A local Martinelli-Bochner formula on hypersurfaces

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1 Introduction

Let M be an oriented real hypersurface of class C^2 in $\mathbb{C}^n$, i.e. $M = \{z \in \theta : \varrho(z) = 0\}$, where θ is an open subset of $\mathbb{C}^n$ and ϱ is a real C^2 function on θ with $d\varrho(z) \neq 0$ for all $z \in \theta$. For $z \in M$ and $\xi \in \mathbb{C}^n$, we denote by $\delta(\xi, z)$ the Euclidean distance between ξ and the complex tangent plane of M at z. The aim of this paper is to prove the following theorem:

Theorem 1.1 *Suppose, for some $z_0 \in M$, the restriction of the Levi form of ϱ at z_0 to the complex tangent plane of M at z_0 has at least one positive and at least one negative eigenvalue. Then there exist an open neighbourhood $M_0 \subseteq M$ of z_0 and a continuous differential form $K(z,\xi)$ defined and continuous for all $(z,\xi) \in \bar{M}_0 \times \bar{M}_0$ with $z \neq \xi$ such that:*

(i) $K(z,\xi)$ is of degree zero in z and of bidegree $(n, n-2)$ in ξ.

(ii) $d_\xi K(z,\xi) = 0$ for all $(z,\xi) \in M_0 \times M_0$ with $z \neq \xi$.

(iii) There is a constant $C > 0$ such that

$$||K(z,\xi)|| \leq C \frac{1 + |\ln|\xi - z||}{(\delta(\xi,z) + |\xi - z|^2)|\xi - z|^{2n-3}} \tag{1}$$

for all $\xi, z \in M_0$ with $\xi \neq z$.

(iv) For each $0 < \alpha < 1$, the coefficients of $K(z,\xi)$ are of class $C^{\alpha,1/2}_{z,\xi}$ for all $(z,\xi) \in M_0 \times M_0$ with $z \neq \xi$ (for the definition of $C^{\alpha,1/2}_{z,\xi}$ cf. the end of Section 2).

(v) Let $\Omega \subset\subset M_0$ be a domain with piecewise C^1 boundary. If f is a continuous function on $\bar{\Omega}$ such that $df(\xi) \wedge d\xi_1 \wedge \ldots \wedge d\xi_n$ is also continuous on $\bar{\Omega}$ then

$$f(z) = \int_{\xi \in b\Omega} f(\xi) K(z,\xi) - \int_{\xi \in \Omega} df(\xi) \wedge K(z,\xi) \tag{2}$$

for all $z \in \Omega$.

Remark 1.2 From estimate (1) it follows that $||K(z,\xi)||$ is integrable with respect to ξ and z. More precisely, it is easy to see that the following estimates hold: Denote by $d\lambda$ the Euclidean volume form on M. Then there is a constant $C > 0$ such that

$$\int\limits_{\substack{\xi\in M_0\\ |\xi-z|<\varepsilon}} ||K(z,\xi)||\, d\lambda(\xi) \leq C\varepsilon(1+|\ln\varepsilon|^2) \tag{3}$$

for all $z \in \bar{M}_0$ and $\varepsilon > 0$, and

$$\int\limits_{\substack{z\in M_0\\ |z-\xi|<\varepsilon}} ||K(z,\xi)||\, d\lambda(z) \leq C\varepsilon(1+|\ln\varepsilon|^2) \tag{4}$$

for all $\xi \in \bar{M}_0$ and $\varepsilon > 0$.

To obtain the kernel $K(z,\xi)$ in Theorem 1.1 we proceed as follows: Consider the Martinelli–Bochner kernel

$$B(z,\zeta) := \frac{(n-1)!}{(2\pi i)^n}\sum_{j=1}^{n}(-1)^{j+1}\frac{\bar{\zeta}_j-\bar{z}_j}{|\zeta-z|^{2n}}d\bar{\zeta}_1\wedge ..\hat{j}..\wedge d\bar{\zeta}_n\wedge d\zeta_1\wedge ...\wedge d\zeta_n \tag{5}$$

and a sufficiently small open ball $U \subseteq \mathbb{C}^n$ centered at z_0. Set $U_+ := \{\zeta \in U : \varrho(\zeta) < 0\}$ and $U_- := \{\zeta \in U : \varrho(\zeta) > 0\}$. Then, in view of the hypothesis on the Levi form of ϱ, it follows from the Andreotti–Grauert theory that, for fixed $z \in M$, one can solve the equations

$$\bar{\partial}K_+(z,\cdot) = -B(z,\cdot) \quad \text{on} \quad U_+$$

and

$$\bar{\partial}K_-(z,\cdot) = -B(z,\cdot) \quad \text{on} \quad U_-.$$

We prove that this can be done with appropriate uniform estimates so that $K_+(z,\xi)$ and $K_-(z,\xi)$ extend to $(U\cap M)\setminus\{z\}$ and $K(z,\xi) := K_+(z,\xi) - K_-(z,\xi)$ has the required properties. For that we use a version of the classical integral operators constructed by GRAUERT/LIEB [G/L], HENKIN [H 1] and W. FISCHER/LIEB [WF/L].

Formula (2) is an analogon of the Martinelli-Bochner formula in $\mathbb{C}^n$. At the end of this paper (Section 6) we want to show that this analogy extends also to some of the applications of the Martinelli–Bochner formula: using the kernel $K(z,\xi)$, we prove strengthened versions of some of the results on the tangential Cauchy–Riemann equation obtained by HENKIN in [H 2] and [H 3] (see the regularity theorems 6.6 and 6.8, the solvability theorem 6.10 for (0,1)-currents with small support, and the Hartogs–Bochner extension theorem 6.11).

2 Preliminaries

Let $K \subset\subset \mathbb{C}^n$ be a compact set. Then $C^0(K)$ is the Banach space of all continuous complex functions on K. For $0 < \alpha < 1$, $C^\alpha(K)$ is the Banach space of all complex functions which are Hölder continuous with exponent α on K. The norm in $C^\alpha(K)$, $0 \leq$

$\alpha < 1$ will be denoted by $||\cdot||_{\alpha,K}$. That means $||\cdot||_{0,K}$ is the max-norm and for $0 < \alpha < 1$, $||\cdot||_{\alpha,K}$ is the Hölder norm with exponent α.

Let $D \subset\subset \mathbb{C}^n$ be a domain and $0 \le \alpha < 1$. Then $C^\alpha_*(\bar{D})$ is the Banach space of differential forms whose coefficients belong to $C^\alpha(\bar{D})$. The norm in $C^\alpha_*(\bar{D})$ will be denoted by $||\cdot||_{\alpha,\bar{D}}$. By $C^\alpha_{(s,r)}(\bar{D})$ we denote the subspace of forms in $C^\alpha_*(\bar{D})$ which are of bidegree (s,r). By $L^1_*(D)$ we denote the Banach space of all differential forms whose coefficients are integrable on D. The norm in $L^1_*(D)$ will be denoted by $||\cdot||_{L^1(D)}$ and $L^1_{(s,r)}(D)$ is the subspace of all forms in $L^1_*(D)$ which are of bidegree (s,r).

Proposition 2.1 *If $A, B \subset\subset \mathbb{C}^n$ are two compact sets, $f(z,\xi)$ is a complex function defined for $(z,\xi) \in A \times B$ and $0 \le \alpha, \beta < 1$ then it is easy to see that the following two conditions are equivalent:*

(i) $f(z,\cdot) \in C^\beta(B)$ for all $z \in A$ and the assignment $A \ni z \to f(z,\cdot)$ is Hölder continuous with exponent α as a map with values in $C^\beta(B)$.

(ii) $f(\cdot,\xi) \in C^\alpha(A)$ for all $\xi \in B$ and the assignment $B \ni \xi \to f(\cdot,\xi)$ is Hölder continuous with exponent β as a map with values in $C^\alpha(A)$.

Let Z be an arbitrary subset of $\mathbb{C}^n \times \mathbb{C}^n$, $f(z,\xi)$ a complex function defined for $(z,\xi) \in Z$ and let $0 \le \alpha, \beta < 1$. Then we say that $f(z,\xi)$ is of class $C^{\alpha,\beta}_{z,\xi}$ on Z if for each pair of compact sets $A, B \subseteq \mathbb{C}^n$ with $A \times B \subseteq Z$ the both equivalent conditions (i) and (ii) in Proposition 2.1 are fulfilled.

3 Local q-convex C^2 domains

If φ is a real C^2 function in some neighbourhood of a point $z \in \mathbb{C}^n$ then we denote by $L_\varphi(z)$ the Levi form and by $H_\varphi(z)$ the Hessian form of φ at z. That means

$$L_\varphi(z)t := \sum_{j,k=1}^{n} \frac{\partial^2 \varphi(z)}{\partial z_j \partial \bar{z}_k} t_j \bar{t}_k \quad \text{for} \quad t \in \mathbb{C}^n$$

and

$$H_\varphi(z)t := \frac{1}{2} \sum_{\nu,\mu=1}^{2n} \frac{\partial^2 \varphi(z)}{\partial x_\nu \partial x_\mu} x_\nu(t) x_\mu(t) \quad \text{for} \quad t \in \mathbb{C}^n$$

where $x_1, ..., x_{2n}$ are the real coordinates on $\mathbb{C}^n$ with $z_j = x_j(z) + i x_{j+n}(z)$ if $z = (z_1, ..., z_n) \in \mathbb{C}^n$.

Definition. Let $0 \le q \le n-1$ be an integer.

(i) If $G \subset\subset \mathbb{C}^n$ is a C^2 domain then we say that G is strictly convex with respect to the real coordinates of $z_1, ..., z_{q+1}$ if there exists a real C^2 function ϱ in a neighbourhood $U_{\bar{G}}$ of $\bar{G}$ such that $G = \{z \in U_{\bar{G}} : \varrho(z) < 0\}$ and $d\varrho(z) \neq 0$ for $z \in bG$ and ϱ is strictly convex with respect to the real coordinates of $z_1, ..., z_{q+1}$, i.e.

$$H_\varrho(\zeta)t > 0 \tag{6}$$

for all $\zeta \in U_{\bar{G}}$ and $t \in \mathbb{C}^n$ with $t_{q+2} = ... = t_n = 0$.

(ii) A local q-convex C^2 domain is a C^2 domain $D \subset\subset \mathbb{C}^n$ for which there exists a biholomorphic map h from a neighbourhood of $\bar{D}$ onto an open set in $\mathbb{C}^n$ such that $h(D)$ is strictly convex with respect to the real coordinates of $z_1, ..., z_{q+1}$.

Lemma 3.1 *Let $0 \leq q \leq n-1$ be an integer. Further let $\theta \subseteq \mathbb{C}^n$ be an open set, ϱ a real C^2 function on θ with $d\varrho(z) \neq 0$ for $z \in \theta$ and let $M = \{z \in \theta : \varrho(z) = 0\}$. Set $\theta_+ = \{z \in \theta : \varrho(z) < 0\}$ and suppose that for some $z_o \in M$ the restriction of $L_\varrho(z_0)$ to the complex tangent plane of M at z_0 has at least q positiv eigenvalues. Then there exist a local q-convex C^2 domain D and a neighbourhood U of z_0 such that*

$$U \cap \theta_+ \subseteq D \subseteq \theta_+. \tag{7}$$

Proof. Choose a real C^2 function φ on θ with $d\varphi(z) \neq 0$ for $z \in \theta$ and $\theta_+ = \{z \in \theta : \varphi(z) < 0\}$ such that $L_\varphi(z_0)$ has at least $q+1$ positive eigenvalues (see Proposition 5.8 in [H/Le 2]). Then the restriction of φ to a certain $(q+1)$-dimensional complex submanifold through z_0 is strictly plurisubharmonic and non-critical. Therefore in view of the Narasimhan lemma (see Theorem 1.4.14 in [H/Le 1]) we may assume that φ is strictly convex with respect to the real coordinates of $z_1, ..., z_{q+1}$. Fix $r > 0$ so small that for the ball $B_r(z_0) := \{z \in \mathbb{C}^n : |z - z_0| < r\}$ we have $\bar{B}_r(z_0) \subseteq \theta$, $d\varphi(z) \neq 0$ for all $z \in \bar{B}_r(z_0)$ and the intersection of $bB_r(z_0)$ and the surface $\{\varphi = 0\}$ is transversal.

Now let $\beta > 0$, $\tau(z) := \max_\beta(\varphi(z), |z - z_0|^2 - r^2)$ and $D := \{z \in \theta : \tau(z) < 0\}$ where $\max_\beta(\cdot,\cdot)$ is the smoothing of the function $\max(\cdot,\cdot)$ from Definition 4.12 in [H/Le 2]. By Lemma 4.13 in [H/Le 2] $\max_\beta(\cdot,\cdot)$ is convex and has non negative first order derivatives at least one of which is positive. Therefore τ is strictly convex with respect to the real coordinates of $z_1, ..., z_{q+1}$ for any $\beta > 0$. Moreover by Lemma 4.13 in [H/Le 2]

$$\max(t_1, t_2) \leq \max_\beta(t_1, t_2) \leq \max(t_1, t_2) + \beta$$

and

$$\max(t_1, t_2) = \max_\beta(t_1, t_2) \quad \text{for} \quad |t_1 - t_2| \geq \beta.$$

Therefore it is clear that for each neighbourhood $U \subset\subset B_r(z_0)$ of z_0 (7) will be satisfied if β is sufficiently small.

It remains to prove that $d\tau(z) \neq 0$ for all $z \in bD$ if β is sufficiently small. For that first we observe that $d\tau$ is a non-trivial linear combination of $d\varphi(z)$ and $d|z - z_0|^2$ (see the proof of Lemma 4.13 in [H/Le 2]). Since the intersection of $bB_r(z_0)$ and $\{\varphi = 0\}$ is transversal this implies that for some neighbourhood V of this intersection $d\tau(z) \neq 0$ for all $z \in V$. Finally we observe that since $\max_\beta(t_1, t_2) = \max(t_1, t_2)$ if $|t_1 - t_2| \geq \beta$ we can choose β so small that for all z in some neighbourhood of $bD \backslash V$ either $\tau(z) = \varphi(z)$ or $\tau(z) = |z - z_0|^2 - r^2$. ■

Lemma 3.2 *Let $G \subset\subset \mathbb{C}^n$ be a C^2 domain which is strictly convex with respect to the real coordinates of $z_1, ..., z_{q+1}$. Let $0 \leq q \leq n-1$ and let $\varrho : U_{\bar{G}} \to \mathbb{R}$ be as in part (i) of the Definition. Further let $\delta > 0$ be so small that the neighbourhood*

$$V_{\bar{G}} := \{z \in U_{\bar{G}} : \varrho(z) < \delta\}$$

of $\bar{G}$ is relatively compact in $U_{\tilde{G}}$. Then there exist constants $\alpha, A > 0$ such that

$$2\text{Re}\sum_{j=1}^{n}\frac{\partial\varrho(\zeta)}{\partial\zeta_j}(\zeta_j - z_j) + A\sum_{j=q+1}^{n}|\zeta_j - z_j|^2 \geq \varrho(\zeta) - \varrho(z) + \alpha|\zeta - z|^2 \qquad (8)$$

for all $z, \zeta \in V_{\tilde{G}}$.

Proof. Set $t' = (t_1, ..., t_{q+1}, 0, ..., 0)$ and $t'' = (0, ..., 0, t_{q+2}, ..., t_n)$ if $t \in \mathbb{C}^n$. Then by (6) there is a constant $\beta > 0$ such that

$$H_\varrho(\zeta)t' \geq 3\beta|t'|^2 \qquad (9)$$

for all $\zeta \in \bar{V}_{\tilde{G}}$ and $t \in \mathbb{C}^n$. Using the inequality $2ab = 2(\varepsilon a)(b/\varepsilon) \leq \varepsilon^2 a^2 + b^2/\varepsilon^2$ we can choose a constant $C > 0$ such that

$$|H_\varrho(\zeta)t - H_\varrho(\zeta)t'| \leq \beta|t'|^2 + (C - 2\beta)|t''|^2 \qquad (10)$$

for $\zeta \in \bar{V}_{\tilde{G}}$ and $t \in \mathbb{C}^n$. Since by Taylor's theorem

$$2\text{Re}\sum_{j=1}^{n}\frac{\partial\varrho(\zeta)}{\partial\zeta_j}(\zeta_j - z_j) = \varrho(\zeta) - \varrho(z) + H_\varrho(\zeta)(\zeta - z) + o(|\zeta - z|^2)$$

it follows from (9) and (10) that for some $\varepsilon > 0$ we have the estimate

$$2\text{Re}\sum_{j=1}^{n}\frac{\partial\varrho(\zeta)}{\partial\zeta_j}(\zeta_j - z_j) + C|\zeta'' - z''|^2 \geq \varrho(\zeta) - \varrho(z) + \beta|\zeta - z|^2 \qquad (11)$$

if $z, \zeta \in \bar{V}_{\tilde{G}}$ with $|\zeta - z| \leq \varepsilon$.

Now let $z, \zeta \in \bar{V}_{\tilde{G}}$ with $|\zeta - z| \geq \varepsilon$ and $\zeta'' = z''$. Set

$$z^\varepsilon = (1 - \frac{\varepsilon}{|\zeta - z|})\zeta + \frac{\varepsilon}{|\zeta - z|}z.$$

Since ϱ is strictly convex with respect to the real coordinates of $z_1, ..., z_q$ we get $z^\varepsilon \in V_{\tilde{G}}$ and

$$\varrho(z^\varepsilon) \leq (1 - \frac{\varepsilon}{|\zeta - z|})\varrho(\zeta) + \frac{\varepsilon}{|\zeta - z|}\varrho(z)$$

and since $|\zeta - z^\varepsilon| = \varepsilon$ it follows from (11) that

$$\begin{aligned} 2\text{Re}\sum_{j=1}^{n}\frac{\partial\varrho(\zeta)}{\partial\zeta_j}(\zeta_j - z_j) &\geq \frac{|\zeta - z|}{\varepsilon}(\varrho(\zeta) - \varrho(z^\varepsilon) + \beta\varepsilon^2) \\ &\geq \varrho(\zeta) - \varrho(z) + \beta\varepsilon|\zeta - z|. \end{aligned}$$

Hence we can find $\delta > 0$ so small that

$$2\text{Re}\sum_{j=1}^{n}\frac{\partial\varrho(\zeta)}{\partial\zeta_j}(\zeta_j - z_j) \geq \varrho(\zeta) - \varrho(z) + \frac{\beta\varepsilon}{2}|\zeta - z|$$

for all $z,\zeta \in \bar{V}_{\bar{G}}$ with $|\zeta - z| \geq \varepsilon$ and $|\zeta'' - z''| \leq \delta$. Clearly this implies that for sufficiently large constants $B > 0$ we have

$$2\mathrm{Re}\sum_{j=1}^{n}\frac{\partial\varrho(\zeta)}{\partial\zeta_j}(\zeta_j - z_j) + B|\zeta'' - z''|^2 \geq \varrho(\zeta) - \varrho(z) + \frac{\beta\varepsilon}{2}|\zeta - z| \tag{12}$$

for all $z,\zeta \in \bar{V}_{\bar{G}}$ with $|\zeta - z| \geq \varepsilon$. (8) now follows from (12) and (11) if we set $A = \max(C,B)$ and $\alpha = \min\limits_{\substack{z,\zeta\in \bar{V}_{\bar{G}} \\ |\zeta - z|\geq\varepsilon}} \frac{\beta\varepsilon}{2|\zeta - z|}$. ■

4 Certain new estimates for $\bar{\partial}$

In this section q is an integer with $0 \leq q \leq n-1$ and $D \subset\subset \mathbb{C}^n$ is a local q-convex C^2 domain. Then we have by definition a C^2 domain $G \subset\subset \mathbb{C}^n$ which is strictly convex with respect to the real coordinates of $z_1, ..., z_{q+1}$ and a biholomorphic map h from a neighbourhood $U_{\bar{D}}$ of $\bar{D}$ onto a neighbourhood $U_{\bar{G}}$ of $\bar{G}$ such that $h(D) = G$. After shrinking these neighbourhoods we may also assume that there is a C^2 function $\varrho : U_{\bar{G}} \to \mathbb{R}$ as in the first part of the Definition in Section 3. Further let $V_{\bar{G}}, A, \alpha$ be as in Lemma 3.2. Before we come to the announced estimates we construct an integral operator which gives a homotopy formula for (n,r)-forms with $n - q \leq r \leq n$.

For all $(\xi,\zeta) \in \mathbb{C}^n \times U_{\bar{G}}$ we set

$$\begin{aligned} w_j(\xi,\zeta) &:= \begin{cases} 2\dfrac{\partial\varrho(\zeta)}{\partial\zeta_j} & \text{for } \ 1 \leq j \leq q+1 \\ 2\dfrac{\partial\varrho(\zeta)}{\partial\zeta_j} + A(\bar{\zeta}_j - \bar{\xi}_j) & \text{for } \ q+2 \leq j \leq n, \end{cases} \\ w(\xi,\zeta) &:= (w_1(\xi,\zeta), ..., w_n(\xi,\zeta)), \\ \Phi(\xi,\zeta) &:= \langle w(\xi,\zeta), \zeta - \xi\rangle - 2\varrho(\zeta). \end{aligned}$$

Then by (8)

$$\mathrm{Re}\,\Phi(\xi,\zeta) \geq -\varrho(\zeta) - \varrho(\xi) + \alpha|\zeta - \xi|^2 \tag{13}$$

for all $\xi,\zeta \in \bar{V}_{\bar{G}}$. In particular $\Phi(\xi,\zeta) \neq 0$ if $\xi,\zeta \in G$ and for all $(\xi,\zeta,\lambda) \in V_{\bar{G}} \times V_{\bar{G}} \times [0,1]$ with $\xi \neq \zeta$ we can define

$$\eta(\xi,\zeta,\lambda) := (1-\lambda)\frac{w(\xi,\zeta)}{\Phi(\xi,\zeta)} + \lambda\frac{\bar{\zeta} - \bar{\xi}}{|\zeta - \xi|^2}$$

and

$$\hat{H}^G(\xi,\zeta,\lambda) := \frac{n!}{(2\pi i)^n} d\eta_1(\xi,\zeta,\lambda) \wedge ... \wedge d\eta_n(\xi,\zeta,\lambda) \wedge d\xi_1 \wedge ... \wedge d\xi_n$$

where $\eta_1, ..., \eta_n$ are the components of η and d is the exterior differential operator with respect to ξ,ζ,λ. For $\zeta \neq \xi$, $\hat{H}^G(\xi,\zeta,\lambda)$ is of class C^∞ in ξ,λ and all derivatives with respect to ξ,λ are continuous in ξ,ζ,λ. Moreover if we consider only the part of $\hat{H}^G(\xi,\zeta,\lambda)$ which is of degree 1 in λ then we see that the singularity at $\xi = \zeta$ of this form is of order $\leq 2n-1$.

Hence for each $g \in L^1_*(G) \cap C^0_*(G)$ the integrals

$$H^G g(\xi) := \int\limits_{(\zeta,\lambda)\in G\times[0,1]} g(\zeta) \wedge \hat{H}^G(\xi,\zeta,\lambda) \quad \text{for} \quad \xi \in G$$

converge (for the definition of such integrals see for instance Section 0.2 in [H/Le 2]) and in this way we obtain a form $H^G g \in C^0_*(G)$. Denote by $\hat{H}(\xi,\zeta,\lambda)$ the pull back of the form $\hat{H}^G(\xi,\zeta,\lambda)$ to $U_D \times U_D \times [0,1]$ with respect to the biholomorphic map h. That means

$$\hat{H}(\xi,\zeta,\lambda) = (h^*_\xi \times h^*_\zeta)\hat{H}^G(\xi,\zeta,\lambda). \tag{14}$$

Further let

$$H = h^* \circ H^G \circ (h^{-1})^*$$

be the pull back of the operator H^G to the domain D with respect to h. Then H is a linear operator from $L^1_*(D) \cap C^0_*(D)$ to $C^0_*(D)$ and for each $f \in L^1_*(D) \cap C^0_*(D)$ we have

$$Hf(\xi) = \int\limits_{(\zeta,\lambda)\in D\times[0,1]} f(\zeta) \wedge \hat{H}(\xi,\zeta,\lambda) \quad \text{for} \quad \xi \in D.$$

Note that for $r = 1, ..., n$

$$H(L^1_{(n,r)}(D) \cap C^0_{(n,r)}(D)) \subseteq C^0_{(n,r-1)}(D). \tag{15}$$

Theorem 4.1 *If $n - q \leq r \leq n$ and if $f \in L^1_{(n,r)}(D) \cap C^0_{(n,r)}(D)$ such that df also belongs to $L^1_*(D) \cap C^0_*(D)$ then*

$$f = \begin{cases} dHf & \text{for} \quad r = n \\ dHf + Hdf & \text{for} \quad n - q \leq r \leq n-1. \end{cases} \tag{16}$$

Theorem 4.2 *There is a constant $C < 0$ such that for each bounded $f \in C^0_*(D)$, Hf is Hölder continuous on $\bar{D}$ and*

$$||Hf||_{1/2,D} \leq C \sup_{\zeta\in D} ||f(\zeta)||.$$

Essentially these theorems are contained already in the works of GRAUERT/LIEB [G/L], HENKIN [H 1] and W. FISCHER/LIEB [WF/L] where certain versions of the operator H with boundary integrals are used. To obtain proofs precisely for the statements formulated here one can use many different sources in the literature. We restrict ourselves to the following remarks: The idea to use operators without boundary integrals is due to HENKIN, LIEB and RANGE (see [L/R] or [H/Le 1]); Theorem 4.1 can be proved by the same arguments as Theorem 4.11 in [La/Le]; Theorem 4.2 can be proved by the same arguments as Theorem 3.1 in [BF].

Theorem 4.2 admits generalisations to forms satisfying different uniform growth conditions ([L/R], [BF]). For example in [BF] the case is studied where for a smooth submanifold N of bD

$$||f(\zeta)|| \leq [\text{dist}(\zeta, N)]^{-\beta} \quad \text{for} \quad \zeta \in D$$

where $0 \le \beta < 2n - \dim_{\mathbb{R}} N$. In the present paper we need the following improvement of this result for the case when N consists only of one point and $\beta = 2n-1$: Set

$$\tau(\xi, z) := \left| \sum_{j=1}^{n} \frac{\partial \varrho \circ h(z)}{\partial z_j}(\xi_j - z_j) \right| \tag{17}$$

for $z \in U_D$ and $\xi \in \mathbb{C}^n$. Note that for $z \in bD$, $\tau(\xi, z)$ is proportional to the Euclidean distance $\delta(\xi, z)$ between ξ and the complex tangent plane of bD at z.

Theorem 4.3 *There is a constant $C > 0$ such that the following holds: If $z \in U_D \backslash D$ (in particular $z \in bD$ is admitted) and $f \in C^0_*(D)$ satisfies the estimate*

$$||f(\zeta)|| \le \frac{1}{|\zeta - z|^{2n-1}} \tag{18}$$

for all $\zeta \in D$ then Hf belongs to $C^{1/2}_*(\bar{D} \backslash \{z\})$ and moreover

$$||Hf(\xi)|| \le C \frac{1 + |\ln|\xi - z||}{(\tau(\xi, z) + |\xi - z|^2)|\xi - z|^{2n-3}} \tag{19}$$

for all $\xi \in \bar{D} \backslash \{z\}$.

Proof. We may assume that $D = G$ and h is the identical map. Let $z \in U_D \backslash D$ and $f \in C^0_*(D)$ with (18) be given. That Hf belongs to $C^{1/2}_*(\bar{D} \backslash \{z\})$ then follows from Theorem 4.2 and the fact that for $\zeta \neq \xi$ the derivatives of $\tilde{H}(\xi, \zeta, \lambda)$ with respect to ξ are continuous in ξ, ζ, λ.

Now we are going to prove estimate (19). During this proof by C, C_1, C_2 we denote positive constants which are independent of f and z. The constant C used in different places may have different values there. Observe that as usual (see for instance Section 3.2.7 in [H/Le 1]) we obtain that

$$||Hf(\xi)|| \le C(I_0(\xi) + I_1(\xi) + I_2(\xi)) \quad \text{for} \quad \xi \in \bar{D} \tag{20}$$

where

$$I_k(\xi) := \int_{\zeta \in D} \frac{d\sigma}{|\Phi(\xi, \zeta)|^k |\zeta - \xi|^{2n-1-k} |\zeta - z|^{2n-1}}$$

and $d\sigma$ is the Lebesgue measure. We omit the elementary arguments which show that

$$|I_0(\xi)| \le \frac{C}{|\xi - z|^{2n-2}} \quad \text{for} \quad \xi \in \bar{D}. \tag{21}$$

To estimate $I_1(\xi)$ and $I_2(\xi)$ we first give some auxiliary estimates. From the definition of Φ it is clear that

$$|\Phi(\xi, z)| \ge 2\tau(\xi, z) - A|\xi - z|^2 \quad \text{for} \quad z \in \bar{D}$$

and

$$|\Phi(\xi, z) - \Phi(\xi, \zeta)| \le C|\zeta - z| \quad \text{for} \quad \xi, \zeta \in \bar{D}.$$

Hence

$$|\Phi(\xi,\zeta)| \geq 2\tau(\xi,z) - C_1(|\zeta - z| + |\xi - z|^2) \tag{22}$$

for all $\xi, \zeta \in \bar{D}$. Further we introduce the abbreviation $t(\xi,\zeta) := \operatorname{Im}\Phi(\xi,\zeta)$ and recall the fact that $d_\zeta t(\xi,\zeta)\big|_{\zeta=\xi} \wedge d\varrho(\zeta) \neq 0$ if $\zeta \in bD$. Choose a neighbourhood U_{bD} of bD and a number $\varepsilon > 0$ so small that

$$d_\zeta t(\xi,\zeta) \wedge d\varrho(\zeta) \neq 0 \tag{23}$$

for all $\xi \in U_{bD}$ and $\zeta \in \mathbb{C}^n$ with $|\zeta - \xi| \leq \varepsilon$. Note also that by (13)

$$|\Phi(\xi,\zeta)| \geq |t(\xi,\zeta)| + |\varrho(\zeta)| + |\varrho(\xi)| + \alpha|\zeta - \xi|^2 \tag{24}$$

for all $\zeta, \xi \in \bar{D}$. It follows from (24) and (21) that

$$I_k(\xi) \leq C I_0(\xi) \leq \frac{C}{|\xi - z|^{2n-2}} \quad \text{for} \quad \xi \in D \backslash U_{bD} \tag{25}$$

and

$$\int\limits_{\substack{\zeta \in D \\ |\zeta - \xi| > \varepsilon}} \frac{d\sigma_{2n}}{|\Phi(\xi,\zeta)|^k |\zeta - \xi|^{2n-1-k} |\zeta - z|^{2n-1}} \leq C \tag{26}$$

for all $\xi \in \bar{D}\backslash\{z\}$ and $k = 1, 2$. Set

$$I_{k,\varepsilon}(\xi) := \int\limits_{\substack{\zeta \in D \\ |\zeta - \xi| < \varepsilon}} \frac{d\sigma_{2n}}{|\Phi(\xi,\zeta)|^k |\zeta - \xi|^{2n-1-k} |\zeta - z|^{2n-1}}$$

for $\xi \in (\bar{D} \cap U_{bD})\backslash\{z\}$. Since

$$\tau(\xi,z) \leq C_2|\xi - z| \tag{27}$$

for all $\xi \in \bar{D}$ now by (20), (21), (25) and (26) it remains to prove that

$$I_{k,\varepsilon}(\xi) \leq C \frac{1 + |\ln|\xi - z||}{|\xi - z|^{2n-1}} \tag{28}$$

and

$$I_{k,\varepsilon}(\xi) \leq C \frac{1 + |\ln|\xi - z||}{\tau(\xi,z)|\xi - z|^{2n-3}} \tag{29}$$

for all $\xi \in (\bar{D} \cap U_{bD})\backslash\{z\}$ and $k = 1, 2$. In doing so we use the following notation: If $\xi \in (\bar{D} \cap U_{bD})\backslash\{z\}$, $W(\xi) \subseteq D$ and $k \in \{1, 2\}$ then

$$I_{k,\varepsilon}(W(\xi)) := \int\limits_{\substack{\zeta \in W(\xi) \\ |\zeta - \xi| < \varepsilon}} \frac{d\sigma_{2n}}{|\Phi(\xi,\zeta)|^k |\zeta - \xi|^{2n-1-k} |\zeta - z|^{2n-1}}.$$

Proof of (28). For $\xi \in (\bar{D} \cap U_{bD})\backslash\{z\}$ we set

$$W'(\xi) = \{\zeta \in D : |\zeta - z| < |\xi - z|/2\}$$

and

$$W''(\xi) = \{\zeta \in D : |\zeta - z| > |\xi - z|/2\}.$$

Then

$$I_{k,\epsilon}(\xi) = I_{k,\epsilon}(W'(\xi)) + I_{k,\epsilon}(W''(\xi)) \tag{30}$$

for all $\xi \in (\bar{D} \cap U_{bD})\backslash\{z\}$ and $k \in \{1,2\}$. Since $|\zeta - \xi| > |\xi - z|/2$ if $\zeta \in W'(\xi)$ and by (24) we have

$$I_{k,\epsilon}(W'(\xi)) \leq \frac{C}{|\xi - z|^{2n-1-k}} \int\limits_{\substack{\zeta \in W'(\xi) \\ |\zeta - \xi| < \epsilon}} \frac{d\sigma_{2n}}{(|t(\xi,\zeta)| + |\varrho(\zeta)| + |\xi - z|^2)^k |\zeta - z|^{2n-1}}$$

for all $\xi \in (\bar{D} \cap U_{bD})\backslash\{z\}$ and $k \in \{1,2\}$. By (23) ϱ and $t(\xi, \cdot)$ may be considered as local coordinates. Hence

$$\begin{aligned} I_{k,\epsilon}(W'(\xi)) &\leq \frac{C}{|\xi - z|^{2n-1-k}} \int\limits_{x \in \mathbb{R}^{2n}} \frac{dx_1 \wedge \ldots \wedge dx_{2n}}{(|x_1| + |x_2| + |\xi - z|^2)^k |x|^{2n-1}} \\ &\leq \frac{C}{|\xi - z|^{2n-1+k}} \int\limits_{\substack{x \in \mathbb{R}^{2n} \\ |x| < |\xi - z|^2}} \frac{dx_1 \wedge \ldots \wedge dx_{2n}}{|x|^{2n-1}} \\ &\quad + \frac{C(1 + |\ln|\xi - z||)}{|\xi - z|^{2n-1-k}} \int\limits_{\substack{x \in \mathbb{R}^{2n-k} \\ |x| > |\xi - z|^2}} \frac{dx_1 \wedge \ldots \wedge dx_{2n-k}}{|x|^{2n-1}} \\ &\leq C \frac{1 + |\ln|\xi - z||}{|\xi - z|^{2n-1}} \end{aligned} \tag{31}$$

for all $\xi \in (\bar{D} \cap U_{bD})\backslash\{z\}$ and $k \in \{1,2\}$. By similar arguments we obtain that

$$\begin{aligned} I_{k,\epsilon}(W''(\xi)) &\leq \frac{C}{|\xi - z|^{2n-1}} \int\limits_{x \in \mathbb{R}^{2n}} \frac{dx_1 \wedge \ldots \wedge dx_{2n}}{(|x_1| + |x_2| + |x|^2)^k |x|^{2n-1-k}} \\ &\leq \frac{C}{|\xi - z|^{2n-1}} \int\limits_{x \in \mathbb{R}^{2n-k}} \frac{(1 + |\ln|x||) dx_1 \wedge \ldots \wedge dx_{2n-k}}{|x|^{2n-1-k}} \\ &\leq \frac{C}{|\xi - z|^{2n-1}} \end{aligned} \tag{32}$$

for all $\xi \in (\bar{D} \cap U_{bD})\backslash\{z\}$ and $k \in \{1,2\}$. Estimate (28) now follows from (30)-(32).

Proof of (29). Let $C_3 = 2(C_1 + C_2)$ where C_1 and C_2 are the same constants as in (22) and (27), and set

$$\begin{aligned} W^0(\xi) &= \{\zeta \in D : |\zeta - z| < \tau(\xi, z)/C_3\}, \\ W^1(\xi) &= \{\zeta \in D : |\zeta - z| > \tau(\xi, z)/C_3\}, \\ W^{10}(\xi) &= \{\zeta \in W^1(\xi) : |\zeta - z| < |\xi - z|/2\}, \\ W^{11}(\xi) &= \{\zeta \in W^1(\xi) : |\zeta - z| > |\xi - z|/2\}, \\ W^{110}(\xi) &= \{\zeta \in W^{11}(\xi) : |\zeta - \xi| < |\xi - z|\}, \\ W^{111}(\xi) &= \{\zeta \in W^{11}(\xi) : |\zeta - \xi| > |\xi - z|\}. \end{aligned}$$

Then

$$I_{k,\epsilon}(\xi) = I_{k,\epsilon}(W^0(\xi)) + I_{k,\epsilon}(W^{10}(\xi)) + I_{k,\epsilon}(W^{110}(\xi)) + I_{k,\epsilon}(W^{111}(\xi)) \tag{33}$$

for all $\xi \in (\bar{D} \cap U_{bD})\backslash\{z\}$ and $k \in \{1,2\}$. Since $|\zeta - \xi| \geq |\xi - z|/2$ if $\zeta \in W^{10}(\xi)$ and by (24) and (23) we obtain that

$$\begin{aligned} I_{k,\epsilon}(W^{10}(\xi)) &\leq \frac{C}{|\xi - z|^{2n-1-k}} \int\limits_{\substack{x \in \mathbb{R}^{2n} \\ \tau(\xi,z)/C_3 < |x| < |\xi - z|/2}} \frac{dx_1 \wedge \ldots \wedge dx_{2n}}{(|x_1| + |x_2| + |\xi - z|^2)^k |x|^{2n-1}} \\ &\leq \frac{C(1 + |\ln|\xi - z||)}{|\xi - z|^{2n-1-k}} \int\limits_{\substack{x \in \mathbb{R}^{2n-k} \\ \tau(\xi,z)/C_3 < |x| < |\xi - z|/2}} \frac{dx_1 \wedge \ldots \wedge dx_{2n-k}}{|x|^{2n-1}} \end{aligned}$$

for all $\xi \in (\bar{D} \cap U_{bD})\backslash\{z\}$ and $k \in \{1,2\}$. Hence

$$\begin{aligned} I_{1,\epsilon}(W^{10}(\xi)) &\leq \frac{C(1 + |\ln|\xi - z||)}{\tau(\xi,z)|\xi - z|^{2n-2}} \int\limits_{\substack{x \in \mathbb{R}^{2n-1} \\ |x| < |\xi - z|/2}} \frac{dx_1 \wedge \ldots \wedge dx_{2n-1}}{|x|^{2n-2}} \\ &\leq \frac{C(1 + |\ln|\xi - z||)}{\tau(\xi,z)|\xi - z|^{2n-3}} \end{aligned} \tag{34}$$

and

$$\begin{aligned} I_{2,\epsilon}(W^{10}(\xi)) &\leq \frac{C(1 + |\ln|\xi - z||)}{|\xi - z|^{2n-3}} \int\limits_{\substack{x \in \mathbb{R}^{2n-2} \\ \tau(\xi,z)/C_3 < |x|}} \frac{dx_1 \wedge \ldots \wedge dx_{2n-2}}{|x|^{2n-1}} \\ &\leq \frac{C(1 + |\ln|\xi - z||)}{\tau(\xi,z)|\xi - z|^{2n-3}} \end{aligned} \tag{35}$$

for all $\xi \in (\bar{D} \cap U_{bD})\backslash\{z\}$. Further it follows from (24), (23) and (27) that

$$\begin{aligned} I_{k,\epsilon}(W^{110}(\xi)) &\leq \frac{C}{|\xi - z|^{2n-1}} \int\limits_{\substack{x \in \mathbb{R}^{2n} \\ |x| < |\xi - z|}} \frac{dx_1 \wedge \ldots \wedge dx_{2n}}{(|x_1| + |x_2| + |x|^2)^k |x|^{2n-1-k}} \\ &\leq \frac{C}{|\xi - z|^{2n-1}} \int\limits_{\substack{x \in \mathbb{R}^{2n-k} \\ |x| < |\xi - z|}} \frac{(1 + |\ln|x||) dx_1 \wedge \ldots \wedge dx_{2n-k}}{|x|^{2n-1-k}} \\ &\leq \frac{C(1 + |\ln|\xi - z||)}{|\xi - z|^{2n-2}} \leq \frac{C(1 + |\ln|\xi - z||)}{\tau(\xi,z)|\xi - z|^{2n-3}} \end{aligned} \tag{36}$$

for all $\xi \in (\bar{D} \cap U_{bD})\backslash\{z\}$ and $k \in \{1,2\}$. Since $|\zeta - z| \geq |\xi - z|/2$ and $|\zeta - \xi| \geq |\xi - z|$ imply $|\zeta - \xi| \geq (1/2)|\zeta - z|$ we get

$$I_{k,\epsilon}(W^{111}(\xi)) \leq C \int\limits_{\zeta \in W^{111}(\xi)} \frac{d\sigma_{2n}}{|\Phi(\xi,\zeta)|^k |\zeta - \xi|^{2n-3} |\zeta - z|^{2n+1-k}}$$

$$\leq \frac{C}{|\xi - z|^{2n-3}} \int\limits_{\substack{x \in \mathbb{R}^{2n} \\ \tau(\xi,z)/C_3 < |x|}} \frac{dx_1 \wedge \ldots \wedge dx_{2n}}{(|x_1| + |x_2| + |x|^2)^k |x|^{2n+1-k}}$$

$$\leq \frac{C(1 + |\ln|\xi - z||)}{|\xi - z|^{2n-3}} \int\limits_{\substack{x \in \mathbb{R}^{2n-k} \\ \tau(\xi,z)/C_3 < |x|}} \frac{dx_1 \wedge \ldots \wedge dx_{2n-k}}{|x|^{2n+1-k}}$$

$$\leq \frac{C(1 + |\ln|\xi - z||)}{\tau(\xi, z)|\xi - z|^{2n-3}} \tag{37}$$

for all $\xi \in (\bar{D} \cap U_{bD}) \backslash \{z\}$ and $k \in \{1, 2\}$. Finally we consider the integrals $I_{k,\epsilon}(W^0(\xi))$. It follows from estimate (28) which is already proved that

$$I_{k,\epsilon}(W^0(\xi)) \leq \frac{CC_3(1 + |\ln|\xi - z||)}{\tau(\xi, z)|\xi - z|^{2n-3}} \tag{38}$$

for all $\xi \in (\bar{D} \cap U_{bD}) \backslash \{z\}$ with $\tau(\xi, z) \leq C_3|\xi - z|^2$ and $k \in \{1, 2\}$. Therefore it remains to estimate $I_{k,\epsilon}(W^0(\xi))$ for all $\xi \in (\bar{D} \cap U_{bD}) \backslash \{z\}$ with

$$\tau(\xi, z) \geq C_3|\xi - z|^2. \tag{39}$$

It follows from (27) that $|\zeta - z| \leq |\xi - z|/2$ and therefore $|\zeta - \xi| \geq |\xi - z|/2$ for all $\xi \in (\bar{D} \cap U_{bD}) \backslash \{z\}$ and $\zeta \in W^0(\xi)$. Moreover it follows from (22) that $|\Phi(\xi, \zeta)| \geq \tau(\xi, z)$ for all $\xi \in (\bar{D} \cap U_{bD}) \backslash \{z\}$ satisfying (39) and $\zeta \in W^0(\xi)$. Hence

$$I_{k,\epsilon}(W^0(\xi)) \leq \frac{C}{(\tau(\xi, z))^k |\xi - z|^{2n-1-k}} \int\limits_{\substack{x \in \mathbb{R}^{2n} \\ |x| < \tau(\xi,z)/C_3}} \frac{dx_1 \wedge \ldots \wedge dx_{2n}}{|x|^{2n-1}}$$

$$\leq \frac{C}{(\tau(\xi, z))^{k-1} |\xi - z|^{2n-1-k}}$$

$$\leq \frac{C}{(\tau(\xi, z))|\xi - z|^{2n-3}} \tag{40}$$

for all $\xi \in (\bar{D} \cap U_{bD}) \backslash \{z\}$ satisfying (39) and $\zeta \in W^0(\xi)$ (for $k = 1$ we used (27)).

Estimate (29) now follows from (33)-(38) and (40). ■

5 Construction of the kernel

We start this section with a corollary to Section 4.

Corollary 5.1 *Let $D \subset\subset \mathbb{C}^n$ be a local 1-convex C^2 domain and let H be the operator constructed in Section 4 for D. Set*

$$K_D(z, \xi) := [H(B(z, \cdot))](\xi)$$

for all $z \in \mathbb{C}^n \backslash D$ and $\xi \in D$ where $B(z, \xi)$ is the Martinelli-Bochner kernel (5). By Theorem 4.3, $K_D(z, \xi)$ is defined and continuous even for all $z \in \mathbb{C}^n \backslash D$ and $\xi \in \bar{D}$ with $z \neq \xi$. Moreover this form has the following properties:

(i) $K_D(z,\xi)$ *is of bidegree* $(n, n-2)$ *in* ξ *and of degree zero in* z.

(ii) $d_\xi K_D(z,\xi) = B(z,\xi)$ *for all* $z \in \mathbb{C}^n \setminus D$ *and* $\xi \in \bar{D}$ *with* $z \neq \xi$.

(iii) There is a constant $C > 0$ *such that*

$$\|K_D(z,\xi)\| \leq C \frac{1 + |\ln|\xi - z||}{(\tau(z,\xi) + |\xi - z|^2)|\xi - z|^{2n-3}} \tag{41}$$

for all $z \in U_D \setminus D$ *and* $\xi \in \bar{D}$ *with* $z \neq \xi$ *where* $\tau(z,\xi)$ *is defined by (17).*

(iv) For each $z \in \mathbb{C}^n \setminus \bar{D}$*, the form* $K_D(z,\cdot)$ *belongs to* $C^{1/2}_{(n,n-2)}(\bar{D})$ *and the assignement* $z \to K_D(z,\cdot)$ *is of class* C^∞ *as a map from* $\mathbb{C}^n \setminus \bar{D}$ *with values in the Banach space* $C^{1/2}_{(n,n-2)}(\bar{D})$.

(v) For any $0 < \alpha < 1$, $K_D(z,\xi)$ *is of class* $C^{\alpha,1/2}_{z,\xi}$ *for all* $z \in \mathbb{C}^n \setminus D$ *and* $\xi \in \bar{D}$ *with* $z \neq \xi$.

Proof. (i) follows from (15), (ii) follows from Theorem 4.1 and (iii) follows from Theorem 4.3. Since the Martinelli–Bochner kernel is of class C^∞ outside the diagonal and, by Theorem 4.2, H acts continuously from $C^0_{(n,n-1)}(\bar{D})$ to $C^{1/2}_{(n,n-2)}(\bar{D})$, (iv) is also clear.

It remains to prove (v). Fix $0 < \alpha < 1$, $z_0 \in \mathbb{C}^n \setminus D$ and $\xi_0 \in \bar{D}$ with $z_0 \neq \xi_0$. Set $\gamma = |z_0 - \xi_0|/5$ and

$$\begin{aligned} B(z_0) &= \{z \in \mathbb{C}^n \setminus D : |z - z_0| < \gamma\}, \\ B(\xi_0) &= \{\xi \in \bar{D} : |\xi - \xi_0| < \gamma\}. \end{aligned}$$

It is sufficient to prove that $K_D(z,\xi)$ is of class $C^{\alpha,1/2}_{z,\xi}$ for $(z,\xi) \in B(z_0) \times B(\xi_0)$. For that we choose a real C^∞ function χ on $\mathbb{C}^n$ with $\chi(\zeta) = 1$ if $|\zeta - \xi_0| < 2\gamma$ and $\chi(\zeta) = 0$ if $|\zeta - \xi_0| > 3\gamma$. Set

$$K^\chi_D(z,\xi) = [H(\chi B(z,\cdot))](\xi),$$
$$K^{1-\chi}_D(z,\xi) = [H((1-\chi)B(z,\cdot))](\xi)$$

for $z \in \mathbb{C}^n \setminus D$ and $\xi \in \bar{D}$ with $z \neq \xi$. Since $\chi(\zeta)B(z,\zeta)$ is of class C^∞ for $(z,\zeta) \in B(z_0) \times \mathbb{C}^n$ and H acts continuously and linearly from $C^0_{(n,n-1)}(\bar{D})$ to $C^{1/2}_{(n,n-2)}(\bar{D})$, we see that the map $z \to K^\chi_D(z,\cdot)$ is C^∞ from $B(z_0)$ to $C^{1/2}_{(n,n-2)}(\bar{D})$. Hence in particular, $K^\chi_D(z,\xi)$ is of class $C^{\alpha,1/2}_{z,\xi}$ for $(z,\xi) \in B(z_0) \times B(\xi_0)$. It remains to prove that $K^{1-\chi}_D(z,\xi)$ is of class $C^{\alpha,1/2}_{z,\xi}$ for $(z,\xi) \in B(z_0) \times B(\xi_0)$. For that we consider the form

$$f(\xi,\zeta) := \int_{\lambda \in [0,1]} (1 - \chi(\zeta))\hat{H}(\xi,\zeta,\lambda)$$

(see (14) for the definition of $\hat{H}(\xi,\zeta,\lambda)$). Since $1-\chi(\zeta)=0$ if $\zeta \in B(\xi_0)$ the map $\xi \to f(\xi,\cdot)$ is C^∞ from $B(\xi_0)$ to $C^0_*(\bar{D})$. Since

$$K_D^{1-\chi}(z,\xi) = \pm \int_{\zeta\in D} f(\xi,\zeta)\wedge B(z,\zeta)$$

for $z \in \mathbb{C}^n\backslash D$ and $\xi \in B(\xi_0)$ and since the Martinell-Bochner integral induces a continuous linear operator from $C^0_*(\bar{D})$ to $C^\alpha_*(\bar{B}(z_0))$ this implies that the map $\xi \to K_D^{1-\chi}(\cdot,\xi)$ is C^∞ from $B(\xi_0)$ to $C^\alpha_*(\bar{B}(z_0))$. This completes the proof. ■

Proof of Theorem 1.1. Choose an open ball $B \subset\subset \mathbb{C}^n$ centered at z_0 so small that $B\backslash M$ consists of precisely two connected components and $B\cap M$ is relatively compact in M. The two connected components of $B\backslash M$ we denote by B_+ and B_- so that on $B\cap M$ the orientations of M and bB_+ coincide. In view of Lemma 3.1 we can find local 1-convex C^2 domains D_+ and D_- and open balls $B_0 \subset\subset B_1 \subset\subset B$ centered at z_0 such that $B_1 \cap B_\pm \subseteq D_\pm \subseteq B_\pm$. Set $M_0 := M\cap B_0$ and denote by H_+ and H_- the operators defined in Section 4 for D_+ and D_- respectively. Set

$$K_\pm(z,\xi) := -[H_\pm B(z,\cdot)](\xi)$$

for all $z\in\mathbb{C}^n\backslash D_\pm$ and $\xi\in D_\pm$ with $z\neq\xi$. By (15) $K_\pm(z,\xi)$ is defined and continuous for all $z\in\mathbb{C}^n\backslash D_\pm$ and $\xi\in D_\pm$ with $z\neq\xi$. Therefore by setting

$$K(z,\xi) := K_+(z,\xi)\big|_{M_0\times M_0} - K_-(z,\xi)\big|_{M_0\times M_0}$$

we obtain a differential form defined and continuous for all $(z,\xi)\in M_0\times M_0$ with $z\neq\xi$. It follows immediately from the statements (i), (ii), (iii) and (v) in Corollary 5.1 that $K(z,\xi)$ has the properties (i)-(iv) formulated in Theorem 1.1.

Now we prove part (v). Let $\Omega \subset\subset M_0$ be a domain with piecewise C^1 boundary. An approximation argument shows that we may restrict ourselves to C^1 functions f.

First we consider a C^1 function f on Ω with compact support. Then there is a C^1 function $\tilde{f}$ on $\mathbb{C}^n$ with $\tilde{f}(\xi)=f(\xi)$ if $\xi\in\Omega$ and

$$\operatorname{supp}\tilde{f} \subset\subset D_+\cup D_-\cup\Omega =: D$$

and since, by Corollary 5.1 (ii), $d_\xi K_\pm(z,\xi) = B(z,\xi)$, it follows from Stokes theorem and the Martinelli–Bochner formula that

$$\begin{aligned} -\int_{\xi\in\Omega} \bar\partial_M f(\xi)\wedge K(z,\xi) &= \int_{\xi\in D_+} \bar\partial\tilde{f}(\xi)\wedge d_\xi K_+(z,\xi) + \int_{\xi\in D_-} \bar\partial\tilde{f}(\xi)\wedge d_\xi K_-(z,\xi) \\ &= -\int_{\xi\in D} \bar\partial f(\xi)\wedge B(z,\xi) = \tilde{f}(z) = f(z) \end{aligned}$$

for all $z\in\Omega$. That is (2) is proved in the case when f has compact support.

Now let f be an arbitrary C^1 function on $\bar\Omega$. Fix $z\in\Omega$ and choose a C^1 function χ_z on M_0 with $\operatorname{supp}\chi_z \subset\subset \Omega$ and $\chi_z\equiv 1$ in some neighbourhood of z.Then $(1-\chi_z)fK(z,\cdot)$

is a continuous form on $\bar{\Omega}$ which is identically zero in a neighbourhood of z and since $d_\xi K(z,\xi) = 0$ for $\xi \neq z$ we have the relation

$$\begin{aligned} d[(1-\chi_z)fK(z,\cdot)] &= d[(1-\chi_z)f] \wedge K(z,\cdot) \\ &= \bar{\partial}_M f \wedge K(z,\cdot) - \bar{\partial}_M(\chi_z f) \wedge K(z,\cdot) \end{aligned}$$

on $\bar{\Omega}$. Therefore $d[(1-\chi_z)fK(z,\cdot)]$ is also continuous on $\bar{\Omega}$ and Stokes theorem implies that

$$\int_{b\Omega} f \wedge K(z,\cdot) = \int_{\Omega} \bar{\partial}_M f \wedge K(z,\cdot) - \int_{\Omega} \bar{\partial}_M(\chi_z f) \wedge K(z,\cdot).$$

Since formula (2) is already proved for $\chi_z f$ and therefore

$$-\int_{\Omega} \bar{\partial}_M(\chi_z f) \wedge K(z,\cdot) = \chi_z(z) f(z) = f(z)$$

this completes the proof of (2). ■

6 Further properties of the kernel $K(z,\xi)$ and applications

In this section we assume that ϱ, M, z_0, M_0 and $K(z,\xi)$ are as in Theorem 1.1 and B_0, B, B_+, B_-, $K_+(z,\xi)$ and $K_-(z,\xi)$ are as in Section 5. Moreover we shall assume that the ball B_0 is chosen sufficiently small so that the following two propositions hold:

Proposition 6.1 *Any continuous CR-function defined on an open set $\Omega \subseteq M_0$ extends to a holomorphic function in some $\mathbb{C}^n$-neighbourhood of Ω.*

Proposition 6.2 *If $B(z) \subseteq B_0$ is an open ball centered at some point $z \in M_0$, then any continuous and closed $(n, n-2)$-form on $\overline{B_+ \cap B(z)}$ respectively $\overline{B_- \cap B(z)}$ can be approximated uniformly on $\overline{B_+ \cap B(z)}$ respectively $\overline{B_- \cap B(z)}$ by $\bar{\partial}$-exact $C^\infty_{(n,n-2)}$-forms on $\mathbb{C}^n$.*

That this is possible follows from the hypothesis on the Levi form of ϱ: Proposition 6.1 is a consequence of the Levi extension theorem (see, e.g., Theorem 1.3.8 in [H/Le 2]), since, in the sense of distributions, any continuous CR-function on a hypersurface is the jump of two holomorphic functions (the latter assertion can be proved by means of the Martinelli–Bochner–Koppelmann formula). Since $\overline{B_\pm \cap B(z)}$ is starshaped if B_0 is sufficiently small, Proposition 6.2 follows from the Andreotti–Grauert–Hörmander approximation theorem (see, e.g., Theorem 8.1 in [H/Le 2]).

Further for each open $\Omega \subseteq M_0$ we use the following notations:

Spaces of forms. $C^k_{(n,r)}(\Omega)$ $(0 \leq r \leq n-1,\ k = 0,1,2)$ is the space of $C^k_{(n,r)}$-forms on Ω endowed with the topology of uniform convergence together with all derivatives of order $\leq k$ on the compact subsets of Ω. By $D^k_{(n,r)}(\Omega)$ we denote the space of all $f \in C^k_{(n,r)}(\Omega)$ with compact support endowed with the test-function-topology of order

k: a sequence f_ν converges in $D^k_{(n,r)}(\Omega)$ if it converges in $C^k_{(n,r)}(\Omega)$ and moreover there is a compact set $\omega \subset\subset \Omega$ with $\operatorname{supp} f_\nu \subseteq \omega$ for all ν. By $L^\infty_{(n,r)}(\Omega)$ $(0 \le r \le n-1)$ we denote the Banach space of (n,r)-forms with bounded measurable coefficients on Ω endowed with the sup-norm.

Spaces of currents. $C^k_{(n,r)}(\Omega)'$ and $D^k_{(n,r)}(\Omega)'$ are the spaces of continuous linear forms on $C^k_{(n,r)}(\Omega)$ and $D^k_{(n,r)}(\Omega)$ respectively, i.e. the elements in $D^k_{(n,r)}(\Omega)'$ are the $(0,n-r-1)$-currents of order k on Ω, and the elements in $C^k_{(n,r)}(\Omega)'$ are the $(0,n-r-1)$-currents of order k with compact support on Ω.

If f is a differential form with locally integrable coefficients and of degree s on Ω then we denote by $\langle f\rangle$ the current in $D^0_{(n,n-s-1)}(\Omega)'$ defined by

$$\langle f\rangle(\varphi) := \int_\Omega f \wedge \varphi \quad \text{for} \quad \varphi \in D^0_{(n,n-s-1)}(\Omega).$$

The operator $\bar\partial_\Omega$: For $0 \le r \le n-1$ and $k = 0, 1$ we denote by $\bar\partial_\Omega$ the operator

$$\bar\partial_\Omega : D^k_{(n,r+1)}(\Omega)' \to D^{k+1}_{(n,r)}(\Omega)'$$

defined by $(\bar\partial_\Omega T)\varphi := (-1)^{n-r-1} T(d\varphi)$ for $T \in D^k_{(n,r+1)}(\Omega)'$ and $\varphi \in D^{k+1}_{(n,r)}(\Omega)$.

Definition. Let $\Omega \subseteq M_0$ be an open set. Set

$$K_\Omega f(\xi) := \int_{z\in\Omega} f(z) \wedge K(z,\xi)$$

for $f \in L^\infty_{(n,n-1)}(\Omega)$ and $\xi \in \Omega$. It follows from estimate (4) that in this way a continuous linear operator

$$K_\Omega : L^\infty_{(n,n-1)}(\Omega) \to C^0_{(n,n-2)}(\Omega)$$

is defined. Denote by K^*_Ω the operator from $C^0_{(n,n-2)}(\Omega)'$ to $D^0_{(n,n-1)}(\Omega)'$ defined by

$$K^*_\Omega T(\varphi) = T(K_\Omega \varphi)$$

for $T \in C^0_{(n,n-2)}(\Omega)'$ and $\varphi \in D^0_{(n,n-1)}(\Omega)$. Denote by $L^1(\Omega)$ the Banach space of integrable functions on Ω and set $\langle L^1(\Omega)\rangle := \{\langle f\rangle : f \in L^1(\Omega)\}$. Then it follows from estimate (4) and the fact that $K(z,\xi)$ is continuous for $z \neq \xi$ that the values of K^*_Ω belong to $\langle L^1(\Omega)\rangle$ and the map

$$K^*_\Omega : C^0_{(n,n-2)}(\Omega)' \to \langle L^1(\Omega)\rangle$$

is continuous if we identify $\langle L^1(\Omega)\rangle$ with $L^1(\Omega)$.

Theorem 6.3 *Let $\Omega \subseteq M_0$ be an open set and $f \in L^\infty_{(n,n-1)}(\Omega)$. Then*

$$dK_\Omega f = f.$$

Proof. If φ is a C^1 function with compact support on Ω then, by formula (2), it is

$$\int_\Omega \varphi f = \int_{z\in\Omega}\int_{\xi\in\Omega} d\varphi(\xi)\wedge K(z,\xi)\wedge f(z) = -\int_\Omega d\varphi\wedge K_\Omega f.$$

■

Lemma 6.4 *(i) Let $\varphi \in D^1_{(n,n-2)}(M_0)$. Then the form $\varphi - K_{M_0}d\varphi$ can be approximated in $C^0_{(n,n-2)}(M_0)$ by $\bar\partial$-exact $C^\infty_{(n,n-2)}$-forms on $\mathbb{C}^n$.*

(ii) Let $z \in M_0$ and $B' \subset\subset B_0$ an open ball such that $z \notin \bar B'$. Then the form $K(z,\cdot)$ can be approximated uniformly on $M_0 \cap \bar B'$ by $\bar\partial$-exact $C^\infty_{(n,n-2)}$-forms on $\mathbb{C}^n$.

Both assertions of this lemma are special cases of an approximation theorem of Henkin for arbitrary continuous $\bar\partial$-closed $(n,n-2)$-forms (see the arguments proving relation (6) in [H 2]). Since the proof of this general theorem is not so easy let us give direct proofs:

Proof of Lemma 6.4 (i). Set

$$K^{\pm}_{M_0}d\varphi(\xi) := \int_{z\in M_0} d\varphi(z)\wedge K_\pm(z,\xi) \quad \text{for} \quad \xi\in B_0\cap B_\pm.$$

Then it follows from estimate (41) that the forms $K^{\pm}_{M_0}d\varphi$ admit continuous extensions onto $(B_0\cap B_\pm)\cup M_0$. Further we set

$$\varphi_\pm(\xi) := \int_{z\in M_0} \varphi(z)\wedge B_1(z,\xi) \quad \text{for} \quad \xi\in B_0\cap B_\pm,$$

where $B_1(z,\xi)$ is the part of the Martinelli–Bochner–Koppelman kernel which is of bidegree (0,1) in z. Since φ is Hölder continuous (it is even C^1) it is well known that also the forms $\varphi_\pm$ admit continuous extensions onto $(B_0\cap B_\pm)\cup M_0$. Moreover it follows from the Martinelli–Bochner–Koppelman formula that $\varphi = \varphi_+|_{M_0} - \varphi_-|_{M_0}$ and therefore

$$\varphi - K_{M_0}d\varphi = (\varphi_+ - K^+_{M_0}d\varphi)|_{M_0} - (\varphi_- - K^-_{M_0}d\varphi)|_{M_0}.$$

Using the relations $d_\xi B_1(z,\xi) = -\bar\partial_z B(z,\xi)$ and $d_\xi K^{\pm}_{M_0}(z,\xi) = -B(z,\xi)$ we see that the forms $\varphi_\pm - K^{\pm}_{M_0}d\varphi$ are $\bar\partial$-closed on $B_0\cap B_\pm$. The required assertion on approximation now follows from Proposition 6.2. ■

Proof of Lemma 6.4 (ii). Since B' is pseudoconvex and $z \notin B'$ we can solve the equation $dG = B(z,\cdot)$ with some continuous $(n,n-2)$-form G on B'. Since $dK_\pm(z,\cdot) = -B(z,\cdot)$ the forms $K_\pm(z,\cdot)+G$ are closed on $B'\cap B_\pm$ and the assertion follows from Proposition 6.2 and the representation

$$K(z,\cdot)|_{M_0\cap B'} = (K_+(z,\cdot)+G)|_{M_0\cap B'} - (K_-(z,\cdot)+G)|_{M_0\cap B'}.$$

■

Definition. Let $\Omega \subseteq M_0$ be an open set and let f be a continuous 1-form with compact support on Ω. Then we define

$$K'_\Omega f(z) := \int_{\xi\in\Omega} f(\xi) \wedge K(z,\xi) \quad \text{for} \quad z \in \Omega.$$

It follows from estimate (3) that $K'_\Omega f$ is a continuous function on Ω.

Remark 6.5 Let $\Omega \subseteq M_0$ be an open set and let f be a continuous 1-form with compact support on Ω. Then it follows from Fubini's theorem that $K^*_\Omega\langle f\rangle = \langle K'_\Omega f\rangle$.

Theorem 6.6 *Let $\Omega \subseteq M_0$ be an open set and let $T \in C^0_{(n,n-1)}(\Omega)'$. If $\bar\partial_\Omega T \in C^0_{(n,n-2)}(\Omega)'$, that means if $\bar\partial_\Omega T$ is also of order 0, then*

$$T = -K^*_\Omega \bar\partial_\Omega T.$$

In particular then T is defined by an L^1 function on Ω.

Proof. If $\varphi \in D^1_{(n,n-1)}(\Omega)$ then by Theorem 6.3

$$T(\varphi) = T(dK_\Omega\varphi) = -\bar\partial_\Omega T(K_\Omega\varphi) = -K^*_\Omega\bar\partial_\Omega T(\varphi).$$

Since $D^1_{(n,n-1)}(\Omega)$ is dense in $D^0_{(n,n-1)}(\Omega)$ this implies the assertion. ∎

Remark 6.7 Let $\Omega \subseteq M_0$ be an open set and let $T \in C^0_{(n,n-2)}(\Omega)'$. Then it is easy to see that

$$f(z) := T(K(z,\cdot)), \qquad z \in \bar\Omega\backslash \operatorname{supp} T,$$

is a continuous function and, on $\bar\Omega\backslash\operatorname{supp}T$, $K^*_\Omega T$ is defined by f. Hence for each $T \in C^0_{(n,n-2)}(\Omega)'$, $K^*_\Omega T$ is defined by an L^1 function on Ω which is continuous on $\bar\Omega\backslash\operatorname{supp}T$.

Theorem 6.8 *Let $\Omega \subseteq M_0$ be an open set and let $T \in D^0_{(n,n-1)}(\Omega)'$. If $\bar\partial_\Omega T$ is defined by a continuous 1-form on Ω then T is defined by a continuous function on Ω.*

Proof. Let $\omega \subset\subset \Omega$ be an open and relatively compact subset of Ω. It is sufficient to find a continuous function g on ω with

$$T(\varphi) = \int_\Omega g\varphi \quad \text{for all} \quad \varphi \in D^0_{(n,n-1)}(\omega). \tag{42}$$

Choose a C^1 function χ with compact support on Ω such that $\chi = 1$ in a neighbourhood of $\bar\omega$. Then by Theorem 6.6 we have

$$T(\varphi) = \chi T(\varphi) = -K^*_\Omega(\bar\partial_\Omega(\chi T))(\varphi) = -K^*_\Omega(\chi\bar\partial_\Omega T)(\varphi) - K^*_\Omega(d\chi \wedge T)(\varphi)$$

for all $\varphi \in D^0_{(n,n-1)}(\omega)$. In view of Remarks 6.5 and 6.7 this implies (42) if we set

$$g(z) = -(K'_\Omega(\chi f))(z) - T(d\chi \wedge K(z,\cdot)) \quad \text{for} \quad z \in \omega,$$

where f is the continuous 1-form defining $\bar\partial_\Omega T$. ∎

Corollary 6.9 *Let $\Omega \subseteq M_0$ be open and $T \in D^0_{(n,n-1)}(\Omega)'$ such that $\bar\partial_\Omega T = 0$. Then T is holomorphic in a $\mathbb{C}^n$-neighbourhood of Ω, that means there exists a holomorphic function h in some $\mathbb{C}^n$-neighbourhood of Ω such that*

$$T(\varphi) = \int_\Omega h\varphi \quad \textit{for all} \quad \varphi \in D^0_{(n,n-1)}(\Omega).$$

Proof. This follows from Theorem 6.8 and Proposition 6.1. ■

Corollary 6.9 was obtained by HENKIN (see Theorem 3 in [H 3]). Note that Theorem 6.8 does not follow from Corollary 6.9 (as the corresponding statement for $\bar\partial$) because under the given hypothesis on the Levi form of ϱ the tangential Cauchy–Riemann equation for (0,1)-currents on M_0 cannot be solved locally (see [A/F/N]).

Theorem 6.10 *Let $T \in C^0_{(n,n-2)}(M_0)'$ such that $\bar\partial_{M_0} T = 0$. Denote by ω_T the connected component of $M_0 \backslash \operatorname{supp} T$ whose boundary contains the boundary of M_0. Then*

$$T = -\bar\partial_{M_0} K^*_{M_0} T \tag{43}$$

and

$$\operatorname{supp} K^*_{M_0} T \subseteq M_0 \backslash \omega_T. \tag{44}$$

That under the hypothesis of Theorem 6.10 there exists a L^1 function u on Ω with $\bar\partial_{M_0}\langle u\rangle = T$ and $\operatorname{supp} u \subseteq M_0\backslash\omega_T$ was proved by HENKIN (see Theorem 1′ in [H 2]). The new information contained in Theorem 6.10 consists in the representation

$$\langle u\rangle = -K^*_{M_0} T. \tag{45}$$

Although the validity of this representation follows immediately from Theorem 6.6 let us give also a proof of Theorem 6.10 which is independent of HENKINs result:

Proof of Theorem 6.10. Since $\bar\partial_{M_0} T = 0$ it follows from Lemma 6.4 (i) that for each $\varphi \in D^1_{(n,n-2)}(M_0)$, $T(\varphi - K_{M_0} d\varphi) = 0$ and therefore

$$-\bar\partial_{M_0} K^*_{M_0} T(\varphi) = T(K_{M_0} d\varphi) = T(\varphi).$$

Since $D^1_{(n,n-2)}(M_0)$ is dense in $D^0_{(n,n-2)}(M_0)$ this proves (43).

From (43) and Corollary 6.9 it follows that on $M_0\backslash\operatorname{supp}T$, $K^*_\Omega T$ is defined by some holomorphic function h. Choose an open ball $B' \subset\subset B_0$ centered at z_0 such that $\operatorname{supp} T \subseteq B'$. Then, by Lemma 6.4 (ii), for each $\varphi \in D^0_{(n,n-1)}(M_0\backslash\bar B')$, the form $K_{M_0}\varphi$ can be approximated uniformly on $M_0 \cap \bar B'$ by $\bar\partial$-exact $C^\infty_{(n,n-2)}$-forms on $\mathbb{C}^n$. Since $\bar\partial_{M_0} T = 0$ and $\operatorname{supp} T \subseteq B'$ this implies that

$$\int_{M_0} h\varphi = K^*_\Omega T(\varphi) = T(K_{M_0}\varphi) = 0$$

for all such φ. Hence $h = 0$ on $M_0\backslash\bar B'$ and, by uniqueness of holomorphic functions, $h = 0$ on ω_T, that means (44) is also proved. ■

It was observed by HENKIN (see Theorem 1 in [H 2]) that in the case of sufficiently smooth functions Theorem 6.10 leads to an Hartogs–Bochner extension theorem on M_0 using the same arguments as in EHRENPREIS' proof of the classical Hartogs extension theorem (see the proof of Theorem 2.3.2 in [Hö]). We want to show that using representation (45) and estimate (1) one can prove this theorem also in the case of Hölder continuous functions. Let $\Omega \subset\subset M_0$ be a domain with C^2-bounday. A continuous function f on $b\Omega$ will be called a CR-function if

$$\int_{b\Omega} f d\varphi = 0 \tag{46}$$

for all $C^\infty_{(n,n-3)}$-forms φ on $\mathbb{C}^n$.

Theorem 6.11 *Suppose $M_0\backslash\bar\Omega$ is connected and let f be a Hölder continuous CR-function on $b\Omega$. Then there exists a (unique) continuous function F on $\bar\Omega$ which extends holomorphically to some $\mathbb{C}^n$-neighbourhood of Ω such that $F(z) = f(z)$ for all $z \in b\Omega$. For $z \in \Omega$ this function is given by*

$$F(z) = \int_{\xi\in b\Omega} f(\xi)K(z,\xi). \tag{47}$$

Proof. (All positive constants will be denoted by the same letter C.) First we note that

$$\int_{\xi\in b\Omega} K(z,\xi) = \begin{cases} 1 & \text{for} \quad z \in \Omega \\ 0 & \text{for} \quad z \in M_0\backslash\bar\Omega. \end{cases} \tag{48}$$

If $z \in \Omega$ this follows from (2) and for $z \in M_0\backslash\bar\Omega$ this follows from Stokes' theorem and the fact that $d_\xi K(z,\xi) = 0$. Denote by $T \in C^0_{(n,n-2)}(M_0)'$ the current defined by

$$T(\varphi) = \int_{b\Omega} f\varphi \quad \text{for} \quad \varphi \in C^0_{(n,n-2)}(M_0).$$

Then by (46), $\bar\partial_{M_0} T = 0$ and it follows from Theorem 6.10 that $\operatorname{supp} K^*_{M_0}T \subseteq \bar\Omega$ ($M_0\backslash\bar\Omega$ is connected) and $T = -\bar\partial_{M_0}K^*_{M_0}T$. Since by Remark 6.7 on $M_0\backslash b\Omega$, $K^*_{M_0}T$ is defined by the function

$$z \to \int_{\xi\in b\Omega} f(\xi)K(z,\xi)$$

this implies that

$$\int_{\xi\in b\Omega} f(\xi)K(z,\xi) = 0 \quad \text{for} \quad z \in M_0\backslash\bar\Omega \tag{49}$$

and, by Corollary 6.9, the function F defined by (47) extends holomorphically to some $\mathbb{C}^n$-neighbourhood of Ω.

It remains to prove that

$$\lim_{\Omega\ni z\to\xi_0} F(z) = f(\xi_0) \quad \text{for all} \quad \xi_0 \in b\Omega.$$

For $z \in M_0$ denote by ξ_z a point in $b\Omega$ with $|z - \xi_z| = \text{dist}(z, b\Omega)$ (ξ_z is uniquely determined if z is close to $b\Omega$). Then $f(\xi_z) \to f(\xi_0)$ if $z \to \xi_0$. Therefore it is sufficient to prove that

$$\lim_{\Omega \ni z \to \xi_0} (F(z) - f(\xi_z)) = 0 \quad \text{for all} \quad \xi_0 \in b\Omega. \tag{50}$$

To prove (50) we fix some $\xi_0 \in b\Omega$. Denote by $B_r(\xi_0)$, $r > 0$ the open ball of radius r centered at ξ_0. Set

$$I_r(z) := \int_{\xi \in b\Omega \cap B_r(\xi_0)} |f(\xi) - f(\xi_z)| \, ||K(z,\xi)|_{b\Omega}|| d\lambda(\xi)$$

for $r > 0$ and $z \in M_0 \backslash (b\Omega \cap B_r(\xi_0))$, where $d\lambda(\xi)$ is the Euclidean volume form of $b\Omega$. Since $|\xi - \xi_z| \leq 2|\xi - z|$ and f is Hölder continuous there exists $0 < \alpha_0 < 1$ with

$$|f(\xi) - f(\xi_z)| \leq C|\xi - z|^{\alpha_0} \tag{51}$$

for all $\xi \in b\Omega$ and $z \in M_0$. Fix $0 < \alpha < \alpha_0$ and prove that then

$$I_r(z) \leq Cr^\alpha \tag{52}$$

for all $r > 0$ and $z \in M_0 \backslash (b\Omega \cap B_r(\xi_0))$.

Proof of estimate (52): Since $K(z,\xi)$ is of maximal holomorphic degree in ξ one has

$$||K(z,\xi)|_{b\Omega}|| \leq C||K(z,\xi)|| \, ||\partial\varrho(\xi)|_{b\Omega}|| \tag{53}$$

for all $\xi \in b\Omega$ and $z \in M_0$ with $z \neq \xi$. Set

$$u(z,\xi) := \text{Im} \sum_{j=1}^{n} \frac{\partial\varrho(\xi)}{\partial\xi_i}(\xi_i - z_i).$$

Then

$$|u(z,\xi)| \leq C\delta(z,\xi) \tag{54}$$

and

$$||\partial\varrho(\xi)|_{b\Omega}|| \leq C(||d_\xi u(z,\xi)|_{b\Omega}|| + |\xi - z|) \tag{55}$$

for all $\xi \in b\Omega$ and $z \in M_0$. Set $\varepsilon = (\alpha_0 - \alpha)/2$. Then it follows from (51)-(55) and (1) that

$$\begin{aligned} I_r(z) \;&\leq\; C \int_{\xi \in b\Omega \cap B_r(\xi_0)} \frac{||d_\xi u(z,\xi)|_{b\Omega}|| d\lambda(\xi)}{(|u(z,\xi)| + |\xi - z|^2)|\xi - z|^{2n-3-\alpha-\varepsilon}} \\ &+C \int_{\xi \in b\Omega \cap B_r(\xi_0)} \frac{d\lambda(\xi)}{|\xi - z|^{2n-2-\alpha}} \end{aligned} \tag{56}$$

for all $r > 0$ and $z \in M_0 \backslash (b\Omega \cap B_r(\xi_0))$. It is clear that the second integral in (56) is bounded by Cr^α. To estimate the first integral we use the trick of RANGE and SIU (see the proof of Proposition (3.7) in [R/S]), which allows us to consider $u(z,\cdot)$ as a local coordinate. So we obtain that this integral is bounded by

$$C \int_{\substack{x\in\mathbb{R}^{2n-2} \\ |x|<r}} \frac{dx_1 \wedge \ldots \wedge dx_{2n-2}}{(|x_1| + |x|^2)|x|^{2n-3-\alpha-\epsilon}}.$$

Integrating first with respect to x_1 we see that the last integral is also bounded by Cr^α. Hence estimate (52) is proved.

End of proof of (50): For $r > 0$ we set

$$H_r(z) = \int_{\xi\in b\Omega\cap B_r(\xi_0)} (f(\xi) - f(\xi_z))K(z,\xi)$$

if $z \in M_0\backslash(b\Omega \cap B_r(\xi_0))$ and

$$G_r(z) = \int_{\xi\in b\Omega\backslash B_r(\xi_0)} (f(\xi) - f(\xi_z))K(z,\xi)$$

if $z \in M_0\backslash(b\Omega\backslash B_r(\xi_0))$. Then by (52) it is

$$|H_r(z)| \leq Cr^\alpha \tag{57}$$

for all $r > 0$ and $z \in M_0\backslash(b\Omega \cap B_r(\xi_0))$. Since by (48) and (49), $H_r(z) + G_r(z) = 0$ if $z \in M_0\backslash\bar{\Omega}$ and G_r is continuous on $M_0 \cap B_r(\xi_0)$ this implies that

$$\left|\lim_{\Omega\ni z\to\xi_0} G_r(z)\right| = |G_r(\xi_0)| \leq Cr^\alpha \tag{58}$$

for all $r > 0$. Moreover it follows from (48) that

$$H_r(z) + G_r(z) = F(z) - f(\xi_z)$$

if $z \in \Omega$. In voew of (57) and (58) this implies that for all $r > 0$ we have

$$\limsup_{\Omega\ni z\to\xi_0} |F(z) - f(\xi_z)| \leq Cr^\alpha.$$

■

Remarks to Theorem 6.11.

(i) It follows from this theorem (by standard arguments) that

$$|F(z)| \leq \max_{\xi\in b\Omega} |f(\xi)| \quad \text{for all} \quad z \in \Omega.$$

Hence the assertion of the theorem holds for each continuous CR-function f on $b\Omega$ which can be approximated uniformly by Hölder continuous CR-functions. It is not clear if this is possible for all continuous CR-functions on $b\Omega$.

(ii) We do not assume that the boundary $b\Omega$ is a CR-manifold. Note however that, by the hypothesis on the Levi form of ϱ, the set of points in $b\Omega$ with complex tangent space is nowhere dense in $b\Omega$.

(iii) The hypothesis that $b\Omega$ is of class C^2 is necessary for the Range–Siu trick in the proof.

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