Multiplicities of discriminants

Paolo Aluffi
Fernando Cukierman

Mathematics Department
University of Kansas
Lawrence KS 66045

USA

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

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FERNANDO CUKIERMAN

§0. INTRODUCTION.

Let $S \subset \mathbf{P}^N$ be a smooth variety over an algebraically closed field of characteristic 0. Denote by $D \subset \check{\mathbf{P}}^N$ the dual variety of S, consisting of all the hyperplanes in $\mathbf{P}^N$ that are tangent to S. The basic question that we consider in this note is: if $X \in D$ is a singular hyperplane section of S, what is the multiplicity $m_X(D)$ of D at X?

Our answer is in the style of the well known formula for the degree of the dual in terms of the Chern classes of S and of the hyperplane bundle $\mathcal{L}$ (see for example [F], p. 63). We associate with each hyperplane section X a zero-dimensional class in the Chow group of S, obtained by capping the Segre class in S of the singular scheme of X by the Chern classes of the bundle $P^1\mathcal{L}$ of principal parts of order one (which are easily expressible in terms of the classes of $\mathcal{L}$ and of the cotangent bundle of S). Assuming that D is a hypersurface, the multiplicity of D at X is then essentially the degree of this class. We give several examples illustrating the 'computability' of the formula in specific instances; in particular, the answer becomes especially simple when S is a surface. For example, if $S = \mathbf{P}^2$, and one writes the section X as $\sum_i m_i X_i$ with m_i natural numbers and X_i irreducible divisors, then $m_X(D)$ has a simple expression in terms of the integers m_i, the degrees of the X_i, and the singularities of $X_{\text{red}} = \sum_i X_i$.

The main formula is stated in §1, Theorem I, in a more general setting also addressing the same question for the 'higher discriminants' $D^{(r)} \subset \check{\mathbf{P}}^N$, consisting of sections of S that have a point of multiplicity at least $r+1$. The formula gives $m_X D^{(r)}$ under a hypothesis of surjectivity onto the bundle $P^r\mathcal{L}$ (automatic in the case of dual varieties) and assuming that the discriminants have the expected dimension. In these hypotheses, the formula also specializes easily to compute the degree of the discriminants (Corollary 1.1).

In fact the class introduced in Theorem I can be used to detect whether the dual variety is not a hypersurface—that is, whether it is 'small': we show (§2, Theorem II) that the class vanishes if and only if the dual is small—again, we prove the corresponding result for all discriminants, under the same surjectivity hypothesis employed in Theorem I. In fact, this exposes a surprising interaction between different hyperplane sections: the vanishing of the degree of the class for one singular section forces the vanishing of the class for *all* sections.

The main difficulty in applying Theorem I to specific situations lies in the computation of the Segre class $s(J^r X, S)$ of the Jacobian schemes of a hyperplane section. One important case occurs when S is a surface, or more generally when the singularities of X_{red} are isolated: then one can write the Segre class—and therefore the multiplicity of $D^{(r)}$ at X—as the sum of an easily computable term and a contribution μ_r due to the singularities of X_{red}. The result is stated in

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Prop. 1.2 for all discriminants. The rôle of μ_1 is clarified in §3, where we discuss the dual of a surface using Lefschetz pencils and show that in this case the contribution μ_1 is the sum of the Milnor numbers of the singularities of X_{red}.

In §4 we provide a few examples of explicit computations of multiplicities. Also, we illustrate Theorem II by showing how to apply it to prove a known criterion for the product of projective spaces $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ to have small dual under the Segre embedding (Example 4.5).

Another approach to the computation of the multiplicity of a dual variety can be found in [**Pa**], generalizing earlier results of Dimca and Némethi ([**D**], [**N**]): there the multiplicity is written in terms of generalized Milnor numbers associated with the singularities of the section and of its intersections with general linear subspaces. Comparing Parusiński's formula with Theorem I yields an expression for the generalized Milnor numbers in terms of Chern and Segre classes, close in spirit to recent work of Parusiński and Pragacz. We plan to explore this connection elsewhere.

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§1. Multiplicities of discriminants.

Let $\mathcal{L}$ be a line bundle on a smooth n-dimensional variety S. We let $P^r\mathcal{L}$ denote the r-th *bundle of principal parts*: if $\mathcal{I}$ is the ideal sheaf of the diagonal in the product $S \times S$, denote by $S(r)$ the subscheme of $S \times S$ defined by $\mathcal{I}^{r+1}$ and by p and q the two projections $S(r) \to S$; then $P^r\mathcal{L}$ is the sheaf on S defined by

$$P^r\mathcal{L} = p_* q^* \mathcal{L} \quad .$$

Then $P^0\mathcal{L} = \mathcal{L}$ and for $r \geq 0$, $P^r\mathcal{L}$ is locally free of rank $\binom{r+n}{n}$. One can think of the fiber of the bundle $P^r\mathcal{L}$ over $p \in S$ as parametrizing the first $r+1$ terms of the Taylor expansion of a germ of a section of $\mathcal{L}$ at p. For all r there are natural maps

$$\tau_r : S \times H^0(S, \mathcal{L}) \to P^r\mathcal{L} \quad ,$$

acting as 'truncated Taylor expansions' ([**G1**], [**G2**], [**Pi**] are good references for facts about bundles of principal parts).

Let now V be a vector space mapping linearly to $H^0(S, \mathcal{L})$. An element X of the projectivization $\mathbf{P}V$ of V determines a divisor on S, or maps to the zero section of $\mathcal{L}$. We say that X 'has multiplicity $\geq r$' at a point p of S if the divisor corresponding to X does: that is, if all terms of degree $< r$ in the Taylor expansion at p of the section of $\mathcal{L}$ corresponding (up to scalar) to X vanish. In particular, note that this will be the case for all r if X corresponds to the zero section. Let now $\mathcal{V} = S \times V$ denote the trivial bundle over S and denote by α_r the composition $\mathcal{V} \to S \times H^0(S, \mathcal{L}) \to P^r\mathcal{L}$: then X has multiplicity $\geq r$ at p if and only if $\alpha_r(p, X) = 0$. For $r \geq 0$ we define the 'r-th discriminant $D^{(r)}$ of $\mathcal{L}$' by

$$D^{(r)} = \{X \in \mathbb{P}V \quad \text{s.t. } X \text{ has a point of multiplicity} > r\} \quad .$$

Thus $D^{(0)} = \mathbb{P}V$, and $D^{(1)}$ is the ordinary discriminant in $\mathbf{P}V$. If $\mathcal{L}$ is very ample, $D^{(1)}$ is the dual variety of $S \subset \mathbf{P}V^\vee$. Given a specific $X \in \mathbf{P}V$, we want to compute the multiplicity $m_X D^{(r)}$ of the r-th discriminant at X.

Let $J^r X$ denote the subscheme of S defined locally by all derivatives up to order r of a local equation of (the divisor corresponding to) X: thus $J^1 X$ is the ordinary Jacobian scheme of X, supported on its singularities; $J^2 X$ is supported on the subset of S along which X has multiplicity > 2, and so on. Finally, let γ_i be the degree (in the sense of [F], Definition 1.4) $\deg[J^i X]$ of the cycle $[J^i X]$ for the *general* element $X \in D^{(i)}$. So in particular $\gamma_i = 0$ if the set of points along which the general X has multiplicity $> i$ has positive (pure) dimension. $\gamma_i = 1$ if the general element of $D^{(i)}$ has exactly one point of multiplicity $i+1$: for example this is the case for $i = 1$ (in char. 0) if $\mathcal{L}$ is very ample and the dual of X is a hypersurface ([K]).

THEOREM I. *With notations as above, suppose that the map $\mathcal{V} \xrightarrow{\alpha_r} P^r\mathcal{L}$ is surjective, and denote by Ω the cotangent bundle of S. Then for $0 < i \le r$*

$$\gamma_i \cdot m_X D^{(i)} = \deg \left\{ c(Sym^i(\Omega \oplus \mathcal{O}) \otimes \mathcal{L}) \cap s(J^i X, S) \right\}_0 \quad .$$

(Here and in the following, $\{\cdot\}_0$ denotes the component of dimension 0 of the class between brackets)

REMARKS. $\mathcal{V}$ surjects onto $P^0\mathcal{L} = \mathcal{L}$ when the natural map $S \to \mathbf{P}V^\vee$ has no base locus; $\mathcal{V}$ surjects onto $P^1\mathcal{L}$ if the same map is locally a closed immersion.

Also, in §2 we will deal more specifically with the case $\gamma_i = 0$.

Before giving a proof of this result, we draw two consequences.

COROLLARY 1.1. *In the hypotheses of the Theorem,*

$$\gamma_i \cdot \deg D^{(i)} = \deg \left\{ c(Sym^i(\Omega \oplus \mathcal{O}) \otimes \mathcal{L}) \cap [S] \right\}_0 \quad .$$

PROOF: Consider the direct sum $V \oplus 1$ of V and a 1-dimensional vector space, with the natural map $V \oplus 1 \to V \to H^0(S, \mathcal{L})$. The discriminant in $\mathbb{P}(V \oplus 1)$ is the cone over the discriminant in $\mathbf{P}V$, with vertex the point X corresponding to the summand 1. The degree of $D^{(i)}$ is then the multiplicity of the new discriminant at X. Now X maps down to the zero-section of $\mathcal{L}$, so $J^i X = S$, so $s(J^i X, S) = [S]$. The statement of the corollary follows then from the theorem. ∎

The statement of the above corollary for $i = 1$ is equivalent to the computation of the degree of the dual variety in [F], p. 63.

Next, denote by $X^{(r)}$ the cycle of codimension 1 in S on which $J^r X$ is supported (so e.g. $X^{(r)} = 0$ if $J^r X$ has no components of codimension 1 in S). That is, if $X = \sum \alpha_i X_i$ with X_i irreducible divisors, then $X^{(r)} = \sum (\alpha_i - r) X_i$, the sum extended over the i's such that $\alpha_i > r$. Abusing notations, we write X for $X^{(0)}$. Thus $X^{(1)} = X - X_{\text{red}}$, and $X^{(r)}$ is the r-th iteration of this operation. Observe that $J^r X$ and $X^{(r)}$ coincide away from the singular locus of X_{red}.

One particular but interesting case is the one in which X_{red} has only *isolated* singularities (of course this is not a restriction in case S is a surface). Then by the above observation and [F], Prop. 9.2:

$$s(J^rX,S) = s(X^{(r)},S) + \mu_r = \sum_{j=1}^{d}(-1)^{j+1}{X^{(r)}}^j + \mu_r$$

where μ_r is a contribution (in dimension 0) supported on the singularities of X_{red}: more precisely, μ_r is the Segre class in S of the residual scheme to $X^{(r)}$ in J^rX. Note that only singular points of X_{red} at which X has multiplicity $> r$ contribute to μ_r.

PROPOSITION 1.2. *Let S be a smooth surface. With the above notation, suppose $\mathcal{V} \longrightarrow P^r\mathcal{L}$ is surjective, and let K denote the canonical divisor of S. Then for $0 < i \leq r$*

$$\gamma_i \cdot m_X D^{(i)} = \left[\binom{i+2}{2}X - X^{(i)} + \binom{i+2}{3}K\right] X^{(i)} + \mu_i \quad .$$

PROOF: As observed above

$$s(J^iX,S) = X^{(i)} - {X^{(i)}}^2 + \mu_i \quad ,$$

where μ_i is supported on points, so this follows immediately from the theorem, after writing out $c_1(\mathrm{Sym}^i(\Omega \oplus \mathcal{O}) \otimes \mathcal{L})$. ▮

Comparing this result with a formula we will obtain in §3, it will follow that in fact μ_1 is the sum of the Milnor numbers of the singularities of X_{red} when S is a surface. It would be interesting to have a similar interpretation for the contributions μ_i, $i > 1$.

PROOF OF THEOREM I: Thinking of X as a point of $D^{(i)}$, one has $s(X, D^{(i)}) = m_X D^{(i)} X$ ([F], 4.3): thus $m_X D^{(i)} = \deg s(X, D^{(i)})$. Now we see $D^{(i)} \subset \mathbf{P}V$ as the projection from $S \times \mathbf{P}V = \mathbf{P}\mathcal{V}$ of the correspondence

$$\mathcal{D}^{(i)} = \{(p,X) \in \mathbf{P}\mathcal{V} \text{ s.t. } X \text{ has multiplicity} > i \text{ at } p\} \quad .$$

To see if (p,X) is in $\mathcal{D}^{(i)}$, one lifts X to any of its representatives in V, maps it to a section $s \in H^0(S,\mathcal{L})$ (near p this will simply be an equation for the divisor corresponding to X), then checks that the Taylor expansion of s at p has no terms in degree $\leq i$. Therefore:

(1) by identifying the fiber of $\mathbf{P}\mathcal{V}$ over X with S, the fiber of $\mathcal{D}^{(i)}$ is identified with J^iX;
(2) $\mathcal{D}^{(i)} = \mathbf{P}\mathcal{N}_i$, where $\mathcal{N}_i$ is the kernel of the 'truncated Taylor map' $\mathcal{V} \longrightarrow P^i\mathcal{L}$;
(3) also, the number γ_i of the statement of the theorem is the degree of the projection $\mathcal{D}^{(i)} \longrightarrow D^{(i)}$.

LEMMA. *In the above hypotheses:*

$$\gamma_i \cdot m_X D^{(i)} = \deg\left\{c(\mathcal{N}_i)^{-1} \cap s(J^iX,S)\right\}_0 \quad .$$

PROOF: If π denotes the projection $\mathcal{D}^{(i)} \to D^{(i)}$, then Proposition 4.2 (a) from [F], (3) and the identification from (1) above give us

$$\gamma_i \cdot s(X, D^{(i)}) = \pi_* s(J^i X, \mathcal{D}^{(i)}) \quad ;$$

taking degrees, we get

$$\gamma_i \cdot m_X D^{(i)} = \deg s(J^i X, \mathcal{D}^{(i)}) \quad .$$

So we are after this latter Segre class. Now both S and $\mathcal{D}^{(i)}$ ($= \mathbf{P}\mathcal{N}_i$ by (2)) are non-singular, so [F], 4.2.6 gives

$$c(T\mathcal{D}^{(i)}) \cap s(J^i X, \mathcal{D}^{(i)}) = c(TS) \cap s(J^i X, S) \quad ,$$

from which

$$s(J^i X, \mathcal{D}^{(i)}) = c(T\mathcal{D}^{(i)}|S)^{-1} \cap s(J^i X, S) \quad .$$

where $T\mathcal{D}^{(i)}|S$ is the relative tangent bundle of $\mathcal{D}^{(i)}$ over S. The Chern class of $T\mathcal{D}^{(i)}|S$ is computed by using the Euler sequence for $\mathcal{D}^{(i)} = \mathbf{P}\mathcal{N}_i$:

$$0 \to \mathcal{O} \to \mathcal{N}_i \otimes \mathcal{O}(1) \to T\mathcal{D}^{(i)}|S \to 0 \quad ,$$

where $\mathcal{O}(1)$ is the canonical line bundle; the restriction of this to the fiber over X is trivial, so in fact we get

$$s(J^i X, \mathcal{D}^{(i)}) = c(\mathcal{N}_i)^{-1} \cap s(J^i X, S) \quad ,$$

which concludes the proof of the lemma.

So proving the theorem amounts to computing the Chern class of $\mathcal{N}_i$ for $0 < i \leq r$, under the hypothesis that $\mathcal{V} \to P^r\mathcal{L}$ is surjective.

CLAIM. *Suppose $\mathcal{V} \to P^r\mathcal{L}$ is surjective for some $r \geq 0$. Then for $0 \leq i \leq r$*

$$c(\mathcal{N}_i)^{-1} = c(Sym^i(\Omega \oplus \mathcal{O}) \otimes \mathcal{L}) \quad .$$

PROOF: If $\mathcal{V} \to P^r\mathcal{L}$ is surjective, then following with the truncation surjections $P^i\mathcal{L} \to P^{i-1}\mathcal{L}$ shows that $\alpha_i : \mathcal{V} \to P^i\mathcal{L}$ is surjective for $0 \leq i \leq r$. Since $\mathcal{N}_i = \ker(\alpha_i)$, Whitney's formula gives $c(\mathcal{N}_i)^{-1} = c(P^i\mathcal{L})$.

To compute the latter, use the standard exact sequence for the bundles of principal parts:

$$0 \to \mathrm{Sym}^k\Omega \otimes \mathcal{L} \to P^k\mathcal{L} \to P^{k-1}\mathcal{L} \to 0$$

implies $c(P^k\mathcal{L})c(P^{k-1}\mathcal{L})^{-1} = c(\mathrm{Sym}^k\Omega \otimes \mathcal{L})$; recalling $P^0\mathcal{L} = \mathcal{L}$, we get

$$\begin{aligned} c(P^i\mathcal{L}) &= c(P^0\mathcal{L}) \cdot \prod_{k=1}^{i} c(P^k\mathcal{L})c(P^{k-1}\mathcal{L})^{-1} \\ &= c(\mathcal{L})c(\Omega \otimes \mathcal{L})c(\mathrm{Sym}^2\Omega \otimes \mathcal{L}) \cdots c(\mathrm{Sym}^i\Omega \otimes \mathcal{L}) \quad ; \end{aligned}$$

but this equals $c(\mathrm{Sym}^i(\Omega \oplus \mathcal{O}) \otimes \mathcal{L})$, so the claim and the theorem are proved. ∎

§2. Small discriminants.

We maintain the notations of section 1. In this section we choose $r > 0$, and *assume that the map $\alpha_r : \mathcal{V} \longrightarrow P^r\mathcal{L}$ of §1 is surjective*; recall that this condition is automatically satisfied for $r = 1$ if $\mathcal{L}$ is very ample (thus, in studying dual varieties).

For every $X \in \mathbb{P}V$ we have considered the class

$$\{c(\mathrm{Sym}^r(\Omega \oplus \mathcal{O}) \otimes \mathcal{L}) \cap s(J^rX, S)\}_0 \quad ,$$

where J^rX denotes the scheme of zeroes of $\alpha_r \circ s$, where s is a section of $\mathcal{V}$ corresponding to X. In the course of the proof, we have realized $D^{(r)}$ as the projection from $\mathbb{P}\mathcal{V} = S \times \mathbb{P}V$ to $\mathbb{P}V$ of a subbundle $\mathbb{P}\mathcal{N}_r$ of $\mathbb{P}\mathcal{V}$: specifically, $\mathcal{N}_r = \ker(\alpha_r)$. Since α_r is assumed to be surjective, the codimension of $\mathbb{P}\mathcal{N}_r$ in $\mathbb{P}\mathcal{V}$ equals the rank of $P^r\mathcal{L}$, that is $\binom{r+n}{n}$. Therefore, the codimension of $D^{(r)}$ in $\mathbb{P}V$ is $\geq \binom{r+n}{n} - n$.

Definition. *We say that the r-th discriminant is 'small' if its codimension in $\mathbb{P}V$ is $> \binom{r+n}{n} - n$.*

It follows from the above discussion that, in our hypotheses, the discriminant is small if and only if the projection $\mathbb{P}\mathcal{N}_r \longrightarrow D^{(r)}$ is not generically finite, that is if and only if the number of points of multiplicity $> r$ in the general element of $D^{(r)}$ is not finite—in other words, if and only if γ_r is 0. We will now show that this happens precisely when the above class *vanishes*.

Theorem II. *Suppose that the map $\mathcal{V} \xrightarrow{\alpha_r} P^r\mathcal{L}$ is surjective. Then the following are equivalent:*

(1) *For all $X \in \mathbb{P}V$*

$$\{c(\mathrm{Sym}^r(\Omega \oplus \mathcal{O}) \otimes \mathcal{L}) \cap s(J^rX, S)\}_0 = 0 \quad ;$$

(2) *For some $X \in D^{(r)}$*

$$\deg\{c(\mathrm{Sym}^r(\Omega \oplus \mathcal{O}) \otimes \mathcal{L}) \cap s(J^rX, S)\}_0 = 0 \quad ;$$

(3)

$$\deg\{c(\mathrm{Sym}^r(\Omega \oplus \mathcal{O}) \otimes \mathcal{L}) \cap [S]\}_0 = 0 \quad ;$$

(4) *the r-th discriminant $D^{(r)}$ is small.*

Proof: The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) are trivial ((3) is a particular case of (2), cf. the proof of Corollary 1.1).

Next, if the discriminant in $\mathbb{P}V$ is *not* small, then $\gamma_r \neq 0$. The degree in (3) is then the degree of the discriminant, multiplied by γ_r (by Corollary 1.1): thus it can't be 0. This shows (3) $\Longrightarrow$ (4).

Thus we simply have to show that (4) implies (1), that is that if the projection $\pi : \mathcal{D}^{(r)} = \mathbb{P}\mathcal{N}_r \longrightarrow D^{(r)}$ is *not* finite, then the class in (1) vanishes for all $X \in D^{(r)}$ (the class vanishes automatically if $X \notin D^{(r)}$). Observe that Theorem I implies immediately that the *degree* of the class is 0 if (4) holds, since in this case $\gamma_r = 0$; proving that the class itself is 0 requires a little more work.

Recall from the proof of Theorem I that the class between brackets in (1) equals the Segre class $s(J^{(r)}, \mathcal{D}^{(r)})$, and that $J^{(r)} = \pi^{-1}(X)$. Thus we have to show that if π is not finite then $\{s(\pi^{-1}(X), \mathcal{D}^{(r)})\}_0 = 0$ (notice that this follows immediately from [F], Proposition 4.2 (b) if π is *flat* over a neighborhood of X). This is a consequence of the following general result.

LEMMA. *Let $\pi : Y' \to Y$ be an onto morphism of irreducible schemes, with $\dim Y' > \dim Y$; let $X \in Y$ be a closed point, and denote by $X' = \pi^{-1}(X)$ the inverse image scheme.*

Then $\{s(X', Y')\}_0 = 0$.

PROOF: Let C, C' resp. denote the normal cones to X in Y, and X' to Y' resp. In this set up we get (see [**F**], proof of Prop. 4.2) an induced morphism

$$\mathbb{P}(C' \oplus 1) \xrightarrow{G} \mathbb{P}(C \oplus 1) \quad ,$$

such that the canonical line bundle on $\mathbb{P}(C' \oplus 1)$ is the pull-back $G^*\mathcal{O}(1)$ of the canonical line bundle on $\mathbb{P}(C \oplus 1)$. If q' denotes the projection from $\mathbb{P}(C' \oplus 1)$ to X', then

$$\{s(X', Y')\}_0 = q'_* \left(c_1(G^*\mathcal{O}(1))^{\dim Y'+1} \cap [\mathbb{P}(C' \oplus 1)] \right) \quad ;$$

but this is necessarily 0: indeed, certainly there exist $\dim Y' + 1$ sections of $\mathcal{O}(1)$ which don't vanish simultaneously anywhere on $\mathbb{P}(C \oplus 1)$ (because $\dim Y < \dim Y'$), and these pull-back to $\dim Y' + 1$ sections of $G^*(\mathcal{O}(1))$ that don't vanish simultaneously anywhere on $\mathbb{P}(C' \oplus 1)$. ∎

This proves the Lemma, and concludes the proof of the Theorem. ∎

REMARK. We find the implications (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1) particularly striking, as they seem to impose (in the hypotheses of the theorem) a strong condition on the Segre classes of the Jacobian schemes of divisors. For example, the vanishing of the class for one singular hyperplane section of a smooth variety $S \subset \mathbb{P}^n$ (in fact, the vanishing of its degree suffices) implies the vanishing of the class for *all* hyperplane sections, and that the dual of S is small. See Example 4.5 for an illustration of this fact.

§3. THE CASE $\dim(S) = 2$.

In this section we give an independent derivation of Proposition (1.2) for $i = 1$, over the complex numbers. In this particular situation we obtain more precise information, namely, that μ_1 is the sum of the Milnor numbers of the singularities of X_{red}.

Let S be a smooth compact algebraic surface over the complex numbers, $|\mathcal{L}|$ a very ample complete linear system on S and $D = D^{(1)} \subset |\mathcal{L}|$ the discriminant hypersurface, consisting of singular members of $|\mathcal{L}|$. It is known [E] that the dual variety D is actually a hypersurface.

Our goal is to determine, for each $X \in |\mathcal{L}|$, the multiplicity $m_X(D)$ of the hypersurface D at the point X. Suppose that

$$X = \sum_{1 \le i \le r} n_i X_i \tag{1}$$

where X_i is reduced and irreducible. Take a general $Y \in |\mathcal{L}|$ (i.e. a Y intersecting $X_{\text{red}} = \sum_{1 \le i \le r} X_i$ transversally) and denote by $L \subset |\mathcal{L}|$ the pencil containing X and Y. Then

$$m_X(D) = \deg\,(D) - s \tag{2}$$

where s is the number of singular members of L different from X (each of these singular members has one node as singular set). In order to determine s we shall blow up S to construct a family parametrized by L and use Lefschetz' formula ([G-H], page 509).

For each $1 \leq i \leq r$, denote by p_{ij} ($1 \leq j \leq Y \cdot X_i$) the points of intersection of Y and X_i. Let $\widehat{S}$ denote the surface obtained from S by blowing up at each p_{ij} n_i times (in the direction of Y). The induced pencil on $\widehat{S}$ is base-point-free and gives a map $f : \widehat{S} \to \mathbb{P}^1$. If E_{ij}^k, $k = 1, \ldots, n_i$ are the exceptional divisors at p_{ij} then the fiber of f at the point 0 corresponding to X is

$$f^*(0) = X + \sum_{i,j} \sum_{1 \leq k \leq n_i - 1} (n_i - k) E_{ij}^k$$

In other words, the special fiber is isomorphic to X with strings of $\mathbb{P}^1$'s (each $\mathbb{P}^1$ with a certain multiplicity) attached at the points p_{ij}; each string has $n_i - 1$ components.

We now denote

$$X' = (f^*(0))_{\mathrm{red}} = X_{\mathrm{red}} + \sum_{i,j} T_{ij}$$

the reduced (i.e. set-theoretic) fiber of f at 0, where $T_{ij} = \sum_{1 \leq k \leq n_i - 1} E_{ij}^k$ is the (reduced) string attached at p_{ij}. The argument in [G-H] works in the present circumstance (the topology does not "see" the multiplicities of the special fiber) and gives

$$\chi(S) = 2\chi(Y) - Y \cdot Y + (\chi(X') - \chi(Y)) + \sum_{1 \leq \lambda \leq s} (\chi(Y_\lambda) - \chi(Y))$$

where χ denotes topological Euler characteristic and Y_λ are the singular fibers for $\lambda \neq 0$. Since $\chi(Y_\lambda) - \chi(Y) = 1$ ([G-H] or (5) below) and $\deg(D) = \chi(S) - 2\chi(Y) + Y.Y$ ([G-H] or Corollary (1.1)), combining with (2) we obtain

(3) $$m_X(D) = \chi(X') - \chi(Y)$$

In order to compute $\chi(X')$, denoting $T = \bigcup_{i,j} T_{ij}$ we have

(4)
$$\begin{aligned}\chi(X') &= \chi(X_{\mathrm{red}} \cup T) = \chi(X_{\mathrm{red}}) + \chi(T) - \chi(X_{\mathrm{red}} \cap T) \\ &= \chi(X_{\mathrm{red}}) + \sum_{i,j} \chi(T_{ij}) - \sum_{i,j} \chi(\{p_{ij}\}) = \chi(X_{\mathrm{red}}) + \sum_{i,j} n_i - \sum_{i,j} 1 \\ &= \chi(X_{\mathrm{red}}) + X \cdot (X - X_{\mathrm{red}})\end{aligned}$$

Now we compute $\chi(X_{\mathrm{red}})$. Let $Z = \sum_{1 \leq i \leq r} Z_i$ be a reduced (connected, for simplicity) curve with normalization

$$\rho : \widetilde{Z} = \coprod_{1 \leq i \leq r} \widetilde{Z}_i \to Z$$

If $p \in Z$ is a singular point, denote $B(p) = \rho^{-1}(p)$ the set of branches of Z at p. Topologically, Z is obtained from the smooth surface $\widetilde{Z}$ by identifying each of the sets $B(p)$ to a point p. Recall from [Gr], page 96, that if X is a topological space and $A \subset X$ is a subspace such that (X, A) is a collared pair then $\chi^{\#}(X) - \chi^{\#}(X/A) = \chi^{\#}(A)$, where X/A is the space obtained from X by identifying A to a point and $\chi^{\#}$ is Euler characteristic for augmented homology. If X is a manifold and A consists of a points then $\chi^{\#}(X) - \chi^{\#}(X/A) = \chi(X) - \chi(X/A) = a - 1$. Applying this for each singular point we obtain

$$\chi(\widetilde{Z}) - \chi(Z) = \sum_{p \in Z} (b(Z,p) - 1) \tag{5}$$

where $b(Z, p)$ is the number of branches of Z at p. Also, from the exact sequence of sheaves

$$0 \to \mathcal{O}_Z \to \rho_* \mathcal{O}_{\widetilde{Z}} \to \rho_* \mathcal{O}_{\widetilde{Z}} / \mathcal{O}_Z \to 0$$

we obtain $1 - r + \sum_{p \in Z} \delta(Z, p) - h^1(\mathcal{O}_Z) + h^1(\mathcal{O}_{\widetilde{Z}}) = 0$, where we let $\delta(Z, p) = \text{length}_p(\rho_* \mathcal{O}_{\widetilde{Z}} / \mathcal{O}_Z)$, and then, combining with (5),

$$\chi(Z) = 2 - 2p_a(Z) + \sum_{p \in Z} \mu(Z, p) \tag{6}$$

where $\mu(Z, p) = 2\delta(Z, p) - b(Z, p) + 1$ is the Milnor number of (Z, p). Combining (3), (4) and (6) we obtain

$$\begin{aligned}
m_X(D) &= \chi(X_{\text{red}}) - \chi(Y) + X \cdot (X - X_{\text{red}}) \\
&= (2 - 2p_a(X_{\text{red}})) - (2 - 2p_a(Y)) + X \cdot (X - X_{\text{red}}) + \sum_{p \in X_{\text{red}}} \mu(X_{\text{red}}, p) \\
&= (K_S + Y) \cdot Y - (K_S + X_{\text{red}}) \cdot X_{\text{red}} + X \cdot (X - X_{\text{red}}) + \sum_{p \in X_{\text{red}}} \mu(X_{\text{red}}, p)
\end{aligned}$$

and rearranging we finally obtain

$$m_X(D) = (X - X_{\text{red}}) \cdot (K_S + 2X + X_{\text{red}}) + \sum_{p \in X_{\text{red}}} \mu(X_{\text{red}}, p) \tag{7}$$

Comparing with Proposition (1.2), $i = 1$, yields $\mu_1 = \sum_{p \in X_{\text{red}}} \mu(X_{\text{red}}, p)$ as claimed at the beginning of this section.

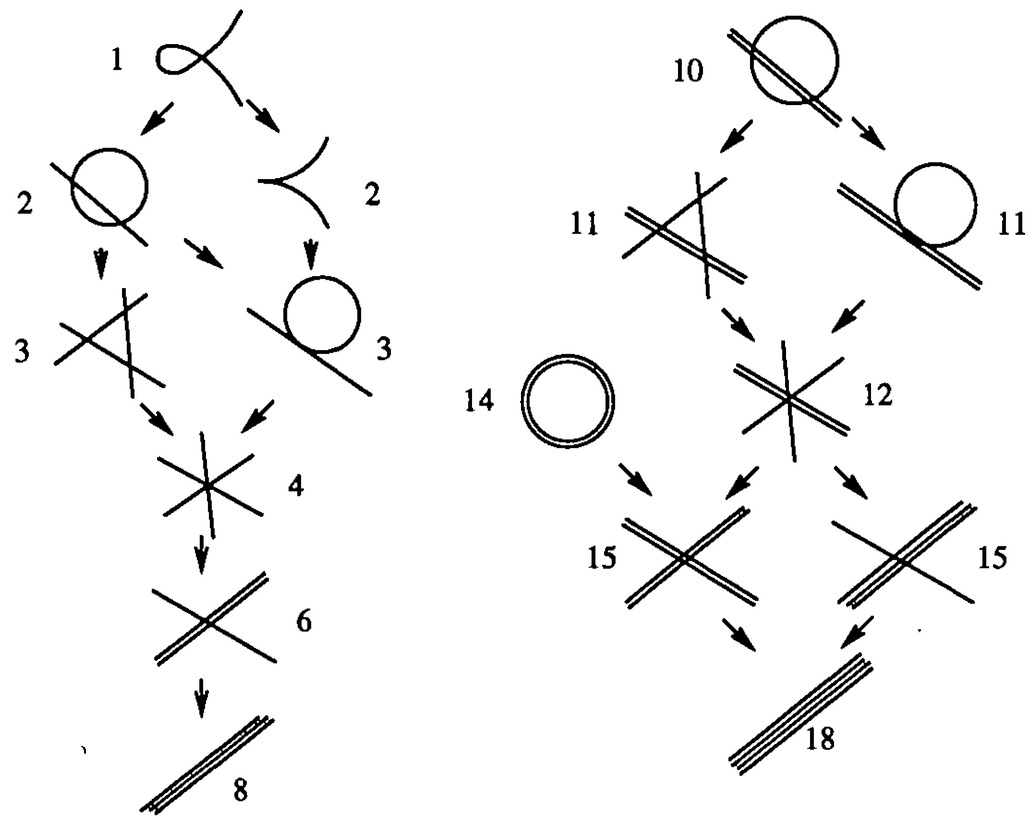
§4. Examples.

In this section we apply the results obtained thus far to a few concrete situations, to illustrate the actual 'computability' of the formulas.

Example 4.1. Applying the result in §3 we may compute the multiplicity of the discriminant of the space of plane curves of a given degree d at a singular curve X. If $d^{(1)}$ is the degree of $X^{(1)} = X - X_{\text{red}}$, then the formula in §3 gives

$$m_X D = \left[3(d-1) - d^{(1)}\right] d^{(1)} + \mu$$

where μ is the sum of the Milnor numbers of the singularities of X_{red}. Thus the multiplicity of the discriminant of plane conics at a double line is 2, while for degree 3 and 4, the following 'kinds' of singular (resp., non-reduced) curves occur, with the indicated multiplicity (arrows denote 'specialization'):



Example 4.2. For higher discriminants, let again $S = \mathbf{P}^2$, $\mathcal{L} = \mathcal{O}(d)$, and $V = H^0(\mathbf{P}^2, \mathcal{O}(d))$. For $0 < i \leq d$ let $d^{(i)} = \deg X^{(i)}$; then Proposition 1.2 gives

$$m_X D^{(i)} = \left[\binom{i+2}{2}(d-i) - d^{(i)}\right] d^{(i)} + \mu_i \quad .$$

(Indeed K has degree -3, and the map $S \times H^0(\mathbf{P}^2, \mathcal{O}(d)) \longrightarrow P^i\mathcal{O}(d)$ is clearly surjective for $i \leq d$. Also, it is clear that $\gamma_i = 1$ in this case.)

For example, for X a d-fold line one has $\mu_i = 0$ for all $i > 0$ (since X_{red} is non-singular), and $d^{(i)} = d - i$; so we get the multiplicity $m_{d,r}$ of the locus of degree-d plane curves with a $\geq r$-tuple point, along the locus of d-fold lines, for $0 < r \leq d$: (let $i = r - 1$ in the above)

$$\begin{aligned} m_{d,r} &= \left(\binom{r+1}{2} - 1\right)(d-r+1)^2 \\ &= \frac{(r+2)(r-1)}{2}(d-r+1)^2 \quad . \end{aligned}$$

In fact the computation runs just as easily to give the multiplicity $m_{n,d,r}$ of the locus degree-d hypersurfaces in $\mathbf{P}^n$ with a $\geq r$-tuple point, along the locus of d-fold hyperplanes. First, for $S = \mathbf{P}^n$ the $\binom{n+i}{n}$ Chern roots of $\mathrm{Sym}^i(\Omega \oplus \mathcal{O})$ are all equal to $-iH$, where H is the hyperplane class (indeed, the $n+1$ Chern roots of $\Omega \oplus \mathcal{O}$ are all equal to $-H$ since $c(\Omega \oplus \mathcal{O}) = (1-H)^{n+1}$) so

$$c(\mathrm{Sym}^i(\Omega \oplus \mathcal{O}) \otimes \mathcal{O}(d)) = (1+(d-i)H)^{\binom{n+i}{n}} \quad ;$$

with the same notation, $X^{(i)} = (d-i)H$; thus, applying the theorem gives

$$\begin{aligned} m_{n,d,r} &= \deg\left\{c(\mathrm{Sym}^{r-1}(\Omega \oplus \mathcal{O}) \otimes \mathcal{O}(d)) \cap s(X^{(r-1)}, \mathbf{P}^n)\right\}_0 \\ &= \text{ coeff. of } H^n \text{ in } \left[(1+(d-r+1)H)^{\binom{n+r-1}{n}} \frac{(d-r+1)H}{1+(d-r+1)H}\right] \\ &= (d-r+1)^n \cdot \binom{\binom{n+r-1}{n}-1}{n-1} \quad . \end{aligned}$$

For $r=2$ this gives the well known multiplicity $n(d-1)^n$ of a d-fold hyperplane in the ordinary discriminant. At the other end of the spectrum, we get the multiplicity of a d-fold hyperplane in the locus of degree-d hypersurfaces with a d-tuple point (i.e., the cones from a point over a degree-d hypersurface of $\mathbf{P}^{n-1}$):

$$\binom{\binom{n+d-1}{n}-1}{n-1} \quad .$$

Example 4.3. To illustrate a case in which the singularities of X_{red} are *not* isolated, consider $S = \mathbb{P}^3$, $\mathcal{L} = \mathcal{O}(d)$, and X = union of three planes, with multiplicities d_1, d_2, d_3 adding up to d. We are going to compute the multiplicity m_X of the ordinary discriminant at X. In this case $c((\Omega \oplus \mathcal{O}) \otimes \mathcal{O}(d)) = (1+(d-1)H)^4$, where H denotes the hyperplane class; so we only need to compute the Segre class of J^1X. One can distinguish several cases: in decreasing order of speciality (and thus necessarily with decreasing multiplicities)

—If $d_2 = d_3 = 0$, then X is simply a d-fold plane, so the previous example gives the multiplicity as

$$3(d-1)^3 \quad .$$

—If $d_1 \neq 0, d_2 \neq 0, d_3 = 0$, then X is the union of two distinct planes, say with equation $x^{d_1}y^{d_2} = 0$. The Jacobian scheme J^1X has ideal $(x^{d_1-1}y^{d_2}, x^{d_1}y^{d_2-1})$, that is the divisor $x^{d_1-1}y^{d_2-1} = 0$ with an embedded component along the line (x,y) at which the planes intersect. The Segre class of J^1X in $\mathbf{P}^3$ can then be obtained for example by applying [F], Prop. 9.2: the reader will check that $s(J^1X, \mathbb{P}^3)$ pushes forward to $\mathbf{P}^3$ to

$$(d-2)H + (1-(d-2)^2)H^2 + ((d-2)^3 - 3(d-2) - 2)H^3$$

and applying the theorem yields

$$m_X = (d-1)^2(3d-4) \quad .$$

—If $d_1 \neq 0, d_2 \neq 0, d_3 \neq 0$ and the three (distinct) planes intersect along a common line, then the same procedure gives for $s(J^1X, \mathbb{P}^3)$

$$(d-3)H + (4-(d-3)^2)H^2 + ((d-3)^3 - 12(d-3) - 16)H^3 \quad ,$$

from which

$$m_X = (d-1)^2(3d-5) \quad .$$

—Finally, if the three planes are in general position, say the equation for X is $x^{d_1}y^{d_2}z^{d_3}$. Then J^1X is the divisor $x^{d_1-1}y^{d_2-1}z^{d_3-1}$ with an embedded component along (xy, xz, yz), supported along three 'coordinate' lines. To compute $s(J^1X, \mathbb{P}^3)$ one can blow-up $\mathbb{P}^3$ at the point common to the lines, then blow-up again along the proper transforms of the lines: the inverse image of J^1X is a Cartier divisor in the top blow-up, and pushing forward the Segre class of this latter to $\mathbb{P}^3$ gives

$$s(J^1X, \mathbb{P}^3) = (d-3)H + (3-(d-3)^2)H^2 + ((d-3)^3 - 9(d-3) - 10)H^3 \quad :$$

from which, applying Theorem I again, we get

$$m_X = (d-2)(d-1)(3d-2) \quad .$$

Example 4.4. Again let $S = \mathbf{P}^3$, and let $\mathcal{L} = \mathcal{O}(d_1 + d_2)$. Let X_1, X_2 be smooth hypersurfaces of degrees d_1, d_2, intersecting along a curve C. If $X = X_1 \cup X_2$, then X is singular along C; the multiplicity m_X of the discriminant at X is then

$$m_X = d_1d_2(3(d_1+d_2) - 4) \quad .$$

Indeed, in this case $J^1X = C$ is regularly embedded in $S = \mathbb{P}^3$, so $s(J^1X, S) = c(N_C\mathbb{P}^3)^{-1} \cap [C]$ pushes forward to $\mathbb{P}^3$ to

$$\frac{d_1d_2H^2}{(1+d_1H)(1+d_2H)} = d_1d_2(H^2 - (d_1+d_2)H^3)$$

(where again H is the hyperplane class in $\mathbb{P}^3$), while $c((\Omega \oplus \mathcal{O}) \otimes \mathcal{O}(d)) = (1 + (d-1)H)^4$: so

$$m_X = \deg\left\{(1 + 4(d_1+d_2-1)H) \cdot d_1d_2(H^2 - (d_1+d_2)H^3)\right\}_0$$

with the above result. More generally, say a complete intersection curve C is (scheme-theoretically) a connected component of the singular scheme of a degree-d hypersurface X in $\mathbf{P}^n$, and $\deg C = r$, $\deg(c_1(T_C)) = 2-2g$; then C 'contributes' to m_X by

$$\deg\left\{(1+(d-1)H)^{n+1}\frac{rH^{n-1} + c_1(T_C)}{(1+H)^{n+1}}\right\}_0 = r(d-2)(n+1) + 2 - 2g \quad .$$

However, at least when C is smooth there are strong constraints on what r, d, n, g can actually be realized, so that for example the genus of the curve is determined by r, d, n. One can show that in this case the multiplicity will necessarily be

$$\frac{r(4 + (d-2)(n+3))}{2} \quad .$$

These constraints arise by comparing Theorem I to Parusiński's results ([**Pa**]). For example, a smooth curve of genus 2 *cannot* appear as the singular scheme of a hypersurface of $\mathbb{P}^n$. We will prove these facts elsewhere.

Example 4.5. Let $S = \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_r}$, let $H_1, \ldots, H_r$ be the pull-backs of the hyperplane classes from the factors, and $\mathcal{L} = \mathcal{O}(H_1 + \cdots + H_r)$—that is, the bundle defining the Segre embedding of S. With this embedding, when is the dual variety of S a hypersurface? The following criterion is proved in [**G-K-Z**], §3, and can be deduced from more general criteria in [**K-M**] (see [**G-K-Z**], Theorem 1.3 and Lemma 3.5). We give here a simple direct argument.

PROPOSITION 4.1. *The dual variety of $S = \mathbf{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ is a hypersurface if and only if $2n_i \le n = \sum_j n_j$ for all i.*

PROOF: Let $x^i = (x_0^i, \ldots, x_{n_i}^i)$ denote homogeneous coordinates on $\mathbb{P}^{n_i}$. If $h \in H^0(S, \mathcal{L})$ then h is a multilinear function in $x^1, \ldots, x^r$, and h is a singular section if and only if the system of equations

$$\frac{\partial h}{\partial x_j^i} = 0 \qquad (1 \le i \le r, 0 \le j \le n_i)$$

has a non-trivial solution (i.e. a solution with $x^i \neq 0$ for all i).

Suppose that the condition $2n_i \le n$ for all i is not satisfied; for simplicity of notation let us assume that it fails for $i = 1$, so that $n_1 > m_1 = \sum_{j>1} n_j$. Let $h \in H^0(S, \mathcal{L})$ denote any singular section. We claim that the singular locus of h is positive-dimensional, and hence the dual of S is small. In fact, let $x = (x^1, x^2, \ldots, x^r)$ denote a singular point of h; it is easy to see from the system of equations above that the singular points of h of the form $(y^1, x^2, \ldots, x^r)$, $y^1 \in \mathbb{P}^{n_1}$, form a family of dimension $n_1 - m_1$.

Conversely, suppose $2n_i \le n$ for all i. We prove by induction on r that there exist sections h with isolated singularities. The initial case $r = 2$ is easy; for $r > 2$, arrange the indices so that $n_i \ge n_{i+1}$ for all i, let $S' = \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_r}$, and $\mathcal{L}' = \mathcal{O}(H_2 + \cdots + H_r)$. We claim that there exist $h_0 \in H^0(S', \mathcal{L}')$ with singularity locus of dimension at most n_1 (notice that $n_1 \le m_1 = \dim(S')$). To see this, we treat two separate cases: first, if $n_2 \le N = \sum_{j>2} n_j$ then by the inductive hypothesis there exists $h_0 \in H^0(S', \mathcal{L}')$ with isolated singularities; second, if $n_2 > N$ take $h_0 = \sum_{0 \le j \le N} x_j^2 f_j(x^3, \ldots, x^r)$ where the f_j are general multilinear forms in the indicated variables. It is easy to see (as in the argument above) that the singular locus of h_0 has dimension $n_2 - N < n_1$. Choose then such an h_0 and choose non-singular sections $h_1, \ldots, h_{n_1} \in H^0(S', \mathcal{L}')$ so that the set $h_1 = \cdots = h_{n_1} = 0$ intersects the singular locus of h_0 in isolated points (since $\mathcal{L}'$ is very ample, this can be achieved). It is now easy to check that $h = \sum x_i^1 h_i$ has isolated singularities. ∎

It is natural to try to use Theorem II to prove (4.1). Curiously, if one uses (3) in Theorem II (or equivalently [F], p. 63), the combinatorics becomes rather involved; but it is easy to show that the dual of S is small if the numerical conditions are not satisfied, using Theorem II (2). By just choosing *one* singular section and showing that the class in (2) is 0, one shows that *all* singular sections must have positive dimensional singular locus.

So we need to produce a singular divisor X on S such that

$$\deg \left\{c((\Omega \oplus \mathcal{O}) \otimes \mathcal{L}) \cap s(J^1X, S)\right\}_0 = 0 \quad ;$$

the trick is to choose X so that J^1X is 'contained in one factor' of the product, so that the computation of the Chern class becomes manageable.

Using the notation above, $n_1 > m_1$. Let $(x_0 : \cdots : x_{n_1})$ be coordinates in $\mathbf{P}^{n_1}$, and choose m_1 sections $h_i \in H^0(S', \mathcal{L}')$ for $0 \leq i < m_1$ that are non-singular and intersect transversally. Then let X be the divisor on S defined by $h = \sum_{0 \leq i < m_1} x_i h_i$: J^1X has ideal $(h_0, \ldots, h_{m_1-1}, x_0, \ldots, x_{m_1-1})$—that is, it consists of the disjoint union of several spaces $\mathbf{P}^{n_1-m_1} = \mathbf{P}^{n_1-m_1} \times \{(p_2, \ldots, p_r)\}$, as $(p_2, \ldots, p_r)$ runs through the list of the points of intersection of $h_0, \ldots, h_{m_1-1}$. It's enough then to show that the above degree is 0 for each of these components. Now notice that $H_2, \ldots, H_r$ are trivial on each component, so with obvious notations

$$\begin{aligned}
&\left\{c((\Omega \oplus \mathcal{O}) \otimes \mathcal{L}) \cap s(\mathbf{P}^{n_1-m_1}, S)\right\}_0 \\
&= \left\{c((\Omega_{\mathbf{P}^{n_1}} \oplus \mathcal{O}^{m_1+1}) \otimes \mathcal{O}(H_1)) \cap \frac{[\mathbf{P}^{n_1-m_1}]}{(1+H_1)^{m_1}}\right\}_0 \\
&= \left\{(1+H_1)^{m_1} \frac{[\mathbf{P}^{n_1-m_1}]}{(1+H_1)^{m_1}}\right\}_0 \\
&= \left\{[\mathbf{P}^{n_1-m_1}]\right\}_0 = 0
\end{aligned}$$

since $n_1 > m_1$. ∎

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Mathematics Department, Florida State University, Tallahassee FL 32306

Mathematics Department, University of Kansas, Lawrence KS 66045