Moduli Spaces of Curves

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1 Introduction

The goal of my talk was to survey some of the recent developments concerning moduli spaces of curves. For a pair (g, n) of nonnegative integers with $2g - 2 + n > 0$, we have the moduli space $M_{g,n}$ of objects $(C, x_1, \ldots, x_n)$, where C is a smooth and irreducible curve of genus g and the x_i are n distinct marked points, and the moduli space $\overline{M}_{g,n}$ of stable n-pointed curves of genus g, where the curves are still connected and reduced and may have ordinary nodes, the x_i are distinct marked nonsingular points, and the objects must have finite automorphism groups. It is known that $\overline{M}_{g,n}$, the Knudsen-Deligne-Mumford compactification of $M_{g,n}$, is a projective variety of dimension $3g - 3 + n$.

$\overline{M}_{g,n}$ has a stratification by topological type. To each stable n-pointed curve, one associates its dual graph: the vertices correspond to the irreducible components of the normalization, whose genera are remembered; for every two points identified in a node, there is an edge; there are n numbered legs for the marked points. The curves with a given graph form a stratum. The stratum belonging to a graph Γ is isomorphic to $(\prod_v M_{g_v,n_v})/\mathrm{Aut}(\Gamma)$. Note that $g = h^1(\Gamma) + \sum_v g_v$. The codimension of the stratum equals the number of edges of the graph (i.e., the number of nodes of the curves).

In codimension one, there is (for $g > 0$) one graph with a single vertex; the closure of the stratum is denoted Δ_{irr} (or Δ_0 when $n = 0$). The other graphs have two vertices, say of genus i with a set I of legs and of genus $g - i$ with I^c. The closure of the stratum is denoted $\Delta_{i,I} = \Delta_{g-i,I^c}$. If $i = 0$ then $|I| \geq 2$ and if $i = g$ then $|I^c| \geq 2$. On $\overline{M}_g$ (i.e., $n = 0$) the empty sets of legs are omitted in the notation.

In dimension one, exactly one factor in the product of moduli spaces is $M_{0,4}$ or $M_{1,1}$; the other factors are $M_{0,3}$. Let us call the closures of these strata the *1-curves*.

Besides the boundary divisors, we have the divisor classes $\psi_i = c_1(\mathbf{L}_i)$, where $\mathbf{L}_i$ is the line bundle with fiber the cotangent space to C at x_i, and λ, the first Chern class of the Hodge bundle $\mathbf{E}$ of rank g. Put

$$\delta_{\dots} = [\Delta_{\dots}]/|\mathrm{Aut}(C_\eta)|,$$

where C_η stands for the generic curve parametrized by the boundary divisor. It is well-known that λ, $\psi_1, \dots, \psi_n$, δ_{irr}, and the $\delta_{i,I}$ generate the Picard group of the moduli functor. In fact, they generate freely for $g \geq 3$, while the relations occurring for $g \leq 2$ are well-understood. It is known that $\kappa_1 = 12\lambda - \delta + \psi$ is ample. Here δ stands for the sum of all the $\delta_{\dots}$ and ψ for the sum of the ψ_i.

2 Nef divisors on $\overline{M}_g$

Cornalba and Harris have shown that $a\lambda - \delta$ is ample exactly when $a > 11$. But it is not known when a general divisor class

$$a\lambda - \sum_{i=0}^{[g/2]} b_i \delta_i$$

is ample. On any projective variety, the ample cone is the interior of the nef cone, where a divisor is said to be nef if it intersects all effective curves nonnegatively. At first sight, it seems quite difficult to describe the effective curves inside $\overline{M}_g$. A very special class of curves is given naturally: the 1-curves mentioned in the introduction. Call a divisor a *1-divisor* if it intersects the 1-curves nonnegatively. Note that it is rather easy to write down explicitly the inequalities in a and the b_i that characterize a 1-divisor: in the corresponding family of curves with $3g-4$ nodes, all nodes outside the component arising from $\overline{M}_{0,4}$ or $\overline{M}_{1,1}$ may be smoothed, which leaves only a few possibilities.

Theorem 1 *On $\overline{M}_g$ for $g \leq 24$ the 1-divisors are exactly the nef divisors.*

This is largely due to Gibney; see [G], [FG], [GKM], [KM], [Fb]. So in these cases we have the remarkable result that the Mori cone of effective curves has finitely many extremal rays, spanned by smooth rational curves, only one of which is $K_{\overline{M}_g}$-negative. As will be recalled below, $\overline{M}_{24}$ is of general type; since the cone of curves is very much related to the birational geometry of a projective variety, the result seems especially remarkable in this case.

3 Effective divisors on $\overline{M}_g$

Farkas talked about this subject at the Arbeitstagung in 2003 and I will try to review what happened since then.

For an effective divisor D, write $[D] = a\lambda - \sum_{i=0}^{[g/2]} b_i\delta_i$ and assume $a > 0$ and $b_i \geq 0$. The *slope* $s(D)$ of such a divisor is defined as $a/(\min_i b_i)$. It is well-known that $s(K_{\overline{M}_g}) = 13/2$. For r and d with $\rho := g - (r+1)(g-d+r) = -1$ the Brill-Noether divisor of curves with a g^r_d has slope $6 + 12/(g+1)$. A single effective divisor with slope $< 13/2$ causes $\overline{M}_g$ to be of general type. Thanks to Eisenbud, Harris, and Mumford, we know that $\overline{M}_g$ is of general type for $g \geq 24$. The *slope conjecture* of Harris and Morrison states in particular that $s(D) \geq 6 + 12/(g+1)$ for all effective D. Two years ago, Farkas discussed the counterexample Popa and he found: the divisor in $\overline{M}_{10}$ of curves lying on a K3 surface. They characterized this divisor in four distinct ways, which enabled them to calculate all the coefficients b_i and complete work of Cukierman. It turns out that some of these characterizations generalize and lead to additional counterexamples.

Recall the properties (N_p) introduced by Green and Lazarsfeld: for a projective variety X embedded in $\mathbf{P}^r$ by the complete linear system of a very ample line bundle L, let I be its ideal in the homogeneous coordinate ring S of $\mathbf{P}^r$ and let

$$0 \to E_{r+1} \to \cdots \to E_1 \to I \to 0$$

be a minimal graded free resolution of S-modules. For $p \geq 1$, property (N_p) is that E_i is a sum of copies of $S(-i-1)$ for $1 \leq i \leq p$, while (N_0) is the property that the map $\mathrm{Sym}^2 H^0(X, L) \to H^0(X, L^{\otimes 2})$ is surjective.

Let me now discuss some of the results in [Fr]. For $g = 2k-2$ even, Farkas defines $Z_{g,i}$ as the locus of curves C having a pencil A of minimal degree k such that $|K_C \otimes A^\vee|$ fails property (N_i). He shows that for all $i \geq 0$ the locus $Z_{6i+10,i}$ is the push-forward from a finite cover of the degeneracy locus of a map between vector bundles of equal rank. Thus $Z_{6i+10,i}$ is expected to be a divisor. If it is indeed a divisor, its closure has the property that $a/b_0 < 6 + 12/(g+1)$. Almost always, b_0 is the smallest of the b_i, so potentially this provides infinitely many counterexamples to the slope conjecture. But it is difficult to show that these loci are divisors. For $i = 0$ (thus $g = 10$), the earlier counterexample is recovered. Both for $i = 1$ and $i = 2$, Farkas shows that $Z_{6i+10,i}$ is a divisor and that b_0 is the smallest of the b_i, establishing two new counterexamples. As corollaries, he shows that the following moduli spaces are of general type: $\overline{M}_{22,n}$ for $n \geq 2$, $\overline{M}_{21,n}$ for $n \geq 5$, and $\overline{M}_{16,n}$ for $n \geq 9$. Another counterexample to the slope conjecture (on $\overline{M}_{21}$) was provided by Khosla [K]. Finally, Farkas has announced that he expects to prove soon that $\overline{M}_{22}$ and $\overline{M}_{23}$ are of general type. Perhaps an appropriate topic for the Arbeitstagung in 2007?

4 Rationality

Farkas's general-type results mentioned above improve upon earlier results of Logan [L]. Logan also has a table of unirational moduli spaces, for $2 \leq g \leq 9$ and $g = 11$, and n at most equal to $f(g)$. For $2 \leq g \leq 6$ and $1 \leq n \leq f(g)$, Casnati and Fontanari [CFn] have shown that $M_{g,n}$ is in fact rational. This was known for $n = 0$ in the same range, by the work of Igusa, Shepherd-Barron and Katsylo. The proofs in the presence of marked points turn out to be considerably simpler. For $g = 1$, Belorousski had earlier shown that $M_{1,n}$ is rational for $n \leq 10$. It is well-known that $\overline{M}_{1,11}$ is not unirational (see below). Bini and Fontanari [BF] proved that it has Kodaira dimension 0, while $\overline{M}_{1,n}$ has Kodaira dimension 1 for $n \geq 12$. Verra [V] recently proved that M_{14} is unirational.

5 Cohomology of $M_{g,n}$ and $\overline{M}_{g,n}$

Harer and Zagier, and Penner, obtained the beautiful result that the orbifold Euler characteristic $\chi(M_{g,1})$ equals $\zeta(1-2g)$. Harer and Zagier also determined $e(M_g)$ and $e(M_{g,1})$. It has long been known that $M_{g,n}$ is the quotient of a smooth variety X by a finite group G; then $\chi = e(X)/|G|$. The corresponding result for $\overline{M}_{g,n}$ is much more recent and was obtained by Looijenga, De Jong, Pikaart, and Boggi. Of course $\chi(M_{g,n})$ can be determined immediately from $\chi(M_{g,1})$. But the other Euler characteristics were not determined. Very recently, Bini and Harer [BH] completed this work by giving recursive formulas for $e(M_{g,n})$ (for $n \geq 2$), $\chi(\overline{M}_{g,n})$, and $e(\overline{M}_{g,n})$.

Recall some of the fundamental results obtained by Harer: the cohomology $H^i(M_{g,n})$ stabilizes as $g \to \infty$; the determination of the virtual cohomological dimension of $M_{g,n}$; and the calculations of $H^i(M_\infty)$ for small i. Recently, Madsen and Weiss [MW], building on work of Tillmann, determined the stable cohomology:

$$H^*(M_\infty) = \mathbf{Q}[\kappa_1, \kappa_2, \ldots].$$

The class κ_i is defined as $\pi_*\psi^{i+1}$, where $\pi : M_{g,1} \to M_g$ is the forgetful map. The κ-classes are called the tautological or Mumford-Morita-Miller classes.

The tautological ring of $\overline{M}_{g,n}$ is by definition generated by decorated strata classes, where each factor in the product of compactified moduli spaces may carry a monomial in κ- and ψ-classes. Denote by $R^*(\overline{M}_{g,n})$ the subalgebra of the rational Chow ring thus obtained and by RH^* its image in rational cohomology. It is known that $c_i(\mathbf{E}) \in R^*$ (Mumford). Recently, Pandharipande and the author showed that (e.g.) $[\overline{H}_g] \in R^*$, the class of the closure of the hyperelliptic locus [FP]. We also showed that ψ_1^g, which is known to vanish on $M_{g,1}$, is tautological boundary. For $g = 3$, this result was used by Kimura and Liu [KL] to give an explicit formula for ψ_1^3; such a so-called topological recursion relation has strong implications in Gromov-Witten theory (some of which were used in [KL] to determine the unknown coefficients).

It is speculated that $R^*(\overline{M}_{g,n})$ satisfies Poincaré duality (so that R^* would equal RH^*), but the evidence is perhaps not so strong, although $R_0(\overline{M}_{g,n})$ is known to be one-dimensional. Maybe the main inspiration for the speculation comes from the author's conjecture for $R^*(M_g)$, which in particular says that it satisfies a form of Poincaré duality with socle in algebraic degree $g-2$. There is a number of general results on $R^*(M_g)$, obtained by Looijenga, Morita, Ionel, Givental, and the author, and in addition the full conjecture is known to be true for $g \leq 21$.

Graber and Pandharipande [GP] have shown that not all algebraic classes are tautological.

For low g and n, $H^*(\overline{M}_{g,n})$ tends to be all tautological. There are general results due to Arbarello and Cornalba, and Polito: $H^1 = H^3 = H^5 = 0$ and H^2 and H^4 are tautological. Once one knows $H^*(\overline{M}_{g,n})$, one may try to show that everything is tautological by intersecting sufficiently many tautological classes; these intersection numbers are determined by the work of Witten and Kontsevich (see [GP]).

But how to determine $H^*(\overline{M}_{g,n})$? A possible strategy is the following: use the stratification by topological type, and compute the Euler characteristic of all strata in a category where the weights of cohomology are remembered. Do this equivariantly for the symmetric groups permuting the marked points; then the theory of modular operads of Getzler and Kapranov allows one to determine the Euler characteristic of $\overline{M}_{g,n}$. Since $H^i(\overline{M}_{g,n})$ is pure of weight i, one knows the cohomology. This was carried out by Getzler for genus 0 and 1, and also for $g = 2$ and $n \leq 3$, in the category of mixed Hodge structures.

One may also work in the category of ℓ-adic Galois representations. Then the trace of the Frobenius F of a finite field $\mathbf{F}_q$ equals the number of points of the moduli space (and the trace of $F\sigma$ with $\sigma \in \Sigma_n$ has an analogous interpretation). The number of points can be counted for small q, or sometimes for almost all q. If the answer is polynomial in q, one is convinced that the Euler characteristic is a polynomial in the class $\mathbf{L} = \mathbf{Q}_\ell(-1)$. But can it be proved? By the recent result of Van den Bogaart and Edixhoven [BE], the answer is yes for a smooth proper stack over $\mathbf{Z}$. If the coarse moduli space is the quotient of a smooth projective variety by a finite group, as is the case for $\overline{M}_{g,n}$, then the corresponding conclusion also holds in the category of mixed Hodge structures.

A variant of this strategy has been used by Bergström and Tommasi [BT] to determine $H^*(\overline{M}_4)$. Tommasi had earlier computed $H^*(M_4)$ in the category of mixed Hodge structures; Bergström has counted $|H_{g,n}(\mathbf{F}_q)|$ for $g \geq 2$, $n \leq 5$, and odd q, and $|Q_n(\mathbf{F}_q)|$ for $n \leq 6$ and all q. Here $H_{g,n}$ is the moduli space of hyperelliptic curves with marked points, and Q_n is the moduli space of smooth plane quartic curves with marked points.

But it is well-known that $|\overline{M}_{g,n}(\mathbf{F}_q)|$ is not always polynomial. The standard example is $\overline{M}_{1,11}$, with $H^{11}(\overline{M}_{1,11}) = S[12]$, the motive of cusp forms for $\mathrm{SL}(2, \mathbf{Z})$ of weight 12 (the work of Eichler and Shimura, Deligne, Scholl, and Getzler). When $S[12]$, or more generally $S[k]$, is viewed as Hodge structure, one knows that $F^{k-1}S[k] = S_k$, the space of holomorphic cusp forms of weight k, and $S[k] = S_k \oplus \overline{S}_k$. Consani and the author recently showed that

$$S[n+1] = \mathrm{Alt}(H^*(\overline{M}_{1,n})) = \mathrm{Alt}(H^n(\overline{M}_{1,n})),$$

the part of the cohomology where Σ_n acts via the alternating representation; this provides an alternative construction of these motives [CFb].

What happens for $g \geq 2$? At the Arbeitstagung in 2003, Van der Geer discussed joint work with the author in which an explicit conjecture for the answer for $g = 2$ was obtained. But what happens for higher g remains mysterious. For example, some years ago M.S. Narasimhan asked the following question: What is the smallest g for which M_g, or $\overline{M}_g$, contains cohomology not of Tate type?

References

[!] As the reader will have noticed, for many results mentioned in the text, no references are provided. Moreover, for lack of time, the references given below are mostly to the preprints at the arXiv, not to the published versions.

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