Families of functional equations for polylogarithms

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Abstract

The polylogarithm function Li_m plays an important role as a regulating map for K-groups of an algebraic number field F in that it gives a map on the generalized Bloch group $B_m(F)$ which is conjectured to be an explicit candidate for the K-group $K_{2m-1}(F)$. This is known—up to torsion—for m=2 and 3 and expected for all m.

One of the most important features of Li_m is that it conjecturally satisfies functional equations which occur in the definition of $B_m(F)$. A good understanding of these functional equations as well as a construction of a new and basic one enabled Goncharov to give a proof of Zagier's conjecture for m=3. This conjecture asserts in general that the Dedekind zeta function for F at the point m is expressible in terms of (a modified version of) Li_m .

Little has been known about (non-trivial) functional equations of higher logarithms. There were few examples given by Kummer, and later also by Lewin and Wechsung, up to order m=5 but until recently no example at all was known for order greater than 5.
We give the first families of functional equations in two variables up to the 6-logarithm.

0. Motivation

The classical polylogarithm $Li_m(z)$ for $m \in \mathbf{N}$ is defined by

$$Li_m(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^m}, \qquad z \in \mathbf{C}, \quad |z| \leq 1,$$

and has an analytic continuation onto the cut plane $\mathbf{C} - (1, \infty)$ (in the case $m = 1$ one recognizes the well-known power series expansion of $-\log(1-z)$).

One can associate to this multi–valued function Li_m a one–valued function P_m (cf. chapter 1) which is defined on all of $\mathbf{C}$ and shares certain important properties with Li_m (e.g. the form of functional equations).

A fundamental invariant in algebraic number theory is the Dedekind zeta function of an algebraic number field F

$$\zeta_F(s) = \sum_{\mathcal{A}} \frac{1}{N(\mathcal{A})^s},$$

where the summation is taken over all integral ideals $\mathcal{A} \neq 0$ in F and N denotes the norm function which associates to each ideal its "volume" (a certain natural number). ζ_F is defined a priori for $\Re(s) > 1$ and can be continued analytically onto the whole complex plane.

The following results provided a first link between the two functions given above:

- Dedekind's classical class number formula relates the residue of ζ_F at the point 1 to Li_1.
- Humbert established a relation between Li_2 and $\zeta_F(2)$ for imaginary quadratic number fields F (cf. [Th], chap. 7).

A conjecture of Zagier connects the value $\zeta_F(m)$ of an algebraic number field F with the function P_m. Roughly speaking, the value $\zeta_F(m)$ for $m \in \mathbf{N}$ should be expressible in terms of products of $P_m(z_i)$, $z_i \in F$.

In order to give a more precise statement, we introduce the following notation: let n be the degree of F over $\mathbf{Q}$, r_1 and $2r_2$ the number of real and complex embeddings, respectively. Then we define $d(m) = r_2$ if m is even, $d(m) = r_1 + r_2$ if m is odd and $m > 1$. We write $\Delta_F \in \mathbf{Z}$ for the discriminant of F and $\mathbf{Z}[F]$ for the free abelian group on F. Finally we extend P_m linearly onto $\mathbf{Z}[F]$.

Then a more precise statement of (a part of) Zagier's conjecture is the

Conjecture (Zagier)

There are formal linear combinations $\xi_j \in \mathbf{Z}[F]$, $j = 1, \ldots, d(m)$, such that for $P_m(\xi_j^\sigma)$

$$\zeta_F(m) = q \frac{\pi^{m(n-d(m))}}{|\Delta_F|^{\frac{1}{2}}} \det \left(P_m(\xi_j^\sigma) \right)_{j,\sigma},$$

where $q \in \mathbf{Q}^\times$ and σ runs through all $d(m)$ embeddings of F into $\mathbf{C}$.

The case $m = 1$, $d(1) = r_1 + r_2 - 1$ is a consequence of Dirichlet's theorem mentioned above, for $m = 2$ the conjecture has been proved by Suslin [Su] and (in a weaker form) by Zagier [Zg-I], and the case $m = 3$ was shown by Goncharov [Go] (and independently by Yang [Ya]).

Borel [Bo] proved a corresponding result for certain fundamental invariants of a field F—the higher algebraic K–groups $K_\nu(F)$ defined by Quillen [Qu]—as well as a regulator map (introduced by Borel) $r_{Bor}^{(m)} : K_{2m-1}(F) \to \mathbf{R}^{d(m)}$, namely (using the notation above)

Theorem (Borel)

$$\zeta_F(m) = q \frac{\pi^{m(n-d(m))}}{|\Delta_F|^{\frac{1}{2}}} \det \left(r_{Bor}^{(m)}(\gamma_j^\sigma) \right)_{j,\sigma}$$

for certain $\gamma_j \in K_{2m-1}(F)$ and a $q \in \mathbf{Q}^\times$.

In his pioneering work [Bl] Bloch investigated the case $m = 2$ and tried to give a constructive version $\mathcal{B}_2(F)$ of $K_3(F)$. He connected $r_{Bor}^{(2)}$ to the function $D = P_2$ given above (the Bloch–Wigner dilogarithm) which reflects the role of the function $r_{Bor}^{(2)}$ on $K_3(F)$.

Suslin [Su] showed that Bloch's map from $\mathcal{B}_2(F)$ to $K_3(F)$ is actually an isomorphism (up to tensoring with $\mathbf{Q}$). Therefore functional equations for the dilogarithm somehow reflect the structure of $K_3(F)$.

In order to generalize this situation for any m it was necessary to produce

1) a single–valued version of Li_m (we have called it P_m above) and
2) a generalized version $\mathcal{B}_m(F)$ of Bloch's constructively given group $\mathcal{B}_2(F)$.

The former has been given by Ramakrishnan (implicitly) and by Zagier [Zg-B] and Wojtkowiak [Wo-A] (explicitly), the latter was found by Zagier [Zg-T] and enabled him to formulate the conjecture named after him which relates $K_{2m-1}(F)$ and $\mathcal{B}_m(F)$. Borel's theorem then implies the conjecture stated above.

The stronger form of the conjecture has undergone a motivic interpretation in the work of Deligne and Beilinson [BD] and they were able to give a partial proof: there exists a canonical map $\gamma : \mathcal{B}_m(F) \to K_{2m-1}(F)$ such that $r_{Bor}^{(m)} \circ \gamma = P_m$. The surjectivity of γ has not been proved yet.

As a consequence, functional equations for P_m reflect some structure of $K_{2m-1}(F)$.

For $m = 3$ Goncharov [Go] gave a complete proof of Zagier's conjecture in the course of which a new functional equation for the trilogarithm P_3 plays a crucial role.

Until recently, only functional equations up to order $m = 5$ were known. The first ones for $m = 4, 5$ have been given by Kummer, some others can be found in papers of Wechsung [We-K] und Lewin (e.g. [Le-S], chap. 6).

The considerations above motivate the quest for further functional equations to obtain more insight into the structure of algebraic K-groups (which are very difficult to handle) as well as the motivic cohomology of a field, their most important feature being that the "right" ones (for which we can't give a candidate so far) should play a dominant role in a proof of Zagier's conjecture.

Using a criterion given by Zagier [Zg-T], the search for functional equations reduces to the search for solutions of an algebraic problem.

In this paper we want to develop an approach for a construction of solutions of the latter problem and therefore for the construction of functional equations of a certain type. We give evidence for its usefulness by constructing whole families of functional equations up to order $m = 6$. This approach slightly resembles an attempt given by Wechsung [We-L].

1. Notations and definitions

In this chapter we introduce our objects and the calculus that we will use in the subsequent chapters.

(1.0) Notation

$\mathbf{P}^1_F$ denotes the projective line over the field F, $\mathbf{P}^1(F)$ the set of F–rational points.

(1.1) Definition

The *dilogarithm function* Li_2 is defined for $z \in \mathbf{C}$, $|z| \leq 1$, by

$$Li_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} .$$

(1.2) Properties

Li_2 can be analytically continued onto $\mathbf{C} - [1, \infty)$ via the integral representation

$$Li_2(z) = -\int_0^z \log(1-t)\frac{dt}{t}, \qquad z \in \mathbf{C} - [1, \infty),$$

It is a multi-valued function which jumps by $2\pi i \log|z|$ if one crosses the line $(1, \infty)$ and it is essentially determined by a functional equation in two variables called the five term relation (or Abel relation) since it relates five Li_2–terms. For various forms of this equation cf. [Le-P], ch. 1.5. In this functional equation certain correction terms (i.e. products of logarithms and a rational multiple of π^2) occur which make it rather cumbersome to deal with.

A one–valued function that satisfies the same functional equation without correction terms can be obtained by the following modification (killing the monodromy of Li_2) where $\Im$ denotes the imaginary part.

(1.3) Definition

The *Bloch-Wigner dilogarithm* D is defined on $\mathbf{P}^1(\mathbf{C})$ as

$$D(z) = P_2(z) = \Im(Li_2(z) + \log|z|\log(1-z)), \qquad z \in \mathbf{C} - \{0, 1\},$$

$$D(0) = D(1) = D(\infty) = 0.$$

(1.4) Properties

The function D

(i) is real-analytic on $\mathbf{C} - \{0, 1\}$,

(ii) is continuous on $\mathbf{P}^1(\mathbf{C})$, and

(iii) satisfies the following functional equations

$$D(x) + D(y) + D\Big(\frac{1-x}{1-xy}\Big) + D(1-xy) + D\Big(\frac{1-y}{1-xy}\Big) = 0, \qquad x, y \in \mathbf{C},\ xy \neq 1,$$

$$D(x) + D\Big(\frac{1}{x}\Big) = 0, \quad D(x) + D(1-x) = 0.$$

(iv) D is characterized (up to a multiple in $\mathbf{C}$) by the properties (i)-(iii) (cf. [Bl], [Du]).

(1.5) Definition

The *classical polylogarithm* $Li_m(z)$ for any $m \in \mathbf{N}$ for $z \in \mathbf{C}, |z| \leq 1$, is defined by

$$Li_m(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^m}.$$

We call m the *order* of Li_m.

(1.6) Property

Li_m has an analytic continuation onto $\mathbf{C} - (1, \infty)$ and can be regarded as a multi-valued function on $\mathbf{C} - \{0, 1\}$.

(1.7) Definition

Zagier's *modified polylogarithm* for $m \in \mathbf{N}$, $|z| \leq 1$, is given by

$$P_m(z) = \Re_m\Big(\sum_{r=0}^{m} \frac{2^r B_r}{r!} Li_{m-r}(z)(\log|z|)^r\Big),$$

where $\Re_m$ denotes the real part for odd m and the imaginary part for even m, and B_r is the r−th Bernoulli number ($B_0 = 1, B_1 = -1/2, B_2 = 1/6, \ldots$). For $|z| \geq 1$, $P_m(z)$ is given via the functional equation

(1.7.1) $$P_m(z) = (-1)^{m-1} P_m\Big(\frac{1}{z}\Big).$$

(1.8) Properties

P_m is real-analytic on $\mathbf{C} - \{0, 1\}$ and can be extended continuously onto $\mathbf{P}^1(\mathbf{C})$.

We put $P_m(0) = P_m(\infty) = 0$ and $P_m(1) = \begin{cases} \zeta(m), & \text{if } m \text{ is odd}, \\ 0 & \text{otherwise}, \end{cases}$

where $\zeta(s)$ denotes the Riemann zeta function.

(1.9) Remark

A similar function with less nice properties (with logarithmic poles in 0 as well as "correction terms" in functional equations) had been given before by Ramakrishnan (implicitly) [Ra] and Zagier (explicitly) [Zg-B]; Wojtkowiak [Wo-A] defined a function that plays the same role in our investigations as the one given above since it agrees on the groups that we consider.

For these P_m, $m \geq 2$, the following conjecture has been stated:

(1.10) Conjecture

(i) Each P_m satisfies a non–trivial functional equation $\sum_i n_i P_m(r_i(t_1, \ldots, t_n))$ where the arguments are rational functions $r_i(t_1, \ldots, t_n)$ in several variables t_i and integer coefficients n_i.

(ii) For each m there is a functional equation that characterizes P_m and implies all other functional equations for P_m.

Here we call a functional equation *trivial* if it is a consequence of the *inversion relation* (1.7.1) and the following *distribution relations* (cf. [Le-P], (7.41)):

(1.10.1) $$P_m(z^n) = n^{m-1} \sum_{\zeta^n=1} P_m(\zeta z), \qquad n \in \mathbf{N}_{>0}.$$

(The inversion relation is in some sense also a distribution relation: put $n = -1$ in the above.)

(1.11) Definition

An element $\sum_x n_x[x]$ in $\mathbf{Z}[X]$, the free abelian group on X, is called *effective*, if $n_x \geq 0 \quad \forall x \in X$ and for some $x_0 \in X$: $n_{x_0} > 0$.

The *support* T of $\sum_x n_x[x]$ is defined to be $T = \{x \in X \mid n_x \neq 0\}$.

Now we give an algebraic counterpart for our function P_m.

(1.12) Definition

For an abelian group A we define the *second wedge power* $\bigwedge^2(A)$ as the following subgroup of the $\mathbf{Z}$–module $A^{\otimes 2} = A \otimes_{\mathbf{Z}} A$:

$$\bigwedge^2(A) = \langle a \otimes b \, - \, b \otimes a \mid a, b \in A \rangle.$$

We write

$$a \wedge b = a \otimes b \, - \, b \otimes a.$$

(1.13) Remark

We have

$$a \wedge b = -b \wedge a, \quad a \wedge a = 0.$$

(1.14) Definition

For any field F we define the following homomorphism β_2 on generators by

$$\begin{aligned} \beta_2 = \beta_2^F : \quad \mathbf{Z}[\mathbf{P}^1(F)] &\to \bigwedge^2(F^\times), \\ [f] &\mapsto f \wedge (1-f), \qquad f \neq 0, 1, \infty, \\ [0], [1], [\infty] &\mapsto 0. \end{aligned}$$

Interesting in our context are elements $\xi \in \mathbf{Z}[\mathbf{P}^1(F)]$ belonging to the kernel of β_2 (modulo 2−torsion). We want to emphasize that we are working in the multiplicative group of a field F, and the terminology "up to 2−torsion" just means that we can neglect the element $-1 \in F^\times$ (e.g. replace $(-a) \wedge b$ by $a \wedge b$).

(1.15) Remark

Up to 2−torsion we have

$$\beta_2([z]) = \beta_2\left(\left[\frac{1}{1-z}\right]\right) = \beta_2\left(\left[\frac{z-1}{z}\right]\right)$$
$$= -\beta_2\left(\left[\frac{1}{z}\right]\right) = -\beta_2([1-z]) = -\beta_2\left(\left[\frac{z}{z-1}\right]\right).$$

(1.16) Definition

For an abelian group A and $k \in \mathbf{N}$ we define the *k-th symmetric power* $\mathrm{Sym}^k(A)$ as the invariants under the symmetric group S_k acting on the k-th tensor power $A^{\otimes k} = A \otimes_{\mathbf{Z}} \ldots \otimes_{\mathbf{Z}} A$ (k factors) by permutation of the factors:

$$\mathrm{Sym}^k(A) := (A \otimes \ldots \otimes A)^{S_k}.$$

We write

$$a^{\odot k} := a \otimes \ldots \otimes a \in \mathrm{Sym}^k(A).$$

After tensoring with $\mathbf{Q}$ we denote the monomial elements in $\mathrm{Sym}^k(A) \otimes \mathbf{Q}$ by

$$a_1 \odot \ldots \odot a_k := \frac{1}{k!} \sum_{\sigma \in S_k} a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(k)} \in \mathrm{Sym}^k(A) \otimes \mathbf{Q}.$$

Occasionally we abbreviate "$a_1 \odot \ldots \odot a_k \in \mathrm{Sym}^k(A) \otimes \mathbf{Q}$" by "$a_1 \odot \ldots \odot a_k \in \mathrm{Sym}^k(A)$".

We use the basic fact that for the elements in $\mathrm{Sym}^k(A) \otimes \mathbf{Q}$ there is a polarisation property, i.e. the $a^{\odot k}$ with $a \in A$ generate $\mathrm{Sym}^k(A) \otimes \mathbf{Q}$.

(1.17) Remark

Let M be a free module over an integral domain R of rank r and with basis $\{\mu_i\}_{i=1}^r$. Let $R[x_1, \ldots, x_r]$ be the polynomial ring over R in r variables.

There is an R−module isomorphism

$$\mathrm{Sym}^*(M) := \bigoplus_{k \geq 0} \mathrm{Sym}^k(M) \cong R[x_1, \ldots, x_r]$$
$$\mu_{i_1} \odot \ldots \odot \mu_{i_k} \mapsto x_{i_1} \cdot \ldots \cdot x_{i_k}, \quad k \in \mathbf{N}.$$

Having chosen such an isomorphism we can identify identities in $\mathrm{Sym}^k(M)$ with identities in a suitable polynomial ring.

(1.19) Remark

(i) From the definitions we immediately get for each $z \in F$

$$\beta_m([z]) = z^{\odot(m-2)} \otimes \beta_2([z]).$$

(ii) For β_m there are the following inversion and distribution relations (ζ runs through all the n-th roots of unity in $\overline{F}$ if $n > 0$, and we put $\zeta = 1$ if $n = -1$):

$$\beta_m([z^n]) = n^{m-1} \sum_{\zeta^n=1} \beta_m([\zeta z]), \qquad n \in \mathbf{N} \text{ or } n = -1.$$

Finally we formulate Zagier's criterion revealing the connection between the analytic function P_m and the algebraic homomorphism β_m which has been fundamental for our investigations. Hereafter we concentrate mainly on β_m.

(1.20) Criterion (Zagier)

Let $n_i \in \mathbf{Z}$, $x_i(t) \in \mathbf{C}(t)^\times$, $i \in I$ (index set).
For $\xi(t) = \sum_i n_i[x_i(t)] \in \mathbf{Z}[\mathbf{C}(t)^\times]$ the following holds

$$\beta_m(\xi(t)) = 0 \quad \Longrightarrow \quad P_m(\xi(t)) = \text{constant}.$$

Here $\beta_m : \mathbf{Z}[\mathbf{C}(t)^\times] \longrightarrow \mathrm{Sym}^{m-2}(\mathbf{C}(t)^\times) \otimes \bigwedge^2(\mathbf{C}(t)^\times)$ is the homomorphism given above and P_m is extended linearly, i.e. $P_m(\sum_i n_i[x_i(t)]) = \sum_i n_i P_m(x_i(t))$.

(1.21) Example $m = 3$,

$$2\beta_3\Big([t] + [\frac{1}{1-t}] + [1 - \frac{1}{t}]\Big) = 0 \quad \text{in } \mathbf{C}(t)^\times \otimes \bigwedge^2(\mathbf{C}(t)^\times).$$

According to the criterion (1.20) above we have

$$P_3(t) + P_3(1/(1-t)) + P_3(1 - 1/t) = C$$

where C is a constant (indeed, $C = \zeta(3)$, which is obtained by putting $t = 0$).

2. Separation of variables

In this chapter we give the basic intermediate result (proposition (2.12)) for the examples examined in chapters 3 and 4. There we always have two (finite) subsets X, Y in a field F as well as arguments "in two variables", i.e. of the form $f(x)g(y)$ for $x \in X$, $y \in Y$. Using the aforementioned result, we succeed in separating the images of these arguments under β_m in a certain way. In the subsequent chapters we will investigate the separated terms obtained in this manner.

(2.0) General assumption

Let F be a field of characteristic 0, $\overline{F}$ an algebraic closure of F.

Denote by $\mathbf{P}^1$ the projective line over F which we identify with the point set $\mathbf{P}^1(\overline{F})$.

For $\alpha \in \mathbf{Z}$ we put

$\alpha^+ = \max(\alpha, 0)$,

$\alpha^- = \min(\alpha, 0)$.

For a rational function $\phi : \mathbf{P}^1 \to \mathbf{P}^1$ we denote the (mapping) degree by $\deg(\phi)$ and the support of the divisor of ϕ by $\mathrm{supp}(\phi) = \phi^{-1}(\{0, \infty\})$.

(2.1) Definition

Let $\phi : \mathbf{P}^1 \to \mathbf{P}^1$ be a rational function (defined over F).

Then $Y \subset \mathbf{P}^1$ is called *full with respect to* ϕ if $\phi^{-1}\phi(Y) = Y$.

We call $Y \subset \mathbf{P}^1$ *truly* full with respect to ϕ if actually $|\phi^{-1}\phi(y)| = \deg(\phi)$ $\forall y \in Y$ (i.e. no element of Y is a critical point of ϕ, $\phi'(y) \neq 0$ for $y \in Y$).

Our goal is to find sets Y and "big" collections Φ of rational functions with given support T such that Y is truly full with respect to ϕ for all $\phi \in \Phi$.

(2.2) Simple properties

Let $\phi : \mathbf{P}^1 \to \mathbf{P}^1$ be a rational function (defined over F), of degree $\deg(\phi)$.

(i) If $Y \subset \mathbf{P}^1$ with $|Y| = \deg(\phi)$ and $|\phi(Y)| = 1$ then Y is (truly) full w.r.t. ϕ.

(ii) If Y_1 and Y_2 are full w.r.t. ϕ then $Y_1 \cup Y_2$ is.

(iii) For $\deg(\phi) = 1$, i.e. $\phi(z) = \frac{az+b}{cz+d}$, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(F)$, each subset $Y \subset \mathbf{P}^1$ is truly full w.r.t. ϕ.

(iv) $Y \subset \mathbf{P}^1$ (truly) full w.r.t. $\phi \implies Y \subset \mathbf{P}^1$ (truly) full w.r.t. $\frac{a\phi+b}{c\phi+d}$, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(F)$.

In particular, $Y \subset \mathbf{P}^1$ (truly) full w.r.t. $\phi \implies Y \subset \mathbf{P}^1$ (truly) full w.r.t. $1/\phi$.
In addition, $\mathrm{supp}(\phi) = \mathrm{supp}(1/\phi)$.

Therefore we shall from now on restrict ourselves to listing only one of the two functions $\phi, 1/\phi$.

(v) For each $f \in \overline{F}$ $\{f, 1/f\}$ is full w.r.t. $\phi(z) = \dfrac{(z-1)^2}{z}$ (and truly full for $f \neq \pm 1$).

(vi) For each $f \in \overline{F}$ $\{f, 1-f\}$ is full w.r.t. $\phi(z) = z(z-1)$ (and truly full for $f \neq 1/2$).

(vii) For each $f \in \overline{F}$ $\{f, f/(f-1)\}$ is full w.r.t. $\phi(z) = \dfrac{z^2}{z-1}$ (and truly full for $f \neq 0, 2$).

(2.3) Examples

(i) (2.2)(iii),(v),(vi),(vii) imply: for $f \in \overline{F}$, $X = \{f, \frac{1}{f}, \frac{f-1}{f}, \frac{f}{f-1}, \frac{1}{1-f}, 1-f\}$ is closed under the involutions $x \mapsto 1/x$, $x \mapsto 1-x$, $x \mapsto x/(x-1)$, therefore
X is full w.r.t. each of the functions $\phi(z) = \dfrac{z}{(z-1)^2}, z(z-1), \dfrac{z^2}{z-1}$,
X is truly full for $f \notin \{-1, 0, 1/2, 1, 2\}$.
Also X is truly full w.r.t. $\phi(z) = z, z-1, \dfrac{z}{z-1}$ ($\deg(\phi) = 1$).

(ii) Let $f \in \overline{F}$, $f \notin \{-1, 0, 1/2, 1, 2\}$, $Y = \left\{\dfrac{f}{1-f+f^2}, \dfrac{1-f}{1-f+f^2}, \dfrac{-f(1-f)}{1-f+f^2}\right\}$ and $\phi(z) = z^2(z-1)$.
Then $\phi(Y) = \left\{-\dfrac{f^2(f-1)^2}{(1-f+f^2)^3}\right\}$ and $|Y| = 3 = \deg(\phi)$.
Thus Y is truly full w.r.t. ϕ.
As in (i) Y is also truly full w.r.t. $\phi(z) = z, z-1, \dfrac{z}{z-1}$.

We are interested in functions with zeros $\phi^{-1}(0)$ and poles $\phi^{-1}(\infty)$ in $\mathbf{P}^1(F)$ or even in $\{0, 1, \infty\}$. They are (up to constant factors) in 1-to-1 correspondence with the elements of $\mathbf{Z}[F]$.

More precisely, for each subset $T \subset F$ there is a correspondence between $\mathbf{Z}[T]$ and the set of those normed rational functions ϕ, for which $\phi^{-1}(\{0, \infty\}) \subset T \cup \{\infty\}$.
This motivates the following definition.

(2.4) Definition

Let $a = \sum_t a_t[t] \in \mathbf{Z}[F]$.

(i) We associate to a the following rational function (on the affine line over F):

$$\phi_a(z) = \prod_{t \in F} (z-t)^{a_t}, \qquad \forall z \in \overline{F}.$$

(ii) We also put

$$\phi_a^{\pm}(z) = \prod_{t \in F} (z-t)^{a_t^{\pm}} \qquad \forall z \in \overline{F},$$

(i.e. ϕ_a^+ = "numerator" of ϕ_a, $1/\phi_a^-$ = "denominator" of ϕ_a)

(iii) as well as

$$d(a) = \max\left(\sum a_t^+, -\sum a_t^-\right) = \deg(\phi_a)$$

(iv) and

$$\chi(a) = \begin{cases} 1, & \text{if } \sum a_t^+ = -\sum a_t^-, \\ 0 & \text{otherwise,} \end{cases}$$

i.e. $\chi(a) = 1$, if $\phi_a(\infty) \notin \{0, \infty\}$.

(v) Finally we define the *support* of $\mathcal{A} \subset \mathbf{Z}[F]$ as

$$\operatorname{supp}(\mathcal{A}) = \bigcup_{a \in \mathcal{A}} \{t \in F \mid a_t \neq 0\}.$$

In many examples we will have $a \in \mathbf{Z}[\{0,1\}]$ and F will be chosen in such a way that Y already lies in F and is full w.r.t. several rational functions. We introduce the following notations for simplicity.

(2.5) Notation and definition

(i) $(r,s) = r[0] + s[1] \in \mathbf{Z}[\{0,1\}]$ for $r, s \in \mathbf{Z}$, therefore $\phi_{(r,s)}(z) = z^r(z-1)^s$.

(ii) $Y \subset F$ is called *(truly) full w.r.t.* $\mathcal{A} \subset \mathbf{Z}[F]$ if Y is (truly) full w.r.t. all ϕ_a, $a \in \mathcal{A}$.

With these notations, example (2.3)(i) can be written as

X is full w.r.t. $\{(1,-2), (1,1), (2,-1), (1,0), (0,1), (1,-1)\}$.

(2.6) Lemma

Let $a = \sum_t a_t[t] \in \mathbf{Z}[F]$, let $C \in F^\times$ such that $Y = \phi_a^{-1}(C) \subset F$ and $|Y| = d(a)$ (therefore Y is truly full w.r.t. ϕ_a).

Then for an indeterminate x and a certain $\lambda_a(C) \in F$

$$1 - \frac{\phi_a(x)}{C} = \phi_a^-(x) \cdot \prod_{y \in Y} (x - y) \cdot \lambda_a(C),$$

where

$$\lambda_a(C) = \begin{cases} -\frac{1}{C}, & \text{if } \phi_a(\infty) = \infty, \\ 1 - \frac{\phi_a(\infty)}{C} & \text{otherwise.} \end{cases} \tag{2.6.1}$$

Also for each $t \in \operatorname{supp}(\{a\})$ one of the following two factorisations holds

$$\lambda_a(C) = \begin{cases} -\dfrac{1}{\phi_a^-(t)} \cdot \displaystyle\prod_{y \in Y} \frac{1}{t - y}, & \text{if } a_t > 0, \\ \dfrac{\phi_a^+(t)}{C} \cdot \displaystyle\prod_{y \in Y} \frac{1}{t - y}, & \text{if } a_t < 0. \end{cases} \tag{2.6.2}$$

Proof For a certain $D \in F$ we obtain

$$1 - \frac{\phi_a(x)}{C} = -\frac{\phi_a^-(x)}{C} \cdot \left(\phi_a^+(x) - C\left(\phi_a^-(x)\right)^{-1}\right) = -\frac{\phi_a^-(x)}{C} \cdot \prod_{y \in Y} (x - y) \cdot D, \tag{2.6.3}$$

since the expression in brackets is a polynomial with zero set Y (note $|Y| = d(a)$).

(2.6.1): $\phi_a(\infty) = \infty :$ $\deg(\phi_a^+) > \deg(1/\phi_a^-)$, the highest coefficient of ϕ_a^+ is 1, therefore $D = 1$.

$\phi_a(\infty) \neq \infty :$ $\lim_{x \mapsto \infty} \left(\phi_a^-(x) \cdot \prod_{y \in Y} (x - y)\right) = 1$, whence $D = \phi_a(\infty) - C$.

(2.6.2): $a_t > 0 \quad \Rightarrow \quad \phi_a(t) = 0,\ D^{-1} = -\frac{\phi_a^-(t)}{C} \cdot \prod_{y \in Y} (t - y)$, therefore

$$\lambda_a(C) = \frac{D}{C} = -\frac{1}{\phi_a^-(t) \prod_{y \in Y} (t - y)}.$$

$a_t < 0 \quad \Rightarrow \quad \left(\phi_a(t)\right)^{-1} = 0,$ transforming (2.6.3) gives

$$\frac{1}{\phi_a(x)} - \frac{1}{C} = -\frac{1}{\phi_a^+(x)} \cdot \prod_{y \in Y} (x - y) \cdot \frac{D}{C},$$

Substituting t yields

$$\lambda_a(C) = \frac{D}{C} = \frac{\phi_a^+(t)}{C} \cdot \prod_{y \in Y} \frac{1}{t - y}. \qquad \diamond$$

(2.7) Proposition

Let $a = \sum_t a_t[t] \in \mathbf{Z}[F]$, T the support of a, $x \in F$ such that $\phi_a(x) \in F^\times$. For $C \in F^\times$ assume $Y = \phi_a^{-1}(C) \subset F$ and $|Y| = d(a)$.

Then the following identity holds in $\bigwedge^2(F^\times)$ (up to 2–torsion):

$$\beta_2\left(\left[\frac{\phi_a(x)}{C}\right]\right) = \frac{\phi_a(x)}{C} \wedge \prod_{y\in Y}(x-y) + \sum_{t\in T}\sum_{y\in Y} a_t\left((y-t)\wedge(x-t)\right)$$
$$+ \sum_{t,u\in T} a_t^+ a_u^- \,\beta_2\left(\left[\frac{x-u}{x-t}\right]\right) - \chi(a)\,\beta_2([C]).$$

Proof Using Lemma (2.6) and (1.13) we get

$$\beta_2\left(\left[\frac{\phi_a(x)}{C}\right]\right) = \frac{\phi_a(x)}{C} \wedge \left(\phi_a^-(x)\prod_{y\in Y}(x-y)\,\lambda_a(C)\right)$$
$$= \phi_a^+(x)\wedge\phi_a^-(x) + \phi_a^-(x)\wedge\phi_a^-(x) - C\wedge\phi_a^-(x) + \frac{\phi_a(x)}{C}\wedge\prod_{y\in Y}(x-y)$$
$$+ \phi_a^+(x)\wedge\lambda_a(C) + \phi_a^-(x)\wedge\lambda_a(C) - C\wedge\lambda_a(C)$$
$$= \phi_a^+(x)\wedge\phi_a^-(x) - C\wedge\phi_a^-(x) + \frac{\phi_a(x)}{C}\wedge\prod_{y\in Y}(x-y)$$
$$+ \sum_{t\in T} a_t^+\cdot(x-t)\wedge\frac{1}{\prod_{y\in Y}(t-y)\,\phi_a^-(t)}$$
$$+ \sum_{t\in T} a_t^-\cdot(x-t)\wedge\frac{\phi_a^+(t)}{C\prod_{y\in Y}(t-y)}$$
$$- C\wedge\begin{cases} -\frac{1}{C}, & \text{if } \phi_a(\infty)=\infty,\\ 1, & \text{if } \phi_a(\infty)=0,\\ 1-\frac{1}{C}, & \text{if } \phi_a(\infty)=1.\end{cases}$$
$$= \prod_{t\in T}(x-t)^{a_t^+}\wedge\prod_{u\in T}(x-u)^{a_u^-} - C\wedge\phi_a^-(x) + \frac{\phi_a(x)}{C}\wedge\prod_{y\in Y}(x-y)$$
$$-\sum_{t\in T}\sum_{y\in Y} a_t\left((x-t)\wedge(y-t)\right) - \sum_{t\in T}(x-t)^{a_t^+}\wedge\phi_a^-(t)$$
$$+\sum_{u\in T}(x-u)^{a_u^-}\wedge\phi_a^+(u) - \prod_{u\in T}(x-u)^{a_u^-}\wedge C - \chi(a)\,\beta_2([C])$$

and because of $\prod_u (x-u)^{a_u^-} = \phi_a^-(x)$ and (1.13) two summands cancel and we get

$$\beta_2\left(\left[\frac{\phi_a(x)}{C}\right]\right) = \frac{\phi_a(x)}{C} \wedge \prod_{y\in Y}(x-y) \;+\; \sum_{t\in T}\sum_{y\in Y} a_t((y-t)\wedge(x-t)) \;-\; \chi(a)\beta_2([C])$$
$$+\sum_{t,u\in T}\left(\frac{x-t}{u-t}\right)^{a_t^+} \wedge \left(\frac{x-u}{t-u}\right)^{a_u^-},$$

since we have the following factorisation up to 2-torsion (using (1.13) again)

$$\sum_{t,u\in T}\left(\frac{x-t}{u-t}\right)^{a_t^+} \wedge \left(\frac{x-u}{t-u}\right)^{a_u^-}$$
$$= \sum_{t,u\in T}(x-t)^{a_t^+}\wedge(x-u)^{a_u^-} - \sum_{t\in T}(x-t)^{a_t^+}\wedge\left(\prod_{u\in T}(t-u)^{a_u^-}\right)$$
$$-\sum_{u\in T}\left(\prod_{t\in T}(u-t)^{a_t^+}\right)\wedge(x-u)^{a_u^-} + \sum_{t,u\in T}(u-t)^{a_t^+}\wedge(t-u)^{a_u^-}$$
$$= \left(\prod_{t\in T}(x-t)^{a_t^+}\right)\wedge\left(\prod_{u\in T}(x-u)^{a_u^-}\right) - \sum_{t\in T}(x-t)^{a_t^+}\wedge\phi_a^-(t)$$
$$-\sum_{u\in T}\phi_a^+(u)\wedge(x-u)^{a_u^-},$$

and the claim follows by

$$\left(\frac{x-t}{u-t}\right)\wedge\left(\frac{x-u}{t-u}\right) = \beta_2\left(\left[\frac{x-t}{u-t}\right]\right). \qquad \diamond$$

(2.8) Corollary

Let $a=\sum_t a_t[t]\in \mathbf{Z}[F]$, T the support of a, $x\in F$ such that $\phi_a(x)\in F^\times$. For $C\in F^\times$ assume $Y=\phi_a^{-1}(C)\subset F$ and $|Y|=d(a)$. Then for each $m\in \mathbf{N}_{>1}$ the following identity holds in $\mathrm{Sym}^{m-2}(F^\times)\otimes\bigwedge^2(F^\times)$ (up to 2-torsion):

$$\beta_m\left(\left[\frac{\phi_a(x)}{C}\right]\right) = \left(\frac{\phi_a(x)}{C}\right)^{\odot(m-2)}\otimes\left(\frac{\phi_a(x)}{C}\right)\wedge\prod_{y\in Y}(x-y)$$
$$+\left(\frac{\phi_a(x)}{C}\right)^{\odot(m-2)}\otimes\sum_{t,y} a_t\left((y-t)\wedge(x-t)\right)$$
$$+\left(\frac{\phi_a(x)}{C}\right)^{\odot(m-2)}\otimes\left(\sum_{t,u\in T} a_t^+a_u^-\,\beta_2\left(\left[\frac{x-u}{x-t}\right]\right) - \chi(a)\beta_2([C])\right).$$

Analogously (interchanging the roles of x and y):
For $B \in F^\times$ assume $X = \phi_a^{-1}(B) \subset F$ and $|X| = d(a)$.
Then for $y \in F$ such that $\phi_a(y) \in F^\times$ the following identity holds (up to 2-torsion) in $\mathrm{Sym}^{m-2}(F^\times) \otimes \bigwedge^2(F^\times)$:

$$\beta_m\left(\left[\frac{D}{\phi_a(y)}\right]\right) = \left(\frac{B}{\phi_a(y)}\right)^{\odot(m-2)} \otimes \left(\frac{B}{\phi_a(y)}\right) \wedge \prod_{x\in X}(x-y)$$
$$+ \left(\frac{B}{\phi_a(y)}\right)^{\odot(m-2)} \otimes \sum_{t,x} a_t\left((y-t)\wedge(x-t)\right)$$
$$- \left(\frac{B}{\phi_a(y)}\right)^{\odot(m-2)} \otimes \left(\sum_{t,u\in T} a_t^+ a_u^- \beta_2\left(\left[\frac{y-u}{y-t}\right]\right) - \chi(a)\,\beta_2([B])\right).$$

◇

(2.9) Remark

Let M be a free module over a ring R. Then an R-module homomorphism $\lambda : M \to R$ induces for all $k \in \mathbf{N}$ an R-module homomorphism

$$M^{\otimes k} \to M^{\otimes(k-1)}$$
$$m_1 \otimes \ldots \otimes m_k \mapsto \sum_{j=1}^{k} m_1 \otimes \ldots \otimes \lambda(m_j) \otimes \ldots \otimes m_k$$

and thus also an R-module homomorphism

$$\mathrm{Sym}^k(M) \to \mathrm{Sym}^{k-1}(M)$$
$$x^{\odot k} \mapsto \lambda(x) \cdot x^{\odot(k-1)}.$$

(2.10) Remark ("tensor derivation")

Let $\mathcal{A} \subset \mathbf{Z}[F]$, $T = \mathrm{supp}(\mathcal{A}) \subset F$ be the support of $\mathcal{A}$ and let $\nu \in \mathbf{N}$.
Let $\{a^{\odot k}\}_{a\in\mathcal{A}} \subset \mathrm{Sym}^k(\mathbf{Z}[T])$ $\mathbf{Z}$–linearly dependent with $\sum_{a\in\mathcal{A}} n(a)\, a^{\odot k} = 0$.
Then for each homogeneous polynomial $p(z)$ of degree $j(p)$ in $z = (z_1, \ldots, z_{|T|})$ we have

$$\sum_{a\in\mathcal{A}} n(a)\, p(a)\, a^{\odot(k-j(p))} = 0.$$

Proof If we evaluate a monomial of degree j in $a \in \mathcal{A}$ then the result can be interpreted as the successive application of j homomorphisms of abelian groups (i.e. $\mathbf{Z}$-modules) as in remark (2.9), and the linear relation in the claim is the homomorphic image of the assumed relation. ◇

(2.11) Notation We use the notation $M\dot{\cup}N$ for the disjoint union of two sets M, N.

(2.12) Proposition

Let $X, Y \subset F$ be finite subsets and $\mathcal{A} = \mathcal{A}_X \dot{\cup} \mathcal{A}_Y \subset \mathbf{Z}[F]$ satisfying

(i) X is truly full w.r.t. $\mathcal{A}_X$,

(ii) Y is truly full w.r.t. $\mathcal{A}_Y$,

(iii) $\dfrac{\phi_a(x)}{\phi_a(y)} \in F^\times \qquad \forall a \in \mathcal{A},\ x \in X,\ y \in Y$.

Let $T = \operatorname{supp}(\mathcal{A})$ and for some $m \in \mathbf{N}_{>1}$ assume the following $\mathbf{Z}$–linear dependence relation:

$$\sum_{a \in \mathcal{A}} n(a)\, a^{\odot m-1} = 0. \tag{2.12.1}$$

Then the following identity in $\operatorname{Sym}^{m-2}(F^\times) \otimes \bigwedge^2(F^\times)$ holds (up to 2-torsion):

$$\sum_{x \in X} \sum_{y \in Y} \sum_{a \in \mathcal{A}} \frac{n(a)}{d(a)}\, \beta_m\left(\left[\frac{\phi_a(x)}{\phi_a(y)}\right]\right)$$

(2.12.2)

$$= \sum_{x,y} \sum_{a \in \mathcal{A}_X} \frac{n(a)}{d(a)} \left(\frac{\phi_a(x)}{\phi_a(y)}\right)^{\odot(m-2)} \otimes \left(\sum_{t,u \in T} a_t^+ a_u^-\, \beta_2\left(\left[\frac{y-u}{y-t}\right]\right) - \chi(a)\, \beta_2([\phi_a(x)])\right)$$
$$- \sum_{x,y} \sum_{a \in \mathcal{A}_Y} \frac{n(a)}{d(a)} \left(\frac{\phi_a(x)}{\phi_a(y)}\right)^{\odot(m-2)} \otimes \left(\sum_{t,u \in T} a_t^+ a_u^-\, \beta_2\left(\left[\frac{x-u}{x-t}\right]\right) - \chi(a)\, \beta_2([\phi_a(y)])\right).$$

Proof

For fixed $x, y \in F$ with property (iii) the map $h : \langle \mathcal{A} \rangle \to F^\times$, $h(a) = \dfrac{\phi_a(x)}{\phi_a(y)}$, is a homomorphism of abelian groups, therefore (2.12.1) also implies

$$\sum_{a \in \mathcal{A}} n(a) \left(\frac{\phi_a(x)}{\phi_a(y)}\right)^{\odot m-1} = 0 \tag{2.12.3}$$

and using remark (2.10)

$$\sum_{a \in \mathcal{A}} n(a)\, a_t \left(\frac{\phi_a(x)}{\phi_a(y)}\right)^{\odot(m-2)} = 0 \quad \forall t \in T. \tag{2.12.4}$$

From (2.12.3) we get

$$0 = \sum_{x \in X} \sum_{y \in Y} \sum_{a \in \mathcal{A}} n(a) \left(\frac{\phi_a(x)}{\phi_a(y)}\right)^{\odot m-1} \otimes (x-y)$$

$$
\begin{aligned}
&= \sum_{y} \sum_{a \in \mathcal{A}_X} n(a) \sum_{x \in X} \frac{1}{d(a)} \sum_{x' \in \phi_a^{-1}\phi_a(x)} \left(\frac{\phi_a(x')}{\phi_a(y)} \right)^{\odot m-1} \otimes (x' - y) \\
(2.12.5) \qquad &+ \sum_{x} \sum_{a \in \mathcal{A}_Y} n(a) \sum_{y \in Y} \frac{1}{d(a)} \sum_{y' \in \phi_a^{-1}\phi_a(y)} \left(\frac{\phi_a(x)}{\phi_a(y')} \right)^{\odot m-1} \otimes (x - y') \\
&= \sum_{x,y} \sum_{a \in \mathcal{A}_X} \frac{n(a)}{d(a)} \left(\frac{\phi_a(x)}{\phi_a(y)} \right)^{\odot m-1} \otimes \prod_{x' \in \phi_a^{-1}\phi_a(x)} (x' - y) \\
&+ \sum_{x,y} \sum_{a \in \mathcal{A}_Y} \frac{n(a)}{d(a)} \left(\frac{\phi_a(x)}{\phi_a(y)} \right)^{\odot m-1} \otimes \prod_{y' \in \phi_a^{-1}\phi_a(y)} (x - y').
\end{aligned}
$$

Analogously we conclude from (2.12.4)

$$
\begin{aligned}
(2.12.6)\; 0 = &\sum_{x,y} \sum_{a \in \mathcal{A}_X} \frac{n(a)}{d(a)} \left(\frac{\phi_a(x)}{\phi_a(y)} \right)^{\odot(m-2)} \otimes \sum_{t \in T} a_t \sum_{x' \in \phi_a^{-1}\phi_a(x)} (y - t) \otimes (x' - t) \\
+ &\sum_{x,y} \sum_{a \in \mathcal{A}_Y} \frac{n(a)}{d(a)} \left(\frac{\phi_a(x)}{\phi_a(y)} \right)^{\odot(m-2)} \otimes \sum_{t \in T} a_t \sum_{y' \in \phi_a^{-1}\phi_a(y)} (y' - t) \otimes (x - t).
\end{aligned}
$$

The homomorphic image of identity (2.12.5) under the map

$$
\begin{gathered}
\mathrm{Sym}^{m-1}(F^\times) \otimes F^\times \to \mathrm{Sym}^{m-2}(F^\times) \otimes \textstyle\bigwedge^2(F^\times), \\
x^{\odot m-1} \otimes y \mapsto x^{\odot(m-2)} \otimes (x \wedge y),
\end{gathered}
$$

yields together with the homomorphic image of (2.12.6) under the map

$$
\begin{gathered}
\mathrm{Sym}^{m-2}(F^\times) \otimes F^\times \otimes F^\times \to \mathrm{Sym}^{m-2}(F^\times) \otimes \textstyle\bigwedge^2(F^\times), \\
x^{\odot(m-2)} \otimes y \otimes z \mapsto x^{\odot(m-2)} \otimes (y \wedge z),
\end{gathered}
$$

the corresponding relations in $\mathrm{Sym}^{m-2}(F^\times) \otimes \bigwedge^2(F^\times)$. Summarising, we get

$$
\begin{aligned}
0 = &\sum_{x,y} \sum_{a \in \mathcal{A}_X} \frac{n(a)}{d(a)} \left(\frac{\phi_a(x)}{\phi_a(y)} \right)^{\odot(m-2)} \otimes \\
&\otimes \left(\frac{\phi_a(x')}{\phi_a(y)} \wedge \prod_{x' \in \phi_a^{-1}\phi_a(x)} (x' - y) + \sum_{t \in T} a_t \sum_{x' \in \phi_a^{-1}\phi_a(x)} (y - t) \wedge (x' - t) \right) \\
+ &\sum_{x,y} \sum_{a \in \mathcal{A}_Y} \frac{n(a)}{d(a)} \left(\frac{\phi_a(x)}{\phi_a(y)} \right)^{\odot(m-2)} \otimes \\
&\otimes \left(\frac{\phi_a(x)}{\phi_a(y')} \wedge \sum_{y' \in \phi_a^{-1}\phi_a(y)} (x - y') + \sum_{t \in T} a_t \sum_{y' \in \phi_a^{-1}\phi_a(y)} (y' - t) \wedge (x - t) \right).
\end{aligned}
$$

The identity in the claim now follows from corollary (2.8). ◇

As an immediate consequence we obtain a simple criterion for elements in $\ker(\beta_m)$ of arbitrary order m.

(2.13) Corollary

Let $X, Y \subset F$ be finite subsets and $\mathcal{A} = \mathcal{A}_X \dot{\cup} \mathcal{A}_Y \subset \mathbf{Z}[F]$ having the properties

(i) X is truly full w.r.t. $\mathcal{A}_X$,

(ii) Y is truly full w.r.t. $\mathcal{A}_Y$,

(iii) $\dfrac{\phi_a(x)}{\phi_a(y)} \in F^\times \qquad \forall a \in \mathcal{A},\, x \in X,\, y \in Y$.

(iv) Each $a \in \mathcal{A}$ is effective.

For an $m \in \mathbf{N}_{>1}$ assume the following $\mathbf{Z}$–linear dependence relation:

$$\sum_{a \in \mathcal{A}} n(a)\, a^{\odot m-1} = 0.$$

Then

$$\sum_{x\in X} \sum_{y \in Y} \sum_{a \in \mathcal{A}} \frac{n(a)}{d(a)}\, \beta_m \left(\left[\frac{\phi_a(x)}{\phi_a(y)} \right] \right) = 0.$$

Proof $\forall a \in \mathcal{A}$ the following properties (i),(ii) are satisfied.

(i) $\forall t, u \in F$: $a_t^+ a_u^- = 0$ and (ii) $\chi(a) = 0$.

Now we use proposition (2.12). ◇

Of course it is our goal to choose sets X, Y in such a way that $\mathcal{A}$ becomes as large as possible since we have much better chances of constructing elements in $\ker \beta_m$ for bigger m.

(2.14) Example $m = 3$, $f, g \in F$, $f \neq -1, 0, 1/2, 1, 2$, $g \neq 1/2$, $T = \{0, 1\}$, with the notation introduced in (2.6). (2.3) and (2.2)(vi) show that

$X = \left\{ \dfrac{f}{1-f+f^2}, \dfrac{1-f}{1-f+f^2}, \dfrac{-f(1-f)}{1-f+f^2} \right\}$ is truly full w.r.t. $\mathcal{A}_X = \{(2,1)\}$,

$Y = \{g, 1-g\}$ is truly full w.r.t. $\mathcal{A}_Y = \{(1,1),\ (1,0),\ (0,1)\}$.

The relation

$$(2,1)^{\odot 2} - 2\,(1,1)^{\odot 2} - 2\,(1,0)^{\odot 2} + (0,1)^{\odot 2} = 0$$

immediately implies the relation

$$\beta_3(\xi) = 0$$

for the divisor

$$\xi = \sum_{\substack{x \in X \\ y \in Y}} \left(\frac{1}{3} \left[\frac{x^2(1-x)}{y^2(1-y)} \right] - \left[\frac{x(1-x)}{y(1-y)} \right] - 2 \left[\frac{x}{y} \right] + \left[\frac{1-x}{1-y} \right] \right),$$

and Zagier's criterion (1.20) yields

$$P_3(\xi) = \text{const}.$$

If we put e.g. $x = \infty$ we can see that the constant must be zero.

(2.15) Remark Since $x^2(x-1)$ takes only one value on X and $y(y-1)$ takes only one value on Y we can write the above divisor in the following form:

$$\xi = \sum_{y \in Y} \left[\frac{x_0^2(1-x_0)}{y^2(1-y)} \right] - 2 \sum_{x \in X} \left[\frac{x(1-x)}{y_0(1-y_0)} \right] - \sum_{\substack{x \in X \\ y \in Y}} \left(2 \left[\frac{x}{y} \right] - \left[\frac{1-x}{1-y} \right] \right),$$

with $x_0 \in X$, $y_0 \in Y$ arbitrary, where the last divisor now has 17 instead of 24 terms.

(2.16) Metaphor In proposition (2.12) we have succeeded to "separate" the two sets X and Y in the last two factors of the tensor product—on the right hand side of (2.12.2) there occur only expressions whose factor in $\bigwedge^2(F^\times)$ depends either on X or on Y. Now we divide the remaining $(m-2)$ factors, i.e. the part in $\mathrm{Sym}^{m-2}(F^\times)$, of the resulting expressions into "mixed terms" like a zipper and investigate $q(X)$ and $r(Y)$ separately.

3. Examples and investigation of mixed terms

(3.1) Example m arbitrary

Let $r \in \mathbf{N}$, $\zeta = \sqrt[r]{1} \in F$.

$\mathcal{A} = \{[0], r[0]\} \subset \mathbf{Z}[\{0\}]$, $\phi_{r[0]}(t) = t^r$,

$x, y \in F$, $X = \{\zeta^j x\}_{j=1}^r$ is full w.r.t. $\mathcal{A}$: $\phi_{r[0]}(\zeta^j x) = x^r \quad \forall j = 1, \ldots, r$.

These assumptions together with (2.13) yield for each $m \in \mathbf{N}_{>1}$ the well-known distribution relations (cf. (1.10.1)) using the following condition which is trivially fulfilled

$$n(r[0]) \cdot r^{m-1} + n([0]) \cdot 1 = 0 \qquad (\text{choose } n(r[0]) = 1,\ n([0]) = r^{m-1}).$$

$$P_m\left(\frac{x^r}{y^r}\right) - r^{m-1} \sum_{j=1}^{r} P_m\left(\zeta^j \frac{x}{y}\right) = \text{const (in } y).$$

We obtain the constant by substituting $y = 0$: it is 0.

(3.2) As an immediate consequence of proposition (2.12) we get examples for elements in $\ker \beta_2$. In this chapter we derive such elements and we formulate the obstruction for extending the results to higher m, i.e. the vanishing of the "mixed terms" (3.11_k).

(3.3) Example m = 2

(i) Let $d(a) = 1 \ \forall a \in \mathcal{A} \subset \mathbf{Z}[F]$, $T = \mathrm{supp}(\mathcal{A})$, and $X, Y \subset F$.

Then each (non-trivial) $\mathbf{Z}$–linear relation $\sum_{a \in \mathcal{A}} n(a)\, a = 0$ induces via proposition (2.12) a (non-trivial) linear relation among the β_2–images of the following set:

$$\left\{ [\frac{\phi_a(x)}{\phi_a(y)}],\ [\frac{x-u}{x-t}],\ [\frac{y-u}{y-t}],\ [\phi_a(x)], [\phi_a(y)] \,\middle|\, x \in X,\ y \in Y,\ a \in \mathcal{A},\ t, u \in T \right\}.$$

(The assumption implies for each $x \in F$: $\{x\}$ is truly full w.r.t. $\mathcal{A}$.)

In particular,

1. $d([t]) = d([t] - [u]) = 1 \ \forall t, u \in T,\ t \neq u$, thus the obvious relation

$$([t] - [u]) - [t] + [u] = 0$$

already produces a (non-trivial) element in $\ker \beta_2$ via (i). More precisely, using proposition (2.12) one obtains the following variant of the five term relation (Abel relation):

$$\beta_2\left([\frac{x-t}{x-u}\,\frac{y-u}{y-t}] - [\frac{x-t}{y-t}] + [\frac{x-u}{y-u}] + [\frac{x-u}{x-t}] - [\frac{y-u}{y-t}] \right) = 0, \qquad x, y \in F.$$

2. Further relations spring to mind: for each $k \in \mathbf{N}$ and each sequence $(t_1, \ldots, t_k)$, $t_i \in T$, we obtain a cycle

$$\sum_i ([t_i] - [t_{i+1}]) = 0, \qquad (i \bmod k),$$

and the β_2–relation that results according to proposition (2.12) is obviously a sum of k five term relations as given in 1.

In the case $k = 3$, $F = \mathbf{Q}(x,y)$, this gives a special case of a relation found by Wechsung [We-L] reproduced in [Zg-A], p.394, in the following way (with P_2 instead of β_2)

$$\text{(3.3.1)} \qquad \sum_{i,j} \beta_2\left(\left[\mathrm{DV}(t_i, t_{i+1}, u_j, u_{j+1})\right]\right) = 0, \qquad \forall t_i, u_j \in \mathbf{P}^1(\mathbf{C}) \quad (i,j \text{ modulo } 3).$$

The specialisation of (3.3.1) mentioned above can be found by putting $\{t_1, t_2, t_3\} \subset T$, $u_1 = x$, $u_2 = y$ and $u_3 = \infty$.
Indeed the result obtained by this specialisation is equivalent to the "full" relation (3.3.1) which we can recover by introducing a new variable u_3 and symmetrising in u_1, u_2, u_3.

(ii) Let $c \in \mathbf{Z}[F]$ with support T, let $Y \subset F$ be a finite subset, truly full w.r.t. c.
Let $x \in F$ and $\mathcal{A} = \{c, [t], [t] - [u] \mid t, u \in T,\ t \neq u\}$.

There is a non-trivial relation $\sum_{a \in \mathcal{A}} n(a)\, a = 0$ where $n(c) \neq 0$, and we obtain a reduction of the expression

$$\sum_{y \in Y} \beta_2\left(\left[\frac{\phi_c(x)}{\phi_c(y)}\right]\right)$$

to "simpler" rational arguments, i.e. each argument is a product of two rational linear transformations (one in x, one in y) —see also the relations found by Rogers [Ro] and Zagier [Zg-D] (in full generality).

(3.4) Example m = 3

Let $c \in \mathbf{Z}[F]$, T the support of c, $\mathcal{A} = \{c, [t], [t] - [u] \mid t, u \in T,\ t \neq u\}$.
We write $e_{tu} = [t] - [u]$ for $t, u \in T$, $t \neq u$.
$c^{\odot 2} \in \mathrm{Sym}^2(\mathbf{Z}[T])$ can be written as a linear combination of the $[t]^{\odot 2}$ and ${e_{tu}}^{\odot 2}$:

$$c^{\odot 2} + \frac{1}{2} \sum_{\substack{t,u \\ t \neq u}} c_t c_u\, {e_{tu}}^{\odot 2} = \Big(\sum_{u \in T} c_u\Big)\Big(\sum_{t \in T} c_t [t]^{\odot 2}\Big).$$

Assume further

(i) c is effective (cf. (1.11)) und
(ii) $Y \subset F$ is a finite subset, Y is truly full w.r.t. c, hence truly full w.r.t. $\mathcal{A}$.

Proposition (2.12) implies for $x \in F$ (we observe $\beta_2([z]) = -\beta_2([1/z])$, $n(e_{tu}) = \frac{1}{2} c_t c_u\, n(c)$ and $\chi(e_{tu}) = 1$)

$$-\sum_{y\in Y}\sum_{a\in\mathcal{A}} \frac{n(a)}{d(a)}\,\beta_3\left(\left[\frac{\phi_a(x)}{\phi_a(y)}\right]\right)$$

$$= \sum_{y\in Y}\sum_{t,u\in T} n(e_{tu})\cdot\left(\frac{x-t}{x-u}\,\frac{y-u}{y-t}\right)\otimes\left(-\beta_2\left(\left[\frac{x-u}{x-t}\right]\right)\;-\;\beta_2\left(\left[\frac{y-t}{y-u}\right]\right)\right)$$

$$= \sum_{y\in Y}\sum_{t,u\in T} n(e_{tu})\cdot\left(\frac{x-t}{x-u}\right)\otimes\left(\beta_2\left(\left[\frac{x-t}{x-u}\right]\right)\;+\;\beta_2\left(\left[\frac{y-u}{y-t}\right]\right)\right)$$

$$+\sum_{y\in Y}\sum_{t,u\in T} n(e_{tu})\cdot\left(\frac{y-u}{y-t}\right)\otimes\left(\beta_2\left(\left[\frac{x-t}{x-u}\right]\right)\;+\;\beta_2\left(\left[\frac{y-u}{y-t}\right]\right)\right)$$

$$(3.4.1)\quad = \frac{1}{2}\sum_{t,u\in T} c_t c_u n(c)\left(d(c)\beta_3\left(\left[\frac{x-t}{x-u}\right]\right)+\sum_{y\in Y}\beta_3\left(\left[\frac{y-u}{y-t}\right]\right)\right)$$

$$-\frac{1}{2}\sum_{t,u\in T} c_t c_u\, n(c)\cdot\left(\frac{x-t}{x-u}\right)\otimes\sum_{y\in Y}\beta_2\left(\left[\frac{y-t}{y-u}\right]\right)$$

$$+\sum_{t,u\in T} n(e_{tu})\cdot\left(\prod_{y\in Y}\frac{y-u}{y-t}\right)\otimes\beta_2\left(\left[\frac{x-t}{x-u}\right]\right).$$

Examining the calculation above we observe that in order to produce a β_3–relation it is sufficient to have the two final sums with mixed terms vanish. Under the assumptions (i) and (ii) both sums are already zero:

(3.5) Lemma

Let $c\in\mathbf{Z}[F]$, c be effective and T be the support of c.
Let $Y\subset F$ be truly full w.r.t. c, $|Y| = d(c)$.

Then (3.5.1) and (3.5.2) hold (modulo 2–torsion):

$$(3.5.1)\qquad \prod_{y\in Y}\frac{y-u}{y-t} = 1 \qquad \forall t,u\in T,$$

$$(3.5.2)\qquad \sum_{t\in T} c_t \sum_{y\in Y}\beta_2\left(\left[\frac{y-u}{y-t}\right]\right) = 0 \qquad \forall u\in T \text{ mit } c_u > 0.$$

Proof

(3.5.1): This follows immediately from (2.6.1) and $\phi_c(\infty) = \infty$, since $\phi_c(t) = 0$, $\phi_c^-(t) = 1$ (analogously for u).

(3.5.2):

$$\sum_{y\in Y}\beta_2\left(\left[\frac{y-u}{y-t}\right]\right) = \sum_y \frac{y-u}{y-t}\wedge\frac{u-t}{y-t}$$
$$= \left(\prod_y \frac{y-u}{y-t}\right)\wedge(u-t) \; - \; \sum_y \frac{y-u}{y-t}\wedge(y-t)$$
$$= -\sum_y (y-u)\wedge(y-t)\,.$$

The final equality uses (3.5.1) and $v\wedge v = 0$.

Now it follows

$$\sum_{t\in T} c_t \sum_{y\in Y}(y-t)\wedge(y-u) = \sum_y\left(\prod_t (y-t)^{c_t}\right)\wedge(y-u)$$
$$= \left(\prod_t (y-t)^{c_t}\right)\wedge\left(\prod_y (y-u)\right),$$

and the left hand side equals the right hand side up to sign in the final expression—this is an immediate consequence of (2.6.2). ◇

(3.6) Corollary

Let $c\in \mathbf{Z}[F]$, c be effective , let T be the support of c.
Let $Y\subset F$ be truly full w.r.t. c, $|Y| = d(c)$, $\mathcal{A} = \{c,[t],[t]-[u] \mid t,u\in T,\, t\neq u\}$.
Then

$$\sum_{y\in Y}\sum_{a\in\mathcal{A}}\frac{n(a)}{d(a)}\,\beta_3\left(\left[\frac{\phi_a(x)}{\phi_a(y)}\right]\right) =$$
$$-\frac{1}{2}\sum_{t,u\in T} c_t c_u\, n(c)\left(d(c)\,\beta_3\left(\left[\frac{x-t}{x-u}\right]\right)+\sum_{y\in Y}\beta_3\left(\left[\frac{y-u}{y-t}\right]\right)\right).$$

Proof The final sum in (3.4.1) vanishes because of (3.5.1).

For the second to last sum we compute

$$
\begin{aligned}
\sum_{t,u\in T} c_t c_u \left(\frac{x-t}{x-u}\right) \otimes \sum_{y\in Y} \beta_2\left(\left[\frac{y-t}{y-u}\right]\right)
&= \sum_{u\in T} c_u \sum_{t\in T} (x-t)^{c_t} \otimes \sum_{y\in Y} \beta_2\left(\left[\frac{y-t}{y-u}\right]\right) \\
&\quad - \sum_{t\in T} c_t \sum_{u\in T} (x-u)^{c_u} \otimes \sum_{y\in Y} \beta_2\left(\left[\frac{y-t}{y-u}\right]\right) \\
&= \sum_{u\in T} c_u \Big(\prod_{t\in T} (x-t)^{c_t}\Big) \otimes \sum_{y\in Y} \beta_2\left(\left[\frac{y-t}{y-u}\right]\right) \\
&\quad - \sum_{t\in T} c_t \Big(\prod_{u\in T} (x-u)^{c_u}\Big) \otimes \sum_{y\in Y} \beta_2\left(\left[\frac{y-t}{y-u}\right]\right) \\
&= \Big(\prod_{t\in T} (x-t)^{c_t}\Big) \otimes \sum_{u\in T} c_u \sum_{y\in Y} \beta_2\left(\left[\frac{y-t}{y-u}\right]\right) \\
&\quad - \Big(\prod_{u\in T} (x-u)^{c_u}\Big) \otimes \sum_{t\in T} c_t \sum_{y\in Y} \beta_2\left(\left[\frac{y-t}{y-u}\right]\right),
\end{aligned}
$$

and (3.5.2) implies that this sum also vanishes. ◊

(3.7) Remark For arbitrary $c \in \mathbf{Z}[F]$ there is a general and rather simple β_3–relation with arguments from the left hand side of the equation in corollary (3.6) (in a complicated form essentially due to Wojtkowiak [Wo-B]) which we are not yet able to deduce in an illuminating way using our approach.

An analogous procedure as in the case $m = 3$ treated above, this time for X and Y, yields for $m = 4$:

(3.8) Proposition

Let $b, c \in \mathbf{Z}[F]$, b, c be effective, $d(b), d(c) > 1$,

let $X \subset F$ be truly full w.r.t. b and $Y \subset F$ be truly full w.r.t. c, $T = \mathrm{supp}(\mathcal{A})$.

Let $\mathcal{A} = \{b, c, [t], [t] - [u] \mid t, u \in T, t \neq u\}$, and for certain $n(a) \in \mathbf{Z}$ let the following equation be satisfied

(3.8.1) $$\sum_{a\in\mathcal{A}} n(a)\, a^{\odot 3} = 0.$$

Then in $\mathrm{Sym}^2(F^\times) \otimes \bigwedge^2(F^\times)$ we have

$$\sum_{x\in X} \sum_{y\in Y} \sum_{a\in\mathcal{A}} \frac{n(a)}{d(a)}\, \beta_4\left(\left[\frac{\phi_a(x)}{C}\right]\right) =$$

(3.8.2) $$= \sum_{x\in X} \sum_{y\in Y} \sum_{t,u\in T} n(e_{tu}) \Big\{ -\beta_4\left(\left[\frac{x-t}{x-u}\right]\right) - \beta_4\left(\left[\frac{y-u}{y-t}\right]\right)$$

$$- 2\left(\frac{x-t}{x-u} \odot \frac{y-u}{y-t}\right) \otimes \left(\beta_2\left(\left[\frac{y-u}{y-t}\right]\right) + \beta_2\left(\left[\frac{x-t}{x-u}\right]\right)\right)$$
$$- \left(\frac{x-t}{x-u}\right)^{\odot 2} \otimes \beta_2\left(\left[\frac{y-u}{y-t}\right]\right) - \left(\frac{y-u}{y-t}\right)^{\odot 2} \otimes \beta_2\left(\left[\frac{x-t}{x-u}\right]\right)\Big\}.$$

Proof As an intermediate step we use proposition (2.12)

$$\sum_{x\in X}\sum_{y\in Y}\sum_{a\in\mathcal{A}} \frac{n(a)}{d(a)}\,\beta_4\left(\left[\frac{\phi_a(x)}{C}\right]\right)$$
$$= \sum_{x\in X}\sum_{y\in Y}\sum_{t,u\in T} n(e_{tu}) \left(\frac{x-t}{x-u}\,\frac{y-u}{y-t}\right)^{\odot 2} \otimes \left(-\beta_2\left(\left[\frac{y-u}{y-t}\right]\right) - \beta_2\left(\left[\frac{x-t}{x-u}\right]\right)\right).$$

Expanding as in (3.4) yields the claim—note that

$$(f(x)g(y))^{\odot 2} = f(x)^{\odot 2} + 2 \cdot f(x) \odot g(y) + g(y)^{\odot 2}\ .$$

◊

(3.9) Remark

In order to obtain an element in $\ker \beta_4$ under the assumptions of proposition (3.8) it is sufficient to satisfy

$$\text{(3.9.1)} \qquad \sum_{x\in X} \frac{x-t}{x-u} \otimes \sum_{y\in Y} \beta_3\left(\left[\frac{y-u}{y-t}\right]\right) = 0 \quad \forall t,u \in T \qquad \text{and}$$

$$\text{(3.9.2)} \qquad \sum_{t,u\in T} n(e_{tu}) \sum_{x\in X} \left(\frac{x-t}{x-u}\right)^{\odot 2} \otimes \sum_{y\in Y} \beta_2\left(\left[\frac{y-u}{y-t}\right]\right) = 0$$

as well as the corresponding equations (3.9.1′) and (3.9.2′) where the roles of x and y are interchanged.

Proof For $v_i, w_i, z_i \in F^\times$ we have

$$\sum_i v_i \otimes w_i \otimes \beta_2([z_i]) = 0 \quad \Rightarrow \quad \sum_i w_i \otimes v_i \otimes \beta_2([z_i]) = 0,$$

hence

$$\sum_i (v_i \odot w_i) \otimes \beta_2([z_i]) = 0,$$

and

$$0 = \sum_{x,y} \frac{x-t}{x-u} \otimes \frac{y-u}{y-t} \otimes \beta_2\left(\left[\frac{y-u}{y-t}\right]\right)$$
$$\Rightarrow \qquad 0 = \sum_{x,y} \left(\frac{x-t}{x-u} \odot \frac{y-u}{y-t}\right) \otimes \beta_2\left(\left[\frac{y-u}{y-t}\right]\right)$$

and using (3.9.1) and (3.9.1′) the second row of the right hand side of (3.8.2) vanishes. ◊

(3.10) Remark Under the assumptions made in proposition (3.8) (i.e. b, c effective) equations (3.9.1) and (3.9.1′) hold (use (3.5.1)), therefore only equations (3.9.2) and (3.9.2′) have to be satisfied.
In the next chapter we will see that the assumption $|T| = 2$ is sufficient to fulfill (3.9.2) and (3.9.2′), and in this case there is an even stronger result than the one in (3.8), namely b, c can be arbitrary elements in $\mathbf{Z}[F]$ (not necessarily effective), cf. theorem (4.5).

In analogy with remark (3.9) we now want to establish additional conditions for the case of arbitrary m that are sufficient to produce an element in $\ker \beta_m$—in the situation of proposition (2.12). (We will unzip in the sense of (2.16).)

(3.11) Notation

(i) For $\mathcal{A} \subset \mathbf{Z}[F]$ we put $\mathcal{A}^- = \{a \in \mathcal{A} \mid \exists t, u \in F \; a_t a_u < 0\}$.

(ii) Let $X, Y \subset F$ be finite subsets, $\mathcal{A} = \mathcal{A}_X \dot\cup \mathcal{A}_Y \subset \mathbf{Z}[F]$ and $T = \mathrm{supp}(\mathcal{A})$, then we define for $m \in \mathbf{N}_{>1}$ and $k = 0, \dots, m-2$ the following "mixed term" in $\mathrm{Sym}^2(F^\times) \otimes \bigwedge^2(F^\times)$

(3.11_k)

$$R_m^{(k)}(X, Y, \mathcal{A}_X, \mathcal{A}_Y) = \sum_{a \in \mathcal{A}} \frac{n(a)}{d(a)} \left(\sum_{x \in X} \phi_a(x)^{\odot k} \right) \odot \sum_{y \in Y} \phi_a(y)^{\odot(m-2-k)} \otimes \beta_2(\xi_{a,y}),$$

where

$$\xi_{a,y} = \begin{cases} \sum_{t,u \in T} a_t^+ a_u^- \left[\frac{y-u}{y-t}\right], & \text{if } a \in \mathcal{A}_X{}^-, \\ \chi(a)\ [\phi_a(y)], & \text{if } a \in \mathcal{A}_Y, \\ [0] & \text{otherwise.} \end{cases}$$

(3.12) Proposition

Let $X, Y \subset F$ be finite subsets and $\mathcal{A} = \mathcal{A}_X \dot\cup \mathcal{A}_Y \subset \mathbf{Z}[F]$ satisfying the properties

(i) X is truly full w.r.t. $\mathcal{A}_X$,

(ii) Y is truly full w.r.t. $\mathcal{A}_Y$,

(iii) $\dfrac{\phi_a(x)}{\phi_a(y)} \in F^\times \quad \forall a \in \mathcal{A},\, x \in X,\, y \in Y$.

For $m \in \mathbf{N}_{>1}$ assume the following $\mathbf{Z}$–linear dependence relation:

$$\sum_{a \in \mathcal{A}} n(a)\, a^{\odot m-1} = 0,$$

and let

$$R_m^{(k)}(X, Y, \mathcal{A}_X, \mathcal{A}_Y) = R_m^{(k)}(Y, X, \mathcal{A}_Y, \mathcal{A}_X) = 0\,, \quad \text{if } k > 0.$$

Then

$$\sum_{x\in X}\sum_{y\in Y}\sum_{a\in\mathcal{A}}\frac{n(a)}{d(a)}\,\beta_m\left(\left[\frac{\phi_a(x)}{\phi_a(y)}\right]\right)=f(\mathcal{A},X,|Y|)+g(\mathcal{A},Y,|X|),$$

where the expressions $f(\mathcal{A},X,|Y|),g(\mathcal{A},Y,|X|)\in \mathrm{Sym}^{m-2}(F^\times)\otimes\bigwedge^2(F^\times)$ depend only on $\mathcal{A}$, Y and $|X|$.

Proof We put $T=\mathrm{supp}(\mathcal{A})$.

We transform the right hand side of (2.12.2) using the following identity

$$\left(\frac{f}{g}\right)^{\odot m-2}=\sum_{k=0}^{m-2}c(m,k)\cdot f^{\odot k}\odot g^{\odot(m-2-k)}$$

where $c(m,k)=(-1)^{m-2-k}\binom{m-2}{k}$. This gives

$$\begin{aligned}
&\sum_{x\in X}\sum_{y\in Y}\sum_{a\in\mathcal{A}}\frac{n(a)}{d(a)}\,\beta_m\left(\left[\frac{\phi_a(x)}{\phi_a(y)}\right]\right)\\
&=\sum_{k=0}^{m-2}c(m,k)\Bigg(\sum_{y\in Y}\sum_{x\in X}\sum_{a\in\mathcal{A}_X}\frac{n(a)}{d(a)}\,\phi_a(x)^{\odot k}\odot\phi_a(y)^{\odot(m-2-k)}\otimes\sum_{t,u\in T}a_t^+a_u^-\,\beta_2([\frac{y-u}{y-t}])\\
&\qquad+\sum_{x\in X}\sum_{y\in Y}\sum_{a\in\mathcal{A}_Y}\frac{n(a)}{d(a)}\,\phi_a(x)^{\odot k}\odot\phi_a(y)^{\odot(m-2-k)}\otimes\chi(a)\,\beta_2\left([\phi_a(y)]\right)\\
&\qquad-\sum_{y\in Y}\sum_{x\in X}\sum_{a\in\mathcal{A}_Y}\frac{n(a)}{d(a)}\,\phi_a(y)^{\odot k}\odot\phi_a(x)^{\odot(m-2-k)}\otimes\sum_{t,u\in T}a_t^+a_u^-\,\beta_2([\frac{x-u}{x-t}])\\
&\qquad-\sum_{x\in X}\sum_{y\in Y}\sum_{a\in\mathcal{A}_X}\frac{n(a)}{d(a)}\,\phi_a(y)^{\odot k}\odot\phi_a(x)^{\odot(m-2-k)}\otimes\chi(a)\,\beta_2\left([\phi_a(x)]\right)\Bigg)\\
&=\sum_{k=0}^{m-2}c(m,k)\left(R_m^{(k)}(X,Y,\mathcal{A}_X,\mathcal{A}_Y)-R_m^{(k)}(Y,X,\mathcal{A}_Y,\mathcal{A}_X)\right)\\
&\quad=c(m,0)\left(R_m^{(0)}(X,Y,\mathcal{A}_X,\mathcal{A}_Y)-R_m^{(0)}(Y,X,\mathcal{A}_Y,\mathcal{A}_X)\right)\quad\text{by the assumption.}
\end{aligned}$$

From this the claim follows since e.g. $R_m^{(0)}(X,Y,\mathcal{A}_X,\mathcal{A}_Y)$ only depends on $\mathcal{A}$, Y and the cardinality of X. (Note that for $a\in\mathcal{A}-\mathcal{A}^-$ and $u\in T$ we have $a_u^-=0$.)

◊

(3.13) Corollary

Let $X, X', Y, Y' \subset F$ be finite subsets and $\mathcal{A} = \mathcal{A}_X \dot{\cup} \mathcal{A}_Y \subset \mathbf{Z}[F]$ having the properties

(i) X, X' are truly full w.r.t. $\mathcal{A}_X$,

(ii) Y, Y' are truly full w.r.t. $\mathcal{A}_Y$,

(iii) $\dfrac{\phi_a(x)}{\phi_a(y)} \in F^\times \quad \forall a \in \mathcal{A},\, x \in X,\, y \in Y,$ and

(iv) $|X| = |X'|,\ |Y| = |Y'|$.

For $m \in \mathbf{N}_{>1}$ assume the following $\mathbf{Z}$-linear dependence relation:

$$\sum_{a\in\mathcal{A}} n(a)\, a^{\odot m-1} = 0,$$

and let

$$R_m^{(k)}(X, Y, \mathcal{A}_X, \mathcal{A}_Y) = R_m^{(k)}(Y, X, \mathcal{A}_Y, \mathcal{A}_X) = 0\,, \quad \text{if } k > 0.$$

Then up to 2–torsion $\beta_m(\xi) = 0$, where

$$\xi = \sum_{a\in\mathcal{A}} \left(\sum_{x\in X} \sum_{y\in Y} \left[\frac{\phi_a(x)}{\phi_a(y)}\right] - \sum_{x\in X} \sum_{y'\in Y'} \left[\frac{\phi_a(x)}{\phi_a(y')}\right] \right.$$
$$\left. - \sum_{x'\in X'} \sum_{y\in Y} \left[\frac{\phi_a(x')}{\phi_a(y)}\right] + \sum_{x'\in X'} \sum_{y'\in Y'} \left[\frac{\phi_a(x')}{\phi_a(y')}\right] \right).$$

◇

4. Families of functional equations up to order $m = 6$

The case $|T| = 2$, where T is the support of a subset $\mathcal{A} \subset \mathbf{Z}[F]$, is of special importance to us since in this case the situation has proved to be simple enough to construct functional equations in 2 variables up to order 6. We want to exhibit these in the following.

(4.0) General assumption

In this chapter we always compute modulo 2-torsion in a field F of characteristic 0 and we let $\mathcal{A} \subset \mathbf{Z}[\{0,1\}]$. (Each set $\{t, u\}$ of 2 elements in F can be transformed into $\{0,1\}$ using a rational linear transformation.)

We first collect some basic facts using notation (2.5).

(4.1) Lemma

Let $Y \subset F$ be truly full w.r.t. $c = (c_0, c_1)$, $c_0 > 0$, $c_1 \notin \{0, -c_0\}$ and $c_2 = -c_0 - c_1$.
Let $|Y| = \deg(c)$ and $C = y^{c_0}(y-1)^{c_1}$ $\forall y \in Y$.
Then the following identities hold

(4.1.1)
$$\prod_{y \in Y} y = \begin{cases} \pm C, & \text{if } c_0 + c_1 > 0, \\ \pm 1 & \text{otherwise,} \end{cases}$$

(4.1.2)
$$\prod_{y \in Y} (1-y) = \begin{cases} \pm C, & \text{if } c_1 > 0, \\ \pm 1, & \text{if } 0 > c_1 > -c_0, \\ \pm \frac{1}{C} & \text{otherwise,} \end{cases}$$

(4.1.3)
$$\prod_{y \in Y} \frac{y}{y-1} = \begin{cases} \pm 1, & \text{if } c_1 > 0, \\ \pm C, & \text{if } c_1 < 0, \end{cases}$$

(4.1.4)
$$\sum_{y \in Y} (y^{c_0}(y-1)^{c_1})^{\odot k} \otimes \beta_2([y]) = 0 \qquad \forall k \in \mathbf{N}_0,$$

(4.1.5)
$$\frac{1}{c_1} \sum_{y \in Y} \beta_3([y]) = \frac{1}{c_0} \sum_{y \in Y} \beta_3\left(\left[\frac{1}{1-y}\right]\right) = \frac{1}{c_2} \sum_{y \in Y} \beta_3\left(\left[1 - \frac{1}{y}\right]\right),$$

(4.1.6)
$$\sum_{y \in Y} \beta_4\left(c_0 c_2 [y] + c_1 c_2 \left[\frac{1}{1-y}\right] + c_0 c_1 \left[1 - \frac{1}{y}\right]\right) = 0.$$

Proof

(4.1.1): For $c_0 + c_1 < 0$ we have $\phi_c(\infty) = 0$, and (2.6.1) implies

$$C - x^{c_0}(x-1)^{c_1} = \prod_{y \in Y} (x-y)\,\phi_c^-(x) \cdot \begin{cases} -1, & \text{if } c_0 + c_1 > 0, \\ C, & \text{if } c_0 + c_1 < 0. \end{cases} \tag{4.1.7}$$

Since $c_0 > 0$ we obtain, putting $x = 0$,

$$C = \prod_y (-y) \cdot \begin{cases} 1, & \text{if } c_1 > 0, \\ (-1)^{c_1} & \text{if } c_0 > -c_1 > 0, \\ (-1)^{c_1} C, & \text{if } c_0 < -c_1. \end{cases}$$

(4.1.2): If $c_1 > 0$ then $\quad C = -\prod_y (1-y)$ (put $x = 1$ in (4.1.7)).

If $c_1 < 0$ then multiply (4.1.7) by $\left(\phi_c^-(x)\right)^{-1} = (x-1)^{-c_1}$ and put $x = 1$. It follows

$$-1 = \prod_y (1-y) \cdot \begin{cases} -1, & \text{if } c_0 + c_1 > 0, \\ C, & \text{if } c_0 + c_1 < 0. \end{cases}$$

(4.1.3): Follows directly from (4.1.1) and (4.1.2).

(4.1.4): Modulo 2-torsion we get

$$\begin{aligned} c_1 \sum_y \beta_2([y]) &= c_1 \sum_y y \wedge (1-y) = \sum_y y \wedge \pm \frac{C}{y^{c_0}} = \sum_y y \wedge C \\ &= \left(\prod_y y\right) \wedge C = \begin{cases} (\pm C) \wedge C, & \text{if } c_0 > |c_1|, \\ (\pm 1) \wedge C & \text{otherwise} \end{cases} \qquad \text{because of (4.1.1)} \\ &= 0, \end{aligned}$$

and by assumption $y^{c_0}(y-1)^{c_1}$ is independent from $y \in Y$.

(4.1.5): Since $\beta_2([y]) = \beta_2([\frac{1}{1-y}])$ we obtain

$$\begin{aligned} \sum_y \beta_3\left(c_0[y] - c_1\left[\frac{1}{1-y}\right]\right) &= \sum_y \left(c_0 y^{\odot 1} - c_1 \left(\frac{1}{1-y}\right)^{\odot 1}\right) \otimes \beta_2([y]) \\ &= \sum_y \left(y^{c_0}(y-1)^{c_1}\right) \otimes \beta_2([y]) = 0 \qquad \text{because of (4.1.4).} \end{aligned}$$

The second equation is treated analogously.

(4.1.6): From remark (1.19) and $\beta_2([y]) = \beta_2([\frac{1}{1-y}]) = \beta_2([1 - \frac{1}{y}])$ we deduce

$$\begin{aligned} &\sum_y \beta_4\left(c_0 c_2 [y] + c_1 c_2 \left[\frac{1}{1-y}\right] + c_0 c_1 \left[1 - \frac{1}{y}\right]\right) \\ &= \sum_y \left(c_0 c_2\, y^{\odot 2} + c_1 c_2 \left(\frac{1}{1-y}\right)^{\odot 2} + c_0 c_1 \left(1 - \frac{1}{y}\right)^{\odot 2}\right) \otimes \beta_2([y]) \\ &= -\sum_y \left(y^{c_0}(y-1)^{c_1}\right)^{\odot 2} \otimes \beta_2([y]) = 0 \qquad \text{by (4.1.4).} \end{aligned}$$

$\diamond$

The following lemma determines the coefficients of an equation of the form

$$\sum_{a\in\mathcal{A}} n(a)\, a^{\odot k} = 0, \qquad \mathcal{A}\subset \mathbf{R}^2.$$

(4.2) Lemma

Let $\mathcal{A}\subset\mathbf{R}^2$, $|\mathcal{A}| = r > 1$, where the elements of $\mathcal{A}$ are pairwise linearly independent. Then we have

(4.2.1)
$$\sum_{a\in\mathcal{A}} \frac{1}{\prod\limits_{a'\in\mathcal{A}-\{a\}} \det(a,a')}\, a^{\odot(r-2)} = 0.$$

where $\det$ denotes the determinant function in $\mathbf{R}^2$.

Proof Lagrange's interpolation formula yields for a set $\{\gamma_i\}_{i=1}^r \in \mathbf{C}$ and a polynomial $f(x)$ of degree $< r$:

(4.2.2)
$$f(x) = \sum_{i=1}^{r} f(\gamma_i) \prod_{j\neq i} \left(\frac{x-\gamma_j}{\gamma_i-\gamma_j}\right).$$

(Proof: Both sides are polynomials of degree $< r$ and coincide for $x = \gamma_i$, $i = 1,\dots r$.)

If we put $f(x) = x^{m-1}$, $m < r$, and compare the coefficient of x^{r-1} on both sides of (4.2.2) we obtain

$$0 = \sum_i \frac{1}{\prod\limits_{j\neq i}(\gamma_i-\gamma_j)}\, \gamma_i^{m-1} .$$

First let $\beta_i \neq 0$ $\forall i$. We put $\gamma_i = \frac{\alpha_i}{\beta_i} + \frac{Y}{X}$ for two variables X, Y , then we get

$$\gamma_i - \gamma_j = \frac{\alpha_i}{\beta_i} - \frac{\alpha_j}{\beta_j} = \begin{vmatrix} \alpha_i & \alpha_j \\ \beta_i & \beta_j \end{vmatrix} / \beta_i\beta_j \qquad \text{and} \qquad f(\gamma_i) = \frac{(\alpha_i X + \beta_i Y)^{m-1}}{(\beta_i X)^{m-1}} .$$

Putting $A_i = \prod_{j\neq i} \begin{vmatrix} \alpha_i & \alpha_j \\ \beta_i & \beta_j \end{vmatrix}$ we deduce

$$0 = \sum_i \frac{\prod_{j\neq i}\beta_j\beta_i}{A_i}\, \frac{(\alpha_i X+\beta_i Y)^{m-1}}{(\beta_i X)^{m-1}} = X^{1-m}\prod_j \beta_j \sum_i \frac{\beta_i^{r-1-m}}{A_i}\, (\alpha_i X + \beta_i Y)^{m-1}.$$

In particular, for $m = r-1$

$$0 = \sum_i \frac{1}{\prod_{j\neq i}\begin{vmatrix} \alpha_i & \alpha_j \\ \beta_i & \beta_j \end{vmatrix}}\, (\alpha_i X + \beta_i Y)^{r-2} .$$

This equation holds also if one of the β_i vanishes by a continuity argument.
Remark (1.17) now proves the claim. $\diamondsuit$

We now come back to our main examples (2.3) for which we can score many useful properties (almost in abundance).

(4.3) Lemma
Let $t \in F$, $X = \{t, \frac{1}{t}, \frac{t-1}{t}, \frac{t}{t-1}, \frac{1}{1-t}, 1-t\} \subset \mathbf{Z}[\mathbf{P}^1(F)]$. Then

$$\sum_{x \in X} \beta_m([x]) = 0 \qquad \text{for } m = 2, 3, 4, 6. \tag{4.3.1}$$

Proof For m even $\beta_m([x] + [x^{-1}]) = 0$.
X is closed under inversion, therefore the claim follows for $m = 2, 4, 6$.
For $m = 3$

$$\beta_3\left([x] + \left[\frac{1}{1-x}\right] + \left[1 - \frac{1}{x}\right]\right) = 0 \quad (\text{cf. } (1.21)) \quad \text{and} \quad \beta_3([x] - [x^{-1}]) = 0\,.$$ $\diamondsuit$

(4.4) Notation
(i) For two divisors $\xi_1, \xi_2 \in \mathbf{Z}[F]$ we write

$$\xi_1 \equiv \xi_2 \qquad (\text{mod } \ker \beta_m)\,, \quad \text{if} \quad \beta_m(\xi_1) = \beta_m(\xi_2)\,.$$

(ii) We have an operation of $\mathcal{S}_3$ on $\mathbf{P}^1(F)$ that is generated by the involutions

$$f \mapsto \frac{1}{f}, \qquad f \mapsto 1 - f.$$

We call this operation the "usual" operation of $\mathcal{S}_3$.
(iii) For $a = (a_0, a_1) \in \mathbf{Z}[\{0,1\}]$ we write $n(a_0, a_1) = n(a)$ and $d(a_0, a_1) = d(a)$.

Now we want to formulate the result indicated already at the end of (3.10).

(4.5) Theorem

Let $b=(b_0,b_1)$, $c=(c_0,c_1)\in \mathbf{Z}^2$ such that $\det(b,c)\neq 0$ and $\frac{b_0}{b_1},\frac{c_0}{c_1}\notin\{0,-1,\infty\}$.
Let $\mathcal{A}=\{b,c,(1,0),(0,1),(1,-1)\}$.
Let $X,Y\subset F$ be finite subsets satisfying the conditions

(i) X is truly full w.r.t. b,

(ii) Y is truly full w.r.t. c and

(iii) $\dfrac{\phi_a(x)}{\phi_a(y)}\in F^\times \qquad \forall a\in\mathcal{A},\ x\in X,\ y\in Y$.

Then up to 2-torsion

$$\sum_{a\in\mathcal{A}}\sum_{x\in X}\sum_{y\in Y}\frac{n(a)}{d(a)}\left[\frac{\phi_a(x)}{\phi_a(y)}\right]\equiv\sum_{x\in X}\sum_{\sigma\in\mathcal{S}_3}B_{x,\sigma}[x^\sigma]+\sum_{y\in Y}\sum_{\sigma\in\mathcal{S}_3}C_{y,\sigma}[y^\sigma] \qquad (\text{mod } \ker\beta_4)$$

for certain coefficients $B_{x,\sigma},C_{y,\sigma}\in\mathbf{Q}$ and $1/n(a)=\prod_{a'\in\mathcal{A}-\{a\}}\det(a,a')$, $a\in\mathcal{A}$.

Proof Let $b_0,c_0>0$ and $|X|=d(b)$, $|Y|=d(c)$.
We denote $B=\phi_b(x)\ \forall x\in X$, $C=\phi_c(y)\ \forall y\in Y$.
We put $\mathcal{A}_X=\{b,(1,0),(0,1),(1,-1)\}$, $\mathcal{A}_Y=\{c\}$, then
$\mathcal{A}_X^-\subset\{b,(1,-1)\}$, $\mathcal{A}_Y^-\subset\{c\}$.

(4.2) implies $\qquad n(a)=\dfrac{1}{\prod_{a'\in\mathcal{A}-\{a\}}\det(a,a')}\quad\forall a\in\mathcal{A}\quad\Longrightarrow\quad\sum_{a\in\mathcal{A}}n(a)\,a^{\odot 3}=0$.

I) First we will show $R_4^{(k)}(X,Y,\mathcal{A}_X,\mathcal{A}_Y)=R_4^{(k)}(Y,X,\mathcal{A}_Y,\mathcal{A}_X)=0$ for $k=1,2$.

(i) $k=1$.

$$\begin{aligned}R_4^{(1)}(X,Y,\mathcal{A}_X,\mathcal{A}_Y)&=\sum_{x\in X}\sum_{y\in Y}\sum_{a\in\mathcal{A}_X^-}\frac{n(a)}{d(a)}\cdot\phi_a(x)\odot\phi_a(y)\otimes a_0^+a_1^-\beta_2([\frac{y-1}{y}])\\&=\frac{n(b)}{d(b)}b_0^+b_1^-\cdot\sum_{x\in X}\phi_b(x)\odot\sum_{y\in Y}\left(y^{b_0}(y-1)^{b_1}\right)\otimes\beta_2([y])\\&\quad+n(1,-1)(-1)\cdot\sum_{x\in X}\frac{x}{x-1}\odot\sum_{y\in Y}\frac{y}{y-1}\otimes\beta_2([y])\\&=\frac{n(b)}{d(b)}b_0\cdot\begin{Bmatrix}0, & b_1>0\\ b_1, & b_1<0\end{Bmatrix}\cdot d(b)\cdot B\odot\sum_{y\in Y}\left(y^{b_0}(y-1)^{b_1}\right)\otimes\beta_2([y])\\&\quad-n(1,-1)\cdot\begin{Bmatrix}\pm 1^{\odot 1}, & b_1>0\\ \pm B^{\odot 1}, & b_1<0\end{Bmatrix}\odot\sum_{y\in Y}\frac{y}{y-1}\otimes\beta_2([y]).\end{aligned}$$

For $b_1>0$ both terms vanish.

For $b_1 < 0$ we deduce from $\sum_{a\in\mathcal{A}} n(a)\, a^{\odot 3} = 0$ and (2.10) the equation

$$n(b)b_0b_1\, b^{\odot 1} + n(c)c_0c_1\, c^{\odot 1} + n(1,-1)(-1)(1,-1)^{\odot 1} = 0\,.$$

Hence

$$\begin{aligned}0 = {} & n(b)b_0b_1 \cdot B \odot \sum_{y\in Y} \left(y^{b_0}(y-1)^{b_1}\right) \otimes \beta_2([y]) \\ & - n(1,-1)\cdot B \odot \sum_{y\in Y} \frac{y}{y-1} \otimes \beta_2([y]) \\ & + n(c)c_0c_1 \cdot B \odot \sum_{y\in Y} \left(y^{c_0}(y-1)^{c_1}\right) \otimes \beta_2([y])\,,\end{aligned}$$

but the final term vanishes because of $y^{c_0}(y-1)^{c_1} = C \quad \forall y \in Y$ and (4.1.4). Analogously one can show $R_4^{(1)}(Y,X,\mathcal{A}_Y,\mathcal{A}_X) = 0$.

(ii) $k=2$. Since $\chi(a) = 0$ for $a \in \mathcal{A}_Y$ we have

$$\begin{aligned}R_4^{(2)}(X,Y,\mathcal{A}_X,\mathcal{A}_Y) &= \sum_{x\in X}\sum_{y\in Y}\sum_{a\in\mathcal{A}_X^-} \frac{n(a)}{d(a)}\,\phi_a(x)^{\odot 2} \otimes a_0^+a_1^-\beta_2\left(\left[\frac{y-1}{y}\right]\right) \\ &= \sum_{x\in X}\sum_{a\in\mathcal{A}_X^-} \frac{n(a)}{d(a)}\,\phi_a(x)^{\odot 2} \otimes a_0^+a_1^- \sum_{y\in Y}\beta_2([y]) \\ &= 0 \qquad \text{by (4.1.4)}.\end{aligned}$$

Also for a certain $q \in \mathrm{Sym}^2(F^\times)$

$$\begin{aligned}R_4^{(2)}(Y,X,\mathcal{A}_Y,\mathcal{A}_X) &= \sum_{x\in X}\sum_{y\in Y}\sum_{a\in\mathcal{A}_Y^-} \frac{n(a)}{d(a)}\,\phi_a(y)^{\odot 2} \otimes a_0^+a_1^-\beta_2\left(\left[\frac{x-1}{x}\right]\right) \\ &\quad - \sum_{x\in X}\sum_{y\in Y} n(1,-1)\left(\frac{y}{y-1}\right)^{\odot 2} \otimes \beta_2\left(\left[\frac{x}{x-1}\right]\right) \\ &= q \otimes \sum_{x\in X}\beta_2([x]) = 0 \qquad \text{by (4.1.4)}.\end{aligned}$$

(II) Finally we want to examine the case $k=0$.

$$\begin{aligned}R_4^{(0)}(X,Y,\mathcal{A}_X,\mathcal{A}_Y) &= |X| \sum_{y\in Y}\sum_{a\in\mathcal{A}_X^-} \frac{n(a)}{d(a)}\,\phi_a(y)^{\odot 2} \otimes a_0^+a_1^-\beta_2\left(\left[\frac{y-1}{y}\right]\right) \\ &= |X| \left(\frac{n(b)}{d(b)}\, b_0^+b_1^- \sum_{y\in Y} \left(y^{b_0}(y-1)^{b_1}\right) \otimes \beta_2\left(\left[\frac{y-1}{y}\right]\right) \right. \\ &\quad \left. - n(1,-1)\sum_{y\in Y}\left(\frac{y}{y-1}\right)^{\odot 2} \otimes \beta_2\left(\left[\frac{y-1}{y}\right]\right) \right).\end{aligned}$$

For $b_1 > 0$: $R_4^{(0)}(X, Y, \mathcal{A}_X, \mathcal{A}_Y) = |X|\, n(1,-1) \sum_{y\in Y} \beta_4\big([\frac{y}{y-1}]\big)$.
For $b_1 < 0$ we deduce from $\sum_{a\in\mathcal{A}} n(a)\, a^{\odot 3}$ and (2.10) the equation

$$n(b)b_1\, b^{\odot 2} + n(c)c_1\, c^{\odot 2} - n(1,-1)(1,-1)^{\odot 2} + n(0,1)(0,1)^{\odot 2} = 0\,.$$

Hence for $b_0 > -b_1$

$$\begin{aligned}
0 = &\frac{n(b)}{d(b)}\, b_0 b_1 \cdot \sum_{y\in Y} \big(y^{b_0}(y-1)^{b_1}\big)^{\odot 2} \otimes \beta_2([y]) \\
&+ \frac{b_0}{d(b)}\, n(c)c_1 \cdot \sum_{y\in Y} C^{\odot 2} \otimes \beta_2([y]) \\
&- \frac{b_0}{d(b)}\, n(1,-1) \cdot \sum_{y\in Y} \big(\frac{y}{y-1}\big)^{\odot 2} \otimes \beta_2([y]) \\
&+ \frac{b_0}{d(b)}\, n(0,1) \cdot \sum_{y\in Y} (y-1)^{\odot 2} \otimes \beta_2([y])\,,
\end{aligned}$$

and since $|X| = d(b)$ we obtain using (4.1.4)

$$\begin{aligned}
|X| &\bigg(\frac{n(b)}{d(b)}\, b_0 b_1 \cdot \sum_{y\in Y} \big(y^{b_0}(y-1)^{b_1}\big)^{\odot 2} \otimes \beta_2([y]) \\
&\quad - n(1,-1) \cdot \sum_{y\in Y} \big(\frac{y}{y-1}\big)^{\odot 2} \otimes \beta_2([y]) \bigg) \\
&= d(b) \bigg(\big(\frac{b_0}{d(b)} - 1\big) n(1,-1) \cdot \sum_{y\in Y} \big(\frac{y}{y-1}\big)^{\odot 2} \otimes \beta_2([y]) \\
&\quad - \frac{b_0}{d(b)}\, n(0,1) \cdot \sum_{y\in Y} (y-1)^{\odot 2} \otimes \beta_2([y]) \bigg)\,.
\end{aligned}$$

$b_0 > -b_1 > 0$ implies $d(b) = b_0$ and on the right hand side we are left with

$$b_0\, n(0,1) \sum_{y\in Y} \beta_4([1-y])\,.$$

If $-b_1 > b_0$ then $d(b) = b_1$ and we deduce in an analogous way

$$b_1\, n(1,0) \sum_{y\in Y} \beta_4([y])\,.$$

Summarising, we have

$$R_4^{(0)}(X, Y, \mathcal{A}_X, \mathcal{A}_Y) = |X| \begin{cases} n(1,-1) \sum_y \beta_4\big([\frac{y}{y-1}]\big), & \text{if } b_1 > 0\,, \\ n(0,1) \sum_y \beta_4([1-y]), & \text{if } b_0 > -b_1 > 0\,, \\ -\, n(1,0) \sum_y \beta_4([y]), & \text{if } 0 < b_0 < -b_1\,. \end{cases}$$

Similar computations show

$$R_4^{(0)}(Y,X,\mathcal{A}_Y,\mathcal{A}_X) = |Y| \begin{cases} n(1,-1)\sum_x \beta_4([\frac{x}{x-1}]), & \text{if } c_1 > 0\,, \\ n(0,1)\sum_x \beta_4([1-x]), & \text{if } c_0 > -c_1 > 0\,, \\ -n(1,0)\sum_x \beta_4([x]), & \text{if } 0 < c_0 < -c_1\,. \end{cases} \qquad \diamondsuit$$

Let us recall a result from the representation theory of $\mathcal{S}_3$, the symmetric group on 3 variables.

(4.6) Lemma

Let sgn be the sign character on $\mathcal{S}_3$. We consider the 2–dimensional irreducible representation V of $\mathcal{S}_3$. Then we have the following dimension formula for the vector space of invariants

$$\dim\left(\mathrm{Sym}^k(V)\right)^{\mathcal{S}_3} = \begin{cases} \lfloor \frac{k}{6} \rfloor, & \text{if } k \equiv 1 \pmod 6\,, \\ \lfloor \frac{k}{6} \rfloor + 1 & \text{otherwise,} \end{cases} \tag{4.6.1}$$

$$\dim\left(\mathrm{Sym}^k(V)\right)^{\mathrm{sgn}} = \begin{cases} \max(0, \lfloor \frac{k-3}{6} \rfloor), & \text{if } k \equiv 4 \pmod 6\,, \\ \lfloor \frac{k-3}{6} \rfloor + 1 & \text{otherwise.} \end{cases} \tag{4.6.2}$$

Lemma (4.6) for the $\mathcal{S}_3$-invariants and the $\mathcal{S}_3$–antiinvariants is only needed in the explicit form given in lemma (4.8) below.

(4.7) Remark

In the theory of modular forms the dimension formula for the vector space of modular forms of weight $2k$ coincides with the dimension formula for the $\mathcal{S}_3$ –invariants given above. An explanation is given by the fact that we can consider the 2–dimensional representation of $\mathcal{S}_3$ on the vector space spanned by the 3 Eisenstein series to an elliptic curve E with a given non-trivial 2–torsion point P. The pairs (E,P) are parametrised by $\Gamma(2)\backslash\mathbf{H}$. $\mathcal{S}_3$ operates as a projective representation on the homogeneous polynomials in 2 variables (the latter correspond to 2 of the 3 Eisenstein series mentioned above).

Further evidence for the fact that good examples for elements in $\ker\beta_m$ might be motivated by the theory of modular functions is provided by the fact that the j–invariant for $\mathrm{SL}_2(\mathbf{Z})$ can be expressed in terms of the "j-invariant" λ for $\Gamma(2)\backslash\mathbf{H}$:

$$j(\lambda) = 2^8\,\frac{(1-\lambda+\lambda^2)^3}{\lambda^2(\lambda-1)^2}.$$

This rational expression plays an exceptional role in the most interesting of our examples: $j(u)-j(t)$ decomposes into linear factors (one factor is $u-t$, the $\mathcal{S}_3$–invariance of the expression implies that each $u-t^\sigma$ for $\sigma\in\mathcal{S}_3$ constitutes a factor, and for reasons of degree this gives the whole decomposition).

The field extension $\mathbf{Q}(j)$ of $\mathbf{Q}(\lambda)$ is rational and Galois.

This suggests looking for (rational) Galois extensions such that j can be expressed in terms of the "j–invariants" for a congruence subgroup Γ of $SL_2(\mathbf{Z})$. This together with a certain set of subgroups U_i of the factor group $SL_2(\mathbf{Z})/\Gamma$ should give arguments for the "main part" of a functional equation for polylogarithms of high order (where the arguments are of the form $\prod_\sigma \frac{x-t^\sigma}{y-t^\sigma}$ $\sigma \in U_i$). The most promising candidates are e.g. $\Gamma(3), \Gamma(5)$.

(4.8) Lemma

Let $a, b, x, y \in F$, let $\mathcal{S}_3$ operate on the set $\{a, b, c = -a-b\}$ by permutation. Let $|\sigma|$ denote the signature of $\sigma \in \mathcal{S}_3$. Then we have

$$(4.8.1) \qquad \sum_{\sigma \in \mathcal{S}_3} \sigma(ax+by)^k = \begin{cases} 6, & \text{if } k=0, \\ 0, & \text{if } k=1, \\ 4(a^2+ab+b^2)(x^2-xy+y^2), & \text{if } k=2, \\ 3ab(a+b)(2x-y)(x+y)(2y-x), & \text{if } k=3. \end{cases}$$

$$(4.8.2) \quad \sum_{\sigma \in \mathcal{S}_3} (-1)^{|\sigma|}\sigma(ax+by)^k = \begin{cases} 0, & \text{if } k=0,1,2,4, \\ 3(a-b)(2a+b)(a+2b)xy(x-y), & \text{if } k=3. \end{cases}$$

Proof (4.6.1) implies that the ($\mathcal{S}_3$–invariant) sum in (4.8.1) vanishes for $k=1$ and decomposes for $k=0,2,3,4,5,7$ into a product of two polynomials (one polynomial in a and b and another one in x and y). An analogous situation holds for the antiinvariant sum in (4.8.2) because of (4.6.2).

One immediately computes the relation given above. ◊

(4.9) Corollary

Let $t \in F^\times - \{1\}$, let $\mathcal{S}_3$ operate on $\mathbf{P}^1(F)$ in the usual way.
Then (up to 2–torsion) we have for $a_0, a_1 \in \mathbf{Z}$:

(4.9.1)

$$\sum_{\sigma \in \mathcal{S}_3} \left((t^\sigma)^{a_0}(t^\sigma-1)^{a_1}\right)^{\odot k} = \begin{cases} 6, & k=0, \\ 0, & k=1, \\ 4(a_0^2+a_0a_1+a_1^2)\cdot\left(t^{\odot 2} - t\odot(t-1) + (t-1)^{\odot 2}\right), & k=2, \\ 3a_0a_1(a_0+a_1)\cdot(\frac{t^2}{t-1})\odot t(t-1)\odot(\frac{(t-1)^2}{t}), & k=3, \end{cases}$$

$$(4.9.2) \qquad \sum_{\sigma \in \mathcal{S}_3} (-1)^{|\sigma|}\left((t^\sigma)^{a_0}(t^\sigma-1)^{a_1}\right)^{\odot k}$$
$$= \begin{cases} 0, & k=0,1,2,4, \\ 3(a_0-a_1)(2a_0+a_1)(a_0+2a_1)\cdot t\odot(t-1)\odot\frac{t}{(t-1)}, & k=3. \end{cases}$$

Proof According to remark (1.17) we can identify identities in $\mathrm{Sym}^*(M)$ for a free module M with the corresponding polynomial identities. We transform (4.8) into the corresponding identity in $\mathrm{Sym}^k(M)$ and set $x=t$ and $y=1-t$. This implies the statement. ◊

(4.10) Lemma For $(\alpha,\beta) \in \{(2,-1),(1,-2),(1,-1)\}$ we have

$$(\alpha-\beta) = \frac{2}{d(\alpha,\beta)}(\alpha^2+\alpha\beta+\beta^2). \tag{4.10.1}$$

◇

After these preparations we now formulate the main theorem.

(4.11) Theorem

Let $c=(c_0,c_1) \in \mathbf{Z}^2$ such that $\frac{c_0}{c_1} \notin \{0,1,\infty,-1/2,-1,-2\}$,
$\mathcal{A} \subset \{(c_0,c_1),(2,-1),(1,-2),(1,1),(1,0),(0,1),(1,-1)\} \subset \mathbf{Z}^2$.
Let $X = \{t, \frac{1}{t}, \frac{t-1}{t}, \frac{t}{t-1}, \frac{1}{1-t}, 1-t\} \subset F$ for a $t \in F$, and let $Y \subset F$ be a finite subset.
If $c \in \mathcal{A}$ then we assume Y is truly full w.r.t. c.
Then for $m = |\mathcal{A}|-1$ the following holds (up to 2–torsion):

$$\sum_{y\in Y}\sum_{x\in X}\sum_{a\in\mathcal{A}} \frac{n(a)}{d(a)}\left[\frac{\phi_a(x)}{\phi_a(y)}\right] \equiv \sum_{x\in X} B_x[x] + \sum_{y\in Y}\sum_{\tau\in\mathcal{S}_3} C_{y,\tau}[y^\tau] \qquad (\mathrm{mod}\ \ker\beta_m)$$

for some coefficients B_x, $C_{y,\tau} \in \mathbf{Q}$. Here $n(a) = 1/\prod_{a'\in\mathcal{A}-\{a\}}\det(a,a')$ and $\mathcal{S}_3$ operates in the usual way.

Remark The coefficients B_x, $C_{y,\tau}$ can be computed effectively as can be seen from the proof. It is not necessary to actually determine their values since we can proceed as in (3.13), i.e. we introduce another pair (X,Y) with the same properties and take the alternating sum of the corresponding expressions for X, X', Y, Y'.

Proof (of theorem) Without loss of generality let $c_0 > 0$.

We put $\mathcal{A}_X = \mathcal{A} - \{c\}$, $\mathcal{A}_Y = \{c\} \cap \mathcal{A}$.

X is truly full w.r.t. $\mathcal{A}_X$ (cf. examples (2.3)). Also we have (with the notation of (3.11))

$$\mathcal{A}_X^- = \{(2,-1),(1,-2),(1,-1)\},\quad \mathcal{A}_Y^- = \begin{cases} \{(c_0,c_1)\} \cap \mathcal{A}, & \text{if } c_0c_1 < 0, \\ \{\} & \text{otherwise.}\end{cases}$$

For the sake of simplicity and without loss of generality we finally assume that Y is *irreducible*, i.e.

$$|Y| = \begin{cases} d(c), & \text{if } c \in \mathcal{A}, \\ 1 & \text{otherwise.}\end{cases} \tag{4.11.1}$$

(The general case now follows by simply combining several such irreducible Y's, thereby adding the corresponding formulas.)

Lemma (4.2) provides us with the relation

$$0 = \sum_{a\in\mathcal{A}} n(a)\, a^{\odot(m-1)}. \tag{4.11.2}$$

Note that for $a \in \mathcal{A}$ we have: $a_0^- = 0$ and $\chi(a) = \begin{cases} 1, & \text{if } a = (1,-1), \\ 0 & \text{otherwise.} \end{cases}$

I) We first show that the following mixed terms vanish

$$R_m^{(k)}(X,Y,\mathcal{A}_X,\mathcal{A}_Y) = R_m^{(k)}(Y,X,\mathcal{A}_Y,\mathcal{A}_X) = 0, \qquad \text{for } k = 1,\ldots,m-2\,.$$

(i) $k = 1\,,\ m \geq 3\,.$

$$R_m^{(1)}(X,Y,\mathcal{A}_X,\mathcal{A}_Y) = 0\,, \quad \text{since} \quad \sum_{x\in X} \left(x^{a_0}(x-1)^{a_1}\right)^{\odot 1} = 0 \quad \forall a \in \mathcal{A} \quad \text{by (4.9)}\,.$$

(ii) $k = 2\,,\ m \geq 4\,.$

$$\begin{aligned} &R_m^{(2)}(X,Y,\mathcal{A}_X,\mathcal{A}_Y) \\ &\quad = \sum_{a\in\mathcal{A}_X^-} \frac{n(a)}{d(a)}\, a_0 a_1 \sum_{x\in X} \left(x^{a_0}(x-1)^{a_1}\right)^{\odot 2} \odot \sum_{y\in Y} \left(y^{a_0}(y-1)^{a_1}\right)^{\odot(m-4)} \otimes \beta_2([y]). \end{aligned}$$

Using the homomorphism
$\mathrm{Sym}^{m-1}(F^\times) \to \mathrm{Sym}^{m-4}(F^\times)$,
$a^{\odot(m-1)} \mapsto a_0 a_1 (a_0 - a_1) \cdot a^{\odot(m-4)}$, we deduce from (4.11.2) and (4.10)

$$\begin{aligned} 0 &= \sum_{a\in\mathcal{A}} n(a)\, a_0 a_1 (a_0 - a_1) \cdot a^{\odot(m-4)} \\ &= \sum_{a\in\mathcal{A}^-} n(a)\, a_0 a_1 (a_0 - a_1) \cdot a^{\odot(m-4)} \\ &= n(c)\, c_0 c_1 (c_0 - c_1) \cdot c^{\odot(m-4)} + 2 \sum_{a\in\mathcal{A}_X^-} n(a) \frac{a_0 a_1}{d(a)} (a_0^2 + a_0 a_1 + a_1^2)\, a^{\odot(m-4)}. \end{aligned}$$

Because of (4.9.1) this implies for a certain $\delta(c) \in \mathbf{Q}$ and each $y_0 \in Y$

$$\begin{aligned} 0 = \frac{1}{2} &\sum_{a\in\mathcal{A}_X^-} \frac{n(a)}{d(a)}\, a_0 a_1 \sum_{x\in X} \left(x^{a_0}(x-1)^{a_1}\right)^{\odot 2} \odot \sum_{y\in Y} \left(y^{a_0}(y-1)^{a_1}\right)^{\odot(m-4)} \otimes \beta_2([y]) \\ &+ \delta(c) \sum_{x\in X} \left(x^{c_0}(x-1)^{c_1}\right)^{\odot 2} \odot \left(y_0^{c_0}(y_0-1)^{c_1}\right)^{\odot(m-4)} \otimes \sum_{y\in Y} \beta_2([y])\,. \end{aligned}$$

If $c \in \mathcal{A}$ then $\sum_{y\in Y} \beta_2([y]) = 0$ because of (4.1.4) and the last summand vanishes.
If $c \notin \mathcal{A}$ then already $n(c) = 0$, hence also $\delta(c) = 0$.
In each case the claim is proved.

(iii) $k=3\,,\ m\geq 5\,.$

$$R_m^{(3)}(X,Y,\mathcal{A}_X,\mathcal{A}_Y)$$
$$= \sum_{a\in\mathcal{A}_X^-}\frac{n(a)}{d(a)}\,a_0a_1\sum_{x\in X}\left(x^{a_0}(x-1)^{a_1}\right)^{\odot 3}\odot\sum_{y\in Y}\left(y^{a_0}(y-1)^{a_1}\right)^{\odot(m-5)}\otimes\beta_2([y]).$$

Using the homomorphism
$\mathrm{Sym}^{m-1}(F^\times)\to\mathrm{Sym}^{m-5}(F^\times)\,,$
$a^{\odot(m-1)}\mapsto a_0a_1(a_0-a_1)(a_0+a_1)\cdot a^{\odot(m-5)}$, we deduce from (4.11.2) and (4.10)

$$0=\sum_{a\in\mathcal{A}} n(a)a_0a_1(a_0-a_1)(a_0+a_1)\,a^{\odot(m-5)}\,.$$

In this equation the coefficients of $(1,0)$, $(0,1)$, $(1,-1)$ and $(1,1)$ are annihilated.
We have: $a_0-a_1=3$ is independent from $a\in\{(2,-1),(1,-2)\}$ and thus

$$0=\ n(c)\,c_0c_1(c_0-c_1)(c_0+c_1)\,c^{\odot(m-5)}+3\sum_{a\in\{(2,-1),(1,-2)\}} n(a)a_0a_1(a_0+a_1)\,a^{\odot(m-5)}\,.$$

We deduce

$$0=n(c)\,c_0c_1(c_0-c_1)(c_0+c_1)\sum_{y\in Y}\left(y^{c_0}(y-1)^{c_1}\right)^{\odot(m-5)}\otimes\beta_2([y])$$
$$+3\sum_{a\in\{(2,-1),(1,-2)\}} n(a)a_0a_1(a_0+a_1)\sum_{y\in Y}\left(y^{a_0}(y-1)^{a_1}\right)^{\odot(m-5)}\otimes\beta_2([y])\,.$$

The first sum vanishes because of (4.1.4), the second one can be transformed (note that $-a_0a_1=d(a)$ is independent from $a\in\{(2,-1),(1,-2)\}$):

$$0=\sum_{a\in\{(2,-1),(1,-2)\}} n(a)\left(3a_0a_1(a_0+a_1)\right)\sum_{y\in Y}\left(y^{a_0}(y-1)^{a_1}\right)^{\odot(m-5)}\otimes\beta_2([y])$$
$$=-\sum_{a\in\mathcal{A}_X^-}\frac{n(a)}{d(a)}\,a_0a_1\left(3a_0a_1(a_0+a_1)\right)\sum_{y\in Y}\left(y^{a_0}(y-1)^{a_1}\right)^{\odot(m-5)}\otimes\beta_2([y])\,,$$

and so, by (4.9.1), $k=3\,,$

$$0=-\sum_{a\in\mathcal{A}_X^-}\frac{n(a)}{d(a)}\,a_0a_1\sum_{x\in X}\left(x^{a_0}(x-1)^{a_1}\right)^{\odot 3}\odot\sum_{y\in Y}\left(y^{a_0}(y-1)^{a_1}\right)^{\odot(m-5)}\otimes\beta_2([y])\,.$$

(iv) $k=m-2\,.$
$R_m^{(m-2)}(X,Y,\mathcal{A}_X,\mathcal{A}_Y)=0$ by (4.1.4).

(v) $R_m^{(k)}(Y,X,\mathcal{A}_Y,\mathcal{A}_X)$

$$= \frac{n(c)}{d(c)}\, c_0^+ c_1^- \sum_{y\in Y} \left(y^{c_0}(y-1)^{c_1}\right)^{\odot k} \odot \sum_{x\in X} \left(x^{c_0}(x-1)^{c_1}\right)^{\odot(m-2-k)} \otimes \beta_2(\left[\frac{x-1}{x}\right])$$
$$+ n(1,-1)(-1) \sum_{y\in Y} \left(\frac{y}{y-1}\right)^{\odot k} \odot \sum_{x\in X} \left(\frac{x}{x-1}\right)^{\odot(m-2-k)} \otimes \beta_2(\left[\frac{x-1}{x}\right])$$
$$=0 \qquad \text{by (4.9.2) and (1.15)},$$

if $m-2-k = 0,1,2,4$.

We still have to examine the case $m-2-k=3$ (i.e. $m=6, k=1$) (we put $C = \phi_c(y)$ for any—and hence all—$y\in Y$).

$R_6^{(1)}(Y,X,\mathcal{A}_Y,\mathcal{A}_X)$

$$= \frac{n(c)}{d(c)}\, c_0^+ c_1^- \sum_{y\in Y} \left(y^{c_0}(y-1)^{c_1}\right)^{\odot 1} \odot \sum_{\sigma\in S_3} (-1)^{|\sigma|}\left((t^\sigma)^{c_0}(t^\sigma-1)^{c_1}\right)^{\odot 3} \otimes \beta_2([t])$$
$$- n(1,-1) \sum_{y\in Y} \left(\frac{y}{y-1}\right)^{\odot 1} \odot \sum_{\sigma\in S_3} (-1)^{|\sigma|}\left(\frac{t^\sigma}{t^\sigma-1}\right)^{\odot 3} \otimes \beta_2([t])$$
$$= \frac{n(c)}{d(c)}\, c_0^+ c_1^- \sum_{y\in Y} C^{\odot 1} \odot \left(3(c_0-c_1)(2c_0+c_1)(c_0+2c_1)\cdot t\odot(t-1)\odot \frac{t}{t-1}\right) \otimes \beta_2([t])$$
$$- n(1,-1) \sum_{y\in Y} \left(\frac{y}{y-1}\right)^{\odot 1} \odot \left(6\cdot t\odot(t-1)\odot\frac{t}{t-1}\right) \otimes \beta_2([t]).$$

Since $m=6$ we must have $|\mathcal{A}|=7$ and c lies in $\mathcal{A}$, so using (4.11.1) we get $d(c) = \sum_{y\in Y} 1$ and using (4.1.3) we have to show

$$0 = n(c)\, c_0 \cdot \begin{Bmatrix} 0, & \text{if } c_1>0 \\ c_1, & \text{if } c_1<0 \end{Bmatrix} \cdot 3(c_0-c_1)(2c_0+c_1)(c_0+2c_1)\cdot C^{\odot 1}$$
$$- n(1,-1)\cdot 3\cdot 2\cdot \begin{Bmatrix} (\pm 1)^{\odot 1}, & \text{if } c_1>0 \\ (\pm C)^{\odot 1}, & \text{if } c_1<0 \end{Bmatrix}.$$

If $c_1>0$ then each of the two summands vanishes.

If $c_1<0$ then the claim follows from (4.11.2) if we apply the following homomorphism: $\mathrm{Sym}^5(F^\times)\to \mathbf{Z}$, $a^{\odot(5)} \mapsto a_0a_1(a_0-a_1)(2a_0+a_1)(a_0+2a_1)$.

II) We want to analyse the case $k=0$ in more detail in order to be able to describe the right hand side of the equation in the theorem. (Note that $\chi(c)=0$.)

(i) $R_m^{(0)}(X,Y,\mathcal{A}_X,\mathcal{A}_Y) = |X| \sum_{a\in\mathcal{A}_X^-} \frac{n(a)}{d(a)}\, a_0a_1 \sum_{y\in Y} \left(y^{a_0}(y-1)^{a_1}\right)^{\odot(m-2)} \otimes \beta_2([y])$.

From (4.11.2) we deduce after applying the homomorphism $\mathrm{Sym}^{m-1}(F^\times)\to \mathrm{Sym}^{m-2}(F^\times)$,

$$a^{\odot(m-1)} \mapsto (a_0 - a_1) \cdot a^{\odot(m-2)} :$$

$$0 = \sum_{a\in\mathcal{A}} n(a)(a_0 - a_1)\, a^{\odot(m-2)} ,$$

and since the coefficients of $(2,-1)$ and $(1,-2)$ are multiplied by the same factor (i.e. 2) we obtain with the abbreviation $\mathcal{A}_2 = \{(2,-1),(1,-2)\}$:

$$3 \sum_{a\in\mathcal{A}_2} n(a)\, a^{\odot(m-2)} = - \sum_{a\in\mathcal{A}-\mathcal{A}_2} n(a)(a_0 - a_1)\, a^{\odot(m-2)} \qquad \Longrightarrow$$

$$\sum_{a\in\mathcal{A}_2} n(a) \sum_{y\in Y} \left(y^{a_0}(y-1)^{a_1}\right)^{\odot(m-2)} \otimes \beta_2([y])$$
$$= -\frac{1}{3} \sum_{a\in\mathcal{A}-\mathcal{A}_2} n(a)(a_0 - a_1) \sum_{y\in Y} \left(y^{a_0}(y-1)^{a_1}\right)^{\odot(m-2)} \otimes \beta_2([y]) .$$

On the right hand side the sums for $a = (1,1)$ and for $a = c$ are annihilated (by (4.1.4)), thus we get because of $d(a) = -a_0 a_1$ for $a \in \mathcal{A}_2$, introducing the abbreviation $\mathcal{A}_1 = \{(1,0),(0,1),(1,-1)\}$:

$$\sum_{a\in\mathcal{A}_2} \frac{n(a)}{d(a)}\, a_0 a_1 \sum_{y\in Y} \left(y^{a_0}(y-1)^{a_1}\right)^{\odot(m-2)} \otimes \beta_2([y])$$
$$= \frac{1}{3} \sum_{a\in\mathcal{A}_1} n(a)(a_0 - a_1) \sum_{y\in Y} \left(y^{a_0}(y-1)^{a_1}\right)^{\odot(m-2)} \otimes \beta_2([y])$$
$$= \frac{1}{3} \sum_{y\in Y} \beta_m \left(n(1,0)[y] + n(0,1)[1-y] - 2\, n(1,-1)[\frac{y}{y-1}] \right) .$$

Altogether we derive the following equation (where $|X| = 6$)

$$R_m^{(0)}(X,Y,\mathcal{A}_X,\mathcal{A}_Y) = 2 \sum_{y\in Y} \beta_m \left(n(1,0)[y] + n(0,1)[1-y] + n(1,-1)[\frac{y}{y-1}] \right) .$$

(ii) For $R_m^{(0)}(Y,X,\mathcal{A}_Y,\mathcal{A}_X)$ the only case left to consider is $m = 5$ (otherwise the expression disappears by (4.9.2)).

$$R_5^{(0)}(Y,X,\mathcal{A}_Y,\mathcal{A}_X) = |Y|\, \frac{n(c)}{d(c)}\, c_0^+ c_1^- \sum_{x\in X} \left(x^{c_0}(x-1)^{c_1}\right)^{\odot 3} \otimes \beta_2([x])$$
$$- |Y|\, n(1,-1) \sum_{x\in X} \left(\frac{x}{x-1}\right)^{\odot 3} \otimes \beta_2([x])$$
$$= |Y|\, \frac{n(c)}{d(c)}\, c_0^+ c_1^- \cdot 3(c_0 - c_1)(2c_0 + c_1)(c_0 + 2c_1) \cdot t \odot (t-1) \odot \frac{t}{t-1} \otimes \beta_2([t])$$
$$+ |Y|\, n(1,-1) \sum_{x\in X} \beta_5([\frac{x}{x-1}]) .$$

We can polarise:

$$t \odot (t-1) \odot \frac{t}{t-1} = \frac{1}{3}\left(t^{\odot 3} + (\frac{t-1}{t})^{\odot 3} + (\frac{1}{1-t})^{\odot 3}\right),$$

and obtain

$$3 \cdot t \odot (t-1) \odot \frac{t}{t-1} \otimes \beta_2([t]) = \beta_5([t] + [\frac{t-1}{t}] + [\frac{1}{1-t}]) = \frac{1}{2}\sum_{x\in X} \beta_5([x]),$$

hence

$$R_5^{(0)}(Y,X,\mathcal{A}_Y,\mathcal{A}_X) = |Y|\left(\frac{n(c)}{2\,d(c)}\, c_0^+ c_1^- (c_0-c_1)(2c_0+c_1)(c_0+2c_1) + n(1,-1)\right)\sum_{x\in X}\beta_5([x]).$$

$\diamondsuit$

(4.12) Examples

(1) Let $c=(2,1)$ and $\mathcal{A}_0 = \{c,(2,-1),(1,-2),(1,1),(1,0),(0,1),(1,-1)\}$.
We have $d(c) = 3$, $d(a) = 2$ for $a \in \{(2,-1),(1,-2),(1,1)\}$ and $d(a) = 1$ for $a \in \{(1,0),(0,1),(1,-1)\}$.
Let $t,u \in F$, $X = \{t, \frac{1}{t}, \frac{t-1}{t}, \frac{t}{t-1}, \frac{1}{1-t}, 1-t\}$ and $Y = \{\frac{u}{1-u+u^2}, \frac{1-u}{1-u+u^2}, \frac{-u(1-u)}{1-u+u^2}\}$.
Y is truly full w.r.t. c, X is truly full w.r.t. $\mathcal{A}_0 - \{c\}$.
Theorem (4.11) yields for $\mathcal{A} \subset \mathcal{A}_0$ and $m \in \mathbf{N}_{>1}$ where $m = |\mathcal{A}|-1$, certain coefficients $n^{(m)}(a)$, $a \in \mathcal{A}$, and $B, C_1, C_2, C_3 \in \mathbf{Q}$

(4.12.1)

$$\sum_{x\in X}\sum_{y\in Y}\sum_{a\in\mathcal{A}} \frac{n^{(m)}(a)}{d(a)}\left[\frac{\phi_a(x)}{\phi_a(y)}\right] + \sum_{x\in X} B[x] + \sum_{y\in Y}\left(C_1[\frac{y}{y-1}] + C_2[1-y] + C_3[\frac{1}{y}]\right) \in \ker \beta_m .$$

(i) $m=6$, $\mathcal{A} = \mathcal{A}_0$. The coefficients $n^{(6)}(a)$ are given by lemma (4.2) (we multiply by the lcm of the denominators). The triple sum in (4.12.1) is independent from X by (4.11), and specialising $u=0$ we obtain an expression for the remaining sums in (4.12.1) with integers B, C_1, C_2, C_3 given in the following table (cf. also [Zg-A]).

a	$(2,1)$	$(1,-2)$	$(2,-1)$	$(1,1)$	$(1,-1)$	$(0,1)$	$(1,0)$	B	C_1	C_2	C_3
$n^{(6)}(a)$	-3	-4	-5	20	60	-90	180	0	120	-180	-360

(ii) For $m < 6$ and certain $\mathcal{A}$, $|\mathcal{A}| = m+1$, we take the coefficients $n_j^{(m)}(a)$ again from lemma (4.2) (multiplied by the lcm of the denominators), and just like in (i) elements $\eta_j^{(m)} = \sum_{a\in\mathcal{A}} \frac{n_j^{(m)}(a)}{d(a)} \left[\frac{\phi_a(x)}{\phi_a(y)}\right]$ in $\ker\beta_m$ emerge. In the following three short tables we give several $\eta_j^{(m)}$ of this kind for $m = 5, 4$ and 3. Here a row corresponds to the coefficients of an element in $\mathbf{Z}[F]$ as in equation (4.12.1).

$m = 5$

a	$(2,1)$	$(1,-2)$	$(2,-1)$	$(1,1)$	$(1,-1)$	$(0,1)$	$(1,0)$	B	C_1	C_2	C_3
$n_1^{(5)}(a)$	−9	4	−5	40	0	−90	180	0	0	180	−360
$n_2^{(5)}(a)$	3	2	5	−10	−30	0	−90	90	60	0	180
$n_3^{(5)}(a)$	3	−8	−5	−20	60	90	0	−180	−120	−180	0

$m = 4$

a	$(2,1)$	$(1,-2)$	$(2,-1)$	$(1,1)$	$(1,-1)$	$(0,1)$	$(1,0)$	B	C_1	C_2	C_3
$n_1^{(4)}(a)$	7	4	5	−20	−20	30	−60	0	0	0	0
$n_2^{(4)}(a)$	−1	0	0	3	−1	−3	6	0	−6	0	0
$n_3^{(4)}(a)$	1	0	3	0	−8	−6	−24	0	0	−36	0
$n_4^{(4)}(a)$	−1	3	0	0	−10	15	15	0	0	0	−90
$n_5^{(4)}(a)$	0	4	5	1	−27	9	−18	0	−42	0	0
$n_6^{(4)}(a)$	0	1	−4	−5	9	18	27	0	0	63	0
$n_7^{(4)}(a)$	0	−5	−1	4	18	−27	−9	0	0	0	126

$m = 3$

a	$(2,1)$	$(1,-2)$	$(2,-1)$	$(1,1)$	$(1,-1)$	$(0,1)$	$(1,0)$	B	C_1	C_2	C_3
$n_1^{(3)}(a)$	1	0	0	−2	0	1	−2	0	0	0	0
$n_2^{(3)}(a)$	1	0	3	0	−4	0	−12	0	0	0	0
$n_3^{(3)}(a)$	1	6	0	0	−10	−15	0	0	0	0	0
$n_4^{(3)}(a)$	9	4	−5	−20	0	0	0	0	0	0	0
$n_5^{(3)}(a)$	1	0	0	0	2	−3	−6	0	−12	0	0

(2) Let $i, t, u \in F$ where $i^2 = -1$, $X = \{t, \frac{1}{t}, \frac{t-1}{t}, \frac{t}{t-1}, \frac{1}{1-t}, 1-t\}$, $c = (3,1)$,

$$U = (U_j)_{j=1}^4 = \Big(1+u+u^2,\ -1-u+i\,u\,(2+u),\ -u\,(1+u)-i\,(1+2\,u),\ u-i\,\left(-1+u^2\right)\Big),$$

$$Y = \left\{ - \frac{\prod_{k\neq j} U_k}{(U_1+U_2)(U_2+U_3)(U_3+U_1)} \right\}_{j=1,..,4}.$$

Y is truly full w.r.t. c.
Just as in (1) we deduce several functional equations for $2 \leq m \leq 6$.

(3) Of course we obtain further elements in $\ker \beta_m$ by taking linear combinations of the expressions in (1) and (2) as well as by specialising one of the "variables" u, t. In both cases we are going to destroy certain symmetries, though. As an example we want to give a specialisation of (1)(i) to $m = 6$, $t = u$.

The following element $\xi \in \mathbf{Z}[\mathbf{Q}(u)]$ is in the kernel of β_6:

$$\begin{aligned} \xi = {} & 6\,B^-_{\{2,-2,3\}} + 8\,B^+_{\{-4,-1,4\}} + 5\,B^-_{\{-2,-2,5\}} - 40\,B^-_{\{2,-1,1\}} - 120\,B^-_{\{-2,-1,3\}} \\ & + 180\,B^-_{\{2,0,-1\}} - 360\,B^+_{\{2,0,-1\}} - 20\,B^-_{\{-2,2,2\}} + 120\,B^+_{\{2,-1,-1\}} - 190\,B^-_{\{2,-1,-2\}} \\ & + 540\,B^-_{\{0,1,0\}} - 360\,B^+_{\{0,1,0\}} + 544\,B^+_{\{1,-1,1\}}, \end{aligned}$$

where we put

$$B^\pm_{\{a_1,a_2,a_3\}} = \sum_{\sigma\in\mathcal{S}_3} [\pm \prod_{j=1}^{3} y_j^{a_{\sigma(j)}}],$$

and $(y_j)_{j=1}^3$ is Y as in (1) with any ordering ($\mathcal{S}_3$ operates by permutations).

(4) Finally, for each element in $\ker \beta_m$ we can produce several elements in $\ker \beta_{m-1}$ via a certain derivation process: for a homomorphism $\phi : F^\times \to \mathbf{Z}$ we associate the linear map $\iota_\phi : \mathbf{Z}[F] \to \mathbf{Z}[F]$ defined on generators by $[x] \mapsto \phi(x)[x]$. Each such map will suffice (cf. [Zg-A], s.1).

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