THETA-FUNCTIONS FOR INDEFINITE POLARIZATIONS

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Abstract. We propose a generalization of the classical theta function to higher cohomology of the polarization line bundle on a family of complex tori with positive index. The constructed cocycles vary horizontally with respect to the (projective) flat connection on this family coming from a heat operator. They also possess modular properties similar to the classical ones.

1. Introduction

The classical theta function is a beautiful entire function of n complex variables

$$\Theta(Z,\Omega) = \sum_{N\in\mathbb{Z}^n} e^{2\pi i N^T Z} e^{\pi i N^T \Omega N},\ Z \in \mathbb{C}^n,$$

where $\Omega \in \mathrm{Mat}_n(\mathbb{C})$ is a symmetric matrix with positive definite $\operatorname{Im}\Omega$. Considered as a section of the polarization line bundle L on an abelian variety, it behaves very nicely in families. Namely, it provides the horizontal section with respect to the flat connection on the push-forward bundle for such a family.

However, if the polarization L of the torus $X = \mathbb{C}^n/(\Omega\mathbb{Z}^n \oplus \mathbb{Z}^n)$ is not positive, which means that $\operatorname{Im}\Omega$ is not positive definite, then the theta-series diverges. The most common way around this is to change the complex structure of the torus such that the line bundle L is positive for this new complex structure. But the choice of this new complex structure is not canonical. Rather, for each complex torus one gets a family of abelian varieties parameterized by some Grassmannian. Another approach (which is morally very similar to the first one) is to represent $H^k(X,L)$ by harmonic forms with values in L (see, e.g., the book by Kempf [K] or by Birkenhake and Lange [BL] for more details).

We propose to identify the space of *theta forms* $H^k(X,L)$ with the group cohomology $H^k(\Lambda, F)$, where $\Lambda \cong \mathbb{Z}^{2n}$ is the fundamental group of X with the appropriate action on $F = H^0(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$, the space of holomorphic functions on $\mathbb{C}^n$. This approach has the advantage that we never have to leave the analytic category. The constructed representatives manifestly satisfy the heat equation of the connection.

The motivation to study this problem came from the holomorphic anomaly equation for $N = 2$ σ-model in 2 dimensions with target a Calabi-Yau 3-fold [BCOV]. Such a theory when twisted gives a topological QFT, which, in turn, after integrating over conformal structures defines a string theory. Integrating over the moduli of Riemann surfaces induces a holomorphic anomaly for the partition functions $\mathcal{F}_g$. The holomorphic anomaly for the generating function for all genera $W = \exp[\sum_{g=1}^{\infty} \lambda^{2g-2}\mathcal{F}_g]$ has the form of a heat equation. In the B-model Witten [Wi] identifies this heat equation with the natural projective connection on the family of intermediate Jacobians over the moduli of Calabi-Yau's. Intermediate

Jacobian is not an abelian variety but rather a complex torus with a principal polarization of type $(1, h^{2,1})$ (the polarization comes from the pairing on the middle cohomology). Thus, the question of finding a horizontal solution arises quite naturally.

In this paper we restrict our attention mostly to the case of principal polarization of index k (i.e. $\operatorname{Im}\Omega$ has k negative eigenvalues). In this simple case $\dim H^k(X, L) = 1$, and all others vanish. In the degenerate case there are also higher cohomology $H^{k+i}(X, L) = H^k(X, L) \otimes H^i(Y, \mathcal{O}_Y)$, where Y is the "degenerate" part of X, and no natural connection is guaranteed to exist. The generalization to non-principal polarizations is straight forward by introducing theta forms with characteristics.

The structure of the paper is as follows. In section 2 we review the general theory of the projective flat connections. We describe the connection in details in the particular case of a family of polarized complex tori. In section 3 we identify the space of theta forms for principal polarizations with the $H^k(\Lambda, H^0(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n}))$ and show that it is 1 dimensional. Using Koszul resolution of $\mathbb{Z}^{2n}$ we write down explicitly a cocycle in the above cohomology. In section 4 we study the action of the modular group on the space of theta forms. Namely, the modular group action affects differentials in the Koszul resolution by acting on the basis of $\mathbb{Z}[\Lambda]$. Establishing the isomorphism between cohomology $H^k(\Lambda, F)$ computed from different resolutions reveals modular properties of particular representatives. Finally, we conclude with an observation that the chosen representatives solve the heat equation already on the level of cocycles, and generalize the construction to non-principal polarizations.

Notations. A^T denotes the transpose matrix of A. In the exponent πi means $\pi\sqrt{-1}$. $\mathbb{Z}[G]$ stands for the group ring of G. $k\langle V_1, \ldots, V_s\rangle$, $k = \mathbb{Z}, \mathbb{R}$ or $\mathbb{C}$, denotes the k-linear span of V_i, where V_i can be free k-modules or abstract elements.

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2. Flat projective connections

In this section we recall the general facts about the flat projective connections coming from heat operators. The main reference here is [BK], and most of the material is borrowed from the notes [DP].

2.1. **Differential operators and projective connections.** Let $V \to S$ be a holomorphic vector bundle over a complex manifold S. Denote by $\mathcal{D}_S(V)$ the sheaf of holomorphic differential operators on V. $\mathcal{D}_S(V)$ is naturally filtered by order

$$0 \subset \mathcal{D}^0_S(V) \subset \mathcal{D}^1_S(V) \subset \ldots,$$

with the associated quotients given by the principal symbol sequences

$$0 \longrightarrow \mathcal{D}^{k-1}_S(V) \longrightarrow \mathcal{D}^k_S(V) \xrightarrow{\sigma_k} S^k T_S \otimes \operatorname{End}(V) \longrightarrow 0. \tag{1}$$

Recall that the Atiyah class is the extension class

$$0 \longrightarrow \mathrm{End}(V) \longrightarrow \mathcal{A}(V) \longrightarrow T_S \longrightarrow 0 \tag{2}$$

from the symbol sequence for $\mathcal{D}^1_S(V)$. That is, $\mathcal{A}(V)$ consists of all differential operators of order ≤ 1, whose symbol lies in $T_S \otimes \mathrm{id} \subset T_S \otimes \mathrm{End}(V)$.

A *projective connection* on V is a holomorphic splitting ∇ of the sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{End}(V)/\mathcal{O}_S \arrow[r] & \mathcal{A}(V)/\mathcal{O}_S \arrow[r] & T_S \arrow[l, bend right, "\nabla"'] \arrow[r] & 0. \end{tikzcd} \tag{3}$$

It lifts to an ordinary connection if this splitting lifts to a splitting of the Atiyah class sequence (2).

Now let $\pi : \mathcal{X} \to S$ be a smooth family of compact complex manifolds. That means that π is a surjective, proper holomorphic map between complex manifolds, which is flat and whose fibers X_s are connected complex manifolds. Let $\mathcal{L}$ be a holomorphic line bundle on $\mathcal{X}$.

Denote by $\mathcal{D}_\pi(\mathcal{L})$ the sheaf of all differential operators on $\mathcal{L}$ acting along the fibers of π (the centralizer of $\pi^{-1}\mathcal{O}_S$ in $\mathcal{D}_\mathcal{X}$). $\mathcal{D}_\pi(\mathcal{L})$ has the order filtration

$$0 \subset \mathcal{D}^0_\pi(\mathcal{L}) \subset \mathcal{D}^1_\pi(\mathcal{L}) \subset \dots,$$

where $\mathcal{D}^k_\pi(\mathcal{L}) := (\sigma_k)^{-1}(S^k T_\pi) \cap D_\pi(\mathcal{L}),\ k \geq 0$ (here we use the natural inclusions $S^k T_\pi \hookrightarrow S^k T_\mathcal{X}$).

There is also a filtration of $\mathcal{D}_\mathcal{X}(\mathcal{L})$ by order along the S direction

$$0 \subset \mathcal{D}^{0,S}_\mathcal{X}(\mathcal{L}) \subset \mathcal{D}^{1,S}_\mathcal{X}(\mathcal{L}) \subset \dots,$$

where we define $\mathcal{D}^{k,S}_\mathcal{X}(\mathcal{L})$ inductively by

$$\begin{gathered} \mathcal{D}^{0,S}_\mathcal{X}(\mathcal{L}) := \mathcal{D}_\pi(\mathcal{L}), \\ \mathcal{D}^{k,S}_\mathcal{X}(\mathcal{L}) := \{D \in \mathcal{D}^k_\mathcal{X} \mid [D, f] \in \mathcal{D}^{k-1,S}_\mathcal{X}(\mathcal{L}),\ \text{for } f \in \pi^{-1}\mathcal{O}_S\}. \end{gathered}$$

Again, the quotients are given by the symbol sequences

$$0 \longrightarrow \mathcal{D}^{k-1,S}_\mathcal{X}(\mathcal{L}) \longrightarrow \mathcal{D}^{k,S}_\mathcal{X}(\mathcal{L}) \xrightarrow{\sigma_{k,S}} \pi^* S^k T_S \otimes \mathcal{D}_\pi(\mathcal{L}) \longrightarrow 0. \tag{4}$$

Definition. A *heat operator* on $\mathcal{L}$ relative to π is a sheaf homomorphism $\mathcal{H} : T_S \to \pi_* \mathcal{D}^{1,S}(\mathcal{L})$, that fits into the commutative diagram

$$\begin{tikzcd} T_S \arrow[r, "\mathcal{H}"] \arrow[dr, "\mathrm{id}\otimes 1"'] & \pi_* \mathcal{D}^{1,S}(\mathcal{L}) \arrow[d, "\sigma_{1,S}"] \\ & T_S \otimes \mathcal{D}_\pi(\mathcal{L}) \end{tikzcd} \tag{5}$$

2.2. **Connections from heat operators.** In some situations a heat operator on $\mathcal{L}$ yields a projective flat connection on the bundles $R^i\pi_*\mathcal{L} \to S$ (see [Hi], [BK]). A projective connection on $V = R^i\pi_*\mathcal{L} \to S$ is a splitting of the exact sequence (3). Equivalently, we need a homomorphism $\nabla : T_S \to \mathcal{D}^1_S(V)/\mathcal{O}_S$ which fits into the commutative diagram

$$\begin{tikzcd} T_S \arrow[r, "\nabla"] \arrow[dr, "\mathrm{id}\otimes 1"'] & \mathcal{D}^1_S(V)/\mathcal{O}_S \arrow[d, "\sigma_1"] \\ & T_S \otimes \mathrm{End}(V) \end{tikzcd} \tag{6}$$

where the symbol map σ_1 is well defined on the quotient, as $\mathcal{O}_S \subset \mathrm{Ker}(\sigma_1)$.

The differential operators $\mathcal{D}_{\mathcal{X}}(\mathcal{L})$ act on the local sections of $\mathcal{L}$. Hence, they act on the local sections of $R^i\pi_*\mathcal{L}$ for all i. In particular, we get a map $\pi_*\mathcal{D}_{\mathcal{X}}(\mathcal{L}) \to \mathcal{D}_S(V)$ which respects the filtration by order along S, and then induces a map

$$\phi : \pi_*\mathcal{D}^{1,S}(\mathcal{L}) \to \mathcal{D}^1_S(V).$$

A projective connection ∇ which *comes from a heat operator* is the composition

$$T_S \longrightarrow \pi_*\mathcal{D}^{1,S}(\mathcal{L})/\mathcal{O}_S \longrightarrow \mathcal{D}^1_S(V)/\mathcal{O}_S.$$

Of course, the existence of such a connection on $R^i\pi_*\mathcal{L}$ depends on the global geometry of the family $(\mathcal{X},\mathcal{L})$. Recall that the Kodaira-Spencer map for this family $\alpha : T_S \longrightarrow R^1\pi_*\mathcal{D}^1_\pi(\mathcal{L})$ is the connecting map in the long exact sequence associated with the push-forward of the sequence

$$0 \longrightarrow \mathcal{D}^1_\pi(\mathcal{L}) \longrightarrow \mathcal{D}^1_{\mathcal{X}}(\mathcal{L}) \longrightarrow \pi^*T_S \longrightarrow 0.$$

Then one has the following

Lemma 2.1. *([BK]) Let $(\mathcal{X},\mathcal{L})$ be a smooth family as before. Suppose*
a) $\pi_*\mathcal{D}_\pi(\mathcal{L}) = \mathcal{O}_S$ and
b) the composition map $T_S \xrightarrow{\alpha} R^1\pi_*\mathcal{D}^1_\pi(\mathcal{L}) \longrightarrow R^1\pi_*\mathcal{D}_\pi(\mathcal{L})$ vanishes.
Then for any $i \geq 0$, the bundle $R^i\pi_*\mathcal{L} \to S$ possesses a unique projective flat connection given by a heat operator.

The connection is constructed as follows. Pushing forward the symbol sequence (4) for $k = 1$, and using the projection formula one gets the long exact sequence of derived images

$$0 \longrightarrow \pi_*\mathcal{D}_\pi(\mathcal{L}) \longrightarrow \pi_*\mathcal{D}^{1,S}(\mathcal{L}) \longrightarrow T_S \otimes \pi_*\mathcal{D}_\pi(\mathcal{L}) \xrightarrow{\beta} R^1\pi_*\mathcal{D}_\pi(\mathcal{L}) \longrightarrow \dots$$

Using $\pi_*\mathcal{D}_\pi(\mathcal{L}) = \mathcal{O}_S$ and the fact that the connecting map $\beta : T_S \to R^1\pi_*\mathcal{D}_\pi(\mathcal{L})$ can be identified with the composition from (b), one gets an isomorphism

$$\pi_*\mathcal{D}^{1,S}(\mathcal{L})/\mathcal{O}_S \simeq T_S.$$

The projective heat operator is the inverse of this isomorphism.

Very often (in fact, in all known cases) the composition in (b) vanishes already in $\mathcal{D}^2_\pi(\mathcal{L})$. That is, the composition

$$T_S \xrightarrow{\alpha} R^1\pi_*\mathcal{D}^1_\pi(\mathcal{L}) \longrightarrow R^1\pi_*\mathcal{D}^2_\pi(\mathcal{L})$$

is already 0. Equivalently, the Kadaira-Spencer map α factors as

$$T_S \longrightarrow \pi_* S^2 T_\pi \xrightarrow{\delta} R^1\pi_*\mathcal{D}^1_\pi(\mathcal{L}),$$

where δ is the connecting map in the push-forward of the symbol sequence

$$0 \longrightarrow \mathcal{D}^1_\pi(\mathcal{L}) \longrightarrow \mathcal{D}^2_\pi(\mathcal{L}) \longrightarrow S^2T_\pi \longrightarrow 0.$$

It will also be so in the case we are interested in.

2.3. **Projective connection on a family of complex tori.** We are going to apply the general theory described above to the special case of polarized complex tori.

Let $(\mathcal{X}, \mathcal{L})$ be a family of complex tori with non-degenerate polarization. This means that $\mathcal{L}$ restricted to any fiber X_s gives a line bundle L_s, such that $c_1(L_s) \in H^2(X_s, \mathbb{R})$ is a non-degenerate 2-form. To show that this family satisfy the conditions of the Lemma 2.1, it is enough to do it for each fiber.

Consider the filtration

$$\mathcal{D}_{X_s}(L_s) \supset \cdots \supset \mathcal{D}^2_{X_s}(L_s) \supset \mathcal{D}^1_{X_s}(L_s) \supset \mathcal{O}_{X_s} \supset 0.$$

There is a spectral sequence associated to this filtration which abuts to $H^i(X_s, \mathcal{D}_{X_s}(L_s))$ with $E_1^{-p,n+p} = H^n(X_s, S^pT_{X_s})$.

$$\begin{array}{ccccccccc}
\cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
0 & \longrightarrow & H^0(X_s, S^2T_{X_s}) & \xrightarrow{d_1} & H^1(X_s, T_{X_s}) & \xrightarrow{d_1} & H^2(X_s, \mathcal{O}_{X_s}) & \longrightarrow & 0 \\
 & & 0 & \longrightarrow & H^0(X_s, T_{X_s}) & \xrightarrow{d_1} & H^1(X_s, \mathcal{O}_{X_s}) & \longrightarrow & 0 \\
 & & & & 0 & \longrightarrow & H^0(X_s, \mathcal{O}_{X_s}) & \longrightarrow & 0 \\
 & & & & & & 0 & &
\end{array}$$

The differential d_1 is the convolution with the class $c_1(L_s) - \frac{1}{2}c_1(T_{X_s}) = c_1(L_s)$ ([Hi], [BK]). Hence, every row is just the Koszul complex. So we have $H^0(X_s, \mathcal{D}_{X_s}(L_s)) = H^0(X_s, \mathcal{O}_{X_s})$, or $\pi_*\mathcal{D}_\pi(\mathcal{L}) = \mathcal{O}_S$.

To show (b) we consider the symbol sequence

$$0 \longrightarrow \mathcal{O}_{X_s} \longrightarrow \mathcal{D}^1_{X_s}(L_s) \longrightarrow T_{X_s} \longrightarrow 0$$

The associated long exact sequence

$$\begin{aligned} H^0(X_s, T_{X_s}) &\xrightarrow{c_1(L_s)} H^1(X_s, \mathcal{O}_{X_s}) \xrightarrow{0} H^1(X_s, \mathcal{D}^1_{X_s}(L_s)) \longrightarrow H^1(X_s, T_{X_s}) \\ &\xrightarrow{c_1(L_s)} H^2(X_s, \mathcal{O}_{X_s}) \longrightarrow \ldots \end{aligned}$$

shows that

$$H^1(X_s, \mathcal{D}^1_{X_s}(L_s)) = \mathrm{Ker}(H^1(X_s, T_{X_s}) \xrightarrow{c_1(L_s)} H^2(X_s, \mathcal{O}_{X_s})).$$

Thus, we deduce from the second Koszul row in the above spectral sequence that $H^1(X_s, \mathcal{D}^1_{X_s}(L_s)) \simeq H^0(X_s, S^2T_{X_s})$. This shows (b). Namely, the Kodaira-Spencer map factors

$$T_S \longrightarrow \pi_* S^2 T_\pi \xrightarrow{\sim} R^1\pi_*\mathcal{D}^1_\pi(\mathcal{L}).$$

In coordinates, the most natural choice of trivialization of $\mathcal{L}$ is to pass to the universal covers of the tori. Let Z_i be the coordinates on the fibers $\mathbb{C}^n$. The base space can be parameterized by symmetric complex matrices $\Omega = \{\omega_{ij}\}$ with non-degenerate imaginary part. Then for the theta line bundle the heat operator which gives the connection can be written as

$$\frac{\partial}{\partial\omega_{ij}} - \frac{1}{4\pi i}\frac{\partial^2}{\partial Z_i \partial Z_j}. \tag{7}$$

For positive definite $\operatorname{Im}\Omega$ this is the heat equation for the classical theta function.

3. Indefinite theta forms

Let $\mathcal{X} \to S$ be a family of n-dimensional complex tori. We assume that the family is parameterized by $\mathfrak{H}(k, n-k)$, the space of non-degenerate $n\times n$ complex symmetric matrices Ω with the quadratic form given by $\operatorname{Im}\Omega$ of fixed signature $(k, n-k)$. Here k, the number of negative directions, is called *index* of the tori. The complex torus over the point Ω is described as $X_\Omega = \mathbb{C}^n/(\Omega\mathbb{Z}^n \oplus \mathbb{Z}^n)$. We denote the universal cover map by $p : \mathbb{C}^n \to X_\Omega$.

Let us now define the sheaf of sections of the theta line bundle L_Ω. A section of L_Ω over an open subset $U \subset X_\Omega$ is a holomorphic function $f(Z)$ on $p^{-1}(U)$ such that

$$f(Z + M + \Omega N) = e^{-2\pi i N^T Z - \pi i N^T\Omega N} f(Z), \text{ all } M, N \in \mathbb{Z}^n.$$

The theta line bundles on X_Ω, put together, form a holomorphic line bundle $\mathcal{L}$ on the family $\mathcal{X}$.

Lets us fix Ω, and drop the subscript from X_Ω and L_Ω. We want to consider the cohomology groups $H^i(X, L)$.

3.1. **Homological algebra.** The fundamental group of $X = \mathbb{C}^n/(\Omega\mathbb{Z}^n \oplus \mathbb{Z}^n)$ splits into direct sum $\Lambda = \Lambda_1 \oplus \Lambda_2 \cong \mathbb{Z}^n \oplus \mathbb{Z}^n$ from the definition. We can define the action of $\Lambda_1 \oplus \Lambda_2$ on $\mathcal{O}_{\mathbb{C}^n}$ by

$$(M, N) : f(Z) \mapsto e^{2\pi i N^T Z} e^{\pi i N^T\Omega N} f(Z + M + \Omega N). \tag{8}$$

According to Grothendieck [Gr], we can view $\mathcal{O}_{\mathbb{C}^n}$ as a Λ-sheaf on $\mathbb{C}^n$, where the group $\Lambda = \mathbb{Z}^{2n}$ acts on $(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$ by translations on $\mathbb{C}^n$ and by (8) on sections. Then L can be identified with $\mathcal{O}^\Lambda_{\mathbb{C}^n}$, the invariants of the Λ-sheaf $p_*\mathcal{O}_{\mathbb{C}^n}$ on X. Since the action is free, there is a spectral sequence ([Gr], Ch. V) which abuts to $H^i(X, L)$ with $E_2^{p,q} = H^p(\Lambda, H^q(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n}))$. Since $H^q(\mathbb{C}^n, \mathcal{O}) = 0$ for $p > 0$, the spectral sequence degenerates at E_2. Thus, $H^i(X, L)$ is naturally isomorphic to $H^i(\Lambda, F)$, where F is the space of all holomorphic functions on $\mathbb{C}^n$ with the Λ-action given by (8). In particular, $H^0(X, L)$ is the space of entire functions on $\mathbb{C}^n$ invariant with respect to the Λ-action.

Before proceeding further, let us make a few comments on notations. We will identify $\Lambda = \Lambda_1 \oplus \Lambda_2$ with $\mathbb{Z}^{2n} = \mathbb{Z}^n \oplus \mathbb{Z}^n$ by fixing the reference basis of Λ from the definition of X. Thus, we will often use the same symbol for an actual element in Λ, the $2n$-column of its coordinates in the reference basis, or for the n-column of its coordinates in cases when this element happens to be in Λ_1 or Λ_2. Also, we will not distinguish between a real symmetric $(n \times n)$ matrix and the quadratic form associated to it in the Λ_1 or Λ_2 part of the reference basis. When it may cause a confusion, we will comment on the specific meaning.

To construct a nice representative in $H^i(\Lambda, F)$ we need to choose a rather special basis in Λ. Given a real non-degenerate symmetric $n \times n$ matrix Q we refer to a basis $\{N_1, \dots, N_n, M_1, \dots, M_n\}$ of $\Lambda = \Lambda_1 \oplus \Lambda_2$ as *split with respect to* Q, or simply as Q*-split*, if the quadratic forms associated to Q in $\Lambda_1 \otimes \mathbb{R}$ and to Q^{-1} in $\Lambda_2 \otimes \mathbb{R}$ are positive definite when restricted to $\Gamma_+ = \mathbb{Z}\langle N_{k+1}, \dots, N_n\rangle \subset \Lambda_1$ and $\mathbb{Z}\langle M_{k+1}, \dots, M_n\rangle \subset \Lambda_2$ respectively. We will denote by $\Gamma_- \subset \Lambda_1$ the sublattice spanned by $\{N_1, \dots, N_k\}$.

Remark. It seems always possible to find a basis such that $Q \leq 0$ on Γ_-. $Q < 0$ is too strong to require with the counterexample given by the standard hyperbolic quadratic form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ in $\mathbb{R}^2$.

¿From now on let us fix a split basis $\{N_1, \dots, N_n, M_1, \dots, M_n\}$ with respect to $\operatorname{Im}\Omega$. We will denote by $(NM) = \left(\begin{smallmatrix} N & 0 \\ 0 & M \end{smallmatrix}\right)$ the $2n \times 2n$ matrix whose columns are (N_j, M_i). It will be convenient to assume also that $N^T = M^{-1}$, that is $(NM) \in Sp(2n, \mathbb{Z})$. This is possible without loss of generality because in some real basis $\operatorname{Im}\Omega = \operatorname{Im}\Omega^{-1} = \operatorname{diag}(-1, \dots, -1, 1, \dots, 1)$. The transformation matrix $\left(\begin{smallmatrix} R & 0 \\ 0 & R^{T(-1)} \end{smallmatrix}\right)$ acts on $\operatorname{Im}\Omega$ and $\operatorname{Im}\Omega^{-1}$ as on quadratic forms, and primitive integral sublattices form a dense subset in $Gr_{\mathbb{R}}(n-k, n)$.

To calculate cohomology of the free abelian group $\Lambda = \mathbb{Z}^{2n}$ with a basis $\{x_i\}$ we will use the Koszul resolution

$$\longrightarrow \mathbb{Z}[\Lambda] \otimes \wedge^2 W \xrightarrow{d} \mathbb{Z}[\Lambda] \otimes W \xrightarrow{d} \mathbb{Z}[\Lambda] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0 \tag{9}$$

Here W is the $2n$-dimensional vector space spanned by $w_1, \dots, w_{2n}$. The differential is given by

$$d(x \otimes w_{p_1} \dots w_{p_k}) = \sum_{i=1}^{k} (-1)^{j+1} x(x_{p_i} - 1) \otimes w_{p_1} \dots \hat{w}_{p_i} \dots w_{p_k}. \tag{10}$$

It is also convenient to decompose

$$W = U_1 \oplus \dots \oplus U_n \oplus V_1 \oplus \dots \oplus V_n$$

with the basis $\{u_1, \dots, u_n, v_1, \dots, v_n\}$ corresponding to the basis $\{N_j, M_i\}$ of Λ. Then, by taking values of $\operatorname{Hom}_\Lambda(\mathbb{Z}[\Lambda] \otimes \wedge^* W, F)$ on the elements $1 \otimes u_{j_1} \wedge \dots \wedge u_{j_r} \wedge v_{i_1} \wedge \dots \wedge v_{i_s}$, the calculation of $H^i(\Lambda, F)$ reduces to the cohomology of the $2n$-tuple complex $\mathcal{C}$ of length 2 in each direction, with every term $\mathcal{C}_{i_1, \dots, i_p; j_1, \dots, j_q}$ equal to F. There are $2n$ differentials $d_i, \delta_j : F \to F,\ i, j = 1, \dots, n$ which act on F by $d_i f = (M_i - 1) \cdot f$ and $\delta_j f = (N_j - 1) \cdot f$. Strictly speaking, one needs to introduce minus signs in about a quarter of all differentials. But since our complex is of length two, this sloppiness should not cause any confusion.

3.2. **Calculation of $H^i(\Lambda, F)$.** Before calculating the cohomology of this $2n$-complex we need the following elementary statement (the author thanks A. Borodin for the elegant proof).

Lemma 3.1. *Let $\sum_{n=1}^{\infty} a_n e^{-n^2}$ be an absolutely convergent series. Then $\sum_{n=1}^{\infty} s_n e^{-n^2}$, where $s_n = \sum_{k=1}^{n} a_k$ are the partial sums, also converges absolutely.*

Proof. The statement is equivalent to the analogous statement for convergence of the integrals of positive functions. Given the convergence of $\int_1^\infty f(x)e^{-x^2}dx$ we want to say the same is true for $f(x)$ replaced by $F(x) = \int_1^x f(t)dt$. By changing the order of integration we can make the following estimate:

$$\begin{aligned}\int_1^\infty \int_1^x f(t)dt\; e^{-x^2}\; dx &= \int_1^\infty f(t) \int_t^\infty e^{-x^2}\; dx\; dt \\ &\leq \int_1^\infty f(t)\frac{e^{-t^2}}{t}\; dt \\ &\leq \int_1^\infty f(t)e^{-t^2}dt,\end{aligned}$$

which proves the statement. ∎

Now we are ready to do the final calculations, which gives another proof of the index theorem for complex tori.

Theorem 3.2. *The cohomology of the total complex associated to the complex $\mathcal{C}$ is $H^i(\mathcal{C}) = 0$, for $i \neq k$, and $H^k(\mathcal{C}) \cong \mathbb{C}$.*

Proof. First, let us note that each of the differentials d_i is surjective. Moreover, d_i is also surjective when restricted to the $\mathrm{Ker}(d_j)$ for any $j \neq i$. This is equivalent to saying that $H^q((\mathbb{C}^*)^n, \mathcal{O}) = 0,\ q > 0$. So the cohomology of the complex $\mathcal{C}$ can be computed from the cohomology of the n-tuple complex with differentials δ_j and terms given by the space of $\mathbb{Z}^n$-periodic holomorphic functions.

Every such function has the Fourier expansion which is convenient to write in the form

$$f(Z) = \sum_{K\in\mathbb{Z}^n} a_K \Theta_K(Z,\Omega), \quad \text{where} \quad \Theta_K(Z,\Omega) = e^{\pi i K^T \Omega K} e^{2\pi i K^T Z}. \tag{11}$$

The form of the Fourier coefficients is motivated by the following simple form of the differential

$$\delta_j f(Z) = N_j \cdot f(Z) - f(Z) = \sum_{K\in\mathbb{Z}^n} (a_{K-N_j} - a_N)\Theta_K(Z,\Omega).$$

The next observation is that δ_j, for $k+1 \leq q \leq n$ are surjective (and remain such on $\mathrm{Ker}(\delta_i)$ for $i \neq j$). Explicitly, if $f(Z)$ is given by (11), then

$$F(Z) = -\sum_{K\in\mathbb{Z}^n} (\sum_{r=0}^{r_j} a_{K-rN_j}) e^{\pi i K^T \Omega K} e^{2\pi i K^T Z}$$

is also convergent by Lemma 3.1 and defines an entire function, and $\delta_j F(Z) = f(Z)$. Here r_j is the coefficient of K written in the basis $\{N_1, \ldots, N_n\}$, i.e., $K = \sum_{i=1}^n r_i N_i$.

So the calculation now reduces to the k-complex with the differentials $\delta_1, \ldots, \delta_k$ and terms in the form

$$f(Z) = \sum_{N_-\in\Gamma_-} a_{N_-} \sum_{K\in\Gamma_+ + N_-} \Theta_K(Z,\Omega)$$

In this complex the differentials $\delta_1, \ldots, \delta_k$ are injective. This easily follows from the fact that vanishing of $(a_{K-N_q} - a_K)$ for $1 \leq q \leq k$ means that a_K are constant into the N_q direction. This forces summing over negative definite sublattice. Hence the Fourier series for $f(Z)$ diverges.

Thus we see that the cohomology of $\mathcal{C}$ reduces to a cocycle concentrated in $\mathcal{C}_{1,2,\ldots,k;\emptyset}$, and given there by a function in the form

$$f(Z) = \sum_{N_- \in \Gamma_-} a_{N_-} \sum_{K \in \Gamma_+ + N_-} \Theta_K(Z, \Omega)$$

with finite number of a_{N_-} non zero. Finally, since

$$N_q \cdot \sum_{K \in \Gamma_+} \Theta_K(Z, \Omega) = \sum_{K \in \Gamma_+ + N_q} \Theta_K(Z, \Omega), \quad \text{for } q = 1, \ldots, k,$$

the cohomology $H^k(\mathcal{C})$ can be represented a multiple of the cocycle

$$c = \begin{cases} \sum_{K \in \Gamma_+} \Theta_K(Z, \Omega) & \text{in } \mathcal{C}_{1,2,\ldots,k;\emptyset}, \\ 0 & \text{in all other terms} \end{cases}$$

□

4. Modular properties

As in the case of the classical theta functions, we define the modular group to be a subgroup of $Sp(2n, \mathbb{Z})$:

$$\Gamma_{1,2} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n, \mathbb{Z}) \mid \operatorname{diag}(A^T C) \quad \text{and} \quad \operatorname{diag}(B^T D) \quad \text{are even} \right\}.$$

We will see in this section that modular properties of the theta forms come most naturally from the fact that one can use different bases to construct the cocycle in $H^k(\Lambda, F)$. Briefly put, the modular group acts on the set of bases of Λ. Changing the basis affects differentials in the Koszul resolution. Using explicit isomorphism for the two resolutions which differ by an element in $\Gamma_{1,2}$ from each other, we will show that the entire functions representing cocycles are modular transforms of each other with respect to that element of $\Gamma_{1,2}$. The strategy is to define the modular group action on $H^k(\Lambda, F)$ via its action on the bases and on functions representing the cocycles for the corresponding resolutions, and check the triviality of this action only for the generators of $\Gamma_{1,2}$.

4.1. **Identification of resolutions.** Let $\{x_i\} = \{N_j, M_l\}$ and $\{x'_i\} = \{N'_j, M'_l\}$ be two bases of Λ related by $x_i = \sum_j S_{ji} x'_j$, where the matrix S is an element in $Sp(2n, \mathbb{Z})$. Then there is an induced chain homomorphism s_* between the two resolutions which is homotopy equivalent to the identity:

$$\begin{array}{ccccccccc} \xrightarrow{d'} & \mathbb{Z}[\Lambda] \otimes \wedge^2 W & \xrightarrow{d'} & \mathbb{Z}[\Lambda] \otimes W & \xrightarrow{d'} & \mathbb{Z}[\Lambda] & \longrightarrow & \mathbb{Z} & \longrightarrow 0 \\ & \uparrow{\scriptstyle s_*} & & \uparrow{\scriptstyle s_*} & & \uparrow{\scriptstyle \mathrm{id}} & & \| & \\ \xrightarrow{d} & \mathbb{Z}[\Lambda] \otimes \wedge^2 W & \xrightarrow{d} & \mathbb{Z}[\Lambda] \otimes W & \xrightarrow{d} & \mathbb{Z}[\Lambda] & \longrightarrow & \mathbb{Z} & \longrightarrow 0 \end{array} \tag{12}$$

In particular it give rise to an isomorphism s^* on the cohomology $H^i(\Lambda, F)$. Writing down s_* can be quite messy and non-canonical (it gives the canonical map only in cohomology). But we will try to be as explicit as possible in a few special cases. The calculations can be somewhat simplified by the fact that our cocycles are concentrated in $\mathcal{C}_{1,2,\dots,k;\emptyset}$. This means that we are interested only in the $1\otimes w_1 \dots w_k$ component of $s_*(1\otimes w_{p_1}\dots w_{p_k})$. With this in mind, we will see below that only the k top rows of S remain relevant.

The calculation involves finding the coefficients R_{ij} in the expressions of the elements $(x_i - 1) = R_{ij}(x'_j - 1)$ in the group ring $\mathbb{Z}[\Lambda]$. Given those, it is easy to see that the collection of $\mathbb{Z}[\Lambda]$-homomorphisms s_* given by

$$g_* : 1\otimes w_{p_1}\dots w_{p_k} \mapsto \sum_{t_1,\dots,t_k} (-1)^{\sigma(\{t_i\})} \det(\{R_{p_i t_j}\}_{i,j=1}^k) \otimes w_{t_1}\dots w_{t_k} \tag{13}$$

is a chain map. In the calculations of R_{ij} below we will write (when possible) the group operation in Λ multiplicatively to distinguish it from the addition in the group ring.

Of course, there are many ways to write the decompositions $(x_i - 1) = R_{ij}(x'_j - 1)$. So let us, first, show the preferred one by example. Let, say, $n = 4$, $k = 2$. Pick, for instance, $x_5 = {x'_1}^3 {x'_2}^2 {x'_3}^4 x'_6$. Then we can write $x_5 - 1 = {x'_1}^3 {x'_2}^2 {x'_3}^4 (x'_6 - 1) + {x'_1}^3 {x'_2}^2 ({x'_3}^3 + {x'_3}^3 + {x'_3}^2 + x'_3 + 1)(x'_3 - 1) + {x'_1}^3 (x'_2 + 1)(x'_2 - 1) + ({x'_1}^2 + x'_1 + 1)(x'_1 - 1)$. Thus, indeed, we see that in this procedure the coefficient at $1 \otimes u_1 \dots u_2$ of $s_*(1\otimes w_{p_1} \otimes w_{p_2})$ depends only on the entries in the top 2 rows of S.

Type I: $S = \left(\begin{smallmatrix} A & 0 \\ 0 & A^{T(-1)} \end{smallmatrix}\right)$. It is clear that there is no interaction between N and M parts. Since we are interested only in $1\otimes u_1 \dots u_k$ component, we can forget about the M part all together.

Type Ia: $A = \left(\begin{smallmatrix} I & 0 \\ * & * \end{smallmatrix}\right)$, where I is the $(k \times k)$ identity matrix. In this case the only contribution to the coefficient of $1\otimes u_1\dots u_k$ comes from $s_*(1\otimes u_1 \dots u_k)$ and equals 1.

Type Ib: $A = E_k$, where E_k is the $(n \times n)$-matrix with $(E_k)_{ij} = 1$ if $i = j$ or $(i, j) = (k-1, k)$, and 0 otherwise. Here $N_j - 1 = N'_j - 1$, $j \neq k$, and $N_k - 1 = N'_k \cdot N'_{k-1} - 1 = (N'_k - 1) + N'_k \cdot (N'_{k-1} - 1)$. Still, after taking the wedge product the only contribution to $1\otimes u_1 \dots u_k$ comes from $s_*(1\otimes u_1 \dots u_k)$ and equals 1.

Type Ic: $A = E_{k+1}$. Similar to the above we have

$$s_*(1\otimes u_1 \dots u_k) = 1\otimes u_1 \dots u_k,$$
$$s_*(1\otimes u_1 \dots u_{k-1} \wedge u_{k+1}) = 1\otimes u_1 \dots u_k + N'_k \otimes u_1 \dots u_{k-1} \wedge u_{k+1}.$$

And no other terms contribute to $1\otimes u_1 \dots u_k$.

Type II: $S = \left(\begin{smallmatrix} I & 0 \\ B & I \end{smallmatrix}\right)$. Here $N_j = N'_j \cdot b(M')$ and $M_j = M'_j$, where the $b(M') \in \mathbb{Z}[\Lambda]$ depends only on M''s. This gives

$$N_j - 1 = (N'_j - 1) + N' \cdot (b(M') - 1),$$
$$M_j - 1 = M'_j - 1.$$

Thus, only contribution to the coefficient of $1\otimes u_1 \dots u_k$ comes from $s_*(1\otimes u_1 \dots u_k)$ and equals to 1.

Type III: $S = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$. Here we have $M_j = N_j'$ and $N_j = (M_j')^{-1}$, which also gives simple expressions for R's:

$$\begin{aligned} M_j - 1 &= N_j' - 1 \\ N_j - 1 &= -(M_j')^{-1}(M_j' - 1). \end{aligned}$$

Thus, $s_*(1 \otimes v_1 \dots v_k) = 1 \otimes u_1 \dots u_k$, and this is the only contribution.

To distinguish between the cohomologies computed from the two resolutions we will denote the corresponding $2n$-complexes by $(\mathcal{C}, d)$ and $(\mathcal{C}', d')$, respectively.

4.2. **Modular group action.** The modular group acts both on the set of bases and on the set of holomorphic functions on $\mathfrak{H}(k, n-k) \times \mathbb{C}^n$. The action on the bases given in the coordinates by

$$g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) : \{N_j, M_i\} \mapsto \{N_j^g, M_i^g\},$$
$$(N_1^g, \dots, N_n^g, M_1^g, \dots, M_n^g) = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} (N_1, \dots, N_n, M_1, \dots, M_n).$$

Thus, in the reference basis the transformation matrix is just $g^{T(-1)}$. Writing the action in the basis $\{N_j, M_i\}$ identifies the matrix S from the previous subsection with $(NM)^{-1} \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)^T (NM)$.

The next lemma defines the classical action of the modular group on the set of functions on $\mathfrak{H}(k, n-k) \times \mathbb{C}^n$.

Lemma 4.1. *The following defines a $\Gamma_{1,2}$-action*

$$g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) : f(Z, \Omega) \mapsto f^g(Z, \Omega), \text{ with}$$
$$f^g(Z, \Omega) = \zeta(g) \det(C\Omega^g + D)^{1/2} e^{\pi i Z^T C (C\Omega^g + D)^T Z} f((C\Omega^g + D)^T Z, \Omega^g),$$

where $\zeta(g)$ is an 8-th root of unity and we denoted $\Omega^g = (D^T\Omega - B^T)(-C^T\Omega + A)^{-1}$, or equivalently $\Omega = (A\Omega^g + B)(C\Omega^g + D)^{-1}$.

Proof. This is quite standard (see, for example, the Mumford's book [Mu] with some minor modifications). Let us, for instance, show that the exponential factor in the transformation of $f(Z, \Omega)$ behaves well with respect to the group multiplication. Let $h = \left(\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right)$, then $hg = \left(\begin{smallmatrix} A'A + B'C & A'B + B'D \\ C'A + D'C & C'B + D'D \end{smallmatrix}\right)$. The exponential factor in $f^{hg}(Z, \Omega)$ is given by

$$\begin{aligned} &\pi i Z^T (C'A + D'C)((C'A + D'C)\Omega^{hg} + (C'B + D'D))^T Z = \\ =&\pi i Z^T ((C'A + D'C)\Omega^{hg} + (C'B + D'D))(C'A + D'C)^T Z. \end{aligned}$$

The exponential factor in $(f^g)^h$ consists of two ingredients:

$$\begin{aligned} h : \pi i Z^T C (C\Omega^g + D) Z &\mapsto \pi i Z^T (C'\Omega^h + D') C (C\Omega^{hg} + D)^T (C'\Omega^h + D')^T Z = \\ &= \pi i Z^T (C'(A\Omega^{hg} + B)(C\Omega^{hg} + D)^{-1} + D')(C\Omega^{hg} + D) C^T (C'\Omega^h + D')^T Z = \\ &= \pi i Z^T ((C'A + D'C)\Omega^{hg} + (C'B + D'D)) C^T (C'\Omega^h + D')^T Z, \end{aligned}$$

and

$$\pi i Z^T C' (C'\Omega^h + D')^T Z.$$

Combining these together and using the identity $(C\Omega^g + D)^T = (-\Omega C + A)^{-1}$ in various forms, matching the two exponential factors reduces to showing the following:

$$C^T + (-\Omega(C'A + D'C) + A'A + B'C)^T C' = (C'A + D'C)^T(-\Omega C' + A').$$

The last identity follows at once using $C'^T A' = A'^T C'$ and $D'^T A' - B'^T C' = I$. ☐

Next we define the modular transformed theta function. For $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ and $\Theta_K(Z,\Omega) = e^{\pi i K^T \Omega K} e^{2\pi i K^T Z}$, $K \in \mathbb{Z}^n$, we define $\Theta_K^g(Z,\Omega)$ according to the action of the above lemma. Explicitly,

$$\Theta_K^g(Z,\Omega) = \zeta \det(C\Omega^g + D)^{1/2} e^{\pi i Z^T C (C\Omega^g + D)^T Z} e^{\pi i K^T \Omega^g K + 2\pi i K^T (C\Omega^g + D)^T Z}.$$

Using direct computations in the spirit of Mumford's book [Mu], we discover the following important property of the functions $\Theta_K^g(Z,\Omega)$:

$$\Theta_K^g(Z + M + \Omega N, \Omega) = e^{-2\pi i N^T Z - \pi i N^T \Omega N} \Theta^g_{K + A^T N + C^T M}(Z,\Omega).$$

Rephrasing the above identity in the language of the Λ-action gives

$$(N, M) \cdot \Theta_K^g(Z,\Omega) = \Theta^g_{K + A^T N + C^T M}(Z,\Omega).$$

In particular, N_j^g and M_i^g act as follows:

$$M_i^g \cdot \Theta_K^g(Z,\Omega) = (-CM_i, AM_i) \cdot \Theta_K^g(Z,\Omega) = \Theta^g_{K + (-A^T C + C^T A) M_i}(Z,\Omega) = \Theta_K^g(Z,\Omega)$$
$$N_j^g \cdot \Theta_K^g(Z,\Omega) = (DN_j, -BN_j) \cdot \Theta_K^g(Z,\Omega) = \Theta^g_{K + (A^T D - C^T B) N_j}(Z,\Omega) = \Theta^g_{K + N_j}(Z,\Omega).$$

Repeating the construction of the cocycle from the previous section, we arrive at the following important observation:

Proposition 4.2. *If $\{N_1, \dots, N_n, M_1, \dots, M_n\}$ is split with respect to $\operatorname{Im}\Omega^g$, then*

$$c^g = \begin{cases} \sum_{K \in \Gamma_+} \Theta_K^g(Z,\Omega) & \text{in } \mathcal{C}^g_{1,2,\dots,k;\emptyset}, \\ 0 & \text{in all other terms} \end{cases}$$

is a cocycle in $\mathcal{C}^g$.

Note that if $\{N_1, \dots, N_n, M_1, \dots, M_n\}$ (allowing permutations) is not a split basis with respect to $\operatorname{Im}\Omega^g$, then the cohomology $H^k(\mathcal{C}^g)$ cannot be represented by a cocycle concentrated in just one place. To avoid dealing with such situations we will consider the cocycles in $\mathcal{C}^{[g]}$ for each left coset $[g] \in \Gamma_{1,2}/(Sl(n,\mathbb{Z}) \cap \Gamma_{1,2})$. The subgroup $Sl(n,\mathbb{Z}) \cap \Gamma_{1,2} = \{\left(\begin{smallmatrix} A & 0 \\ 0 & A^{T(-1)} \end{smallmatrix}\right)\} \subset \Gamma_{1,2}$ can be thought of as the stabilizer of the fixed Lagrangian splitting $\Lambda = \Lambda_1 \oplus \Lambda_2$. Hence, the left cosets can be identified with the set of such splittings. Given a splitting $\Lambda = \Lambda_1^{[g]} \oplus \Lambda_2^{[g]}$ we choose $g \in [g]$, such that the basis $\{N_1^g, \dots, N_n^g, M_1^g, \dots, M_n^g\}$ respects this splitting and $\{N_1, \dots, N_n, M_1, \dots, M_n\}$ is split with respect to $\operatorname{Im}\Omega^g$. Thus for any $[g]$ and a choice of a good representative g we get a cohomology class in $H^k(\Lambda, F)$ which is represented by the cocycle c^g in $\mathcal{C}^g$.

The next statement shows that this class is, in fact, independent of the choice of $g \in [g]$.

Proposition 4.3. *Let $g \in [\mathrm{id}]$ be such that $\{N_j, M_i\}$ is split with respect to $\operatorname{Im}\Omega^g$. Equivalently, the basis $\{N_j^g, M_i^g\}$ respects $\Lambda = \Lambda_1 \oplus \Lambda_2$ and is $\operatorname{Im}\Omega$-split. Then, c and $g^*(c^g)$ are homologous in $\mathcal{C}$.*

Proof. It is easy to see from linear algebra that the passage from $\{N_j, M_i\}$ to $\{N_j^g, M_i^g\}$ can be factored into a sequence of transformations of types (Ia), (Ib) and (Ic) such that on each intermediate step the basis remains $\operatorname{Im}\Omega$-split. More precisely, any positive primitive sublattice Γ_+^g can be transformed to Γ_+ in at most $(n-k)$ steps by applying the transformations of type (Ic) (with possible conjugation by types (Ia) and (Ib)). This, in turn, is a consequence of the following assertion: given a positive primitive sublattice Γ_+ and a positive vector r, there is a positive primitive sublattice Γ'_+ which contains r and $\dim(\Gamma'_+ \cap \Gamma'_+) = n-k-1$. Since the primitive integral sublattices form a dense subset in the real Grassmannian of positive $(n-k)$-subspaces, it is enough to show the last statement over $\mathbb{R}$. Then we can take $\Gamma'_+ = \langle r, \Gamma_+^r \rangle$, the subspace spanned by r and the $(n-k-1)$-dimensional part of Γ_+ orthogonal to r. Obviously, the quadratic form restricted to Γ'_+ is positive definite.

According to our previous calculations, the transformations of types (Ia) and (Ib) have no visible impact neither on the cocycle c nor on the complex $\mathcal{C}$. Hence, we are reduced to show the statement of the proposition for the transformation of type (Ic).

Recall that for g of the type (Ic) we have $N_j^g = N_j$ for all j, except for $N_{k+1}^g = N_{k+1} - N_k$. Let $\Gamma_+^g \subset \Lambda_1$ denote the positive sublattice of the basis $\{N_1^g, \ldots, N_n^g\}$. Next we consider the following entire function

$$f = \sum_{r\geq 0} \sum_{K \in \Gamma_+^g + rN_k} \Theta_K - \sum_{r \geq 0} \sum_{K\in \Gamma_+ + rN_k} \Theta_K. \tag{14}$$

Though each of the two sums are divergent, their difference is the convergent sum over the lattice points in between Γ_+ and Γ_+^g. On the picture below, showing the $\langle N_k, N_{k+1}\rangle$ slice of the lattice, those are represented by bold dots. The hollow dots represent terms taken with negative sign and $\operatorname{Im}\Omega$ is negative in the shaded area.

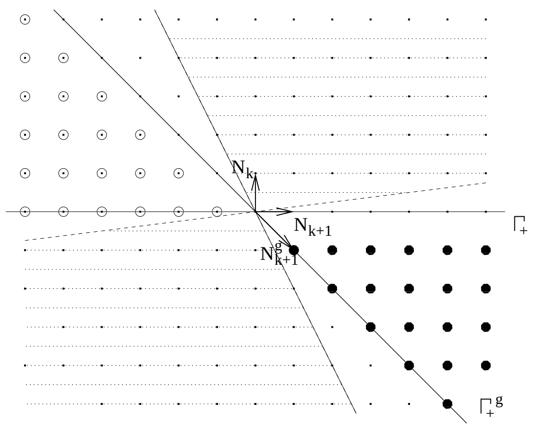

FIGURE 1. The lattice points in $\mathbb{Z}\langle N_k, N_{k+1}\rangle$ for the function f.

The cochain represented by this function in $\mathcal{C}_{1,2,\ldots,k-1;\emptyset}$ is a coboundary between c and $g^*(c^g)$. Indeed, all differential vanish when applied to f except for (N_k-1) and $(N_{k+1}-1)$:

$$(N_k-1)\cdot f = \sum_{K\in\Gamma_+}\Theta_K - \sum_{K\in\Gamma_+^g}\Theta_K \quad \text{in } \mathcal{C}_{1,2,\ldots,k;\emptyset},$$
$$(N_{k+1}-1)\cdot f = -\sum_{K\in\Gamma_+^g}\Theta_K \quad \text{in } \mathcal{C}_{1,2,\ldots,k-1,k+1;\emptyset}.$$

¿From the calculation of (Ic) we recall that $g^*(c^g)$ has components only in $\mathcal{C}_{1,2,\ldots,k;\emptyset}$ and $\mathcal{C}_{1,2,\ldots,k-1,k+1;\emptyset}$ both given by $\sum_{K\in\Gamma_+}\Theta_K^g = \sum_{K\in\Gamma_+^g}\Theta_K$. This completes the proof. □

Remark. The function f can be considered as a secondary object. In fact, it has a life of its own. Similar functions appeared, for instance, in the work of Göttsche and Zagier [GZ] in connection with the wall crossing phenomenon for Donaldson invariants. In general, the theta sum over the points in some simplicial k-cone can be thought of as an k-ary object. These "multi" theta functions appear in Fukaya's study of m_k products among the affine Lagrangian submanifolds in a symplectic $2n$-torus [Fu]. It seems interesting to explore this connection further.

The above Proposition and Lemma 4.1 allow us to define the action of the modular group on $H^k(\Lambda, F)$ via its action on $H^k(\mathcal{C}^{[g]})$. An element $g\in\Gamma_{1,2}$ acts on $H^k(\Lambda,F)$ as follows. Pick an element $h\in[g]$ such that $\{N_j, M_i\}$ is $\operatorname{Im}\Omega^h$-split. Then define

$$h:\ H^k(\mathcal{C}) \to H^k(\mathcal{C}^h)$$
$$h\cdot[c] \mapsto [c^h],$$

where we canonically identify $H^k(\mathcal{C})$ and $H^k(\mathcal{C}^h)$ with $H^k(\Lambda,F)$.

Finally, we are ready to prove the main result of this section.

Theorem 4.4. *The above action is trivial on $H^k(\Lambda,F)$.*

Proof. Since $\Gamma_{1,2}$ defines an action, we need to check the triviality for the set of generators only.

Case 1: $g=\left(\begin{smallmatrix} A^T & 0\\ 0 & A^{-1}\end{smallmatrix}\right)$. The action is defined exactly modulo the transformations of this type. Hence in this case the statement holds trivially.

Case 2: $g=\left(\begin{smallmatrix} I & B\\ 0 & I\end{smallmatrix}\right)$. Here we have $\Omega^g=\Omega-B$, hence $\operatorname{Im}\Omega=\operatorname{Im}\Omega^g$. Also the functions representing the cocycles c and c^g are identical: $\sum_{K\in\Gamma_+}\Theta_K(Z,\Omega)=\sum_{K\in\Gamma_+}\Theta_K^g(Z,\Omega)$. The matrix for the resolution identification is

$$S=(NM)\left(\begin{smallmatrix} I & 0\\ B^T & I\end{smallmatrix}\right)(NM)^{-1}=\left(\begin{smallmatrix} I & 0\\ MB^TN^{-1} & I\end{smallmatrix}\right).$$

Using the calculation for the transformation of type II we conclude that $g^*(c^g)=c$ already on the level of cocycles.

Case 3: $g=\left(\begin{smallmatrix} 0 & -I\\ I & 0\end{smallmatrix}\right)$. This seems to be the most interesting case. As for the classical theta functions the proof involves application of Fourier transform and Poisson summation formula. We will demonstrate the basic idea on the simple example of an 1-dimensional torus $X=\mathbb{C}/\tau\mathbb{Z}+\mathbb{Z}$, with $\operatorname{Im}\tau<0$ (X is not an elliptic curve since the polarization is negative). According to the matching map on resolutions of type III we have to show that two cocycles of the double complex $\mathcal{C}$

$$
\begin{array}{ccc} 0 & \xrightarrow{\mathbf{1}-1} & 0 \\ {\scriptstyle \boldsymbol{\tau}-1}\big\downarrow & & \big\downarrow \\ e^{\pi i\tau k^2+2\pi ikz} & \longrightarrow & 0 \end{array}
\quad \text{and} \quad
\begin{array}{ccc} 0 & \xrightarrow{\mathbf{1}-1} & \zeta\frac{1}{\sqrt{\tau}}e^{-\pi i(z+n)^2/\tau} \\ \big\downarrow{\scriptstyle \boldsymbol{\tau}-1} & & \big\downarrow \\ 0 & \longrightarrow & 0 \end{array}
\tag{15}
$$

are homologous in $H^1(\mathcal{C})$. To see this we define the entire function $f \in \mathcal{C}_{0;0}$ by

$$
f(z,\tau) = \int_{\operatorname{Re} y = k-\frac{1}{2}} \frac{e^{\pi i\tau y^2}e^{2\pi i(z+n)y}}{e^{2\pi iy}-1}dy. \tag{16}
$$

Then

$$
\begin{aligned}
(\mathbf{1}-1)\cdot f(z,\tau) &= f(z+1,\tau) - f(z,\tau) = \\
&= \int_{\operatorname{Re} y = k-\frac{1}{2}} \frac{e^{\pi i\tau y^2}e^{2\pi i(z+n)y}(e^{2\pi iy}-1)}{e^{2\pi iy}-1}\,dy = \zeta\frac{1}{\sqrt{\tau}}e^{-\pi i(z+n)^2/\tau}
\end{aligned} \tag{17}
$$

and

$$
\begin{aligned}
(\boldsymbol{\tau}-1)\cdot f(z,\tau) &= e^{\pi i\tau+2\pi iz}f(z+\tau,\tau) - f(z,\tau) = \\
&= \int_{\operatorname{Re} y = k-\frac{1}{2}} \frac{e^{\pi i\tau (y+1)^2}e^{2\pi i(z+n)(y+1)}}{e^{2\pi iy}-1}\,dy - \int_{\operatorname{Re} y = k-\frac{1}{2}} \frac{e^{\pi i\tau y^2}e^{2\pi i(z+n)y}}{e^{2\pi iy}-1}\,dy \\
&= -2\pi i \operatorname{Res}_{y=k}\Big(\frac{e^{\pi i\tau y^2}e^{2\pi i(z+n)y}}{e^{2\pi iy}-1}\Big) = -e^{\pi i\tau k^2+2\pi ikz}.
\end{aligned} \tag{18}
$$

The minus sign appearing here reflects the choice of orientation on the line $\operatorname{Re} y = k$. This ambiguity is suppressed by the choice of 8-th root of unity for the modular transform. With all that, we see that the difference of the above cocycles is the coboundary of the function $f(z,\tau)$.

To deal with the higher dimensional case, first, we need to make some comments on the choice of the original split basis $\{N_j, M_i\}$. Namely, using the transformation of the Case 1 above, we may assume without loss of generality that the reference basis is $\operatorname{Im}\Omega$-split, that is $\{N_j, M_i\}$ can be chosen to be the reference basis.

In such a basis the matrix S has the form $S = g^T = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$. So we can apply calculation of type III to conclude that $g^*(c^g)$ is represented by $\sum_{K\in\Gamma_+}\Theta_K^g$ in $\mathcal{C}_{\emptyset;1,2,\dots,k}$.

Next, we need to choose a totally real vector subspace in $\mathbb{C}^n = \mathbb{C}\langle N_1,\dots,N_n\rangle$ such that the imaginary part of Ω restricted to V is positive definite. By induction we may also assume that V has a codimension one filtration $V = V_0 \supset V_1 \supset V_2 \supset \cdots \supset V_k = \Gamma_+ \otimes \mathbb{R}$, such that $V_i \otimes \mathbb{C} = \mathbb{C}\langle N_{i+1},\dots,N_n\rangle$, $i = 1,\dots,k$. The convenient property of this filtration is that V_i can be cut out by i equations $\{N_j^T Y = 0, j = 1,\dots,i\}$.

Following the analogy with the example above we define the Fourier transform of the function $e^{\pi i Y^T\Omega Y}$ by using the partial Vick rotation:

$$
\hat{f}(X) = \int_V f(Y)e^{2\pi iY^TX}dY.
$$

Then one can use the usual theory of Fourier transform for positive definite $\operatorname{Im}\Omega$. In particular, the operators of translation and multiplication by the corresponding character interchange. Also, using the real coordinates on $V = A\cdot\mathbb{R}^n$ one has

$$\int_V e^{\pi i Y^T\Omega Y}dY = \int_{\mathbb{R}^n} e^{\pi i X^T(A^T\Omega A)X}\,(\det A)dX = \zeta(\det A)(\det(A^T\Omega A))^{-\frac{1}{2}} = \zeta(\det\Omega)^{-\frac{1}{2}}.$$

The same identity holds if the integration cycle V is shifted by any vector in $\mathbb{C}^n$.

We define a cochain f in $\mathcal{C}^{k-1}$ by the functions $f_i \in \mathcal{C}^{k-1}_{1,2,\dots,i-1;i+1,\dots,k}$, $1\le i\le k$:

$$f_i(Z) = \sum_{K\in\Gamma_+}\int_{V_{i-1}-\frac{1}{2}\sum_{j=i}^k N_j} \frac{e^{\pi i Y^T\Omega Y + 2\pi i Y^T(Z+K)}}{e^{2\pi i Y^T N_i}-1}\,d^{n-i+1}Y.$$

We want to show that the total differential applied to this cochain gives the difference between $g^*(c^g)$ and c in $\mathcal{C}$:

$$\begin{tikzcd}
 & & f_1 \arrow[r, "M_1-1"] \arrow[d, "N_1-1"] & \zeta(\det\Omega)^{-\frac{1}{2}}\sum_{K\in\Gamma_+} e^{-\pi i(Z+K^T)\Omega^{-1}(Z+K)} \\
 & \cdots \arrow[d, "N_{k-1}"] & \cdots & \\
f_k \arrow[r, "M_k-1"] \arrow[d, "N_k-1"] & 0 & & \\
-\sum_{K\in\Gamma_+} e^{\pi i K^T\Omega K + 2\pi i K^T Z} & & &
\end{tikzcd}$$

Since $f_i \in \mathcal{C}^{k-1}_{1,2,\dots,i-1;i+1,\dots,k}$, the differentials $(M_s-1)\cdot f_i$ vanish trivially for $s=i+1,\dots,k$. Also, $f_i(Z)$ is clearly periodic with respect to M_s, $s=k+1,\dots,n$. For the rest we have:

$$\begin{aligned}(M_s-1)\cdot f_i &= f_i(Z+M_s) - f_i(Z) = \\ &= \sum_{K\in\Gamma_+}\int_{V_{i-1}-\frac{1}{2}\sum_{j=i}^k N_j} \frac{e^{\pi i Y^T\Omega Y+2\pi i Y^T(Z+K)}(e^{2\pi i Y^T M_s}-1)}{e^{2\pi i Y^T N_i}-1}\,d^{n-i+1}Y.\end{aligned}$$

But $Y \in V_{i-1}+\frac{1}{2}\sum_{j=i}^k N_j$ means that $Y^T M_s = 0$ for $s<i$. Thus the only non-trivial d-differential is given by

$$(M_i-1)\cdot f_i = \sum_{K\in\Gamma_+}\int_{V_{i-1}-\frac{1}{2}\sum_{j=i}^k N_j} e^{\pi i Y^T\Omega Y+2\pi i Y^T(Z+K)}\,d^{n-i+1}Y.$$

In particular for $i = 1$ we have:

$$
\begin{aligned}
(M_1 - 1) \cdot f_1 = & \sum_{K \in \Gamma_+} e^{-\pi i (Z+K)^T \Omega^{-1} (Z+K)} \int_{V - \frac{1}{2} \sum_{j=1}^k N_j} e^{\pi i (Y + \Omega^{-1}(Z+K))^T \Omega (Y + \Omega^{-1}(Z+K))} d^n Y = \\
= & \sum_{K \in \Gamma_+} e^{-\pi i (Z+K^T) \Omega^{-1} (Z+K)} (\zeta (\det \Omega)^{-\frac{1}{2}}) = \sum_{K \in \Gamma_+} \Theta_K^g \quad \text{in } \mathcal{C}_{\emptyset; 1,2,\ldots,k}.
\end{aligned}
$$

Similarly, the differentials $(N_s - 1) \cdot f_i$ vanish for trivial reasons for $s = 1, \ldots, i-1$. On the other hand

$$
\begin{aligned}
N_s \cdot f_i = & e^{2\pi i N_s^T Z + \pi i N_s^T \Omega N_s} f_s (Z + \Omega N_s) = \\
= & \sum_{K \in \Gamma_+} \int_{V_{i-1} - \frac{1}{2} \sum_{j=i}^k N_j} \frac{e^{\pi i (Y+N_s)^T \Omega (Y+N_s) + 2\pi i (Y+N_s)^T (Z+K)}}{e^{2\pi i Y^T N_i} - 1} d^{n-i+1} Y \\
= & \sum_{K \in \Gamma_+} \int_{V_{i-1} + N_s - \frac{1}{2} \sum_{j=i}^k N_j} \frac{e^{\pi i Y^T \Omega Y + 2\pi i Y^T (Z+K)}}{e^{2\pi i Y^T N_i} - 1} d^{n-i+1} Y.
\end{aligned}
$$

For $s = k+1, \ldots, n$ the domain of integration remains the same, hence $N_s \cdot f_i = f_i$. Also, for $s = i+1, \ldots, k$, we have $Y^T N_i = -\frac{1}{2} + at \neq 0$, where a is a fixed complex number with non-zero imaginary part, and $t \in \mathbb{R}$. So the integrant is a holomorphic function in some open neighborhood of the subspace $\mathbb{R}\langle V_{i-1}, N_s \rangle$ in $\mathbb{C}^n$. Hence the only non-trivial δ-differential is given by

$$
\begin{aligned}
(N_i - 1) \cdot f_i = & \sum_{K \in \Gamma_+} \Big(\int_{V_{i-1} + \frac{1}{2} N_i - \frac{1}{2} \sum_{j=i+1}^k N_j} \frac{e^{\pi i Y^T \Omega Y + 2\pi i Y^T (Z+K)}}{e^{2\pi i Y^T N_i} - 1} d^{n-i+1} Y - \int_{V_{i-1} - \frac{1}{2} \sum_{j=i}^k N_j} \ldots \Big) \\
\overset{P.R.}{=} & - \sum_{K \in \Gamma_+} \int_{V_i - \frac{1}{2} \sum_{j=i+1}^k N_j} e^{\pi i Y^T \Omega Y + 2\pi i Y^T (Z+K)} d^{n-i} Y.
\end{aligned}
$$

The last equality is given by the Poincaré residue map evaluated on the corresponding cycles. Thus we see that $(N_s - 1) \cdot f_s + (M_{s-1} - 1) \cdot f_{s-1} = 0$. Finally, using Poisson summation formula for the function

$$
F(X) = e^{\pi i X^T \Omega X + 2\pi i X^T Z}, \text{ with } X_1 = \cdots = X_k = 0,
$$

of $(n-k)$ real variables and the lattice $\Gamma_+ \subset V_+^{\mathbb{R}}$, we deduce that

$$
\begin{aligned}
(N_k - 1) \cdot f_k = & - \sum_{K \in \Gamma_+} \int_{V_k} e^{\pi i Y^T \Omega Y + 2\pi i Y^T Z + 2\pi i Y^T K} d^{n-k} Y = \\
= & - \sum_{K \in \Gamma_+} e^{\pi i K^T \Omega K + 2\pi i K^T Z} = - \sum_{K \in \Gamma_+} \Theta_K \quad \text{in } \mathcal{C}_{1,2,\ldots,k;\emptyset}.
\end{aligned}
$$

This completes the proof of the theorem. ∎

5. The heat equation

The construction in the group cohomology $H^k(\Lambda, F)$ can be easily replaced by the one in Čech cohomology. Namely, one can use the covering of X by one open patch overlapping itself at the boundary of the fundamental domain. Then the condition of a k-cochain being a cocycle translates exactly into the vanishing of the differential in the group cohomology complex. One can also use étale cohomology for the étale cover $\mathbb{C}^n \to X$. One way or another we can identify the connection on the bundle $R^k\pi_*\mathcal{L}$ with the heat operator (7):

$$\mathcal{H} = \frac{\partial}{\partial\omega_{ij}} - \frac{1}{4\pi i}\frac{\partial^2}{\partial Z_i \partial Z_j},$$

acting on the elements of F, the entire functions on $\mathbb{C}^n$.

A priori, the connection is defined only projectively. But for the families parameterized by $\mathfrak{H}(k, n-k)$ there are no monodromy obstructions. Hence, the projective connection can be lifted to an ordinary one. The heat operator $\mathcal{H}$ defines this ordinary connection.

Clearly, $\mathcal{H}(\Theta_K) = 0$, hence the family of the classes $[c(\Omega)] \in H^k(\Lambda, F_\Omega)$ defines a horizontal section of $R^k\pi_*\mathcal{L}$ already on the level of cocycles. Moreover, a straight forward calculation shows that $\mathcal{H}(\Theta_K^g) = 0$, for any $g \in \Gamma_{1,2}$. So the classes $[c^g(\Omega)] \in H^k(\Lambda, F_\Omega)$ also provide a covariantly constant section.

¿From the point of view of projective connection an one-dimensional vector bundle is not very interesting. So one would like to generalize the construction to non-principal polarizations. This can be done by introducing theta forms with characteristics. In fact, every step in the two previous sections goes through with just minor modifications.

Let $X_\Omega = \mathbb{C}^n/\Omega\mathbb{Z}^n \oplus \Delta\mathbb{Z}^n$ be a complex torus with polarization of type $\Delta = \mathrm{diag}(\delta_1, \dots, \delta_n)$, with $\delta_1|\delta_2|\dots|\delta_n$ positive integers. The theta line bundle L_Ω is defined as before by its sheaf of sections. A section of L_Ω over an open subset $U \subset X$ is a holomorphic function $f(Z)$ on $p^{-1}(U)$ such that

$$f(Z + \Delta M + \Omega N) = e^{-2\pi i N^T Z - \pi i N^T \Omega N} f(Z), \text{ all } M, N \in \mathbb{Z}^n.$$

Again, $H^k(X_\Omega, L_\Omega)$ can be canonically identified with the group cohomology $H^k(\Lambda, F_\Omega)$, with the action of $\Lambda \simeq \mathbb{Z}^n \oplus \mathbb{Z}^n$ on F_Ω given by

$$(M, N) : f(Z) \mapsto e^{2\pi i N^T Z} e^{\pi i N^T \Omega N} f(Z + \Delta M + \Omega N).$$

Now $H^k(\Lambda, F_\Omega)$ is of dimension $\det \Delta = \delta_1\delta_2 \cdot \ldots \delta_n$, and it can be represented by the functions in $\mathcal{C}^k_{1,2,\dots,k;\emptyset}$:

$$\Theta[a](Z, \Omega) = \sum_{K \in \Gamma_+} e^{\pi i (K+a)^T \Omega (K+a) + 2\pi i (K+a)^T Z},$$

where a is an element in the lattice $\Delta^{-1}\mathbb{Z}^n$. Note that shifting of a by an element of $\mathbb{Z}^n$ changes the cocycle defined by $\Theta[a](Z, \Omega)$ at most by a coboundary. Hence, the cohomology class of $c[a] \in H^k(\Lambda, F_\Omega)$ is well defined for each $a \in \Delta^{-1}\mathbb{Z}^n/\mathbb{Z}^n$, and the collection of these form a basis.

Clearly, $\mathcal{H}(\Theta[a](Z, \Omega)) = 0$, hence each of the families $c[a](\Omega) \in H^k(\Lambda, F_\Omega)$ defines a horizontal section of $R^k\pi_*\mathcal{L}$ with respect to the flat connection given by the heat operator $\mathcal{H}$.

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