GENERIC PROPERTIES OF THE ADJUNCTION MAPPING FOR SINGULAR SURFACES AND APPLICATIONS

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INTRODUCTION. Let Σ be an irreducible surface embedded in some complex projective space $\mathbb{P}^r$ and let $\eta : S \longrightarrow \Sigma$ be the normalization of Σ with L the pullback to S under η of $\mathcal{O}_{\mathbb{P}^r}(1)$. Let $\pi : S' \longrightarrow S$ be the minimal desingularization of S and let $\mathcal{K}_S = \pi_* K_S$ be the Grauert-Riemenschneider canonical sheaf of S. In [A-S1] the first and last authors showed that $K_{S'} \otimes L'$, $L' = \pi^* L$, is nef and big except when (S',L') and (S,L) are of a very restricted type. In the case when $K_{S'} \otimes L'$ is nef, $h^0(K_{S'} \otimes L') = h^0(\mathcal{K}_S \otimes L) \neq 0$ (see (0.7)) and it makes sense to look at the meromorphic map $\phi : S \longrightarrow \mathbb{P}^N$ associated to $\Gamma(\mathcal{K}_S \otimes L)$. We call this the <u>small adjunction map</u> (for the reason behind this name see §1). The meromorphic map associated to $\Gamma(K_S \otimes L)$ is called the <u>big adjunction map</u> or the <u>adjunction map</u>, for short.

In §1 we use Reider's theorem ([B],[R]) to show that ϕ is birational if $K_{S'} \otimes L'$ is nef with $c_1(L)^2 \geq 10$ or $c_1(L)^2 \geq 9$ and $L' \not\approx 3D$ for some effective divisor D such that $D \cdot D = 1$. The analogue of this for a smooth S goes back to Van de Ven [V].

In §2 we prove a number of inequalities that are standard tools in the smooth theory [S1]. E.g., if the Kodaira dimension of S' is non-negative and $c_1(L)^2 \geq 10$, then (see 2.1)

$$c_1(K_{S'})^2 + 2g(L) \geq d + 2(p_g(S') - q(S'))$$

where $p_g(S') = h^{2,0}(S')$ and $q(S') = h^{1,0}(S')$ are the geometric genus and the irregularity of S' and $g(L)$ is the sectional genus of L.

In §3 we give some simple applications. For the main one we derive a result, on when $|L|$ contains even one smooth hyperelliptic curve, that generalizes results of Sommese [S1] and Van de Ven [V]. The result is the following (see (3.1)).

THEOREM. *Assume* $c_1(L)^2 \geq 10$ *or* $c_1(L)^2 = 9$ *and* $L' \not\approx 3D$ *for some effective divisor* D *with* $D \cdot D = 1$ *where* (S,L) *and* (S',L') *are as in the first paragraph of this introduction. Then if there is a smooth hyperelliptic curve* $C \in |L|$ *it follows that* $h^{2,0}(S') = 0$, $d \geq g(L) + 2$ *and either*

i) $h^{1,0}(S') > 0$, $h^0(L) = h^0(L') = 4$;

ii) (S,L) *is a cone or a scroll; or*

iii) (S',L') *is a conic bundle over a smooth curve.*

□

We would note that in [A-S2] the first and last authors give very precise results on the set where $K_S \otimes L$ is spanned by its global sections.

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§0. Background material

We work over the complex number field $\mathbb{C}$. By variety we mean an irreducible and reduced projective scheme X of dimension n. We denote its structure sheaf by $\mathcal{O}_X$. For any coherent sheaf $\mathcal{G}$ on X, $h^i(\mathcal{G})$ denotes the complex dimension of $H^i(X,\mathcal{G})$.

If X is normal, the canonical sheaf K_X is defined to be $j_*K_{Reg}(X)$ where $j : Reg(X) \longrightarrow X$ is the inclusion of the smooth points of X and $K_{Reg(X)}$ is the canonical sheaf of the holomorphic n-forms. Note that K_X is a line bundle if X is Gorenstein.

Let $\mathcal{L}$ be a line bundle on X. $\mathcal{L}$ is said to be numerically effective, nef for short, if $\mathcal{L} \cdot C \geq 0$ for each irreducible curve C on S, and in this case $\mathcal{L}$ is said to be big if $c_1(\mathcal{L})^n > 0$, where $c_1(\mathcal{L})$ is the first Chern class of $\mathcal{L}$. We shall denote by $|\mathcal{L}|$ the complete linear system associated to $\mathcal{L}$ and by $\Gamma(\mathcal{L})$ the space of its global sections. We say that $\mathcal{L}$ is spanned if it is spanned by $\Gamma(\mathcal{L})$.

(0.1) We fix some more notation.

$\sim$ (resp. $\approx$) the numerical (resp. linear) equivalence of divisors;

$\chi(\mathcal{L}) = \Sigma(-1)^i h^i(\mathcal{L})$, the Euler characteristic of a line bundle $\mathcal{L}$;

$\kappa(X)$, the <u>Kodaira dimension</u> of X, that is the Kodaira dimension of a nonsingular model of X.

<u>Abuses.</u> Line bundles and divisors are used with little or no distinction. Hence we shall freely switch from the multiplicative to the additive notation and viceversa.

(0.2) Throughout the paper, S always denotes an irreducible projective normal surface. Let $\pi : S' \longrightarrow S$ be the <u>minimal desingularization</u> of S, i.e. S' is the unique desingularization of S which is minimal in the sense that the fibres of π contain no smooth rational curves C satisfying $C^2 = -1$. If L is a line bundle on S we will denote by L' the inverse image, π^*L. We shall briefly say that (S',L') is the minimal desingularization of the pair (S,L). If D is a (Weil) divisor we will denote by D' the proper transform of D. For every Weil divisor D and line bundle L on S the intersection $L \cdot \mathcal{O}_S(D) = L \cdot D = L' \cdot D'$ is well defined.

(0.3) Let $\pi : S' \longrightarrow S$ be the minimal resolution of the singularities of S and let $\Delta = \pi^{-1}(\mathrm{Irr}(S))$, where $\mathrm{Irr}(S)$ denotes the <u>irrational locus</u> of S. We say that (S,L) is a-<u>minimal</u> if there are no smooth rational curves E on $S' - \Delta$, with $E \cdot E = -1$ and $\pi^*L \cdot E = 0$. Note that the pair (S',L') in (0.2) is clearly a-minimal if L is ample; this allows us to apply to (S',L') the results of [A-S1].

(0.4) The genus formula. Let L be a nef and big line bundle on a normal surface S. Then the sectional genus, $g(L)$, of L is defined by the equality $2g(L) - 2 = (K_S + L) \cdot L$.

It can be easily seen that $g(L)$ is an integer. Furthermore if there exists an irreducible reduced curve C in $|L|$, $g(L)$ is simply the arithmetic genus $p_a(C) = 1 - \chi(\mathcal{O}_C)$ of C. Note also that $g(L) = g(L')$, where (S',L') is the minimal desingularization of (S,L).

(0.5) Let S be a normal surface and let L be a nef and big line bundle on S. We say that the (generically) polarized pair (S,L) is geometrically ruled if S is a $\mathbb{P}^1$-bundle, $p : S \longrightarrow R$, over a nonsingular curve R and the restriction L_f of L to a fibre f of p is $\mathcal{O}_f(1)$. We say that (S,L) is a scroll (resp. a conic bundle) over a nonsingular curve R if there is a surjective morphism with connected fibres $p : S \longrightarrow R$, with the property that L is relatively ample with respect to p and there exist some $k > 0$ and some very ample line bundle M on R such that $(K_S \otimes L^2)^k \approx p^*M$ (resp. $(K_S \otimes L)^k \approx p^*M$); here $K_S^k = (K_S^{\otimes k})^{**}$.

□

The following result will be used several times through the paper.

(0.6) LEMMA. *Let* S *be a nonsingular surface and let* L *be a nef and big line bundle on* S. *Assume* (S,L) *is* a-*minimal*. *Then the following are equivalent*

(0.6.1) $h^0(K_S + L) \neq 0$;

(0.6.2) $h^0((K_S + L)^N) \neq 0$ *for some* $N > 0$;

(0.6.3) $K_S + L$ *is nef*;

(0.6.4) $g(L) \geq 1$ *and* $(K_S + L)^2 \geq 0$.

Proof. The equivalence between (0.6.2) and (0.6.3) is proved in [A-S1], (2.5), while (0.6.1) ⇒ (0.6.2) and (0.6.3) ⇒ (0.6.4) are clear. So let us prove that (0.6.4) implies (0.6.1). Now we have

$$h^0(K_S + L) = \chi(K_S + L) = g(L) - 1 + \chi(\mathcal{O}_S).$$

Hence if (0.6.1) would be false, then $\chi(\mathcal{O}_S) \leq 0$. Let $g(L) = 1$. Therefore $(K_S + L) \cdot L = 0$ and the Hodge index theorem combined with $(K_S + L)^2 \geq 0$ gives $K_S \sim -L$ whence $\chi(\mathcal{O}_S) > 0$, a contradiction. So $g(L) > 1$ and $\chi(\mathcal{O}_S) < 0$. It thus follows that S is ruled; further we claim that $g(L) = q(S)$. Indeed, since $p_g(S) = 0$, we have

$$0 = h^0(K_S + L) = \chi(K_S + L) = \chi(L^{-1}) = g(L) - q(S).$$

Let $d = L \cdot L$. Then the assumption $(K_S + L)^2 \geq 0$ and genus formula (0.4) yield

$$K_S^2 + 4g(L) - 4 \geq d.$$

Therefore, since S is ruled and $g(L) = q(S) > 1$,

$$d \leq 8(1 - q(S)) + 4q(S) - 4 = 4 - 4q(S)$$

a contradiction.

□

(0.7) COROLLARY. Let S be a normal surface, L a nef and big line bundle on S and let (S',L') be the minimal desingularization of (S,L). Then $K_{S'} + L'$ is nef if and only if $h^0(K_S + L) >$ length $(K_S/\mathcal{K}_S)$, where $\mathcal{K}_S$ denotes the Grauert-Riemenschneider canonical sheaf.

Proof. Look at the exact sequence

$$0 \longrightarrow \mathcal{K}_S \otimes L \longrightarrow K_S \otimes L \longrightarrow \mathcal{S} \otimes L \longrightarrow 0$$

and note $H^1(\mathcal{K}_S \otimes L) = (0)$ by the Grauert-Riemenschneider vanishing theorem and $h^0(\mathcal{S} \otimes L) =$ length $(K_S/\mathcal{K}_S)$. Now the statement is an immediate consequence of Lemma (0.6).

□

In section 1 we shall use Reider's result for separating general points in the following form

(0.8) THEOREM (Reider, [R]). *Let* L *be a nef and big line bundle on a smooth surface* S. If $L \cdot L \geq 9$ *and the map associated to* $\Gamma(K_S + L)$ *is not a birational morphism, then there exists an effective divisor* D *on* S *such that*

$$L \cdot D = 0, \quad D^2 = -1;$$
$$L \cdot D = 0, \quad D^2 = -1 \text{ or } 0;$$
$$L \cdot D = 2, \quad D^2 = 0; \text{ or}$$
$$L \sim 3D, \quad D^2 = 1.$$

(0.9) *Castelnuovo's bound.* Let X be a n-dimensional normal variety and let L be a big and spanned line bundle on X. Further assume that the map $\varphi : X \longrightarrow \mathbb{P}^N$ associated to $\Gamma(L)$ is generically one to one. Let C be a smooth curve obtained as transversal intersection of $n - 1$ general members of $|L|$ and write $d = L^n$. Then

$$g(C) \leq \left[\frac{d-2}{N-n}\right](d - N + n - 1 -(\left[\frac{d-2}{N-n}\right] - 1)\frac{N-n}{2}).$$

Indeed C is nothing but the normalization of $C' = \varphi(C)$, so $\deg C' = d$ and inequality above is a consequence of the usual Castelnuovo's bound for the embeddings $C' \subset \varphi(X) \subset \mathbb{P}^N$.

□

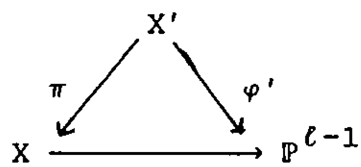
Finally, let us give the following general results we use in the sequel.

(0.10) LEMMA (Nef and big degree Lemma). *Let* X *be a normal variety of dimension* n *and let* $\mathcal{L}$ *be a nef and big line bundle on* X. *Denote by* φ *the rational map associated to* $|\mathcal{L}|$ *and let* $\#_s$ *be the sheet number of the Stein factorization of* φ . *Then*

(0.10.1) $c_1(\mathcal{L})^n \geq \#_s$ (deg $\varphi(X)$);

(0.10.2) $c_1(\mathcal{L})^n \geq 2(h^0(\mathcal{L}) - n)$ *if* $\kappa(X) \geq 0$ *and* dim $\varphi(X) = n$.

Proof. Look at a resolution of the fundamental locus of φ



where $\ell = h^0(\mathcal{L})$. Then $\mathcal{L}' = \pi^*\mathcal{L} \approx \mathcal{M} + F$ where $\mathcal{M}$ is spanned and φ' is the morphism associated to $|\mathcal{M}|$. Further we can assume X' to be nonsingular. Take the Remmert-Stein factorization $s \circ r : X' \longrightarrow Y \longrightarrow \mathbb{P}^{\ell-1}$ of φ'. Therefore $\mathcal{M} \approx r^*M$ for some ample line bundle M on Y. Let $\#_s$ be the degree of s and $m = \dim Y$. Then

*) $r^*c_1(M)^m = c_1(\mathcal{M})^m$ and $c_1(M)^m[Y] = \#_s(\deg s(Y))$.

If $m = n$,

$$**)\quad c_1(\mathcal{L}')^n = (\mathcal{M} + F) \cdot c_1(\mathcal{L}')^{n-1} \geq \mathcal{M} \cdot c_1(\mathcal{L}')^{n-1} \geq c_1(\mathcal{M})^n$$

and *), **) yield (0.10.1). If $m < n$,

$$***)\qquad c_1(\mathcal{L}')^n \geq c_1(\mathcal{M})^m \cdot c_1(\mathcal{L}')^{n-m} \geq c_1(\mathcal{M})^m$$

and (0.10.1) follows now from *), ***).

As a consequence of (0.10.1) we get

$$c_1(\mathcal{L})^n \geq \#_s(h^0(\mathcal{L}) - n).$$

Whenever $\#_s \geq 2$, (0.10.2) is proved. If $\#_s = 1$, $\varphi(X)$ has a desingularization of non-negative Kodaira dimension. It thus follows that the general surface section $S \subset \mathbb{P}^{\ell+1-n}$ has a desingularization of non-negative Kodaira dimension. Now a standard argument shows that $\deg(S) \geq 2(N-1) = 2(h^0(\mathcal{L}) - n)$ (see also [L-S], §0).

(0.11) LEMMA. Let $\mathcal{L}, L$ be two line bundles on an irreducible variety X. Assume that L is spanned and big and $h^0(\mathcal{L}) \geq 2$. Then, given a general element $D \in |L|$, the restriction

$$\Gamma(\mathcal{L}) \longrightarrow \Gamma(\mathcal{L}_D)$$

has an image of dimension ≥ 2.

Proof. Look at the exact sequence

$$0 \longrightarrow \mathcal{L} \otimes L^{-1} \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}_D \longrightarrow 0.$$

If the statement is not true, then there exists a non-zero element $t \in \Gamma(\mathcal{L} \otimes L^{-1})$. Consider the restriction map

$$\delta : \Gamma(\mathcal{L} \otimes L^{-1}) \longrightarrow \Gamma(\mathcal{L} \otimes L^{-1}_{|D}).$$

Since L is spanned and big we can find non-trivial $t_1, t_2 \in h^0(L)$ whose restrictions on D are not multiples of one another. If $\delta(t) \neq 0$, then $t_1 \otimes t$, $t_2 \otimes t$ are not multiples of one another on D and we are done. Otherwise we would have $\delta(t) \otimes (t_1 - t_2)_D = 0$ in $\Gamma(\mathcal{L}_D)$ after possibly multiplying the t_i by non zero constants. Since $(t_1 - t_2)_D \neq 0$ by the above, this leads to a contradiction; here we use that D is irreducible since it is general. Hence δ is the zero-map and therefore $\Gamma(\mathcal{L} \otimes L^{-2}) = \Gamma(\mathcal{L} \otimes L^{-1}) \neq (0)$. By repeating the same argument we find that $\Gamma(\mathcal{L} \otimes L^{-m}) \neq (0), m >> 0$, again a contradiction.

□

For any further background material we refer to [A-S1] and [A-S2].

§1. *The birationality theorem*

Let L be an ample and spanned line bundle on a normal surface S. Let (S',L') denote the minimal desingularization of (S,L) and let $\mathcal{K}_S = \pi_* K_{S'}$ be the Grauert-Riemenschneider canonical sheaf. Then the following can be proved

(1.1) THEOREM. *Let* (S,L), (S',L') *be as above with* $K_{S'} + L'$ *nef and big*. *Further assume that* $\Gamma(L)$ *gives a generically one to one map*. *If* $c_1(L)^2 \geq 9$ *then* $\Gamma(\mathcal{K}_S + L)$ *gives a birational map unless possibly* $L' \sim 3D$, *for some effective divisor* D *with* $D \cdot D = 1$.

Proof. From Lemma (0.6) we see that $h^0(K_{S'} + L') = h^0(\mathcal{K}_S + L) > 0$. Hence looking at the meromorphic maps ϕ, Ψ associated to $\Gamma(\mathcal{K}_S + L)$, $\Gamma(K_{S'} + L')$ respectively and from the commutative diagram

$$\begin{array}{ccc} S' & \xrightarrow{\pi} & S \\ & \overset{\phi}{\searrow} & \downarrow \Psi \\ & & \mathbb{P}^N \end{array}$$

we see that it suffices to work with ϕ on S'. To go on assume that Ψ is not birational. Then given a general point x of S' there is a general point y of S' such that the morphism

$$\Gamma(K_{S'} \otimes L') \longrightarrow K_{S'} \otimes L' \otimes (\mathbb{C}_x \oplus \mathbb{C}_y)$$

is not onto, where $\mathbb{C}_x \oplus \mathbb{C}_y$ is the skyscraper sheaf $\mathcal{O}_{S'}/m_x \oplus m_y$. By Reider's theorem (0.8) there exists on S' an effective divisor D passing through x,y such that $L' \cdot D = 0,1$ or 2 or $L' \cdot D = 3$ and $D \cdot D = 1$.

The case $L' \cdot D = 0$ can be easily ruled out. Indeed, if $L' \cdot D = 0$, then $\pi(D)$ is a finite set. Therefore x,y belong to some positive dimensional fibre of π, so that either x nor y is a general point; a contradiction.

The case $L' \cdot D = 3$ with $D \cdot D = 1$ gives $(L' - 3D) \cdot D = 0$. Hence $L' \sim 3D$ or $(L' - 3D)^2 < 0$ by the Hodge index theorem; since

$$(L' - 3D)^2 = L'^2 - 6L' \cdot D + 9D \cdot D = L'^2 - 9 \geq 0$$

it has to be $L' \sim 3D$.

Finally, let $L' \cdot D = 2$ or 1. Since L is ample and spanned and $\Gamma(L)$ gives a generically one to one map it thus follows that $\pi(D)$ is either a smooth line or a (possibly singular) conic. Then the proper transform D' of $\pi(D)$ (or of a reduced component of $\pi(D)$) under π is a nonsingular rational curve. Since x,y are general points we find in this way an uncountable set of distinct nonsingular rational curves on S'. Now the general theory on the Hilbert scheme says that $\mathrm{Hilb}(D',S')$ has countably many components. Therefore there is an irreducible component T of the Hilbert scheme $\mathrm{Hilb}(D',S')$ with a subset corresponding to uncountably many of these singular rational curves. Fix a curve $\ell \in T$. Then $\ell^2 \geq 0$ on S' and hence $H^1(S',N_\ell) = (0)$ where N_ℓ is the normal bundle of ℓ in S'. It thus follows that there exist irreducible projective varieties $\mathscr{C}$ and $\mathscr{X}$ with $\mathscr{X} \subset S' \times \mathscr{C}$ and if $p : \mathscr{X} \longrightarrow \mathscr{C}$ and $q : \mathscr{X} \longrightarrow S'$ denote the maps induced by the product projections, then p is a flat surjection and q identifies $p^{-1}(c)$ with ℓ for a general point $c \in \mathscr{C}$. Therefore we have $K_{S'} \cdot \ell \leq -2$ and hence $(K_{S'} + L') \cdot \ell \leq 0$. Since $K_{S'} + L'$ is nef and big, this leads to a contradiction by the Hodge index theorem.

(1.2) REMARK. Note that $K_S + L$ could also be considered to obtain an analogous result to that of the Theorem above. However, the exact sequence

$$0 \longrightarrow \mathscr{K}_S \otimes L \longrightarrow K_S \otimes L \longrightarrow \mathscr{S} \otimes L \longrightarrow 0$$

gives an inclusion $\Gamma(\mathscr{K}_S \otimes L) \subset \Gamma(K_S \otimes L)$, so that the birational results proved for $\mathscr{K}_S \otimes L$ imply birationality results for the adjunction mapping associated to $\Gamma(K_S + L)$.

§2. Some inequalities

The first two theorems we prove below generalize some results contained in [S4], §3.

The following is a consequence of Theorem 1.1.

(2.1) THEOREM. *Let* L *be a nef and big line bundle on a normal surface* S *and let* (S',L') *be the minimal desingularization of* (S,L). *Suppose* $K_{S'} + L'$ *to be nef and big. Further assume* $\kappa(S) \geq 0$ *and let* $c_1(L)^2 \geq 10$ *or* $c_1(L)^2 \geq 9$ *and* $L' \not\approx 3D$, D *effective divisor with* $D \cdot D = 1$. *Then*

$$(K_{S'} + L')^2 \geq 2(g(L) - q(S') + p_g(S') - 2)$$

or, equivalently,

$$K_{S'}^2 + 2g(L) \geq 2(p_g(S') - q(S')) + d.$$

Proof. Under the hypotheses made the map Ψ associated to $\Gamma(K_{S'} + L')$ is birational by Theorem (1.1). Then Lemma (0.10.2) yields

$$(K_{S'} + L')^2 \geq 2(h^0(K_{S'} + L') - 2)$$

Now

$$h^0(K_{S'} + L') = \chi(K_{S'} + L') = g(L) - q(S') + p_g(S')$$

so we are done.

(2.2) THEOREM. _Let_ L _be a nef and big line bundle on a normal surface_ S _and let_ (S',L') _be the minimal desingularization of_ (S,L). _Assume_ $K_{S'} + L'$ _to be nef and big_. _Then_

(2.2.1) $(K_{S'} + L')^2 \geq g(L) - q(S') + p_g(S') - 2;$

(2.2.2) $(K_{S'} + L')^2 \geq g(L) + q(S') - 2.$

Proof. Look at the map Ψ associated to $\Gamma(K_{S'} + L')$. Then Lemma (0.10.1) gives us

$$(K_{S'} + L')^2 \geq \operatorname{cod} \Psi(S') + 1 = h^0(K_{S'} + L') - 2$$

and again $h^0(K_{S'} + L') = \chi(K_{S'} + L') = g(L) - q(S') + p_g(S')$, this leading to (2.2.1).

To prove (2.2.2) note that there exists an effective member $C' \in |K_{S'} + L'|$ in view of Lemma (0.6). Then the exact sequence

$$0 \longrightarrow K_{S'} \longrightarrow 2K_{S'} \otimes L' \longrightarrow \omega_{C'} \longrightarrow 0,$$

where $\omega_{C'}$ denotes the dualizing sheaf of C', gives a surjective morphism

$$H^0(C',\omega_{C'}) \longrightarrow H^1(S',K_{S'}) \cong H^1(S',\mathcal{O}_{S'})$$

since $H^1(S',2K_{S'}+L') = (0)$ by the Kawamata-Viehweg vanishing theorem. Now, $h^1(-L) = 0$ since L is nef and big, so that $h^0(\mathcal{O}_{C'}) = h^0(\mathcal{O}_S) = 1$ and hence $h^0(\omega_{C'}) = h^1(\mathcal{O}_{C'}) = g(K_{S'} + L')$. Therefore

$$g(K_{S'} + L') \geq q(S')$$

so by the genus formula we find

$$2q(S') - 2 \leq 2g(K_{S'} + L') - 2 = (2K_{S'} + L')\cdot(K_{S'} + L') = \\ = 2(K_{S'} + L')^2 - L' \cdot (K_{S'} + L'),$$

that is

$$(K_{S'} + L')^2 \geq g(L) + q(S') - 2.$$

(2.3) COROLLARY. <u>Let</u> (S,L), (S',L') <u>be as in Theorem (2.2) and let</u> $d = L \cdot L$. <u>Further assume that</u> $q(S') > 0$ <u>and</u> $p_g(S') = 0$. <u>Then</u>

$$g(L) \geq d/3 + 3q(S') - 2.$$

<u>Proof.</u> From [S1] (0.8.2) we know that $K_{S'} \cdot K_{S'} \leq 8(1 - q(S'))$ and the genus formula reads

$$(K_{S'} + L')^2 + d = K_{S'} \cdot K_{S'} + 4g(L) - 4.$$

Hence

$$8(1 - q(S')) \geq (K_{S'} + L')^2 + d - 4g(L) + 4.$$

By combining the inequality above with (2.2.2) we get the result.

(2.4) REMARK. Note that whenever S' is birationally ruled and $|L'|$ contains a smooth curve C which meets a general fibre of the ruling $S' \longrightarrow R$, R nonsingular curve, in t points, then the Hurwitz theorem gives us

$$g(L) \geq 1 + t(q(S') - 1).$$

However, such an inequality is usually weaker than (2.3) for t around 3.

(2.5) COROLLARY. _Let_ (S,L), (S',L') _be as in Corollary_ (2.3). _Further assume_ $g(L) \leq 6$. _Then_ $q(S') \leq 2$. _If_ $q(S') = 2$, _then either_ $g(L) = 5$ _with_ $d \leq 3$ _or_ $g(L) = 6$ _with_ $d \leq 6$. _Furthermore if_ L _is spanned and_ $\Gamma(L)$ _gives a generically one to one map then_ $q(S') = 2$ _implies that_ $h^0(L) = 4, g(L) = 6$ _and_ $d = 5$ _or_ 6.

Proof. Indeed $q(S') \geq 3$ implies $g(L) > 7$ by (2.3) above, so that $q(S') \leq 2$. Again (2.3) and $q(S') = 2$ imply $g(L) = 5$ or 6 with the stated bound for d.

If L is spanned and $\Gamma(L)$ gives a generically one to one map then Castelnuovo's bound (0.9) shows $d \geq 4$ if $g(L) = 5$. Therefore $g(L) = 6$. By combining (2.3) and (0.9) we find $d = 5$ or 6 and $h^0(L') = h^0(L) = 4$.

□

Finally let us give an easy but useful generalization of some of Sommese's results contained in [S3].

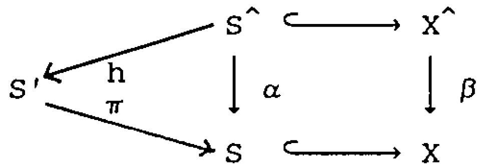
(2.6) THEOREM. Let X be an irreducible variety of dimension n. Let L be a spanned and big line bundle on X. Let $S \subset X$ be a general surface section obtained as transversal intersection of $n - 2$ general members of $|L|$ and let (S',L') be the minimal desingularization of (S,L). If $\kappa(X) \geq 0$ then one has

(2.6.1) $K_{S'} \cdot K_{S'} \geq (n - 3)K_{S'} \cdot L'_{S'} + (n - 2)L'_{S'} \cdot L'_{S'}$

(2.6.2) $K_{S'} \cdot L'_{S'} \geq (n - 2)L'_{S'} \cdot L'_{S'}$

Note $K_S \cdot L_S = K_{S'} \cdot L'_{S'}$ and $L'_{S'} \cdot L'_{S'} = L_S \cdot L_S$. Furthermore if either inequality is an equality then $\kappa(X) = 0$.

Proof. By using Bertini's type theorems and the fact that S is general one sees that there exists a commutative diagram



where α, β are desingularizations, $S^\wedge = \beta^{-1}(S)$, $\alpha = \beta_{|S^\wedge}$ and it factorizes through some morphism h since π is the minimal desingularization of S. Note that by hypothesis $K_{X^\wedge}$ is $\mathbb{Q}$-effective. Note also that $\kappa(S') = 2$ since $\kappa(X) \geq 0$, and hence $K_{S'} + L'$ is nef and big by [A-S1]. Let $L^\wedge = \beta^* L$. From the commutativity of the diagram it thus follows that $L^\wedge_{S^\wedge} \approx h^* L'_{S'}$. Then we can compute:

$$0 \leq h^*(K_{S'} + L'_{S'}) \cdot (K_{X^\wedge} \cdot L^{\wedge\, n-2}) =$$

$$= h^*(K_{S'} + L'_{S'}) \cdot (K_{S^\wedge} - (n-2)L^\wedge_{S^\wedge}) =$$

$$= (K_{S'} + L'_{S'}) \cdot K_{S'} - (n-2)(K_{S'} + L'_{S'}) \cdot L'_{S'} =$$

$$= K_{S'} \cdot K_{S'} - (n-2)L'_{S'} \cdot L'_{S'} - (n-3)K_{S'} \cdot L'_{S'}$$

which leads to (2.6.1). Similarly one has

$$K_{S'} \cdot L'_{S'} = h^* K_{S'} \cdot L^\wedge_{S^\wedge} = K_{S^\wedge} \cdot L^\wedge_{S^\wedge} = (K_{X^\wedge|S^\wedge} + (n-2)L^\wedge_{S^\wedge}) \cdot L^\wedge_{S^\wedge} \geq (n-2)L^\wedge_{S^\wedge} \cdot L^\wedge_{S^\wedge}$$

that is (2.6.2). To prove the last part of the statement, note that the equality in (2.6.1) or (2.6.2) gives respectively

$$K_{X^\wedge}|_{S^\wedge} \cdot h^*(K_{S'} + L'_{S'}) = 0$$

or

$$K_{X^\wedge}|_{S^\wedge} \cdot \hat{L}_{S^\wedge} = 0.$$

Now if $h^0(K_{X^\wedge}^N) \geq 2$ for some $N \geq 1$, then $h^0((K_{X^\wedge}|_{S^\wedge})^N) \geq 2$ by Lemma (0.11). Therefore since $h^*(K_{S'} + L'_{S'})$ and $L'_{S'}$ are nef and big a straightforward check shows that the intersection numbers $K_{X^\wedge}|_{S^\wedge} \cdot h^*(K_{S'} + L'_{S'})$ and $K_{X^\wedge}|_{S^\wedge} \cdot \hat{L}_{S^\wedge}$ must be positive. It thus follows that $h^0(K_{X^\wedge}^N) \leq 1$ for all $N > 0$, whence $\kappa(X) = 0$.

□

The following consequence of the Theorem above is a slight generalization of (0.5.1) in [L-S].

(2.7) COROLLARY. <u>Let</u> (X,L) <u>be as in Theorem</u> (2.6) <u>and let</u> $d = L^{\cdot N}$. <u>Then</u>

$$d \leq \frac{2(g(L)-1)}{n-1}$$

<u>with equality only if</u> $\kappa(X) = 0$.

<u>Proof.</u> From (2.6.2) and the genus formula we get

$$2g(L)-2 = K_{S'}\cdot K_{S'} + L'_{S'}\cdot L'_{S'} \geq (n-1)L'_{S'}\cdot L'_{S'} = (n-1)d.$$

§3 An application to hyperelliptic hyperplane sections

First of all note that it is equivalent to consider pairs (S,L) where S is a normal surface with L an ample and spanned line bundle such that $\Gamma(L)$ gives a generically one to one map and pairs (S,L) where S is the normalization $\eta : S \longrightarrow \Sigma$ of an irreducible surface $\Sigma \subset \mathbb{P}^r$ and $L \approx \eta^*\mathcal{O}(1)$. Indeed, $\eta^*\mathcal{O}(1)$ is ample and spanned and $\Gamma(\eta^*\mathcal{O}(1))$ gives a generically one to one map.

Now let (S',L') be the minimal desingularization of a pair (S,L) as above. The following is the analogue of a result of Sommese's working in the case when $q(S') = 0$ (see [S1], §4) and of a result of Van de Ven's where L has to verify the two extra conditions $h^0(L) \geq 7$ and $L \cdot L \geq 10$ (see [V], Cor. IV).

(3.1) THEOREM. With the notation as above, let L be an ample and spanned line bundle on a normal surface S. Further assume that $\Gamma(L)$ gives a generically one to one map and $c_1(L)^2 \geq 10$ or $c_1(L)^2 \geq 9$ and $L' \not\approx 3D$, D effective divisor with $D \cdot D = 1$. If there exists a smooth hyperelliptic curve $C \in |L|$ then $p_g(S') = 0$, $d = c_1(L)^2 \geq g(L) + 2$ and either

(3.1.1) $q(S') > 0$, $h^0(L) = 4$, $g(L) + 2 \geq 3q(S') + d/3$

<u>and there exist at most finitely many smooth curves in</u> $|L|$;

(3.1.2) (S,L) <u>is a cone or a scroll</u>; <u>or</u>

(3.1.3) (S',L') <u>is a conic bundle over a smooth curve</u>.

<u>Proof.</u> Let C be a nonsingular hyperelliptic curve belonging to $|L|$. It should be noted that $C' = \pi^{-1}(C)$ is a nonsingular hyperelliptic curve in $|L'|$ since C does not pass through the singular points of S; viceversa, given any smooth hyperelliptic curve $C' \in |L'|$, $\pi(C') = C$ is an hyperelliptic curve in $|L|$ since $L' = \pi^{*}L$.

From now on, we can assume that $K_{S'} + L'$ is nef and big. Otherwise in view of [A-S1], (2.5), (2.7), (S',L') is either a conic bundle or a scroll over a nonsingular curve or the mininimal desingularization of a quadric cone. Now an easy argument shows that if $(S,L) \neq (S',L')$ and (S',L') is a scroll then (S,L) is a cone. Thus we fall in one of classes (3.1.2) or (3.1.3).

First, note the fact that there exist at most finitely many smooth hyperelliptic curves C in $|L|$ is clear. Otherwise, if $C' = \pi^{-1}(C)$, $K_{C'} \approx (K_{S'} + L')_{|C'}$ and hence the map associated to $\Gamma(K_{S'} + L')$ would be at least 2 to 1 on a dense set of curves, this contradicting Theorem 1.1. To go on, we need the following

CLAIM. Let $x \in C$ be a ramification point for the canonical map associated to $\Gamma(K_C)$. If $q(S') = 0$ and a smooth $C' \in |L' - x|$ is tangent to C at x or if $q(S') > 0$ and a smooth $C' \in |L' - x|$ is tangent to C at x of the 2^{nd} order, then C' is hyperelliptic.

<u>Proof of the Claim.</u> Note that the proof in [S2], (4.2) works with almost no change to give the $q(S') > 0$ result. We give here the proof of the stronger statement when $q(S') = 0$. Take an element $A \in |K_{S'} + L' - x|$. Then the local intersection multiplicity $(A.C)_x$ at x is nothing but the zero's order of a 1-form belonging to $\Gamma(K_C)$, therefore $(A \cdot C)_x \geq 2$. It thus follows that $(A \cdot C')_x \geq 2$ also. Indeed, if $(A \cdot C')_x = 1$ then A would be smooth at x and transverse to C' at x and hence to any smooth curve C tangent to C' at x. Thus, since the map

$$\Gamma(K_{S'} + L') \longrightarrow \Gamma(K_{C'}) \longrightarrow 0$$

is onto $q(S')$ being zero, we see that any 1-form $\omega \in \Gamma(K_{C'})$ which vanishes at x, vanishes to the 2^{nd} order at x. This means that C' is hyperelliptic (see again [S2]).

□

From the Claim we infer that if $q(S') = 0$ and $h^0(L') = 4$ or $q(S') > 0$ and $h^0(L') \geq 5$ there is a pencil of smooth hyperelliptic curves $C' \in |L'|$ on S'. Again, looking at the restriction $(K_{S'} + L')_{|C'} \approx K_{C'}$ the same argument as above leads to a contradiction in view of Theorem 1.1. Note $h^0(L) \geq 4$ since $h^0(L) = 3$ would imply $(S,L) \cong (\mathbb{P}^2, \mathcal{O}(1))$, by Zariski's Main Theorem, contradicting $L \cdot L \geq 9$. Thus it has to be $q(S') > 0$ and $h^0(L) = 4$.

To prove that $p_g(S') = 0$ the same argument as in [S1], (0.8.3) works. We recall it for reader's convenience. First, $h^1(L'_{C'}) = h^0(K_{C'} - L'_{C'}) = 0$ since C' is hyperelliptic. Otherwise, let s be a non-zero element in $\Gamma(C', K_{C'} - L'_{C'})$. Then $s \otimes \Gamma(L)$ is a subspace V of $\Gamma(K_{C'})$ with the property that the map associated to V is generically one to one on $\{x \in C',\ s(x) \neq 0\}$, a contradiction. Thus easily it follows that $h^1(L'^{t}_{C'}) = h^0(K_{C'} - tL'_{C'}) = 0$ for all $t \geq 1$. Now, since clearly $h^2(L'^{t}) = h^0(K_{S'} - tL') = 0$ for $t >> 0$, the long exact cohomology sequence associated to

$$0 \longrightarrow L'^{t} \longrightarrow L'^{t+1} \longrightarrow L'^{t+1}|_{C'} \longrightarrow 0,$$

for $t \geq 0$, shows that $p_g(S') = 0$.

Moreover, $h^0(L_C) \geq 3$ because $h^0(L) \geq 4$. Therefore $\chi(L_C) = d - g(L) + 1 \geq 3$ since $h^1(L_C) = 0$, which gives $d \geq g(L) + 2$.

Finally we apply Corollary (2.3) to get $g(L) + 2 \geq d/3 + 3q(S')$ whenever $q(S') > 0$ and this completes the proof.

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