A REMARK ON GROUP ACTIONS WITH DIVISORIAL FIXED POINT SETS

by

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§0. Introduction

Let X be a (possibly non-Kähler) compact complex n-dimensional connected manifold and D a 1-codimensional irreducible reduced analytic subset of X. We then put

Θ_X := sheaf of germs of holomorphic vector fields on X,

$\mathrm{Aut}^0(X)$:= identity component of the group of holomorphic automorphisms of X,

V_k := $H^0(X,\Theta_X(-kD))$ $(\subset H^0(X,\Theta_X))$, $k = 1,2,\ldots$.

Assume that $V_1 \neq \emptyset$, i.e., X admits a holomorphic vector field on X which vanishes on D. We then fix an arbitrary nonzero element v of V_1, and consider the 1-dimensional complex Lie subgroup $G (= \exp(\mathbb{C}v))$ of $\mathrm{Aut}^0(X)$ corresponding to the complex Lie subalgebra $\mathbb{C}v$ of $H^0(X,\Theta_X)$. Let $\{O_\mu \mid \mu \in M\}$ be the set of all G-orbits O_μ in X satisfying the following conditions:

(a) $O_\mu \cong \mathbb{C}$ as complex manifolds.

(b) The closure $\bar{O}_\mu$ of O_μ in X (in terms of the Euclidean topology) is an analytic subset of X such that $\bar{O}_\mu - O_\mu$ is a single point (denoted by p_μ). In particular, $\bar{O}_\mu$ is a rational curve (possibly with singularity at p_μ).

(c) $O_\mu \subset X - D$ and $p_\mu \in D$.

The theorem we shall prove in this note is now stated as follows:

Theorem A. Assume that $M \neq \phi$. Then

(0.1) $\bar{O}_\mu \cong \mathbb{P}^1(\mathbb{C})$ (i.e., $\bar{O}_\mu$ is nonsingular) for every $\mu \in M$.

(0.2) $(\bar{O}_\mu \circ D) = 1$ or 2 for each $\mu \in M$, where $(\bar{O}_\mu \circ D)$ denotes the intersection number of $\bar{O}_\mu$ and D in X.

(0.3) Let $N := \{\nu \in M \mid (\bar{O}_\nu \circ D) = 2\}$. Then the mapping $N \ni \nu \longmapsto p_\nu \in D$ is injective.

(0.4) $v \notin V_3$.

(0.5) Suppose $v \in V_2$. Then $(\bar{O}_\mu \circ D) = 1$ for all $\mu \in M$, and the mapping $M \ni \mu \longmapsto p_\mu \in D$ is injective.

For the sake of completeness, we here add a theorem treating a typical case of $M = \phi$. Note that this added one is no more than a reformulation of the generalized Bialynicki-Birula's decomposition of Fujiki [2], Carrell and Sommese [1].

Theorem B. Suppose that G is, as a complex Lie group, isomorphic to $\mathbb{C}^*$. (Then it is well-known that D is nonsingular.) Fix a natural inclusion $\mathbb{C}^* (= \mathbb{P}^1(\mathbb{C}) - \{(0:1),(1:0)\}) \subset \mathbb{P}^1(\mathbb{C})$. We furthermore assume that our G-action

$$\begin{array}{ccc} \mathbb{C}^* (= G) \times X & \longrightarrow & X \\ (\;g, \quad x) & \longmapsto & g \cdot x \end{array}$$

<u>extends to a meromorphic map:</u> $\mathbb{P}^1(\mathbb{C}) \times X \longrightarrow X$ <u>(and this is always the case if</u> X <u>is Kähler</u> (cf. Sommese [4; Corollary II-A])). <u>Then the zero section:</u> $D \longrightarrow N_{D/X}$ <u>of the normal bundle</u> $N_{D/X}$ <u>of</u> D <u>in</u> X <u>naturally extends to a</u> G-<u>equivariant biholomorphic map of an open neighbourhood</u> X_0 <u>of</u> D <u>in</u> X <u>onto</u> $N_{D/X}$. <u>Furthermore if</u> X <u>is Kähler, then</u> X_0 <u>is Zariski open in</u> X.

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§1. Comments on Theorem B.

Before getting into the proof of Theorem A, we shall explain how results of Fujiki [2], for instance, imply Theorem B. In this section, we identify G with $\mathbb{C}^*$. First, by virtue of [2;(2.1)], there exist i) a locally closed $\mathbb{C}^*$-invariant submanifold X_0 of X and ii) a $\mathbb{C}^*$-invariant retraction $\pi : X_0 \longrightarrow D$ such that, with respect to π, X_0 is a fibre bundle over D with typical fibre $\mathbb{C}$ on which $\mathbb{C}^*$ linearly acts. Since $\dim X = \dim X_0$, one immediately sees that X_0 is open in X. On the other hand, in view of ii) above, D is written as a union $\cup_{\alpha \in A} U_\alpha$ of open subsets such that, for each α, there is a $\mathbb{C}^*$-equivariant biholomorphic map

$$j_\alpha : \pi^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{C}$$
$$x \longmapsto j_\alpha(x) = (\pi(x), x_\alpha),$$

where $\mathbb{C}^*$ acts on the second factor $\mathbb{C}$ by

$$\mathbb{C}^* \times \mathbb{C} \longrightarrow \mathbb{C}$$
$$(t, x_\alpha) \longmapsto t^m x_\alpha .$$

Note that m is an integer independent of α. (Since $\mathbb{C}^*$ acts effetively on X, we have $m = 1$ or -1.) If $U_\alpha \cap U_\beta \neq \phi$, then x_α/x_β is a $\mathbb{C}^*$-invariant function on $\pi^{-1}(U_\alpha \cap U_\alpha) - D$, i.e., $x_\alpha = f_{\alpha\beta} x_\beta$ on $\pi^{-1}(U_\alpha \cap U_\beta)$ for some non-vanishing holomorphic function $f_{\alpha\beta}$ on $U_\alpha \cap U_\beta$. Since $\{x_\alpha = 0\}$ locally defines D in $\pi^{-1}(U_\alpha)$, the line bundle $(f_{\alpha\beta}) \in H^1(D, \mathcal{O}^*)$ is nothing but the normal bundle $N_{D/X}$. Hence there exists a G-equivariant biholomorphic map of X_0 onto $N_{D/X}$ which is a natural extension of the zero section: $D \longrightarrow N_{D/X}$. If X is Kähler, then by [2;(2.2)], X_0 is Zariski open in X.

Remark (1.1). If X is of class $\mathcal{C}$ (i.e., X is bimeromorphic to a compact Kähler manifold), we always have a meromorphic extension: $\mathbb{P}^1(\mathbb{C}) \times X \longrightarrow X$ of the G-action: $G \times X \longrightarrow X$ by a result of Fujiki [3;(6.10)] combined with Step 1 of §3.

§2. Proof of Theorem A.

In this section, we often use the identification

$$\begin{array}{ccc} \mathbb{P}^1(\mathbb{C}) & \cong & \mathbb{C} \cup \{\infty\} \\ (1:z) & \longleftrightarrow & z \ . \end{array}$$

Fix an arbitrary $\mu \in M$. We then choose a normalization

$$\sigma : \mathbb{P}^1(\mathbb{C}) \ (= \mathbb{C} \cup \{\infty\}) \longrightarrow \bar{O}_\mu$$

of $\bar{O}_\mu$ such that $\sigma(0) = p_\mu$. Let U be an open neighbourhood of p_μ in X with holomorphic local coordinates $w = (w_1, w_2, \ldots, w_n)$ centered at p_μ. Then for a sufficiently small disc $\Delta := \{|z| < \varepsilon\}$ $(\subset \mathbb{P}^1(\mathbb{C}))$, we have $\sigma(\Delta) \subset U$, and each $w_i(z)$ $(:= w_i \circ \sigma(z))$ is written as

$$(2.1) \qquad w_i(z) = z^{\alpha_i} h_i(z) \qquad (z \in \Delta)$$

for some positive integer α_i and some holomorphic function h_i on Δ such that $h_i(0) \neq 0$. Put $\alpha := \mathrm{Min}\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ and $J := \{i \mid \alpha_i = \alpha\}$. Since the holomorphic G-action on $\bar{O}_\mu$ can naturally be lifted to the one on $\mathbb{P}^1(\mathbb{C})$, the restriction $v|_{O_\mu}$ of v to O_μ naturally extends to a holomorphic vector field on $\mathbb{P}^1(\mathbb{C})$ via the identification

$$\begin{array}{ccc} \mathbb{P}^1(\mathbb{C}) - \{0\} & \cong & O_\mu \ (= \bar{O}_\mu - \{p_\mu\}) \\ z & \longleftrightarrow & \sigma(z). \end{array}$$

Note that this vector field on $\mathbb{P}^1(\mathbb{C})$ has the only zero at $z = 0$. Hence, multiplying v by some nonzero constant, we may assume without loss of generality that

$$v|_{O_\mu} = z^2\,\partial/\partial z\ .$$

Regard this (resp. z) as an element of $H^0(\bar{O}_\mu, \Theta_X|_{\bar{O}_\mu})$ (resp. a meromorphic function on $\bar{O}_\mu$). Then, in a neighbourhood of p_μ,

$$(2.2)\qquad \begin{aligned} v|_{\bar{O}_\mu} &= z^2 \sum_{i=1}^n (\partial w_i(z)/\partial z)\ \partial/\partial w_i \\ &= \alpha\, z^{\alpha+1} \sum_{j\in J} h_j(0)\ \partial/\partial w_j + \text{higher order term of } z\ . \end{aligned}$$

On the other hand, by setting $m := \mathrm{Max}\{k \mid v \in V_k\}\ (\geq 1)$, one can express v on U as

$$(2.3)\qquad v = \varphi(w)^m \sum_{i=1}^n f_i(w)\ \partial/\partial w_i$$

with $\varphi, f_i \in H^0(U,\mathcal{O})$, where "$\varphi = 0$" is the local defining equation of D on U. Recall that $p_\mu \in D$. Hence $\varphi(0) = 0$. We next put $a_{ij} := (\partial f_i/\partial w_j)(0)$. Now, in a neighbourhood of p_μ,

$$(2.4)\ \begin{cases} \varphi(w) = \varphi_d(w) + \text{higher order term of } w_1,\dots,w_n\ , \\ f_i(w) = f_i(0) + \sum_{j=1}^n a_{ij}\, w_j + \text{higer order term of } w_1,\dots,w_n\ , \end{cases}$$

with some nonzero homogeneous polynomial $\varphi_d(w)$ of $w_1, w_2, \ldots, w_n$ of degree $d \geqq 1$. In view of (2.1), the order β of the zero of $\varphi(w(z))$ $(:= \varphi(w \circ \sigma(z)))$ at $z = 0$ satisfies $\beta \geqq \alpha d$. Moreover, restricting (2.3) to $\bar{O}_\mu$, we can write it in a neighbourhood of p_μ as follows (cf. (2.1), (2.4)):

$$(2.5) \quad v|_{\bar{O}_\mu} = \varphi(w(z))^m \sum_{i=1}^{n} (f_i(0) + \sum_{j \in J} a_{ij} z^\alpha h_j(0))\, \partial/\partial w_i$$

$$+ \text{ term whose order in } z \text{ exceeds } \alpha + m\beta .$$

Put $\gamma := (\bar{O}_\mu \circ D)$ for simplicity. Now the following cases are possible:

(Case 1) $f_i(0) \neq 0$ for some $i \in \{1, 2, \ldots, n\}$: In this case, taking U and Δ smaller if necessary, we may assume that $\tilde{v} := \sum_{i=1}^{n} f_i(w)\, \partial/\partial w_i$ is nonvanishing on U. Let C_μ $(\subset U)$ be the holomorphic integral curve of $\tilde{v}$ through p_μ. Since on $U - D$, holomorphic vector fields v and $\tilde{v}$ generate the same holomorphic integral curves, one has

$$(2.6) \qquad \bar{O}_\mu|_U = C_\mu .$$

Note that C_μ is nonsingular. In particular, p_μ is a simple point of $\bar{O}_\mu$, i.e., $\alpha = 1$. By comparing (2.2) with (2.5), we now obtain

$$md = m\alpha d \leqq m\beta = \alpha + 1 = 2$$

Since $\alpha = 1$, the identity $\gamma = \beta$ holds. Hence

$$(m,d,\alpha,\gamma) = (1,1,1,2),\ (2,1,1,1) \text{ or } (1,2,1,2).$$

(Case 2) $f_i(0) = 0$ for all $i \in \{1,2,\dots,n\}$: Then again by comparing (2.2) with (2.5), we obtain

$$1 \leq m\alpha d \leq m\beta \leq 1 .$$

Hence $m = \alpha = d = 1$. From $\alpha = 1$, it follows that $\gamma = \beta$. Thus,

$$(m,d,\alpha,\gamma) = (1,1,1,1) .$$

In view of these possible cases, we can now conclude by the following observations:

(0.1) follows from $\alpha = 1$; (0.2) holds by $\gamma = 1$ or 2; (0.3) is an easy consequence of (2.6), because $\gamma = 2$ occurs only in Case 1; (0.4) is true by $m \leq 2$; the former half of (0.5) is obtained from the fact that $m = 2$ implies $\gamma = 1$; the latter half of (0.5) follows from (2.6), because $m = 2$ occurs only in Case 1.

Remark (2.7). In Theorem A, replace $(\bar{O}_\mu \circ D)$ (resp. $(\bar{O}_\nu \circ D)$) by the intersection multiplicity $i(\bar{O}_\mu \circ D; p_\mu)$ (resp. $i(\bar{O}_\nu \circ D; p_\nu)$) of $\bar{O}_\mu$ (resp. $\bar{O}_\nu$) and D at the point p_μ (resp. p_ν). Then even if X is noncompact, Theorem A is valid as long as the holomorphic local 1-parameter group

$$\{ \exp(tv) \mid t \in \mathbb{C} \}$$

(of local transformations of X) generated by v defines a complex 1-parameter subgroup of $\mathrm{Aut}^0(X)$.

§3. Some application.

A combination of Theorems A and B gives

Theorem C. Let Y be a compact complex connected manifold of class $\mathcal{C}$, and D a 1-codimensional irreducible reduced analytic subset of Y such that $H^0(Y,\Theta_Y(-2D)) \neq \{0\}$. Take an arbitrary nonzero element v of $H^0(Y,\Theta_Y(-2D))$. Then the group $G := \exp(\mathbb{C}v)\ (\subset \mathrm{Aut}^0(Y))$ is isomorphic to $\mathbb{C}$, and for every $p \in Y - D$, the closure $C_{p,v} := \overline{G \cdot p}$ of its orbit is either a point or a (possibly singular) rational curve with $(C_{p,v} \circ D) \leq 1$. If in addition $v \in H^0(Y,\Theta_Y(-3D))$, then $C_{p,v} \cap D = \phi$ for every $p \in X - D$.

Proof: Let I be the ideal sheaf of D in $\mathcal{O}_Y$. We then put

$$H := \{g \in \mathrm{Aut}^0(Y) \mid g_{|D} = \mathrm{id}_D\}$$

$$N := \{h \in H \mid h \text{ acts identically on } I/I^2\} .$$

The proof is now divided into two steps.

Step 1. Recall that the Albanese map $\alpha : Y \longrightarrow \mathrm{Alb}(Y)$ naturally induces the homomorphism

$$\tilde{\alpha} : \mathrm{Aut}^0(Y) \longrightarrow \mathrm{Aut}^0(\mathrm{Alb}(Y)) \quad (\cong \mathrm{Alb}(Y)) .$$

Then by a theorem of Fujiki [3;(5.8)], $\mathrm{Ker}\,\tilde{\alpha}$ has the natural structure of a linear algebraic group. We now have $H \subset \operatorname{Ker}\tilde{\alpha}$ because for any $h \in H$, the corresponding $\tilde{\alpha}(h)$ fixes all points of $\alpha(D)$ and hence $\tilde{\alpha}(h) = id_{Alb(Y)}$. Consider the homomorphism

$$\varphi : \operatorname{Ker}\tilde{\alpha} \longrightarrow \operatorname{Aut}^0(D)$$
$$g \longmapsto g_{|D} \quad .$$

By virtue of a result of Fujiki [3; Proposition 2.3], one easily sees that $H (= \operatorname{Ker}\varphi)$ is an algebraic subgroup of $\operatorname{Ker}\tilde{\alpha}$.

Step 2: For each $p \in D$, we put $E_p := (I/I^2) \otimes_{\mathcal{O}_{D,P}} \mathbb{C}$, and have the isotropy representation

$$\psi_p : H \longrightarrow GL(E_p) \quad .$$

Then $N (= \cap_{p \in D} \operatorname{Ker}\psi_p)$ is an algebraic subgroup of H (and hence of $\operatorname{Ker}\tilde{\alpha}$) with the corresponding Lie algebra $H^0(Y, \Theta_Y(-2D))$. Let $0 \neq v \in H^0(Y, \Theta_Y(-2D))$. Then $G := \exp(\mathbb{C}v)$ can not be an algebraic subgroup of N isomorphic to $\mathbb{G}_m$ ($= \mathbb{C}^*$ as a complex Lie group), because otherwise, Theorem B combined with the linearity of the G-action on the fibres of $N_{D/X}$ would imply $v \notin H^0(Y, \Theta_Y(-2D))$ in contradiction. Hence the identity component of N is a unipotent algebraic group, and in particular for v and G as above, it automatically follows that G is an algebraic subgroup of N isomorphic to $\mathbb{G}_a$ ($= \mathbb{C}$ as a complex Lie group). The rest of the proof is straightforward from Theorem A.

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