FUNCTORIALITY IN RESOLUTION OF SINGULARITIES

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To Heisuke Hironaka for his 77th birthday,
in celebration of Kiju — *joy and long life!*

Contents

1. Introduction

Hironaka's theorem on resolution of singularities in characteristic zero [Hi1, 1964] is one of the monuments of twentieth century mathematics. Canonical or functorial versions of Hironaka's theorem were established more than fifteen years ago, by Villamayor [V1, V2] and by the authors [BM3, BM4]. Two new treatments of canonical resolution of singularities have appeared just in the last couple of years, by Włodarcyzk [W] and Kollár [Ko]. These papers reflect a continuing interest in a better understanding of desingularization, in large part with the goal of discovering techniques that extend to positive characteristic (cf. [Hi3, Hi4, Ka]). They also show that certain aspects have remained mysterious, even in characteristic zero. Our goal here is to clarify these issues (and raise some questions).

The various proofs of desingularization have a lot in common, but also important differences that are related to the notion of *functoriality* or *canonicity* involved. They all involve reducing the problem to "canonical desingularization" of a collection of local resolution data, originating in Hironaka's idea of an *idealistic exponent*, and called a *presentation* [BM4], *basic object* [V2] or *marked ideal* [W] (with variations in the meaning of these objects). In all cases, canonical desingularization is proved by induction on dimension. Functoriality (apart from its intrinsic interest) plays two important roles: (i) in the statements of the theorems that can be proved as consequences of canonical desingularization of a marked ideal; (ii) in the proof

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itself — it is sometimes easier to prove a stronger theorem by induction, because we can make a stronger inductive assumption. (We use Włodarcyzk's terminology "marked ideal", though our notion is a little more general. See §1.1.)

We begin with the theorem of *canonical resolution of singularities* in characteristic zero (not the most general statement, but formulated in a way that is easy to state and includes the most useful conditions). For simplicity, we restrict our attention to algebraic varieties (or reduced separated schemes of finite type). (There are more general theorems for nonreduced schemes; see [Hi1], [BM4, Sect. 11]. Moreover, all results below apply to analytic spaces, where the sequences of blowings-up are finite over any relatively compact open set.)

Theorem 1.1. *Given an algebraic variety X over a field of characteristic zero, there is finite sequence of blowings-up σ_j with smooth centres,*

$$X = X_0 \xleftarrow{\sigma_1} X_1 \longleftarrow \cdots \xleftarrow{\sigma_t} X_t\,,$$

such that:

1. *X_t is smooth and the exceptional divisor E_t in X_t has only normal crossings.*
2. *The morphism σ given by the composite of the σ_j is an isomorphism over $X \setminus \operatorname{Sing} X$.*
3. *The resolution morphism $\sigma_X = \sigma : X_t \to X$ (or the entire sequence of blowings-up σ_j) can be associated to X in a way that is functorial (at least with respect to étale or smooth morphisms, and field extensions).*

This theorem can be proved with following stronger version of the condition (2): For each j, let $C_j \subset X_j$ denote the centre of the blowing-up $\sigma_{j+1} : X_{j+1} \to X_j$, and let E_j denote the exceptional divisor of $\sigma_1 \circ \cdots \circ \sigma_j$. Then:

(2′) *For each j, either $C_j \subset \operatorname{Sing} X_j$ or X_j is smooth and $C_j \subset \operatorname{supp} E_j$.*

Theorem 1.1 is proved in [BM4], in [Hi1] without the functoriality condition (3), and in [V2] assuming Hironaka's reduction to the case of an idealistic exponent [Hi2]. A weaker version of the theorem is proved in [EV1, W, Ko], where the blowings-up σ_j do not necessarily have smooth centres. (We recall that every birational morphism of quasi-projective varieties is a blowing-up with centre that is not necessarily smooth [Ha, Thm. 7.17].) We do not know of a proof that provides the condition of smooth centres without also giving (2′). (See §1.3 below.)

There is no version of Theorem 1.1 that is functorial with respect to all morphisms. Consider the morphism φ from $\mathbb{A}^2$ to the quadratic cone $X := \{uv - w^2 = 0\} \subset \mathbb{A}^3$ given by $\varphi(x, y) = (x^2, y^2, xy)$ [Ko, Ex. 3.4]. Then $\sigma_{\mathbb{A}^2}$ is the identity morphism (by functoriality with respect to translations), so functoriality with respect to φ would imply that φ lifts to the blowing-up $\sigma : X' \to X$ over the origin (for example, by the universal mapping property of blowing up). But this is clearly false.

1.1. **Reduction to desingularization of a marked ideal.** For the sake of an inductive proof of Theorem 1.1, it is natural to deduce the result from an *embedded* version: We assume that $X \subset M$, where M is smooth. Then the successive blowings-up of X are given by restricting a sequence of blowings-up of M to the corresponding strict transforms of X.

All currently available proofs of canonical embedded resolution of singularities follow Hironaka's idea of reducing the result to "functorial desingularization" of an ideal with associated multiplicity [Hi2]:

A *marked ideal* is a quintuple $\underline{\mathcal{I}} = (M, N, E, \mathcal{I}, d)$, where N is a smooth subvariety of M, E is a normal crossings divisor, $\mathcal{I}$ is a coherent ideal (sheaf) on N, and $d \in \mathbb{N}$. $\underline{\mathcal{I}}$ encodes the data we have at each step of resolution: N will be a *maximal contact* subvariety and E an exceptional divisor. A marked ideal has a transformation law that is simpler than strict transform: Say that a blowing-up $\sigma : M' \to M$ is *admissible* if its centre $C \subset \operatorname{cosupp} \underline{\mathcal{I}} := \{x \in N : \operatorname{order}_x \mathcal{I} \geq d\}$, and C, E have only normal crossings. The *(weak) transform* $\mathcal{I}'$ of $\mathcal{I}$ by σ is generated by $y_{\mathrm{exc}}^{-d} f \circ \sigma$, for all $f \in \mathcal{I}$, where y_{exc} is a local generator of the ideal of $\sigma^{-1}(C)$. We define a *resolution of singularities* of $\underline{\mathcal{I}}$ as a finite sequence of admissible blowings-up, after which the cosupport is empty. (See Section 2 for detailed definitions.)

The definition of a marked ideal above is a slight generalization of that of [W] and corresponds to a *presentation* in [BM4]. Włodarczyk's marked ideal would be given by $(N, E|_N, \mathcal{I}, d)$.

We prove desingularization of a marked ideal by induction on $\dim N$: We reduce the dimension by passing to an *equivalent coefficient ideal* on a smooth *maximal contact hypersurface*. (See Defns. 2.5 and Section 4.) Maximal contact exists locally in characteristic zero, but is not unique. Coefficient ideals are defined using ideals of derivatives of $\mathcal{I}$ (The notion goes back to [Hi2, G], and is used in [V2]). An idea in [BM4, BM5] was to use derivatives preserving the exceptional divisor E (i.e., derivatives logarithmic with respect to the exceptional variables). The idea is natural to the strong version of functoriality in Theorem 1.3 below (see §3.4), and is crucial to Kawanoue's proposal for desingularization in positive characteristic [Ka]. The use of derivatives preserving E in resolution of singularities occurs already in [C].

There are at least two approaches to proving Theorem 1.1 using desingularization of a marked ideal. Given $X \subset M$, let $\mathcal{I}_X \subset \mathcal{O}_M$ denote the ideal of X. First, one can desingularize the marked ideal $\underline{\mathcal{I}} = (M, M, \emptyset, \mathcal{I}_X, 1)$ and restrict the sequence of blowings-up of M that are involved to the successive strict transforms of X. This provides embedded resolution of singularities of X in the case that $X \subset M$ is a hypersurface (i.e., of codimension 1), but only a weaker theorem in general (cf. [EV1, W]): Consider the sequence of blowings-up $\{\sigma_j\}$ of M that resolves the singularities of $\underline{\mathcal{I}}$ (and let $\underline{\mathcal{I}}_j$ denote the corresponding transforms of $\underline{\mathcal{I}}$). For some r, the centre C_r of σ_{r+1} coincides with $\operatorname{cosupp} \underline{\mathcal{I}}_r$. Then X_r is smooth and has only normal crossings with E_r (by Defns. 2.2). We get conditions (1)–(3) of Theorem 1.1, but the intersections $C_j \cap X_j$ are not necessarily smooth; i.e., the induced sequence of blowings-up of X does not necessarily have smooth centres (nor do we necessarily get $C_j \cap X_j \subset \operatorname{Sing} X_j$). See Example 8.2.

To get Theorem 1.1 in full, we follow a second approach (of origin in [Hi2]) which requires a stronger idea of functoriality in desingularization of a marked ideal. This approach involves *equisingularity* of the centres of blowing up with respect to the *Hilbert-Samuel function*; Theorem 1.1 is obtained using a marked ideal (different from that considered above) which is constructed using a distinguished set of local generators of $\mathcal{I}_X$. We know of no other way to get Theorem 1.1 in full. (See §1.3 and Question 1.6.)

1.2. **Test sequences and functorial desingularization.** Hironaka [Hi2] already proposed to deal with the preceding issue by using blowings-up not only as building blocks in resolution of singularities, but also to test for equisingularity of marked ideals. Define a *weak test sequence* for $\underline{\mathcal{I}}$ as a sequence of morphisms, each either an admissible blowing-up or a projection $M \times \mathbb{A}^1 \to M$. Say that two marked ideals on M are *weak equivalent* if they have the same weak test sequences (Defns. 2.4).

Functoriality in a general sense should mean that "equivalent" marked ideals undergo the same resolution process. In [Hi2, §9], Hironaka asserts (without proof) that a marked ideal can be desingularized functorially, not only with respect to smooth morphisms, but also with respect to weak equivalence classes (i.e., weak equivalent marked ideals have the same resolution sequences).

Question 1.2. Is this true?

The question is interesting partly in view of the role played by weak equivalence in [Hi3], which develops techniques for use in positive characteristic. We do not even know whether the algorithms of [BM4, EV1, W, Ko] are functorial in the sense of Hironaka's assertion.

In [BM4, BM5], we realize Hironaka's philosophy by enlarging the class of test transformations to include *exceptional blowing-ups* (centre = an intersection of two components of the exceptional divisor). Say that two marked ideals on M are *equivalent* if they have the same sequences of test transformations. (See Defns. 2.5.)

Theorem 1.3 (Functorial resolution of singularities of a marked ideal)**.** *Let* $\underline{\mathcal{I}} = (M, N, E, \mathcal{I}, d)$ *denote a marked ideal. Set* $\underline{\mathcal{I}}_0 := \underline{\mathcal{I}}$, $\underline{\mathcal{I}}_0 = (M_0, N_0, E_0, \mathcal{I}_0, d_0)$. *Then there is a finite sequence of blowings-up with smooth centres,*

$$M = M_0 \xleftarrow{\sigma_1} M_1 \longleftarrow \cdots \xleftarrow{\sigma_t} M_t\,, \tag{1.1}$$

where σ_1 *is admissible for* $\underline{\mathcal{I}}_0$ *and each successive* σ_{j+1} *is admissible for the transform* $\underline{\mathcal{I}}_j$ *of* $\underline{\mathcal{I}}_{j-1}$, *such that*

$$\operatorname{cosupp} \underline{\mathcal{I}}_t = \emptyset\,.$$

Moreover, a resolution sequence (1.1) can be associated to every marked ideal $\underline{\mathcal{I}}$ *in a way that is functorial with respect to equivalence classes of marked ideals and smooth morphisms.*

The *functoriality* assertion here means that, if $\underline{\mathcal{I}} = (M, N, E, \mathcal{I}, d)$, $\underline{\mathcal{I}}_1 = (M_1, N_1, E_1, \mathcal{I}_1, d_1)$ are marked ideals and $\varphi : M_1 \to M$ is a smooth morphism such that $\underline{\mathcal{I}}_1$ is equivalent to $\varphi^*(\underline{\mathcal{I}})$, then φ lifts to smooth morphisms throughout the resolution towers for $\underline{\mathcal{I}}_1$, $\underline{\mathcal{I}}$. Functoriality with respect to smooth morphisms in [W, Ko] is the weaker assertion that φ lifts to smooth morphisms throughout the resolution towers for $\varphi^*(\underline{\mathcal{I}})$ and $\underline{\mathcal{I}}$. When we say that φ lifts to smooth morphisms throughout the resolution sequences, we allow the possibility of inserting trivial blowings-up (i.e., identity morphisms) in the resolution tower for $\underline{\mathcal{I}}_1$. (For example, if $\varphi : U \hookrightarrow M$ is an embedding of an open subvariety U, then the centre of a given blowing-up in the resolution sequence for $\underline{\mathcal{I}}$ may have no points over U.) This issue is treated in Section 7.

We give a complete proof of Theorem 1.3 in Sections 5–7. The proof is essentially a repetition of that in [BM4], but we use the very clear inductive framework of [W]. Part of our purpose is to clarify the relationship between various proofs, particularly regarding the role of functoriality and the different notions of derivative ideal

(Sect. 5). We also want to isolate the precise role of a desingularization invariant (Sect. 7) and to show that the idea of equivalence makes it possible not only to prove the strong version of Theorem 1.1, but also to compare the actual recipes for choosing the centres of blowing up in different versions of desingularization of a marked ideal. In particular, we get the following corollary of our proof of Theorem 1.3.

Corollary 1.4. *The algorithm for resolution of singularities of marked ideals of Włodarczyk* [W] *or Kollár* [Ko] *coincides with that of* [BM4].

Here of course the blowings-up of M have smooth centres; our discussion in §1.1 above concerns the way that desingularization of a marked ideal is applied to prove Theorem 1.1. Corollary 1.4 is proved in §8.1. We know of no way to prove it other than using the strong notion of functoriality in Theorem 1.3. Our proof of Theorem 1.3 shows that the association of a resolution sequence to a marked ideal is also functorial with respect to field extensions and to closed embeddings $\iota : M \hookrightarrow M'$ in the case that $d = 1$ (as in [W, Ko]), but we do not explicitly discuss these issues.

Remark 1.5. There is another application of our notion of functoriality, raised by remarks in [Hi2] and in an earlier version of [Ko]. Given two ideal sheaves one of which is integral over the other, it is easy to see that the associated marked ideals are equivalent, so that Theorem 1.3 associates the same resolution sequence to them (cf. [EV2]).

Theorem 1.3 is proved by induction on $\dim \underline{\mathcal{I}} := \dim N$. The inductive step breaks up into two distinct steps (Section 5):

I. Functorial desingularization of a marked ideal of dimension $n-1$ implies functorial desingularization of a marked ideal of *maximal order*, of dimension n. (See Defns. 2.1.)

II. Functorial desingularization of a marked ideal of maximal order of dimension n implies functorial desingularization of a general marked ideal of dimension n.

Step I involves passing to an equivalent coefficient ideal on a local maximal contact hypersurface. We have to show that the centres of blowing up given by canonical desingularization of the coefficient ideals (which holds by induction) in overlapping coordinate charts, glue together to define a global centre of blowing up for the original marked ideal. Our proof and those of [W, Ko] treat this problem in rather different ways: In our induction, the coefficient ideals in overlapping charts are equivalent, so that Step I follows from the inductive assumption of strong functorial desingularization in lower dimension – this is the reason for our notion of equivalence. But Step II requires some technical work. On the other hand, in [W, Ko], most of the work is required for Step I (passage to a “homogenized ideal”, to reduce to a situation where maximal contact is uniquely determined up to an étale automorphism), while II is simpler.

1.3. **Use of the Hilbert-Samuel function.** We can deduce Theorem 1.1 (including (2′)) from Theorem 1.3, following Hironaka’s philosophy in [Hi2]. We use the *Hilbert-Samuel function*

$$H_{X,x}(k) := \text{length}\, \frac{\mathcal{O}_{X,x}}{\mathfrak{m}_{X,x}^{k+1}}\,, \quad k \in \mathbb{N}\,.$$

The Hilbert-Samuel function $H_{X,\cdot} : X \to \mathbb{N}^{\mathbb{N}}$ has the following basic properties, established by Bennett [Be] (see [BM2, BM4] for simple proofs): (1) $H_{X,\cdot}$ distinguishes smooth and singular points; (2) $H_{X,\cdot}$ is Zariski upper-semicontinuous; (3) $H_{X,\cdot}$ is *infinitesimally upper-semicontinuous* (i.e., $H_{X,\cdot}$ cannot increase after blowing-up with centre on which it is constant); (4) Any decreasing sequence in the value set of the Hilbert-Samuel function stabilizes.

The Hilbert-Samuel function $H_{X,\cdot}$ is a local invariant that plays the same role with respect to strict transform of a variety X as the order with respect to (weak) transform of a (marked) ideal. More precisely, for all $a \in X$, there is a local embedding $X \hookrightarrow M$ at a and a marked ideal $\underline{\mathcal{I}} = (M, N, \emptyset, \mathcal{I}, d)$ which has the same *test sequences* as $\underline{X} := (M, \emptyset, X, H)$, where $H = H_{X,a}$. (We define a *test sequence* for $\underline{X} = (M, E, X, H)$ by analogy with that for a marked ideal (Defns. 2.5), but where a blowing-up $\sigma : M' \to M$ with smooth centre C is *admissible* if $C \subset \operatorname{supp} \underline{X} := \{x \in X : H_{X,x} \geq H\}$ and C, E have only normal crossings, and where X transforms by *strict transform*.) We call $\underline{\mathcal{I}}$ a *presentation* of $H_{X,\cdot}$ at a. If a is a maximum point of $H_{X,\cdot}$, then the maximal value of $H_{X,\cdot}$ decreases after desingularization of $\underline{\mathcal{I}}$, and Theorem 1.1 follows (using the basic properties of $H_{X,\cdot}$ above and) using functoriality in the same way as in Step I of the proof of Theorem 1.3. (See [BM4, Ch. III] for these results.)

We are interested in equisingularity with respect to the Hilbert-Samuel function as a *necessary* rather than only sufficient condition on the centres of blowing up:

Question 1.6. Does any local invariant $\iota_{X,\cdot}$ which satisfies the basic properties (1)–(4) above and which admits a presentation determine the Hilbert-Samuel function $H_{X,\cdot}$? For example, is the stratification of X by the values of $\iota_{X,\cdot}$ a refinement of that by $H_{X,\cdot}$?

2. Marked ideals and test transformations

In this section we give detailed definitions of the notions concerning marked ideals, test transormations and equivalence that have been introduced informally in Section 1.

Definitions 2.1. A *marked ideal* $\underline{\mathcal{I}}$ is a quintuple $\underline{\mathcal{I}} = (M, N, E, \mathcal{I}, d)$, where:

(1) M, N are smooth, and $N \subset M$ is a closed subvariety;
(2) $E = \sum_{i=1}^{s} H_i$ is a simple normal crossings divisor on M which is tranverse to N and *ordered*. (The H_i are smooth hypersurfaces in M, not necessarily irreducible, with ordered index set as indicated.)
(3) $\mathcal{I}$ is an ideal (i.e., a coherent ideal sheaf) in $\mathcal{O}_N$.
(4) $d \in \mathbb{N}$.

We define the *cosupport* $\operatorname{cosupp} \underline{\mathcal{I}}$ of a marked ideal $\underline{\mathcal{I}} = (M, N, E, \mathcal{I}, d)$ as

$$\operatorname{cosupp} \underline{\mathcal{I}} := \{x \in N : \operatorname{ord}_x \mathcal{I} \geq d\},$$

where $\operatorname{ord}_x \mathcal{I}$ denotes the largest nonnegative integer q such that $\mathcal{I}_x \subset \mathfrak{m}_{N,x}^q$, with $\mathfrak{m}_{N,x}$ the maximal ideal of $\mathcal{O}_{N,x}$.

We say that a marked ideal $\underline{\mathcal{I}} = (M, N, E, \mathcal{I}, d)$ is of *maximal order* if $d = \max\{\operatorname{ord}_x \mathcal{I} : x \in \operatorname{cosupp} \underline{\mathcal{I}}\}$.

Definitions 2.2. Let $\underline{\mathcal{I}} = (M, N, E, \mathcal{I}, d)$ denote a marked ideal. A blowing-up $\sigma : M' \to M$ (with smooth centre C) is *admissible* for $\underline{\mathcal{I}}$ if $C \subset \operatorname{cosupp} \underline{\mathcal{I}}$, and C, E have only normal crossings.

We define the *transform* of $\underline{\mathcal{I}}$ by an admissible blowing-up $\sigma : M' \to M$ as the marked ideal $\underline{\mathcal{I}}' = (M', N', E', \mathcal{I}', d' = d)$, where:

(1) N' is the strict transform of N by σ.
(2) $E' = \sum_{i=1}^{s+1} H_i'$, where H_i' denotes the strict transform of H_i, for each $i = 1, \dots, s$, and $H_{s+1}' := \sigma^{-1}(C)$ (the exceptional divisor of σ, introduced as the last member of E').
(3) $\mathcal{I}' := \mathcal{I}_{\sigma^{-1}(C)}^{-d} \cdot \sigma^*(\mathcal{I})$ (where $\mathcal{I}_{\sigma^{-1}(C)} \subset \mathcal{O}_{N'}$ denotes the ideal of $\sigma^{-1}(C)$). I.e., $\mathcal{I}'$ is generated locally by $y_{\mathrm{exc}}^{-d} f \circ \sigma$, for all local generators f of $\mathcal{I}$, where y_{exc} denotes a local generator of $\mathcal{I}_{\sigma^{-1}(C)}$.

In this definition, note that $\mathcal{I}'$ is divisible by $\mathcal{I}_{\sigma^{-1}(C)}^{d}$ and E' is a normal crossings divisor transverse to N', because σ is admissible.

Definition 2.3. Let $\underline{\mathcal{I}} = (M, N, E, \mathcal{I}, d)$ denote a marked ideal, and let $\varphi : M_1 \to M$ be a smooth morphism. We define the *pull-back* $\varphi^*\underline{\mathcal{I}}$ of $\underline{\mathcal{I}}$ as the marked ideal $\varphi^*\underline{\mathcal{I}} := (M_1, \varphi^{-1}(N), \varphi^{-1}(E), \varphi^*(\mathcal{I}), d)$.

Definitions 2.4. Let $\underline{\mathcal{I}} = (M, N, E, \mathcal{I}, d)$ denote a marked ideal as above. Consider the product of M with a line, $M' := M \times \mathbb{A}^1$, and let $\pi : M' \to M$ denote the projection morphism. We define the *transform* $\underline{\mathcal{I}}'$ of $\underline{\mathcal{I}}$ by π as the marked ideal $\underline{\mathcal{I}}' = (M', N', E', \mathcal{I}', d' = d)$, where $N' := \pi^{-1}(N)$, $\mathcal{I}' := \pi^*(\mathcal{I})$, and $E' = \sum_{i=1}^{s+1} H_i'$, where $H_i' := \pi^{-1}(H_i)$, for each $i = 1, \dots, s$, and H_{s+1}' denotes the *horizontal divisor* $D := M \times \{0\}$ (included as the last member of E'). (Of course, π is a smooth morphism, but $\underline{\mathcal{I}}'$ differs from $\pi^*(\underline{\mathcal{I}})$ in the term E'.)

We define a *weak test sequence* for $\underline{\mathcal{I}}_0 = \underline{\mathcal{I}}$ as a sequence of morphisms

$$M = M_0 \xleftarrow{\sigma_1} M_1 \longleftarrow \cdots \xleftarrow{\sigma_t} M_t\,, \tag{2.1}$$

where each successive morphism σ_{j+1} is either an admissible blowing-up (for the transform $\underline{\mathcal{I}}_j$ of $\underline{\mathcal{I}}_{j-1}$), or the projection from the product with a line.

We say that two marked ideals $\underline{\mathcal{I}}$ and $\underline{\mathcal{I}}_1 = (M_1 = M, N_1, E_1 = E, \mathcal{I}_1, d_1)$ (with the same ambient variety M and the same normal crossings divisor E) are *weak-equivalent* if they have the same weak test sequences; i.e., every weak test sequence for one is a weak test sequence for the other.

Definitions 2.5. Let $\underline{\mathcal{I}} = (M, N, E, \mathcal{I}, d)$ be a marked ideal. A blowing-up $\sigma : M' \to M$ will be called an *exceptional blowing-up* for $\underline{\mathcal{I}}$ if its centre $C = H_i \cap H_j$, where H_i, H_j are distinct hypersurfaces in E.

We define the *transform* $\underline{\mathcal{I}}'$ of $\underline{\mathcal{I}}$ by an exceptional blowing-up $\sigma : M' \to M$ (with centre C, say) as the marked ideal $\underline{\mathcal{I}}' = (M', N', E', \mathcal{I}', d')$, where N' the strict transform of N ($N' = \sigma^{-1}(N)$, in this case), E' is defined as in Definitions 1.3 above, $\mathcal{I}' = \sigma^*(\mathcal{I})$, and $d' = d$.

We define a *test sequence* for $\underline{\mathcal{I}}_0 = \underline{\mathcal{I}}$ as a sequence of morphisms (2.1), where each successive σ_{j+1} is either an admissible blowing-up, the projection from a product with a line, or an exceptional blowing-up. (Morphisms of these three kinds will be called *test transformations* or *test morphisms*.)

We say that two marked ideals $\underline{\mathcal{I}}$ and $\underline{\mathcal{I}}_1$ (with the same ambient variety M and the same normal crossings divisor E) are *equivalent* if they have the same test sequences.

3. Differential calculus

Let N denote a smooth variety. If $(x_1, \ldots, x_n)$ are local coordinates at a point of N, then the partial derivatives $\partial/\partial x_1, \ldots, \partial/\partial x_n$ are well-defined as local generators of the sheaf of derivations Der_N of $\mathcal{O}_N$.

3.1. **Transformation of differential operators by blowing up.** Let $\sigma : N' \to N$ denote the blowing-up with centre a smooth subvariety C of N. Choose a local coordinate chart U of N with coordinates $(x_1, \ldots, x_n)$ in which $C = \{x_r = \cdots = x_n = 0\}$. Then $\sigma^{-1}(C)$ is covered by coordinate charts $U_{x_r}, \ldots, U_{x_n}$, where, for example in the x_r-*chart* U_{x_r}, we can choose coordinates $(y_1, \ldots, y_n)$ with $y_1 = x_1, \ldots, y_r = x_r$, $y_{r+1} = x_{r+1}/x_r, \ldots, y_n = x_n/x_r$. The following lemma is a simple but crucial exercise using the chain rule.

Lemma 3.1. *With coordinates chosen as above, we have:*

$$\begin{aligned} \frac{\partial}{\partial x_j} &= \frac{\partial}{\partial y_j}, \qquad j = 1, \ldots, r-1, \\ x_r \frac{\partial}{\partial x_r} &= y_r \frac{\partial}{\partial y_r} - \sum_{j=r+1}^{n} y_j \frac{\partial}{\partial y_j}, \\ x_j \frac{\partial}{\partial x_j} &= y_j \frac{\partial}{\partial y_j}, \qquad j = r+1, \ldots, n. \end{aligned}$$

The transformation formulas in Lemma 3.1 go back to [G], and are usually written a little differently. As arranged here, they show that it is logarithmic differential operators which transform naturally by blowing up.

3.2. **Derivative ideals.** Let $\mathcal{I} \subset \mathcal{O}_N$ denote a (coherent) ideal. Let $\mathcal{D}(\mathcal{I}) \subset \mathcal{O}_N$ denote the coherent sheaf of ideals generated by all first derivatives of local sections of $\mathcal{I}$ (so that $\mathcal{I} \subset \mathcal{D}(\mathcal{I})$); i.e., $\mathcal{D}(\mathcal{I})$ is the image of the natural morphism $\mathrm{Der}_N \times \mathcal{I} \to \mathcal{O}_N$.

If $f_1, \ldots, f_q$ are local generators of $\mathcal{I}$ in a chart with coordinates $(x_1, \ldots, x_n)$, then f_i, $\partial f_i/\partial x_j$, where $i = 1, \ldots, q$, $j = 1, \ldots, n$, are local generators of $\mathcal{D}(\mathcal{I})$.

We also define *higher-derivative ideals* inductively by

$$\mathcal{D}^{j+1}(\mathcal{I}) := \mathcal{D}(\mathcal{D}^j(\mathcal{I})), \quad j = 1, \ldots.$$

Now let E denote a normal-crossings divisor on N. We define $\mathcal{D}_E(\mathcal{I}) \subset \mathcal{O}_N$ as the ideal generated by all local sections of $\mathcal{I}$ and first derivatives which preserve the ideal $\mathcal{I}_E$ of E. (If $N \subset M$, where M is smooth, and E is a normal-crossings divisor on M which is transverse to N, then we will simply write $\mathcal{D}_E(\mathcal{I})$ instead of $\mathcal{D}_{E|_N}(\mathcal{I})$.)

Consider a divisor $E = \sum_{i=1}^r \alpha_i H_i$, where each $\alpha_i \in \mathbb{N}$. Each point of N has a neighbourhood U with a coordinate system $(x_1, \ldots, x_q, \ldots, x_n)$, $q \le r$, such that $x_1, \ldots, x_q$ generate the ideals $\mathcal{I}_H|_U$ for those $H = H_i$ which intersect U; write $x_j = x_H$, where H is the corresponding divisor. If f_i are local generators of $\mathcal{I}$, then $\mathcal{D}_E(\mathcal{I})$ is generated locally by all

$$f_i, \quad x_H \frac{\partial f_i}{\partial x_H}, \quad \text{and} \quad \frac{\partial f_i}{\partial x_j}, \quad q+1 \le j \le n.$$

We again define

$$\mathcal{D}_E^{j+1}(\mathcal{I}) := \mathcal{D}_E(\mathcal{D}_E^j(\mathcal{I})), \quad j = 1, \ldots.$$

Now consider a marked ideal $\underline{\mathcal{I}} = (M, N, E, \mathcal{I}, d)$. For simplicity, we will write $\underline{\mathcal{I}} = (\mathcal{I}, d)$ when the remaining entries are evident. We define

$$\underline{\mathcal{D}}(\underline{\mathcal{I}}) := (M, N, E, \mathcal{D}(\mathcal{I}), d-1) = (\mathcal{D}(\mathcal{I}), d-1),$$
$$\underline{\mathcal{D}}_E(\underline{\mathcal{I}}) := (M, N, E, \mathcal{D}_E(\mathcal{I}), d-1) = (\mathcal{D}_E(\mathcal{I}), d-1).$$

and

$$\underline{\mathcal{D}}_E^{j+1}(\underline{\mathcal{I}}) := \underline{\mathcal{D}}_E(\underline{\mathcal{D}}_E^j(\underline{\mathcal{I}})) = (\mathcal{D}_E^{j+1}(\mathcal{I}), d-j-1), \quad j = 1, \ldots.$$

Lemma 3.2. *Let $\underline{\mathcal{I}} = (M, N, E, \mathcal{I}, d)$ be a marked ideal. Then:*

(1) $\mathcal{D}_E^k(\mathcal{D}_E^l(\mathcal{I})) = \mathcal{D}_E^{k+l}(\mathcal{I})$.
(2) $\mathcal{D}_E^k(\mathcal{I} \cdot \mathcal{J}) \subset \sum_{j=0}^k \mathcal{D}_E^j(\mathcal{I}) \cdot \mathcal{D}_E^{k-j}(\mathcal{J})$.
(3) $\operatorname{cosupp} \underline{\mathcal{I}} \subset \operatorname{cosupp} \underline{\mathcal{D}}_E^j(\underline{\mathcal{I}})$, $j = 0, 1, \ldots,$ *with equality if* $E = \emptyset$.
(4) *If $\varphi : P \to N$ is a smooth morphism, then $\mathcal{D}_{\varphi^{-1}(E)}(\varphi^*(\mathcal{I})) = \varphi^*(\mathcal{D}_E(\mathcal{I}))$.*

Proof. These statements are immediate from the definitions. (It is enough to check (4) on completions.) □

Lemma 3.3. *Let $\underline{\mathcal{I}} = (M, N, E, \mathcal{I}, d)$ be a marked ideal. Let $\sigma : M' \to M$ denote a test transformation for $\underline{\mathcal{I}}$ (i.e., either an admissible or exceptional blowing-up, or a projection $M \times \mathbb{A}^1 \to M$), and let $\underline{\mathcal{I}}' = (M', N', E', \mathcal{I}', d')$ denote the transform of $\underline{\mathcal{I}}$ by σ. (See Sect. 2.) Then σ is a test transformation for $\underline{\mathcal{D}}_E(\underline{\mathcal{I}})$, and $\mathcal{D}_E(\mathcal{I})' \subset \mathcal{D}_{E'}(\mathcal{I}')$.*

Proof. The first assertion follows from Lemma 3.2(3), and the second is an exercise using the transformation formulas of Lemma 3.1. □

The following three corollaries are easy consequences of Lemma 3.3.

Corollary 3.4. *With the hypotheses of Lemma 3.3, if $j \le d-1$, then $\mathcal{D}_E^j(\mathcal{I})' \subset \mathcal{D}_{E'}^j(\mathcal{I}')$.*

Corollary 3.5. *If $j \le d-1$, then any sequence of test transformations of $\underline{\mathcal{I}}$ is a sequence of test transformations of $\underline{\mathcal{D}}_E^j(\underline{\mathcal{I}})$ and, after any sequence of test transformations*

$$M = M_0 \xleftarrow{\sigma_1} M_1 \longleftarrow \cdots \xleftarrow{\sigma_t} M_t\,,$$

we have an inclusion $\mathcal{D}_E^j(\mathcal{I})_t \subset \mathcal{D}_{E_t}^j(\mathcal{I}_t)$.

Of course, we can also define $\underline{\mathcal{D}}^{j+1}(\underline{\mathcal{I}})$, $j = 1, \ldots$, by iterating $\underline{\mathcal{D}}$, and we have analogues of Lemmas 3.2, 3.3 and Corollaries 3.4, 3.5 for the $\underline{\mathcal{D}}^j(\underline{\mathcal{I}})$. We will use the following to prove Corollary 1.4.

Corollary 3.6. *Suppose that $E = \emptyset$. Under the hypotheses of Lemma 3.3, if $j \le d-1$, then $\mathcal{D}^j(\mathcal{I})' \subset \mathcal{D}_{E'}^j(\mathcal{I}')$. Likewise after any sequence of test transformations, as in Corollary 3.5.*

3.3. **Aside on sums and products of marked ideals.** We define sums and products, and give two simple lemmas.

Consider marked ideals $\underline{\mathcal{I}} = (M, N, E, \mathcal{I}, d) = (\mathcal{I}, d)$ and $\underline{\mathcal{J}} = (M, N, E, \mathcal{J}, d) = (\mathcal{J}, d)$. Define $\underline{\mathcal{I}} \cdot \underline{\mathcal{J}} := (\mathcal{I} \cdot \mathcal{J}, d+e)$.

Lemma 3.7. (1) $\operatorname{cosupp} \underline{\mathcal{I}} \cap \operatorname{cosupp} \underline{\mathcal{J}} \subset \operatorname{cosupp} \underline{\mathcal{I}} \cdot \underline{\mathcal{J}}$ *(whereas* $\operatorname{cosupp} \underline{\mathcal{I}}^k = \operatorname{cosupp} \underline{\mathcal{I}}$.*)*

(2) *Let* $\sigma : M' \to M$ *be a test morphism for both* $\underline{\mathcal{I}}$ *and* $\underline{\mathcal{J}}$*. Then* σ *is a test morphism for* $\underline{\mathcal{I}} \cdot \underline{\mathcal{J}}$ *and the transforms satisfy* $\underline{\mathcal{I}}' \cdot \underline{\mathcal{J}}' = (\underline{\mathcal{I}} \cdot \underline{\mathcal{J}})'$*.*
(3) *Multiplication of marked ideals is associative.*

Define $\underline{\mathcal{I}} + \underline{\mathcal{J}} := (\underline{\mathcal{I}}^{l/d} + \underline{\mathcal{J}}^{l/e}, l)$, where $l = \mathrm{lcm}(d, e)$. Likewise, for any finite sum. Addition is not associative, but $\underline{\mathcal{I}} + \underline{\mathcal{J}}$ is equivalent to $(\underline{\mathcal{I}}^e + \underline{\mathcal{J}}^d, de)$. (See Lemma 3.8(3).)

Lemma 3.8. (1) $\operatorname{cosupp}(\underline{\mathcal{I}} + \underline{\mathcal{J}}) = \operatorname{cosupp} \underline{\mathcal{I}} \cap \operatorname{cosupp} \underline{\mathcal{J}}$.
(2) $\sigma : M' \to M$ *is a test morphism for* $\underline{\mathcal{I}} + \underline{\mathcal{J}}$ *if and only if* σ *is a test morphism for both* $\underline{\mathcal{I}}$ *and* $\underline{\mathcal{J}}$*, and the transforms by such a test morphism satisfy* $\underline{\mathcal{I}}' + \underline{\mathcal{J}}' = (\underline{\mathcal{I}} + \underline{\mathcal{J}})'$*.*
(3) *Addition of marked ideals is associative up to equivalence.*

3.4. **Logarithmic derivatives determine equivalent ideals.** Consider a marked ideal $\underline{\mathcal{I}} = (M, N, E, \mathcal{I}, d)$. We define

$$\underline{\mathcal{C}}^k_E(\underline{\mathcal{I}}) := \sum_{j=0}^{k} \underline{\mathcal{D}}^j_E(\underline{\mathcal{I}}), \quad k \le d-1.$$

Write $\underline{\mathcal{C}}^k_E(\underline{\mathcal{I}}) = \Big(M, N, E, \mathcal{C}^k_E(\underline{\mathcal{I}}), d_{\underline{\mathcal{C}}^k_E(\underline{\mathcal{I}})}\Big)$.

Lemma 3.9. $\operatorname{cosupp} \underline{\mathcal{I}} = \operatorname{cosupp} \underline{\mathcal{C}}^k_E(\underline{\mathcal{I}})$.

Proof. $\operatorname{cosupp} \underline{\mathcal{I}} \subset \operatorname{cosupp} \underline{\mathcal{C}}^k_E(\underline{\mathcal{I}})$, by Lemmas 3.2(3) and 3.8(1). But $\operatorname{cosupp} \underline{\mathcal{C}}^k_E(\underline{\mathcal{I}}) \subset \operatorname{cosupp} \underline{\mathcal{I}}$, since $\underline{\mathcal{D}}^0_E(\underline{\mathcal{I}}) = \underline{\mathcal{I}}$. □

Theorem 3.10. *Consider any sequence of test morphisms for* $\underline{\mathcal{I}}$*,*

$$M = M_0 \xleftarrow{\sigma_1} M_1 \longleftarrow \cdots \xleftarrow{\sigma_t} M_t\,. \tag{3.1}$$

Then (3.1) is a sequence of test morphisms for $\underline{\mathcal{C}}^k_E(\underline{\mathcal{I}})$*, and for the transforms, we have*

$$\operatorname{cosupp} \underline{\mathcal{C}}^k_{E_t}(\underline{\mathcal{I}}_t) = \operatorname{cosupp} \underline{\mathcal{C}}^k_E(\underline{\mathcal{I}})_t.$$

Proof. By Corollary 3.5, any sequence of test morphisms of $\underline{\mathcal{I}}$ is a sequence of test morphisms of $\underline{\mathcal{C}}^k_E(\underline{\mathcal{I}})$, and $\operatorname{cosupp} \underline{\mathcal{C}}^k_{E_t}(\underline{\mathcal{I}}_t) \subset \operatorname{cosupp} \underline{\mathcal{C}}^k_E(\underline{\mathcal{I}})_t$. Since $\underline{\mathcal{C}}^k_E(\underline{\mathcal{I}}) = \underline{\mathcal{I}} + \sum_{j=1}^k \underline{\mathcal{D}}^j_E(\underline{\mathcal{I}})$, we have

$$\begin{aligned}
\operatorname{cosupp} \underline{\mathcal{C}}^k_E(\underline{\mathcal{I}})_t &\subset \operatorname{cosupp} \underline{\mathcal{I}}_t \\
&\subset \bigcap_{j=0}^{k} \operatorname{cosupp} \underline{\mathcal{D}}^j_{E_t}(\underline{\mathcal{I}}_t), \quad \text{by Lemma 3.2(3)}, \\
&= \operatorname{cosupp} \underline{\mathcal{C}}^k_{E_t}(\underline{\mathcal{I}}_t).
\end{aligned}$$

□

Corollary 3.11. $\underline{\mathcal{I}}$ *and* $\underline{\mathcal{C}}^k_E(\underline{\mathcal{I}})$ *are equivalent.*

Proof. They have the *same* test sequences, by Lemma 3.9 and Theorem 3.10. □

Lemma 3.12. *If* $\varphi : M' \to M$ *is smooth, then* $\varphi^*(\underline{\mathcal{C}}^k_E(\underline{\mathcal{I}})) = \underline{\mathcal{C}}^k_{\varphi^{-1}(E)}(\varphi^*(\underline{\mathcal{I}}))$.

Proof. This follows from Lemma 3.2(4). □

4. Maximal contact and coefficient ideals

Corollary 4.1. *Let $\underline{\mathcal{I}} = (M, N, E, \mathcal{I}, d)$ denote a marked ideal. Let z denote a section of $\mathcal{D}_E^{d-1}(\mathcal{I})$ on $N|_U$, where U is an open subset of M. Suppose that z has maximum order 1 and that z is transverse to E (equivalently, $z \cdot \mathcal{I}_E|_N$ has only normal crossings and $\mathcal{I}_E|_N \not\subset (z)$, where $(z) \subset \mathcal{O}_N|_U$ is the ideal generated by z). Let $P = V(z)$. Then, after any sequence of test transformations of $\underline{\mathcal{I}}|_U$,*

$$\operatorname{cosupp} \underline{\mathcal{I}}_t \subset P_t \subset N_t,$$

where $P_t = V((z)_t)$ and $(z)_t$ denotes the transform of (z) (cf. notation of Thm. 3.10). Moveover, if $\operatorname{cosupp} \underline{\mathcal{I}} \subsetneq P$, then $(z)_t$ has maximum order 1 and is transverse to E_t.

Proof. This follows from Theorem 3.10 and Corollary 3.11. □

In the situation of Corollary 4.1, we call the hypersurface P of $N|_U$ a *hypersurface of maximal contact* for $\underline{\mathcal{I}}$. Note that the existence of z with maximum order 1 means that $\underline{\mathcal{I}}$ is of maximal order on U. In the setting of Corollary 3.1, we define the *coefficient (marked) ideal*

$$\underline{\mathcal{C}}_{E,P}(\underline{\mathcal{I}}) := \left(U, P, E, \mathcal{C}_E^{d-1}(\underline{\mathcal{I}})|_P, d_{\underline{\mathcal{C}}_E^{d-1}(\underline{\mathcal{I}})}\right). \tag{4.1}$$

Corollary 4.2. $\underline{\mathcal{C}}_{E,P}(\underline{\mathcal{I}})$ *is equivalent to* $\underline{\mathcal{I}}|_U$.

Remark 4.3. $\underline{\mathcal{C}}_{E,P}$ commutes with smooth morphisms. In other words, if $\varphi : M' \to M$ is smooth, then $P' := \varphi^{-1}(P) = V(\varphi^*(z))$ is a maximal contact hypersurface for $\varphi^*(\underline{\mathcal{I}})$, and $\varphi^*(\underline{\mathcal{C}}_{E,P}(\underline{\mathcal{I}})) = \underline{\mathcal{C}}_{E',P'}(\varphi^*(\underline{\mathcal{I}}))$ (by Lemma 3.12).

Exercises 4.4. The following exercises give alternative ways to define coefficient ideals. These approaches are used in [BM2], [BM4], and can simplify explicit calculations.

(1) Under the hypotheses of Corollary 4.1, we can find a system of local coordinates $(x_1, \ldots, x_n)$ for N such that $x_n = z$ and the components of E are given by $x_i = 0$, $i = 1, \ldots, r < n$ (in this chart). Let $\mathcal{D}_z(\mathcal{I})$ denote the ideal generated by f, $\partial f/\partial z$, for all $f \in \mathcal{I}$ (so that $\mathcal{I} \subset \mathcal{D}_z(\mathcal{I}) \subset \mathcal{D}_E(\mathcal{I})$), and set $\underline{\mathcal{C}}_z^{d-1}(\underline{\mathcal{I}}) := \sum_{j=0}^{d-1} \underline{\mathcal{D}}_z^j(\underline{\mathcal{I}})$. Then $\underline{\mathcal{C}}_{z,P}(\underline{\mathcal{I}}) := \left(U, P, E, \mathcal{C}_z^{d-1}(\underline{\mathcal{I}})|_P, d_{\underline{\mathcal{C}}_z^{d-1}(\underline{\mathcal{I}})}\right)$ is equivalent to $\underline{\mathcal{I}}$ (in the chart).

(2) Suppose that $\mathcal{I} \subset \mathcal{O}_M$ is a principal ideal. Let g be a local generator of $\mathcal{I}$ in a neighbourhood U of $a \in M$ with coordinates $(x_1, \ldots, x_m)$. Let $d = \operatorname{ord}_a \mathcal{I}$. (By making a linear coordinate change), we can assume that $\partial^d g/\partial x_m^d$ is nonvanishing. Let $z := \partial^{d-1} g/\partial x_n^{d-1}$. Then $N := V(z)$ is a hypersurface of maximal contact for $\underline{\mathcal{I}}|_U := (U, U, \emptyset, \mathcal{I}|_U, d)$, and $\underline{\mathcal{I}}|_U$ is equivalent to $\underline{\mathcal{C}}_{x_n,N}(\underline{\mathcal{I}}) := \left(U, N, \emptyset, \mathcal{C}_{x_n}^{d-1}(\mathcal{I})|_N, d_{\underline{\mathcal{C}}_{x_n,N}(\underline{\mathcal{I}})}\right)$, where $\mathcal{C}_{x_n}^{d-1}(\mathcal{I}) := \sum_{k=0}^{d-1} \left((\partial^k g/\partial x_n^k), d-k\right)$.

5. Proof of resolution of singularities of a marked ideal

In this section, we prove Theorem 1.3. Let $\underline{\mathcal{I}} = (M, N, E, \mathcal{I}, d)$ denote a marked ideal, where $d > 0$. The proof is by induction on $\dim \underline{\mathcal{I}} := \dim N$.

First suppose $\dim N = 0$. If $\mathcal{I} = 0$, then $\operatorname{cosupp} \underline{\mathcal{I}} = N$ and we can blow up with centre N to resolve the singularities of $\underline{\mathcal{I}}$. If $\mathcal{I} \neq 0$, then $\operatorname{cosupp} \underline{\mathcal{I}} = \emptyset$ (since $d > 0$), so that $\underline{\mathcal{I}}$ is already resolved.

The inductive step breaks up into the two independent steps I and II formulated in Section 1.

Step I. Let $\underline{\mathcal{I}} = (M, N, E, \mathcal{I}, d)$ be a marked ideal of maximal order.

Case A. $E = \emptyset$**.** Let $a \in \operatorname{cosupp} \underline{\mathcal{I}}$. By Corollary 4.1, there is a hypersurface of maximal contact P for $\underline{\mathcal{I}}$ at a; $P \subset N \cap U$, where U is a neighbourhood of a in M. Define

$$\underline{\mathcal{C}}_U(\underline{\mathcal{I}}) := \underline{\mathcal{C}}_{\emptyset,P}(\underline{\mathcal{I}}) = (U, P, \emptyset, \mathcal{C}_\emptyset^{d-1}(\underline{\mathcal{I}})|_P, d_{\underline{\mathcal{C}}}), \tag{5.1}$$

where $d_{\underline{\mathcal{C}}} = d_{\underline{\mathcal{C}}_\emptyset^{d-1}(\underline{\mathcal{I}})}$. Then $\underline{\mathcal{C}}_U(\underline{\mathcal{I}})$ is equivalent to $\underline{\mathcal{I}}|_U = (U, N \cap U, \emptyset, \mathcal{I}|_{N\cap U}, d)$, according to Corollary 4.2. Therefore, a resolution of singularities of $\underline{\mathcal{C}}_U(\underline{\mathcal{I}})$ (which exists by induction) is a resolution of singularities of $\underline{\mathcal{I}}|_U$.

The marked ideals $\underline{\mathcal{C}}_U(\underline{\mathcal{I}})$ and $\underline{\mathcal{C}}_V(\underline{\mathcal{I}})$ defined in two overlapping charts U and V are equivalent in $U \cap V$ (since both are equivalent to $\underline{\mathcal{I}}_{U\cap V}$; therefore, by functoriality in dimension $n-1$, their resolution sequences are the same over $U \cap V$ (not counting blowings-up that restrict to isomorphisms over $U \cap V$).

Claim 5.1. *It follows that the resolution sequences for the $\underline{\mathcal{C}}_U(\underline{\mathcal{I}})$ (defined locally) patch together to give functorial resolution of singularities of a marked ideal $\underline{\mathcal{I}}$.*

The problem here is that, to glue together the local centres of blowing up, we have to know which nontrivial centres in the various charts U should be taken first, etc. We leave this claim to Section 7 below, where we give two different proofs.

Case B. General maximal order case. Let $\underline{\mathcal{I}}_\emptyset$ denote the marked ideal $(M, N, \emptyset, \mathcal{I}, d)$. Let $a \in \operatorname{cosupp} \underline{\mathcal{I}} = \operatorname{cosupp} \underline{\mathcal{I}}_\emptyset$. By Corollary 4.1, there exists an (irreducible) hypersurface of maximal contact P for $\underline{\mathcal{I}}_\emptyset$, defined in a neighbourhood U of a. We introduce $\underline{\mathcal{C}} := \underline{\mathcal{C}}_U(\underline{\mathcal{I}}_\emptyset) = (U, P, \emptyset, \mathcal{C}, d_{\underline{\mathcal{C}}})$, as in Case A above.

If $x \in N$, set

$$s(x) := \#\{H \in E : x \in H\}$$

(where "$H \in E$" means "H is a component of E", and $\#$ denotes the cardinality of a finite set). After any sequence of test transformations of $\underline{\mathcal{I}}$, we will continue to write E (as opposed to E_t) for the divisor whose components are the strict transforms of those of E, with the same ordering as in E, and we will continue to write $s(x)$ for $\#\{H \in E : x \in H\}$.

Let $s := \max\{s(x) : x \in \operatorname{cosupp} \underline{\mathcal{I}}\}$, and let $\operatorname{Sub}(E, s)$ denote the set of s-element subsets of E (i.e., of the set of components of E). Let $\underline{\mathcal{E}}$ denote the marked ideal $\underline{\mathcal{E}} := \big(U, P, \emptyset, \mathcal{E}, d_{\underline{\mathcal{C}}}\big)$, where

$$\mathcal{E} := \prod_{\Lambda \in \operatorname{Sub}(E,s)} \sum_{H \in \Lambda} \mathcal{I}_H^{d_{\mathcal{C}}} \cdot \mathcal{O}_P.$$

Set $\underline{\mathcal{J}} := \underline{\mathcal{C}} + \underline{\mathcal{E}}$.

Clearly, $\operatorname{cosupp} \underline{\mathcal{E}} = \{x \in P : s(x) \geq s\}$, so that

$$\operatorname{cosupp} \underline{\mathcal{J}} = \operatorname{cosupp} \underline{\mathcal{I}}|_U \bigcap \{x \in P : s(x) = s\},$$

(and likewise after any sequence of test transformations of $\underline{\mathcal{J}}$). Thus any sequence of test transformations of $\underline{\mathcal{J}}$ is a sequence of test transformations of $\underline{\mathcal{I}}|_U$ (and the centre C of each admissible blowing-up lies inside the intersection of the components of E that pass through a point of C). Therefore, the equivalence class of $\underline{\mathcal{J}}$ depends only on that of $\underline{\mathcal{I}}|_U$, and blowings-up which are admissible for $\underline{\mathcal{J}}$ are also admissible for $\underline{\mathcal{I}}|_U$.

By induction on dimension, there is a resolution of singularities of $\underline{\mathcal{J}}$ We can define $\underline{\mathcal{J}}$ for each element of a finite covering $\{U\}$ of M and, as in Claim 4.1, the resolution sequences for the $\underline{\mathcal{J}}$ patch together to define a sequence of admissible blowings-up of $\underline{\mathcal{I}}$. (See Section 7.)

If this sequence of blowings-up resolves $\underline{\mathcal{I}}$, then we have finished Case B. Otherwise, $\operatorname{cosupp}\underline{\mathcal{I}}_j$ and $\{x : s(x) = s\}$ become disjoint for some index j of the blowings-up sequence, say for $j = q_1$ (i.e., $s(x) < s$, for all $x \in \operatorname{cosupp}\underline{\mathcal{I}}_j$).

We can now repeat the process above, using $\underline{\mathcal{C}}_{q_1} = \underline{\mathcal{C}}_U(\underline{\mathcal{I}}_\emptyset)_{q_1}$, $\underline{\mathcal{C}}_{q_2} = \underline{\mathcal{C}}_U(\underline{\mathcal{I}}_\emptyset)_{q_2}$, etc., in place of $\underline{\mathcal{C}}$, and using the new value of $\max s(x)$ in place of s. After finitely many steps, $\underline{\mathcal{I}}$ is resolved, so we have completed Case B and therefore Step I.

Step II. General case. Let $\underline{\mathcal{I}} = (M, N, E, \mathcal{I}, d)$ be an arbitrary marked ideal.

First suppose that $\mathcal{I} = 0$ (so that $\operatorname{cosupp}\underline{\mathcal{I}} = N$). Then we can blow up with centre N to resolve singularities.

Now suppose that $\mathcal{I} \neq 0$. Write

$$\mathcal{I} = \mathcal{M}(\underline{\mathcal{I}}) \cdot \mathcal{N}(\underline{\mathcal{I}}), \tag{5.2}$$

where $\mathcal{M}(\underline{\mathcal{I}})$ is a product of prime ideals defining irreducible components of the elements of E, and $\mathcal{N}(\underline{\mathcal{I}})$ is divisible by no such prime ideal. We consider two cases.

Case A. Monomial case $\mathcal{I} = \mathcal{M}(\underline{\mathcal{I}})$. For any sequence of admissible blowings-up σ_j of $\underline{\mathcal{I}}$, let us order the collection of subsets of each E_j as follows: Suppose that $E = \{H_1, \dots, H_q\}$. Write $E_0 := E = \{H_1, \dots, H_q\} =: \{H_1^0, \dots, H_q^0\}$. For each $j = 1, \dots$, write $E_j = \{H_1^j, \dots, H_{q+j}^j\}$, where H_i^j denotes the strict transform of H_i^{j-1} by σ_j, if $i < q + j$, and $H_{q+j}^j = \sigma_j^{-1}(C_{j-1})$. Now, we associate to each subset I of E_j the finite sequence $(\delta_1, \dots, \delta_{q+j})$, where $\delta_i = 0$ if $H_i^j \notin I$, and $\delta_i = 1$ if $H_i^j \in I$. Then we order the subsets I of E_j, for *all* j, by the lexicographic ordering of such sequences.

Let $a \in \operatorname{cosupp}\underline{\mathcal{I}}$. In a neighbourhood of a, we can write $\mathcal{I} = \mathcal{I}_{H_{i_1}}^{\alpha_1} \cdots \mathcal{I}_{H_{i_r}}^{\alpha_r}$ (where $a \in H_{i_1} \cap \dots \cap H_{i_r}$ and where we write $\mathcal{I}_{H_{i_k}}$ instead of $\mathcal{I}_{H_{i_k}} \cdot \mathcal{O}_N$ to simplify the notation); in particular, $\alpha_1 + \dots + \alpha_r > d$. (In Theorem 6.1 below, we will see that $\mu_a(\underline{\mathcal{I}}) = (\alpha_1 + \dots + \alpha_r)/d$ is a local invariant of the equivalence class of $\underline{\mathcal{I}}$.) Then (in the neighbourhood above) $\operatorname{cosupp}\underline{\mathcal{I}} = \cup Z_I$, where each $Z_I := N \cap \bigcap_{H \in I} H$, and I runs over the *smallest* subsets of $\{H_{i_1}, \dots, H_{i_r}\}$ such that $\sum_{l \in I} \alpha_l \geq d$; in other words, I runs over the subsets of $\{H_{i_1}, \dots, H_{i_r}\}$ such that

$$0 \leq \sum_{l \in I} \alpha_l - d < \alpha_k, \quad \text{for all } k \in I.$$

(We have simplified the notation by identifying subsets of $\{H_{i_1}, \dots, H_{i_r}\}$ with subsets of $\{1, \dots, r\}$.)

Let $J(a)$ denote the maximum of these subsets I (with respect to the order above). Clearly, $J(x)$ is Zariski upper-semicontinuous on $\operatorname{cosupp}\underline{\mathcal{I}}$, and the maximum locus of $J(x)$ consists of at most one irreducible component of $\operatorname{cosupp}\underline{\mathcal{I}}$ through each point.

Consider the blowing-up σ with centre C given by the maximum locus of $J(\cdot)$. If $a \in C$, then (in a neighbourhood as above), $C = N \cap \bigcap_{l \in J(a)} H_{i_l}$. We can choose local coordinates $(x_1, \dots, x_n)$ for N at a such that, for each $k \in J(a)$, x_k is a local generator $x_{H_{i_k}}$ of the ideal $\mathcal{I}_{H_{i_k}}|_N$. Then, in the x_k-chart of the blowing-up σ, the transform of $\mathcal{I}$ is given by

$$\mathcal{I}' = \mathcal{I}_{H'_{i_1}}^{\alpha_1} \cdots \mathcal{I}_{\sigma^{-1}(C)}^{\sum_{J(a)} \alpha_l - d} \cdots \mathcal{I}_{H'_{i_r}}^{\alpha_r},$$

(with $\mathcal{I}_{\sigma^{-1}(C)}^{\sum_{J(a)} \alpha_l - d}$ in the k'th place). Since $\sum_l \alpha_l - d < \alpha_k$, $\mu_{a'}(\underline{\mathcal{I}}') = \mu_a(\underline{\mathcal{I}}) - p/d$, where p is a positive integer, over all points a of some component of $\operatorname{cosupp} \underline{\mathcal{I}}$. We therefore resolve singularities after a finite number of steps.

Case B. General case. Set

$$\operatorname{ord} \mathcal{N}(\underline{\mathcal{I}}) := \max_{x \in \operatorname{cosupp} \underline{\mathcal{I}}} \operatorname{ord}_x \mathcal{N}(\underline{\mathcal{I}}),$$

and define marked ideals

$$\underline{\mathcal{N}}(\underline{\mathcal{I}}) := (\mathcal{N}(\underline{\mathcal{I}}), \operatorname{ord} \mathcal{N}(\underline{\mathcal{I}})),$$
$$\underline{\mathcal{M}}(\underline{\mathcal{I}}) := (\mathcal{M}(\underline{\mathcal{I}}), d - \operatorname{ord} \mathcal{N}(\underline{\mathcal{I}})).$$

We define the *companion ideal* of $\underline{\mathcal{I}} = (\mathcal{I}, d)$ as

$$\underline{\mathcal{G}}(\underline{\mathcal{I}}) := \begin{cases} \underline{\mathcal{N}}(\underline{\mathcal{I}}) + \underline{\mathcal{M}}(\underline{\mathcal{I}}), & \text{if } \operatorname{ord} \mathcal{N}(\underline{\mathcal{I}}) < d; \\ \underline{\mathcal{N}}(\underline{\mathcal{I}}), & \text{if } \operatorname{ord} \mathcal{N}(\underline{\mathcal{I}}) \geq d. \end{cases}$$

Then $\underline{\mathcal{G}}(\underline{\mathcal{I}})$ is of maximal order and, for any sequence of test transformations of $\underline{\mathcal{G}}(\underline{\mathcal{I}})$,

$$\begin{aligned} \operatorname{cosupp} \underline{\mathcal{G}}(\underline{\mathcal{I}})_j &= \operatorname{cosupp} \underline{\mathcal{N}}(\underline{\mathcal{I}})_j \bigcap \operatorname{cosupp} \underline{\mathcal{M}}(\underline{\mathcal{I}})_j \\ &\subset \operatorname{cosupp} \underline{\mathcal{N}}(\underline{\mathcal{I}})_j \bigcap \operatorname{cosupp} \underline{\mathcal{I}}_j. \end{aligned}$$

By Lemma 3.7, $\underline{\mathcal{I}}_j = \underline{\mathcal{M}}(\underline{\mathcal{I}})_j \cdot \underline{\mathcal{N}}(\underline{\mathcal{I}})_j$.

Remark 5.2. For any sequence of *admissible* blowings-up of $\underline{\mathcal{G}}(\underline{\mathcal{I}})$, it is clear that $\underline{\mathcal{N}}(\underline{\mathcal{I}})_j = \underline{\mathcal{N}}(\underline{\mathcal{I}}_j)$, $\underline{\mathcal{M}}(\underline{\mathcal{I}})_j = \underline{\mathcal{M}}(\underline{\mathcal{I}}_j)$, and $\operatorname{cosupp} \underline{\mathcal{G}}(\underline{\mathcal{I}})_j = \operatorname{cosupp} \underline{\mathcal{N}}(\underline{\mathcal{I}})_j \cap \operatorname{cosupp} \underline{\mathcal{I}}_j$.

In Theorems 6.1 and 6.2 below, we will prove that, if $a \in \operatorname{cosupp} \underline{\mathcal{I}}$, then

$$\mu_a(\underline{\mathcal{I}}) := \frac{\operatorname{ord}_a \mathcal{I}}{d} \quad \text{and} \quad \mu_{H,a}(\underline{\mathcal{I}}) := \frac{\operatorname{ord}_{H,a} \mathcal{I}}{d}, \; H \in E, \tag{5.3}$$

depend only on the equivalence class of $\underline{\mathcal{I}}$ and $\dim N$. ($\operatorname{ord}_{H,a} \mathcal{I}$ denotes the *order* of $\mathcal{I} \subset \mathcal{O}_N$ *along* $H|_N$ at a; i.e., the largest $\mu \in \mathbb{N}$ such that $\mathcal{I}_a \subset \mathcal{I}^\mu_{H|_N, a}$.) The following is a corollary of these results.

Corollary 5.3. *The equivalence class of* $\underline{\mathcal{G}}(\underline{\mathcal{I}})$ *depends only on the equivalence class of* $\underline{\mathcal{I}}$ *and* $\dim N$.

Proof. The divisor (with rational coefficients) $\mathcal{M}(\underline{\mathcal{I}})^{1/d}$ depends only on the equivalence class of $\underline{\mathcal{I}}$, by Theorems 6.1 and 6.2. Over any sequence of test transformations of $\underline{\mathcal{G}}(\underline{\mathcal{I}})$,

$$\begin{aligned} \operatorname{cosupp} \underline{\mathcal{G}}(\underline{\mathcal{I}})_j &= \big\{x \in \operatorname{cosupp} \underline{\mathcal{I}}_j : \operatorname{ord}_x \mathcal{N}(\underline{\mathcal{I}})_j \geq \operatorname{ord} \mathcal{N}(\underline{\mathcal{I}}), \\ &\qquad\qquad \operatorname{ord}_x \mathcal{M}(\underline{\mathcal{I}})_j \geq d - \operatorname{ord} \mathcal{N}(\underline{\mathcal{I}})\} \\ &= \Big\{x \in \operatorname{cosupp} \underline{\mathcal{I}}_j : \mu_x(\underline{\mathcal{I}}_j) - \frac{1}{d} \operatorname{ord}_x \mathcal{M}(\underline{\mathcal{I}})_j \geq \frac{1}{d} \operatorname{ord} \mathcal{N}(\underline{\mathcal{I}}), \\ &\qquad\qquad \frac{1}{d} \operatorname{ord}_x \mathcal{M}(\underline{\mathcal{I}})_j \geq 1 - \frac{1}{d} \operatorname{ord} \mathcal{N}(\underline{\mathcal{I}}) \Big\}. \end{aligned}$$

All terms in the latter condition depend only on the equivalence class of $\underline{\mathcal{I}}$ (and $\dim N$). $\square$

Now, we can resolve the singularities of the marked ideal of maximal order $\underline{\mathcal{G}}(\underline{\mathcal{I}})$, using Step I. The blowings-up involved are admissible for $\underline{\mathcal{I}}$. Such a resolution leads, after a finite number of steps (say r_1 steps) to a marked ideal $\underline{\mathcal{I}}_{r_1}$ such that either $\mathrm{ord}\,\mathcal{N}(\underline{\mathcal{I}}_{r_1}) < \mathrm{ord}\,\mathcal{N}(\underline{\mathcal{I}})$, or $\mathrm{cosupp}\,\underline{\mathcal{I}}_{r_1} = \emptyset$. (See Remark 5.2.) In the latter case, we have resolved the singularities of $\underline{\mathcal{I}}$. In the former case, we can repeat the process for $\underline{\mathcal{I}}_{r_1} = (\mathcal{I}_{r_1}, d), \ldots$. We get marked ideals $\underline{\mathcal{I}}_{r_1}, \ldots, \underline{\mathcal{I}}_{r_k}, \ldots$, such that

$$\mathrm{ord}\,\mathcal{N}(\underline{\mathcal{I}}) > \mathrm{ord}\,\mathcal{N}(\underline{\mathcal{I}}_{r_1}) > \cdots > \mathrm{ord}\,\mathcal{N}(\underline{\mathcal{I}}_{r_k}) > \cdots .$$

The process terminates after a finite number of steps, when either we have $\mathrm{ord}\,\mathcal{N}(\underline{\mathcal{I}}_{r_k}) = 0$ (i.e., we have reduced to the monomial case $\underline{\mathcal{I}}_{r_k} = \underline{\mathcal{M}}(\underline{\mathcal{I}}_{r_k})$), or we have $\mathrm{cosupp}\,\underline{\mathcal{I}}_{r_k} = \emptyset$ (i.e., we have resolved singularities).

This will complete the proof of functorial resolution of singularities of a marked ideal (Theorem 1.3) once we establish Claim 5.1 and prove Theorems 6.1, 6.2. $\square$

6. Invariants of a marked ideal

If $\underline{\mathcal{I}} = (M, N, E, \mathcal{I}, d)$ is a marked ideal and $a \in \mathrm{cosupp}\,\underline{\mathcal{I}}$, we define $\mu_a(\underline{\mathcal{I}}) := \mathrm{ord}_a \mathcal{I}/d$ and $\mu_{H,a}(\underline{\mathcal{I}}) := \mathrm{ord}_{H,a}\mathcal{I}/d$, $H \in E$, as in (5.3). The following theorem is due to Hironaka [Hi2].

Theorem 6.1. *Let $\underline{\mathcal{I}} = (M, N, E, \mathcal{I}, d)$ and $\underline{\mathcal{J}} = (M, P, E, \mathcal{J}, e)$ be marked ideals. Suppose that $\underline{\mathcal{I}}$ and $\underline{\mathcal{J}}$ are weak-equivalent (Defns. 2.4). Let $a \in \mathrm{cosupp}\,\underline{\mathcal{I}} = \mathrm{cosupp}\,\underline{\mathcal{J}}$. If $\dim N = \dim P$, then $\mu_a(\underline{\mathcal{I}}) = \mu_a(\underline{\mathcal{J}})$. If $\dim N > \dim P$, then $\mu_a(\underline{\mathcal{I}}) = 1$.*

Proof. We have $\mu_a(\underline{\mathcal{I}}) = \infty$ if and only if $\mathcal{I} = 0$ at a, i.e., $\mathrm{cosupp}\,\underline{\mathcal{I}}$ is smooth of dimension $= \dim N$ at a.

Suppose that $\mu_a(\underline{\mathcal{I}}) < \infty$. Set $\underline{\mathcal{I}}_0 := \underline{\mathcal{I}}$, $\underline{\mathcal{I}}_0 = (M_0, N_0, E_0, \mathcal{I}_0, d)$. Let $\sigma_1 : M_1 := M_0 \times \mathbb{A}^1 \to M_0$ denote the projection, and let $\underline{\mathcal{I}}_1 = (M_1, N_1, E_1, \mathcal{I}_1, d)$ denote the transform of $\underline{\mathcal{I}}_0$ by σ_1 (Defns. 2.4). Let $a_1 := (a, 0)$ and let $\Gamma_1 := \{a\} \times \mathbb{A}^1 \subset \mathrm{cosupp}\,\underline{\mathcal{I}}_1 = \sigma_1^{-1}(\mathrm{cosupp}\,\underline{\mathcal{I}}_0)$. Consider a sequence of admissible blowings-up

$$M_1 \xleftarrow{\sigma_2} M_2 \longleftarrow \cdots \longleftarrow M_j \xleftarrow{\sigma_{j+1}} M_{j+1} \longleftarrow \cdots$$

and the corresponding transforms $\underline{\mathcal{I}}_j = (M_j, N_j, E_j, \mathcal{I}_j, d)$ of $\underline{\mathcal{I}}_1$, where, for each $j \geq 1$, σ_{j+1} denotes the blowing-up with center a_j, and the a_j are defined inductively as follows: For each $j \geq 1$, $a_{j+1} := \Gamma_{j+1} \cap D_{j+1}$, where D_{j+1} is the exceptional divisor $\sigma_{j+1}^{-1}(a_j)$ and Γ_{j+1} is the strict transform of Γ_j.

Clearly, each $\Gamma_{j+1} \subset \mathrm{cosupp}\,\underline{\mathcal{I}}_{j+1}$, since $\Gamma_{j+1} \setminus \{a_{j+1}\} \subset \mathrm{cosupp}\,\underline{\mathcal{I}}_{j+1}$.

We introduce a subset S of $\mathbb{N} \times \mathbb{N}$ depending only on $\dim N$ and the weak-equivalence class of $\underline{\mathcal{I}}$, as follows: Let $r = \dim M - \dim N$. Note that, for each $j \geq 1$, $D_{j+1} \cap N_{j+1}$ is smooth, and has codimension r in D_{j+1} and codimension 1 in N_{j+1}. We say that $(j, 0) \in S$, $j \geq 1$, if, after j blowings-up $\sigma_2, \ldots, \sigma_{j+1}$ as above, $D_{j+1} \cap \mathrm{cosupp}\,\underline{\mathcal{I}}_{j+1}$ contains (i.e., is) a smooth subvariety of codimension r in D_{j+1} at a_{j+1}.

In this case, $D_{j+1} \cap \mathrm{cosupp}\,\underline{\mathcal{I}}_{j+1} = D_{j+1} \cap N_{j+1}$ at a_{j+1}, and the condition $D_{j+1} \cap N_{j+1} \subset \mathrm{cosupp}\,\underline{\mathcal{I}}_{j+1}$ means that $\mathrm{ord}_{D_{j+1}, a_{j+1}} \underline{\mathcal{I}}_{j+1} \geq d$. The latter inequality is equivalent to $j(\mu_a(\underline{\mathcal{I}}) - 1)d \geq d$, i.e., $j(\mu_a(\underline{\mathcal{I}}) - 1) \geq 1$. (This can be proved by induction on j, by a simple formal Taylor series calculation: For each j, $\mathcal{I}_{j+1, a_{j+1}} =$

$\mathcal{I}_{D_{j+1},a_{j+1}}^{-jd} \cdot (\mathcal{I}_a \cdot \mathcal{O}_{N_{j+1},a_{j+1}})$. But $\mathrm{ord}_{a_{j+1}}\underline{\mathcal{I}}_{j+1} = \mathrm{ord}_{a_j}\underline{\mathcal{I}}_j = \cdots = \mathrm{ord}_a\underline{\mathcal{I}}$; therefore, $\mathcal{I}_a \cdot \mathcal{O}_{N_{j+1},a_{j+1}}$ is divisible by $\mathcal{I}_{D_{j+1},a_{j+1}}^{j\mathrm{ord}_a\mathcal{I}}$ and not by any higher power. Since $\mathrm{ord}_a\mathcal{I} = \mu_a(\underline{\mathcal{I}})d$, we have $\mathcal{I}_{j+1,a_{j+1}} = \mathcal{I}_{D_{j+1},a_{j+1}}^{j(\mu_a(\underline{\mathcal{I}})-1)d} \cdot \tilde{\mathcal{I}}_{a_{j+1}}$, where $\tilde{\mathcal{I}}_{a_{j+1}}$ is not divisible by $\mathcal{I}_{D_{j+1},a_{j+1}}$.)

In particular, since $\mu_a(\underline{\mathcal{I}}) \geq 1$, $(j,0) \notin S$ for all $j \geq 1$ if and only if $\mu_a(\underline{\mathcal{I}}) = 1$.

It follows that if $\underline{\mathcal{I}}$ is weak-equivalent to $\underline{\mathcal{J}}$, where $\dim P < \dim N$, then $\mu_a(\underline{\mathcal{I}}) = 1$ (since $D_{j+1} \cap \mathrm{cosupp}\,\underline{\mathcal{I}}_{j+1} = D_{j+1} \cap \mathrm{cosupp}\,\underline{\mathcal{J}}_{j+1} \subset D_{j+1} \cap P_{j+1}$ cannot contain a smooth subvariety of codimension r in D_{j+1}.)

Now suppose that $(j,0) \in S$, for some $j \geq 1$. Then we can consider the blowing-up $\sigma_{j+2} : M_{j+2} \to M_{j+1}$ with centre $C_{j+1} = D_{j+1} \cap N_{j+1}$. Note that $\sigma_{j+2}|_{N_{j+2}} : N_{j+2} \to N_{j+1}$ is the identity! Set $G_{j+2} :=$ strict transform of G_{j+1} and $a_{j+2} := G_{j+2} \cap D_{j+2}$. We say that $(j,1) \in S$ if $D_{j+2} \cap \mathrm{cosupp}\,\underline{\mathcal{I}}_{j+2}$ contains a smooth subvariety of codimension r in D_{j+2} at a_{j+2}.

If so, then again $D_{j+2} \cap \mathrm{cosupp}\,\underline{\mathcal{I}}_{j+2} = D_{j+2} \cap N_{j+2}$. Since $\mathcal{I}_{j+1} = \mathcal{I}_{D_{j+1}}^{j(\mu_a(\underline{\mathcal{I}})-1)d} \cdot \tilde{\mathcal{I}}_{j+1}$ at a_{j+1}, where $\tilde{\mathcal{I}}_{j+1}$ is not divisible by $\mathcal{I}_{D_{j+1}}$, we see that $\mathcal{I}_{j+2} = \mathcal{I}_{D_{j+2}}^{j(\mu_a(\underline{\mathcal{I}})-1)d-d} \cdot \tilde{\mathcal{I}}_{j+2}$ at a_{j+2}, where $\tilde{\mathcal{I}}_{j+2} = \tilde{\mathcal{I}}_{j+1}$ is not divisible by $\mathcal{I}_{D_{j+2}}$. Therefore, $(j,1) \in S$ if and only if $j(\mu_a(\underline{\mathcal{I}}) - 1)d - d \geq d$.

We continue inductively: If $i \geq 1$ and $(j, i-1) \in S$, let $\sigma_{j+i+1} : M_{j+i+1} \to M_{j+i}$ denote the blowing-up with centre $C_{j+i} = D_{j+i} \cap N_{j+i}$. Set $G_{j+i+1} :=$ strict transform of G_{j+i} and $a_{j+i+1} := G_{j+i+1} \cap D_{j+i+1}$. We say that $(j,i) \in S$ if $D_{j+i+1} \cap \mathrm{cosupp}\,\underline{\mathcal{I}}_{j+i+1}$ is smooth and of codimension r in D_{j+i+1}; i.e., $(j,i) \in S$ if and only if $j(\mu_a(\underline{\mathcal{I}}) - 1) - i \geq 1$.

According to its definition, S depends only on $\dim N$ and the weak-equivalence class of $\underline{\mathcal{I}}$ (and the point a). We have shown that $S = \emptyset$ if and only if $\mu_a(\underline{\mathcal{I}}) = 1$, and, if $S \neq \emptyset$, then

$$S = \{(j,i) \in \mathbb{N} \times \mathbb{N} : j(\mu_a(\underline{\mathcal{I}}) - 1) - i \geq 1\};$$

therefore, $\mu_a(\underline{\mathcal{I}})$ is uniquely determined by S. ($\mu_a(\underline{\mathcal{I}}) = 1 + \sup_{(j,i)\in S}(i+1)/j$.) $\square$

Theorem 6.2. *Let $\underline{\mathcal{I}} = (M, N, E, \mathcal{I}, d)$ be a marked ideal and let $a \in \mathrm{cosupp}\,\underline{\mathcal{I}}$. Consider $H \in E$. Then $\mu_{H,a}(\underline{\mathcal{I}})$ depends only on the equivalence class of $\underline{\mathcal{I}}$ and the dimension of N at a. (See Defns. 2.5.)*

Proof. Suppose that $H \ni a$. Set Set $\underline{\mathcal{I}}_0 := \underline{\mathcal{I}}$, $\underline{\mathcal{I}}_0 = (M_0, N_0, E_0, \mathcal{I}_0, d)$. We can choose local coordinates $(x_1, \ldots, x_n)$ for N at a, such that $H|_{N_0} = \{x_1 = 0\}$. Let $\sigma_1 : M_1 = M_0 \times \mathbb{A}^1 \to M_0$ denote the projection, let $a_1 = (a, 0)$ and let $\underline{\mathcal{I}}_1 = (M_1, N_1, E_1, \mathcal{I}_1, d)$ denote the transform of $\underline{\mathcal{I}}_0$ by σ_1. (Thus $(x_1, \ldots, x_n, t)$ are local coordinates for N_1 at a_1.) Let $H_0^1 := M_0 \times \{0\}$ (so that $H_0^1|_{N_1} = \{t = 0\}$) and let $H_1^1 := \sigma^{-1}(H)$ ((so that $H_1^1|_{N_1} = \{x_1 = 0\}$.) Set $\Gamma_1 := \{a\} \times \mathbb{A}^1$.

We follow σ_1 by a sequence of exceptional blowings-up

$$M_1 \xleftarrow{\sigma_2} M_2 \longleftarrow \cdots \longleftarrow M_j \xleftarrow{\sigma_{j+1}} M_{j+1} \longleftarrow \cdots,$$

where (for all $j \geq 1$) we inductively define σ_{j+1} as the blowing-up with centre $C_j := H_0^j \cap H_1^j$, and set $H_0^{j+1} := \sigma_{j+1}^{-1}(C_j)$, $H_1^{j+1} :=$ strict transform of H_1^j by σ_{j+1}, $\Gamma_{j+1} :=$ strict transform of Γ_j, $a_{j+1} := \Gamma_{j+1} \cap H_0^{j+1} = C_{j+1} \cap \sigma_{j+1}^{-1}(a_j)$. Then, for each $j \geq 1$, N_{j+1} has a chart with coordinates $(x,t) = (x_1, \ldots, x_n, t)$ at a_{j+1} in which $(\sigma_2 \circ \cdots \circ \sigma_{j+1})(x,t) = (t^j x_1, x_2, \ldots, x_n, t)$.

Write $\mathcal{I} = \mathcal{I}_H^{\mu_{H,a}(\underline{\mathcal{I}})d} \cdot \mathcal{J}$ at a, where $\mathcal{J}$ is not divisible by $\mathcal{I}_H$. Set $\tau_{j+1} := \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_{j+1}$. Then $\tau_{j+1}^*(\mathcal{J})$ is not divisible by $\mathcal{I}_{H_1^{j+1}}$ at a_{j+1}, and

$$\tau_{j+1}^*(\mathcal{I}) = \mathcal{I}_{H_0^{j+1}}^{j\mu_{H,a}(\underline{\mathcal{I}})d} \cdot \mathcal{I}_{H_1^{j+1}}^{\mu_{H,a}(\underline{\mathcal{I}})d} \cdot \tau_{j+1}^*(\mathcal{J}).$$

(Consider the formal expansion of the pullback of $f(x) = x_1^{\mu_{H,a}(\underline{\mathcal{I}})d} g(x) \in \mathcal{I}$.)

Therefore, there exists $i \geq 1$ such that $\mathrm{ord}_{a_{i+1}} \tau_{i+1}^*(\mathcal{J}) = \mathrm{ord}_{a_{j+1}} \tau_{j+1}^*(\mathcal{J})$, for all $j \geq i$. (We can take i to be the least order of a monomial not divisible by x_1 in the Taylor expansions of a set of generators of $\mathcal{J}$ at a.) Clearly, if $j \geq i$, then

$$\mu_{H,a}(\underline{\mathcal{I}}) = \mu_{a_{j+1}}(\underline{\mathcal{I}}_{j+1}) - \mu_{a_j}(\underline{\mathcal{I}}_j),$$

so the result follows from Theorem 6.1. □

7. Passage from local to global

In this section, we give two different proofs of Claim 5.1 in Step I.A in Section 5 (and of the analogous claim in Step I.B). The first proof is the method described by Kollár [Ko], which we present briefly here. (See [Ko, Thm. 105] for an axiomatization of the procedure.) The second proof is the method of [BM4, BM5], which involves proving a stronger result than Theorem 1.3: we give an algorithm for functorial resolution of singularities, where each successive centre of blowing up is the maximum locus of a *desingularization invariant*. We use the notation of Section 4.

The invariant is essentially already present in the proof of Theorem 1.3 in Section 5; we just make it explicit below. The invariant is defined using a sequence of pairs — it begins with $(\mathrm{ord}\,\mathcal{N}(\underline{\mathcal{I}}), s)$ and the following pairs are determined in decreasing dimension, by induction. (The first pair is the same as that denoted (d, s) in our first version of resolution of singularities [BM1, §4].)

7.1. **First proof.** For simplicity, assume that N has constant dimension n. Choose a finite open covering $M = \cup U_i$ such that, for each i, we have a coefficient ideal $\underline{\mathcal{C}}_{U_i}(\underline{\mathcal{I}})$ as in Step I.A. Set $M' := \coprod_i U_i$ (the disjoint union) and let $\gamma : M' \to M$ denote the natural morphism. (γ is smooth.) The fibre-product $M'' := M' \times_M M'$ can be identified with the disjoint union $\coprod_{i \leq j} U_i \cap U_j$, so that, if $\tau_1,\ \tau_2 : M'' \to M'$ are the two projections, then $\tau_1|_{U_i \cap U_j}$ and $\tau_2|_{U_i \cap U_j}$ are the inclusions $U_i \cap U_j \hookrightarrow U_i$ and $U_i \cap U_j \hookrightarrow U_j$ (respectively).

Set $\underline{\mathcal{I}}' := \gamma^*(\underline{\mathcal{I}})$ and $\underline{\mathcal{I}}'' := \tau_1^*(\underline{\mathcal{I}}') = \tau_2^*(\underline{\mathcal{I}}')$. The coefficient ideals $\underline{\mathcal{C}}_{U_i}(\underline{\mathcal{I}})$ induce a marked ideal $\underline{\mathcal{C}}'$ of dimension $n-1$ on M'. Set $\underline{\mathcal{C}}_1'' := \tau_1^*(\underline{\mathcal{C}}')$, $\underline{\mathcal{C}}_2'' := \tau_2^*(\underline{\mathcal{C}}')$. Then $\underline{\mathcal{C}}'$ is equivalent to $\underline{\mathcal{I}}'$, and $\underline{\mathcal{C}}_1''$ is equivalent to $\underline{\mathcal{C}}_2''$.

By the inductive assumption of functorial resolution of singularities in dimension $n-1$, there are resolution sequences associated to $\underline{\mathcal{C}}'$, $\underline{\mathcal{C}}_1''$ and $\underline{\mathcal{C}}_2''$ such that: (1) the resolution sequences for $\underline{\mathcal{C}}_1''$ and $\underline{\mathcal{C}}_2''$ are the same; (2) the morphism τ_1 (or τ_2) lifts throughout the resolution sequences for $\underline{\mathcal{C}}_1''$ and $\underline{\mathcal{C}}'$ (or $\underline{\mathcal{C}}_2''$ and $\underline{\mathcal{C}}'$).

Since $\underline{\mathcal{C}}'$ is equivalent to $\underline{\mathcal{I}}'$, the resolution sequence for $\underline{\mathcal{C}}'$ is a resolution sequence for $\underline{\mathcal{I}}'$. Consider the centre $C_0' \subset M_0' := M'$ of the first blowing-up in this sequence. For each i, let C_{0i}' denote the restriction of C_0' to U_i. Then $C_{0i}'|U_i \cap U_j = C_{0j}'|U_i \cap U_j$, for all $i \leq j$.

Therefore, the C_{0i}' glue together to define a smooth closed subvariety C_0 of $M_0 := M$. We thus obtain the centre of the first global blowing-up, and can continue in the same way to get the entire functorial resolution sequence for $\underline{\mathcal{I}}$.

The same argument works for the analogous claim in Step I.B.

7.2. **Second proof.** Given a marked ideal $\underline{\mathcal{I}}$, we will construct a resolution sequence and an invariant (with values in an ordered set) defined at the points of the $\operatorname{cosupp}\underline{\mathcal{I}}_j$ (where the $\underline{\mathcal{I}}_j$ are the successive transforms of $\underline{\mathcal{I}}$), such that the invariant is upper-semicontinuous, we blow up its maximum locus at each step, and the invariant decreases with (finitely many) blowings-up. The following theorem makes this precise. Let $\mathbb{Q}_{>0}$ (or $\mathbb{Q}_{\geq 0}$) denote the set of positive (or nonnegative) rational numbers.

Theorem 7.1. *Let $\underline{\mathcal{I}} = (M, N, E, \mathcal{I}, d)$ denote a marked ideal, where $d > 0$. Then $\underline{\mathcal{I}}$ admits a resolution of singularities*

$$M = M_0 \overset{\sigma_1}{\longleftarrow} M_1 \longleftarrow \cdots \overset{\sigma_t}{\longleftarrow} M_t\,,$$

such that, if $\underline{\mathcal{I}}_j$ denotes the j'th transform of $\underline{\mathcal{I}}$, then the following properties are satisfied.

(1) *There are Zariski upper-semicontinuous functions $\operatorname{inv}_{\underline{\mathcal{I}}}$, $\mu_{\underline{\mathcal{I}}}$ and $J_{\underline{\mathcal{I}}}$ defined on $\operatorname{cosupp}\underline{\mathcal{I}}_j$, for all j, where $\operatorname{inv}_{\underline{\mathcal{I}}}$ takes values in the set of sequences consisting of finitely many pairs in $\mathbb{Q}_{>0} \times \mathbb{N}$ followed by 0 or ∞ (ordered lexicographically), $\mu_{\underline{\mathcal{I}}}$ takes values in $\mathbb{Q}_{\geq 0} \cup \{\infty\}$, and $J_{\underline{\mathcal{I}}}$ takes values in the set of subsets of E_j, for all j (ordered as in Step II.A in Section 5).*

(2) *Each centre of blowing up $C_j \subset M_j$ is given by the maximum locus of $(\operatorname{inv}_{\underline{\mathcal{I}}}, J_{\underline{\mathcal{I}}})$ on $\operatorname{cosupp}\underline{\mathcal{I}}_j$ (where pairs are ordered lexicographically).*

(3) *Let $a \in \operatorname{cosupp}\underline{\mathcal{I}}_{j+1}$ and $b = \sigma_{j+1}(a)$. If $b \notin C_j$, then*

$$\operatorname{inv}_{\underline{\mathcal{I}}}(a) = \operatorname{inv}_{\underline{\mathcal{I}}}(b), \quad \mu_{\underline{\mathcal{I}}}(a) = \mu_{\underline{\mathcal{I}}}(b), \quad J_{\underline{\mathcal{I}}}(a) = J_{\underline{\mathcal{I}}}(b).$$

If $b \in C_j$, then

$$\big(\operatorname{inv}_{\underline{\mathcal{I}}}(a), \mu_{\underline{\mathcal{I}}}(a)\big) < \big(\operatorname{inv}_{\underline{\mathcal{I}}}(b), \mu_{\underline{\mathcal{I}}}(b)\big)\,.$$

(4) *Suppose that $\varphi : M' \to M$ is a smooth morphism and that $\underline{\mathcal{I}}' = (M', N', E', \mathcal{I}', d')$ is a marked ideal such that $\underline{\mathcal{I}}'$ is equivalent to $\varphi^*(\underline{\mathcal{I}})$ (in particular, $E' = \varphi^{-1}(E)$) and $\dim N' = \dim \varphi^{-1}(N)$. Then φ lifts to smooth morphisms φ_j throughout the resolution towers for $\underline{\mathcal{I}}'$ and $\underline{\mathcal{I}}$, and, for each j, $\operatorname{inv}_{\underline{\mathcal{I}}'} = \operatorname{inv}_{\underline{\mathcal{I}}} \circ \varphi_j$, $\mu_{\underline{\mathcal{I}}'} = \mu_{\underline{\mathcal{I}}} \circ \varphi_j$ and $J_{\underline{\mathcal{I}}'} = J_{\underline{\mathcal{I}}} \circ \varphi_j$.*

Proof. We follow the steps of the proof of Theorem 1.3 in Section 5.

Step I. $\underline{\mathcal{I}}$ of maximal order.

Case A. $E = \emptyset$**.** The coefficient ideals $\underline{\mathcal{C}}_U(\underline{\mathcal{I}})$ admit resolution sequences and invariants as required, by induction. Over each U, the desingularization of $\underline{\mathcal{I}}$ is realized by that of $\underline{\mathcal{C}}_U(\underline{\mathcal{I}})$. If x is a point of $\operatorname{cosupp}\underline{\mathcal{I}}_j$ lying over U, we set

$$\overline{\operatorname{inv}}_{\underline{\mathcal{I}}}(x) := \big(0, \operatorname{inv}_{\underline{\mathcal{C}}_U(\underline{\mathcal{I}})}(x)\big)\,, \quad \overline{\mu}_{\underline{\mathcal{I}}}(x) := \mu_{\underline{\mathcal{C}}_U(\underline{\mathcal{I}})}(x), \quad \overline{J}_{\underline{\mathcal{I}}}(x) := J_{\underline{\mathcal{C}}_U(\underline{\mathcal{I}})}(x).$$

Case B. General maximal order case. The marked ideal $\underline{\mathcal{J}}$ (defined using $\underline{\mathcal{C}}_U(\underline{\mathcal{I}}_\emptyset)$) has empty exceptional divisor. We set

$$\overline{\operatorname{inv}}_{\underline{\mathcal{I}}}(x) := \big(s, \operatorname{inv}_{\underline{\mathcal{J}}}(x)\big)\,, \quad \overline{\mu}_{\underline{\mathcal{I}}}(x) := \mu_{\underline{\mathcal{J}}}(x), \quad \overline{J}_{\underline{\mathcal{I}}}(x) := J_{\underline{\mathcal{J}}}(x),$$

for $x \in \operatorname{cosupp}\underline{\mathcal{J}}_j$, until $\underline{\mathcal{J}}$ is resolved. If the sequences of blowings-up of the $\underline{\mathcal{J}}$ do not resolve $\underline{\mathcal{I}}$, then we repeat the process using the new $s = \max s(x)$, as in Section 5.

Note that if $x \in \operatorname{cosupp}\underline{\mathcal{I}}_j$ and $s(x) < s$, then all blowings-up in the desingularization tower of $\underline{\mathcal{I}}$ are isomorphisms over x, until we reach a year q_k where $s(x)$ is the maximum value. Then the values of the invariants at x equal their values in year q_k (by definition; see also Step II.B below).

Step II. General $\underline{\mathcal{I}}$. First suppose that $\mathcal{I} = 0$. If $x \in \operatorname{cosupp} \underline{\mathcal{I}} = N$, then we set

$$\operatorname{inv}_{\underline{\mathcal{I}}}(x) := \infty, \quad \mu_{\underline{\mathcal{I}}}(x) := \infty, \quad J_{\underline{\mathcal{I}}}(x) := \emptyset.$$

We blow up with centre N to resolve singularities.

Now suppose that $\mathcal{I} \neq 0$.

Case A. Monomial case. If $x \in \operatorname{cosupp} \underline{\mathcal{I}}_j$, then we set

$$\operatorname{inv}_{\underline{\mathcal{I}}}(x) := 0, \quad \mu_{\underline{\mathcal{I}}}(x) := \mu_x(\underline{\mathcal{I}}_j) = \frac{\operatorname{ord}_x \mathcal{I}_j}{d},$$

and we let $J_{\underline{\mathcal{I}}}(x)$ denote the maximum among the subsets of E_j that define the components of $\operatorname{cosupp} \underline{\mathcal{I}}_j$ at x. The required properties in this case have been proved in Section 4.

Case B. General case. We resolve the singularities of the companion ideal $\underline{\mathcal{G}} := \underline{\mathcal{G}}(\underline{\mathcal{I}})$ (which is of maximal order) in order to reduce the maximum order of the ideal $\mathcal{N}(\underline{\mathcal{I}})$. If $x \in \operatorname{cosupp} \underline{\mathcal{G}}_j$, then we set

$$\operatorname{inv}_{\underline{\mathcal{I}}}(x) := \left(\frac{\operatorname{ord} \mathcal{N}(\underline{\mathcal{I}})}{d}, \overline{\operatorname{inv}}_{\underline{\mathcal{G}}}(x) \right), \quad \mu_{\underline{\mathcal{I}}}(x) := \overline{\mu}_{\underline{\mathcal{G}}}(x), \quad J_{\underline{\mathcal{I}}}(x) := \overline{J}_{\underline{\mathcal{G}}}(x).$$

If $x \in \operatorname{cosupp} \underline{\mathcal{I}} \setminus \operatorname{cosupp} \underline{\mathcal{G}}$, then $\operatorname{ord}_x \mathcal{N}(\underline{\mathcal{I}})$ will be the maximum order of $\mathcal{N}(\underline{\mathcal{I}})$ in some neighbourhood of x, so we can define the invariant in the same way over such a neighbourhood. (When we resolve the singularities of $\underline{\mathcal{G}}(\underline{\mathcal{I}})$, either $\operatorname{cosupp} \underline{\mathcal{I}}_{r_1} = \emptyset$ or $\operatorname{ord} \mathcal{N}(\underline{\mathcal{I}}_{r_1}) < \operatorname{ord} \mathcal{N}(\underline{\mathcal{I}})$ after r_1 steps, and we repeat the process using $\big(\mathcal{N}(\underline{\mathcal{I}}_{r_1}), \operatorname{ord} \mathcal{N}(\underline{\mathcal{I}}_{r_1})\big)$. If $x \in \operatorname{cosupp} \underline{\mathcal{G}}(\underline{\mathcal{I}}_{r_1})$ maps to the complement of $\operatorname{cosupp} \underline{\mathcal{G}}(\underline{\mathcal{I}})$, then all previous blowings-up are isomorphisms at the successive images of x, so the values of the invariants at these points are the same as at x.)

All properties required in the theorem follow by induction and semicontinuity of ord_x. □

8. Comparison of algorithms and an example

In this final section, we prove Corollary 1.4 and we give an example to show that the versions of canonical desingularization (Theorem 1.1) in [EV1, W, Ko] involve blowings-up with centres that are not necessarily smooth (see also [Ma, Ex. 11-5]).

8.1. **Włodarczyk's method.** Let $\underline{\mathcal{I}} = (M, N, E, \mathcal{I}, d)$ denote a marked ideal. The objects constructed in Sections 3 and 4 using our logarithmic derivative ideals $\underline{\mathcal{D}}_E^j(\underline{\mathcal{I}})$ of course have analogues defined using the standard derivative ideals $\underline{\mathcal{D}}^k(\underline{\mathcal{I}})$ (cf. [V2]): We define $\underline{\mathcal{C}}^k(\underline{\mathcal{I}}) := \sum_{j=0}^k \underline{\mathcal{D}}^j(\underline{\mathcal{I}})$, $k \leq d-1$. Then $\underline{\mathcal{I}}$ and $\underline{\mathcal{C}}^k(\underline{\mathcal{I}})$ are weak-equivalent (cf. Cor. 3.11), but in general they are not equivalent [BM5, Ex. 5.14]. For $\underline{\mathcal{I}}$ of maximal order, Włodarczyk introduces a *homogenized ideal* $\underline{\mathcal{H}}(\underline{\mathcal{I}}) = (M, N, E, \mathcal{H}(\underline{\mathcal{I}}), d)$, where

$$\underline{\mathcal{H}}(\underline{\mathcal{I}}) := \sum_{j=0}^{d-1} \mathcal{D}^j(\mathcal{I}) \cdot (\mathcal{D}^{d-1}(\mathcal{I}))^j.$$

Then $\underline{\mathcal{H}}(\underline{\mathcal{I}})$ is weak-equivalent to $\underline{\mathcal{I}}$ [W, Lemma 3.5.2]. In general, $\underline{\mathcal{H}}(\underline{\mathcal{I}})$ is not equivalent to $\underline{\mathcal{I}}$. It follows from Corollary 3.6, however, that if $E = \emptyset$, then $\underline{\mathcal{C}}^k(\underline{\mathcal{I}})$

and $\underline{\mathcal{H}}(\underline{\mathcal{I}})$ are both equivalent to $\underline{\mathcal{I}}$. (The same statements hold for the variant $\underline{\mathcal{W}}(\underline{\mathcal{I}})$ of $\underline{\mathcal{H}}(\underline{\mathcal{I}})$ introduced by Kollár [Ko].)

The desingularization algorithm as described in Section 5 allows many minor variations. (See [BM5].) For example, Step I.B (moving apart E and $\operatorname{cosupp}\underline{\mathcal{I}}$) is described in a different way in [W] that may mean separating the components of E from $\operatorname{cosupp}\underline{\mathcal{I}}$ in a somewhat different order. We will ignore this minor difference. The essential difference in Włodarczyk's treatment is his use of the homogenized ideal. Instead of using the ideal $\underline{\mathcal{C}}_E^{d-1}(\underline{\mathcal{I}})$ of §3.4, he uses $\underline{\mathcal{C}}^{d-1}(\underline{\mathcal{H}}(\underline{\mathcal{I}}))$. Following the algorithm as presented in Section 5, this introduces a change in Step I, where, instead of our coefficient ideal $\underline{\mathcal{C}}_{\emptyset,P}(\underline{\mathcal{I}})$ (see (4.1)), Włodarczyk would use

$$\underline{\mathcal{C}}_P(\underline{\mathcal{I}}) := \left(U, P, \emptyset, \mathcal{C}^{d-1}(\underline{\mathcal{H}}(\underline{\mathcal{I}}))|_P, d_{\underline{\mathcal{C}}}\right).$$

(Likewise in Kollár's version, using $\underline{\mathcal{W}}(\underline{\mathcal{I}})$.)

A priori, this change might result in a change in the actual recipe for choosing the centres of blowing up. However, since $\underline{\mathcal{C}}^{d-1}(\underline{\mathcal{H}}(\underline{\mathcal{I}}))$ is equivalent to $\underline{\mathcal{I}}$ when $E = \emptyset$, it follows that $\underline{\mathcal{C}}_P(\underline{\mathcal{I}})$ is equivalent to $\underline{\mathcal{C}}_{\emptyset,P}(\underline{\mathcal{I}})$. We thus obtain Corollary 1.4.

8.2. **Example.** Let X denote the subvariety of $\mathbb{A}^4$ defined by the ideal $\mathcal{I}$ generated by $y^2 - x^3$ and $x^4 + xz^2 - w^3$. We consider resolution of singularities (Section 5) of the corresponding marked ideal $\underline{\mathcal{I}} = (\mathbb{A}^4, \mathbb{A}^4, \emptyset, \mathcal{I}, 1)$ (which we write simply as $(\emptyset, \mathcal{I}, 1)$). The ideal $\mathcal{I}$ has maximum order 2 precisely at the origin, so that $C_0 := \{0\}$ is the centre of the first blowing-up σ_1. Let $\underline{\mathcal{I}}_1 = (E_1, \mathcal{I}_1, 1)$ denote the transform of $\underline{\mathcal{I}}$ by σ_1. In the x-chart of σ_1,

$$\mathcal{I}_1 = \left(x - y^2,\ x(x + z^2 - w^3)\right) = \left(x - y^2,\ y^2(y^2 + z^2 - w^3)\right),$$

and E_1 is given by (x). There is no nontrivial monomial part to factor out, and $\underline{\mathcal{I}}_1$ is of maximal order. According to Step I.B, the centre C_1 of the next blowing-up σ_2 is given by $x = y = 0$.

But the ideal of the strict transform X_1 of X in the x-chart above is generated by

$$x - y^2, \quad y^2 + z^2 - w^3.$$

Therefore, $C_1 \cap X_1$ is given by

$$x = y = 0, \quad z^2 - w^3 = 0.$$

Moreover, $\operatorname{Sing} X_1 = \{0\}$.

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