Isoparametric geodesic spheres and a conjecture of Osserman concerning the Jacobi operator

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ISOPARAMETRIC GEODESIC SPHERES AND A CONJECTURE OF OSSERMAN CONCERNING THE JACOBI OPERATOR

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1. INTRODUCTION

Let (M^n, g) be Riemannian manifold of dimension n, with curvature tensor R. The *Jacobi operator* R_X is the symmetric endomorphism of T_pM defined by $R_X(Y) = R(X,Y)X$ and we will usually restrict X to lie in the unit sphere S_pM. In [29], based on results obtained in [30,7], Osserman made the following conjecture:

Conjecture (Osserman). *If the eigenvalues of the Jacobi operator R_X are independent of the choice of $X \in S_pM$ and of the choice of $p \in M$, then either M is locally a rank-one symmetric space or M is flat.*

It may be checked directly that the Jacobi operator of a rank-one symmetric space satisfies the hypotheses of the conjecture; indeed, such a space is two-point homogeneous so the isometry group is transitive on the unit sphere bundle. Chi [12] has already proved the Osserman Conjecture in dimensions $n \equiv 1 \pmod 2$, $n \equiv 2 \pmod 4$ and $n = 4$ (and obtained further results in [13,14]); however, in [21] examples were given of metrics which are not locally symmetric but which nevertheless have a point p over which the eigenvalues of the Jacobi operator are independent of X. This leads to a pointwise version of the above conjecture:

Question 1.1. *If the eigenvalues of the Jacobi operator R_X only depend on $p \in M$ and not on the choice of $X \in S_pM$ and M is not flat, is M locally isometric to a rank-one symmetric space?*

Manifolds satisfying these hypotheses on the Jacobi operator will be said to be *pointwise Osserman* in contrast to the *globally Osserman* manifolds of the Osserman conjecture.

The main purpose of this note is to study the relationship between pointwise Osserman and globally Osserman manifolds. In §2 we give some elementary properties of pointwise Osserman manifolds, relate the pointwise Osserman condition to the notion of k-stein manifold and give four-dimensional examples of pointwise Osserman manifolds which are not globally Osserman. In the following section we show that various conjectures related to isoparametric spheres lead to the global version of the Osserman conjecture.

The rest of the paper is deals with links between Osserman conditions and Clifford structures. In §4, the results of [21] are used to outline a two-step approach to solving

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the Osserman conjecture. This immediately deals with the conjecture when the dimension of M is odd and in §5 we rework Chi's arguments [12,14] for the case $\dim M \equiv 2 \pmod 4$ to fit into this scheme. In §6, we show that the candidates for pointwise Osserman curvatures tensors in §4 are often necessarily globally Osserman. These results are used in §7 to show that if a Riemannian manifold (M, g) has a curvature tensor built in a non-trivial way from a quaternionic structure as in §4, then M is necessarily locally isometric to $\mathbb{H}\mathrm{P}(n)$ or its non-compact dual.

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2. Pointwise Osserman Manifolds

If ∇ is the Levi-Civita connection of g, our convention for the sign of the curvature tensor R is that $R(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$. Let $\lambda_1 < \lambda_2 < \cdots < \lambda_r$ be the eigenvalues of the Jacobi operator R_X with multiplicities $\sigma_1, \ldots, \sigma_r$. Since we always have $R_X X = 0$ we will ignore this eigenvector, so $\sum_i \sigma_i = n-1$. In the following it will be important to consider the functions $\mu_k = \sum_i(\sigma_i {\lambda_i}^k) = \mathrm{Tr}({R_X}^k)$. Indeed the pointwise Osserman condition is equivalent to saying that μ_k is independent of $X \in S_pM$ for all k and the manifold is globally Osserman if in addition each μ_k is globally constant. Further, in the terminology of [10], a manifold for which there exist functions f_ℓ such that

$$\mathrm{Tr}({R_X}^\ell) = f_\ell \|X\|^{2\ell}, \qquad \text{for all } X \in T_pM \text{ and for all } \ell \leqslant k,$$

is said to be *k-stein*. Note that 1-stein is the same as the Einstein condition. By taking the functions f_ℓ to be μ_ℓ we have:

Proposition 2.1. *A pointwise Osserman manifold is k-stein for all k. In particular a pointwise Osserman manifold is Einstein and hence analytic in normal coordinates.*

Lemma 2.2. [13] *A (locally) reducible pointwise Osserman manifold is flat.*

Chi [13] gives a direct proof, but this also follows from [10] where it is shown that reducible 2-stein manifolds are flat or from [11, Thm. 6.22].

Lemma 2.3. *If M is pointwise Osserman and also locally symmetric, then either M is flat or M is locally a rank-one symmetric space.*

Proof. (cf. [17]) By the previous lemma, if M is not flat then it is irreducible, so the universal cover of M is a symmetric space G/K for some semi-simple Lie group G and some compact group K. The Lie algebra $\mathfrak{g}$ of G splits as $\mathfrak{g} \cong \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ is isomorphic to $T_e(G/K)$. Let $\mathfrak{a}$ be a maximal Abelian subalgebra of $\mathfrak{p}$. By definition, the rank of G/K is the dimension of $\mathfrak{a}$. Let α be any root of $\mathfrak{a}$, this is a linear function such that for some non-zero x in $\mathfrak{g}^{\mathbf{C}}$ we have

$$\mathrm{ad}(v)x = [v, x] = \alpha(v)x, \qquad \text{for all } v \in \mathfrak{a}.$$

If $\dim \mathfrak{a} > 1$ then there exist $v_1, v_2 \in \mathfrak{a}$ such that $\alpha(v_1) = 0$ and $\alpha(v_2) \neq 0$. Now the curvature of G/K is given by $R(x,y) = -[[x,y],z]$ (see [24]), so the Jacobi operator is

$$R_v = -\,\mathrm{ad}(v)^2.$$

In particular $R_v x = -\alpha(v)^2 x$ and the eigenvalue $-\alpha(v)^2$ is a non-constant function on the unit circle $\{ v = \cos(\theta)v_1 + \sin(\theta)v_2 \}$. Furthermore, the eigenvalues of R_v on $T_{\mathbb{C}}M$ are just those of R_v on TM with the multiplicities σ_i doubled. So $\dim \mathfrak{a} > 1$ implies that M is not pointwise Osserman. □

Note that an indirect proof of this may be found in [10] where a case-by-case argument shows that the only non-flat symmetric spaces which are k-stein for all k are those of rank one. The result also follows from the fact that the hypotheses imply that M is a locally symmetric harmonic space and hence locally isometric to a two-point homogeneous space (see for example [5]).

Whenever one has an equation such as $\mathrm{Tr}(R_X) = \mu_1\|X\|^2$, for all X and for some function μ_1, there is always the question of whether μ_1 is necessarily a constant. Such problems are called Schur-like problems and in this context we have the following known results.

Theorem 2.4. *On a pointwise Osserman manifold M^n we have:*

(1) *if $n \neq 2$, then μ_1 is constant;*
(2) *if $n \neq 2, 4$, then μ_2 is constant.*

Proof. (1) is a direct consequence of the corresponding well-known statement for Einstein manifolds (see for example [6, Thm. 1.97]).

For (2), let $\{E_i\}$ be an orthonormal basis of T_pM. Then

$$\mathrm{Tr}({R_X}^2) = R(X,i,X,j)R(X,j,X,i) = \mu_2\|X\|^4,$$

where we have written i for E_i when it is an argument of R and repeated indices are summed. We may polarise this identity to obtain

$$\begin{aligned} R(X,i,X,j)R(Y,j,Y,i) + R(X,i,Y,j)R(X,j,Y,i) + R(X,i,Y,j)R(Y,j,X,i) \\ = \mu_2\left(g(X,X)g(Y,Y) + 2g(X,Y)g(X,Y)\right). \end{aligned}$$

Putting $Y = E_k$ and summing over k gives

$$R(X,i,X,j)\rho_{ji} + R(X,i,k,j)R(X,j,k,i) + R(X,i,k,j)R(k,j,X,i) = \mu_2(n+2)\|X\|^2,$$

where ρ_{ij} is the Ricci tensor of g. However, M is Einstein with $\rho = \mu_1 g$, so the first term is $\mu_1^2\|X\|^2$. Now using the Bianchi identity one has

$$\begin{aligned} R(X,i,k,j)R(X,j,k,i) &= -R(X,i,k,j)R(X,k,i,j) - R(X,i,k,j)R(X,i,j,k) \\ &= R(X,i,j,k)R(X,j,k,i) + R(X,i,k,j)R(X,i,k,j), \end{aligned}$$

and hence

$$\xi(X,X) := R(X,i,k,j)R(X,i,k,j) = 2R(X,i,k,j)R(X,j,k,i).$$

Thus

$$\xi(X,X) = \tfrac{2}{3}(\mu_2(n+2) - \mu_1^2)\|X\|^2 = \tfrac{1}{n}\|R\|^2\|X\|^2, \tag{2.1}$$

where $\|R\|^2 = R_{ijkl}R_{ijkl}$. Thus M is ***super-Einstein*** in the terminology of [23], that is M is Einstein and $\xi = fg$ for some function f (see also [5, p. 165]). Now differentiating, we have

$$\xi_{ab;b} = R_{aijk;b}R_{bijk} + R_{aijk}R_{bijk;b}.$$

Since M is Einstein, $R_{bijk;b} = -R_{bikb;j} - R_{bibj;i} = \rho_{ik;j} - \rho_{ij;k} = 0$. But we also have

$$R_{aijk;b}R_{bijk} = -R_{ibjk;a}R_{bijk} - R_{bajk;i}R_{bijk} = \tfrac{1}{2}\|R\|^2_{;a} - R_{aijk;b}R_{bijk}.$$

Thus

$$\xi_{ab;b} = \tfrac{1}{4}\|R\|^2_{;a}.$$

On the other hand, (2.1) implies

$$\xi_{ab;b} = \tfrac{1}{n}\|R\|^2_{;b}\delta_{ab} = \tfrac{1}{n}\|R\|^2_{;a}.$$

So if $n \neq 4$, $\|R\|^2$ is constant and hence μ_2 is constant. (See also [5, p. 165] for another proof.) □

Corollary 2.5. *Suppose M is pointwise Osserman. If R_X has at most two eigenvalues, then these eigenvalues do not depend on p and M is globally Osserman.*

Theorem 2.6. *For a four-dimensional manifold M the following are equivalent:*

(1) *M is pointwise Osserman;*
(2) *M is 2-stein;*
(3) *locally there is a choice of orientation of M for which the metric is self-dual and Einstein.*

Proof. We have already seen that (1) implies (2) in general. The equivalence of (2) and (3) is proved in [33, §3]. This uses a result of [25] which states that M is 2-stein if and only if at each point $p \in M$ there are local almost complex structures I, J and K with the following three properties: firstly, they satisfy the quaternion identities $I^2 = J^2 = K^2 = -1$ and $IJ = K = -JI$; secondly, the metric g is Hermitian with respect to each of I, J and K; and thirdly, for any $X \in S_pM$, the curvature tensor R regarded as a symmetric endomorphism of Λ^2T^*M is diagonal with respect to the basis dual to

$$\begin{gathered} X\wedge IX + JX\wedge KX, \quad X\wedge JX + KX\wedge IX, \quad X\wedge KX + IX\wedge JX, \\ X\wedge IX - JX\wedge KX, \quad X\wedge JX - KX\wedge IX, \quad X\wedge KX - IX\wedge JX, \end{gathered}$$

and

$$R = \operatorname{diag}(a-s, b-s, c-s, s, s, s),$$

where $s = (a+b+c)/6$ and a, b and c only depend on p. Using this one may check directly that R_X has eigenvalues a, b and c and so M is pointwise Osserman. □

Note that the fact that (1) implies (3) is equivalent to the equation (3.5) in [12]. Also the deduction of (3) from (2) may be found in [31].

Corollary 2.7. *There exist pointwise Osserman four-manifolds which are not globally Osserman and hence are not locally isometric to rank-one symmetric spaces. Also there are examples where the function $\|R\|^2$ is not constant.*

Proof and Discussion. Since the Osserman conjecture is true in four-dimensions, for the first part it suffices to give an example of a self-dual Einstein four-manifold which is not locally symmetric. A compact example may be obtained by taking M to be a $K3$-surface. By Yau's proof of the Calabi conjecture, this carries a hyperKähler metric so in particular it is self-dual and Ricci-flat (see for example [6]). Local examples with non-zero scalar curvature may be obtained by taking a smooth open set in say one of the quaternionic Kähler orbifolds constructed in [18] or [19] and further hyperKähler examples with many possibilities for the eigenfunctions may be found in [26].

An example where $\|R\|^2$ is clearly non-constant is given by the Calabi metric on $T^*\mathbb{C}P(1)$, see [8,9,20]. This is a complete hyperKähler metric which has an action of $U(2)$ such that the central $U(1)$ fixes a complex structure I and rotates J and K. Thus at any given point two of the eigenvalues of the Jacobi operator are equal and μ_1 and μ_2 generate all the symmetric functions of the eigenvalues of the Jacobi operator. Now μ_1 is constant as M is Einstein (in fact Ricci-flat). However, μ_2 can not be constant otherwise M would be globally Osserman and hence locally symmetric. Thus μ_2 and hence $\|R\|^2$ are non-constant functions for this metric. A second example with $\|R\|^2$ non-constant which arises from the work of Olszak [28] will be discussed at the end of section 5.

[Since the hyperKähler metrics on $K3$-surfaces form a 57-dimensional family it would seem reasonable to suppose that compact examples with $\|R\|^2$ non-constant may be found amongst them. However, this seems hard to verify, partly because no explicit expressions for these metrics are known.] □

3. Isoparametric Geodesic Spheres

If we fix a point p of our Riemannian manifold M then we may look at small geodesic spheres $G_p(r)$ centred on p. For $x \in G_p(r)$ one can consider the second fundamental form $\mathrm{I\!I}(x,p,r)$ of $G_p(r) \subset M$. The following is due to Tricerri & Vanhecke [38].

Isoparametric Conjecture. *Suppose the eigenvalues of $\mathrm{I\!I}(x,p,r)$ depend only on p and r, then either M is locally a rank one-symmetric space or M is flat.*

This conjecture is already known to hold in dimensions 2, 4 and 1 (mod 2) [38]. The following proposition shows that the Isoparametric Conjecture is also valid in dimensions 2 (mod 4) by [12] (and also under the other hypotheses shown by Chi [13,14] to imply the Osserman conjecture).

Proposition 3.1. *If M is a manifold for which the eigenvalues of $\mathrm{I\!I}(x,p,r)$ depend only on p and r, then M is globally Osserman.*

Proof. Fix a point p and a tangent vector $\xi \in S_pM$ and let $x = \exp_p(r\xi)$. From [11] we have the following series expansion relating the second fundamental form and the Jacobi operator

$$\mathrm{I\!I}(x,p,r) = \tfrac{1}{r}\,\mathrm{Id} - \tfrac{r}{3}R_\xi - \tfrac{r^2}{4}\nabla_\xi R_\xi + O(r^3),$$

where ∇ is the Levi-Civita connection and all terms on the right are evaluated at p. Under our hypotheses, $\mathbb{I}(x,p,r)$ has eigenvalues independent of ξ so this also holds for

$$f(\xi,p,r) = R_\xi + \tfrac{3}{4}r\nabla_\xi R_\xi + O(r^2)$$

and any power of f. In particular,

$$\mathrm{Tr}(f(\xi,p,r)^k) = \mathrm{Tr}\left({R_\xi}^k + \tfrac{3}{4}kr{R_\xi}^{k-1}\nabla_\xi R_\xi + O(r^2)\right)$$

is independent of ξ, which implies that $\mathrm{Tr}({R_\xi}^k)$ and $\mathrm{Tr}({R_\xi}^{k-1}\nabla_\xi R_\xi)$ are also. The first of these implies that M is pointwise Osserman. Defining μ_k as in the previous section, we have that

$$\xi\mu_k = \xi\,\mathrm{Tr}({R_\xi}^k)$$

is independent of ξ, so replacing ξ by $-\xi$ shows that $\xi\mu_k = 0$. Hence the eigenvalues are constant and the manifold is globally Osserman. □

Note that although under the hypothesis of Proposition 3.1, M is necessarily harmonic, the recent examples of Damek & Ricci [15] of harmonic manifolds which are not symmetric do not provide counterexamples to the Isoparametric Conjecture [38, Prop. 6.1].

The Isoparametric Conjecture considers geodesic spheres centred on a point p. Alternatively one may consider geodesic spheres passing through p. In [40] it is shown that if s is the geodesic symmetry about p, then $\mathbb{I}(p,x,r) = \mathbb{I}(p,s(x),r)$ as endomorphisms of T_pM for all x,p sufficiently close if and only if M is locally symmetric.

Proposition 3.2. *If M is a manifold for which the eigenvalues of $\mathbb{I}(p,x,r)$ depend only on p and r then M is globally Osserman.*

Proof. Write $x = \exp_p(r\xi)$, then [39] (see also [4]) shows that

$$\mathbb{I}(p,x,r) = \tfrac{1}{r}\,\mathrm{Id} - \tfrac{r}{3}R_\xi - \tfrac{r^2}{12}\nabla_\xi R_\xi + O(r^3).$$

The result now follows exactly as in the previous proposition. □

Thus non-flat manifolds with the eigenvalues of $\mathbb{I}(p,x,r)$ depending only on p and r are rank-one symmetric if their dimension is 4, 1 (mod 2) or 2 (mod 4) (and other cases as in [13,14]). Also, these manifolds are necessarily harmonic (see [4, Prop. 1] where these spaces arise as particular examples of $\mathfrak{TC}$-spaces).

An intrinsic version of the Isoparametric Conjecture may be obtained by considering the Ricci endomorphism $\tilde{Q}(x,p,r)$ at x of the induced connection on the geodesic sphere of radius r centred on p.

Proposition 3.3. *If M is a manifold of dimension $n > 2$ for which the eigenvalues of $\tilde{Q}(x,p,r)$ depend only on p and r, then M is globally Osserman.*

Proof. For $x = \exp_p(r\xi)$, from [16] we have

$$\begin{aligned}\tilde{Q}(x,p,r) = {}& \tfrac{n-2}{r^2}\,\mathrm{Id} + \left\{Q - \rho(\xi,\cdot)\xi - \tfrac{1}{3}\rho(\xi,\xi)\,\mathrm{Id} - \tfrac{n}{3}R_\xi\right\} \\ & + r\left\{\nabla_\xi Q - (\nabla_\xi\rho)(\xi,\cdot)\xi - \tfrac{1}{4}(\nabla_\xi\rho)(\xi,\xi)\,\mathrm{Id} - \tfrac{n+1}{4}\nabla_\xi R_\xi\right\} + O(r^2),\end{aligned}$$

where Q is the Ricci endomorphism of M. Hence, the first two terms of $\mathrm{Tr}(\tilde{Q})$ are

$$\tfrac{(n-1)(n-2)}{r^2} + s - \tfrac{2(n+1)}{3}\rho(\xi,\xi),$$

where s is the scalar curvature of M. By the hypothesis, these terms are independent of ξ, so M is Einstein. The second term in the expansion of $\tilde{Q}$ is now a linear combination of Id and R_ξ and the third is $-r(n+1)\nabla_\xi R_\xi/4$. Thus we may apply the same arguments as in the previous two propositions to show M is globally Osserman. □

Since manifolds satisfying the hypotheses of Proposition 3.1 or Proposition 3.3 are harmonic [11], we have:

Corollary 3.4. *Under the hypotheses of Proposition 3.1 or Proposition 3.4 the spaces are locally isometric to two-point homogeneous spaces in all cases where the Osserman conjecture or the Lichnerowicz conjecture on harmonic spaces are proved to be true.*

For discussion of the Lichnerowicz conjecture see [35].

4. Clifford Structures

We recall the discussion of [21]. If we assume M is pointwise Osserman then in each fibre T_pM the eigenspaces of R_X define distributions in TS^{n-1}. There are topological restrictions to the existence of such distributions and these are linked to the existence of Clifford structures on TM.

Theorem 4.1. [3, Prop. 15.14; 1, Thm. 1.1; 34, Thm. 27.16, p.144] *Let $n = 2^r n_0$ with $2 \nmid n_0$ and define $\nu(r)$ by $\nu(i) = 2^i - 1$, for $i = 0,1,2,3$, and $\nu(i+4) = \nu(i) + 8$. Then*

(1) *$\mathbb{R}^n$ admits a $\mathrm{Cliff}(\nu)$-module structure if and only if $\nu \leqslant \nu(r)$;*
(2) *TS^{n-1} admits a q-dimensional distribution for $2q \leqslant n-1$ if and only if $q \leqslant \nu(r)$.*

Note that the curvature tensor R^c of a metric of constant curvature is given up to scale by

$$R^c(X,Y,Z) = g(X,Z)Y - g(Y,Z)X$$

and that, if I is an almost complex structure making g Hermitian, then the curvature tensor of (a multiple of) the Fubini-Study metric on $\mathbb{C}\mathrm{P}(n/2)$ is $R^c + R^I$, where

$$R^I(X,Y,Z) = g(Y,IZ)IX - g(X,IZ)IY - 2g(X,IY)IZ.$$

The Jacobi operator of R^c has the single eigenvalue 1 and R^I_X has eigenvalues 3 and 0 with multiplicities 1 and $n-2$, and moreover the eigenvector corresponding to the eigenvalue 3 is IX.

Theorem 4.2. [21] *Suppose there is a $\mathrm{Cliff}(\nu)$-structure on $\mathbb{R}^n$ and consider a set of generators $\{I_1,\dots,I_\nu\}$ such that $I_iI_j + I_jI_i = -2\delta_{ij}$. Let $\lambda_0,\dots,\lambda_\nu$ be any real numbers. Then*

$$R = \lambda_0 R^c + \sum_{i=1}^{\nu} \tfrac{1}{3}(\lambda_i - \lambda_0)R^{I_i}$$

is an algebraic curvature tensor such that

$$R_X(I_iX) = \lambda_i I_i X \quad \text{and} \quad R_X Y = \lambda_0 Y, \tag{4.1}$$

where $X, Y \in S_0\mathbb{R}^n$ and Y is orthogonal to $X, I_1X, \dots, I_\nu X$.

The theory of normal coordinates shows that such algebraic curvature tensors may be realised at a point as the curvature tensor of a Riemannian metric defined in a neighbourhood of that point (see for example [6]).

The theorem suggests the following two-step approach to solving the Osserman conjecture. First, show that the pointwise Osserman condition implies the existence of a Clifford structure so that the eigenspaces of the Jacobi operator are as in (4.1). One then has the following fact which was also observed by Neda Bokan and Novica Blažić:

Proposition 4.3. *For a given Clifford structure with generators $\{I_1, \dots, I_\nu\}$ and given $\lambda_0, \dots, \lambda_\nu$ there is precisely one algebraic curvature tensor satisfying (4.1).*

Proof. If there are two such curvature tensors, then their difference is an algebraic curvature tensor all of whose Jacobi operators are identically zero. □

The second step is then to decide which of these algebraic curvature tensors may be realised as curvature tensors of a Riemannian manifold.

The first case to consider is when M is pointwise Osserman and $\dim M = n = 2m + 1$. Theorem 4.1 says that R_X can only have one eigenvalue, so M is globally Osserman and has constant curvature. This is exactly the argument given by Chi [12]. He also deals with the case when $\dim M \equiv 2 \pmod 4$. In the next section we rework his arguments to make explicit the relationship with almost complex structures.

5. The Pointwise Osserman Conjecture for $\dim M \equiv 2 \pmod 4$

If M is pointwise Osserman and $\dim M = n = 4m + 2 > 0$, then Theorem 4.1 implies that R_X has at most two eigenvalues b and c, with b of multiplicity $n-2$ and c of multiplicity one. First note that if λ is either b or c, then

$$\begin{aligned} R_X Y = \lambda Y \quad &\text{if and only if} \quad (R(X,Y)X, Y) = \lambda \\ &\text{if and only if} \quad R_Y X = \lambda X. \end{aligned}$$

Let L be the line bundle over SM corresponding to the eigenvalue c and without loss of generality assume that M is contractible. As any line bundle over $S^{n-1} \cong S_pM$ is trivial (see for example [34]), we can find a (smooth) global unit section $IX = I(X)$.

Lemma 5.1. *$I^2(X) = -X$ and $I(-X) = -I(X)$.*

Proof. Let V_X be the span of X and IX and note that $Y \perp V_X$ if and only if $X \perp V_Y$. If U is an element of V_X of unit length, then $U = \cos(\theta)X + \sin(\theta)IX$ and we may define $Z(U)$ to be $-\sin(\theta)X + \cos(\theta)IX$. The sectional curvature of V_X is

$$(R(U, Z(U))U, Z(U)) = (R(X, IX)X, IX) = c,$$

so $R_U Z(U) = cZ(U)$ and $Z(U) = \pm IU$. By continuity the sign is independent of θ. Thus $I^2(X) = Z^2(X) = -X$ and $I(-X) = \pm Z(-X) = \mp Z(X) = -I(X)$. □

We now have a well-defined extension of I to the whole of TM via the formula $I(rX) = rI(X)$, for $r \in \mathbb{R}$.

Theorem 5.2. *The extension of I to TM is linear.*

Proof. It is sufficient to show $I(X+Y) = IX + IY$ for $Y \perp V_X$, because if we consider $I(A+B)$, we may rescale so that A has unit length and write $B = A' + B'$, where $A' \in V_A$ and $B' \perp V_A$, and then obtain

$$I(A+B) = I(A+A'+B') = I(A+A')+IB' = IA+IA'+IB' = IA+I(A'+B') = IA+IB.$$

By rescaling we need only show

$$I(\cos(\theta)X + \sin(\theta)Y) = \cos(\theta)IX + \sin(\theta)IY, \tag{5.1}$$

for $\|X\| = \|Y\| = 1$ and $Y \perp V_X$. Fix such a pair X, Y, let $\alpha = \cos(\theta)$, $\beta = \sin(\theta)$ and for convenience define $I' = \epsilon I$ for some $\epsilon = \pm 1$. Let

$$A_\theta = \alpha X + \beta Y, \quad B_\theta = \alpha IX + \beta I'Y.$$

Suppose we can show that for some choice of ϵ

$$\lambda_\theta := (R(A_\theta, B_\theta)A_\theta, B_\theta) = c \tag{5.2}$$

for all θ. Then $B_\theta = \pm I(A_\theta)$ and by continuity the sign is constant. Calculating at $\theta = 0$ shows $B_\theta = I(A_\theta)$, whereas taking $\theta = \pi/2$ shows $I'Y = IY$, $\epsilon = 1$, proving (5.1).

To prove (5.2) first note that the curvature identities and the eigenvalue property imply that $R(\cdot,\cdot,\cdot,\cdot) = 0$ when three of the arguments lie in $\{X, IX\}$ and one lies in $\{Y, I'Y\}$, or vice versa. Thus, when we expand λ_θ in terms of X, IX, Y and $I'Y$, the coefficients of $\alpha^3\beta$ and $\alpha\beta^3$ vanish. Also $R(X, I'Y, X, I'Y) = R(Y, IX, Y, IX) = b$, so

$$\lambda_\theta = (\alpha^4 + \beta^4)c + 2\alpha^2\beta^2\{R(X, IX, Y, I'Y) + b + R(X, I'Y, Y, IX)\} \tag{5.3}$$

and to prove (5.2) it suffices to show that the last bracket equals c, that is

$$R(X, IX, Y, I'Y) + R(X, I'Y, Y, IX) = c - b. \tag{5.4}$$

Lemma 5.3. (1) *$R(S,U)T = -R(T,U)S$ when $S \perp T$ and $S, T \in V_U^\perp$;*
(2) *$R(S,T)U = 0$ when $S \perp V_T$ and $S, T \in V_U^\perp$;*
(3) *$2R(X, Y, IX, I'Y) = R(X, IX, Y, I'Y)$;*
(4) *$2R(X, I'Y, IX, Y) = -R(X, IX, Y, I'Y)$.*

Proof. Let $W = \cos(\phi)S + \sin(\phi)T$. Then U is orthogonal to V_W, so $R(W,U)W = bU$. Expanding this identity gives

$$(\cos^2\phi + \sin^2\phi)bU + \cos\phi\sin\phi(R(S,U)T + R(T,U)S) = bU,$$

proving (1). For (2) we see that the rôles of S, T, U are symmetric, so we may use (1) and the Bianchi identities to get

$$R(S,T)U = -R(T,U)S - R(U,S)T = R(U,T)S + R(T,S)U = -2R(S,T)U,$$

as required. For (3), using (1) we have

$$\begin{aligned} R(X, Y, IX, I'Y) &= -R(X, IX, I'Y, Y) - R(X, I'Y, Y, IX) \\ &= R(X, IX, Y, I'Y) - R(IX, I'Y, X, Y) \end{aligned}$$

and (4) follows from

$$R(X, I'Y, IX, Y) = -R(IX, I'Y, X, Y) = -R(X, Y, IX, I'Y) = -\tfrac{1}{2}R(X, IX, Y, I'Y). \quad \square$$

Lemma 5.4. $R(X, IX, Y, I'Y) = \pm 2(b - c)/3$.

Proof. Let $A = A_{\pi/4} = (X + Y)/\sqrt{2}$, $B = B_{\pi/4} = (IX + I'Y)/\sqrt{2}$ and define

$$C = (X - Y)/\sqrt{2}, \quad D = (IX - I'Y)/\sqrt{2}.$$

Then $\{A, B, C, D\}$ is an orthonormal basis for $V_X \oplus V_Y$ and we now prove that B is an eigenvector of R_A.

If Z is orthogonal to $V_X \oplus V_Y$, then

$$\begin{aligned}
&2^{3/2}(R(A, B)A, Z)\\
&\quad = R(X, IX, X, Z) + R(X, IX, Y, Z) + R(X, I'Y, X, Z) + R(X, I'Y, Y, Z)\\
&\qquad + R(Y, IX, X, Z) + R(Y, IX, Y, Z) + R(Y, I'Y, X, Z) + R(Y, I'Y, Y, Z).
\end{aligned}$$

As $R(X, IX)X = cIX$, the first term is zero and $R(X, IX, Y, Z) = -R(Y, Z, X, IX) = 0$, so the second term is also zero. The remaining five terms may be dealt with in a similar way and so $(R(A, B)A, Z) = 0$.

Now compute

$$\begin{aligned}
&4R(A, B, A, C)\\
&\quad = -2R(X + Y, IX + I'Y, X, Y)\\
&\quad = -2\,(R(X, IX, X, Y) + R(X, I'Y, X, Y) + R(Y, IX, X, Y) + R(Y, I'Y, X, Y))\\
&\quad = 0
\end{aligned}$$

and

$$\begin{aligned}
&4R(A, B, A, D)\\
&\quad = R(X, IX, X, IX) - R(X, IX, X, I'Y) + R(X, IX, Y, IX) - R(X, IX, Y, I'Y)\\
&\qquad + R(X, I'Y, X, IX) - R(X, I'Y, X, I'Y) + R(X, I'Y, Y, IX) - R(X, I'Y, Y, I'Y)\\
&\qquad + R(Y, IX, X, IX) - R(Y, IX, X, I'Y) + R(Y, IX, Y, IX) - R(Y, IX, Y, I'Y)\\
&\qquad + R(Y, I'Y, X, IX) - R(Y, I'Y, X, I'Y) + R(Y, I'Y, Y, IX) - R(Y, I'Y, Y, I'Y)\\
&\quad = c - 0 + 0 - R(X, IX, Y, I'Y) + 0 - b + R(X, I'Y, Y, IX) - 0\\
&\qquad + 0 - R(Y, IX, X, I'Y) + b - 0 + R(Y, I'Y, X, IX) - 0 + 0 - c\\
&\quad = 0.
\end{aligned}$$

Thus $R(A, B)A = \lambda' B$ for some λ', as claimed. As in (5.3) we now have

$$\lambda' = \tfrac{1}{2}(c + b) + \tfrac{3}{4}R(X, IX, Y, I'Y)$$

and the Lemma follows. □

The proof of Theorem 5.2 is completed by choosing $I' = \epsilon I$ so that $R(X, IX, Y, I'Y) = -2(b - c)/3$. This means that (5.4) is satisfied, as required. □

Corollary 5.5. *The curvature tensor of M is given by*

$$R = bR^c + c'R^I,$$

where $c' = (c - b)/3$.

However very few of these tensors occur as the curvature tensor of a Riemannian manifold, as the next result shows.

Theorem 5.6. [37] *If M is an almost Hermitian manifold of dimension strictly greater than four with curvature tensor of the form $R = bR^c + c'R^I$, for some functions b, c' and $c' \not\equiv 0$, then $b = c'$ and M is locally isometric to $\mathbb{C}P(n/2)$ or its non-compact dual.*

Applying this to a pointwise Osserman manifold gives the following generalisation of one of Chi's results [12].

Theorem 5.7. *If M is a pointwise Osserman manifold of dimension $n = 4m + 2 > 2$, then M is either flat or locally a rank-one symmetric space.*

Note that Theorem 5.6 is also valid for $\dim M = 4$ when b or c' is assumed to be constant [37]. Olszak [28] showed that this additional assumption is necessary by constructing examples of four-dimensional almost Hermitian manifolds with curvature tensor $bR^c + c'R^I$ with b and c non-constant. Since in this situation the Jacobi operator only has two eigenvalues, his construction provides us with further examples for Corollary 2.7 of pointwise Osserman four-manifolds with $\|R\|^2$ non-constant.

6. The Global Osserman Condition and Clifford Structures

In this section we discuss the relationship between the global Osserman condition and the pointwise Osserman algebraic curvature tensors which were associated to a Clifford algebra structure in §4. This is partly technical preparation for the computations of the next section where we will concentrate on the cases of Cliff(2)- and certain Cliff(3)-structures.

Suppose we have a Cliff(ν)-structure with generators $\{I_1, \dots, I_\nu\}$ as in Theorem 4.2. We assume that the curvature tensor R is in the form

$$R = \lambda_0 R^c + \sum_{i=1}^{\nu} \lambda_i R^{I_i},$$

for some functions λ_i.

Proposition 6.1. *Let $n = \dim M$ and assume the* Cliff(ν)*-structure on M satisfies one of the following six conditions:*

(a) $\nu > 9$;
(b) $n > 4\nu$;
(c) $\nu = 3$, $n > 8$ *and TM decomposes as a direct sum of isomorphic irreducible* Cliff(3)*-modules (or, equivalently, $I_3 = \pm I_1 I_2$);*
(d) $\nu = 5$ *and* $n > 16$;
(e) $\nu = 6$ *and* $n > 16$;
(f) $\nu = 7$, $n > 16$ *and TM decomposes as a direct sum of isomorphic irreducible* Cliff(7)*-modules.*

Then we have:

(1) *for all $X \in TM$ and all $i = 1, \dots, \nu$, the vector field $\lambda_i(\nabla_X I_i)X$ is in the linear span of $\{I_j X : j \neq i\}$;*
(2) *the functions $\lambda_0, \dots, \lambda_\nu$ are constant.*

Proof. In this and following calculations we will make repeated use of the differential Bianchi identity

$$\begin{aligned} 0 &= \underset{A,B,C}{\mathfrak{S}} (\nabla_A R)(B, C, D) \\ &= \mathcal{A}^0 + \sum_{j=1}^{\nu} \mathcal{A}_j^1 + \lambda_j \mathcal{A}_j^2 + \lambda_j \mathcal{A}_j^3, \end{aligned} \tag{6.1}$$

where

$$\begin{aligned} \mathcal{A}^0 &= \underset{A,B,C}{\mathfrak{S}} A(\lambda_0)\left(g(B,D)C - g(C,D)B\right), \\ \mathcal{A}_j^1 &= \underset{A,B,C}{\mathfrak{S}} A(\lambda_j)\left(g(C,I_jD)I_jB - g(B,I_jD)I_jC - 2g(B,I_jC)I_jD\right), \\ \mathcal{A}_j^2 &= \underset{A,B,C}{\mathfrak{S}} g(C,(\nabla_A I_j)D)I_jB - g(B,(\nabla_A I_j)D)I_jC - 2g(B,(\nabla_A I_j)C)I_jD, \\ \mathcal{A}_j^3 &= \underset{A,B,C}{\mathfrak{S}} g(C,I_jD)(\nabla_A I_j)B - g(B,I_jD)(\nabla_A I_j)C - 2g(B,I_jC)(\nabla_A I_j)D. \end{aligned}$$

We first prove (1). Fix i and let X be any unit vector. Let Y be a unit vector orthogonal to X and I_jX, $j = 1, \dots, \nu$ and choose a unit vector Z such that I_jZ and I_iI_jZ are orthogonal to both X and Y for $j = 1, \dots, \nu$. This is a total of at most 4ν conditions on Z, so this may certainly be satisfied under hypothesis (b).

For condition (a), recall that for $\nu \not\equiv 3 \pmod 4$, Cliff(ν) has a unique irreducible representation over $\mathbb{R}$ and this representation is of dimension $2^{(\nu+\delta)/2}$, where $\nu \equiv 3+(3-\delta)\varepsilon \pmod 8$ for $0 \leqslant \delta \leqslant 2$ and $\varepsilon = \pm 1$. On the other hand for $\nu \equiv 3 \pmod 4$, there are two non-isomorphic irreducible representations, each of dimension $2^{(\nu-\varepsilon)/2}$, where $\nu \equiv 5 + 2\varepsilon \pmod 8$ and $\varepsilon = \pm 1$ (see for example [3]). One may now directly check that condition (a) implies condition (b) and so once again the conditions on Z may be satisfied.

The other four hypotheses are just the cases where the bound $n > 4\nu$ may be lowered. The hypotheses (c) and (f) both imply that for $j \neq k$ and any $Y \in TM$, I_jI_kY lies in the linear span of $\{I_\ell Y : \ell \neq j, k\}$. Thus in these cases we only have $2(\nu + 1)$ conditions on Z. Our hypotheses on n are now exactly those required for the existence of such a Z. Hypotheses (d) and (e) now follow from the fact that Cliff(5) and Cliff(6)-structures give rise to Cliff(7)-structures of the type described in (f).

Now consider (6.1) with $A = Z$, $B = X$, $C = I_iZ$ and $D = X$. The assumptions on Y and Z immediately imply

$$g(\mathcal{A}^0, Y) = g(\mathcal{A}_j^1, Y) = g(\mathcal{A}_j^2, Y) = 0,$$

for $j = 1, \ldots, \nu$. Thus the Bianchi identity gives

$$\begin{aligned} 0 &= \sum_{j=1}^{\nu} \lambda_j g(\mathcal{A}_j^3, Y) = -\sum_{j=1}^{\nu} 2\lambda_j g(I_i Z, I_j Z) g((\nabla_X I_j)X, Y) \\ &= -2\lambda_i g((\nabla_X I_i)X, Y). \end{aligned}$$

Thus $\lambda_i(\nabla_X I_i)X$ lies in the linear span $\langle X, I_j X\rangle$. However, as g is Hermitian with respect to I_i, we have

$$g((\nabla_X I_i)X, X) = 0 = g((\nabla_X I_i)X, I_i X),$$

so $\lambda_i(\nabla_X I_i)X \in \langle I_j X : j \neq i\rangle$, as required.

For (2) we first show that λ_0 is constant. Fix $i > 0$, let Y be an arbitrary unit vector. Let $Z = I_i Y$ and choose X to be a unit vector with $I_j X$ and $I_i I_j X$ orthogonal to Z. We now consider the same Bianchi identity as before but this time take the inner product with Z. Thus

$$\begin{gathered} g(\mathcal{A}^0, Z) = -(I_i Z)(\lambda_0) = Y(\lambda_0), \qquad g(\mathcal{A}_j^1, Z) = 0, \\ g(\mathcal{A}_j^2, Z) = -g(X, (\nabla_Z I_j)X) g(I_j I_i Z, Z) = 0, \\ g(\mathcal{A}_j^3, Z) = -2g(I_i Z, I_j Z) g((\nabla_X I_j)X, Z) = 0 \end{gathered}$$

and summing, the Bianchi identity implies $Y(\lambda_0) = 0$, as required.

To show λ_i is constant, first fix a unit vector X and then choose Z to be a unit vector with $I_j Z$ and $I_i I_j Z$ orthogonal to X. Again consider the same Bianchi identity, but now take the inner product with $I_i X$. This gives

$$\begin{gathered} g(\mathcal{A}^0, I_i X) = 0, \qquad g(\mathcal{A}_j^1, I_i X) = -2X(\lambda_j) g(I_i Z, I_j Z)\delta_{ij} = -2X(\lambda_i)\delta_{ij}, \\ g(\mathcal{A}_j^3, I_i X) = -2g(I_i Z, I_j Z) g((\nabla_X I_j)X, I_i X) = 0 \end{gathered}$$

and

$$\begin{aligned} g(\mathcal{A}_j^2, I_i X) &= \big(g(I_i Z, (\nabla_Z I_j)X) - 2g(X, (\nabla_Z I_j) I_i Z) - 3g(Z, (\nabla_{I_i Z} I_j)X) \\ &\qquad - 2g(I_i Z, (\nabla_X I_j)Z)\big)\delta_{ij} \\ &= \big(-g((\nabla_Z I_i) I_i Z, X) + 2g(X, I_i(\nabla_Z I_i)Z) + 3g((\nabla_{I_i Z} I_i)Z, X) + 0\big)\delta_{ij} \\ &= \big(-3g((\nabla_Z I_i)Z, X) + 3g((\nabla_{I_i Z} I_i) I_i Z, I_i X)\big)\delta_{ij} \\ &= 0. \end{aligned}$$

Thus $X(\lambda_i) = 0$ and the proof is complete. □

If we assume that λ_0 is constant, but relax the restrictions on ν, we may obtain:

Proposition 6.2. *Let $n = \dim M$ and assume the* Cliff(ν)*-structure on M satisfies either (a') $\nu > 8$ or (b') $n > 2\nu$. Now suppose that λ_0 is constant. Then we have:*

1. *for all $X \in TM$ and all $i = 1, \ldots, \nu$, the vector field $\lambda_i(\nabla_X I_i)X$ is in the orthogonal complement of $I_i X$ in the linear span of $\{I_j X, I_j I_i X : j \neq i\}$;*
2. *the functions $\lambda_1, \ldots, \lambda_\nu$ are also constant.*

Proof. The proof of (1) is almost exactly as before. The only change is that we take $Y = Z$ and only require Z to be such that $I_j Z$ and $I_i I_j Z$ are orthogonal to X. This is now at most 2ν conditions on Z and analysis similar to before leads to the hypotheses (a') and (b'). One now uses the same Bianchi identity to obtain the desired conclusion.

(2) is deduced using the same method as in part (2) of the previous proposition. □

7. Cliff(2)-structures and Certain Quaternionic Structures

In this section we consider the special case of a pointwise Osserman manifold which admits a compatible Cliff(2)-structure with generators I, J satisfying $I^2 = -1 = J^2$ and $IJ = -JI$. It will be helpful to define $K = IJ$ and consider the more general case arising from the resulting Cliff(3)-structure. The curvature tensor may then be written as

$$R = \lambda_0 R^c + \lambda_1 R^I + \lambda_2 R^J + \lambda_3 R^K,$$

where λ_i are real functions and I, J, K satisfy the quaternion identities $I^2 = J^2 = K^2 = -1$ and $IJ = K = -JI$. (Note that for an arbitrary Cliff(3)-structure it is not possible to choose generators in this way.) In the case of a Cliff(2)-structure we have that λ_3 is identically zero.

Theorem 7.1. *Let M be a pointwise Osserman manifold with a quaternionic structure as above. If one of the following two conditions is satisfied*

(1) $\dim M \geqslant 12$, *or*
(2) $\dim M \geqslant 8$ *and λ_0 is constant,*

then M is either flat or locally isomorphic to a rank-one symmetric space of real, complex or quaternionic type.

Note that in the case that $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3$, since the manifold M is Einstein, $\lambda_0, \dots, \lambda_3$ are automatically constant and the theorem reduces to a result of Marchiafava [27].

Proof. First, by Propositions 6.1 and 6.2 we have that $\lambda_0, \dots, \lambda_3$ are constant. We may now assume that $\lambda_1 \lambda_2 \neq 0$, otherwise we are reduced to the case with just one complex structure considered in section 5. The Propositions also give that $(\nabla_X I)X \in \langle JX, KX \rangle$. Calculating the coefficients of JX and KX from the Bianchi identity gives

$$\begin{aligned}2\lambda_1(\nabla_X I)X &= \lambda_2 JX\big(2g((\nabla_X I)Y, JY) - 3g(Y, (\nabla_{IY} J)X) + 3g(IY, (\nabla_Y J)X)\big)\\ &\quad + \lambda_3 KX\big(2g((\nabla_X I)Y, KY) - 3g(Y, (\nabla_{IY} K)X) + 3g(IY, (\nabla_Y K)X)\big).\end{aligned} \tag{7.1}$$

Lemma 7.2. *The constant λ_3 is not zero.*

Proof. An almost Hermitian manifold $(N, g, \mathcal{I})$ is said to be *nearly Kähler* if $(\nabla_X \mathcal{I})X = 0$ for all X. Gray [22] shows that a nearly Kähler manifold of constant holomorphic sectional curvature and dimension strictly bigger than 6 is either flat or locally isometric to $\mathbb{C}P(n/2)$ or its non-compact dual. We will prove the Lemma by showing that $\lambda_3 = 0$ implies that M is nearly Kähler and of constant holomorphic sectional curvature.

Consider the Bianchi identity (6.1) with $A = X$, $B = IX$, $C = JX$ and $D = Y$, where X and Y have unit length and Y is orthogonal to the quaternionic span $\langle X\rangle_{\mathbb{H}}$. Taking the inner product with X gives

$$-\lambda_1 g(JX, (\nabla_X I)Y) + \lambda_2 g(IX, (\nabla_X J)Y) + 3\lambda_1 g((\nabla_{JX} I)Y, X) - 3\lambda_2 g(X, (\nabla_{IX} J)Y) = 0.$$

Since, for each X, $(\nabla_X I)X$ lies in the quaternionic span of X, we have $g((\nabla_X I)JX, Y) = -g((\nabla_{JX} I)X, Y)$. Using this identity, the previous equation reduces to

$$4(\lambda_1 + \lambda_2)g((\nabla_X K)X, Y) = 0.$$

However, $\lambda_3 = 0$ implies $g((\nabla_X K)X, IX) = -g(KX, (\nabla_X I)X) = 0$, by (7.1). So, if $\lambda_1 \neq -\lambda_2$, we conclude that $(\nabla_X K)X = 0$ and (M, g, K) is a nearly Kähler manifold. The holomorphic sectional curvature is λ_0, so Gray's result [22] implies $R = \frac{1}{4}\lambda_0(R^c + R^K)$, which contradicts the assumption $\lambda_1\lambda_2 \neq 0$.

If $\lambda_1 = -\lambda_2$ and $\lambda_3 = 0$, we will prove that (M, g, I) is nearly Kähler. By (7.1), we only need to show that $(\nabla_X I)X$ has no JX-component. Consider the Bianchi identity (6.1) with $A = X$, $B = IX$, $C = JX$, $D = X$ and take the inner product with IX. This gives

$$\begin{aligned} 0 &= -3\lambda_1 g(IX, (\nabla_X I)JX) - 3\lambda_1 g(JX, (\nabla_{IX} I)X) - 3\lambda_2 g((\nabla_{IX} J)X, IX) \\ &= 0 - 3(\lambda_1 - \lambda_2)g(JX, (\nabla_{IX} I)X) \\ &= 3(\lambda_1 - \lambda_2)g(JZ, (\nabla_Z I)Z), \end{aligned}$$

where $Z = IX$. Thus, we have $(\nabla_X I)X = 0$. The holomorphic sectional curvature of (M, g, I) is $\lambda_0 + 3\lambda_1$ so, by [22], R is a multiple of $R^c + R^I$. However, this contradicts the assumption $\lambda_1\lambda_2 \neq 0$. □

We now have $\lambda_3 \neq 0$ and may define 1-forms a_{IJ}, etc., by

$$\begin{aligned} (\nabla_X I)X &= a_{IJ}(X)JX + a_{IK}(X)KX, \\ (\nabla_X J)X &= a_{JI}(X)IX + a_{JK}(X)KX, \\ (\nabla_X K)X &= a_{KI}(X)IX + a_{KJ}(X)JX. \end{aligned}$$

Since $g((\nabla_X I)X, JX) + g(IX, (\nabla_X J)X) = 0$, we have $a_{IJ} = -a_{JI}$, etc.,

Lemma 7.3. *If $\lambda_1\lambda_2 \neq 0$ then M is quaternionic Kähler.*

Proof. We need to show that

$$\nabla_X I = a_{IJ}(X)J + a_{IK}(X)K, \qquad \text{etc.}$$

Now

$$\begin{aligned} (\nabla_X I)IX &= -I(\nabla_X I)X = -a_{IJ}(X)KX + a_{IK}(X)JX \\ &= (a_{IJ}(X)J + a_{IK}(X)K)IX \end{aligned}$$

and

$$\begin{aligned}(\nabla_X I)JX &= (\nabla_X K)X - I(\nabla_X J)X \\ &= a_{KI}(X)IX + a_{KJ}(X)JX + a_{JI}(X)X + a_{JK}(X)JX \\ &= a_{KI}(X)IX + a_{JK}(X)X \\ &= (a_{IJ}(X)J + a_{IK}(X)K)JX.\end{aligned}$$

The computation for $(\nabla_X I)KX$ is similar.

Let Y be a unit vector orthogonal to the quaternionic span of X. Then expanding $(\nabla_{X+Y}I)(X+Y)$ and using the expressions for $(\nabla_X I)X$ and $(\nabla_Y I)Y$ gives

$$(\nabla_X I)Y + (\nabla_Y I)X = a_{IJ}(X)JY + a_{IJ}(Y)JX + a_{IK}(X)KY + a_{IK}(K)KX. \tag{7.2}$$

Considering the differential Bianchi identity (6.1) with $A = Y$, $B = X$, $C = IX$ and $D = X$ gives

$$\lambda_1\,(3(\nabla_Y I)X - (\nabla_X I)Y) \in \langle JX, KX, JY, KY\rangle.$$

Together with (7.2) this implies that $(\nabla_X I)Y \in \langle JX, KX, JY, KY\rangle$. However,

$$g((\nabla_X I)Y, JX) = -g(Y, (\nabla_X I)JX) = -g(Y, (\nabla_X K)X) + g(Y, I(\nabla_X J)X) = 0,$$

so $(\nabla_X I)Y \in \langle JY, KY\rangle$ and the result follows from (7.2). $\square$

Since M is quaternionic Kähler we have $R = \lambda R^{\mathbb{HP}} + R^0$, where $R^{\mathbb{HP}}$ is the curvature of quaternionic projective space $\mathbb{HP}(n)$ and R^0 has the symmetries of a curvature tensor of a hyperKähler manifold [32,2,36]. In particular, $R^0(A,B,IC) = IR^0(A,B,C)$ and $R^0(A,B,JC) = JR^0(A,B,C)$. Now

$$R^{\mathbb{HP}} = c(R^c + R^I + R^J + R^K),$$

so $R^0 = R - \lambda R^{\mathbb{HP}} = \alpha_0 R^c + \alpha_1 R^I + \alpha_2 R^J + \alpha_3 R^K$ for some constants α_i. This gives

$$\begin{aligned}R^0(A,B,IC) = \alpha_0\,&(g(A,IC)B - g(B,IC)A) \\ &+ \alpha_1\,(-g(B,C)IA + g(A,C)IB + 2g(A,IB)C) \\ &+ \alpha_2\,(-g(B,KC)JA + g(A,KC)JB + 2g(A,JB)KC) \\ &+ \alpha_3\,(g(B,JC)KA - g(A,JC)KB - 2g(A,KB)JC)\end{aligned}$$

and

$$\begin{aligned}IR^0(A,B,C) = \alpha_0\,&(g(A,C)IB - g(B,C)IA) \\ &+ \alpha_1\,(-g(B,IC)A + g(A,IC)B + 2g(A,IB)C) \\ &+ \alpha_2\,(g(B,JC)KA - g(A,JC)KB - 2g(A,JB)KC) \\ &+ \alpha_3\,(-g(B,KC)JA + g(A,KC)JB + 2g(A,KB)JC).\end{aligned}$$

Now take A orthogonal to the quaternionic span of C. From the coefficient of A we have $\alpha_0 = \alpha_1$. The coefficient of KA gives $\alpha_2 = \alpha_3$, whereas that of KC gives $\alpha_2 = -\alpha_3$. Therefore $\alpha_2 = 0 = \alpha_3$ and repeating the computation with I replaced by J shows $R^0 \equiv 0$. This completes the proof of the theorem. $\square$

References

1. J. F. Adams, *Vector fields on spheres*, Ann. Math. **75** (1962), 603–632.
2. D. V. Alekseevskiĭ, *Riemannian manifolds with exceptional holonomy groups*, Funkts. Anal. Prilozh. **2**(2) (1968), 1–10; English translation, Funct. Anal. Appl. **2** (1968), 97–105.
3. M. F. Atiyah, R. Bott and A. Shapiro, *Clifford modules*, Topology **3** (1964), suppl. 1, 3–38.
4. J. Berndt and L. Vanhecke, *Geodesic spheres and generalizations of symmetric spaces*, Boll. Un. Mat. Ital. (to appear).
5. A. L. Besse, *Manifolds all of whose geodesics are closed*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 93, Springer, Berlin, 1978.
6. ______, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, 10, Springer, Berlin, Heidelberg and New York, 1987.
7. K. Burns and A. Katok, *Manifolds with non-positive curvature*, Ergodic Theory Dynamical Systems **5** (1985), 307–317.
8. E. Calabi, *Métriques kählériennes et fibrés holomorphes*, Ann. Scient. Ec. Norm. Sup., 4^e sér **12** (1979), 269–294.
9. E. Calabi, *Isometric Families of Kähler Structures*, The Chern Symposium, 1979 (W.-Y. Hsiang, S. Kobayashi, I. M. Singer, A. Weinstein, J. Wolf and H.-H. Wu, eds.), Springer-Verlag, 1980, pp. 23–39..
10. P. Carpenter, A. Gray and T. J. Willmore, *The curvature of Einstein symmetric spaces*, Quart. J. Math. Oxford **33** (1982), 45–64.
11. B.-Y. Chen and L. Vanhecke, *Differential geometry of geodesic spheres*, J. Reine Angew. Math. **325** (1981), 29–67.
12. Q.-S. Chi, *A curvature characterization of certain locally rank-one symmetric spaces*, J. Differ. Geom. **28** (1988), 187–202.
13. ______, *Quaternionic Kähler manifolds and a characterization of two-point homogeneous spaces*, Illinois J. Math. **35** (1991), 408–418.
14. ______, *Curvature characterization and classification of rank-one symmetric spaces*, Pacific J. Math. **150** (1991), 31–42.
15. E. Damek and F. Ricci, *A class of nonsymmetric harmonic Riemannian spaces*, Bull. Amer. Math. Soc. **27** (1992), 139–142.
16. M. Djorić and L. Vanhecke, *Almost Hermitian geometry, geodesic spheres and symmetries*, Math. J. Okayama Univ. **32** (1990), 187–206.
17. J.-H. Eschenberg, *A note on symmetric and harmonic spaces*, J. Lond. Math. Soc. **21** (1980), 541–543.
18. K. Galicki and H. B. Lawson, *Quaternionic reduction and quaternionic orbifolds*, Math. Ann. **282** (1988), 1–21.
19. K. Galicki and T. Nitta, *Non-zero scalar curvature generalizations of the ALE hyperkähler metrics*, J. Math. Phys. **33** (1992), 1765–1771.
20. G. W. Gibbons and C. N. Pope, *The positive action conjecture and asymptotically Euclidean metrics in quantum gravity*, Commun. Math. Phys. **66** (1979), 267–290.
21. P. B. Gilkey, *Manifolds whose curvature operator has constant eigenvalues at the basepoint*, J. Geom. Anal. (to appear).
22. A. Gray, *Classification des variétés approximativement kählériennes de courbure sectionelle holomorphe constante*, C. R. Acad. Sci. Paris **279** (1974), 797–800.
23. A. Gray and T. J. Willmore, *Mean-value theorems for Riemannian manifolds*, Proc. Roy. Soc. Edinburgh **92A** (1982), 343–364.
24. S. Helgason, *Differential geometry, Lie groups and symmetric spaces*, Academic Press, New York, 1978.
25. O. Kowalski and L. Vanhecke, *Ball-homogeneous and disk-homogeneous Riemannian manifolds*, Math. Z. **180** (1982), 429–444.
26. P. B. Kronheimer, *The construction of ALE spaces as hyper-Kähler quotients*, J. Differ. Geom. **29** (1989), 665–683.
27. S. Marchiafava, *Variétés riemanniennes dont le tenseur de courbure est celui d'un espace symétrique de rang un*, C. R. Acad. Sci. Paris **295** (1982), 463-466.
28. Z. Olszak, *On the existence of generalized complex space forms*, Israel J. Math. **65** (1989), 214–218.

29. R. Osserman, *Curvature in the eighties*, Amer. Math. Monthly **97** (1990), 731–756.
30. R. Osserman and P. Sarnak, *A new curvature invariant and entropy of geodesic flows*, Invent. Math. **77** (1984), 455–462.
31. S. M. Salamon, *Harmonic 4-spaces*, Math. Ann. **269** (1984), 169–178.
32. S. M. Salamon, *Quaternionic Kähler manifolds*, Invent. Math. **67** (1982), 143–171.
33. K. Sekigawa and L. Vanhecke, *Volume preserving geodesic symmetries on four-dimensional Kähler manifolds*, Differential Geometry Peñíscola, 1985, Proceedings (A. M. Naveira, A. Ferrández and F. Mascaró, eds.), Lecture Notes in Math., 1209, Springer, pp. 275–290.
34. N. Steenrod, *Topology of Fibre Bundles*, Princeton, 1965.
35. Z. I. Szabó, *Lichnerowicz conjecture on harmonic manifolds*, J. Differ. Geom. **31** (1990), 1–28.
36. F. Tricerri and L. Vanhecke, *Decomposition of a space of curvature tensors on a quaternionic Kähler manifolds and spectral theory*, Simon Stevin **53** (1979), 163–173.
37. ______, *Curvature tensors on almost Hermitian manifolds*, Trans. Amer. Math. Soc. **267** (1981), 365–398.
38. ______, *Geometry of a class of non-symmetric harmonic manifolds*, Proc. Conf. Differential Geometry and its Applications, Opava 1992 (to appear).
39. L. Vanhecke, *Geometry in normal and tubular neighbourhoods*, Proc. Workshop on Differential Geometry and Topology, Cala Gonone (Sardinia), Rend. Sem. Fac. Sci. Univ. Cagliari, supp. al vol. 58, 1988, pp. 73–176.
40. L. Vanhecke and T. J. Willmore, *Interaction of tubes and spheres*, Math. Ann. **263** (1983), 31–42.

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