Integral Motives and Special Values of Zeta Functions

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ABSTRACT. For each field k, we define a category of rationally decomposed mixed motives with $\mathbb{Z}$-coefficients. When k is finite, we show that the category is Tannakian, and we prove formulas relating the behaviour of zeta functions near integers to certain Ext groups.

CONTENTS

Introduction. For each field k, there is conjecturally a category of mixed isomotives whose full subcategory of semisimple objects is the category $\mathcal{M}(k;\mathbb{Q})$ of pure isomotives. In this paper, we define a $\mathbb{Z}$-category $\mathcal{M}(k;\mathbb{Z})$ of motives whose quotient by its subcategory of torsion objects is $\mathcal{M}(k;\mathbb{Q})$. Thus, $\mathcal{M}(k;\mathbb{Z})$ is the abelian category of mixed motives over k whose weight filtrations split modulo torsion. As Grothendieck observed, when k is finite, the weight filtrations always split modulo torsion, and so, in this case, $\mathcal{M}(k;\mathbb{Z})$ is the full category of mixed motives over k.

The category $\mathcal{M}(k;\mathbb{Z})$ depends on our choice of $\mathcal{M}(k;\mathbb{Q})$. For definiteness, take $\mathcal{M}(k;\mathbb{Q})$ to be the category of isomotives based on the smooth projective varieties over k whose Künneth projectors are algebraic with the correspondences taken to be the numerical equivalence classes of algebraic classes. This category is abelian and

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semisimple (Jannsen 1992), hence Tannakian (Deligne 1990), and when we assume[1] numerical equivalence coincides with homological equivalence, it admits canonical l-adic fibre functors for each l. The category $\mathcal{M}(k;\mathbb{Z})$ we construct is noetherian, abelian, and, at least when k is finite, Tannakian.

The Ext groups in the full category $\mathcal{M}(k;\mathbb{Z})$ seem to be pathological, but those in the subcategory $\mathcal{M}^{+}(k;\mathbb{Z})$ of effective motives are of interest. When k is finite, we show that the Tate conjecture implies[2] relations between these Ext groups and the behaviour of zeta functions near integers (Theorems 10.1, 10.5). Moreover, to a smooth projective variety V satisfying a certain degeneration condition on its crystalline cohomology, we are able to attach "Weil" motivic cohomology groups $H^i_{\mathrm{mot}}(V,\mathbb{Z}(r))$ satisfying the spectral sequence

$$H^i(\Gamma_0, H^j_{\mathrm{mot}}(\bar{V},\mathbb{Z}(r)) \Longrightarrow H^{i+j}_{\mathrm{mot}}(V,\mathbb{Z}(r)) \tag{1}$$

posited by Lichtenbaum (2002), and deduce from Theorem 10.5 that

$$\zeta(V,s) \sim \pm\chi^{\times}(V,\mathbb{Z}(r)) \cdot q^{\chi(V,\mathcal{O}_V,r)} \cdot (1-q^{r-s})^{\rho} \text{ as } s \to r \tag{2}$$

(Theorem 10.7). Here Γ_0 is the subgroup of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ generated by the Frobenius element (so $\Gamma_0 \cong \mathbb{Z}$), $\chi^{\times}(V,\mathbb{Z}(r))$ is the alternating product of the orders of the cohomology groups of a complex

$$\cdots \to H^i_{\mathrm{mot}}(V,\mathbb{Z}(r)) \to H^{i+1}_{\mathrm{mot}}(V,\mathbb{Z}(r)) \to \cdots$$

arising from (1), $\chi(V,\mathcal{O}_V,r)$ is as in Milne 1986a, and $\rho = \sum(-1)^i i\cdot\mathrm{rank}\, H^i_{\mathrm{mot}}(V,\mathbb{Z}(r))$.

The category $\mathcal{M}(k;\mathbb{Z})$ is defined in §5 as a full subcategory of the fibre product of the $\mathbb{Q}$-category $\mathcal{M}(k;\mathbb{Q})$ with a certain $\hat{\mathbb{Z}}$-category of realizations. The construction requires that we show that the various l-adic fibre functors on $\mathcal{M}(k;\mathbb{Q})$ define an adèlic fibre functor — the proof of this depends crucially on the theorem of Gabber (1983). In §4, we show that the usual Tate conjecture implies an adèlic version. For example, for a variety over a finite field, if the l-adic Tate conjecture holds for a single l, then it holds l-integrally for almost all l.

The category $\mathcal{M}(\mathbb{F}_q;\mathbb{Z})$ is shown to be Tannakian in §6, and the main results on Exts and motivic cohomology groups are obtained in §10 after preliminaries in §§7–9.

This article is part of a project whose ultimate goal is to define a triangulated category of motives with t-structure whose heart is the category $\mathcal{M}(k;\mathbb{Z})$ defined above, and to show that the Tate conjecture for smooth projective varieties implies the formula (2) for *arbitrary* varieties over $\mathbb{F}_q$.

We note that our approach to motives is opposite to that of other authors (Hanamura 1995, 1999; Levine 1998; Voevodsky 2000) who define a triangulated category of motives with the goal of putting a t-structure on it whose heart will be the abelian category of motives. While our approach can be expected to give the full category of mixed motives only when the ground field is $\mathbb{F}_q$ or $\mathbb{F}$, we do expect to obtain the correct p-structure, as illustrated by (2). Most of the difficulties in the present work involve the p-part in characteristic p.

[1]See (5.26) for an alternative choice of $\mathcal{M}(k;\mathbb{Q})$ that does not require any unproven conjectures.
[2]When $\mathcal{M}(k;\mathbb{Q})$ is as in (5.26), these relations are hold unconditionally.

Notations. Throughout, l is a prime number, p is the characteristic (or characteristic exponent) of the ground field, and ℓ is a prime number $\neq p$. The l-adic valuations $|\ \ |_l$ and ord_l are normalized so that $|l|_l = l^{-1}$ and $\mathrm{ord}_l(l) = 1$. The expression "for almost all l" means "for all but (possibly) a finite number of l".

The profinite completion of $\mathbb{Z}$ is denoted $\hat{\mathbb{Z}}$ and $\mathbb{A}_f = \mathbb{Q} \otimes \hat{\mathbb{Z}}$. Thus, $\hat{\mathbb{Z}} = \prod \mathbb{Z}_l$, and we let $\mathbb{Z}^p = \prod_{l \neq p} \mathbb{Z}_l$ and $\mathbb{A}_f^p = \mathbb{Q} \otimes \mathbb{Z}^p$. When $p = 1$, $\mathbb{Z}^p = \hat{\mathbb{Z}}$ and $\mathbb{A}_f^p = \mathbb{A}_f$.

We use M_m and $M^{(m)}$ respectively to denote the kernel and cokernel of multiplication by m on M. For a prime l, $M(l) = \bigcup M_{l^n}$ is the l-primary component of M, and $M_{\mathrm{tors}} = \bigcup M_m$ is the torsion subobject.

For a module M endowed with an action of a group Γ, M^Γ (resp. M_Γ) is the largest submodule (resp. quotient module) of a Γ-module M on which Γ acts trivially.

Lattice always means full lattice: when R is a ring and V is a free $\mathbb{Q} \otimes R$-module of finite rank, an R-lattice in V is a finitely generated R-submodule Λ of V such that the inclusion $\Lambda \hookrightarrow V$ induces an isomorphism $\mathbb{Q} \otimes \Lambda \to V$.

For a field k, k^{al} denotes an algebraic closure of k, k^{sep} a separable (algebraic) closure of k, and k^{pf} a perfect closure of k (for example, $k^{\mathrm{pf}} = k^{p^{-\infty}} \subset k^{\mathrm{al}}$).

For a perfect field k of characteristic p, $W(k)$ denotes the ring of Witt vectors with coefficients in k, and $B(k)$ denotes the field of fractions of $W(k)$. The unique lift of the automorphism $x \mapsto x^p \colon k \to k$ to $W(k)$ is denoted σ. We let A denote the skew polynomial ring $W[F, \sigma]$ (polynomials in F with the relation $F \cdot a = \sigma(a) \cdot F$, $a \in W$).

Generally, rings are assumed to be commutative. For a ring R, $\mathbf{Mod}(R)$ is the category of R-modules and $\mathbf{Modf}(R)$ the category of finitely presented R-modules, and for a field k, $\mathbf{Vec}(k)$ is the category of finite dimensional k-vector spaces. We usually abbreviate $M \otimes_R S$ to M_S. For an R-linear category (alias, R-category) $\mathcal{M}$, $\mathcal{M}_S$ denotes the category with the same objects as $\mathcal{M}$, but with $\mathrm{Hom}_{\mathcal{M}_S}(X, Y) = \mathrm{Hom}_{\mathcal{M}}(X, Y)_S$.

For an affine group scheme G over a noetherian ring R, $\mathbf{Rep}(G; R)$ denotes the category of linear representations of G on finitely generated R-modules.

Given functors $F \colon \mathcal{A} \to \mathcal{C}$ and $G \colon \mathcal{B} \to \mathcal{C}$, the *fibre product category* $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ has as objects the triples (A, B, γ) with A an object of $\mathcal{A}$, B an object of $\mathcal{B}$, and $\gamma \colon F(A) \to G(B)$ an isomorphism; the morphisms are the pairs $a \colon A \to A'$ and $b \colon B \to B'$ giving rise to a commutative diagram in $\mathcal{C}$.

A variety V is a geometrically reduced separated scheme of finite type over a field. The group of algebraic cycles of codimension r on V is denoted $Z^r(V)$, and $Z^r_\sim(V)$ is the quotient of $Z^r(V)$ by some adequate relation $\sim$. We let $H^*(V, F) = \oplus_i H^i(V, F)$.

An equivalence class containing x is denoted $[x]$. We also use $[S]$ to denote the cardinality of a finite set S. For a homomorphism $f \colon M \to N$ of abelian groups whose kernel and cokernel are finite, $z(f)$ is defined to be $[\mathrm{Ker}(f)]/[\mathrm{Coker}(f)]$.

Isomorphisms are denoted $\approx$ and canonical isomorphisms $\cong$.

1. The Realization Categories

The categories $\mathcal{R}$ away from p. Throughout this subsection, k is a field of characteristic exponent p, and R is a topological ring, for example, $R = \mathbb{Z}$, $\mathbb{Z}_\ell$, $\mathbb{Q}_\ell$, $\mathbb{Z}^p$, or $\mathbb{A}_f^p$ with its natural topology (discrete for $\mathbb{Z}$).

DEFINITION 1.1. (a) For a profinite group Γ, $\mathcal{R}(\Gamma; R)$ is the category of continuous linear representations of Γ on finitely presented R-modules.

(b) When k is finitely generated over the prime field,

$$\mathcal{R}(k; R) = \mathcal{R}(\Gamma; R), \quad \Gamma = \mathrm{Gal}(k^{\mathrm{sep}}/k).$$

Otherwise, $\mathcal{R}(k; R) = \varinjlim \mathcal{R}(k'; R)$ where k' runs over the subfields of k finitely generated over the prime field.

REMARK 1.2. Let Γ be a profinite group. Two continuous linear representations ρ_1 and ρ_2 of open subgroups U_1 and U_2 of Γ on an R-module M are said to be *related* if they agree on an open subgroup of $U_1 \cap U_2$. This is an equivalence relation, and an equivalence class will be called a *germ* of a continuous linear representation of Γ. A *morphism* $(M, [\rho_U]) \to (M', [\rho'_{U'}])$ *of germs* is an R-linear map $\alpha \colon M \to M'$ that is equivariant for some open subgroup of $U \cap U'$. When k is a separable closure of a field k_0 finitely generated over the prime field, an object of $\mathcal{R}(k; R)$ can be identified with a germ of a continuous linear representation of $\mathrm{Gal}(k/k_0)$.

LEMMA 1.3. *For each object M of $\mathcal{R}(\Gamma; \mathbb{Z}_\ell)$, there is an open subgroup $U \subset \Gamma$ such that M decomposes into the sum of a torsion object and a torsion-free object in $\mathcal{R}(U; \mathbb{Z}_\ell)$. Consequently, when k is separably closed, for each M in $\mathcal{R}(k; \mathbb{Z}_\ell)$, M_{tors} is a direct summand of M.*

PROOF. Let $M = M_{\mathrm{tors}} \oplus M_1$ as $\mathbb{Z}_\ell$-modules, and let $\gamma \in \Gamma$ act on M as the matrix $\left(\begin{smallmatrix} a(\gamma) & b(\gamma) \\ 0 & c(\gamma) \end{smallmatrix}\right)$. The map $\gamma \mapsto a(\gamma)$ is the homomorphism describing the action of Γ on M_{tors}. After replacing Γ with an open subgroup U, we may suppose this action to be trivial. The map $\gamma \mapsto c(\gamma)$ defines an action of U on M_1, and hence an action of U on $\mathrm{Hom}(M_1, M_{\mathrm{tors}})$, which, again, we may suppose to be trivial. Then $\gamma \mapsto b(\gamma)$ is a homomorphism $U \to \mathrm{Hom}(M_1, M_{\mathrm{tors}})$, and after we replace U with the kernel of this map, $M = M_{\mathrm{tors}} \oplus M_1$ will be a decomposition of U-modules. □

For M in $\mathcal{R}(\Gamma; \mathbb{Z}_\ell)$, set $M^\vee = \mathrm{Hom}_{\mathbb{Z}_\ell}(M, \mathbb{Z}_\ell)$ with $\gamma \in \Gamma$ acting according to the rule $(\gamma f)(m) = f(\gamma^{-1} m)$. We say M is *reflexive* if the canonical map $M \to M^{\vee\vee}$ is an isomorphism.

PROPOSITION 1.4. (a) *An object of $\mathcal{R}(\Gamma; \mathbb{Z}_\ell)$ is reflexive if and only if it is torsion-free.*

(b) *Every object of $\mathcal{R}(\Gamma; \mathbb{Z}_\ell)$ is a quotient of a reflexive object.*

PROOF. (a) As $M^\vee$ is torsion-free, the condition is necessary, and it is obviously sufficient.

(b) If M decomposes in $\mathcal{R}(\Gamma; \mathbb{Z}_\ell)$ into the direct sum of a torsion module M_t with trivial Γ-action and a torsion-free module M_1, then M will be the quotient of $\mathbb{Z}_\ell^r \oplus M_1$ for some r. According the lemma, there is an open subgroup U of Γ such that M has such a decomposition in $\mathcal{R}(U; \mathbb{Z}_\ell)$, and hence there will be a surjection $\mathbb{Z}_\ell^r \oplus M_1 \to M$ in $\mathcal{R}(U; \mathbb{Z}_\ell)$ with M_1 free. Now

$$\mathrm{Ind}_U^\Gamma(\mathbb{Z}_\ell^r \oplus M_1) \twoheadrightarrow \mathrm{Ind}_U^\Gamma(M) \xrightarrow{\varphi \mapsto \sum_{s \in \Gamma/U} s\varphi(s^{-1})} M$$

realizes M as the quotient of a torsion-free Γ-module. Here $\mathrm{Ind}_U^\Gamma(N)$ denotes the induced module. □

REMARK 1.5. (a) The category $\mathcal{R}(k;\mathbb{Q}_\ell)$ is denoted $\underline{\text{Tate}}(k)$ in Saavedra 1972, VI A4.2, A4.3, and its objects are called Tate modules. It is a neutral Tannakian category over $\mathbb{Q}_\ell$.

(b) The functor $M \mapsto (M_\ell)_{\ell\neq p}\colon \mathcal{R}(\Gamma;\mathbb{Z}^p) \to \prod_{\ell\neq p}\mathcal{R}(\Gamma;\mathbb{Z}_l)$ is exact, full, and faithful, with essential image the objects $(M_\ell)_{\ell\neq p}$ such that $\dim_{\mathbb{F}_\ell} M_\ell/\ell M_\ell$ is bounded.

The categories $\mathcal{R}$ at p. An *F-isocrystal over a perfect field k* is a finite dimensional $B(k)$-vector space M endowed with a σ-linear bijection $F_M\colon M \to M$. With the obvious structures, the F-isocrystals over k form a Tannakian category $\mathbf{Isoc}(k)$ over $\mathbb{Q}_p$, and the forgetful functor is a $B(k)$-valued fibre functor (Saavedra 1972, VI 3.2). The identity object is $\mathbb{1} = (B(k),\sigma)$, and the Tate object is $\mathbb{T} = (B(k), p^{-1}\cdot\sigma)$.

An F-isocrystal (M,F_M) is *effective* if F_M stabilizes a $W(k)$-lattice in M. Let $\mathbf{Isoc}^+(k)$ denote the full subcategory of $\mathbf{Isoc}(k)$ whose objects are the effective F-isocrystals. Every object of $\mathbf{Isoc}(k)$ is of the form $X\otimes\mathbb{T}^{\otimes m}$ with X effective. In other words, for each (M,F_M) in $\mathbf{Isoc}(k)$, there exists an $m\in\mathbb{Z}$ such that p^mF_M stabilizes a lattice (Saavedra 1972, VI 3.1.3, 3.2.1).

An *F-crystal*[3] *over a perfect field k* is a finitely generated $W(k)$-module Λ endowed with a σ-linear map $F_\Lambda\colon \Lambda\to\Lambda$ such that the kernel of F_Λ is torsion. The F-crystals over k form an abelian tensor category $\mathbf{Crys}^+(k)$ over $\mathbb{Z}_p$. For an F-crystal (Λ,F_Λ), $(\Lambda,F_\Lambda)_\mathbb{Q} =_{\mathrm{df}} (\Lambda_\mathbb{Q}, F_\Lambda\otimes 1_\mathbb{Q})$ is an F-isocrystal, and the functor

$$(\Lambda,F_\Lambda)\mapsto(\Lambda,F_\Lambda)_\mathbb{Q}\colon \mathbf{Crys}^+(k)_\mathbb{Q}\to\mathbf{Isoc}(k)$$

is fully faithful with essential image $\mathbf{Isoc}^+(k)$.

Let $\mathbb{L} = (W(k), p\cdot\sigma)$ be the Lefschetz F-crystal. The functor

$$-\otimes\mathbb{L}\colon \mathbf{Crys}^+(k)\to\mathbf{Crys}^+(k)$$

is faithful (full and faithful on torsion-free objects). We define $\mathbf{Crys}\,(k)$ to be the tensor category obtained from $\mathbf{Crys}^+(k)$ by inverting $\mathbb{L}$. Thus, the objects of $\mathbf{Crys}\,(k)$ are pairs (Λ,m) with Λ an F-crystal and $m\in\mathbb{Z}$, and

$$\mathrm{Hom}((\Lambda,m),(\Lambda',m')) = \varinjlim_{N\geq m,m'} \mathrm{Hom}(\Lambda\otimes\mathbb{L}^{\otimes N-m}, \Lambda'\otimes\mathbb{L}^{\otimes N-m'}).$$

The tensor product is defined by

$$(\Lambda,m)\otimes(\Lambda',m') = (\Lambda\otimes_{W(k)}\Lambda', m+m').$$

The *Tate object* $\mathbb{T}$ of $\mathbf{Crys}(k)$ is $(\mathbb{1},1)$.

The functor

$$\Lambda\mapsto(\Lambda,0)\colon \mathbf{Crys}^+(k)\to\mathbf{Crys}(k)$$

is faithful, and it is full on torsion-free objects. The forgetful functor defines an equivalence of the full subcategory of $\mathbf{Crys}(k)$ of torsion objects with the category of $W(k)$-modules of finite length.

DEFINITION 1.6. Let k be a field of characteristic $p\neq 0$. If k is finitely generated over $\mathbb{F}_p$,

$$\mathcal{R}(k;\mathbb{Q}_p) = \mathbf{Isoc}(k^{\mathrm{pf}}),\quad \mathcal{R}(k;\mathbb{Z}_p) = \mathbf{Crys}(k^{\mathrm{pf}}).$$

[3]Some authors use "crystal" to mean free crystal, i.e., an M that is torsion-free (as a $W(k)$-module).

Otherwise,

$$\mathcal{R}(k;\mathbb{Z}_p) = \varinjlim_{k'\subset k} \mathcal{R}(k';\mathbb{Z}_p), \quad \mathcal{R}(k;\mathbb{Q}_p) = \varinjlim_{k'\subset k} \mathcal{R}(k';\mathbb{Q}_p),$$

where the limits are over the subfields of k finitely generated over $\mathbb{F}_p$. We define $\mathcal{R}^+(k;\mathbb{Q}_p)$ and $\mathcal{R}^+(k;\mathbb{Z}_p) = \varinjlim_{k'\subset k} \mathbf{Crys}^+(k'^{\mathrm{pf}})$ similarly.

LEMMA 1.7. *For each object M of $\mathcal{R}^+(k;\mathbb{Z}_p)$, there is an n such that $M\otimes\mathbb{L}^{\otimes n}$ decomposes into a direct sum of a torsion object and a torsion-free object. Consequently, for each M in $\mathcal{R}(k;\mathbb{Z}_p)$, M_{tors} is a direct summand of M.*

PROOF. Let $M = M_{\mathrm{tors}} \oplus M_1$ as $W(k'^{\mathrm{pf}})$-modules, and let F_M act as $\left(\begin{smallmatrix} a & b \\ 0 & c\end{smallmatrix}\right)$. Then $F_{M\otimes\mathbb{L}^{\otimes n}}$ acts as $\left(\begin{smallmatrix} p^n a & p^n b \\ 0 & p^n c\end{smallmatrix}\right)$. Choose n so that $p^n M_{\mathrm{tors}} = 0$, and then $p^n b = 0$. This proves the first statement, and the second follows because $-\otimes\mathbb{L}$ is an equivalence of categories on $\mathcal{R}(k;\mathbb{Z}_p)$. □

Let (Λ, F) be an F-crystal. For some N there exists a σ^{-1}-linear map $V\colon \Lambda\to\Lambda$ such that $FV = p^N = VF$. Let $\Lambda' = \mathrm{Hom}_{W(k)}(\Lambda, W(k))$ and define $F'\colon \Lambda'\to\Lambda'$ by $(F'f)(\lambda) = f(V\lambda)$. For an object (Λ, F, m) of $\mathcal{R}(k;\mathbb{Z}_p)$, define $(\Lambda,F,m)^\vee = (\Lambda', F', N-m)$. We say that (Λ, F, m) is *reflexive* if the canonical map $(\Lambda,F,m)\to(\Lambda,F,m)^{\vee\vee}$ is an isomorphism.

PROPOSITION 1.8. (a) *An object of $\mathcal{R}(k;\mathbb{Z}_p)$ is reflexive if and only if it is torsion-free.*

(b) *Every object of $\mathcal{R}(k;\mathbb{Z}_p)$ is a quotient of a reflexive object.*

PROOF. (a) is a obvious. After (1.7) it suffices to prove (b) for a torsion F-crystal N. Such an N is a quotient $M\to N$ of a free finitely generated W-module M. Endow M with the F-crystal structure such that $F_M = \sigma$. After $M\to N$ has been tensored with a high power of $\mathbb{L}$, it will be a morphism of F-crystals. □

REMARK 1.9. (a) There is a canonical functor $\varinjlim \mathbf{Isoc}(\mathbb{F}_q) \to \mathbf{Isoc}(\mathbb{F})$, which is faithful and essentially surjective (Demazure 1972, p. 85) but **not** full ($W(\mathbb{F}) \neq \varinjlim W(\mathbb{F}_q)$).

(b) Ekedahl 1986, p36, defines a *virtual F-crystal* to be a triple (M, F, Λ) with (M, F) an isocrystal and Λ a W-submodule of M such that $B(k)\cdot\Lambda = M$; a virtual F-crystal is of *finite type* if Λ is finitely generated. The functor $((\Lambda, F_\Lambda), m)\mapsto (\Lambda_{\mathbb{Q}}, F_\Lambda\otimes p^{-m}, \Lambda)$ defines an equivalence of the full subcategory of $\mathbf{Crys}(k)$ of torsion-free objects with the category of virtual F-crystals of finite type.

The categories $\mathcal{R}$ in general.

DEFINITION 1.10. For a field k of characteristic $p\neq 0$,

$$\begin{aligned} \mathcal{R}(k;\hat{\mathbb{Z}}) &= \mathcal{R}(k;\mathbb{Z}^p)\times\mathcal{R}(k;\mathbb{Z}_p), \quad \mathcal{R}(k;\mathbb{A}_f) = \mathcal{R}(k;\mathbb{A}_f^p)\times\mathcal{R}(k;\mathbb{Q}_p) \\ \mathcal{R}^+(k;\hat{\mathbb{Z}}) &= \mathcal{R}(k;\mathbb{Z}^p)\times\mathcal{R}^+(k;\mathbb{Z}_p), \quad \mathcal{R}^+(k;\mathbb{A}_f) = \mathcal{R}(k;\mathbb{A}_f^p)\times\mathcal{R}^+(k;\mathbb{Q}_p) \end{aligned}$$

For a field of characteristic zero,

$$\mathcal{R}(k;\hat{\mathbb{Z}}) = \mathcal{R}(k;\mathbb{Z}^p), \quad \mathcal{R}(k;\mathbb{A}_f) = \mathcal{R}(k;\mathbb{A}_f^p).$$

Note that $\mathcal{R}(k;\mathbb{A}_f)\cong\mathcal{R}(k;\hat{\mathbb{Z}})_{\mathbb{Q}}$. For notational convenience, we sometimes use $\mathcal{R}^+(k;\mathbb{Z}_l)$ to denote $\mathcal{R}(k;\mathbb{Z}_l)$ when $l\neq p$.

The motivic subcategory of $\mathcal{R}(k;\mathbb{Q}_\ell)$. Let $\mathcal{M}(k;\mathbb{Q}_\ell)$ be the strictly full subcategory of $\mathcal{R}(k;\mathbb{Q}_\ell)$ whose objects are subquotients of a Tate twist of a tensor power of an ℓ-adic étale cohomology group of a smooth projective variety over k. When $k=\mathbb{F}_q$, $\mathcal{M}(k;\mathbb{Q}_\ell)$ consists of the semisimple representations of $\Gamma=\mathrm{Gal}(k^{\mathrm{al}}/k)$ for which the eigenvalues of the Frobenius generator of Γ are Weil q-numbers: the Weil conjectures (proved by Grothendieck and Deligne) imply that every object of $\mathcal{M}(k;\mathbb{Q}_\ell)$ has this property, but all such representations arise already in the cohomology of abelian varieties (cf. Milne 1994, 3.7). When k is a number field, Conjecture 1 of Fontaine and Mazur (1995) suggests a description of $\mathcal{M}(k;\mathbb{Q}_\ell)$.

2. Adèlic Cohomology

Modules over $\mathbb{A}_f^p$ and lattices. Recall that $\mathbb{Z}^p=\prod_{\ell\neq p}\mathbb{Z}_\ell$ and $\mathbb{A}_f^p$ is the restricted direct product $\mathbb{A}_f^p=\prod_{\ell\neq p}(\mathbb{Q}_\ell,\mathbb{Z}_\ell)$.

2.1. Endow $(\mathbb{A}_f^p)^m$ with its topology as the restricted direct product of the spaces $\mathbb{Q}_\ell^m$ relative to the subspaces $\mathbb{Z}_\ell^m$. Every $\mathbb{A}_f^p$-linear bijection $(\mathbb{A}_f^p)^m\to(\mathbb{A}_f^p)^m$ is bicontinuous. This allows us to endow any free $\mathbb{A}_f^p$-module M of finite rank with a topology by choosing an isomorphism $\alpha\colon M\to(\mathbb{A}_f^p)^m$.

2.2. Let M be a free $\mathbb{A}_f^p$-module of finite rank m, so that $M_\ell=_{\mathrm{df}}M\otimes_{\mathbb{A}_f^p}\mathbb{Q}_\ell$ is a $\mathbb{Q}_\ell$-vector space of dimension m for all $\ell\neq p$. A *lattice* in M is a finitely generated $\mathbb{Z}^p$-submodule Λ such that the inclusion $\Lambda\hookrightarrow M$ induces an isomorphism $\Lambda_{\mathbb{A}_f^p}\to M$; then $\Lambda_\ell=_{\mathrm{df}}\Lambda\otimes_{\mathbb{Z}^p}\mathbb{Z}_\ell$ is a $\mathbb{Z}_\ell$-lattice in M_ℓ for all $\ell\neq p$, and Λ is the free $\mathbb{Z}^p$-submodule spanned by a basis for M. Clearly, every lattice in M is compact and open, and so any two lattices Λ and Λ' in M are *commensurable*, that is, both quotients $\Lambda/\Lambda\cap\Lambda'$ and $\Lambda'/\Lambda\cap\Lambda'$ are finite (hence, $\Lambda_\ell=\Lambda'_\ell$ for almost all ℓ). Let M_0 be a $\mathbb{Q}$-structure on M. There is a canonical one-to-one correspondence between the lattices Λ in M and the lattices Λ_0 in M_0, namely,

$$\Lambda\mapsto\Lambda\cap M_0,\ \Lambda_0\mapsto\text{its closure in }M.$$

2.3. Let $(M_\ell)_{\ell\neq p}$ be a family of vector spaces over the fields $\mathbb{Q}_\ell$, all of the same finite dimension. Two families $(\Lambda_\ell)_{\ell\neq p}$ and $(\Lambda'_\ell)_{\ell\neq p}$ of lattices in the M_ℓ will be said to be *equivalent* if $\Lambda_\ell=\Lambda'_\ell$ for almost all ℓ. To give a free $\mathbb{A}_f^p$-module of finite rank is the same as to give a family $(M_\ell)_{\ell\neq p}$ together with an equivalence class of families $(\Lambda_\ell)_{\ell\neq p}$: given a family $(M_\ell,\Lambda_\ell)_{\ell\neq p}$, let $M=\prod(M_\ell,\Lambda_\ell)$; given M, choose an isomorphism $M\to(\mathbb{A}_f^p)^m$, and let Λ_ℓ be the inverse image of $\mathbb{Z}_\ell^m$. Let M and M' be the free $\mathbb{A}_f^p$-modules of finite rank defined by systems $(M_\ell,\Lambda_\ell)_{\ell\neq p}$ and $(M'_\ell,\Lambda'_\ell)_{\ell\neq p}$ respectively. A family of $\mathbb{Q}_\ell$-linear maps $\alpha_\ell\colon M_\ell\to M'_\ell$ defines an $\mathbb{A}_f^p$-linear map $\alpha\colon M\to M'$ if and only if $\alpha_\ell(\Lambda_\ell)\subset\Lambda'_\ell$ for almost all ℓ, in which case α is injective if and only if α_ℓ is injective for all ℓ and it is surjective if and only if α_ℓ is surjective for all ℓ and maps Λ_ℓ onto Λ'_ℓ for almost all ℓ.

2.4. An $\mathbb{A}_f^p$-linear map $\alpha\colon M\to M'$ of free $\mathbb{A}_f^p$-modules of finite rank will be said to be of *constant rank* if the rank of $\alpha_\ell=_{\mathrm{df}}\alpha\otimes_{\mathbb{A}_f^p}\mathbb{Q}_\ell$ is independent of ℓ. Then the kernel, cokernel, image, and co-image of α are again free $\mathbb{A}_f^p$-modules of finite rank.

Adèlic cohomology (characteristic $p \neq 0$). Throughout this subsection, all varieties are smooth and projective over a separably closed field k of characteristic $p \neq 0$. For such a variety V, $H^r(V, -)$ denotes an étale cohomology group and $\Lambda^i_\ell(V, r)$ denotes the image of $H^i(V, \mathbb{Z}_\ell(r))$ in $H^i(V, \mathbb{Q}_\ell(r))$. Thus,

$$\Lambda^i_\ell(V, r) \cong H^i(V, \mathbb{Z}_\ell(r))/\{\text{torsion}\}.$$

PROPOSITION 2.5. (a) *The torsion subgroup of $H^i(V, \mathbb{Z}_\ell)$ is finite for all ℓ, and is zero for almost all ℓ.*

(b) *Assume V is connected of dimension d_V and fix a prime $\ell \neq p$. If the cohomology groups $H^j(V, \mathbb{Z}_\ell)$ of V are torsion-free for all j, then the pairing*

$$H^i(V, \mathbb{Q}_\ell) \times H^{2d_V - i}(V, \mathbb{Q}_\ell(d_V)) \to H^{2d_V}(V, \mathbb{Q}_\ell(d_V)) \cong \mathbb{Q}_\ell$$

induces a perfect[4] *pairing on the cohomology groups with coefficients in $\mathbb{Z}_\ell$.*

(c) *Let W be a second variety over k and fix a prime $\ell \neq p$. If the groups $H^i(V, \mathbb{Z}_\ell)$ and $H^j(W, \mathbb{Z}_\ell)$ are torsion-free for all i and j, then the Künneth isomorphism*

$$H^r(V \times W, \mathbb{Q}_\ell) \cong \bigoplus_{i+j=r} H^i(V, \mathbb{Q}_\ell) \otimes H^j(W, \mathbb{Q}_\ell)$$

induces an isomorphism on the groups with coefficients in $\mathbb{Z}_\ell$.

PROOF. (a) The finiteness of the torsion subgroup of $H^i(V, \mathbb{Z}_\ell)$ is a standard result, and the rest of statement (a) is proved in Gabber 1983.

(b) The pairing on the cohomology groups with coefficients in $\mathbb{Z}_\ell$ is the unique pairing for which the diagrams

$$\begin{array}{ccccc}
H^i(V, \mathbb{Z}_\ell) & \times & H^{2d_V - i}(V, \mathbb{Z}_\ell(d_V)) & \longrightarrow & H^{2d_V}(V, \mathbb{Z}_\ell(d_V)) \cong \mathbb{Z}_\ell \\
\downarrow & & \downarrow & & \downarrow \\
H^i(V, \mathbb{Z}/\ell^m\mathbb{Z}) & \times & H^{2d_V - i}(V, (\mathbb{Z}/\ell^m\mathbb{Z})(d_V)) & \longrightarrow & H^{2d_V}(V, (\mathbb{Z}/\ell^m\mathbb{Z})(d_V)) \cong \mathbb{Z}/\ell^m\mathbb{Z}
\end{array}$$

commute for all m. The lower pairing is perfect and the vertical maps are the cokernels of multiplication by ℓ^m on $H^i(V, \mathbb{Z}_\ell)$ and $H^{2d_V - i}(V, \mathbb{Z}_\ell(d_V))$ because we are assuming there is no torsion. This implies (even with $m = 1$) that the discriminant of the top pairing is an ℓ-adic unit, and so the top pairing is also perfect.[5]

(c) This is a standard result. □

DEFINITION 2.6. For V as above, $H^i(V, \mathbb{A}^p_f(r))$ is the restricted direct product of the spaces $H^i(V, \mathbb{Q}_\ell(r))$ with respect to the lattices $\Lambda^i(V, r)$. Equivalently,

$$H^i(V, \mathbb{A}^p_f(r)) = H^i(V, \mathbb{Z}^p(r)) \otimes \mathbb{Q}$$

where $H^i(V, \mathbb{Z}^p(r)) = \varprojlim_{(n,p)=1} H^i(V, \mu_n^{\otimes r})$.

REMARK 2.7. (a) Because the Betti numbers of a smooth projective variety are independent of ℓ, $H^i(V, \mathbb{A}^p_f)$ is a free $\mathbb{A}^p_f$-module of finite rank. A regular map $\alpha \colon V \to W$ defines an $\mathbb{A}^p_f$-linear map $\alpha^* \colon H^i(W, \mathbb{A}^p_f) \to H^i(V, \mathbb{A}^p_f)$.

(b) For a connected V, the proposition implies the following:

[4] A bilinear form $M \times N \to R$ of R-modules is *nondegenerate* if its left and right kernels are zero, and *perfect* if the maps $M \to \mathrm{Hom}_R(N, R)$ and $N \to \mathrm{Hom}_R(M, R)$ it defines are isomorphisms.

[5] We thank M. Nori for this proof, which is simpler than our original.

the pairings in (b) of the proposition define a perfect pairing

$$H^i(V,\mathbb{A}_f^p)\times H^{2d_V-i}(V,\mathbb{A}_f^p(d_V))\to H^{2d_V}(V,\mathbb{A}_f^p(d_V))\cong\mathbb{A}_f^p$$

of free $\mathbb{A}_f^p$-modules of finite rank.

In other words, Poincaré duality holds for $\mathbb{A}_f^p$-cohomology. This implies that the map α^* in (a) has an $\mathbb{A}_f^p$-linear adjoint

$$\alpha_*\colon H^j(V,\mathbb{A}_f^p(d_V))\to H^{j+2d_W-2d_V}(W,\mathbb{A}_f^p(d_W)),\quad j=2d_V-i.$$

(c) Part (c) of the theorem implies the following: there is a canonical isomorphism of free $\mathbb{A}_f^p$-modules of finite rank

$$H^r(V\times W,\mathbb{A}_f^p)\cong\bigoplus_{i+j=r}H^i(V,\mathbb{A}_f^p)\otimes H^j(W,\mathbb{A}_f^p).$$

(d) Assume (for simplicity) that V is connected. Let f be an algebraic correspondence of degree r from V to W with rational coefficients, i.e., an element of $Z^{d_V+r}(V\times W)_{\mathbb{Q}}$. For all but finitely many ℓ, f will be ℓ-integral, and so its ℓ-adic cohomology classes define an element $cl(f)\in H^{2d_V+2r}(V\times W,\mathbb{A}_f^p(d_V+r))$. Combining this remark with (b) and (c), we obtain a map (also denoted f)

$$x\mapsto q_{W*}(cl(f)\cup q_V^*x)\colon H^i(V,\mathbb{A}_f^p(m))\to H^{i+2r}(W,\mathbb{A}_f^p(m+r))$$

for all m. Here q_W and q_V are the projection morphisms.

Remark 2.8. The main theorem of de Jong 1996 shows that Gabber's theorem (Gabber 1983) holds also for the cohomology with compact support of arbitrary varieties over separably closed fields. Thus, appropriately interpreted, so do the remaining results of this subsection.

Adèlic cohomology (characteristic zero). Let V be a variety (not necessarily smooth or projective) over an algebraically closed field k of characteristic zero. For any model V_0/k_0 of V over an algebraically closed subfield k_0 of k and embedding $k_0\hookrightarrow\mathbb{C}$,

$$H^i(V,\mathbb{Z}_l)\cong H^i(V_0,\mathbb{Z}_l)\cong H^i(V_0(\mathbb{C}),\mathbb{Z})_{\mathbb{Z}_l}.$$

The group $H^i(V_0(\mathbb{C}),\mathbb{Z})$ is finitely generated, and so $H^i(V,\mathbb{Z}_l)$ is torsion-free for almost all l. The results in the last subsection hold *mutatis mutandis* for the group $H^i(V,\mathbb{A}_f(r))$ defined to be the restricted topological product of the $H^i(V,\mathbb{Q}_l)$ with respect to the $\Lambda_l^i(V,r)$. For any choice of a pair V_0/k_0, $k_0\hookrightarrow\mathbb{C}$ as above,

$$H^i(V,\mathbb{A}_f)\cong H^i(V_0,\mathbb{A}_f)\cong H^i(V_0(\mathbb{C}),\mathbb{Q})_{\mathbb{A}_f}.$$

3. An Adèlic Tate Conjecture

Throughout this section, V is a smooth projective variety of dimension d over a field k. For $\ell\neq\mathrm{char}(k)$, $H^i(V,\mathbb{Z}_\ell(r))$ denotes the étale cohomology group. We let $\bar{k}$ $=k^{\mathrm{sep}}$ and $\bar{V}=V_{/\bar{k}}$. Finally, $\Gamma=\mathrm{Gal}(\bar{k}/k)$.

Conjectures on algebraic cycles in ℓ-adic cohomology. The group $Z^r_{\ell\text{-hom}}(V)$ is the co-image of the cycle class map $Z^r(V) \to H^{2r}(\bar{V}, \mathbb{Z}_\ell(r))$, and $Z^r_{\text{num}}(V)$ is the quotient of $Z^r(V)$ by numerical equivalence.

Recall (Tate 1994) that, when k is finitely generated over the prime field, there are the following conjectures.

- $\mathbf{T}^r(\mathbf{V}/\mathbf{k},\ell)$**:** The cycle map $Z^r_{\ell\text{-hom}}(V)_{\mathbb{Q}_\ell} \to H^{2r}(\bar{V}, \mathbb{Q}_\ell(r))^\Gamma$ is surjective.
- $\mathbf{E}^r(\mathbf{V}/\mathbf{k},\ell)$**:** The quotient map $Z^r_{\ell\text{-hom}}(V)_{\mathbb{Q}} \to Z^r_{\text{num}}(V)_{\mathbb{Q}}$ is an isomorphism.
- $\mathbf{I}^r(\mathbf{V}/\mathbf{k},\ell)$**:** The cycle map $Z^r_{\ell\text{-hom}}(V)_{\mathbb{Q}_\ell} \to H^{2r}(\bar{V}, \mathbb{Q}_\ell(r))$ is injective.
- $\mathbf{S}^r(\mathbf{V}/\mathbf{k},\ell)$**:** The map $H^{2r}(\bar{V}, \mathbb{Q}_\ell(r))^\Gamma \to H^{2r}(\bar{V}, \mathbb{Q}_\ell(r))_\Gamma$ induced by the identity map is bijective.

An adèlic Tate conjecture.

LEMMA 3.1. *Assume k is finitely generated over the prime field. If $T^r(V/k,\ell)$, $T^{d-r}(V/k,\ell)$, and $S^r(V/k,\ell)$ hold for almost all ℓ, then the cycle class map*

$$Z^r_{\ell\text{-}hom}(V)_{\mathbb{Z}_\ell} \to H^{2r}(\bar{V}, \mathbb{Z}_\ell(r))^\Gamma$$

is surjective for almost all ℓ.

PROOF. According to Tate 1994, 2.9 (c)⇒(d), the conditions imply that $E^r(V/k,\ell)$ and $E^{d-r}(V/k,\ell)$ hold for almost all ℓ.

Note that $E^r(V/k,\ell)$ implies $Z^r_{\ell\text{-hom}}(V)/\{\text{torsion}\} \cong Z^r_{\text{num}}(V)$, which is finitely generated (see, for example, Milne 1980, VI 11.7). Therefore, for those ℓ for which $E^r(V/k,\ell)$ and $E^{d-r}(V/k,\ell)$ hold,

$$Z^r_{\ell\text{-hom}}(V)/\{\text{torsion}\} \times Z^{d-r}_{\ell\text{-hom}}(V)/\{\text{torsion}\} \to \mathbb{Z}$$

is a nondegenerate pairing of finitely generated abelian groups whose discriminant D is independent of ℓ.

Consider the diagram

$$\begin{array}{ccccc} Z^r_{\ell\text{-hom}}(V)_{\mathbb{Z}_\ell} & \times & Z^{d-r}_{\ell\text{-hom}}(V)_{\mathbb{Z}_\ell} & \xrightarrow{\bullet} & \mathbb{Z}_\ell \\ \downarrow & & \downarrow & & \| \\ H^{2r}(\bar{V}, \mathbb{Z}_\ell(r))^\Gamma & \times & H^{2d-2r}(\bar{V}, \mathbb{Z}_\ell(d-r))^\Gamma & \xrightarrow{\cup} & \mathbb{Z}_\ell. \end{array}$$

For those ℓ for which $T^r(V/k,\ell)$ and $T^{d-r}(V/k,\ell)$ hold, the vertical arrows have finite cokernel.

For those ℓ for which

- the groups $H^{2r}(\bar{V}, \mathbb{Z}_\ell(r))$ and $H^{2d-2r}(\bar{V}, \mathbb{Z}_\ell(d-r))$ are torsion-free,
- $E^r(V/k,\ell)$ and $E^{d-r}(V/k,\ell)$ hold,
- $T^r(V/k,\ell)$ and $T^{d-r}(V/k,\ell)$ hold, and
- ℓ does not divide D,

the vertical maps are surjective. Since these conditions hold for almost all primes ℓ, this proves the lemma. □

THEOREM 3.2. *Let k be as in the lemma. If $T^r(V/k,\ell)$, $T^{d-r}(V/k,\ell)$, and $S^r(V/k,\ell)$ hold for all $\ell \neq p$, then the cycle class map defines a bijection*

$$Z^r_{num}(V)_{\mathbb{A}^p_f} \to H^{2r}(\bar{V}, \mathbb{A}^p_f(r))^\Gamma.$$

PROOF. We have to show that the cycle class map $Z^r(V) \to H^{2r}(\bar{V}, \mathbb{Z}_\ell(r))$ defines

(a) a bijection $Z^r_{\text{num}}(V)_{\mathbb{Q}_\ell} \to H^{2r}(\bar{V}, \mathbb{Q}_\ell(r))^\Gamma$ for all $\ell \neq p$, and
(b) a surjection $Z^r_{\text{num}}(V)_{\mathbb{Z}_\ell} \to \Lambda^{2r}_\ell(\bar{V}, r)^\Gamma$ for almost all ℓ.

Applying Tate 1994, 2.9, we see that $E^r(V/k, \ell)$, $I^r(V/k, \ell)$, and $T^r(V/k, \ell)$ hold for all $\ell \neq p$. This implies (a), and (b) follows from the lemma. □

COROLLARY 3.3. *Let V be a (smooth projective) variety over $\mathbb{F}_q$, and let r be an integer. The following statements are equivalent:*

(a) *for some prime $\ell \neq p$, $\dim_{\mathbb{Q}} Z^r_{num}(V) = \dim_{\mathbb{Q}_\ell} H^{2r}(V, \mathbb{Q}_\ell(r))^\Gamma$;*
(b) *the order of the pole of the zeta function $Z(V, t)$ of V at $t = q^{-r}$ is equal to the rank of $Z^r_{num}(V)$;*
(c) *the cycle class map defines a bijection $Z^r_{num}(V)_{\mathbb{A}^p_f} \to H^{2r}(\bar{V}, \mathbb{A}^p_f(r))^\Gamma$.*

PROOF. Apply Tate 1994, 2.9, and the theorem. □

Counterexamples to an integral Tate conjecture. Assume k is finitely generated over the prime field, and consider the cycle class map

$$Z^r(V)_{\mathbb{Z}_\ell} \to H^{2r}(\bar{V}, \mathbb{Z}_\ell(r))^\Gamma. \tag{3}$$

Over infinite fields k there exist V's without a rational zero cycle of degree 1 (for example, any curve of genus zero without a rational point or any curve[6] of genus 1 with index $\neq 1$) and for such a variety (3) will fail to be surjective for $r = d$ and suitable ℓ. The argument that Atiyah and Hirzebruch (1962) used to prove that not all torsion cohomology classes on a complex variety are algebraic (contradicting Hodge's original conjecture) also applies over finite fields[7] and proves that (3) is not surjective on torsion classes.

Griffiths and Harris (1985) conjecture that if V is a very general hypersurface[8] of degree $d \geq 6$ in $\mathbb{P}^4$, then d divides the degree of every curve on V. Kollár has obtained some results in the direction of this conjecture (see Ballico et al. 1992, p. 135). Thus, it is known that, if k is an uncountable algebraically closed field and $\ell > 3$ is distinct from the characteristic of k, then there exist smooth hypersurfaces $V \subset \mathbb{P}^4$ such that the cycle class map

$$Z^2(V)_{\mathbb{Z}_\ell} \to H^4(V, \mathbb{Z}_\ell(2)) \cong \mathbb{Z}_\ell$$

is not surjective (cf. Schoen 1998, 0.6).

We do not know of an example with k finite for which (3) is not surjective modulo torsion (however, Schoen 1998, 2.1, gives an example with V singular). For $r = 1$ and k finite, one sees from the Kummer sequence that $T^1(V/k, \ell)$ implies that the map (3) is surjective.

[6]For such a curve, even $CH^1(\bar{V})^\Gamma \xrightarrow{\text{degree}} \mathbb{Z}$ fails to be surjective (Lang and Tate 1958).

[7]For this, one needs to use that the odd-dimensional Steenrod operations with values in $H^*(\bar{V}, \mathbb{Z}/\ell\mathbb{Z})$ vanish on algebraic cycles (Brosnan 1999).

[8]A statement is said to hold for a very general hypersurface of degree d if it holds for all hypersurfaces in the complement of countably many proper subvarieties of the space of hypersurfaces of degree d.

4. The Adèlic Fibre Functor

Isomotives. Fix a field k of characteristic exponent p, and let $\mathcal{S}$ be a class of smooth projective varieties over k satisfying the following condition:

4.1. $\mathcal{S}$ is closed under passage to a connected component and under the formation of products and disjoint unions, and it contains the zero-dimensional varieties and the projective spaces.

4.2. Fix an adequate equivalence relation $\sim$. Recall that, for smooth projective varieties V and W over k, the group of correspondences of degree i from V to W is

$$\mathrm{Corr}^i(V,W) = \oplus_j Z_{\sim}^{\dim(V_j)+i}(V_j \times W)_{\mathbb{Q}}$$

where the V_j are the equidimensional components of V.

4.3. The category $\mathcal{CV}^0_{\sim}(k)$ of correspondences based on $\mathcal{S}$, has one object hV for each $V \in \mathcal{S}$, and

$$\mathrm{Hom}_{\mathcal{CV}^0_{\sim}(k)}(hV, hW) = \mathrm{Corr}^0(V,W)_{\mathbb{Q}}.$$

With the definition

$$hV \otimes hW = h(V \times W) \tag{4}$$

and the obvious constraints, $\mathcal{CV}^0_{\sim}(k)$ becomes a tensor category over $\mathbb{Q}$ (Saavedra 1972, VI 4.1.1).

4.4. The category of effective isomotives $\mathcal{M}^+_{\sim}(k;\mathbb{Q};\mathcal{S})$ is the pseudo-abelian hull of $\mathcal{CV}^0_{\sim}(k)$. Its objects are pairs (hV, e) with e an idempotent in $\mathrm{End}(hV)$. It is a pseudo-abelian tensor category over $\mathbb{Q}$.

4.5. In $\mathcal{M}^+_{\sim}(k;\mathbb{Q};\mathcal{S})$ there is a canonical decomposition

$$h\mathbb{P}^1 = \mathbb{1} \oplus \mathbb{L},$$

and the category of isomotives $\mathcal{M}_{\sim}(k;\mathbb{Q};\mathcal{S})$ is obtained from $\mathcal{M}^+_{\sim}(k;\mathbb{Q};\mathcal{S})$ by inverting $\mathbb{L}$ (Saavedra 1972, VI 4.1.3.1). Its objects are pairs $((hV,e),m)$ with (hV,e) an effective motive and $m \in \mathbb{Z}$. It is a pseudo-abelian rigid tensor category over $\mathbb{Q}$ (Saavedra 1972, VI 4.1.3.5). We sometimes write $h(V,e,m)$ for $((hV,e),m)$.

4.6. We say that *the Künneth projectors are algebraic* for a smooth projective variety V if this is so for the l-adic étale cohomologies for all $l \neq p$ and the crystalline cohomology (characteristic $p \neq 0$). There are then well-defined elements $\pi^i \in Z^i_{\mathrm{num}}(V)_{\mathbb{Q}}$ (independent of the cohomology theory) that give these projectors for each cohomology theory (when k is finite, this follows from the Riemann hypothesis, and when k is arbitrary it can be proved by a specialization argument).

When the Künneth projectors are algebraic for all V in $\mathcal{S}$, we use them to modify the commutativity constraints in $\mathcal{M}^+_{\sim}(k;\mathbb{Q};\mathcal{S})$ and $\mathcal{M}_{\sim}(k;\mathbb{Q};\mathcal{S})$ (Saavedra 1972, VI 4.2.1.5).

For the remainder of this section, we assume that $\sim$ is not coarser than ℓ-adic homological equivalence on $Z^r(V)_{\mathbb{Q}}$ for any $\ell \neq p$, so that the formula

$$\omega_\ell(h(V,e,m)) = e\left(H^*(\bar{V}, \mathbb{Q}_\ell(m))\right), \quad \bar{V} = V_{k^{\mathrm{sep}}},$$

defines a tensor functor $\omega_\ell\colon \mathcal{M}_{\sim}(k;\mathbb{Q};\mathcal{S}) \to \mathbf{Vec}(\mathbb{Q}_\ell)$, and we drop $\sim$ from the notation.

The adèlic fibre functor. With the notation of (2.6), define

$$\omega^p(hV) = H^*(\bar{V}, \mathbb{A}_f^p).$$

Because of (2.7d) and our assumption on $\sim$, a correspondence f of degree 0 from V to W defines a homomorphism $\omega^p(f)\colon \omega^p(hV) \to \omega^p(hW)$, and one sees immediately from the definitions that $\omega^p(f \circ g) = \omega^p(f) \circ \omega^p(g)$. The isomorphisms

$$\omega^p(hV \otimes hW) \overset{(4)}{=} \omega^p(h(V \times W)) \overset{2.7c}{\cong} \omega^p(hV) \otimes \omega^p(hW)$$

provide ω^p with the structure of a tensor functor $\mathcal{CV}^0(k) \to \mathbf{Mod}(\mathbb{A}_f^p)$. It is $\mathbb{Q}$-linear (in particular, additive).

THEOREM 4.7. *The functor $\omega^p\colon \mathcal{CV}^0(k) \to \mathbf{Mod}(\mathbb{A}_f^p)$ extends (essentially uniquely) to a $\mathbb{Q}$-linear tensor functor $\omega^p\colon \mathcal{M}(k;\mathbb{Q};\mathcal{S}) \to \mathbf{Mod}(\mathbb{A}_f^p)$. For each $\ell \neq p$, $\omega^p \otimes_{\mathbb{A}_f^p} \mathbb{Q}_\ell \cong \omega_\ell$.*

PROOF. As $\mathbf{Mod}(\mathbb{A}_f^p)$ is abelian, ω^p extends (essentially uniquely) to $\mathcal{M}^+(k;\mathbb{Q};\mathcal{S})$ because of the universality of the pseudo-abelian hull. Because $\omega^p(\mathbb{L}) = \mathbb{A}_f^p(-1)$ is invertible in $\mathbf{Mod}(\mathbb{A}_f^p)$, ω^p has an essentially unique extension from $\mathcal{M}^+(k;\mathbb{Q};\mathcal{S})$ to $\mathcal{M}(k;\mathbb{Q};\mathcal{S})$. For $V \in \mathcal{S}$, $\omega^p(hV) \otimes_{\mathbb{A}_f^p} \mathbb{Q}_\ell \cong \omega_\ell(hV)$, and this property is preserved under each extension. □

REMARK 4.8. There is the following formula,

$$\omega^p(h(V,e,m)) = e(H^*(\bar{V}, \mathbb{A}_f^p(m)))$$

where, on the right, e denotes the map defined in (2.7d). For a motive $X = h(V,e,m)$, define $\lambda_\ell(X)$ to be the image of

$$H^*(\bar{V}, \mathbb{Z}_\ell(m)) \to H^*(\bar{V}, \mathbb{Q}_\ell(m)) \xrightarrow{e} H^*(\bar{V}, \mathbb{Q}_\ell(m)).$$

Then, $\omega^p(X)$ is the restricted direct product

$$\omega^p(X) = \prod_{\ell \neq p} (\omega_\ell(X), \lambda_\ell(X)).$$

PROPOSITION 4.9. *If $\sim$ is numerical equivalence and the Künneth projectors of all varieties in $\mathcal{S}$ are algebraic, then for all isomotives X,*

(a) *$\omega^p(X)$ is a free $\mathbb{A}_f^p$-module of finite rank;*
(b) *for all morphisms α of isomotives, $\omega^p(\alpha)$ has constant rank.*

PROOF. The hypotheses imply that $\mathcal{M}(k;\mathbb{Q};\mathcal{S})$ is a Tannakian category over $\mathbb{Q}$ (Jannsen 1992). For any fibre functor ω on a Tannakian category with values in the vector spaces over a field, and any morphism $\alpha\colon X \to Y$ in the category

$$\begin{aligned} \dim(\omega(X)) &= \operatorname{rank}(X) \\ \operatorname{rank}(\omega(\alpha)) &= \operatorname{rank} X - \operatorname{rank}\operatorname{Ker}(\alpha). \end{aligned}$$

In particular, $\dim(\omega(X))$ and $\operatorname{rank}(\omega(\alpha))$ are independent of ω. When we apply this statement to the functors ω_ℓ, we obtain the theorem. □

REMARK 4.10. Because we are assuming $\sim$ is not coarser than ℓ-adic homological equivalence, the hypotheses of the proposition are that, for all $\ell \neq p$, numerical equivalence coincides with ℓ-adic homological equivalence on algebraic cycles with $\mathbb{Q}$-coefficients and that the Künneth projectors are algebraic. Without these hypotheses, the conclusion is not known (see the discussion Katz 1994, p. 28, (3)).

The functor $\omega^p\colon \mathcal{M}(k;\mathbb{Q};\mathcal{S}) \to \mathcal{R}(k;\mathbb{A}_f^p)$.

THEOREM 4.11. *The functor ω_l has a canonical factorization through the forgetful functor $\mathcal{R}(k;\mathbb{Q}_\ell) \to \mathbf{Vec}(\mathbb{Q}_\ell)$. Under the hypotheses of 4.9, the functor ω^p has a canonical factorization through the forgetful functor $\mathcal{R}(k;\mathbb{A}_f^p) \to \mathbf{Mod}(\mathbb{A}_f^p)$.*

PROOF. Let $V \in \mathcal{S}$. When k is finitely generated over the prime field, there is a canonical continuous action of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ on $H^i(\bar{V}, \mathbb{Q}_\ell(m))$, which therefore lies in $\mathcal{R}(k;\mathbb{Q}_\ell)$. For any correspondence $f \in \mathrm{Corr}^r(V,W)$, the map

$$f\colon H^i(\bar{V}, \mathbb{Q}_\ell(m)) \to H^{i+r}(\bar{W}, \mathbb{Q}_\ell(m+r))$$

respects the actions of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$. Therefore, the functor

$$\omega_\ell\colon \mathcal{M}(k;\mathbb{Q}) \to \mathbf{Vec}(\mathbb{Q}_\ell)$$

has a canonical factorization through the forgetful functor $\mathcal{R}(k;\mathbb{Q}_\ell) \to \mathbf{Vec}(\mathbb{Q}_\ell)$. The argument for ω^p is similar.

For an arbitrary field k, any varieties V, W and correspondence f will have models V_1, W_1, f_1 over some subfield k_1 of k finitely generated over the prime field. The image of $\omega_l(f_1)\colon \omega_l(V_1) \to \omega_l(W_l)$ under the functor $\mathcal{R}(k_1;\mathbb{Q}_l) \to \mathcal{R}(k;\mathbb{Q}_l)$ is independent of the choice of k_1, V_1, W_1, f_1. Thus, the statements hold also for k. $\square$

5. The Category of Motives

We fix a field k of characteristic exponent p, and a class $\mathcal{S}$ of smooth projective varieties over k satisfying the condition (4.1) and such that

(a) for all V in $\mathcal{S}$, the Künneth projectors are algebraic, and
(b) for all V in $\mathcal{S}$ and all l (including p), numerical equivalence coincides with l-adic homological equivalence for algebraic cycles with $\mathbb{Q}$-coefficients.

Let $\mathcal{M}^+(k;\mathbb{Q})$ and $\mathcal{M}(k;\mathbb{Q})$ be the categories $\mathcal{M}^+_{\mathrm{num}}(k;\mathbb{Q};\mathcal{S})$ and $\mathcal{M}_{\mathrm{num}}(k;\mathbb{Q};\mathcal{S})$ defined in 4.6. Recall that they are semisimple abelian $\mathbb{Q}$-categories (Jannsen 1992) and that $\mathcal{M}(k;\mathbb{Q})$ is Tannakian (Deligne 1990).

Because ω_l is a fibre functor on $\mathcal{M}(k;\mathbb{Q})$, the map

$$\mathrm{Hom}_{\mathcal{M}(k;\mathbb{Q})}(X,Y)_{\mathbb{Q}_l} \to \mathrm{Hom}_{\mathcal{R}(k;\mathbb{Q}_l)}(\omega_l(X), \omega_l(Y)) \tag{5}$$

defined by ω_l is injective (Deligne 1990, 2.13). It follows that it is also injective when $\mathcal{M}$ and $\mathcal{R}$ are replaced by $\mathcal{M}^+$ and $\mathcal{R}^+$. We say that the *l-adic Tate conjecture holds* for $\mathcal{M}(k;\mathbb{Q})$ or $\mathcal{M}^+(k;\mathbb{Q})$ if the corresponding map is an isomorphism for l.

While conditions (a) and (b) are conjectured to hold for all smooth projective varieties, they have been proved for very few varieties (see Kleiman 1994 for a list). In the final subsection of this section (Variants, p22), we list some alternative categories $\mathcal{M}^+(k;\mathbb{Q})$ and $\mathcal{M}(k;\mathbb{Q})$ to which our constructions apply, but which require no unproven conjectures.

Our construction of $\mathcal{M}(k;\mathbb{Z})$ is suggested by the following simple observation: for a finitely generated $\mathbb{Z}$-module M, $M \cong M_{\hat{\mathbb{Z}}} \times_{M_{\mathbb{A}_f}} M_{\mathbb{Q}}$; therefore, any category whose Hom-sets are finitely generated $\mathbb{Z}$-modules admits a fully faithful functor into the fibre product of a $\hat{\mathbb{Z}}$-category with a $\mathbb{Q}$-category over a $\mathbb{A}_f$-category.

The category of effective motives $\mathcal{M}^+(k;\mathbb{Z})$.

DEFINITION 5.1. The *category of effective motives* $\mathcal{M}^+(k;\mathbb{Z})$ over k is the full subcategory of the fibre product category

$$\mathcal{R}^+(k;\hat{\mathbb{Z}}) \times_{\mathcal{R}^+(k;\mathbb{A}_f)} \mathcal{M}^+(k;\mathbb{Q})$$

whose objects (X_f, X_0, x_f) are those for which the torsion subgroup of X_f is finite.

Thus, an effective motive X over k is a triple (X_f, X_0, x_f) consisting of

(a) an object $X_f = (X_l)_l$ of $\mathcal{R}^+(k;\hat{\mathbb{Z}})$ such that X_l is torsion-free for almost all l,
(b) an effective isomotive X_0, and
(c) an isomorphism $x_f\colon (X_f)_{\mathbb{Q}} \to \omega_f(X_0)$ in $\mathcal{R}^+(k;\mathbb{A}_f)$.

A morphism of effective motives $\alpha\colon X \to Y$ is a pair of morphisms

$$(\alpha_f\colon X_f \to Y_f,\ \alpha_0\colon X_0 \to Y_0)$$

such that

$$\begin{array}{ccc} X_f & \xrightarrow{\alpha_f} & Y_f \\ \downarrow{\scriptstyle x_f} & & \downarrow{\scriptstyle y_f} \\ \omega_f(X_0) & \xrightarrow{\omega_f(\alpha_0)} & \omega_f(Y_0) \end{array}$$

commutes. We say that a motive $X = (X_f, X_0, x_f)$ is *finite* if $X_0 = 0$.

Note that to give (X_f, x_f) amounts to giving an X_l in $\mathcal{R}(k;\mathbb{Z}_l)$ for each l together with homomorphisms $x_l\colon X_l \to \omega_l(X_0)$ that induce isomorphisms $X_l \to \lambda_l(X_0)$ for almost all l and isomorphisms $(X_l)_{\mathbb{Q}} \to \omega_l(X_0)$ for all l. We usually regard x_f as a map $X_f \to \omega_f(X_0)$ with finite kernel. We use the notations: $X_f = X^p \times X_p$, $X^p = \prod_{l\neq p} X_l$, $x_f = x^p \times x_p$, $\alpha_f = \alpha^p \times \alpha_p$.

PROPOSITION 5.2. *The category $\mathcal{M}^+(k;\mathbb{Z})$ is abelian. A sequence*

$$X \to Y \to Z$$

in $\mathcal{M}^+(k;\mathbb{Z})$ is exact if and only if the sequences

$$X_0 \to Y_0 \to Z_0 \text{ and } X_f \to Y_f \to Z_f$$

are exact in $\mathcal{M}^+(k;\mathbb{Q})$ and $\mathcal{R}^+(k;\hat{\mathbb{Z}})$ respectively.

PROOF. Both statements are true for the fibre product category, and so it remains to show that $\mathcal{M}^+(k;\mathbb{Z})$ is closed under the formation of kernels and cokernels in the larger category. Any subobject of an object in $\mathcal{M}^+(k;\mathbb{Z})$ is also in $\mathcal{M}^+(k;\mathbb{Z})$. Therefore, $\mathcal{M}^+(k;\mathbb{Z})$ is closed under the formation of kernels, and, in proving that the cokernel of α is in $\mathcal{M}^+(k;\mathbb{Z})$, we may assume that α is injective. Thus, given a short exact sequence

$$0 \to X \xrightarrow{\alpha} Y \to Z \to 0$$

in the fibre product category with X, Y in $\mathcal{M}^+(k;\mathbb{Z})$, we have to show that Z is in $\mathcal{M}^+(k;\mathbb{Z})$. As $\mathcal{M}^+(k;\mathbb{Q})$ is semisimple, the sequence of $\mathcal{M}^+(k;\mathbb{Q})$-components splits. But ω^p is additive, and so

$$0 \to \omega^p(X_0) \xrightarrow{\omega^p(\alpha_0)} \omega^p(Y_0) \to \omega^p(Z_0) \to 0$$

is split-exact, which implies that

$$0 \to \lambda_\ell(X_0) \to \lambda_\ell(Y_0) \to \lambda_\ell(Z_0) \to 0$$

is split-exact for almost all ℓ. In turn, this implies that

$$0 \to X_\ell \overset{\alpha_\ell}{\to} Y_\ell \to Z_\ell \to 0$$

is split-exact for almost all ℓ. Thus, the torsion subgroup of Z_ℓ is zero for almost all ℓ and finite for all ℓ, which shows that Z is in $\mathcal{M}^+(k;\mathbb{Z})$. □

COROLLARY 5.3. *Let X and X' be submotives of an effective motive Y, and let $X \cap X' = X \times_Y X'$. If $X_0 = Y_0$, then the cokernel Z of $X \cap X' \to X'$ is finite.*

PROOF. Because $X \mapsto X_0$ is exact, $(X \times_Y X' \to X')_0 \cong (X_0 \times_{Y_0} X'_0 \to X'_0)$, which is an isomorphism if $X_0 = Y_0$. □

PROPOSITION 5.4. *The category $\mathcal{M}^+(k;\mathbb{Z})$ is noetherian (i.e., its objects are noetherian).*

PROOF. Let

$$X(1) \subset X(2) \subset X(3) \subset \cdots$$

be a sequence of a submotives of an effective motive Y. Eventually, the sequence $X(1)_0 \subset X(2)_0 \subset \cdots$ becomes stationary, equal, say, to $X(\infty)_0 \subset Y_0$. Let $X(\infty)_f = Y_f \times_{\omega_f(Y_0)} \omega_f(X(\infty)_0)$. Then $X(\infty) = (X(\infty)_f, X(\infty)_0, \cdot)$ is an effective motive, and $X(1) \subset X(2) \subset \cdots$ is a sequence of submotives of $X(\infty)$ such that $X(r)_0 = X(\infty)_0$ for $r \geq r_1$, some r_1. Now $X(\infty)/X(r_1)$ is finite (5.3), and so the sequence $X(1) \subset X(2) \subset \cdots$ becomes stationary. □

LEMMA 5.5. *Given a morphism $\beta\colon X_0 \to Y_0$ of effective isomotives, there exists an integer m such that $m\beta$ is in the image of* $\mathrm{Hom}(X,Y) \to \mathrm{Hom}(X_0, Y_0)$.

PROOF. We have to find an integer m and morphisms α_l such that the diagrams

$$\begin{array}{ccc} X_l & \xrightarrow{\alpha_l} & Y_l \\ \downarrow{\scriptstyle x_l} & & \downarrow{\scriptstyle y_l} \\ \omega_l(X_0) & \xrightarrow{\omega_l(m\beta)} & \omega_l(Y_0) \end{array}$$

commute. Because $\omega_l(\beta)$ is the l-component of a homomorphism $\omega^p(X_0) \to \omega^p(Y_0)$, it maps $x_l(X_l)$ into $y_l(Y_l)$ for almost all l, and so there exists an integer m such that $m\omega_l(\beta)$ maps $x_l(X_l)$ into $y_l(Y_l)$ for all l. Choose an integer n that kills the torsion in Y_l for all l. Then the map

$$X_l \xrightarrow{x_l} x_l X_l \xrightarrow{\omega_l(mn\beta)} y_l Y_l = Y_l/\text{torsion}.$$

lifts to a map $X_l \to Y_l$. □

PROPOSITION 5.6. *For all X, Y in $\mathcal{M}^+(k;\mathbb{Z})$,* $\mathrm{Hom}(X,Y)$ *is a finitely generated $\mathbb{Z}$-module (modulo p-torsion when k is infinite and $p \neq 1$).*

PROOF. As $\mathrm{Hom}(X,Y) \subset \mathrm{End}(X \oplus Y)$, it suffices to show that all endomorphism rings are finitely generated $\mathbb{Z}$-modules (modulo ...). Because ω^p is faithful, $X \mapsto X_f$ is faithful. In particular, the map

$$\alpha \mapsto \alpha_f \colon \mathrm{End}(X) \to \mathrm{End}(X_f)$$

is injective, and so the torsion subgroup of $\mathrm{End}(X)$ is contained in that of $\mathrm{End}(X_f)$, which is finite except possibly for the p-torsion when k is infinite and $p \neq 1$. The kernel of the map

$$\alpha \mapsto \alpha_0 \colon \mathrm{End}(X) \to \mathrm{End}(X_0)$$

is obviously torsion, and so it remains to show that its image R is finitely generated over $\mathbb{Z}$. If $\beta \in \mathrm{End}(X_0)$ is in R, then $\omega_l(\beta)$ stabilizes the lattice $x_l(X_l)$ for all l. Thus, the characteristic polynomial of β has coefficients in $\mathbb{Z} \subset \mathbb{Q}$. Let $\alpha_1, \ldots, \alpha_n$ be a basis for $\mathrm{End}(X_0)$ as a $\mathbb{Q}$-vector space. According to Lemma 5.5, after possibly replacing each α_i with $m\alpha_i$ for some integer m, we may suppose that each $\alpha_i \in R$. Because $\mathcal{M}^+(k;\mathbb{Q})$ is semisimple, $\mathrm{End}(X_0)$ is a semisimple $\mathbb{Q}$-algebra, and so the trace pairing $\alpha, \beta \mapsto \mathrm{Tr}_{\mathrm{End}(X_0)/\mathbb{Q}}(\alpha\beta)$ is nondegenerate. Therefore, there is a $\mathbb{Q}$-basis $\beta_1, \ldots, \beta_n$ for $\mathrm{End}(X_0)$ dual to $\alpha_1, \ldots, \alpha_n$. Now the usual argument shows that

$$R \subset \mathbb{Z}\beta_1 + \cdots + \mathbb{Z}\beta_n.$$

Namely, let $\beta \in R$. Then β can be written uniquely as a linear combination $\beta = \sum b_j \beta_j$ of the β_j with coefficients $b_j \in \mathbb{Q}$, and we have to show that each $b_j \in \mathbb{Z}$. As α_i and β are in R, so also is $\beta \cdot \alpha_i$, and so $\mathrm{Tr}(\beta \cdot \alpha_i) \in \mathbb{Z}$. But

$$\mathrm{Tr}(\beta \cdot \alpha_i) = \mathrm{Tr}(\sum_j b_j \beta_j \alpha_i) = \sum_j b_j \,\mathrm{Tr}(\beta_j \alpha_i) = b_i. \qquad \square$$

REMARK 5.7. (a) If Y has no p-torsion (i.e., Y_p is torsion-free), then $\mathrm{Hom}(X,Y)$ has no p-torsion and hence is finitely generated over $\mathbb{Z}$.

(b) Let $p \neq 1$. The effective motive X with $X_0 = 0$ and $X_f = X_p = k$ (F acting as zero) has $\mathrm{End}(X) = k$. The same phenomenon occurs with subgroup schemes of abelian varieties: a supersingular elliptic curve has $\alpha_p = \mathrm{Spec}\, k[T]/(T^p)$ as a subgroup, and $\mathrm{End}(\alpha_p) = k$.

PROPOSITION 5.8. *The functor $X \mapsto X_0 \colon \mathcal{M}^+(\mathbb{F};\mathbb{Z}) \to \mathcal{M}^+(\mathbb{F};\mathbb{Q})$ defines an equivalence of categories*

$$\mathcal{M}^+(\mathbb{F};\mathbb{Z})_{\mathbb{Q}} \to \mathcal{M}^+(\mathbb{F};\mathbb{Q}).$$

PROOF. Because the kernel of

$$\alpha \mapsto \alpha_0 \colon \mathrm{Hom}(X,Y) \to \mathrm{Hom}(X_0,Y_0)$$

is torsion, the map

$$r \otimes \alpha \mapsto r\alpha_0 \colon \mathbb{Q} \otimes \mathrm{Hom}(X,Y) \to \mathrm{Hom}(X_0,Y_0)$$

is injective, and Lemma 5.5 shows that it is surjective. Thus, the functor is faithful and full. Given an X in $\mathcal{M}^+(k;\mathbb{Q})$, clearly there exists an X^p in $\mathcal{R}^+(k;\mathbb{Z}^p)$ such that $(X^p)_{\mathbb{Q}} \approx \omega^p(X)$. Using this, one sees that $X \mapsto X_0$ is essentially surjective. $\square$

REMARK 5.9. An object X of $\mathcal{M}^{+}(k;\mathbb{Z})$ is said to be *torsion* if $mX = 0$ for some m (equivalently, X is finite). The torsion objects form a thick subcategory of $\mathcal{M}^{+}(k;\mathbb{Z})$ and Proposition 5.8 shows that the functor

$$X \mapsto X_0 \colon \mathcal{M}^{+}(k;\mathbb{Z}) \to \mathcal{M}^{+}(k;\mathbb{Q})$$

realizes $\mathcal{M}^{+}(k;\mathbb{Q})$ as the quotient of $\mathcal{M}^{+}(k;\mathbb{Z})$ by its subcategory of torsion objects.

PROPOSITION 5.10. *The functor*

$$\mathcal{M}^{+}(k;\mathbb{Z})_{\mathbb{Z}_l} \to \mathcal{R}^{+}(k;\mathbb{Z}_l)$$

defined by $X \mapsto X_l$ is faithful; it is full if and only if the l-adic Tate conjecture holds for $\mathcal{M}^{+}(k;\mathbb{Q}_l)$.

PROOF. We first show that

$$z \otimes \alpha \mapsto z\alpha_l \colon \mathbb{Z}_l \otimes_{\mathbb{Z}} \operatorname{Hom}_{\mathcal{M}^{+}(k;\mathbb{Z})}(X,Y) \to \operatorname{Hom}_{\mathcal{R}^{+}(k;\mathbb{Z}_l)}(X_l,Y_l)$$

is injective. The torsion subgroup of $\mathbb{Z}_l \otimes \operatorname{Hom}_{\mathcal{M}^{+}(k;\mathbb{Z})}(X,Y)$ is equal to the l-primary component of the torsion subgroup of $\operatorname{Hom}_{\mathcal{M}^{+}(k;\mathbb{Z})}(X,Y)$, and this maps injectively to $\operatorname{Hom}_{\mathcal{R}^{+}(k;\mathbb{Z}_l)}(X_l,Y_l)$ (because the functor $X \mapsto X_f$ is faithful). Thus, the kernel of $z \otimes \alpha \mapsto z\alpha_l$ is torsion-free. Consider the diagram

$$\begin{array}{ccc}
\mathbb{Z}_l \otimes_{\mathbb{Z}} \operatorname{Hom}(X,Y) & \longrightarrow & \operatorname{Hom}_{\mathcal{R}^{+}(k;\mathbb{Z}_l)}(X_l,Y_l) \\
\downarrow & & \downarrow \\
\mathbb{Q}_l \otimes_{\mathbb{Q}} \operatorname{Hom}(X_0,Y_0) & \longrightarrow & \operatorname{Hom}_{\mathcal{R}^{+}(k;\mathbb{Q}_l)}(\omega_l(X_0),\omega_l(Y_0)).
\end{array}$$

The left vertical arrow is obtained from

$$\operatorname{Hom}(X,Y) \to \operatorname{Hom}(X_0,Y_0)$$

by tensoring with $\mathbb{Z}_l$, and hence has torsion kernel (5.8). The bottom arrow is injective (5). Hence, the kernel of the top horizontal arrow is torsion, but we have already shown it to be torsion-free; it is therefore zero.

We next show that the cokernel of $z \otimes \alpha \mapsto z\alpha_l$ is torsion-free. Suppose that $\beta \in \operatorname{Hom}_{\mathcal{R}^{+}(k;\mathbb{Z}_l)}(X_l,Y_l)$ is such that $l^m\beta = z\alpha_l$ for some $m \in \mathbb{N}$, $\alpha \in \operatorname{Hom}(X,Y)$, and $z \in \mathbb{Z}_l$. Because $z \otimes \alpha \mapsto z\alpha_l$ is (obviously) surjective on torsion, we may suppose that z is not divisible by l. Then α_l is divisible by l^m in $\operatorname{Hom}(X_l,Y_l)$. For $l' \neq l$, l^m is a unit in $\mathbb{Z}_{l'}$, and so $\alpha_{l'}$ is also divisible by l^m. Now α', with

$$\alpha'_0 = l^{-m}\alpha_0, \quad \alpha'_l = z^{-1}\beta, \quad \alpha'_{l'} = l^{-m}\alpha_{l'} \text{ for } l' \neq l$$

is a morphism $X \to Y$ such that $z\alpha'_l = \beta$.

Finally, we assume the l-adic Tate conjecture for $\mathcal{M}^{+}(k;\mathbb{Q})$, and we prove that the cokernel of $z \otimes \alpha \mapsto z\alpha_l$ is torsion. For this, it suffices to prove that

$$\mathbb{Q}_l \otimes_{\mathbb{Z}} \operatorname{Hom}(X,Y) \to \mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Hom}_{\mathcal{R}^{+}(k;\mathbb{Z}_l)}(X_l,Y_l)$$

is surjective. But this follows from the isomorphisms

$$\mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Hom}(X,Y) \overset{(5.8)}{\cong} \operatorname{Hom}(X_0,Y_0),$$

and

$$\mathbb{Q}_l \otimes_{\mathbb{Q}} \operatorname{Hom}(X_0,Y_0) \overset{\text{Tate}}{\to} \operatorname{Hom}_{\mathcal{R}^{+}(k;\mathbb{Q}_l)}(\omega_l(X_0),\omega_l(Y_0)) \cong \mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Hom}_{\mathcal{R}^{+}(k;\mathbb{Z}_l)}(X_l,Y_l).$$

$\square$

PROPOSITION 5.11. *The category*

$$\mathcal{M}^+(k;\mathbb{Z}) = \varinjlim_{k'} \mathcal{M}^+(k';\mathbb{Z})$$

where k' runs over the subfields of k finitely generated over the prime field.

PROOF. The statement is true for each of the categories $\mathcal{M}^+(k;\mathbb{Q})$, $\mathcal{R}^+(k;\hat{\mathbb{Z}})$, and $\mathcal{R}^+(k;\mathbb{A}_f)$, and it follows easily for their fibre product and its subcategory $\mathcal{M}^+(k;\mathbb{Z})$. □

The category of motives.

DEFINITION 5.12. The *category of motives* $\mathcal{M}(k;\mathbb{Z})$ is the full subcategory of

$$\mathcal{R}(k;\hat{\mathbb{Z}}) \times_{\mathcal{R}(k;\mathbb{A}_f)} \mathcal{M}(k;\mathbb{Q})$$

whose objects (X_f, X_0, x_f) are those for which the torsion subgroup of X_f is finite.

The results in the previous subsection hold *mutandis mutatis* for $\mathcal{M}(k;\mathbb{Z})$ — in particular, $\mathcal{M}(k;\mathbb{Z})$ is a noetherian abelian category whose quotient by its torsion subcategory is $\mathcal{M}(k;\mathbb{Q})$.

PROPOSITION 5.13. *The evident functor $\mathcal{M}^+(k;\mathbb{Z}) \to \mathcal{M}(k;\mathbb{Z})$ is faithful, and it is full on torsion-free objects (on all objects when k has characteristic zero). It realizes $\mathcal{M}(k;\mathbb{Z})$ as the category obtained from $\mathcal{M}^+(k;\mathbb{Z})$ by inverting $\mathbb{L}$.*

PROOF. Straightforward. □

REMARK 5.14. (a) When $k = \mathbb{Q}$ there is a canonical functor from $\mathcal{M}(k;\mathbb{Z})$ to the category of systems of realizations over $\mathbb{Q}$ with integer coefficients defined in Deligne 1989, 1.23.

(b) Let k be a subfield of $\mathbb{C}$, and let $\mathcal{M}(k;\mathbb{Q})$ be the category based on a class $\mathcal{S}$ satisfying (4.1) but using the "motivated" classes as correspondences (André 1996, 4.2). In this case, our definition of $\mathcal{M}(k;\mathbb{Z})$ is equivalent to that of André (ibid. 8.1).

Artin motives. Fix a separable algebraic closure $\bar{k}$ of k, and let $\Gamma = \mathrm{Gal}(\bar{k}/k)$. The category of Artin isomotives is the full subcategory $\mathcal{M}^{\mathrm{Artin}}(k;\mathbb{Q})$ of $\mathcal{M}^+(k;\mathbb{Q})$ of objects of the form $X = hV$ with V a variety of dimension 0. For such an X, define $\pi_0(X) = \pi_0(\bar{V})$ — it is a finite set with a continuous action of Γ. The functor $H\colon X \mapsto \mathbb{Q}^{\pi_0(X)} =_{\mathrm{df}} \mathrm{Map}(\pi_0(X),\mathbb{Q})$ defines an equivalence of $\mathcal{M}^{\mathrm{Artin}}(k;\mathbb{Q})$ with the category $\mathcal{R}(\Gamma;\mathbb{Q})$ of continuous representations of Γ on finite-dimensional $\mathbb{Q}$-vector spaces (cf. Deligne and Milne 1982, 6.17). Hence, $\mathcal{M}^{\mathrm{Artin}}(k;\mathbb{Q})$ is a Tannakian category.

We define the category of *Artin motives* to be the full subcategory $\mathcal{M}^{\mathrm{Artin}}(k;\mathbb{Z})$ of $\mathcal{M}^+(k;\mathbb{Z})$ of objects X such that X_0 is an Artin isomotive and (when $p \neq 1$) F_{X_p} is bijective.

Recall (Saavedra 1972, VI 3.1.2; Demazure 1972, p. 69) that there is a fully faithful functor

$$\gamma\colon \mathcal{R}(\Gamma;\mathbb{Z}_p) \to \mathbf{Crys}^+(k^{\mathrm{pf}}) = \mathcal{R}^+(k;\mathbb{Z}_p) \tag{6}$$

whose essential image consists of the crystals with bijective F. Let X be an Artin motive. When $p \neq 1$, we choose $X_p' \in \mathcal{R}(\Gamma;\mathbb{Z}_p)$ so that $\gamma(X_p') = X_p$, and we let $\omega_p'(X_0) = (X_p')_{\mathbb{Q}}$; otherwise we set $X_p' = 0 = \omega_p'(X_0)$. Define $H(X)$ so that the right hand square in the following diagram is cartesian:

$$\begin{array}{ccccccc}
0 & \longrightarrow & H(X)_{\text{tors}} & \longrightarrow & H(X) & \longrightarrow & H(X_0) \\
 & & \downarrow\cong & & \downarrow & & \downarrow \\
0 & \longrightarrow & X^p_{\text{tors}} \times X'_{p\text{tors}} & \longrightarrow & X^p \times X'_p & \longrightarrow & \omega^p(X_0) \times \omega'_p(X_0)
\end{array}$$

The diagram is exact and commutative, and $H(X)$ is a finitely generated $\mathbb{Z}$-module with a continuous action of Γ (discrete topology on $H(X)$).

PROPOSITION 5.15. *The functor $H\colon \mathcal{M}^{Artin}(k;\mathbb{Z}) \to \mathcal{R}(\Gamma;\mathbb{Z})$ is an equivalence of categories.*

PROOF. The functor $H\colon \mathcal{M}^{\text{Artin}}(k;\mathbb{Q}) \to \mathcal{R}(\Gamma;\mathbb{Q})$ defines an equivalence of categories

$$\mathcal{R}(\Gamma;\mathbb{Z}) \times_{\mathcal{R}(\Gamma;\mathbb{Q})} \mathcal{M}^{\text{Artin}}(k;\mathbb{Q}) \to \mathcal{R}(\Gamma;\mathbb{Z}) \times_{\mathcal{R}(\Gamma;\mathbb{Q})} \mathcal{R}(\Gamma;\mathbb{Q}) = \mathcal{R}(\Gamma;\mathbb{Z})$$

and γ defines a full faithful functor

$$\mathcal{R}(\Gamma;\mathbb{Z}) \times_{\mathcal{R}(\Gamma;\mathbb{Q})} \mathcal{M}^{\text{Artin}}(k;\mathbb{Q}) \to \mathcal{R}(k;\hat{\mathbb{Z}}) \times_{\mathcal{R}(k;\mathbb{A}_f)} \mathcal{M}^{\text{Artin}}(k;\mathbb{Q})$$

whose essential image is $\mathcal{M}^{\text{Artin}}(k;\mathbb{Z})$. $\square$

Rationally decomposed one-motives. Recall that a one-motive is a triple (G, X, u) with G a semi-abelian variety, X a finitely generated free $\mathbb{Z}$-module on which Γ acts continuously, and u a Γ-homomorphism $X \to G(k^{\text{sep}})$. A one-motive $X \xrightarrow{u} G$ is said to be *decomposed* if $u = 0$ and the extension

$$0 \to T \to G \to A \to 0 \quad (T \text{ a torus, } A \text{ an abelian variety})$$

splits, and it is *rationally decomposed* if it is isogenous to a decomposed one-motive. Over a finite field, every one-motive is rationally decomposed. The one-motives form an additive category $\mathcal{M}_1(k)$ with the decomposed and rationally decomposed one-motives as additive subcategories $\mathcal{M}_1^{\text{d}}(k) \subset \mathcal{M}_1^{\text{rd}}(k) \subset \mathcal{M}_1(k)$.

Choose a quasi-inverse $F\colon \mathcal{R}(\Gamma;\mathbb{Z}) \to \mathcal{M}^{\text{Artin}}(k;\mathbb{Z})$ to the functor H of (5.15).

PROPOSITION 5.16. *For $L = (X \xrightarrow{0} T \times A)$ in $\mathcal{M}_1^d(k)$, define hL to be the effective motive $F(X) \oplus h^1_{\mathbb{Z}}A \oplus F(X_*(T)(1))$. Then h extends to a functor $h\colon \mathcal{M}_1^{rd}(k) \to \mathcal{M}^+(k;\mathbb{Z})$. (Here $h^1_{\mathbb{Z}}A$ denotes h^1A endowed with the $\mathbb{Z}$-structure provided by the maps $H^1(A,\mathbb{Z}_l) \to H^1(A,\mathbb{Q}_l)$.)*

PROOF. Let L be a rationally decomposed one-motive. There exists a decomposed one-motive L' and an isogeny $\alpha\colon L \to L'$ such that (L', α) is universal, and α induces isomorphisms $\omega^p(L) \cong \omega^p(L')$ and $\omega_p(L) \cong \omega_p(L')$. We define hL to be the isomotive $(hL')_0$ equipped with the structures

$$\begin{array}{rcl}
\lambda^p L & \to & \omega^p(L) \cong \omega^p(L') \cong \omega^p((hL')_0) \\
\lambda_p L & \to & \omega_p(L) \cong \omega_p(L') \cong \omega_p((hL')_0).
\end{array} \qquad \square$$

PROPOSITION 5.17. *The functor $h\colon \mathcal{M}_1^{rd}(k) \to \mathcal{M}^+(k;\mathbb{Z})$ is fully faithful (except possibly when k has characteristic 2 and is not algebraic over $\mathbb{F}_2$).*

PROOF. It suffices to prove this in the case that k is finitely generated over the prime field, in which case it is a consequence of the following theorem of Tate (1966a), Zarhin (1975), Faltings (Faltings and Wüstholz 1984), and de Jong (1998): for abelian varieties A and B over k, the map $\mathrm{Hom}_k(A,B)_{\mathbb{Z}_l} \to \mathrm{Hom}_{\mathcal{R}(k;\mathbb{Z}_l)}(T_lA, T_lB)$ is bijective (except possibly when k has characteristic 2 ...). □

REMARK 5.18. Propositions 5.16 and 5.17 show that, for any two rationally decomposed one-motives L, M, there is an injective homomorphism

$$\mathrm{Ext}^1_{\mathcal{M}_1(k)}(L,M) \to \mathrm{Ext}^1_{\mathcal{M}^+(k;\mathbb{Z})}(hL, hM).$$

This map will not in general be surjective because an extension of two $W[F]$-modules for which there exist maps V with $FV = p = VF$ need not be similarly endowed. In fact, the formula Milne 1968, Theorem 3, for the order of the Ext^1 of abelian varieties disagrees with that in Theorem 10.1 below.

The category $\mathcal{M}(\mathbb{F}_q;\mathbb{Z})$. For a motive $X = h(V,e,m)$ over $\mathbb{F}_q$, the Frobenius endomorphism of $V/\mathbb{F}_q$ defines a canonical Frobenius element $\pi_X \in \mathrm{End}(X)$. For a motive X over $\mathbb{F}$, each model $X_1/\mathbb{F}_{p^n}$ of X defines a pair (π, n) with π equal to π_{X_1} regarded as an endomorphism of X. Any two pairs (π,n), (π',n') arising in this way are equivalent in the sense that $\pi^{n'm} = \pi'^{nm}$ for some integer $m > 0$. The *germ of a Frobenius endomorphism of* X is the equivalence class of pairs containing the pair arising from one (hence, every) model of X over a finite field.

PROPOSITION 5.19. *Assume the Tate conjecture holds for $\mathcal{M}(\mathbb{F};\mathbb{Q})$. Then*

$$X \mapsto X_{/\mathbb{F}} \colon \mathcal{M}(\mathbb{F}_{p^n};\mathbb{Z}) \to \mathcal{M}(\mathbb{F};\mathbb{Z})$$

defines an equivalence of $\mathcal{M}(\mathbb{F}_{p^n};\mathbb{Z})$ with the category of pairs (X,π) consisting of a motive X in $\mathcal{M}(\mathbb{F};\mathbb{Z})$ and an endomorphism π of X such that (π,n) represents π_X. A similar statement holds for effective motives.

PROOF. Easy consequence of the similar statement for isomotives (Milne 1994, 3.5). □

Remarks on the definition of $\mathcal{M}(k;\mathbb{Z})$.

5.20. As André notes (1996, 8.1), in view of the counterexamples of Atiyah and Hirzebruch (1962) to the original Hodge conjecture, it would be "peu judicieux" to define a category of integral motives by using algebraic cycles with integer coefficients in Grothendieck's construction, in other words, with

$$\mathrm{Hom}(\mathbb{1}, h_{\mathbb{Z}}V) = Z^{\dim V}_{\mathrm{num}}(V \times V).$$

There is also the difficulty of knowing what to replace the pseudo-abelian hull with.

With our definitions, a smooth projective variety V defines a complex of motives, rather than a motive (see an article in preparation), but we can make the ad hoc definitions: $h^r_{\mathbb{Z}}V$ is the isomotive h^rV with the $\mathbb{Z}$-structure provided the maps $H^r(V,\mathbb{Z}_l) \to H^r(V,\mathbb{Q}_l)$, and $h_{\mathbb{Z}}V = \oplus h^r_{\mathbb{Z}}V$. If V has torsion-free cohomology, then

$$\mathrm{Hom}(\mathbb{1}, hV) = Z^{\dim V}_{\mathrm{num}}(V \times V) \cap \mathrm{End}(H^*(V,\mathbb{Z}_l)).$$

There is a map from $Z^{\dim V}_{\mathrm{num}}(V \times V)$ to this group, but it will not always be onto (p11).

5.21. J-M. Fontaine has pointed out to us that there is an incompatibility between our definitions in characteristic 0 and in characteristic p: in the first case, we use the étale cohomology to define the integral structure at every prime, whereas in characteristic p we use the crystalline cohomology at the prime p; but the crystalline cohomology in characteristic p corresponds to the de Rham cohomology in characteristic zero, and the integral structures on the de Rham cohomology and the p-adic étale cohomology do not agree. We do not know how to resolve this problem (or even whether it is resolvable).

Variants. We list some alternative categories $\mathcal{M}(k;\mathbb{Q})$ to which our construction applies.

5.22. Let $\mathcal{S}$ consist of the smooth projective varieties over k for which the Künneth projectors are algebraic. This class satisfies the condition 4.1, includes all abelian varieties and surfaces, and includes all smooth projective varieties when k is finite. Let $\mathcal{M}_l(k;\mathbb{Q})$ and $\mathcal{M}(k;\mathbb{Q})$ be the categories of motives based on $\mathcal{S}$ using respectively the algebraic classes modulo l-adic homomological equivalence and numerical equivalence as the correspondences. There is a tensor functor

$$\mathcal{M}_l(k) \to \mathcal{M}(k)$$

which, according to André and Kahn 2001, has a tensor section s_l. Therefore, $\omega_l \circ s_l$ is a fibre functor on $\mathcal{M}(k;\mathbb{Q})$. Moreover, s_l can be chosen so that $(\omega_l \circ s_l)(hV) = H^*(V,\mathbb{Q}_l)$. If, as seems likely, s_l can be chosen[9] so that the action of $Z^{\dim V}_{\mathrm{num}}(V \times V) \subset \mathrm{End}(hV)$ on $(\omega_l \circ s_l)(hV)$ preserves the image of $H^*(V,\mathbb{Z}_l)$ in $H^*(V,\mathbb{Q}_l)$ for all V with torsion-free cohomology, then the $\omega_l \circ s_l$ for $l \neq p$ are the l-components of an $\mathbb{A}^p_f$-valued fibre functor on $\mathcal{M}(k;\mathbb{Q})$, and we can construct $\mathcal{M}^+(k;\mathbb{Z})$ and $\mathcal{M}(k;\mathbb{Z})$ as above.

5.23. Let $\mathcal{S}$ consist of all smooth projective varieties over k. For any V in $\mathcal{S}$, the Künneth projectors are almost algebraic (in the sense of Tate 1994, p76). Therefore, the discussion in 5.22 applies to the category $\mathcal{M}(k;\mathbb{Q})$ defined using numerical equivalence classes of almost algebraic classes rather than algebraic classes.

5.24. Recall that a Lefschetz class modulo $\sim$ on a variety V is an element of the $\mathbb{Q}$-subalgebra of $\oplus Z^r_\sim(V)$ generated by divisor classes. Let $\mathcal{S}$ be the smallest class of varieties over k satisfying (4.1) and including the abelian varieties. Then it is possible to define a category $\mathcal{M}(k;\mathbb{Q})$ of isomotives based on $\mathcal{S}$ using the Lefschetz classes modulo numerical equivalence as correspondences; moreover, $\mathcal{M}(k;\mathbb{Q})$ is a semisimple Tannakian category over $\mathbb{Q}$ and there exist canonical functors ω_l (Milne 1999, §1).

5.25. For any field k of characteristic zero, define $\mathcal{M}(k;\mathbb{Q})$ to be the category of motives based on some class $\mathcal{S}$ satisfying (4.1) and using the absolute Hodge classes as correspondences (Deligne and Milne 1982, §6).

5.26. Let $\mathbf{CM}(\mathbb{Q}^{\mathrm{al}})$ be the category of motives based on the abelian varieties of CM-type over $\mathbb{Q}^{\mathrm{al}}$ using the absolute Hodge classes as the correspondences, and let $\mathcal{M}(\mathbb{F};\mathbb{Q})$ be the quotient of $\mathbf{CM}(\mathbb{Q}^{\mathrm{al}})$ corresponding to a fibre functor on $\mathbf{CM}(\mathbb{Q}^{\mathrm{al}})^P$ which is correct at every prime (see Milne 2002; here P is the Weil-number protorus). As above, we can define $\mathcal{M}^+(\mathbb{F};\mathbb{Z})$ and $\mathcal{M}(\mathbb{F};\mathbb{Z})$. Almost by definition, the Tate conjecture holds for $\mathcal{M}(\mathbb{F};\mathbb{Q})$.

6. Tannakian Properties

Throughout this section, R is a commutative noetherian ring.

[9] According to André (email 25.03.02) it does not appear possible to prove this by the methods of their paper.

Tannakian categories over rings. In Saavedra 1972, Tannakian categories are defined only over a base field. We adapt his definition (ibid. III 1.1.1) of an ind-Tannakian category to obtain the notion of a Tannakian category over an arbitrary noetherian ring.

Recall that a *tensor category* $\mathcal{C}$ is a category together with a bifunctor $\otimes\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ and compatible associativity and commutativity constraints for which there exists an identity object $\mathbb{1}$. When $\mathrm{End}(\mathbb{1})=R$, we call $(\mathcal{C},\otimes)$ a tensor category over R or a tensor R-category. Let R' be an R-algebra. An *R'-valued fibre functor* is an R-linear exact tensor functor $\mathcal{C}\to\mathbf{Mod}(R')$.

Some linear algebra. Let $\mathcal{C}$ be an R-linear category, and let R' be an R-algebra. When $\mathcal{C}$ has arbitrary direct limits, Saavedra (1972, I 1.5) constructs an R'-linear category $\mathcal{C}_{(R')}$ by "extension of the base ring". We adapt his construction to the case that R' is a finite R-algebra and $\mathcal{C}$ has finite direct limits (for example, $\mathcal{C}$ is abelian).

Thus, let $\mathcal{C}$ be an abelian R-linear category. Let R' be a finite R-algebra, and define $\mathcal{C}_{(R')}$ to be the category whose objects are pairs (X,i_X) comprising an object X of $\mathcal{C}$ and a homomorphism $i_X\colon R'\to\mathrm{End}_R(X)$ of R-algebras (Saavedra 1972, II 1.5).

6.1. For any finitely generated R-module M, there is a functor

$$X\mapsto M\otimes_R X\colon \mathcal{C}\to\mathcal{C}$$

and an isomorphism (natural in X and Y)

$$\mathrm{Hom}_{\mathcal{C}}(M\otimes_R X,Y)\to\mathrm{Hom}_R(M,\mathrm{Hom}_{\mathcal{C}}(X,Y))$$

(Saavedra 1972, II 1.5.1.1).

6.2. The category $\mathcal{C}_{(R')}$ is an abelian R'-linear category. A sequence

$$(X',i_{X'})\to(X,i_X)\to(X'',i_{X''})$$

in $\mathcal{C}_{(R')}$ is exact if and only if

$$X'\to X\to X''$$

is exact in $\mathcal{C}$. The functor $j^{R'/R}\colon\mathcal{C}\to\mathcal{C}_{(R')}$ sending X to $R'\otimes_R X$ (with its canonical R'-structure) is left adjoint to the functor $j_{R/R}\colon\mathcal{C}_{(R')}\to\mathcal{C}$ sending (X,i_X) to X (Saavedra 1972, II 1.5.2). For any $X,Y\in\mathrm{ob}(\mathcal{C})$,

$$\mathrm{Hom}_{\mathcal{C}_{(R')}}(j^{R'/R}X,j^{R'/R}Y)\cong\mathrm{Hom}_{\mathcal{C}}(X,Y)\otimes_R R',\tag{7}$$

and so $j^{R'/R}$ realizes $\mathcal{C}_{R'}$ as a full subcategory of $\mathcal{C}_{(R')}$.

6.3. When R' is a flat R-algebra, the functor $j^{R'/R}\colon\mathcal{C}\to\mathcal{C}_{(R')}$ is exact.

6.4. Let $\mathcal{C}$ be an abelian tensor category over R such that $\otimes$ is right exact. The category $\mathcal{C}_{(R')}$ has a canonical structure of a tensor category over R' for which $j^{R'/R}$ and $j_{R'/R}$ are tensor functors; moreover $\otimes$ on $\mathcal{C}_{(R')}$ is right exact (Saavedra 1972, II 1.5.4). Let $\mathcal{D}$ be an abelian tensor category over R' for which $\otimes$ is right exact. The map $u\mapsto u\circ j^{R'/R}$ defines an equivalence of the category of right exact R'-linear tensor functors $\mathcal{C}_{(R')}\to\mathcal{D}$ with the category of right exact R-linear tensor functors $\mathcal{C}\to\mathcal{D}$. When R' is a flat R-algebra, $u\mapsto u\circ j^{R'/R}$ carries exact functors to exact functors. (Ibid. II 1.5.3, 1.5.4.) (But it is not known that $u\circ j^{R'/R}$ exact implies u exact, even when R' is faithfully flat over R, except in the case that k and k' are fields, when the proof requires the main theorem of Deligne 1990).

Definitions. We say that an abelian tensor R-category $\mathcal{C}$ is *finitely generated* if there exists a finite set of objects of $\mathcal{C}$ contained in no proper strictly full abelian tensor R-subcategory of $\mathcal{C}$.

DEFINITION 6.5. (a) A *neutral Tannakian category over* R is a tensor category over R that is tensor equivalent to $\mathbf{Rep}(G;R)$ for some flat affine group scheme G over R.

(b) An *algebraic Tannakian category over* R is a finitely generated abelian tensor category $\mathcal{C}$ over R such that $\mathcal{C}_{(R')}$ becomes a neutral Tannakian category for some faithfully flat finite R-algebra R'.

(c) A *Tannakian category over* R is an abelian tensor category $\mathcal{C}$ over R that is a filtered union of algebraic Tannakian categories.

Note that, in a Tannakian category, the Homs are finitely generated R-modules, because

$$\mathrm{Hom}_{\mathcal{C}}(X,Y)\otimes_R R' \overset{(7)}{\cong} \mathrm{Hom}_{\mathcal{C}_{(R')}}(j^{R'/R}X, j^{R'/R}Y) \subset \mathrm{Hom}_{R'}(\omega(X),\omega(Y)).$$

A criterion to be a Tannakian category over a Dedekind domain. Let $\mathcal{C}$ be an abelian tensor category over a noetherian ring R. When internal Homs exist in $\mathcal{C}$ (Saavedra 1972, I 3), we let $X^{\vee}=\underline{\mathrm{Hom}}(X,1\!\!1)$. There is a canonical morphism $X\to X^{\vee\vee}$, and X is said to be *reflexive* when this morphism is an isomorphism.

For example, when G is an affine group scheme over R, an object X of $\mathbf{Rep}(G;R)$ is reflexive if and only if its underlying R-module is projective. Serre (1968, 2.2) proves that, when R is a Dedekind domain, every object in $\mathbf{Rep}(G;R)$ is a quotient of a representation of G on a projective R-module. Thus every object in $\mathbf{Rep}(G;R)$ is a quotient of a reflexive object. This property can be used to characterize Tannakian categories over Dedekind domains.

PROPOSITION 6.6. *Let $\mathcal{C}$ be an abelian tensor category over a Dedekind domain R such that every object of $\mathcal{C}$ is a quotient of a reflexive object. If there exists an R'-valued fibre functor on $\mathcal{C}_{(R')}$ for some Dedekind domain R' finite and flat over R, then $\mathcal{C}$ is Tannakian.*

PROOF. Let $\omega\colon \mathcal{C}_{(R')}\to \mathbf{Mod}(R')$ be such a fibre functor. To prove that $\mathcal{C}$ is Tannakian, we shall show that the functor $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ of R'-algebras is representable by a flat affine group scheme G over R', and that ω defines an equivalence of R'-tensor categories $\mathcal{C}_{(R')}\to\mathbf{Rep}(G;R')$.

When $R'=R$, this is proved in Saavedra 1972, II 4.1. In detail: ibid. II 4.1.1 shows that $\underline{\mathrm{End}}^{\otimes}(\omega)$ is represented by a flat affine monoid scheme G over R and that ω defines an equivalence of R-linear tensor categories $\mathcal{C}\to\mathbf{Rep}(G;R)$; but $\underline{\mathrm{End}}^{\otimes}(\omega)\cong\underline{\mathrm{End}}^{\otimes}(\omega_0)$ where ω_0 is the restriction of ω to the full subcategory of reflexive objects; this last category is rigid, and so every endomorphism of ω_0 is an automorphism (ibid. I 5.2.3), which implies that G is a group scheme.

According to (6.2) and (6.4), $\mathcal{C}_{(R')}$ is an abelian tensor category over R'. Thus, the general case will follow from the neutral case once we show that $\mathcal{C}_{(R')}$ inherits the property that every object is a quotient of a reflexive object. Let (Y,i_Y) be an object of $\mathcal{C}_{(R')}$, and let $q\colon X\to Y$ express Y as a quotient of a reflexive object. The composite

$$j^{R'/R}X \xrightarrow{j^{R'/R}(q)} j^{R'/R}Y \to (Y, i_Y)$$

is again surjective, and $j^{R'/R}X$ is reflexive because X is. □

EXAMPLE 6.7. For $\ell \neq p$, the category $\mathcal{R}(k;\mathbb{Z}_\ell)$ is Tannakian over $\mathbb{Z}_\ell$ (apply 1.4).

EXAMPLE 6.8. (a) Let k be a finite field, and let ω be the forgetful functor $\mathcal{R}(k;\mathbb{Z}_p) \to \mathbf{Mod}(W(k))$. For (X, i_X) in $\mathcal{R}(k;\mathbb{Z}_p)_{(W(k))}$, there are two actions of $W(k)$ on $\omega(X)$, namely, that coming from the action i_X of $W(k)$ on X and that coming from the action of $W(k)$ on ω, and we define $\omega'(X, i_X) = W(k) \otimes_{W(k)\otimes_{\mathbb{Z}_p} W(k)} \omega(X)$. Then ω' is exact. Indeed, $W(k) \otimes_{\mathbb{Z}_p} W(k)$ is isomorphic to a product of copies of $W(k)$ indexed by the elements of $\mathrm{Gal}(k/\mathbb{F}_p)$, and the map $W(k) \otimes_{\mathbb{Z}_p} W(k) \to W(k)$ corresponds to the projection to the "id"-component. Therefore ω' is a $W(k)$-valued fibre functor on $\mathcal{R}(k;\mathbb{Z}_p)_{(W(k))}$, and $\mathcal{R}(k;\mathbb{Z}_p)$ is Tannakian (apply 1.8(b)).

(b) When k has nonzero characteristic p and is infinite, then $\mathcal{R}(k;\mathbb{Z}_p)$ is not Tannakian (at least according to our definition, which may be too strict), because there exist objects X for which $\mathrm{End}(X)$ is not a finitely generated $\mathbb{Z}_p$-module.

Existence of fibre functors. We refer to Deligne 1989, §6, for the notion of the fundamental group of a Tannakian category $\mathcal{T}$: it is an affine group scheme $\pi(\mathcal{T})$ in $\mathrm{Ind}\,\mathcal{T}$ acting on the objects of $\mathcal{T}$; each fibre functor ω carries $\pi(\mathcal{T})$ to $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ and the action of $\pi(\mathcal{T})$ on an object X to the action of $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ to $\omega(X)$. When $\pi(\mathcal{T})$ is commutative, it lies in $\mathrm{Ind}\,\mathcal{T}^0$ where $\mathcal{T}^0$ is the full subcategory of trivial objects. Since $\mathrm{Hom}(\mathbb{1}, -)\colon \mathcal{T}^0 \to \mathbf{Vec}(k)$ is an equivalence of categories, in this case $\pi(\mathcal{T})$ can be identified with an affine group scheme in the usual sense.

PROPOSITION 6.9. *Let $\mathcal{M}$ be a Tannakian category over $\mathbb{Q}$ whose fundamental group T is an algebraic group of multiplicative type, and let ω^p be a $\mathbb{Q}$-linear tensor functor $\mathcal{M} \to \mathbf{Mod}(\mathbb{A}_f^p)$ such that $\omega_\ell =_{df} \omega^p \otimes_{\mathbb{A}_f^p} \mathbb{Q}_\ell$ is a $\mathbb{Q}_\ell$-valued fibre functor for all $\ell \neq p$. Then there exists a finite field extension L of $\mathbb{Q}$, a fibre functor $\omega\colon \mathcal{M} \to \mathbf{Vec}(L)$, and an isomorphism $\omega \otimes_L \mathbb{A}_{f,L}^p \to \omega^p \otimes_{\mathbb{A}_f^p} \mathbb{A}_{f,L}^p$ of tensor functors $\mathcal{M} \to \mathbf{Mod}(\mathbb{A}_{f,L}^p)$.*

PROOF. The hypotheses imply that $\omega^p(X)$ is a free $\mathbb{A}_f^p$-module for all X (see the proof of 4.9).

Assume first that T is split over $\mathbb{Q}$ and that there exists a fibre functor $\omega\colon \mathcal{M} \to \mathbf{Vec}(\mathbb{Q})$. We show in this case that there is an isomorphism $\omega \otimes_{\mathbb{Q}} \mathbb{A}_f^p \to \omega^p$. We use ω to identify $\mathcal{M}$ with $\mathbf{Rep}(T;\mathbb{Q})$. Let Ξ be a basis for the group $X^*(T)$ of characters of T, and let X be the representation $X = \bigoplus_{\lambda\in\Xi} X_\lambda$ where X_λ is a one-dimensional representation of T with character λ. Choose a graded lattice Λ in X, i.e., a $\mathbb{Z}$-lattice such that $\Lambda = \bigoplus_{\lambda\in\Xi} \Lambda \cap X_\lambda$. Likewise, choose a graded lattice Λ_f in $\omega^p(X)$. Then $\Lambda_\ell =_{\mathrm{df}} \Lambda_f \otimes_{\mathbb{A}_f^p} \mathbb{Z}_\ell$ is a graded lattice in $\omega_\ell(X)$. As T is a split torus, $H^1(\mathbb{Q}_\ell, T) = 0$, and so the theory of Tannakian categories shows that there exists an isomorphism of fibre functors $\alpha_\ell\colon \omega \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \to \omega_\ell$ which is uniquely determined up to an element of $T(\mathbb{Q}_\ell)$. We may choose α_ℓ to map $\Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ onto Λ_ℓ. Now the family (α_ℓ) defines an isomorphism $\omega \otimes_{\mathbb{Q}} \mathbb{A}_f^p \to \omega^p$.

In the general case, there will exist a finite field extension L of $\mathbb{Q}$ such that T splits over L and such that there exists a fibre functor $\omega\colon \mathcal{M} \to \mathbf{Vec}(L)$. Then ω defines a fibre functor $\mathcal{M}_{(L)} \to \mathbf{Vec}(L)$ (Deligne and Milne 1982, 3.11), and the previous argument applied to $\mathcal{M}_{(L)}$ proves the general case of the proposition. □

The category $\mathcal{M}(k;\mathbb{Z})$. Fix a field k, and let $\mathcal{M}(k;\mathbb{Q})$ and $\mathcal{M}(k;\mathbb{Z})$ be as in §5.

LEMMA 6.10. *The category $\mathcal{M}(k;\mathbb{Z})$ has internal Homs.*

PROOF. Since each of the categories $\mathcal{R}(k;\hat{\mathbb{Z}})$, $\mathcal{R}(k;\mathbb{A}_f^p)$, $\mathcal{M}(k;\mathbb{Q})$ has internal Homs, it is straightforward to check this. □

LEMMA 6.11. *Every motive in $\mathcal{M}(k;\mathbb{Z})$ is a quotient of a reflexive motive.*

PROOF. The proof is similar to that of Proposition 1.8: a motive is reflexive if and only if it is torsion-free; every motive is a direct sum of a torsion motive and a torsion-free motive; every torsion motive is a quotient of $\mathbb{1}(r)^s$ for some r and s. □

THEOREM 6.12. *The category $\mathcal{M}(\mathbb{F}_q;\mathbb{Z})$ is Tannakian.*

PROOF. We apply Proposition 6.6. It remains to prove the existence of fibre functors. We replace $\mathcal{S}$ with a subset generated by a finite set of varieties, and prove that $\mathcal{M}(k;\mathbb{Z};\mathcal{S})_{(R')}$ admits an R'-valued fibre functor for some Dedekind domain R' finite and flat over R. For this, we can apply Proposition 6.9 because the fundamental group of $\mathcal{M} =_{\mathrm{df}} \mathcal{M}(k;\mathbb{Q};\mathcal{S})$ is of multiplicative type: it is reductive because $\mathcal{M}$ is semisimple, and it is commutative because its extension to $\mathbb{Q}_\ell$ is contained in the algebraic group fixing the Tate classes, which is commutative.

According to Proposition 6.9, there exists a finite field extension L of $\mathbb{Q}$ splitting T, a fibre functor $\omega_0\colon \mathcal{M} \to \mathbf{Vec}(L)$, and an isomorphism $\xi^p\colon L\otimes_{\mathbb{Q}}\omega^p \to \mathbb{A}_f^p\otimes_{\mathbb{Q}}\omega_0$. After possibly extending L, we may suppose that there is also an isomorphism $\xi_p\colon L\otimes_{\mathbb{Q}}\omega_p \to \mathbb{Q}_p\otimes_{\mathbb{Q}}\omega_0$ where ω_p is $X \mapsto X_p$ (regarded as a $B(\mathbb{F}_q)$-module). Let R be the integral closure of $\mathbb{Z}$ in L.

For X_0 in $\mathcal{M}_{(L)}$, there are two actions of L on $\omega_0(X_0)$, namely, that coming from the action of L on ω_0 and that coming from the action of L on X_0. Define

$$\omega_1(X_0) = L\otimes_{L\otimes_{\mathbb{Q}}L}\omega_0(X_0).$$

Then ω_1 is an L-valued fibre functor on $\mathcal{M}_{(L)}$ (Deligne and Milne 1982, 3.11).

For (X, i_X) in $\mathcal{M}_{(R)}$, set

$$\omega_2(X) = (X^p, \omega'(X_p))$$

where ω' is the functor defined (6.8a). Then ω_2 is an $\hat{R}$-valued fibre functor on $\mathcal{M}_{(R)}$, and ξ defines an isomorphism $L\otimes_{\mathbb{Q}}\omega_2 \to \mathbb{A}_f\otimes_{\mathbb{Q}}\omega_1$.

Thus, we have defined an exact tensor functor

$$X \mapsto (\omega_2(X), \omega_1(X_0), \xi)\colon \mathcal{M}_{(R)} \to \mathbf{Modf}(\hat{R}) \times_{\mathbf{Modf}(\mathbb{A}_f\otimes R)} \mathbf{Modf}(L).$$

Its image lies in the full subcategory of triples whose $\mathbf{Modf}(\hat{R})$-component has finite torsion, which can be identified with the category $\mathbf{Modf}(R)$. It is therefore an R-valued fibre functor on $\mathcal{M}(\mathbb{F}_q;\mathbb{Z})_{(R)}$. □

REMARK 6.13. If $\mathcal{M}(k;\mathbb{Q})$ is as in 5.24 or 5.26 and k is finite, or as in 5.25, then $\mathcal{M}(k;\mathbb{Z})$ is Tannakian. In the first two cases, this can be proved as in Theorem 6.12, and in the third case, there is a Betti fibre functor on $\mathcal{M}(k;\mathbb{Z})$.

Stacks of categories of motives. We refer the reader to Saavedra 1972, I 4.5, for the notion of *tensor fibred category* $p\colon \mathcal{M} \to \mathcal{E}$. Briefly, it is a fibred category $p\colon \mathcal{M} \to \mathcal{E}$ endowed with an $\mathcal{E}$-bifunctor $\otimes\colon \mathcal{M} \times_{\mathcal{E}} \mathcal{M} \to \mathcal{M}$ and compatible associativity and commutativity constraints for which there exists a cartesian section to p giving identity objects in each fibre; thus, for each E in $\mathcal{E}$, the fibre $\mathcal{M}_E$ is a tensor category. When $\mathcal{E}$ is endowed with a topology (in the sense of Grothendieck) and $p\colon \mathcal{M} \to \mathcal{E}$ is a stack, we call it a *stack of tensor categories.* When the fibres are abelian categories and the inverse image functors are exact, we call $p\colon \mathcal{M} \to \mathcal{E}$ a *stack of abelian tensor categories.* Finally, when $\operatorname{End}(\mathbb{1}) = R$ for each identity object $\mathbb{1}$ in each fibre, we call $p\colon \mathcal{M} \to \mathcal{E}$ a *stack of abelian tensor R-categories.* In particular, then each fibre is an abelian tensor R-category.

For a scheme $S = \coprod \operatorname{Spec} k_i$, finite and étale over $\mathbb{F}_q$, let $\mathcal{M}(S;\mathbb{Z}) = \prod \mathcal{M}(k_i;\mathbb{Z})$.

THEOREM 6.14. *Let $\mathbf{Et}_{\mathbb{F}_q}$ be the category of all schemes finite and étale over $\mathbb{F}_q$, and endow $\mathbf{Et}_{\mathbb{F}_q}$ with the étale topology. For $k = \mathbb{F}_q$ or $\mathbb{F}$, let $\mathcal{M}(k;\mathbb{Q})$ be as at the start of §5 and assume the Tate conjecture holds for $\mathcal{M}(k;\mathbb{Q})$. The categories $\mathcal{M}(-;\mathbb{Z})$ form (in a natural way) the fibres of a stack of abelian tensor $\mathbb{Z}$-categories over $\mathbf{Et}_{\mathbb{F}_q}$.*

PROOF. Immediate consequence of Proposition 5.19. □

7. Some Spectral Sequences

Abstract spectral sequences. We prove an abstract version of Tate's spectral sequence (Milne 1986b, I 0.3). We fix a field k, and let $\mathbf{Et}_k$ be the category of all schemes finite and étale over k. Also, we fix a ring R, and let $\mathcal{M} \to \mathbf{Et}_k$ be a stack of noetherian abelian tensor R-categories with internal Homs over $\mathbf{Et}_k$.

7.1. When X and Y are objects of an abelian category $\mathcal{A}$, $\operatorname{Ext}^r(X,Y)$ denotes the Yoneda extension group (Mitchell 1965, VII). It agrees with the group defined in terms of the derived category (Verdier 1996, III 3.2.12). When $\mathcal{A}$ is noetherian, the Yoneda extension group also agrees with that defined using injective resolutions in the ind-category $\operatorname{Ind}(\mathcal{A})$ (Oort 1964, p. 235; see also Huber 1993).

7.2. An object of a tensor category is *trivial* if it is a quotient of a finite direct sum of copies of $\mathbb{1}$, and an object Y of an abelian tensor category is *flat* if the functor $X \mapsto X \otimes Y$ is exact. When X and Y are objects of an abelian tensor category over R, we write

$$\operatorname{Tor}_i^R(X,Y) = 0 \text{ for all } i > 0$$

to mean Y has a resolution

$$\cdots \to Y^1 \to Y^0 \to Y \to 0$$

by flat objects which remains exact when it has been tensored by X.

7.3. Let k' be a finite Galois extension of k with Galois group Γ, and write $X \mapsto X'$ for an inverse image functor $\mathcal{M}(k) \to \mathcal{M}(k')$. Because $\mathcal{M} \to \mathbf{Et}_k$ is a stack, the functor

$$X \mapsto N(X) \stackrel{\text{df}}{=} \operatorname{Hom}(\mathbb{1}, X')$$

defines an equivalence from the full subcategory of $\mathcal{M}(k)$ of objects becoming trivial in $\mathcal{M}(k')$ to the category $\mathbf{Modf}(R[\Gamma])$ with quasi-inverse $N \mapsto N \otimes \mathbb{1}$.

7.4. Let k' be a finite étale k-algebra. The theory of "Weil restriction of scalars" (Weil 1982, 1.3) applies in the present situation and gives an exact functor $\mathcal{M}(k') \to \mathcal{M}(k)$ that is both left and right adjoint to the inverse image functor $\mathcal{M}(k) \to \mathcal{M}(k')$.

PROPOSITION 7.5. *Let k' be a finite Galois extension of k with Galois group Γ. Write T' for the inverse image in $\mathcal{M}(k')$ of an object T of $\mathcal{M}(k)$. For all X, Y, Z in $\mathcal{M}(k)$ such that X' is trivial and $\mathrm{Tor}_i^R(X,Y) = 0$ for all $i > 0$, there is a spectral sequence*

$$\mathrm{Ext}^r_{R[\Gamma]}(N(X), \mathrm{Ext}^s_{\mathcal{M}(k')}(Y', Z')) \Longrightarrow \mathrm{Ext}^{r+s}_{\mathcal{M}(k)}(X \otimes Y, Z).$$

Consider the functors (for fixed X, Y)

$$\begin{array}{ll} \alpha\colon \mathcal{M}(k) \to \mathbf{Mod}(R[\Gamma]), & \alpha(Z) = \mathrm{Hom}_{\mathcal{M}(k')}(Y', Z') \\ \beta\colon \mathbf{Mod}(R[\Gamma]) \to \mathbf{Ab} & \beta(M) = \mathrm{Hom}_{R[\Gamma]}(N(X), M) \\ \gamma\colon \mathcal{M}(k) \to \mathbf{Ab} & \gamma(Z) = \mathrm{Hom}_{\mathcal{M}(k)}(X \otimes Y, Z). \end{array}$$

To prove the proposition, we shall show:

(a) $\beta \circ \alpha \cong \gamma$;
(b) the functor α takes injective objects in $\mathrm{Ind}(\mathcal{M}(k))$ to β-acyclic $R[\Gamma]$-modules;
(c) the functor $Z \mapsto Z'\colon \mathcal{M}(k) \to \mathcal{M}(k')$ takes injective objects in $\mathrm{Ind}(\mathcal{M}(k))$ to injective objects in $\mathrm{Ind}(\mathcal{M}(k'))$.

Statements (a) and (b) imply that there is a spectral sequence

$$(R^r\beta \circ R^s\alpha)(Z) \Longrightarrow (R^{r+s}\gamma)(Z);$$

hence,

$$\mathrm{Ext}^r_{R[\Gamma]}(N(X), (R^s\alpha)(Z)) \Longrightarrow \mathrm{Ext}^{r+s}_{\mathcal{M}(k)}(X \otimes Y, Z).$$

Statement (c), together with the exactness of $Z \mapsto Z'$, shows that $(R^s\alpha)(Z) = \mathrm{Ext}^s_{\mathcal{M}(k')}(Y', Z')$.

PROOF OF (c). The functor $Y \mapsto Y'$ has an exact left adjoint (7.4). □

PROOF OF (a). We have to prove

$$\mathrm{Hom}_{R[\Gamma]}(N(X), \mathrm{Hom}_{\mathcal{M}(k')}(Y', Z')) \cong \mathrm{Hom}_{\mathcal{M}(k)}(X \otimes Y, Z)$$

functorially in Z. From the definition of $\underline{\mathrm{Hom}}$ (Saavedra 1972, I 3.1.1)

$$\mathrm{Hom}_{\mathcal{M}(k')}(X' \otimes Y', Z') \cong \mathrm{Hom}_{\mathcal{M}(k')}(X', \underline{\mathrm{Hom}}_{\mathcal{M}(k')}(Y', Z')).$$

But $X' \cong N(X) \otimes \mathbb{1}$, and so

$$\begin{aligned} \mathrm{Hom}_{\mathcal{M}(k')}(X' \otimes Y', Z') &\overset{6.1}{\cong} \mathrm{Hom}_R(N(X), \mathrm{Hom}(\mathbb{1}, \underline{\mathrm{Hom}}_{\mathcal{M}(k')}(Y', Z'))) \\ &\cong \mathrm{Hom}_R(N(X), \mathrm{Hom}_{\mathcal{M}(k')}(Y', Z')). \end{aligned}$$

On taking Γ-invariants we obtain (a). □

PROOF OF (b). We write Hom_k and $\mathrm{Hom}_{k'}$ for $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{M}(k))}$ and $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{M}(k'))}$ respectively. We have to show that if I is injective in $\mathrm{Ind}(\mathcal{M}(k))$, then

$$\mathrm{Ext}^r_{R[\Gamma]}(N(X), \mathrm{Hom}_{k'}(Y', I')) = 0 \text{ for } r > 0.$$

We first show that, if Y is flat and I is injective, then $\mathrm{Hom}_{k'}(Y', I')$ is an injective $R[\Gamma]$-module. For this, we must show that the functor

$$\mathrm{Hom}_{R[\Gamma]}(-, \mathrm{Hom}_{k'}(Y', I'))\colon \mathbf{Mod}(R[\Gamma]) \to \mathbf{Ab}$$

is exact, at least on finitely generated $R[\Gamma]$-modules, but (a) shows that the composite of this functor with the equivalence $X \mapsto N(X)$ is the composite of the exact functors $-\otimes Y$ and $\mathrm{Hom}_k(-, I)$.

We now prove (b). Let

$$Y^\bullet \to Y$$

be a flat resolution of Y such that $X \otimes (Y^\bullet \to Y)$ is exact, and let I be an injective object of $\mathrm{Ind}(\mathcal{M}(k))$. Then, by (c), I' is injective, and so

$$\mathrm{Hom}_{k'}(Y', I') \to \mathrm{Hom}_{k'}(Y^{\bullet\prime}, I')$$

is an exact complex; in fact, it is an injective resolution of $\mathrm{Hom}_{k'}(Y', I')$. We shall use it to compute the groups $\mathrm{Ext}^r_{R[\Gamma]}(X, \mathrm{Hom}_{k'}(Y', I'))$. Statement (a) shows that the complex

$$\mathrm{Hom}_{R[\Gamma]}(N(X), \mathrm{Hom}_{k'}(Y', I')) \to \mathrm{Hom}_{R[\Gamma]}(N(X), \mathrm{Hom}_{k'}(Y^{\bullet\prime}, I'))$$

is isomorphic to the complex

$$\mathrm{Hom}_k(X \otimes Y, I) \to \mathrm{Hom}_k(X \otimes Y^\bullet, I)$$

which is exact because $X \otimes (Y \to Y^\bullet)$ is exact and I is injective. □

COROLLARY 7.6. *Let $\bar{k}$ be a Galois extension of k (possibly infinite), and let $\mathcal{M}(\bar{k}) = \varinjlim_{k'\subset\bar{k}} \mathcal{M}(k')$ where k' runs over the finite Galois extensions of k. Denote the inverse image functor $\mathcal{M}(k) \to \mathcal{M}(\bar{k})$ by $X \mapsto \bar{X}$, and let $\Gamma = \mathrm{Gal}(\bar{k}/k)$. Let Y be an object of $\mathcal{M}(k)$ that admits a resolution by flat objects. For all Z in $\mathcal{M}(k)$, there is a spectral sequence*

$$H^r(\Gamma, \mathrm{Ext}^s_{\mathcal{M}(\bar{k})}(\bar{Y}, \bar{Z})) \Longrightarrow \mathrm{Ext}^{r+s}_{\mathcal{M}(k)}(Y, Z).$$

PROOF. The condition on Y is that $\mathrm{Tor}_i^R(\mathbb{1}, Y) = 0$ for all $i > 0$. When $\bar{k}$ is finite over k, the two definitions of $\mathcal{M}(\bar{k})$ coincide, and we obtain the spectral sequence by taking $M = \mathbb{1}$ in Proposition 7.5. When $\bar{k}$ is of infinite degree over k, we pass to the limit over subfields $k' \subset \bar{k}$ that are finite and Galois over k. □

Spectral sequences at ℓ.

PROPOSITION 7.7. *Let $\Gamma = \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$. For M, N in $\mathcal{R}(\mathbb{F}_q; \mathbb{Z}_\ell)$, there is a spectral sequence*

$$H^r(\Gamma, \mathrm{Ext}^s_{\mathcal{R}(\mathbb{F};\mathbb{Z}_\ell)}(M, N)) \Longrightarrow \mathrm{Ext}^{r+s}_{\mathcal{R}(\mathbb{F}_q;\mathbb{Z}_\ell)}(M, N). \tag{8}$$

PROOF. Proposition 1.4 shows M has a resolution by flat objects, $i > 0$, and so this is a special case of Corollary 7.6. □

Denote Exts in $\mathbf{Mod}(\mathbb{Z}_\ell)$ by $\mathrm{Ext}_{\mathbb{Z}_\ell}$.

PROPOSITION 7.8. *Let $\Gamma = \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$. For M, N in $\mathcal{R}(\mathbb{F}_q; \mathbb{Z}_\ell)$, there is a spectral sequence*

$$H^r_{cts}(\Gamma, \mathrm{Ext}^s_{\mathbb{Z}_\ell}(M, N)) \Longrightarrow \mathrm{Ext}^{r+s}_{\mathcal{R}(\mathbb{F}_q;\mathbb{Z}_\ell)}(M, N). \tag{9}$$

Here H^r_{cts} is computed using continuous cochains relative to the ℓ-adic topology on $\mathrm{Ext}^s_{\mathbb{Z}_\ell}(M, N)$.

Proof. When N is finite, (9) will become (8) once we have shown that the natural map

$$\operatorname{Ext}^j_{\mathcal{R}(\mathbb{F};\mathbb{Z}_\ell)}(M,N)) \to \operatorname{Ext}^j_{\mathbb{Z}_\ell}(M,N)$$

is an isomorphism. When M is finite, this is obvious (with no condition on N), because it is true for $j=0$ and the forgetful functor $\operatorname{Ind}\mathcal{R}(\mathbb{F};\mathbb{Z}_\ell) \to \mathbf{Mod}(\mathbb{Z}_\ell)$ is exact and preserves injectives. The $\operatorname{Ext}(M,-)$ sequences of

$$\begin{aligned} 0 &\to M \xrightarrow{\ell^n} M \to M/\ell^n M \to 0\,, \quad \ell^n N = 0, \\ 0 &\to M_{\text{tors}} \to M \to M/M_{\text{tors}} \to 0 \end{aligned} \tag{10}$$

allow us to deduce it, first for all torsion-free M, and then for all M.

There are spectral sequences

$$\begin{aligned} E_2^{ij} &= \operatorname{Ext}^i_{\mathcal{R}(\mathbb{F}_q;\mathbb{Z}_\ell)}(M, \varprojlim{}^j N^{(\ell^n)}) \\ E_2^{i,j} &= \varprojlim{}^i \operatorname{Ext}^j_{\mathcal{R}(\mathbb{F}_q;\mathbb{Z}_\ell)}(M, N^{(\ell^n)}), \end{aligned}$$

with isomorphic final terms (cf. Jensen 1972, 4.3). The higher inverse limits vanish because the groups $\operatorname{Ext}^j_{\mathcal{R}(\mathbb{F}_q;\mathbb{Z}_\ell)}(M,N^{(\ell^n)})$ and $N^{(\ell^n)}$ are finite. Moreover, $\varprojlim N^{(\ell^n)} \cong N$, and so this shows that

$$\operatorname{Ext}^i_{\mathcal{R}(\mathbb{F}_q;\mathbb{Z}_\ell)}(M,N) \cong \varprojlim_n \operatorname{Ext}^i_{\mathcal{R}(\mathbb{F}_q;\mathbb{Z}_\ell)}(M,N^{(\ell^n)}). \tag{11}$$

Similarly,

$$\operatorname{Ext}^j_{\mathbb{Z}_\ell}(M,N) \cong \varprojlim_n \operatorname{Ext}^j_{\mathbb{Z}_\ell}(M,N^{(\ell^n)}),$$

and so (Tate 1976, 2.2)

$$H^i_{\text{cts}}(\Gamma, \operatorname{Ext}^j_{\mathbb{Z}_\ell}(M,N)) \cong \varprojlim_n H^i_{\text{cts}}(\Gamma, \operatorname{Ext}^j_{\mathbb{Z}_\ell}(M,N^{(\ell^n)})).$$

Therefore, (9) can be obtained by passing to the limit in the spectral sequences

$$H^r_{\text{cts}}(\Gamma, \operatorname{Ext}^s_{\mathbb{Z}_\ell}(M,N^{(\ell^n)})) \Longrightarrow \operatorname{Ext}^{r+s}_{\mathcal{R}(\mathbb{F}_q;\mathbb{Z}_\ell)}(M,N^{(\ell^n)}).$$

□

Spectral sequences at p.

Lemma 7.9. *For torsion-free objects M,N in $\mathcal{R}^+(\mathbb{F};\mathbb{Z}_p)$, $\operatorname{Ext}^2_{\mathcal{R}^+(\mathbb{F};\mathbb{Z}_p)}(M,N)$ is torsion-free.*

Proof. From the exact sequences

$$0 \to \operatorname{Ext}^1_{\mathcal{R}^+}(M,N)^{(p^n)} \to \operatorname{Ext}^1_{\mathcal{R}^+}(M,N^{(p^n)}) \to \operatorname{Ext}^2_{\mathcal{R}^+}(M,N)_{p^n} \to 0$$

we obtain a surjection

$$\varinjlim_n \operatorname{Ext}^1_{\mathcal{R}^+}(M,N^{(p^n)}) \to \operatorname{Ext}^2_{\mathcal{R}^+}(M,N)_{\text{tors}}.$$

Let $\mathcal{R}^+_{p^n}$ denote the full subcategory of $\mathcal{R}^+(\mathbb{F};\mathbb{Z}_p)$ of objects killed by p^n. The same argument as in the proof pp34–34 shows that the map

$$s\colon\ \operatorname{Ext}^1_{\mathcal{R}^+}(M,N^{(p^n)}) \to \operatorname{Ext}^1_{\mathcal{R}^+_{p^n}}(M^{(p^n)},N^{(p^n)})$$

that replaces each term E in a short exact sequence with $E^{(p^n)}$ is an isomorphism. Therefore, it remains to show that

$$\varinjlim_n \operatorname{Ext}^1_{\mathcal{R}^+_{p^n}}(M^{(p^n)}, N^{(p^n)}) = 0. \tag{12}$$

We claim (i) that the map

$$\operatorname{Ext}^1_{\mathbf{Crys}^+(\mathbb{F})}(M \otimes W(\mathbb{F}), N \otimes W(\mathbb{F})) \to \operatorname{Ext}^1_{\mathcal{R}^+_{p^n}}(M^{(p^n)}, N^{(p^n)})$$

that replaces each term E in a short exact sequence by $E^{(p^n)}$ is surjective, and (ii) that $\operatorname{Ext}^1_{\mathbf{Crys}^+(\mathbb{F})}(M, N)$ is a torsion group. Together, (i) and (ii) imply (12) , because (i) gives a surjection

$$\operatorname{Ext}^1_{\mathbf{Crys}^+(\mathbb{F})}(M \otimes W(\mathbb{F}), M \otimes W(\mathbb{F})) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \varinjlim_n \operatorname{Ext}^1_{\mathcal{R}^+_{p^n}}(M^{(p^n)}, N^{(p^n)})$$

and (ii) implies that the first group is zero.

To prove (i), note that an element of either group splits when regarded as an extension of W-modules, and is therefore determined by the σ-linear map describing the action of the Frobenius. Since every σ-linear map $M^{(p^n)} \to N^{(p^n)}$ lifts to a σ-linear map $M \to N$, (i) is clear. According to a theorem of Manin (Demazure 1972, p85), the category $\mathbf{Isoc}^+(k^{\mathrm{al}})$ is semisimple, which implies (ii). □

LEMMA 7.10. *When $k = \mathbb{F}$ or $\mathbb{F}_q$, every object in $\mathcal{R}^+(k; \mathbb{Z}_p)$ is a quotient of a torsion-free object.*

PROOF. Suppose that the object M is an extension of objects M' and M'' each of which is a quotient of a torsion-free objects, say, $N' \twoheadrightarrow M'$, $N'' \twoheadrightarrow M''$. Pull back the original extension by $N'' \to M''$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & E & \longrightarrow & N'' & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \end{array}$$

If the class of E in $\operatorname{Ext}^1(N'', M')$ is the image of an element of $\operatorname{Ext}^1(N'', N')$, then M also is a quotient of an torsion-free object. The obstruction to this is an element of $\operatorname{Ext}^2(N'', K)$ where K is the kernel of $N' \to M'$. Certainly, when $k = \mathbb{F}_q$, this group will be zero if there exists a resolution

$$0 \to A \xrightarrow{\cdot\lambda} A \to N'' \to 0$$

(because in this case Exts in $\mathcal{R}^+_p$ agree with those in $\mathbf{Mod}(A)$ — see 7.1).

It suffices to prove the lemma in the case $k = \mathbb{F}_q$, and, for a given M, an argument using restriction of scalars allows us to replace $\mathbb{F}_q$ by a finite extension. After such an extension, a torsion M will have a composition series whose quotients are isomorphic to one of (k, σ) or $(k, 0)$. Each of these is a quotient

$$\begin{array}{ccccccccc} 0 & \to & (W, \sigma) & \xrightarrow{p} & (W, \sigma) & \to & (k, \sigma) & \to & 0 \\ 0 & \to & (W, p\sigma) & \xrightarrow{p} & (W, p\sigma) & \to & (k, 0) & \to & 0. \end{array}$$

Since $(W, \sigma) \cong A/A(F - \sigma)$ and $(W, p\sigma) \cong A/A(F - p\sigma)$ the above remarks prove the lemma for M. Let M be an arbitrary object, and let $M_1 = M/M_{\mathrm{tors}}$. The obstruction to extending the statement from M_{tors} to M is a torsion element of $\mathrm{Ext}^2(M_1, K)$, which (see 7.9) becomes zero after a finite extension of $\mathbb{F}_q$. □

PROPOSITION 7.11. *Let $\Gamma = \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$. For M, N in $\mathcal{R}^+(\mathbb{F}_q;\mathbb{Z}_p)$, there is a spectral sequence*

$$H^r(\Gamma, \mathrm{Ext}^s_{\mathcal{R}^+(\mathbb{F};\mathbb{Z}_p)}(\bar{M}, \bar{N})) \Longrightarrow \mathrm{Ext}^{r+s}_{\mathcal{R}^+(\mathbb{F}_q;\mathbb{Z}_p)}(M, N). \tag{13}$$

PROOF. Lemma 7.10 shows that M has a flat resolution. We can not apply Corollary 7.6 directly, because $\mathcal{R}^+(\mathbb{F}_{q'};\mathbb{Z}_p)$ does not have internal Homs. However, this is only used in the proof of (a) in 7.5, namely, in showing that

$$\mathrm{Hom}_{\mathbb{Z}_p[\Gamma]}(P, \mathrm{Hom}_{\mathcal{R}^+(\mathbb{F}_{q'};\mathbb{Z}_p)}(M', N')) \cong \mathrm{Hom}_{\mathcal{R}^+(\mathbb{F}_q;\mathbb{Z}_p)}(P \otimes M, N)$$

where $\Gamma = \mathrm{Gal}(\mathbb{F}_{q'}/\mathbb{F}_q)$ and P is an object of $\mathcal{R}^+(\mathbb{F}_q;\mathbb{Z}_p)$ that becomes isomorphic to a direct sum of copies of (W, σ) in $\mathcal{R}^+(\mathbb{F}_{q'};\mathbb{Z}_p)$. This can be shown directly. □

PROPOSITION 7.12. *Let $\Gamma = \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$. For M, N in $\mathcal{R}^+(\mathbb{F}_q;\mathbb{Z}_p)$, there is a spectral sequence*

$$H^r_{cts}(\Gamma, \mathrm{Ext}^s_{\mathbf{Crys}^+(\mathbb{F})}(M_{W(\mathbb{F})}, N_{W(\mathbb{F})})) \Longrightarrow \mathrm{Ext}^{r+s}_{\mathcal{R}^+(\mathbb{F}_q;\mathbb{Z}_p)}(M, N). \tag{14}$$

Here H^r_{cts} is computed using continuous cochains relative to the p-adic topology on $\mathrm{Ext}^s_{W(\mathbb{F})}(M, N)$.

PROOF. When N (or M) is finite, (14) becomes the (13) — see the proof of Proposition 7.8. The remainder of the proof of Proposition 7.8 applies to this case, except that $N^{(p^n)}$ and $\mathrm{Ext}^j_{\mathbf{Crys}^+(\mathbb{F})}(M, N^{(p^n)})$ need no longer be finite. However, $N^{(p^n)}$ is of finite length over W, and so we can apply Jensen 1972, 7.2, to see that $\varprojlim{}^i N^{(p^n)} = 0$ for $i > 0$. Moreover, there exists an N such that $p^N \cdot \mathrm{Ext}^j_{\mathbf{Crys}^+(\mathbb{F})}(M, N^{(p^n)})$ is finite for all n, and so the inverse system $\mathrm{Ext}^j_{\mathbf{Crys}^+(\mathbb{F})}(M, N^{(p^n)})$ satisfies the Mittag-Leffler condition. □

8. Homological Algebra in the Category of Motives

In this section, we assume that the l-adic Tate conjectures hold for $\mathcal{M}^+(k;\mathbb{Q})$ (see p14).

Homological algebra in $\mathcal{M}^+(k;\mathbb{Z})$.

THEOREM 8.1. *Let X and Y be effective motives in $\mathcal{M}^+(k;\mathbb{Z})$.*

(a) *The map $(\alpha, \beta) \mapsto \alpha - \beta$,*

$$\mathrm{Hom}_{\mathcal{M}^+(k;\mathbb{Q})}(X_0, Y_0) \times \mathrm{Hom}_{\mathcal{R}^+(k;\hat{\mathbb{Z}})}(X_f, Y_f) \to \mathrm{Hom}_{\mathcal{R}^+(k;\mathbb{A}_f)}(\omega_f(X_0), \omega_f(Y_0)),$$

is surjective with kernel $\mathrm{Hom}_{\mathcal{M}^+(k;\mathbb{Z})}(X, Y)$.

(b) *For $i > 0$, $\mathrm{Ext}^i_{\mathcal{M}^+(k;\mathbb{Z})}(X, Y)$ is torsion and the sequence*

$$0 \to \mathrm{Ext}^i_{\mathcal{M}^+(k;\mathbb{Z})}(X, Y)_{\mathbb{Z}_l} \to \mathrm{Ext}^i_{\mathcal{R}(k;\mathbb{Z}_l)}(X_l, Y_l) \to T_l\,\mathrm{Ext}^{i+1}_{\mathcal{M}^+(k;\mathbb{Z})}(X, Y) \to 0 \tag{15}$$

is exact (all l); hence,

$$\mathrm{Ext}^i_{\mathcal{M}^+(k;\mathbb{Z})}(X, Y) \cong \mathrm{Ext}^i_{\mathcal{R}^+(k;\hat{\mathbb{Z}})}(X_f, Y_f)_{tors}.$$

(c) *If k is separably closed, then* $\mathrm{Ext}^i_{\mathcal{M}^+(k;\mathbb{Z})}(X,Y)=0$ *for all $i>2$, and for all $i>1$ when X and Y are torsion-free.*

PROOF. For any $\mathbb{Z}$-module M of finite rank, the sequence

$$0\to M\to M_{\hat{\mathbb{Z}}}\times M_{\mathbb{Q}}\to M_{\mathbb{A}_f}\to 0$$

is exact. We obtain (a) by taking $M=\mathrm{Hom}(X,Y)$ in this sequence and applying (5.6, 5.8, 5.10).

The remainder of the proof will require several steps.

$\mathrm{Ext}^i_{\mathcal{M}^+(k;\mathbb{Z})}(X,Y)$ IS TORSION FOR $i>0$: It suffices to prove that $\mathrm{Ext}^1_{\mathcal{M}^+(k;\mathbb{Z})}(X,Y)$ is torsion. Moreover, we may assume that X and Y are torsion-free. Consider an exact sequence

$$0\to Y\to E\xrightarrow{\pi}X\to 0$$

in $\mathcal{M}^+(k;\mathbb{Z})$. Because $\mathcal{M}^+(k;\mathbb{Q})$ is semisimple, there exists a morphism $\alpha_0\colon X_0\to E_0$ such that $\pi_0\circ\alpha_0=\mathrm{id}_{X_0}$. For some $m\in\mathbb{Z}$, $m\alpha_0\in\mathrm{Hom}_{\mathcal{M}^+(k;\mathbb{Z})}(X,E)$ (see 5.5). Now the equation $\pi\circ(m\alpha_0)=m\,\mathrm{id}_X$ implies that the class of the sequence is killed by m.

Write $f^i(X,Y)$ for the natural map

$$\mathrm{Ext}^i_{\mathcal{M}^+(k;\mathbb{Z})}(X,Y)_{\mathbb{Z}_l}\to\mathrm{Ext}^i_{\mathcal{R}^+(k;\mathbb{Z}_l)}(X_l,Y_l).$$

Recall (5.10) that $f^0(X,Y)$ is an isomorphism.

THE MAPS $f^i(X,Y)$ ARE ISOMORPHISMS WHEN X AND Y ARE FINITE. The functor $X\mapsto X_f$ defines an equivalence of the category of torsion effective motives with the category of torsion objects in $\mathcal{R}^+(k;\hat{\mathbb{Z}})$, and so it suffices to prove that the Exts in the torsion subcategories coincide with the Exts in the full categories. We do this for $\mathcal{M}^+(k;\mathbb{Z})$ since $\mathcal{R}^+(k;\mathbb{Z}_l)$ is similar. Because $\mathcal{M}^+(k;\mathbb{Z})$ is noetherian, $\mathrm{Ind}(\mathcal{M}^+(k;\mathbb{Z}))$ has enough injectives, and the Exts computed using injective resolutions coincide with the Yoneda Exts (7.1). Because $n\colon\mathbb{1}\to\mathbb{1}$ is injective, every injective object in $\mathrm{Ind}(\mathcal{M}^+(k;\mathbb{Z}))$ is divisible, and so an injective resolution $Y\to I^\bullet$ in $\mathrm{Ind}(\mathcal{M}^+(k;\mathbb{Z}))$ gives an injective resolution $Y\to I^\bullet_{\mathrm{tors}}$ in the subcategory of torsion objects. As $\mathrm{Hom}(X,I^\bullet_{\mathrm{tors}})\cong\mathrm{Hom}(X,I^\bullet)$, this proves that the Exts in the two categories agree.

THE MAPS $f^i(X,Y)$ ARE ISOMORPHISMS WHEN Y IS FINITE. It suffices to prove this when X is torsion-free and Y is l-torsion, say $l^nY=0$. The Ext sequences of

$$0\to X\xrightarrow{l^n}X\to X/l^nX\to 0$$

give a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^{i-1}_{\mathcal{M}^+}(X,Y) \arrow[r] \arrow[d, "f^{i-1}(X,Y)"] & \mathrm{Ext}^i_{\mathcal{M}^+}(X/l^nX,Y) \arrow[r] \arrow[d, "\cong"] & \mathrm{Ext}^i_{\mathcal{M}^+}(X,Y) \arrow[r] \arrow[d, "f^i(X,Y)"] & 0 \\
0 \arrow[r] & \mathrm{Ext}^{i-1}_{\mathcal{R}^+}(X_l,Y_l) \arrow[r] & \mathrm{Ext}^i_{\mathcal{R}^+}(X_l/l^nX_l,Y_l) \arrow[r] & \mathrm{Ext}^i_{\mathcal{R}^+}(X_l,Y_l) \arrow[r] & 0
\end{tikzcd}$$

from which it follows that $f^{i-1}(X,Y)$ is injective and $f^i(X,Y)$ is surjective. Since this holds for all i, $f^i(X,Y)$ is an isomorphism.

THE MAPS $f^i(X,Y)$ ARE ISOMORPHISMS WHEN X IS FINITE. Similar to the preceding case.

THE SEQUENCE (15) IS EXACT. Because $f^i(X,Y)$ is an isomorphism when Y is finite, $f^i(X,Y^\bullet)$ (map of hyper-Exts) is an isomorphism when $Y^\bullet$ is a bounded complex with finite cohomology, for example, when $Y^\bullet = Y^{\{l^n\}} =_{\mathrm{df}} Y \xrightarrow{l^n} Y$:

$$\mathrm{Ext}^i_{\mathcal{M}^+}(X, Y^{\{l^n\}}) \cong \mathrm{Ext}^i_{\mathcal{R}^+}(X_l, Y_l^{\{l^n\}}).$$

There are exact sequences

$$0 \to \mathrm{Ext}^i_{\mathcal{M}^+}(X,Y)^{(l^n)} \to \mathrm{Ext}^i_{\mathcal{M}^+}(X, Y^{\{l^n\}}) \to \mathrm{Ext}^i_{\mathcal{M}^+}(X,Y)_{l^n} \to 0, \tag{16}$$

and

$$\varprojlim \mathrm{Ext}^i_{\mathcal{M}^+}(X, Y^{\{l^n\}}) \cong \varprojlim \mathrm{Ext}^i_{\mathcal{R}^+}(X_l, Y_l^{\{l^n\}}) \cong \varprojlim \mathrm{Ext}^i_{\mathcal{R}^+}(X_l, Y_l^{(l^n)}) \cong \mathrm{Ext}^i_{\mathcal{R}^+}(X_l, Y_l)$$

(for the third isomorphism, see the proofs of 7.8 and 7.12). Thus, on passing to the inverse limit in (16), we obtain (15).

This completes the proof of (b). We now assume that $k = k^{\mathrm{sep}}$.

$\mathrm{Ext}^2_{\mathcal{M}^+}(X,Y) = 0$ IF X AND Y ARE TORSION-FREE. For an integer m, let $\mathcal{M}^+_m$ (resp. $\mathcal{R}^+_m$) denote the subcategory of $\mathcal{M}^+(k;\mathbb{Z})$ (resp. $\mathcal{R}^+(k;\hat{\mathbb{Z}})$) of objects killed by m. On passing to the direct limit in the sequence

$$0 \to \mathrm{Ext}^1_{\mathcal{M}^+}(X,Y)^{(m)} \to \mathrm{Ext}^1_{\mathcal{M}^+}(X, Y^{(m)}) \to \mathrm{Ext}^2_{\mathcal{M}^+}(X,Y)_m \to 0$$

and using that $\mathrm{Ext}^1_{\mathcal{M}^+}(X,Y)$ and $\mathrm{Ext}^2_{\mathcal{M}^+}(X,Y)$ are torsion, we find that

$$\varinjlim_m \mathrm{Ext}^1_{\mathcal{M}^+}(X, Y^{(m)}) \cong \mathrm{Ext}^2_{\mathcal{M}^+}(X,Y).$$

Consider the diagram

$$\begin{tikzcd}
\mathrm{Ext}^1_{\mathcal{M}^+}(X^{(m)}, Y^{(m)}) \arrow[rr, "\pi"] & & \mathrm{Ext}^1_{\mathcal{M}^+}(X, Y^{(m)}) \arrow[dl, "s"] \\
& \mathrm{Ext}^1_{\mathcal{M}^+_m}(X^{(m)}, Y^{(m)}) \arrow[ul, "i"] &
\end{tikzcd}$$

in which π is the surjection induced by $X \to X^{(m)}$, i is the obvious injection, and s applied to a short exact sequence replaces each motive E in the sequence with $E^{(m)}$. One checks easily that $s \circ (\pi \circ i) = \mathrm{id}$ and $(\pi \circ i) \circ s = \mathrm{id}$, and so s is an isomorphism. Thus

$$\mathrm{Ext}^2_{\mathcal{M}^+}(X,Y) \cong \varinjlim_m \mathrm{Ext}^1_{\mathcal{M}^+_m}(X^{(m)}, Y^{(m)}) \cong \varinjlim_m \mathrm{Ext}^1_{\mathcal{R}^+_m}(X_f^{(m)}, Y_f^{(m)}).$$

For $l \neq p$, $\mathrm{Ext}^1_{\mathcal{R}^+_{l^n}}(X_f^{(l^n)}, Y_f^{(l^n)})$ is obviously zero, and so it remains to show that

$$\varinjlim_{p^n} \mathrm{Ext}^1_{\mathcal{R}^+_{p^n}}(X_p^{(p^n)}, Y_p^{(p^n)}) = 0. \tag{17}$$

But this is shown in the proof of (7.9) (for $k = \mathbb{F}$, but the argument works generally).

$\mathrm{Ext}^i_{\mathcal{M}^+}(X,Y) = 0$ FOR ALL $i > 2$. This follows from (b) and the fact that $\mathrm{Ext}^i_{\mathcal{R}^+(k;\mathbb{Z}_l)}(M,N) = 0$ for $i > 2$ (for $l \neq p$, this is obvious, and for $l = p$ it can be proved by the arguments used to prove the p-case of Theorem 9.1 below). $\square$

REMARK 8.2. The group $\mathrm{Ext}^1_{\mathcal{R}^+(\mathbb{F};\mathbb{Z}_\ell)}(M,N)$ need not be torsion. Consider, for example, the extension

$$0 \to \mathbb{Z}_\ell \to \mathbb{Z}_\ell \oplus \mathbb{Z}_\ell \to \mathbb{Z}_\ell \to 0$$

on which the Frobenius element in $\Gamma = \mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)$ acts on the middle term as the matrix $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ with a a nonzero element of $\mathbb{Z}_\ell$. Multiplying the class of the extension by m corresponds to multiplying a by m, and so this represents a nontorsion element of $\mathrm{Ext}^1_{\mathcal{R}^+(\mathbb{F}_p;\mathbb{Z}_\ell)}(M,N)$, which remains nontorsion in $\mathrm{Ext}^1_{\mathcal{R}^+(\mathbb{F};\mathbb{Z}_\ell)}(M,N)$. Thus, in general,

$$\mathrm{Ext}^1_{\mathcal{M}^+(\mathbb{F};\mathbb{Z})}(X,Y)(\ell) \not\approx \mathrm{Ext}^1_{\mathcal{R}(\mathbb{F};\mathbb{Z}_\ell)}(X_\ell, Y_\ell).$$

Similarly, the group $\mathrm{Ext}^1_{\mathcal{R}^+(\mathbb{F};\mathbb{Z}_p)}(M,N)$ need not be torsion (because the category $\mathcal{R}^+(\mathbb{F};\mathbb{Z}_p)$, in contrast to $\mathbf{Isoc}^+(\mathbb{F})$, is not semisimple).

Homological algebra in $\mathcal{M}^+(\mathbb{F}_q;\mathbb{Z})$.

LEMMA 8.3. *Every effective motive over $\mathbb{F}_q$ or $\mathbb{F}$ is a quotient of a torsion-free effective motive.*

PROOF. Straightforward adaptation of the proof of Lemma 7.10. □

THEOREM 8.4. *Let $\Gamma = \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$. For all motives X and Y over $\mathbb{F}_q$, there is a spectral sequence:*

$$H^r(\Gamma, \mathrm{Ext}^s_{\mathcal{M}^+(\mathbb{F};\mathbb{Z})}(X_{/\mathbb{F}}, Y_{/\mathbb{F}})) \implies \mathrm{Ext}^{r+s}_{\mathcal{M}^+(\mathbb{F}_q;\mathbb{Z})}(X,Y).$$

PROOF. Lemma 8.3 shows that X has a flat resolution. We can not apply Corollary 7.6 directly because $\mathcal{M}^+(\mathbb{F}_q;\mathbb{Z})$ does not have internal Homs, but its proof can be adapted as in the proof of 7.11. □

Homological algebra in $\mathcal{M}(k;\mathbb{Z})$. The results of the last two subsections hold mutatis mutandis for motives.

Examples.

EXAMPLE 8.5. We apply (8.1) to compute $\mathrm{Ext}^1_{\mathcal{M}^+(\mathbb{F};\mathbb{Z})}(\mathbb{Z},\mathbb{Z}(r))$ (we are writing $\mathbb{Z}$ for $\mathbb{1}$ and $\mathbb{Z}(r)$ for $\mathbb{T}^{\otimes r}$).

For $r=0$, $\mathrm{Hom}_{\mathcal{M}^+(\mathbb{F};\mathbb{Z})}(\mathbb{Z},\mathbb{Z}(0)) \cong \mathbb{Z}$ and $\mathrm{Ext}^i_{\mathcal{M}^+(\mathbb{F};\mathbb{Z})}(\mathbb{Z},\mathbb{Z}(0)) = 0$ for $i = 1, 2..$

For $r \neq 0$, $\mathrm{Hom}_{\mathcal{M}^+(\mathbb{F};\mathbb{Z})}(\mathbb{Z},\mathbb{Z}(r)) = 0$, and so

$$\mathrm{Hom}_{\mathcal{M}^+(\mathbb{F};\mathbb{Z})}(\mathbb{Z},\mathbb{Z}(r)^{(m)}) \xrightarrow{\cong} \mathrm{Ext}^1_{\mathcal{M}^+(\mathbb{F};\mathbb{Z})}(\mathbb{Z},\mathbb{Z}(r))_m.$$

For $m = p^n$, the left hand side is zero, and for $m = \ell^n$ it is $(\mathbb{Z}^{(\ell^n)})(r)$. Therefore, on passing to the direct limit over m, we find that

$$\mathrm{Ext}^1_{\mathcal{M}^+(\mathbb{F};\mathbb{Z})}(\mathbb{Z},\mathbb{Z}(r)) \cong \left(\bigoplus_{\ell \neq p} \mathbb{Q}_\ell/\mathbb{Z}_\ell\right)(r)$$

where $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)(r)$ is $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ with the Frobenius element of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ acting as q^r. Moreover, $\mathrm{Ext}^i_{\mathcal{M}^+(\mathbb{F},\mathbb{Z})}(\mathbb{Z},\mathbb{Z}(r)) = 0$ for $i \neq 0, 2$. (Without the $+$, these statement holds only modulo p-torsion.)

EXAMPLE 8.6. We compute $\mathrm{Ext}^1_{\mathcal{M}^+(\mathbb{F}_q,\mathbb{Z})}(\mathbb{Z},\mathbb{Z}(r))$ for $r>0$. The spectral sequence in (8.4) shows that $\mathrm{Ext}^1_{\mathcal{M}^+(\mathbb{F}_q,\mathbb{Z})}(\mathbb{Z},\mathbb{Z}(r))$ and $\mathrm{Ext}^2_{\mathcal{M}^+(\mathbb{F}_q,\mathbb{Z})}(\mathbb{Z},\mathbb{Z}(r))$ are the kernel and cokernel respectively of

$$\oplus_{\ell\neq p}\mathbb{Q}_\ell/\mathbb{Z}_\ell \xrightarrow{x\mapsto(q^r-1)x} \oplus_{\ell\neq p}\mathbb{Q}_\ell/\mathbb{Z}_\ell.$$

Therefore $\mathrm{Ext}^1_{\mathcal{M}^+(\mathbb{F}_q;\mathbb{Z})}(\mathbb{Z},\mathbb{Z}(r))$ is cyclic of order q^r-1, and $\mathrm{Ext}^i_{\mathcal{M}^+(\mathbb{F}_q;\mathbb{Z})}(\mathbb{Z},\mathbb{Z}(r))=0$ for $i\neq 0,2$.

EXAMPLE 8.7. Let A be an abelian variety over $\mathbb{F}$, and let h_1A be the isomotive $h_1(A)_0$ endowed with the $\mathbb{Z}$-structure provided by the maps $H_1(A,\mathbb{Z}_l)\to H_1(A,\mathbb{Q}_l)$. Since $\mathrm{Hom}(\mathbb{Z},h_1A)=0$,

$$\mathrm{Hom}_{\mathcal{M}^+(\mathbb{F};\mathbb{Z})}(\mathbb{Z},(h_1A)^{(m)})\cong\mathrm{Ext}^1_{\mathcal{M}^+(\mathbb{F};\mathbb{Z})}(\mathbb{Z},h_1A)_m$$

for all m, and so

$$\mathrm{Hom}_{\mathcal{M}^+(\mathbb{F};\mathbb{Z})}(\mathbb{Z},h_1A\otimes\mathbb{Q}/\mathbb{Z})\cong\mathrm{Ext}^1_{\mathcal{M}^+(\mathbb{F};\mathbb{Z})}(\mathbb{Z},h_1A).$$

But

$$\mathrm{Hom}_{\mathcal{M}^+(\mathbb{F};\mathbb{Z})}(\mathbb{Z},h_1A\otimes\mathbb{Q}/\mathbb{Z})\cong\mathrm{Hom}_{\mathcal{R}^+(\mathbb{F};\hat{\mathbb{Z}})}(\mathbb{Z},(h_1A)_{f\mathbb{Q}}/(h_1A)_f)\cong A(\mathbb{F}).$$

Thus,

$$\mathrm{Ext}^1_{\mathcal{M}^+(\mathbb{F};\mathbb{Z})}(\mathbb{Z},h_1A)\cong A(\mathbb{F}).$$

When A is defined over $\mathbb{F}_q$, $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ acts, and, on taking invariants, we find that

$$\mathrm{Ext}^1_{\mathcal{M}^+(\mathbb{F}_q;\mathbb{Z})}(\mathbb{Z},h_1A)\cong A(\mathbb{F}_q).$$

A similar argument using that $\mathrm{Hom}_{\mathcal{M}^+(\mathbb{F};\mathbb{Z})}((h_1A)(-1),\mathbb{Z})=0$ shows that

$$\mathrm{Hom}_{\mathcal{M}^+(\mathbb{F};\mathbb{Z})}((h_1A)(-1),\mathbb{Q}/\mathbb{Z})\cong\mathrm{Ext}^1_{\mathcal{M}^+(\mathbb{F};\mathbb{Z})}((h_1A)(-1),\mathbb{Z}).$$

But

$$\mathrm{Hom}_{\mathcal{M}^+(\mathbb{F};\mathbb{Z})}((h_1A)(-1),\mathbb{Q}/\mathbb{Z})\cong\mathrm{Hom}_{\mathcal{R}^+(\mathbb{F};\hat{\mathbb{Z}})}((h_1A)_f(-1),\mathbb{Q}/\mathbb{Z})\cong A^\vee(\mathbb{F})$$

where $A^\vee$ is the dual abelian variety. For an abelian variety A over $\mathbb{F}_q$, this becomes

$$\mathrm{Ext}^1_{\mathcal{M}^+(\mathbb{F}_q;\mathbb{Z})}((h_1A)(-1),\mathbb{Z})\cong A^\vee(\mathbb{F}_q).$$

REMARK 8.8. It is generally conjectured, that for a smooth variety V over a field F,

$$H^i_{\mathrm{mot}}(V,\mathbb{Q}(r))\cong(K_{2r-i}V)^{(r)}_{\mathbb{Q}} \tag{18}$$

and

$$H^i_{\mathrm{mot}}(V,\mathbb{Q}(r))\cong\mathrm{Ext}^i_{\mathcal{MM}(F;\mathbb{Q})}(\mathbb{1},(hV)(r)); \tag{19}$$

see, for example, Deligne 1994, 3.7, 3.3. Here $\mathrm{Ext}^i_{\mathcal{MM}(F;\mathbb{Q})}(\mathbb{1},(hV)(r))$ is computed in the conjectural category of mixed isomotives over F and $(K_{2r-i}V)^{(r)}_{\mathbb{Q}}$ is the subspace of $(K_{2r-1}V)_{\mathbb{Q}}$ on which the Adams operators ψ^k act as k^r.

In particular, for F itself, it is conjectured that

$$\mathrm{Ext}^1_{\mathcal{MM}(F;\mathbb{Q})}(\mathbb{Q},\mathbb{Q}(r))\cong(K_{2r-1}F)^{(r)}_{\mathbb{Q}} \tag{20}$$

For $F=\mathbb{F}_q$ or $\mathbb{F}$, the statement (20) is true because the K-groups are torsion and, under any definition, the category of mixed isomotives will be semisimple, so both sides are 0.

With $\mathbb{Z}$-coefficients, statement (20) is expected to be false in general, and (18) is definitely false[10]. However, Quillen (1972) shows that $K_{2r-1}\mathbb{F} \approx \oplus_{\ell\neq p}\mathbb{Q}_\ell/\mathbb{Z}_\ell$ with the Frobenius automorphism acting as p^r and $K_{2r-1}\mathbb{F}_q = (K_{2r-1}\mathbb{F})^{\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)}$, and Hiller (1981) and Kratzer (1980) show that the Adams operator ψ^k acts on $K_{2r-1}\mathbb{F}$ as multiplication by k^r, and so an integral version of (20) *does* hold when $F = \mathbb{F}_q$ or $\mathbb{F}$:

$$\mathrm{Ext}^1_{\mathcal{M}^+(F;\mathbb{Z})}(\mathbb{Z}, \mathbb{Z}(r)) \approx (K_{2r-1}F)^{(r)}. \tag{21}$$

The isomorphism (21) can be made canonical: Soulé (1979, IV.2, p. 284) proves that the Chern class map $c_{i,1}\colon K_{2i-1}(\mathbb{F}_q)_{\mathbb{Z}_l} \to H^1(\mathbb{F}_q, \mathbb{Z}_l(i))$ is an isomorphism if $i < l$; later Dyer and Friedlander (1985) defined a Chern character $ch_{i,1}\colon K_{2i-1}(\mathbb{F}_q)_{\mathbb{Z}_l} \to H^1(\mathbb{F}_q, \mathbb{Z}_l(i))$ that is related the Chern class map by $(-1)^{i-1}(i-1)!ch_{i,1} = c_{i,1}$ (Soulé 1999, 3.2.2, p. 267).

9. Extensions and Zeta Functions: Local Case

We use the notations: $\Gamma = \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$, $q = p^a$, $W = W(\mathbb{F}_q)$, $B = B(\mathbb{F}_q)$, $A = W[F,\sigma]$, $\mathcal{R}_\ell = \mathcal{R}(\mathbb{F}_q;\mathbb{Z}_\ell) =_{\mathrm{df}} \mathcal{R}(\Gamma;\mathbb{Z}_\ell)$, $\bar{\mathcal{R}}_\ell = \mathbf{Mod}(\mathbb{Z}_\ell)$, $\mathcal{R}_p = \mathcal{R}^+(\mathbb{F}_q, \mathbb{Z}_p) =_{\mathrm{df}} \mathbf{Crys}^+(\mathbb{F}_q)$, $\bar{\mathcal{R}}_p = \mathbf{Crys}^+(\mathbb{F})$.

Statement of the theorem. Let M be an object of $\mathcal{R}_l$. The Frobenius endomorphism π_M of M is the map induced by the element $x \mapsto x^q$ of Γ when $l \neq p$ and by F^a when $l = p$. We denote the characteristic and minimum polynomials of π_M acting on $M_{\mathbb{Q}_l}$ by $m_M(t)$ and $P_M(t)$ respectively. They are monic polynomials with coefficients in $\mathbb{Z}_l$ (see Demazure 1972, p89, for the case $l = p$). For M in $\mathcal{R}_p$, let $r(M)$ denote the rank of M and $s(M)$ the sum of the slopes of M; thus if

$$P_M(T) = t^h + \cdots + c$$

then $r(M) = h$ and $s(M) = \mathrm{ord}_p(c)/a$ (cf. Demazure 1972, p90). For M in $\mathcal{R}_\ell$ we set $s(M) = 0$.

Let M, N be objects of $\mathcal{R}_l$. From the spectral sequences (7.8, 7.12) we obtain a map

$$\mathrm{Hom}_{\bar{\mathcal{R}}_l}(\bar{M}, \bar{N})_\Gamma \to \mathrm{Ext}^1_{\mathcal{R}_l}(M, N).$$

Define

$$f = f(M,N)\colon\ \mathrm{Hom}_{\mathcal{R}_l}(M,N) \to \mathrm{Ext}^1_{\mathcal{R}_l}(M,N)$$

to be the composite of this with the obvious map $\mathrm{Hom}_{\mathcal{R}_l}(M,N) \to \mathrm{Hom}_{\bar{\mathcal{R}}_l}(\bar{M}, \bar{N})_\Gamma$.

THEOREM 9.1. *If $m_M(T)$ and $m_N(T)$ have no multiple root in common, then $z(f)$ is defined and satisfies*

$$z(f) \cdot [\mathrm{Ext}^2_{\mathcal{R}_l}(M,N)] = \left| q^{s(M)\cdot r(N)} \cdot \prod_{a_i \neq b_j} \left(1 - \frac{b_j}{a_i}\right) \right|_l . \tag{22}$$

where $(a_i)_{1\leq i\leq r(M)}$ and $(b_j)_{1\leq j\leq r(N)}$ are the roots of $P_M(t)$ and $P_N(t)$ respectively.

REMARK 9.2. (a) When $l \neq p$, the term $q^{s(M)\cdot r(N)} = 1$.

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[10]For a projective smooth variety over a field k,

$$H^3_{\mathrm{mot}}(V, \mathbb{Z}(1)) = H^2(V, \mathbb{G}_m),$$

which is a nonzero torsion group in general whereas $K_{-1}V = 0$. See also Bloch and Esnault 1996, p. 304, which explains why a closely related conjecture should not be true integrally.

(b) In the course of the proof, we shall show that $\mathrm{Ext}^i_{\mathcal{R}_l}(M,N)=0$ for $i>2$ (without any condition on M and N) and that $\mathrm{Ext}^2_{\mathcal{R}_l}(M,N)$ is finite.

(c) When one of M or N is finite, the theorem simply states that the groups $\mathrm{Ext}^i_{\mathcal{R}_l}(M,N)$ are finite and the alternating product of their orders is 1.

(d) Let $P_{M^\vee\otimes N}(t)=\prod\left(1-\frac{b_j}{a_i}t\right)$, and let $\rho(M,N)$ be the rank of $\mathrm{Hom}_{\mathcal{R}_l}(M,N)$. Then $\rho(M,N)$ is the number of pairs (i,j) with $a_i=b_j$, and (22) becomes the equation

$$z(f)\cdot[\mathrm{Ext}^2(M,N)]=\left|q^{s(M)\cdot r(N)}\cdot\lim_{t\to 1}\frac{P_{M^\vee\otimes N}(t)}{(1-t)^{\rho(M,N)}}\right|_l.$$

(e) When the groups $\mathrm{Ker}(f)$, $\mathrm{Coker}(f)$, $\mathrm{Ext}^2_{\mathcal{R}_l}(M,N)$, $\mathrm{Ext}^3_{\mathcal{R}_l}(M,N)$, $\ldots$ are finite and almost all zero, then we say that $\chi^\times(M,N)$ is defined, and we set it equal to the alternating product of the orders of the groups. Granted (b), Theorem 9.1 is equivalent to the statement: if $m_M(T)$ and $m_N(T)$ have no multiple root in common, then $\chi^\times(M,N)$ is defined and equals

$$\left|q^{s(M)\cdot r(N)}\cdot\prod_{a_i\neq b_j}\left(1-\frac{b_j}{a_i}\right)\right|_l \tag{23}$$

(f) Let

$$0\to N'\to N\to N''\to 0$$

be an exact sequence in $\mathcal{R}_l$. If $\chi^\times(M,N')$ and $\chi^\times(M,N'')$ are defined, then so also is $\chi^\times(M,N)$, and

$$\chi^\times(M,N)=\chi^\times(M,N')\cdot\chi^\times(M,N'').$$

Since a similar statement holds for exact sequences in M and the expression (23) is multiplicative, we see that it suffices to prove Theorem 9.1 for M and N running through a set of generators for the Grothendieck group $K(\mathcal{R}_l)$ of $\mathcal{R}_l$.

Even more is true: both $\chi^\times$ and the expression in (23) take values in $\mathbb{Q}^\times_+$, the group of positive rational numbers, which is torsion-free. Therefore, in order to prove Theorem 9.1, it suffices to verify it for M and N running through a set of generators for $K(\mathcal{R}_l)\otimes\mathbb{Q}$.

Proof of Theorem 9.1 in the case $l\neq p$. From Proposition 7.8, we obtain an exact sequence and isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{R}_\ell}(M,N) &\cong \mathrm{Hom}_{\mathbb{Z}_\ell}(M,N)^\Gamma\\
0\to\mathrm{Hom}_{\mathbb{Z}_\ell}(M,N)_\Gamma &\xrightarrow{c} \mathrm{Ext}^1_{\mathcal{R}_\ell}(M,N)\to\mathrm{Ext}^1_{\mathbb{Z}_\ell}(M,N)^\Gamma\to 0\\
\mathrm{Ext}^1_{\mathbb{Z}_\ell}(M,N)_\Gamma &\cong \mathrm{Ext}^2_{\mathcal{R}_l}(M,N)\\
\mathrm{Ext}^i_{\mathcal{R}_l}(M,N) &= 0 \text{ for } i>2.
\end{aligned}$$

Let f_0 be the map

$$\mathrm{Hom}_{\mathbb{Z}_\ell}(M,N)^\Gamma\to\mathrm{Hom}_{\mathbb{Z}_\ell}(M,N)_\Gamma$$

induced by the identity map. Then $f=c\circ f_0$, and so (see 9.8)

$$z(f)=z(f_0)/[\mathrm{Ext}^1_{\mathbb{Z}_\ell}(M,N)^\Gamma]. \tag{24}$$

Let γ be the generator $x \mapsto x^q$ of Γ. The family of eigenvalues of γ on $\mathrm{Hom}_{\mathbb{Z}_\ell}(M, N)$ is $(\frac{b_i}{a_j})_{i,j}$. Because of our hypothesis on $m_M(t)$ and $m_N(t)$, (9.9) applies, and shows that

$$z(f_0) = \left| \prod\nolimits_{a_i \neq b_j} \left(1 - \frac{b_i}{a_j}\right) \right|_\ell .$$

Because $\mathrm{Ext}^1_{\mathbb{Z}_\ell}(M, N)$ is finite,

$$[\mathrm{Ext}^1_{\mathbb{Z}_\ell}(M, N)^\Gamma] = [\mathrm{Ext}^1_{\mathbb{Z}_\ell}(M, N)_\Gamma]$$

and so

$$[\mathrm{Ext}^2_{\mathcal{R}_l}(M, N)] = [\mathrm{Ext}^1_{\mathbb{Z}_\ell}(M, N)^\Gamma].$$

On substituting for $z(f_0)$ and $[\mathrm{Ext}^1_{\mathbb{Z}_\ell}(M, N)^\Gamma]$ in (24), we obtain (22).

Proof of Theorem 9.1 in the case $l = p$. An A-module M will be said to be *special* if it is cyclic, $M = A/A \cdot \lambda$, and λ lies in the centre $\mathbb{Z}_p[F^a]$ of A, in which case $\lambda(F) = m_M(F^a)$ and $P_M(t) = m_M(t)^a$. We let k denote the A-module with underlying W-module $\mathbb{F}_q$ and with F acting as 0.

Let $\mathcal{R}_{p,f}$ be the full subcategory of $\mathcal{R}_p$ of finite objects. Then Exts computed in $\mathcal{R}_{p,f}$ coincide with those in $\mathcal{R}_p$ (see the argument p33) and with those in Pro-$\mathcal{R}_{p,f}$ ($\mathcal{R}_{p,f}$ is Artinian and so we can apply the opposite of 7.1). There is a canonical functor $\mathcal{R}_p \to$Pro-$\mathcal{R}_{p,f}$ and by using the usual exact sequences (10) one deduces that Exts in $\mathcal{R}_p$ coincide with those in Pro-$\mathcal{R}_{p,f}$. This allows us to use resolutions by free finitely generated A-modules to compute Exts in $\mathcal{R}_p$.

The case M or N is finite with F acting invertibly. Because of (6), this case is essentially the same as the non-p case. From the spectral sequence (7.12), we obtain exact sequences

$$0 \to \mathrm{Ext}^{i-1}_{\bar{\mathcal{R}}_p}(M_{W(\mathbb{F})}, N_{W(\mathbb{F})})_\Gamma \to \mathrm{Ext}^i_{\mathcal{R}_p}(M, N) \to \mathrm{Ext}^i_{\bar{\mathcal{R}}_p}(M_{W(\mathbb{F})}, N_{W(\mathbb{F})})^\Gamma \to 0.$$

In this case, the groups $\mathrm{Ext}^i_{\bar{\mathcal{R}}_p}(M_{W(\mathbb{F})}, N_{W(\mathbb{F})})$ are finite, and it follows immediately that the alternating product of the orders of the $\mathrm{Ext}^i_{\mathcal{R}_p}(M, N)$ is 1.

The case of $M = k$ and N finite. We have the following Koszul-type resolution of k

$$0 \to A \xrightarrow{t} A \oplus A \xrightarrow{s} A \to k \to 0 \tag{25}$$

where $t(\alpha) = (\alpha F, p\alpha)$ and $s(x, y) = px - yF$. For any N, the groups $\mathrm{Ext}^i(k, N)$ are the cohomology groups of the complex

$$\mathrm{Hom}(A, N) \xrightarrow{\circ s} \mathrm{Hom}(A \oplus A, N) \xrightarrow{\circ t} \mathrm{Hom}(A, N) \to 0 \to \cdots \tag{26}$$

Therefore, $\mathrm{Ext}^i(k, N) = 0$ for $i > 2$, and if N is finite, then

$$\mathrm{Ext}^0(k, N) \cdot \mathrm{Ext}^2(k, N) = \mathrm{Ext}^1(k, N)$$

because the groups in (26) are finite.

The case M cyclic and $N = k$. The resolution,

$$0 \to A \xrightarrow{\cdot\lambda} A \to M \to 0,$$

gives an exact sequence

$$0 \to \mathrm{Hom}(M,k) \to \mathrm{Hom}(A,k) \to \mathrm{Hom}(A,k) \to \mathrm{Ext}^1(M,k) \to 0$$

and equalities $\mathrm{Ext}^i(M,k) = 0$ $(i > 1)$. As $\mathrm{Hom}(A,k) \cong k$ is finite, this implies that

$$[\mathrm{Ext}^0(M,k)] = [\mathrm{Ext}^1(M,k)].$$

The case $M = k$ and N cyclic. From (25) and (26), we find that $\mathrm{Ext}^i(k,A) = 0$ unless $i = 2$ in which case $\mathrm{Ext}^2(k,A) = k$. Now the $\mathrm{Ext}(k,-)$-sequence of

$$0 \to A \xrightarrow{\cdot\lambda} A \to M \to 0$$

gives an exact sequence

$$0 \to \mathrm{Ext}^1(k,N) \to \mathrm{Ext}^2(k,A) \to \mathrm{Ext}^2(k,A) \to \mathrm{Ext}^2(k,N) \to 0$$

and equalities $\mathrm{Ext}^i(k,N) = 0$ $(i \neq 1,2)$. As $\mathrm{Ext}^2(k,A)$ is finite, the sequence gives that

$$[\mathrm{Ext}^1(k,N)] = [\mathrm{Ext}^2(k,N)].$$

The case M, N special, m_M and m_N relatively prime. Let

$$\begin{array}{rcl} M & = & A/A\lambda_1, \quad \lambda_1 = m_M(F^a) \\ N & = & A/A\lambda_2, \quad \lambda_2 = m_N(F^a) \end{array}$$

with $m_M(t)$ and $m_N(t)$ relatively prime. From the resolution

$$0 \to A \xrightarrow{\cdot\lambda_1} A \to M \to 0$$

we obtain an exact sequence

$$0 \to \mathrm{Hom}(M,N) \to N \xrightarrow{\lambda_1\cdot} N \to \mathrm{Ext}^1(M,N) \to 0 \to \dots.$$

As $\cdot\lambda_1$ is injective, we see that $\mathrm{Ext}^i(M,N) = 0$ for $i \neq 1$, and so

$$z(f) \cdot [\mathrm{Ext}^2(M,N)] = z(\lambda_1\cdot) \overset{9.7}{=} |\det(\lambda_1\cdot)|_p^a.$$

Here $\det(\lambda_1\cdot)$ can be computed as the determinant of $m_M(F^a)$ acting on the B-vector space $N_{\mathbb{Q}}$. But $P_M(t) = m_M(t)^a$, and so

$$\det(m_M(F^a)|N_{\mathbb{Q}})^a = \prod\nolimits_{\substack{1\le i\le r(M) \\ 1\le j\le r(N)}} (a_i - b_j).$$

Thus,

$$z(f) \cdot [\mathrm{Ext}^2(M,N)] = \left|\prod (a_i - b_j)\right|_p = \left| q^{s(M)\cdot r(N)} \prod \left(1 - \frac{b_j}{a_i}\right)\right|_p$$

as required.

The case M and N are special and equal. We first give an explicit description of the map

$$f\colon \operatorname{Hom}(M,N) \to \operatorname{Ext}^1(M,N)$$

in the case that $M = A/A\cdot\lambda_1$ and $N = A/A\cdot\lambda_2$ are special but not necessarily equal. We use $^-$ to denote $-\otimes_{W(\mathbb{F}_q)} W(\mathbb{F})$, and we let σ_k be the canonical generator of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$. The $\operatorname{Ext}(-,N)$ sequence of

$$0 \to A \xrightarrow{\cdot\lambda_1} A \to M \to 0$$

is

$$0 \to \operatorname{Hom}(M,N) \to N \xrightarrow{\lambda_1\cdot} N \to \operatorname{Ext}^1(M,N) \to 0$$

and the $\operatorname{Ext}^r(-,\bar{N})$ sequence of

$$0 \to \bar{A} \xrightarrow{\cdot\lambda_1} \bar{A} \to \bar{M} \to 0$$

is

$$0 \to \operatorname{Hom}(\bar{M},\bar{N}) \to \bar{N} \xrightarrow{\lambda_1\cdot} \bar{N} \to \operatorname{Ext}^1(\bar{M},\bar{N}) \to 0.$$

The map f can be described as follows: let $u \in \operatorname{Hom}(M,N)$ and regard u as an element of N such that $\lambda_1 u = 0$; then u can be written $u = (\sigma_k - 1)v$ for some $v \in \bar{N}$; now

$$(\sigma_k - 1)(\lambda_1 v) = \lambda_1(\sigma_k - 1)v = \lambda_1 u = 0,$$

and so $\lambda_1 v \in \bar{N}^\Gamma = N$; the image of $\lambda_1 v$ under $N \to \operatorname{Ext}^1(M,N)$ is $f(u)$.

Now take $N = M$, so that multiplication by $\lambda = \lambda_1 = \lambda_2$ is zero on M. Thus

$$\operatorname{Hom}(M,N) = A/A\lambda = \operatorname{Ext}^1(M,N)$$

and f is an endomorphism of $A/A/\lambda$. Since $A/A\lambda$ is torsion-free, $z(f)$ is defined if and only if

$$f_{\mathbb{Q}}\colon A_{\mathbb{Q}}/A_{\mathbb{Q}}\lambda \to A_{\mathbb{Q}}/A_{\mathbb{Q}}\lambda$$

has nonzero determinant (9.7), in which case

$$z(f) = |\det(f)|_p^a.$$

Let $u \in A_{\mathbb{Q}}/A_{\mathbb{Q}}\lambda$ and choose $v \in \bar{A}_{\mathbb{Q}}/\bar{A}_{\mathbb{Q}}\lambda$ such that $u = \sigma_k v - v$. Then $\sigma_k^i v = iu + v$ for all i. Let

$$\lambda(F) = F^{ma} + b_{ma-a}F^{ma-a} + \cdots + b_0 = m(F^a).$$

Then

$$\begin{aligned} f(u) &= \lambda(F)v \\ &= muF^{ma} + (m-1)b_{ma-a}uF^{ma-a} + \cdots + 0 \qquad (\text{as } v\lambda = 0) \\ &= uF^a\frac{d}{dF^a}(m(F^a)) \\ &= F^a\frac{d}{dF^a}(m(F^a))u. \end{aligned}$$

From (9.7) we find that

$$z(f) = \left|\det\left(\frac{d}{dF^a}(m(F^a))\right)\right|_p^a \cdot |\det(F^a)|_p^a.$$

But

$$|\det(F^a)|_p^a = |q^{a\cdot s(N)}|_p$$

and

$$\left|\det\left(\frac{d}{dF^a}(m(F^a))\right)\right|_p^a = \left|\prod_{a_i\neq b_j}(a_i-b_j)\right|_p = \left|q^{s(N)(r(N)-a)}\prod_{a_i\neq b_j}\left(1-\frac{b_j}{a_i}\right)\right|_p.$$

Thus,

$$z(f) = \left|q^{r(N)\cdot s(N)}\prod_{a_i\neq b_j}\left(1-\frac{b_j}{a_i}\right)\right|_p.$$

As $\mathrm{Ext}^2(M,N)=0$ this proves Theorem 9.1 in this case.

The general case. An isogeny (denoted $\sim$) of objects in $\mathcal{R}_p$ is a homomorphism whose kernel and cokernel have finite length over W. We say an object in $\mathcal{R}_p$ is indecomposable if it is not isogenous to a direct sum of two nonzero objects.

LEMMA 9.3. *Every object of $\mathcal{R}_p$ is isogenous to a direct sum of indecomposable objects, and the decomposition is unique up to isogeny.*

PROOF. Apply the Krull-Schmidt theorem in the category of $A_{\mathbb{Q}}$-modules. □

LEMMA 9.4. *If M in $\mathcal{R}_p$ is indecomposable, then $M_{\mathbb{Q}} \approx A_{\mathbb{Q}}/A_{\mathbb{Q}}\lambda$ for some $\lambda \in A_{\mathbb{Q}}$; moreover, the minimum polynomial $m_M(t)$ of the Frobenius endomorphism of M is a power of a $\mathbb{Z}_p$-irreducible polynomial, and there exists an integer e such that $\oplus^e M \sim A/A\cdot m_M(t)$.*

PROOF. The ring $A_{\mathbb{Q}}$ is the skew polynomial ring $B[F,\sigma]$. It is therefore a principal ideal domain (Jacobson 1943, III 1, p30), and every finitely generated module over such a ring is isomorphic a direct sum of cyclic modules (ibid. III, Theorem 19, p44). This proves the first part.

For the second, we use that the two-sided ideals in $A_{\mathbb{Q}}$ are precisely those generated by a polynomial in the centre $\mathbb{Q}_p[F^a]$ of $A_{\mathbb{Q}}$ (ibid. III 5, p38). It follows that $A_{\mathbb{Q}}\cdot m(F^a)$ is the largest two-sided ideal contained in $A_{\mathbb{Q}}\cdot\lambda$, i.e., it is the bound of the ideal $A_{\mathbb{Q}}\cdot\lambda$ in the sense of ibid. III 6, p38. Because $A_{\mathbb{Q}}/A_{\mathbb{Q}}\lambda$ is indecomposable, this implies that $m(t)$ is a power of a $\mathbb{Z}_p$-irreducible polynomial (ibid. III Theorem 13, p40). Finally, when $A_{\mathbb{Q}}/A_{\mathbb{Q}}\cdot m(F^a)$ is decomposed into a direct sum of indecomposable modules $A_{\mathbb{Q}}/A_{\mathbb{Q}}\lambda'$, each of the ideals $A_{\mathbb{Q}}\lambda'$ has $A_{\mathbb{Q}}\cdot m(F^a)$ as its bound, and is therefore isomorphic to M (ibid. III Theorem 20, p45). □

PROPOSITION 9.5. *Objects of the following types generate $K(\mathcal{R}_p)_{\mathbb{Q}}$:*

(a) *$k=(\mathbb{F}_q,0)$;*
(b) *(M,F_M) with M finite and F_M invertible;*
(c) *M special and $m_M(t)$ is a power of an irreducible polynomial.*

PROOF. For every finite module M, there exists an n such that $M=\mathrm{Im}(F^n)\oplus\mathrm{Ker}(F^n)$ (Fitting's Lemma). It follows that such an M has a composition series with quotients of type (a) or (b). The general statement now follows from Lemma 9.4. □

Therefore (see 9.2(f)), it suffices to prove Theorem 9.1 when each of M and N is of type (a), (b), or (c). Moreover, because of the condition on the minimum polynomials, in case (c) we can assume that either m_M and m_N have no common factor or they are irreducible and equal. This we have done.

Restatement of the theorem for complexes. We sketch a generalization of Theorem 9.1 to complexes. Let $M^\bullet$ and $N^\bullet$ be bounded complexes of A-modules such that $H^i(M^\bullet)$ lies in $\mathcal{R}_p$ for all i. Define

$$P_{M^{\bullet\vee}\otimes N^\bullet}(t)=\prod P_{H^i(M^\bullet)^\vee\otimes H^j(N^\bullet)}(t)^{(-1)^{i+j}}.$$

Let $f^i\colon \operatorname{Ext}^i(M^\bullet,N^\bullet)\to\operatorname{Ext}^{i+1}(M^\bullet,N^\bullet)$ be the map rendering

$$\begin{array}{ccc}\operatorname{Ext}^i(M^\bullet,N^\bullet) & \xrightarrow{f^i} & \operatorname{Ext}^{i+1}(M^\bullet,N^\bullet)\\ \downarrow & & \downarrow\\ \operatorname{Ext}^i(\bar{M}^\bullet,\bar{N}^\bullet)^\Gamma & \xrightarrow{\cup\theta_p} & \operatorname{Ext}^i(\bar{M}^\bullet,\bar{N}^\bullet)_\Gamma\end{array}$$

commutative. Here the vertical maps arise from a spectral sequence (cf. 7.12) and the lower map is cup-product with the canonical generator $\theta_p\in H^1(\Gamma,W)$ (equal to the map induced by the identity map on $\operatorname{Ext}^i(\bar{M}^\bullet,\bar{N}^\bullet)$). Because $\theta_p^2=0$,

$$\operatorname{Ext}^\bullet(M^\bullet,N^\bullet)\colon\qquad \cdots\to\operatorname{Ext}^i(M^\bullet,N^\bullet)\xrightarrow{f^i}\operatorname{Ext}^{i+1}(M^\bullet,N^\bullet)\to\cdots$$

is a complex. Define

$$\chi^\times(M^\bullet,N^\bullet)=\prod[H^i(\operatorname{Ext}^\bullet(M^\bullet,N^\bullet))]^{(-1)^i}$$

when these numbers are finite.

THEOREM 9.6. *Let $M^\bullet$ and $N^\bullet$ be bounded complexes of A-modules such that $H^i(M^\bullet)$ and $H^i(N^\bullet)$ are semisimple and finitely generated over W for all i.*

(a) *The groups $\operatorname{Ext}^i(M^\bullet,N^\bullet)$ are finitely generated $\mathbb{Z}_p$-modules for all i.*
(b) *The alternating sum $\sum(-1)r_i$ of the ranks of the $\operatorname{Ext}^i(M^\bullet,N^\bullet)$ is zero.*
(c) *The order of zero of $P(M^{\bullet\vee}\otimes N^\bullet,t)$ at $t=1$ is equal to the secondary Euler characteristic $\sum(-1)^{i+1}\cdot i\cdot r_i$.*
(d) *The groups $H^i(\operatorname{Ext}^\bullet(M^\bullet,N^\bullet))$ are finite, and*

$$\left|q^{\chi(M^\bullet,N^\bullet)}\cdot\lim_{s\to1}\frac{P_{M^{\bullet\vee}\otimes N^\bullet}(t)}{(1-t)^{\rho(M^\bullet,N^\bullet)}}\right|_l=\chi^\times(M^\bullet,N^\bullet)$$

where $\chi(M^\bullet,N^\bullet)=\sum(-1)^{i+j}s(H^i(M^\bullet))\cdot r(H^j(N^\bullet))$.

PROOF. Consider first the case that $M^\bullet$ and $N^\bullet$ are modules M and N regarded as complexes concentrated in degree 0.

(a) Because M and N are semisimple, $m(M,t)$ and $m(N,t)$ have only simply roots, and therefore the hypothesis of Theorem 9.1 hold. Hence, the groups $\operatorname{Ext}^i(M,N)$ are finitely generated $\mathbb{Z}_p$-modules for all i, finite for $i=2$, and zero for $i>2$.

(b) Moreover, $f\colon \operatorname{Hom}(M,N)\to\operatorname{Ext}^1(M,N)$ has finite kernel and cokernel, and so $\operatorname{Hom}(M,N)$ and $\operatorname{Ext}^1(M,N)$ have the same rank.

(c) In fact, one sees easily that the rank of $\operatorname{Hom}(M,N)$ is the number of pairs (i,j) such that $a_i=b_j$ (see 9.2d).

(d) The map f of Theorem 9.1 coincides with f^0, and so (d) is a restatement of the theorem.

The general case case now follows from the usual induction argument on the number of nonzero terms in the complexes $M^\bullet$ and $N^\bullet$ (cf. 9.2f). □

There is a similar Theorem for $l \neq p$, whose formulation and proof we leave to the reader.

Why $\mathcal{R}^+(\mathbb{F}_q;\mathbb{Z}_p)$ rather than $\mathcal{R}(\mathbb{F}_q;\mathbb{Z}_p)$? Write $\mathcal{R}^+$ and $\mathcal{R}$ for $\mathcal{R}^+(\mathbb{F}_q;\mathbb{Z}_p)$ and $\mathcal{R}(\mathbb{F}_q;\mathbb{Z}_p)$ respectively. Even the torsion subgroup of $\mathrm{Ext}^1_{\mathcal{R}}(M,N)$ can be infinite. Suppose, for example, that M and N are torsion-free, and consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{R}^+}(M,N)^{(p^n)} \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathcal{R}^+}(M,N^{(p^n)}) \arrow[r] \arrow[d] & \mathrm{Ext}^1_{\mathcal{R}^+}(M,N)_{p^n} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(M,N)^{(p^n)} \arrow[r] & \mathrm{Hom}_{\mathcal{R}}(M,N^{(p^n)}) \arrow[r] & \mathrm{Ext}^1_{\mathcal{R}}(M,N)_{p^n} \arrow[r] & 0.
\end{tikzcd}$$

Since $[\mathrm{Hom}_{\mathcal{R}}(M,N^{(p^n)})] = (q^n)^{r(M)r(N)}$ is unbounded, whereas $[\mathrm{Hom}_{\mathcal{R}^+}(M,N^{(p^n)})]$ is bounded if $\mathrm{Hom}_{R^+}(M,N)=0$, we see that $\mathrm{Ext}^1_{\mathcal{R}}(M,N)(p)$ is frequently infinite. From a different perspective, it follows from Theorem 9.1 that $\mathrm{Ext}^1_{\mathcal{R}^+}(M,N)$ grows when M and N are replaced by $M(1)$ and $N(1)$ because $s(M(1)) > s(M)$, but the functor $M \mapsto M(1)$ is an equivalence on the category $\mathcal{R}(\mathbb{F}_q;\mathbb{Z}_p)$.

Appendix: Lemmas on abelian groups. Recall that, for a homomorphism $f\colon M \to N$ of abelian groups, we define $z(f) = \frac{[\mathrm{Ker}(f)]}{[\mathrm{Coker}(f)]}$ when both the top and bottom are finite. There are the following elementary statements (Tate 1966b, §5). We let $W = W(\mathbb{F}_q)$, $q = p^a$.

9.7. *Let M and N be finitely generated $\mathbb{Z}$-modules (resp. W-modules) of the same rank and let (x_i) and (y_i) be bases for M and N respectively modulo torsion. Suppose $f(x_i) \equiv \sum a_{ij}y_j$ modulo torsion. Then $z(f)$ is defined if and only if* $\det(a_{ij}) \neq 0$, *in which case*

$$z(f) = \frac{[M_{tors}]}{|\det(a_{ij})|[N_{tors}]}$$

(resp.

$$z(f) = \frac{[M_{tors}]|\det(a_{ij})|_p^a}{[N_{tors}]}).$$

Note that, when $M = N$, $\det(a_{ij})$ is the determinant of $f_{\mathbb{Q}}\colon M_{\mathbb{Q}} \to M_{\mathbb{Q}}$.

9.8. *Consider maps $f\colon M \to N$ and $g\colon N \to P$. If any two of the three numbers $z(f)$, $z(g)$, $z(g\circ f)$ are defined, then so is the third, and*

$$z(g\circ f) = z(g)\cdot z(f).$$

9.9. *Let $\Gamma = \hat{\mathbb{Z}}$, and let γ be its canonical topological generator. Let M be a Γ -module that is finitely generated as a $\mathbb{Z}$-module, and let $f\colon M^{\Gamma} \to M_{\Gamma}$ be the map induced by the identity map on M. Then $z(f)$ is defined if and only if the minimum polynomial of γ on $M_{\mathbb{Q}}$ does not have 1 as a multiple root, in which case*

$$z(f)\left|\prod_{a_i\neq 1}(1-a_i)\right| = 1$$

where $a_1, a_2, \ldots$ is the family of eigenvalues of γ acting on $M_{\mathbb{Q}}$.

10. Extensions and Zeta Functions: Global Case

In this section, $\mathcal{M}(\mathbb{F};\mathbb{Q})$ will be one of the following categories:

(a) the category $\mathcal{M}_{\mathrm{num}}(\mathbb{F};\mathbb{Q};\mathcal{S})$ where $\mathcal{S}$ consists of all smooth projective varieties over $\mathbb{F}$ and we assume that the Tate conjecture holds for all $V\in\mathcal{S}$;

(b) the category $\mathcal{M}(\mathbb{F};\mathbb{Q})$ defined in (5.26).

In fact, if the category in (a) exists and the Hodge conjecture holds for complex abelian varieties of CM-type, then these categories are essentially the same: the reduction functor $\mathbf{CM}(\mathbb{Q}^{\mathrm{al}})\to\mathcal{M}_{\mathrm{num}}(\mathbb{F};\mathbb{Q};\mathcal{S})$ defines a fibre functor ω on $\mathbf{CM}(\mathbb{Q}^{\mathrm{al}})^P$, and the quotient of $\mathbf{CM}(\mathbb{Q}^{\mathrm{al}})$ corresponding to ω is canonically equivalent with $\mathcal{M}_{\mathrm{num}}(\mathbb{F};\mathbb{Q};\mathcal{S})$.

The main theorem for extensions.

THEOREM 10.1. *For all effective motives X and Y in $\mathcal{M}^{+}(\mathbb{F}_q;\mathbb{Z})$, the groups* $\mathrm{Ext}^1(X,Y)$ *and* $\mathrm{Ext}^2(X,Y)_{cotors}$ *are finite, and*

$$q^{\chi(X,Y)}\cdot\zeta(X^{\vee}\otimes Y,s)\sim\pm\frac{[\mathrm{Ext}^1(X,Y)]\cdot D(X,Y)}{[\mathrm{Hom}(X,Y)_{tors}]\cdot[\mathrm{Ext}^2(X,Y)_{cotors}]}\cdot(1-q^{-s})^{\rho(X,Y)}\quad \textit{as } s\to 0$$

where $\chi(X,Y)=r(X_p)s(Y_p)$ (notations as on p37), $D=D(X,Y)$ is the discriminant of the pairing

$$\mathrm{Hom}(Y,X)\times\mathrm{Hom}(X,Y)\xrightarrow{\circ}\mathrm{End}(Y)\xrightarrow{trace}\mathbb{Z},\tag{27}$$

and $\rho(X,Y)$ is the rank of $\mathrm{Hom}(X,Y)$.

REMARK 10.2. Let X and Y be effective motives over $\mathbb{F}_q$.

(a) Because $\mathcal{M}^{+}(\mathbb{F}_q;\mathbb{Q})$ is a semisimple category, $\mathrm{End}(X_0\times Y_0)$ is a semisimple ring. As it is also finite dimensional over $\mathbb{Q}$, the trace pairing

$$\mathrm{End}(X_0\times Y_0)\times\mathrm{End}(X_0\times Y_0)\to\mathbb{Q}$$

is nondegenerate: $D(X,Y)\neq 0$.

(b) Below we prove that $[\mathrm{Ext}^2(X,Y)_{\mathrm{cotors}}]=[\mathrm{Hom}(Y,X)_{\mathrm{tors}}]$. This makes the formula more symmetric — note that $D(X,Y)=D(Y,X)$.

(c) When we replace X by $X(r)$ or Y by $Y(r)$ ($r\in\mathbb{N}$) in Theorem 10.1, we obtain a description of the behaviour of $\zeta(X^{\vee}\otimes Y,s)$ near $-r$ or $+r$.

Proof of Theorem 10.1 with $z(\varepsilon)$ for $D(X,Y)$. Recall (8.1) that for all l, there is an exact sequence

$$0\to\mathrm{Ext}^1_{\mathcal{M}^{+}(\mathbb{F}_q;\mathbb{Z})}(X,Y)_{\mathbb{Z}_l}\xrightarrow{c_l}\mathrm{Ext}^1_{\mathcal{R}^{+}(\mathbb{F}_q;\mathbb{Z}_l)}(X_l,Y_l)\to T_l\,\mathrm{Ext}^2_{\mathcal{M}^{+}(\mathbb{F}_q;\mathbb{Z}_l)}(X,Y)\to 0.$$

Define

$$\varepsilon=(\varepsilon_l)_l\colon\ \mathrm{Hom}_{\mathcal{M}^{+}(\mathbb{F}_q;\mathbb{Z})}(X,Y)_{\hat{\mathbb{Z}}}\to T\,\mathrm{Ext}^2_{\mathcal{M}^{+}(\mathbb{F}_q;\mathbb{Z})}(X,Y)$$

to be the map making

$$\begin{array}{ccc}\mathrm{Hom}_{\mathcal{M}^{+}(\mathbb{F}_q;\mathbb{Z})}(X,Y)_{\mathbb{Z}_l} & \xrightarrow{\varepsilon_l} & T_l\,\mathrm{Ext}^2_{\mathcal{M}^{+}(\mathbb{F}_q;\mathbb{Z})}(X,Y)\\ \Big\downarrow{\cong} & & \Big\uparrow{c_l}\\ \mathrm{Hom}_{\mathcal{R}^{+}(\mathbb{F}_q;\mathbb{Z}_l)}(X_l,Y_l) & \xrightarrow{f_l} & \mathrm{Ext}^1_{\mathcal{R}^{+}(\mathbb{F}_q;\mathbb{Z}_l)}(X_l,Y_l)\end{array}$$

commute. Here $f_l = f(X_l, Y_l)$ is the map defined on p37. According to 9.8 and 9.1

$$z(\varepsilon_l) = z(c_l) \cdot z(f_l) = \frac{[\mathrm{Ext}^1(X,Y)(l)]}{[\mathrm{Ext}^2(X_l,Y_l)]} \cdot \left| q^{s(X_l)\cdot r(Y_l)} \cdot \prod_{a_i \neq b_j} \left(1 - \frac{b_j}{a_i}\right) \right|_l . \tag{28}$$

The a_i, b_j are algebraic numbers independent of l. Because $\mathrm{Ext}^i(X_l, Y_l) = 0$ for $i \geq 3$, (8.1b) shows that $\mathrm{Ext}^i(X, Y) = 0$ for $i \geq 3$, and it follows (from 8.1b again) that

$$\mathrm{Ext}^2(X,Y)_{\mathrm{cotors}}(l) \cong \mathrm{Ext}^2(X_l, Y_l).$$

On multiplying the equations (28) for the different l and applying the product formula, we find that

$$q^{\chi(X,Y)} \cdot \prod_{a_i \neq b_j} \left(1 - \frac{b_j}{a_i}\right) \cdot z(\varepsilon) = \pm \frac{[\mathrm{Ext}^1(X,Y)]}{[\mathrm{Ext}^2(X,Y)_{\mathrm{cotors}}]}.$$

This will become Theorem 10.1 once we have shown that

$$z(\varepsilon) = \pm \frac{[\mathrm{Hom}(X,Y)_{\mathrm{tors}}]}{D(X,Y)}.$$

Comparison of $z(\varepsilon)$ with $D(X,Y)$. Let N be a finitely generated torsion-free discrete Γ-module, and let $N^\vee = \mathrm{Hom}(N, \mathbb{Z})$. The pairing

$$N^\Gamma \times H^2(\Gamma, N^\vee) \to H^2(\Gamma, \mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$$

realizes the compact group $(N^\Gamma)_{\hat{\mathbb{Z}}}$ as the Pontryagin dual of the discrete group $H^2(\Gamma, N^\vee)$ (e.g., Milne 1986b, I 1.10). There is therefore a canonical isomorphism

$$\mathrm{Hom}(N^\Gamma, \hat{\mathbb{Z}}) \to \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, H^2(\Gamma, N^\vee)) \stackrel{\mathrm{df}}{=} TH^2(\Gamma, N^\vee).$$

Suppose we are also given a nondegenerate pairing

$$\psi\colon N \times M \to \mathbb{Z}$$

where M is a discrete Γ-module such that M_{tors} is killed by some integer and M/M_{tors} is finitely generated. This defines a map $M \to N^\vee$ with torsion kernel and cokernel, which induces isomorphisms $H^2(\Gamma, M) \to H^2(\Gamma, N^\vee)$ and $TH^2(\Gamma, M) \to TH^2(\Gamma, N^\vee)$. Hence, there is a canonical isomorphism

$$\mathrm{Hom}(N^\Gamma, \hat{\mathbb{Z}}) \to TH^2(\Gamma, M).$$

When combined with the map

$$M^\Gamma \to \mathrm{Hom}(N^\Gamma, \mathbb{Z})$$

defined by ψ, this gives a homomorphism

$$\varepsilon(\psi)\colon (M^\Gamma)_{\hat{\mathbb{Z}}} \to TH^2(\Gamma, M).$$

Note that $z(\varepsilon(\psi))$ is defined if and only if M^Γ_{tors} is finite and the restriction ψ^Γ to a pairing $N^\Gamma \times M^\Gamma \to \mathbb{Z}$ is nondegenerate, in which case

$$z(\varepsilon(\psi)) = \pm \frac{[M^\Gamma_{\mathrm{tors}}]}{\det(\psi^\Gamma)}.$$

When we apply this remark to the pairing

$$\mathrm{Hom}(\bar{Y}, \bar{X})/\{\mathrm{torsion}\} \times \mathrm{Hom}(\bar{X}, \bar{Y}) \to \mathbb{Q},$$

we obtain a homomorphism

$$\varepsilon\colon \operatorname{Hom}(X,Y)_{\hat{\mathbb{Z}}} \to TH^2(\Gamma,\operatorname{Hom}(X,Y)),$$

and

$$z(\varepsilon) = \pm\frac{[\operatorname{Hom}(X,Y)_{\mathrm{tors}}]}{D(X,Y)}.$$

Since ε coincides with the map defined in the preceding subsubsection, this completes the proof of Theorem 10.1.

Comparison of $\operatorname{Hom}(Y,X)_{\mathbf{tors}}$ with $\operatorname{Ext}^2(X,Y)_{\mathbf{cotors}}$. In this subsection, we prove that

$$[\operatorname{Hom}(Y,X)_{\mathrm{tors}}] = [\operatorname{Ext}^2(X,Y)_{\mathrm{cotors}}].$$

It follows from Theorem 8.1 that

$$\operatorname{Hom}(Y,X)(l) \cong \operatorname{Hom}(Y_l,X_l)_{\mathrm{tors}}, \quad \operatorname{Ext}^2(X,Y)_{\mathrm{cotors}}(l) \cong \operatorname{Ext}^2(X_l,Y_l)_{\mathrm{cotors}},$$

and so it suffices to show that, for M and N in $\mathcal{R}^+(\mathbb{F}_q;\mathbb{Z}_l)$,

$$[\operatorname{Hom}(N,M)_{\mathrm{tors}}] = [\operatorname{Ext}^2(M,N)_{\mathrm{cotors}}].$$

Because $\operatorname{Ext}^3 = 0$ in $\mathcal{R}^+(\mathbb{F}_q;\mathbb{Z}_l)$, $\operatorname{Ext}^2(M/M_{\mathrm{tors}},N)$ is divisible and

$$\operatorname{Ext}^2(M,N)_{\mathrm{cotors}} \cong \operatorname{Ext}^2(M_{\mathrm{tors}},N).$$

As $\operatorname{Hom}(N,M_{\mathrm{tors}}) \cong \operatorname{Hom}(N,M)_{\mathrm{tors}}$, this shows that it suffices to prove that

$$[\operatorname{Hom}(N,M)] = [\operatorname{Ext}^2(M,N)] \tag{29}$$

when M is torsion. In fact, we shall show that

$$\operatorname{Hom}(N,M) \cong \operatorname{Ext}^2(M,N)^* \tag{30}$$

when N is torsion-free and M is torsion. Here $*$ denotes $\operatorname{Hom}(-,\mathbb{Q}/\mathbb{Z})$. This implies (29) because an arbitrary N has a resolution by torsion-free objects (1.4b, 7.10).

Define

$$\operatorname{Ext}^i(M,N\otimes\mathbb{Q}/\mathbb{Z}) = \varinjlim_n \operatorname{Ext}^i(M,N^{(l^n)}).$$

Then

$$\operatorname{Ext}^1(M,N\otimes\mathbb{Q}/\mathbb{Z}) \cong \operatorname{Ext}^2(M,N)$$

and so we have to show that

$$\operatorname{Hom}(N,M) \cong \operatorname{Ext}^1(M,N\otimes\mathbb{Q}/\mathbb{Z})^* \tag{31}$$

(M torsion, N torsion-free).

Proof of (31) in the case $l \neq p$. We let $\bar{M}$ and $\bar{N}$ denote M and N regarded as $\mathbb{Z}_l$-modules. There is a perfect pairing of finite $\mathbb{Z}_l$-modules

$$\operatorname{Hom}(\bar{N},\bar{M})\times\operatorname{Hom}(\bar{M},\bar{N}\otimes\mathbb{Q}/\mathbb{Z}) \xrightarrow{\circ} \operatorname{Hom}(\bar{N},\bar{N}\otimes\mathbb{Q}/\mathbb{Z}) \cong \operatorname{End}(\bar{N})\otimes\mathbb{Q}/\mathbb{Z} \xrightarrow{\operatorname{Tr}\otimes 1} \mathbb{Q}/\mathbb{Z}.$$

This gives a perfect pairing of Γ-cohomology groups

$$\operatorname{Hom}(\bar{N},\bar{M})^{\Gamma}\times\operatorname{Hom}(\bar{M},\bar{N}\otimes\mathbb{Q}/\mathbb{Z})_{\Gamma} \to \mathbb{Q}/\mathbb{Z},$$

which can be identified with (31).

Proof of (31) in the case $l = p$. If F acts as an isomorphism on M, this can be proved as in the case $l \neq p$. Thus, we may assume that F act nilpotently on M, which therefore has a composition series whose quotients are isomorphic to $k =_{\mathrm{df}} (\mathbb{F}_q, 0)$.

Any extension of $N^{(p^n)}$ by $N^{(p^n)}$ splits as an extension of W-modules, and therefore is described by a σ-linear endomorphism α of $N^{(p^n)}$. The trace of α acting on $N^{(p^n)}$ is an element of W/p^nW, and the trace of this element is a well-defined element of $\mathbb{Z}_p/p^n\mathbb{Z}_p$. Thus, we have a homomorphism

$$\mathrm{Ext}^1(N^{(p^n)}, N^{(p^n)}) \to \mathbb{Z}_p/p^n\mathbb{Z}_p \cong p^{-n}\mathbb{Z}_p/\mathbb{Z}_p.$$

On passing to the limit, we obtain a homomorphism

$$t\colon \ \mathrm{Ext}^1(N, N \otimes \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}.$$

We claim that the pairings

$$\mathrm{Ext}^i(N, M) \times \mathrm{Ext}^{1-i}(M, N \otimes \mathbb{Q}/\mathbb{Z}) \to \mathrm{Ext}^1(N, N \otimes \mathbb{Q}/\mathbb{Z}) \xrightarrow{t} \mathbb{Q}/\mathbb{Z} \qquad (32)$$

are perfect for $i \geq 0$. When N is cyclic, $N = A/A \cdot \lambda$, and $M = k$, this can be proved directly: either both groups are zero, or the pairing is the trace pairing

$$\mathbb{F}_q \times \mathbb{F}_q \to \mathbb{F}_p.$$

Consider an exact sequence

$$0 \to N' \to N \to k \to 0.$$

If the pairings (32) are perfect for N, k, and $i \geq 0$, then the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(k,k) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}(N,k) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}(N',k) \arrow[r] \arrow[d] & \cdots \arrow[d, "\cong"] \\
0 \arrow[r] & \mathrm{Ext}^2(k,k)^* \arrow[r] & \mathrm{Ext}^1(k, N \otimes \mathbb{Q}/\mathbb{Z})^* \arrow[r] & \mathrm{Ext}^1(k, N' \otimes \mathbb{Q}/\mathbb{Z})^* \arrow[r] & \cdots
\end{tikzcd}$$

shows that they are perfect for N', k, and $i \geq 0$ (note that (24) shows that $\mathrm{Ext}^i(k, k)$ equals $k, k \oplus k, k$ respectively for $i = 0, 1, 2$). A similar argument applies when N/N' is a finite module on which F acts invertibly. Since an arbitrary torsion-free N can be realized as a submodule of a direct sum of cyclic modules (9.3), this proves that the pairings (32) are perfect when $M = k$. An induction argument on the length of M extends this to an arbitrary torsion M. The case $i = 0$ completes the proof of (31).

The main theorem for Weil extensions. Let Γ_0 be the subgroup of Γ generated by the Frobenius element (so $\Gamma_0 \cong \mathbb{Z}$). Inspired by Lichtenbaum 2002, we define "Weil" extension groups satisfying an analogue of the spectral sequence (1).

As we noted in 5.19, to give a motive over $\mathbb{F}_q$ amounts to giving a motive X over $\mathbb{F}$ together with an endomorphism π of X that represents its Frobenius germ. The π can be considered a descent datum on X, and the condition on it a continuity requirement. We now define the category of *Weil motives* over $\mathbb{F}_q$, $\mathcal{M}(\mathbb{F}_q; \mathbb{Z})_W$ to be the category of pairs (X, π) consisting of a motive X over $\mathbb{F}$ and an endomorphism π ("noncontinuous descent datum"). Equivalently, $\mathcal{M}(\mathbb{F}_q; \mathbb{Z})_W$ is the category of motives over $\mathbb{F}$ with a $\mathbb{Z}$-action. It contains $\mathcal{M}(\mathbb{F}_q; \mathbb{Z})$ as a full subcategory.

Similarly, we define the category $\mathcal{M}^+(\mathbb{F}_q;\mathbb{Z})_W$ of *effective Weil motives.* For effective motives X and Y over $\mathbb{F}_q$, let

$$\operatorname{Ext}^i(X,Y)_0 = \operatorname{Ext}^i_{\mathcal{M}^+(\mathbb{F}_q;\mathbb{Z})_W}(X,Y).$$

PROPOSITION 10.3. *For effective motives X, Y over $\mathbb{F}_q$, there is a spectral sequence*

$$H^i(\Gamma_0, \operatorname{Ext}^j_{\mathcal{M}^+(\mathbb{F};\mathbb{Z})}(\bar{X},\bar{Y})) \Longrightarrow \operatorname{Ext}^{i+j}(X,Y)_0$$

PROOF. The proof of Theorem 8.4 can be adapted to the present situation. □

LEMMA 10.4. *The groups $\operatorname{Ext}^i(X,Y)_0$ are finitely generated for all i, torsion for $i \geq 2$, and zero for $i \geq 3$.*

PROOF. The groups $\operatorname{Ext}^j_{\mathcal{M}^+(\mathbb{F};\mathbb{Z})}(\bar{X},\bar{Y})$ have composition series whose quotients are finitely generated $\mathbb{Z}$-modules or $\mathbb{F}$ with its natural Γ_0-action. Moreover, they are torsion for $i \geq 1$ and zero for $i \geq 2$. Thus the statement follows from 10.3. □

Let $f\colon \operatorname{Hom}(M,N)_0 \to \operatorname{Ext}^1(M,N)_0$ be the map rendering

$$\begin{array}{ccc} \operatorname{Hom}(M,N)_0 & \xrightarrow{f} & \operatorname{Ext}^1(M,N)_0 \\ \downarrow & & \uparrow \\ \operatorname{Hom}_{\mathcal{M}^+(\mathbb{F};\mathbb{Z})}(\bar{M},\bar{N})^{\Gamma_0} & \longrightarrow & \operatorname{Hom}_{\mathcal{M}^+(\mathbb{F};\mathbb{Z})}(\bar{M},\bar{N})_{\Gamma_0} \end{array}$$

commutative. Here the vertical maps arise from a spectral sequence (10.3) and the lower map is induced by the identity map on $\operatorname{Hom}(\bar{M},\bar{N})$ (equal to cup-product with the canonical generator of $H^1(\Gamma_0,\mathbb{Z}) \cong \mathbb{Z}$).

THEOREM 10.5. *For effective motives X and Y over $\mathbb{F}_q$, $z(f)$ is defined and*

$$q^{\chi(X,Y)} \cdot \zeta(X^\vee \otimes Y, s) \sim \pm z(f) \cdot [\operatorname{Ext}^2(X,Y)_0] \cdot (1-q^{-s})^{\rho(X,Y)} \qquad \text{as } s \to 0.$$

(Notations as in 10.1.)

PROOF. If either X or Y is finite, then the spectral sequence 10.3 coincides with 8.4 and so

$$\operatorname{Ext}^i(X,Y) \cong \operatorname{Ext}^i(X,Y)_0.$$

Now the same argument as in §8 proves that, for arbitrary X, Y, there are exact sequences

$$0 \to \operatorname{Ext}^i(X,Y)_0 \otimes \mathbb{Z}_l \to \operatorname{Ext}^i(X_l,Y_l) \to T_l \operatorname{Ext}^{i+1}(X,Y)_0 \to 0.$$

Lemma 10.4 shows that $\operatorname{Ext}^{i+1}(X,Y)_0$ has no l-divisible elements, and so these sequences reduce to isomorphisms

$$\operatorname{Ext}^i(X,Y)_0 \otimes \mathbb{Z}_l \cong \operatorname{Ext}^i(X_l,Y_l).$$

The spectral sequence (10.3) is compatible with the spectral sequences (7.8) and (7.12), and so the diagrams

$$\begin{array}{ccc} \operatorname{Hom}(X,Y)_0 \otimes \mathbb{Z}_l & \xrightarrow{f\otimes 1} & \operatorname{Ext}^1(X,Y)_0 \otimes \mathbb{Z}_l \\ \downarrow{\cong} & & \downarrow{\cong} \\ \operatorname{Hom}(X_l,Y_l) & \xrightarrow{f_l} & \operatorname{Ext}^1(X_l,Y_l) \end{array}$$

commute. Here f_l is the map in Theorem 9.1. Therefore, Theorem 10.5 follows from Theorem 9.1 and the product formula. □

REMARK 10.6. The advantage of statement 10.5 over 10.1 is that it has a natural extension to complexes of motives — see the sequel to this paper. Compare also Theorem 9.6.

Motivic cohomology. Let V be a smooth projective variety[11] over $\mathbb{F}_q$. In contrast to the pure case, V should define, not a mixed motive, but rather a complex which will not (in general) decompose when there is torsion, the torsion at p being particularly complicated. Nevertheless, for an $r \geq 0$ such that the truncated de Rham-Witt cohomology groups $H^i(V, W\Omega^{\leq r-1})$ are finitely generated over W, we make the ad hoc definition: $h^iV(r)$ is the isomotive $h_0^iV(r)$ endowed with the $\mathbb{Z}$-structure provided by the maps $H^i(V, \mathbb{Z}_l(r)) \to H^i(V, \mathbb{Q}_l(r))$ (étale cohomology for $l \neq p$ and crystalline cohomology $l = p$). Define the (Weil) motivic cohomology groups by[12]

$$H^j_{\mathrm{mot}}(V, \mathbb{Z}(r)) = \oplus_i \operatorname{Ext}^i(\mathbb{1}, h^{j-i}V(r))_0. \tag{33}$$

Let

$$H^{\bullet}_{\mathrm{mot}}(V, \mathbb{Z}(r)) = \cdots \to H^j_{\mathrm{mot}}(V, \mathbb{Z}(r)) \xrightarrow{f^j} H^{j+1}_{\mathrm{mot}}(V, \mathbb{Z}(r)) \to \cdots$$

be the complex with

$$\begin{tikzcd}
H^j_{\mathrm{mot}}(V, Z(r)) \arrow[d, "f^j"'] \arrow[dr, phantom, "="] & \operatorname{Ext}^0(\mathbb{1}, h^j(V)(r))_0 \oplus \arrow[dr, "f"] & \operatorname{Ext}^1(\mathbb{1}, h^{j-1}(V)(r))_0 \oplus \arrow[dr, "f"] & \operatorname{Ext}^2(\mathbb{1}, h^{j-2}(V)(r))_0 \\
H^{j+1}_{\mathrm{mot}}(V, Z(r)) & \operatorname{Ext}^0(\mathbb{1}, h^{j+1}(V)(r))_0 \oplus & \operatorname{Ext}^1(\mathbb{1}, h^{j}(V)(r))_0 \oplus & \operatorname{Ext}^2(\mathbb{1}, h^{j-1}(V)(r))_0
\end{tikzcd}$$

The maps f at right are as in Theorem 10.5.

THEOREM 10.7. *Let V be a smooth projective variety over $\mathbb{F}_q$, and let $r \geq 0$ be such that the groups $H^i(V, W\Omega^{\leq r-1})$ are finitely generated over W.*

(a) *The groups $H^i_{mot}(V, \mathbb{Z}(r))$ are finitely generated abelian groups.*
(b) *The alternating sum $\sum(-1)^i r_i$ of the ranks r_i of the groups $H^i_{mot}(V, \mathbb{Z}(r))$ is zero.*
(c) *The order of the zero of $\zeta(V, s)$ at $s = r$ is equal to the secondary Euler characteristic $\rho(V, r) =_{df} \sum(-1)^i i r_i$.*
(d) *The cohomology groups of the complex $(H^{\bullet}_{mot}(V, \mathbb{Z}(r)), f)$ are finite, and the alternating product of their orders $\chi^{\times}(V, \mathbb{Z}(r))$ satisfies*

$$\zeta(V, s) \sim \pm \chi^{\times}(V, \mathbb{Z}(r)) \cdot q^{\chi(V, \mathcal{O}, r)} \cdot (1 - q^{r-s})^{\rho} \text{ as } s \to r$$

[11] If the category $\mathcal{M}(\mathbb{F}; \mathbb{Q})$ at the start of this section is taken to be as in 5.26 (case (b)), then V must be taken to be an abelian variety.

[12] According to Deligne 1994, 3.2.1, the motivic cohomology groups ($\mathbb{Q}$-coefficients) should be the final term of a spectral sequence

$$E_2^{ij} = \operatorname{Ext}^i(\mathbb{1}, h^j(V)) \Longrightarrow H^{i+j}_{\mathrm{mot}}(V, \mathbb{Q}(r)).$$

Over $\mathbb{F}_q$ it is natural to expect this also with $\mathbb{Z}$-coefficients and (following Lichtenbaum's ideas) replace the Exts with the Weil Exts:

$$E_2^{ij} = \operatorname{Ext}^i(\mathbb{1}, h^j(V))_0 \Longrightarrow H^{i+j}_{\mathrm{mot}}(V, \mathbb{Z}(r)).$$

If this spectral sequence degenerates, then we arrive at (33). In the sequel to this paper, we shall take a less ad hoc approach.

where

$$\chi(V,\mathcal{O},r) = \sum_{\substack{0\le i\le r\\ 0\le j\le \dim V}} (-1)^{i+j}(r-i)\cdot \dim_{\mathbb{F}_q} H^j(X,\Omega^i).$$

PROOF. Immediate consequence of Theorem 10.5 and Milne 1986a, 4.1 (the condition on the de Rham-Witt groups implies that the numbers $d^i(r-1)$ in the second reference are zero). □

REMARK 10.8. When $r=0$, the condition on the de Rham-Witt groups is satisfied vacuously, and the groups $H^i_{\mathrm{mot}}(V,\mathbb{Z}(0))$ coincide with the Weil-étale groups $H^i_W(V,\mathbb{Z})$ of Lichtenbaum 2002. Thus, in this case the theorem coincides with the smooth projective case of ibid., Theorem 8.2.

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