, each U_i' is open in X. Moreover, consider

$$f \in \bigcap_{i=1}^{l} N_{K_i,U_i'}.$$

If $f \in Y^V$ then

$$f \in \bigcap_{i=1}^{l} N_{K_i,U_i'} \cap Y^V = \bigcap_{i=1}^{l} N_{K_i,U_i' \cap Y} = \bigcap_{i=1}^{l} N_{K_i,U_i} \subset U \cap Y^V,$$

hence $f \in U$. On the other hand, if $f \notin Y^V$, then there is a unique $f'' \colon K \to M \setminus A$ such that $f = \Phi^V(f'')$. Since for every i $fK_i = hf''K_i \subset U_i'$, we have

$$f''K_i \subset \Phi^{-1}U_i' = U_i'',$$

thus

$$f'' \in \bigcap_{i=1}^{l} N_{K_i,U_i''} \subset U''.$$

Hence $f = \Phi^V(f'') \in \Phi^V U'' = \Phi^V (\Phi^V)^{-1}U = U$. Therefore

$$\bigcap_{i=1}^{l} N_{K_i,U_i'} \subset U,$$

and the proof is complete. *q.e.d.*

(9.4) Lemma. *Assume that (M, A) is a NNEP-pair, Y is a $\mathfrak{F}$-family and $h\colon A \to Y$ is a $\mathfrak{F}$-map. Let X be the push-out $X = M \cup_h Y$. Then the following diagram*

$$\begin{tikzcd} A^V \arrow[r, "h^V"] \arrow[d, tail] & Y^V \arrow[d, tail] \\ M^V \arrow[r, "\Phi^V"] & X^V \end{tikzcd}$$

is a push-out.

Proof. By lemma (9.3) (X, Y) is a NNEP-pair. Let $\mathcal{K}$ denote the generating family of compact sets of V. Let P denote the push-out $M^V \cup_{h^V} Y^V$. By the push-out property there is a continuous map $P \to X^V$. It is easy to see that it is bijective, so that, by identifying P and X^V, the only thing to prove is that if U is an open in the push-out topology, then it is open in the compact-open topology. This is true if and only if for every $f_0 \in U$ there is a subset $U' \subset X^V$ open in the compact-open topology such that $f_0 \subset U' \subset U$.

Consider first the case $f_0 \notin Y^V$. Then there exists a unique $f'_0 \in M^V \smallsetminus A^V$ such that $\Phi^V(f'_0) = f_0$. Since A^V is closed in M^V, f'_0 is contained in an open neighborhood $U''_0 \subset M^V$ such that $U''_0 \cap A^V = \emptyset$ and U''_0 is of type

$$\bigcap_{i=1}^{l} N_{K_i, U''_i}$$

for some $K_i \subset V$ and some U''_i in $M \smallsetminus A$. For each i the image $U'_i = \Phi U''_i$ is open in X, hence the set

$$\bigcap_{i=1}^{l} N_{K_i, U'_i}$$

is open in X^V, contains f_0 and is contained in U.

Now consider $f_0 \in Y^V$. The intersection $Y^V \cap U$ is open in Y^V and hence there is a neighborhood of f_0 in Y^V contained in $Y^V \cap U$. Because Y^V has the compact-open topology this means that there are l compact subsets $K_i \subset V$ and open sets $U_i \subset Y$ such that

$$f_0 \in \bigcap_{i=1}^{l} N_{K_i, U_i} \subset U, \tag{9.5}$$

where as above N_{K_i, U_i} denotes the set of all the maps in Y^V such that $f K_i \subset U_i$. Without loss of generality we can assume that K_i belongs to the generating family $\mathcal{K}$ for every $i = 1 \ldots l$.

Consider the sets

$$U'' = (\Phi^V)^{-1} U \subset M^V$$

and

$$\bigcap_{i=1}^{l} N_{K_i, h^{-1}U_i} \subset A^V \cap U''.$$

Since (M, A) is a NNEP-pair, by definition (9.1) there are open sets $U_i'' \subset M$ such that $U_i'' \cap A = h^{-1}U_i$ and

$$\bigcap_{i=1}^{l} N_{K_i, U_i''} \subset U''.$$

Let $U_i' = U_i \bigcup \Phi U_i''$ for every $i = 1, \ldots, l$. As in the proof of lemma (9.3), they are l open subsets of X with the property that $U_i' \cap Y = U_i$, hence $f_0 \in N_{K_i, U_i'}$ for every i. And, again as in the proof of lemma (9.3),

$$U' = \bigcap_{i=1}^{l} N_{K_i, U_i'} \subset U.$$

This concludes the proof. *q.e.d.*

(9.6) Lemma. *Consider a sequence of $\mathfrak{F}$-families*

$$X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots \subset X = \lim_{n\to\infty} X_n$$

such that for every $n \geq 1$ (X_n, X_{n-1}) is a NNEP-pair. Then (X, X_0) is a NNEP-pair and

$$X^V = \lim_{n\to\infty} X_n^V$$

Proof. Let $U \subset X^V$ be open, and $\bigcap_{i=1}^{l} N_{K_i, U_i^0} \subset U \cap X_0^V$ a subset of X_0^V, with $K_i \subset V$ compact containing bases and U_i^0 open in X_0. By induction, it is possible to define sequences of sets U_i^n open in X_n such that for every ≥ 1

$$U_i^n \cap X_{n-1} = U_i^{n-1}$$

$$\bigcap_{i=1}^{l} N_{K_i, U_i^n} \subset X_n^V \cap U.$$

Now, because $X = \lim_{n\to\infty} X_n$, the sets $U_i^\infty = \bigcup_{n=0}^\infty U_i^n$ are open in X, hence

$$\bigcap_{i=1}^{l} N_{K_i,U_i^\infty} \subset U.$$

is an open set. Moreover, by construction $U_i^\infty \cap X_0 = U_i^0$, therefore X_0 has the NNEP in X. The same construction shows that $X^V = \lim_{n\to\infty} X_n^V$. *q.e.d.*

10 NKC-categories

The crucial property of a NKC-category as defined in (6.5) is the next result.

(10.1) Proposition. *Assume that $\mathfrak{F}$ is NKC. Then for every $W \in \mathfrak{F}$ the pair $(W \times D^n, W \times S^{n-1})$ is a NNEP-pair.*

The proof of proposition (10.1) will take the rest of the section.

(10.2) Lemma. *Let R be a space and V and W objects of $\mathfrak{F}$. Then*

$$(W \times R)^V = W^V \times R.$$

Proof. Let pr_1 and pr_2 denote the projections onto the first and the second factor of $W \times R$. The map $\mathrm{pr}_2\epsilon_0 \colon (W \times R)^V \to R$ sending $f \colon V \to W \times R$ to $\mathrm{pr}_2 f(0) \in R$ is continuous. Furthermore, the map $\mathrm{pr}_1^V \colon (W \times R)^V \to W^V$ is continuous. Therefore the map $F = (\mathrm{pr}_1^V, \mathrm{pr}_2\epsilon_0) \colon (W \times R)^V \to W^V \times R$ is continuous. Consider now the function $G \colon W^V \times R \to (W \times R)^V$ defined by $G(f, r)(v) = (f(v), r)$ for every $v \in V$ and every $r \in R$. It is the adjoint of the map $V \times W^V \times R \to W \times R$ defined by $(v, f, r) \mapsto (f(v), r)$, which is continuous since the evaluation map is continuous. Therefore G is continuous. It is readily seen that $GF = 1$ and $FG = 1$. *q.e.d.*

Let $D_\epsilon^n = \{x \in D^n : |x| > 1 - \epsilon\}$, for any ϵ small, and let $\rho_\epsilon \colon D_\epsilon^n \to S^{n-1}$ be the retraction defined by $x \mapsto \frac{x}{|x|}$. With an abuse of notation we use the same symbol for the induced retraction $\rho_\epsilon = 1_W \times \rho_\epsilon \colon W \times D_\epsilon^n \to W \times S^{n-1}$. For every $W \in \mathfrak{F}$ there is an induced retraction

$$\rho_\epsilon^V \colon (W \times D_\epsilon^n)^V \to (W \times S^{n-1})^V.$$

(10.3) Lemma. *If C is a compact in $(W \times S^{n-1})^V$, and C is contained in an open set A of $(W \times D^n)^V$, then there exists $\epsilon > 0$ such that $(\rho_\epsilon^V)^{-1} C \subset A$.*

Proof. If ϵ is small, we have $D^n_\epsilon = S^{n-1} \times [0,\epsilon)$, and hence $(W \times D^n_\epsilon)^V = (W \times S^{n-1})^V \times [0,\epsilon)$ and ρ^V_ϵ is the projection onto the first factor. Each $x \in C$ is a point of A, therefore there exists an open neighborhood of x of type $O_x \times [0,\epsilon_x)$ contained in A. Being C compact there is a finite number of points x such that C is covered by $O_x \times [0,\epsilon_x)$. Therefore there exists a $\epsilon > 0$ such that $\cup_x O_x \times [0,\epsilon)$ is an open subset of A containing C and thus $(\rho^V_\epsilon)^{-1} C \subset A$. *q.e.d.*

Now we can start the proof of proposition (10.1). Let U be an open subset $U \subset (W \times D^n)^V$. Let $\mathcal{K}$ be the generating family of compact subsets of V of the NKC property. Let $K_i \subset V$ be compact subsets in $\mathcal{K}$ and $U_i \subset W \times S^{n-1}$ open subsets, such that

$$\bigcap_{i=1}^{l} N_{K_i,U_i} \subset U \cap (W \times S^{n-1})^V. \tag{10.4}$$

By lemma (7.14) there is a sequence of open sets $A^k_i \subset W \times S^{n-1}$ with $k \geq 1$ such that, for every k, $A^k_i \subset A^{k+1}_i$,

$$\bigcup_{k\geq 1} A^k_i = U_i \tag{10.5}$$

and the closure $\bar{A}^k_i$ is a compact subset of U_i.

(10.6) Lemma. *For every $i = 1,\ldots,l$ the space $N_{K_i,\bar{A}^k_i}$ is compact.*

Proof. It is closed, because it is equal to the intersection of all the spaces $N_{\{x\},\bar{A}^k_i}$, with $x \in K_i$, and they are closed because pre-images of the closed set $\bar{A}^k_i$ under the evaluation map $(W \times S^{n-1})^V \times \{x\} \to W \times S^{n-1}$, which is continuous because of lemma (6.11). Now we prove that it is compact. Consider the projection $p\colon W \times S^{n-1} \to W$ onto the first factor. Since $\bar{A}^k_i$ is compact, its image $p\bar{A}^k_i$ is compact in W. Moreover, it is easy to see that

$$p^V(N_{K_i,\bar{A}^k_i}) \subset N_{K_i,p\bar{A}^k_i},$$

and hence that

$$N_{K_i,\bar{A}^k_i} \subset (p^V)^{-1} N_{K_i,p\bar{A}^k_i} = N_{K_i,p\bar{A}^k_i} \times S^{n-1}.$$

Now, since K_i and $p\bar{A}^k_i$ are compact and $\mathfrak{F}$ is an NKC-category, $N_{K_i,p\bar{A}^k_i}$ is compact. Therefore $N_{K_i,\bar{A}^k_i}$ is a closed subset of a compact space, and hence compact. *q.e.d.*

Consider for every $k \geq 1$ the set

$$C_k = \bigcap_{i=1}^{l} N_{K_i, A_i^k} \subset (W \times S^{n-1})^V. \tag{10.7}$$

It is open, by definition of compact-open topology. Let C_∞ denote the union $\bigcup_{k \geq 1} C_k$.

(10.8) Lemma. *We have that*

$$C_\infty = \bigcap_{i=1}^{l} N_{K_i, U_i} \subset (W \times S^{n-1})^V \cap U.$$

Proof. A $\mathfrak{F}$-map $f \in (W \times S^{n-1})^V$ belongs to C_∞ if and only if there is $k \geq 1$ such that $f \in C_k$, that is if and only if there is $k \geq 1$ such that $fK_i \subset A_i^k$ for every $i = 1, \dots, l$. Of course this implies that $fK_i \subset U_i$ for every i. On the other hand, if $fK_i \subset U_i$, then by equation (10.5) there is $k_i \geq 1$ such that $fK_i \subset A_i^k$ for every $k \geq k_i$. This implies that there is k (take the maximum of all k_i) such that $fK_i \subset A_i^k$ for every $i = 1, \dots, l$. *q.e.d.*

(10.9) Lemma. *The closure $\bar{C}_k \subset (W \times S^{n-1})^V$ is compact and contained in C_∞.*

Proof. Because of its definition $\bar{C}_k \subset \cap_{i=1}^{l} \overline{N_{K_i, A_i^k}}$. Because for every $i = 1, \dots, l$ the closure $\overline{N_{K_i, A_i^k}}$ is contained in $N_{K_i, \bar{A}_i^k}$, which is compact by lemma (10.6), we have that $\bar{C}_k$ is compact. Furthermore, by assumption $\bar{A}_i^k$ is contained in U_i, and hence $N_{K_i, \bar{A}_i^k}$ is contained in N_{K_i, U_i}. Therefore the closure of C_k is contained in C_∞. *q.e.d.*

Because of lemma (10.9) and lemma (10.8) the compact set $\bar{C}_k$ is contained in U in $(W \times D^n)^V$. By lemma (10.3) it is possible to find an $\epsilon_k > 0$ such that

$$(\rho_{\epsilon_k}^V)^{-1} C_k \subset U. \tag{10.10}$$

Without loss of generality we can assume that the sequence ϵ_k, with $k \geq 1$, is decreasing.

Now consider for $i = 1, \dots, l$ the following sets in $(W \times D^n)^V$:

$$U_i' = \bigcup_{k \geq 1} \rho_{\epsilon_k}^{-1} A_i^k. \tag{10.11}$$

Since it is the union of open sets, for every $i = 1, \ldots, l$, the set U'_i is open in $W \times D^n$. Hence the set

$$U' = \bigcap_{i=1}^{l} N_{K_i, U'_i} \subset (W \times D^n)^V, \tag{10.12}$$

is open.

(10.13) Lemma. *For every $i = 1, \ldots, l$, $U'_i \cap (W \times S^{n-1}) = U_i$.*

Proof. Because of (10.11), it suffices to prove that for every i and every $k \geq 1$

$$\rho_{\epsilon_k}^{-1} A_i^k \cap (W \times S^{n-1}) \subset U_i,$$

and this is true because $\rho_{\epsilon_k}^{-1} A_i^k \cap (W \times S^{n-1}) = A_i^k$. *q.e.d.*

(10.14) Lemma. $U' \subset U$.

Proof. If $f \in U' \cap (W \times S^{n-1})^V$, then $f K_i \subset (W \times S^{n-1}) \cap U'_i$, and by lemma (10.13) this implies $f K_i \subset U_i$. Thus, by (10.4), $f_0 \in U$. On the other hand, assume that $f \in (W \times e^n)^V$. For every $i = 1, \ldots, l$ the image of the compact $f(K_i)$ is in U'_i and hence, because of (10.11), for every i there exists an integer $k_i \geq 1$ (depending on i) such $f'(K_i) \subset \rho_{\epsilon_{k_i}}^{-1} A_i^{k_i}$. In particular, for every $i = 1, \ldots, l$,

$$f'(K_i) \subset W \times D^n_{\epsilon_{k_i}},$$

and thus, because f is a $\mathfrak{F}$-map and ϵ_k is a decreasing sequence,

$$f'(V) \subset W \times D^n_{\epsilon_m},$$

where m denotes the maximum of $k_1, \ldots, k_l$. Moreover, the sequence of spaces A_i^k is increasing with $k \geq 1$, and therefore $A_i^{k_i} \subset A_i^m$ for every $i = 1, \ldots, l$. Thus for every $i = 1, \ldots, l$,

$$f'(K_i) \subset \rho_{\epsilon_m}^{-1} A_i^m.$$

But this means that for every i

$$f \in N_{K_i, \rho_{\epsilon_m}^{-1} A_i^m} = \left(\rho_{\epsilon_m}^V\right)^{-1} N_{K_i, A_i^m},$$

hence that

$$f \in (\rho_{\epsilon_m}^V)^{-1} C_m.$$

Because of equation (10.10) this implies that $f \in U$. As claimed, we have proved that $U' \subset U$. *q.e.d.*

This is the end of the proof of proposition (10.1). We can now draw the consequences which are needed.

(10.15) Lemma. *Let $\mathfrak{F}$ be a NKC-category. Let Y be a $\mathfrak{F}$-family, Z a $\mathfrak{F}$-set and $h\colon Z \times S^{n-1} \to Y$ a $\mathfrak{F}$-map, with $n \geq 1$. If X is a family obtained as a push-out*

$$\begin{tikzcd} Z \times S^{n-1} \arrow[r, "h"] \arrow[d, tail] & Y \arrow[d, tail] \\ Z \times D^n \arrow[r, "\Phi"] & X \end{tikzcd}$$

then the following diagram

$$\begin{tikzcd} Z^V \times S^{n-1} \arrow[r, equal] \arrow[d, tail] & (Z \times S^{n-1})^V \arrow[r, "h^V"] \arrow[d, tail] & Y^V \arrow[d] \\ Z^V \times D^n \arrow[r, equal] & (Z \times D^n)^V \arrow[r, "\Phi^V"] & X^V \end{tikzcd}$$

is a push-out, and Y has the N-neighborhood extension property in X.

Proof. By lemma (9.2) and proposition (10.1) if Z is a $\mathfrak{F}$-set the pair $(Z \times D^n, Z \times S^{n-1})$ is a NNEP-pair. Hence, by lemma (9.3) and lemma (9.4) X^V is the push-out of $(Z \times D^n)^V$ and Y^V and Y has the N-neighborhood extension property in X. *q.e.d.*

(10.16) Corollary. *Let (X, D) be a relative $\mathfrak{F}$-complex where $\mathfrak{F}$ is a NKC-category. Let $\Phi : Z_n \times D^n \to X$ be its characteristic map of n-cells and let h_n denote the n-attaching map of X, i.e. the restriction of Φ to $Z_n \times S^{n-1}$. For every $n \geq 1$ and every $V \in \mathfrak{F}$ the diagram*

$$\begin{tikzcd} Z_n^V \times S^{n-1} \arrow[r, equal] \arrow[d, tail] & (Z_n \times S^{n-1})^V \arrow[r, "h_n^V"] \arrow[d, tail] & X_{n-1}^V \arrow[d] \\ Z_n^V \times D^n \arrow[r, equal] & (Z_n \times D^n)^V \arrow[r, "\Phi^V"] & X_n^V \end{tikzcd}$$

is a push-out (in **Top***). Moreover,*

$$X^V = \lim_{n \to \infty} X_n^V.$$

Proof. The first part is a direct consequence of lemma (10.15). Then, by applying (9.6) we obtain that (X, X_n) is a NNEP-pair for every n and that $X^V = \lim_{n\to\infty} X_n^V$. *q.e.d.*

(10.17) Corollary. *Let (X, D) be a relative $\mathfrak{F}$-complex (or a $\mathfrak{F}$-CW-pair) with $\mathfrak{F}$ NKC. Then (X, D) is a NNEP-pair.*

(10.18) Corollary. *Let $\mathfrak{F}$ be NKC. Let (M, A) be a $\mathfrak{F}$-CW-pair, Y a $\mathfrak{F}$-family and $h \colon A \to Y$ a $\mathfrak{F}$-map. If X is the push-out $X = M \cup_h Y$ then (X, Y) is a NNEP-pair.*

Proof. By corollary (10.17) (M, A) is a NNEP-pair, hence by lemma (9.3) and lemma (9.4) (X, Y) is a NNEP-pair. *q.e.d.*

11 Examples of NKC categories

(11.1) Proposition. *If the fibre functor F is faithful and the Hom-sets* $\hom_{\mathfrak{F}}(V, W)$ *are compact spaces then $\mathfrak{F}$ is a NKC category.*

The proof of the previous proposition is simple. The interesting fact is that there are NKC structure categories with non-compact hom-sets, as a consequence of the following proposition.

(11.2) Proposition. *If $\mathfrak{F}$ is a closed subcategory of the category* **Vect** *of finitely dimensional $\mathbb{R}$-vector spaces and linear maps, then $\mathfrak{F}$ has the NKC property.*

Proof. See [4], lemma (4.2). *q.e.d.*

This result yields the principal stratified bundle associated to a $\mathfrak{F}$-stratified vector bundle.

References

[1] Atiyah, M. F. *K-theory.* W. A. Benjamin, Inc., New York-Amsterdam, 1967.

[2] Baas, N. A. On bordism theory of manifolds with singularities. *Math. Scand. 33* (1973), 279–302 (1974).

[3] Baues, H.-J. *Combinatorial foundation of homology and homotopy.* Springer-Verlag, Berlin, 1999.

[4] Baues, H.-J., and Ferrario, D. L. K-theory of stratified vector bundles, 2002. Preprint.

[5] Botvinnik, B. I. *Manifolds with singularities and the Adams-Novikov spectral sequence.* Cambridge University Press, Cambridge, 1992.

[6] Davis, J. F., and Lück, W. Spaces over a category and assembly maps in isomorphism conjectures in K- and L-theory. *K-Theory 15*, 3 (1998), 201–252.

[7] Davis, M. Smooth G-manifolds as collections of fiber bundles. *Pacific J. Math. 77*, 2 (1978), 315–363.

[8] Dugundji, J. *Topology.* Allyn and Bacon Inc., Boston, Mass., 1966.

[9] Gray, B. *Homotopy theory.* Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975. An introduction to algebraic topology, Pure and Applied Mathematics, Vol. 64.

[10] Husemoller, D. *Fibre bundles.* McGraw-Hill Book Co., New York, 1966.

[11] Kreck, M. Differentiable algebraic topology, 2002. Pre-print.

[12] MacLane, S. *Categories for the working mathematician.* Springer-Verlag, New York, 1971. Graduate Texts in Mathematics, Vol. 5.

[13] Rudyak, Y. B. *On Thom spectra, orientability, and cobordism.* Springer-Verlag, Berlin, 1998. With a foreword by Haynes Miller.

[14] Steenrod, N. *The Topology of Fibre Bundles.* Princeton University Press, Princeton, N. J., 1951.

[15] Sullivan, D. On the Hauptvermutung for manifolds. *Bull. Amer. Math. Soc. 73* (1967), 598–600.

[16] tom Dieck, T. *Transformation groups.* Walter de Gruyter & Co., Berlin, 1987.

[17] Vershinin, V. V. *Cobordisms and spectral sequences.* American Mathematical Society, Providence, RI, 1993. Translated from the Russian manuscript by M. Farber, Translation edited by David Louvish.